Excluding Induced Paths: Graph Structure and Coloring

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ABSTRACT

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An *induced subgraph* of a given graph is any graph which can be obtained by successively deleting vertices, possible none. In this thesis, we present several new structural and algorithmic results on a number of different classes of graphs which are closed under taking induced subgraphs.

The first result of this thesis is related to a conjecture of Hayward and Nastos [30] on the structure of graphs with no induced four-edge path or four-edge antipath. They conjectured that every such graph which is both prime and perfect is either a split graph or contains a certain useful arrangement of simplicial and antisimplicial vertices. We give a counterexample to their conjecture, and prove a slightly weaker version. This is joint work with Maria Chudnovsky, and first appeared in *Journal of Graph Theory* [7].

The second result of this thesis is a decomposition theorem for the class of all graphs with no induced four-edge path or four-edge antipath. We show that every such graph can be obtained from pentagons and split graphs by repeated application of complementation, substitution, and split graph unification. Split graph unification is a new graph operation we introduced, which is a generalization of substitution and involves "gluing" two graphs along a common induced split graph. This is a combination of joint work with Maria Chudnovsky and Irena Penev [8], together with later work of Louis Esperet, Laetitia Lemoine and Frederic Maffray [15], and first appeared in [6].

The third result of this thesis is related to the problem of determining the complexity of coloring graphs which do not contain some fixed induced subgraph. We show that three-

coloring graphs with no induced six-edge path or triangle can be done in polynomial-time. This is joint work with Maria Chudnovsky and Mingxian Zhong, and first appeared in [10]. Working together with Flavia Bonomo, Oliver Schaudt, and Maya Stein, we have since simplified and extended this result [3].

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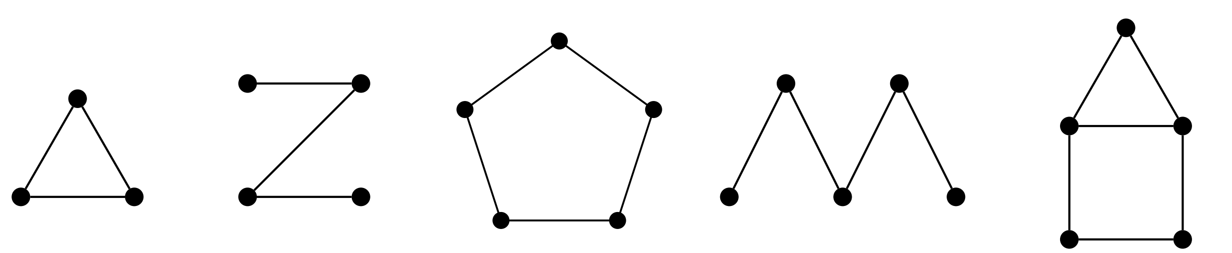
Chapter 1

Introduction

1.1 Perfect Graphs

All graphs in this thesis are finite and simple. Several small graphs are drawn below.



Triangle Three-edge path Pentagon Four-edge path Four-edge antipath

Before discussing the results of this thesis, we begin with some definitions and history. We say two vertices of a graph are *adjacent* if they are joined by an edge, and *non-adjacent* otherwise. A *clique* in a graph is a set of vertices all pairwise adjacent, and a *stable set* is a set of vertices all pairwise non-adjacent. The *clique number* of a graph is the size of the largest clique in the graph. The *complement* of a given graph is the graph with the same vertex set such that two vertices are adjacent if and only the same two vertices are non-adjacent in the original graph. An *induced subgraph* of a given graph is any graph which can be obtained by deleting vertices, possible none. It is important to note that this is a more restrictive notion of containment than that of a *subgraph*, where we are allowed to delete both edges and vertices. A general question in structural graph theory is to try and understand how forbidding a specific induced subgraph in a graph impacts its "global" structure.

