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Abstract

In repeated games with public monitoring, the consideration of behavior strategies makes relevant the distinction between public and private strategies. Recently, Kandori and Obara [6] and Mailath, Matthews and Sekiguchi [8] have provided examples of games with equilibrium payoffs in private strategies which lie outside the set of Public Perfect Equilibrium payoffs. The present paper focuses on another distinction, that between mixed and behavior strategies. It is shown that, as far as with mixed strategies one is concerned, the restriction to public strategies is not a restriction at all.

1 Introduction

Repeated Games with public monitoring are repeated games where, at the end of each stage, players do not observe each others' action, but only a public outcome, which is stochastically related to those. Abreu, Pearce and Stacchetti (henceforth, APS) [1] studied the set of pure-strategy (sequential) equilibrium payoff set of such games. Among other things, they showed that for any pure-strategy equilibrium payoff vector v, one can find a pure-strategy equilibrium profile σ which supports v and which is public in that strategies in σ depend on the publicly observed variable only. Hence, as far as with pure strategies one is concerned, the distinction between public and private strategies (strategies which, in addition, depend on the players' private information) does not matter.

The dynamic programming method of Abreu, Pearce and Stacchetti can be readily extended to study the equilibria associated to a special class of behavior

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strategies, namely those that depend on the publicly observed variable alone (for instance, Fudenberg, Levine and Maskin [4]). Such a set is typically referred to as the set of Public Perfect Equilibria (PPE). However, as soon as behavior strategies are introduced, the distinction between public and private strategies becomes relevant. In fact, Kandori and Obara [6] and Mailath, Matthews and Sekiguchi [8] have recently provided examples of games with equilibrium payoffs in private strategies which lie outside the set of PPE payoffs.

The present paper focuses on another distinction, that between mixed and behavior strategies, which appears to be relevant for the class of games under consideration. This point of view complements that of Kandori and Obara [6] and Mailath, Matthews and Sekiguchi [8] by showing that, as far as with mixed strategies one is concerned, the restriction to public strategies is not a restriction at all. Specifically, I show that if v is an equilibrium payoff vector supported by private behavior strategies, then there exists an equilibrium profile in public mixed strategies which supports v. The converse is true as well.

The main argument is based on Dalkey's theorem (Dalkey [3]), which states that in games with effectively perfect recall it is immaterial whether or not a player recalls his own past actions. Given a game with imperfect private monitoring satisfying the same assumptions as in Abreu, Pearce and Stacchetti [1], I consider a game that differs from the original one only in that players do not observe their own past actions. Then, after checking some measurability conditions, I use Dalkey's theorem to show that this game and the original one have the same equilibria in mixed strategies. Two corollaries and a couple of remarks complete the exposition.

The paper proceeds as follows. Section 2 describes the model. Section 3 contains the proof of the equivalence stated above. Section 4 concludes by observing that such equivalence does not extend beyond the model of section 2.

2 The Model

The model described below is essentially the same as in APS [1] with the assumption about the existence of a pure-strategy equilibrium for the stage game removed. When other assumptions are weakened, or additional interpretations proposed, a detailed discussion is given.

2.1 The Stage Game G

This is a finite-player game. Each player $i \in I$ has a finite set of actions, A_i; $a \in A = \times_i A_i$ denotes a profile of pure actions. The payoffs in the game depend on the action profile being played and on the realization $\omega \in \Omega$ of a state of the world. Such a state is randomly selected according to a measure $\mu(a)$ when profile a is played. I assume

R1 Ω is a compact subset of a Polish space. $(\Omega, \mathcal{B}_\Omega)$ is a measurable space, with $\mathcal{B}_\Omega$ referring to the Borel σ-field.

R2 The measures $\{\mu(a)\}_{a\in A}$ are mutually absolutely continuous, and *supp* $\mu(a) = \Omega$.

R2 implies that there exists a fixed measure μ such that $\mu(a) << \mu$, $\forall a \in A$.

Let $g_i(\omega, a)$ be the payoff that player i receives when a is played and ω realized. I assume also,

P1 $g_i(\cdot, a)$ is continuous in ω for every $a \in A$.

Note that R1-P1 imply bounded payoffs, that is $\exists c \in R$ such that $g_i(\cdot,\cdot) \in [-c, c]$.