The *chromatic number* of a graph is the minimum number of colors necessary to color the vertices of the graph so that no two adjacent vertices receive the same color. Since when coloring a graph every vertex in a clique must be colored differently, it follows that the chromatic number of a graph is always at least as large as its clique number. A graph is called *perfect* if for each of its induced subgraphs the chromatic number is equal to the clique number. Perfect graphs were introduced by Claude Berge [2] and are a central object of study in graph theory. An important initial result of Lovász [29], know as The Weak Perfect Graph Theorem, is the following:

1.1.1. *The complement of a perfect graph is also perfect.*

This result implies many famous min-max theorems in combinatorics, such as Dilworth's theorem for partially ordered sets and König's theorem. A graph is called *Berge* if neither it nor its complement contain an induced cycle with odd length at least five. Since any induced subgraph of a perfect graph is also perfect and every cycle of odd length at least five, such as the pentagon, is not perfect, it follows that any perfect graph must be Berge. The Strong Perfect Graph Theorem, conjectured in the 1960s by Berge and proven about a decade ago by Chudnovsky, Robertson, Seymour and Thomas [12], states that:

1.1.2. *A graph is perfect if and only if it is Berge.*

The crux of their proof was a more general result describing the structure of all Berge graphs. That is, they showed every Berge graph either belongs to one of a few well-understood families of basic graphs or admits a certain useful decomposition. This result is "one of the great achievements in discrete mathematics" [36] and, in many ways, is where the work of this thesis begins.

Independent of the proof of the Strong Perfect Graph Theorem, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [5] gave a polynomial-time algorithm to test if a given graph is Berge. Combined, these results imply it is possible to efficiently recognize perfect graphs. Since prior work of Fulkerson [18] had established many deep connections between perfect graphs and integer programming, this perfect graph recognition algorithm has many important computational implications. However, we still do not know how to explicitly "build" perfect graphs. Ideally, we would like to be able to say that a graph is Berge if and only if it can be constructed by piecing together graphs from some collection of basic building blocks in a way that preserves the property of being Berge. As a step in this direction, researchers have been exploring the structure and construction of certain classes of graphs closely related to Berge graphs.

One approach to studying the classes of graphs surrounding perfect graphs is to consider graphs which do not contain long induced paths. The simple reason being that if a graph does not contain long induced paths, then it will not contain long induced cycles. A graph is called a *cograph* if it does not contain the three-edge path as an induced subgraph. Cographs are an important subclass of perfect graphs with many interesting algorithmic properties as a result of their "nice" recursive structure. For much of this thesis, we consider the class $\mathcal{G}$ of all graphs which do not contain the four-edge path or its complement as induced subgraphs. Since the three-edge path is an induced subgraph of the four-edge path, it follows that $\mathcal{G}$ contains all cographs. However, $\mathcal{G}$ contains the pentagon, and so is not a subclass of perfect graphs.

Substitution is a graph operation that "replaces" a vertex v of a base graph G by another graph H in such a way that every vertex adjacent to v in G becomes adjacent to every vertex of H in the new graph, and every vertex non-adjacent to v in G becomes non-adjacent to every vertex of H. An important fact is that substituting one perfect graph for a vertex in another perfect graph yields a larger perfect graph [29]. A theorem of Fouquet [17] tells us that every graph in $\mathcal{G}$ can be obtained by substitution starting from pentagons and smaller perfect graphs contained in $\mathcal{G}$. That is, pentagons and perfect graphs are essentially the basic building blocks of all the graphs in $\mathcal{G}$. And so, we are interested in understanding the structure of those perfect graphs contained in $\mathcal{G}$.

A graph is called a *split graph* if its vertices can be partitioned into a clique and a stable set. Földes and Hammer [16], showed that split graphs are characterized by excluding cycles of length four and five, and their complements as induced subgraphs. From this characterization, it follows that split graphs are a subclass of perfect graphs properly contained in $\mathcal{G}$. Hayward and Nastos [30] conjectured that every perfect graph in $\mathcal{G}$ which cannot be obtained by substitution is either a split graph or contains certain "special" vertices. In [7] with Maria Chudnovsky, we gave a counterexample to their conjecture, and proved a slightly weaker version. Chapter 3 of this thesis is based on [7].

Later working with Maria Chudnovsky and Irena Penev [8], we were able to prove that a graph belongs to $\mathcal{G}$ if and only if it can be obtained from pentagons and split graphs by repeated application of complementation, substitution, and "split graph unification." *Split graph unification* is a new graph operation we introduced, and is a generalization of substitution and involves "gluing" two graphs along a common induced split graph. Essentially, by understanding the structure of those perfect graphs contained in $\mathcal{G}$, we obtained our "decomposition theorem" and saw how to construct such graphs. Recently, working with Louis Esperet, Laetitia Lemoine and Frederic Maffray [15], we were able to strengthen our result from [8]. Chapter 4 comes from merging [8] and [15], and first appeared in [6].

1.2 Coloring Graphs with Forbidden Induced Subgraphs

Efficiently coloring a graph is a fundamental and notoriously difficult algorithmic problem. The problem of determining the chromatic number of an arbitrary graph is one of the initial problems Karp [26] showed to be NP-complete. For a fixed integer $k \geq 1$, we say a graph is *k-colorable* if its chromatic number is at most k, and the algorithmic problem of determining if a given graph is k-colorable is referred to as the *k-COLORING problem.* Determining whether a graph is two-colorable can done in linear-time via breadth first search, however

Stockmeyer [35] showed that for any fixed integer $k \geq 3$, the k-COLORING problem is NP-complete. Let $k \geq 1$ be an integer. By definition, a perfect graph is k-colorable if and only if its clique number is at most k. And so, by simply checking if all subsets of $k+1$ vertices contain a non-adjacent pair of vertices, it follows that the k-COLORING problem restricted to the class of perfect graphs can be solved in polynomial-time. Gröschen, Lovász and Schrijver [20] proved, a much stronger and deeper result, that when restricted to the class of perfect graphs the chromatic number can be determined in polynomial-time. However, their algorithm makes use of the ellipsoid method for linear programming, and so there is great interest in developing a purely combinatorial algorithm to color perfect graphs.

In hopes of developing techniques with which to attack the more general problem of coloring perfect graphs, we focus on the restricted problem of determining the complexity of coloring graphs which do not contain some single fixed induced subgraph. Combining results of Hoyler, Leven and Galil [24; 28], with those of Kamiński and Lozin [31], it follows that unless the excluded induced subgraph is a disjoint union of paths, for any fixed integer $k \geq 3$ the k-COLORING problem remains NP-complete. Recently, much work has been done to try and determine the complexity of this problem when the excluded induced subgraph is a path. For k-coloring with $k \geq 4$, almost all cases have been settled, and the problem is usually NP-complete. The most interesting case is the problem of determining whether a graph without long induced paths is three-colorable. Working with Maria Chudnovsky and Mingxian Zhong [10; 11], we resolved the first open case for three-coloring. Specifically, we showed that three-coloring graphs with no induced six-edge path can be done in polynomial-time, a question which had been open for over ten years [19]. In [10], we consider the case that the input graph with no induced six-edge path does not contain a triangle. In this case, by 1.1.2, it follows that if the input graph is not perfect, then it must contain a cycle of length five or seven as an induced subgraph. This observation allows one to describe the structure of all such graphs, and then explicitly construct a coloring if one exists. Chapter 5 of this thesis is based on [10].

1.3 Outline

In this section, we provide a brief outline of this thesis.