To complete the description of the stage game, denote by m_i a mixed action for player i. The set of mixed profiles is $\Delta(A) = \times_i \Delta(A_i)$, and a generic element in $\Delta(A)$ is denoted by m. The family $\{\mu(a)\}_{a\in A}$ is extended to a family $\{\mu(m)\}_{m\in\Delta(A)}$ in the obvious way. Finally, when profile m is played, player i's expected payoff is $\int g_i(\omega, m) d\mu(m) = E[g_i(\omega, m)]$.

By Nash, G has an equilibrium (in mixed strategies).

Remark 1 *One can replace the assumption about the finiteness of the A_i's with A_i compact for each i and $g_i(\omega, \cdot)$ continuous in a, $\forall \omega \in \Omega$. Then, existence of equilibria is ensured by Glicksberg's fixed point theorem.*

2.2 The Repeated Game $G^\infty(\delta)$ and the Signalling Structure

I denote by $G^\infty(\delta)$ the game consisting of the infinite repetition of G, where all players discount their payoffs at the common discount factor $\delta \in [0, 1)$. The signalling structure in $G^\infty(\delta)$ is described by the following assumptions

S1 At the end of each stage t, $t \in N$, ω is realized and observed by all players.

S2 In each stage t, the distribution on Ω depends only on the (mixed) action profile played in that stage. In other words, transition probabilities are state-independent.

S3 At the end of each stage t, player i observes the payoff he achieved in that stage. However, as in APS, I assume that such a payoff depends on the actions a_{-i} of player i's opponents only indirectly through the effect that these actions have on the distributions on Ω.

The last assumption was introduced by APS in [1], and is recurrent in the applied work on games with public information. It imposes that payoffs, though observed, do not carry player i additional information other than that derived from the realization of ω and the knowledge of (the realization of) his own action. It might be worth noticing that this is a "global" assumption. Payoffs do not carry additional information both on and off the equilibrium path. Because of this, one can replace S3 by the following alternative assumption, which still preserves the same information structure, but does not impose any restriction on the technology of the payoff functions.

S3' Stage payoffs are unobservable.

Summarizing, the information that player i has at the beginning of stage t consists of a t-vector $\omega^t = (\omega_0, \omega_1, ..., \omega_{t-1})$ of realizations of the publicly observed state along with a t-vector $a_i^t = (a_{i,1}, ..., a_{i,t-1})$ of realizations of his own (mixed) actions.

2.3 Strategies

A pure-strategy for player i in $G^\infty(\delta)$ is a sequence of measurable maps $\sigma_i = \{\sigma_{i,t}\}_0^\infty$, with $\sigma_{i,t} : \Omega^t \times A_i^{t-1} \longrightarrow A_i$. A mixed-strategy is a distribution over pure strategies. A behavior-strategy for player i is a sequence of measurable maps $\sigma_i = \{\sigma_{i,t}\}_0^\infty$, with $\sigma_{i,t} : \Omega^t \times A_i^{t-1} \longrightarrow \Delta(A_i)$. Finally, a general strategies is a distribution over behavior strategies. $\Omega^t \times A_i^{t-1}$ is a measurable space with the product σ-algebra $\left(\overset{t}{\otimes}\mathcal{B}_{\Omega^k}\right) \otimes \left(\overset{t-1}{\otimes}\mathcal{P}(A_i)\right)$, and $\Delta(A_i)$ has its usual Borel structure. The set of player i's pure strategies (mixed, behavior, general) in $G^\infty(\delta)$ is denoted by Σ_i ($\mathcal{M}_i, \mathcal{B}_i, \mathcal{G}_i$, respectively). Here, I have adopted the convention of denoting by $\cdot_t$ the signal received after stage t is played. To ease the notation, I have defined $\sigma_{i,0}$ on ω_0 which is, clearly, arbitrary (similarly, APS [1]). Such a choice has, obviously, no consequences for the analysis.

We conclude the section with the following

Definition 2 *A strategy (pure or behavior) σ_i is said to be a public strategy if for any $\omega^t \in \Omega^t$ and any $t \in N$, $\sigma_{i,t}(\omega^t, a_i^t) = \sigma_{i,t}(\omega^t, \bar{a}_i^t)$ for any pair $(a_i^t, \bar{a}_i^t) \in A_i^t \times A_i^t$.*

3 Equivalence of Public Mixed-Strategies and Private Behavior-Strategies

The following definitions and theorem are well-known (see, for instance, Mertens, Sorin and Zamir [9]). They are included here for the ease of the reader.

Definition 3 *A game is said to have* ***(effectively) perfect recall*** *for player i if player i* ***(knowing the pure-strategy he is using)*** *can deduce from any signal he may get along some feasible play, the sequence of previous signals he got along that play. A game is said to have (effectively) perfect recall if it has (effectively) perfect recall for each player $i \in I$.*

For games with effectively perfect recall for player i, we have the following important theorem due to Dalkey [3].

Theorem 4 ***(Dalkey [3])*** *In a game with effectively perfect recall for player i, player i's pure-strategy set is the same (up to duplications) whether or not he recalls his own past moves.*

To prove the theorem ([9], p. 64), start by noticing that – as in Section 2.3 – a pure-strategy for player i has the form $a_n = \sigma(\omega_0, \omega_1, ..., \omega_{n-1}; a_1, ..., a_{n-1})$, $a_k \in A_i$ for each k. We want to define a new strategy, ζ, which does not depend on $(a_1, ..., a_{n-1})$. For each initial signal ω_0, define $a_1 = \sigma(\omega_0) = \zeta(\omega_0)$. Proceed by defining $a_2 = \sigma(\omega_0, \omega_1; a_1) = \zeta(\omega_0, \omega_1; \zeta(\omega_0))$. Then, inductively, ζ is defined. Finally, note that, whatever the other players' strategy, ζ generates the same probability distribution on plays as σ.

Now, let us return to our problem. Alongside with $G^{\infty}(\delta)$, let us introduce a new game, $G^{\infty}_{P}(\delta)$, which is such that players do not observe the realizations of their own mixed actions, and is otherwise identical to $G^{\infty}(\delta)$. Observe that, by construction, there are no private strategies in $G^{\infty}_{P}(\delta)$. Though patently artificial, this line of reasoning is useful to establish the desired equivalence by means of Dalkey's theorem. In order to use it in the our setting, one has only to show that, if σ_i is a pure-strategy in $G^{\infty}(\delta)$, the strategy ζ_i defined like in the above proof is indeed a sequence of measurable maps.

Lemma 5 ζ_i *is measurable*

Proof. ζ_i is defined by means of the following composition

$$\begin{tikzcd} \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1} \arrow[d, "i^t \otimes \left(\overset{t-1}{\otimes}[\zeta_k \circ \pi^k]\right)"'] \arrow[dr, "\zeta_{i,t}"] & \\ \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1} \times A_1 \times \cdots \times A_{t-1} \arrow[r, "\sigma_{i,t}"] & A \end{tikzcd}$$

where

i^t is the identity on $\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1}$

$\pi^k : \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1} \longrightarrow \Omega_0 \times \Omega_1 \times \cdots \times \Omega_k$ is the projection on the first $k+1$ factors

$\otimes$ denotes tensor products

Start by noticing that $(\zeta_{i,1}, ..., \zeta_{i,t-1})$ Borel $\Longrightarrow \zeta_{i,t}$ Borel [$\zeta_{i,k}$ is a map from $\left(\overset{k}{\times}\Omega_j, \overset{k}{\otimes}\mathcal{B}_{\Omega^j}\right) \longrightarrow (A, \mathcal{P}(A))$]. In fact, i^t and π^k are measurable (product σ-algebra). Hence, each $\zeta_{i,k} \circ \pi^k$ is Borel being the composition of Borel maps, and so is $\xi = i^t \otimes \left(\overset{t-1}{\otimes}[\zeta_{i,k} \circ \pi^k]\right)$ being a tensor product of measurable maps. Hence, $\zeta_{i,t} = \sigma_{i,t} \circ \xi$ is measurable. In particular, $\zeta_{i,t}$ is $\overset{t}{\times}\mu$–measurable. In fact, since $\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1}$ is Polish, every Borel subset of $\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1}$ is universally measurable.

Now, observe that $\zeta_{i,1}$ is trivially measurable since $\zeta_{i,1} = \sigma_{i,1}$. Hence, by induction, the proof is complete. ■

Let V_P be the equilibrium payoff set of $G_P^\infty(\delta)$, and let V be that of $G^\infty(\delta)$. Both sets are nonempty as the infinite repetition of the stage game equilibrium is an equilibrium (the same) in both cases. We have,

Theorem 6 *For the model described in Section 2,*

$$V = V_P$$

Proof. Start by observing that when defining pure-strategies in $G^\infty(\delta)$ as mappings $\sigma_{i,t} : \Omega^t \times A_i^t \longrightarrow A_i$, we have implicitly assumed both that a player does not forget the signals he has received and that the stage is known to him. Hence, $G^\infty(\delta)$ has effectively perfect recall.