- In chapter 2, we give several graph theory definitions that will appear throughout this thesis.
- In chapter 3, we prove the existence of certain useful arrangements of simplicial and antisimplicial vertices in graphs with no induced four-edge path or four-edge antipath. This is joint work with Maria Chudnovsky, and first appeared in *Journal of Graph Theory* [7].
- In chapter 4, we prove a decomposition theorem for graphs with no induced four-edge path or antipath. This is a combination of joint work with Maria Chudnovsky and Irena Penev [8], together with later work of Louis Esperet, Laetitia Lemoine and Frederic Maffray [15], and first appeared in [6].
- In chapter 5, we give a polynomial-time algorithm for three-coloring triangle-free graphs with no induced six-edge path. This is joint work with Maria Chudnovsky and Mingxian Zhong, and first appeared in [10].
- In chapter 6, we pose several open questions related to the work of this thesis.

Chapter 2

Definitions

In this chapter, we present several graph theory definitions that will appear throughout this thesis.

2.1 Graph Definitions

A *graph* G consists of a set $V(G)$, called the *vertex set of* G, together with a set of subsets $E(G) \subseteq V(G) \times V(G)$, called the *edge set of* G. We refer to the elements of $V(G)$ as *the vertices of* G, and the elements of $E(G)$ as *the edges of* G. A graph G is called *finite* provided that $V(G)$ is a finite set. A graph G is called *simple* provided that $\{v, v\} \notin E(G)$ for all $v \in V(G)$. All graphs in this thesis are finite and simple.

Let G be a graph. We say two vertices $x, y \in V(G)$ are *adjacent* if $\{x, y\} \in E(G)$, and *non-adjacent* otherwise. A *clique* in a graph is a set of vertices all pairwise adjacent, and a *stable set* is a set of vertices all pairwise non-adjacent. The *clique number* of G, denoted by $\omega(G)$, is the size of the largest clique in the graph. The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$, such that two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. The *neighborhood* of a vertex $v \in V(G)$ is the set of all vertices adjacent to v, and is denoted by $N(v)$. The *degree* of a vertex $v \in V(G)$ is $|N(v)|$, and is denoted *deg*(v). A vertex $v \in V(G)$ is called *simplicial* if $N(v)$ is a clique. A vertex $v \in V(G)$ is *antisimplicial* if $V(G) \setminus N(v)$ is a stable set, that is, if and only if v is a simplicial vertex in the complement.

Let G be a graph and X a subset of $V(G)$. We denote by $G[X]$ *the subgraph of* G *induced by* X, that is, the subgraph of G with vertex set X such that two vertices are adjacent in $G[X]$ if and only if they are adjacent in G. We denote by $G \setminus X$ the graph $G[V(G) \setminus X]$. If $X = \{v\}$ for some $v \in V(G)$, we write $G \setminus v$ instead of $G \setminus \{v\}$. Let H be a graph. If G has no induced subgraph isomorphic to H, then we say that G is H*-free*. For a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. If G is not H-free, then we say that G *contains* H. A *copy of* H *in* G is an induced subgraph of G isomorphic to H. If $G[X]$ is isomorphic to H, then we say that X *is an* H *in* G.

For $n \geq 0$, we denote by P_{n+1} *the path with* $n+1$ *vertices and* n *edges*, that is, the graph with distinct vertices $\{p_0, p_1, ..., p_n\}$ such that p_i is adjacent to p_j if and only if $|i-j| = 1$. We sometimes refer to the complement of a path as an *antipath*. For $n \geq 3$, we denote by C_n *the cycle of length* n, that is, the graph with distinct vertices $\{c_1, ..., c_n\}$ such that c_i is adjacent to c_j if and only if $|i-j| = 1$ or $n-1$. Let G be a graph. By convention, when explicitly describing a path or a cycle in G, we always list the vertices in order. That is, when $G[\{p_0, p_1, ..., p_n\}]$ is a copy of P_{n+1} in G, we say that $p_0 - p_1 - ... - p_n$ *is a* P_{n+1} *in* G. Similarly, when $G[\{c_1, c_2, ..., c_n\}]$ is a copy of C_n in G, we say that $c_1 - c_2 - ... - c_n - c_1$ *is a* C_n *in* G. For $n \geq 3$, we also refer to a copy of C_n in G as an *n-gon*. We sometimes also refer to a cycle of length three as a *triangle*, and a cycle of length five as a *pentagon*.