Next, observe that the game obtained from $G^\infty(\delta)$ by assuming that players do not recall their past actions is identical to $G_P^\infty(\delta)$. From Dalkey's theorem it follows that $G^\infty(\delta)$ and $G_P^\infty(\delta)$ have the same (reduced) normal-form. Hence, they have the same equilibrium payoff set in mixed strategies.

To conclude, it suffices to observe that, since both games are linear (Isbell, [5]), they have the same equilibria in general strategies. ■

An immediate consequence of the result just found is

Corollary 7 *Every $v \in V$ can be obtained by public strategies.*

We also have,

Corollary 8 *Let $v \in V$ be supported by a private behavior-strategy profile, σ_{pr}. Then, there exists a public mixed-strategy equilibrium profile, σ_{pm}, which supports v. Conversely, if $v \in V$ is supported by a public mixed-strategy profile, then there exists a private behavior-strategy profile which supports it.*

Proof. Let σ_{pr} be an equilibrium profile in private behavior-strategies, and let $v \in V$ be the corresponding equilibrium payoff vector. Since the game is linear, there exists (Isbell, [5]) an equivalent (equilibrium) profile in private mixed strategies. Finally, by theorem 6, there exists an equivalent profile in public mixed strategies.

Conversely, given an equilibrium profile σ_{pm} in public mixed strategies, by theorem 6 there exists an equivalent profile in private mixed strategies. Hence, since $G^\infty(\delta)$ has perfect recall, by Kuhn-Aumann theorem ([7] and [2]) there exists an equivalent profile in private behavior-strategies. ■

It is worth noting an asymmetry in the latter proof. Given an equilibrium private behavior-strategy, in order to establish the existence of an equivalent public mixed-strategy, we only need (in addition to Dalkey's theorem) that the game be linear. The converse requires a stronger property, that is that the game have perfect recall. The reason for such an asymmetry can be seen by looking at our games. $G^\infty(\delta)$ has perfect recall, while $G_P^\infty(\delta)$ does not. In fact, for a game to have perfect recall, it is necessary that every message recall both the last message and the last action ([9], p. 64). Clearly, this does not occur in $G_P^\infty(\delta)$. Because of this, generally speaking (i.e., for arbitrary payoff functions and discount factors), the set of behavior-strategy equilibrium payoffs of $G_P^\infty(\delta)$ is a proper subset of V^P. Since such a set is exactly the set of PPE payoffs (because of assumption R2, Section 2.1), one can reinterpret the examples of Kandori and Obara [6] and Mailath, Matthews and Sekiguchi [8] as exploiting exactly this fact.

4 Conclusion

In this paper, we have shown the equivalence, from the viewpoint of the equilibrium payoffs, between public mixed strategies and private behavior strategies. In the proof, it was crucial for the use of Dalkey's theorem that players' information patterns in $G^\infty(\delta)$ and $G_P^\infty(\delta)$ differ only because of a player's knowledge of his own actions, a property that was delivered by either assumption **S3** or **S3'** of Section 3.

To see that the equivalence result does not extend if either assumption is removed, consider the following example[1]. Suppose that players are engaged in a repeated Bertrand competition, that $\mu(a) = \mu$, $\forall a \in A$ and that payoffs are independent of ω, but assume that payoffs are observable. Suppose also that the discount factor is sufficiently high so that the monopolistic outcome is an equilibrium outcome. Clearly, such an outcome emerges as an equilibrium only because, by observing his per-period payoff, a player can infer with certainty whether or not the other players conformed to the monopolistic outcome. In fact, it is supported by a profile where player i cooperates at $t+1$ if the other did so at t, and reverts to the static Nash equilibrium if cooperation did not take place at t. It is clear that such a profile cannot be replicated by any profile in public strategies, since that would violate the measurability condition of a player's strategy with respect to his information. In other words, for the model just described – where both **S3** and **S3'** are violated – the equivalence between $G^\infty(\delta)$ and $G_P^\infty(\delta)$ fails since we have at least one element in $G^\infty(\delta)$ that cannot be obtained in $G_P^\infty(\delta)$.

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