Let G be a graph, and let A and B be disjoint subsets of $V(G)$. For a vertex $b \in V(G) \setminus A$, we say that b *is complete to* A if b is adjacent to every vertex of A, and that b *is anticomplete to* A if b is non-adjacent to every vertex of A. If every vertex of A is complete to B, we say A *is complete to* B, and if every vertex of A is anticomplete to B, we say that A *is anticomplete to* B. If $b \in V(G) \setminus A$ is neither complete nor anticomplete to A, we say that b *is mixed on* A.

A *partition* of a set S is a collection of non-empty disjoint subsets of S whose union is S. Let G be a graph. We say G is *connected* if $V(G)$ cannot be partitioned into two sets anticomplete to each other. If $\overline{G}$ is connected, we say that G is *anticonnected*. Let X be a subset of $V(G)$. We say that X is *connected* if $G[X]$ is connected, and that X is *anticonnected* if $G[X]$ is anticonnected. A *component* of X is a maximal connected subset of X, and an *anticomponent* of X is a maximal anticonnected subset of X. Let Y be a subset set of X. We say Y is a *connected subset of* X if $G[Y]$ is connected, and that Y is an *anticonnected subset of* X if $G[Y]$ is anticonnected.

A *k-coloring* of a graph G is a mapping $c : V(G) \to \{1, ..., k\}$ such that if $x, y \in V(G)$ are adjacent, then $c(x) \neq c(y)$. If a k-coloring exists for a graph G, we say that the G is *k-colorable*. The *chromatic number* of a graph G is the smallest integer k for which G is k-colorable. A graph is called *perfect* if for each of its induced subgraphs the chromatic number is equal to the clique number. A graph is called *imperfect* if it is not perfect. A graph G is called *bipartite* if there is a partition $V(G) = A \cup B$ such that both A and B are stable sets. A graph G is a *split graph* if there is a partition $V(G) = A \cup B$ such that A is a stable set and B is a clique. Note that, both bipartite and split graphs are subclasses of perfect graphs.

A *homogeneous set* in a graph G is a subset X of $V(G)$ with $1 < |X| < |V(G)|$ such that no vertex of $V(G)\backslash X$ is mixed on X. We say that a graph is *prime* if it has at least four vertices, and no homogeneous set. We now define the *substitution* operation. Given graphs H_1 and H_2, on disjoint vertex sets, each with at least two vertices, and $v \in V(H_1)$, we say that H *is obtained from* H_1 *by substituting* H_2 *for* v, or *obtained from* H_1 *and* H_2 *by substitution* (when the details are not important) if:

- $V(H) = (V(H_1) \cup V(H_2)) \setminus \{v\}$,
- $H[V(H_2)] = H_2$,
- $H[V(H_1) \setminus \{v\}] = H_1[V(H_1) \setminus \{v\}]$, and
- $u \in V(H_1)$ is adjacent in H to $w \in V(H_2)$ if and only if w is adjacent to v in H_1.

By construction, it follows that a graph can be obtained via substitution from smaller graphs if and only if it is not prime. From this observation, it also follows that if H_1 and H_2 are

P-free graphs, where P is a prime graph, then any graph obtained from H_1 and H_2 by substitution is also P-free.

Chapter 3

Ups and Downs of the Four-Edge Path

Let $\mathcal{G}$ be the class of all graphs with no induced four-edge path or four-edge antipath. Hayward and Nastos [30] conjectured that every prime graph in $\mathcal{G}$ not isomorphic to the cycle of length five is either a split graph or contains a certain useful arrangement of simplicial and antisimplicial vertices. In this chapter, we give a counterexample to their conjecture, and prove a slightly weaker version. Additionally, applying a result of Chudnovsky and Seymour [13] we give a short proof of Fouquet's result [17] on the structure of the subclass of bull-free graphs contained in $\mathcal{G}$. This is joint work with Maria Chudnovsky, and first appeared in *Journal of Graph Theory* [7].

3.1 Introduction

A theorem of Fouquet [17] tells us that:

3.1.1. *Any $\{P_5, \overline{P_5}\}$-free graph that contains C_5 is either isomorphic to C_5 or has a homogeneous set.*

That is, C_5 is the unique prime $\{P_5, \overline{P_5}\}$-free graph that contains C_5, and so we concern ourselves with prime $\{P_5, \overline{P_5}, C_5\}$-free graphs, the main subject of this chapter.

In [30] Hayward and Nastos proved:

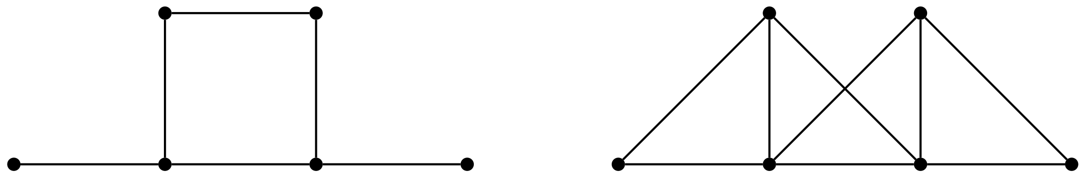


Figure 3.1: H_6 and $\overline{H_6}$.

3.1.2. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph, then there exists a copy of P_4 in G whose vertices of degree one are simplicial, and whose vertices of degree two are antisimplicial.*

Recall, that a graph G is a *split graph* if there is a partition $V(G) = A \cup B$ such that A is a stable set and B is a clique. Földes and Hammer [16] showed:

3.1.3. *A graph G is a split graphs if and only if G is a $\{C_4, \overline{C_4}, C_5\}$-free graph.*

Drawn in Figure 3.1 with its complement, H_6 is the graph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_6, v_5v_6\}$.

Hayward and Nastos conjectured the following:

3.1.4 (The H_6-Conjecture)**.** *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and whose two vertices of degree three are antisimplicial.*

First, in Figure 3.2 we provide a counterexample to 3.1.4. On the other hand, we prove the following slightly weaker version:

3.1.5. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is antisimplicial.*

We say that a graph G *admits a 1-join*, if $V(G)$ can be partitioned into four non-empty pairwise disjoint sets (A, B, C, D), where A is anticomplete to $C \cup D$, and B is complete to C and anticomplete to D. In trying to use 3.1.5 to improve upon 3.1.1 we conjectured the following:

3.1.6. *If G is a $\{P_5, \overline{P_5}\}$-free graph, then either*

- *G is isomorphic to C_5, or*
- *G is a split graph, or*
- *G has a homogeneous set, or*
- *G or $\overline{G}$ admits a 1-join.*

However, 3.1.6 does not hold, and we give a counterexample in Figure 3.3. Lastly, applying a result of the Chudnovsky and Seymour [13] we give a short proof of 3.1.1, and Fouquet's result [17] on the structure of $\{P_5, \overline{P_5}$,bull$\}$-free graphs.

This chapter is organized as follows. Section 3.2 contains results about the existence of simplicial and antisimplicial vertices in $\{P_5, \overline{P_5}\}$-free graphs. In Section 3.3 we give a counterexample to the H_6-conjecture 3.1.4, and prove 3.1.5, a slightly weaker version of the conjecture. We also give a simpler proof of 3.1.2, and provide a counterexample to 3.1.6. Finally, in Section 3.4 we give a new proof of 3.1.1, and a structure theorem for $\{P_5, \overline{P_5}$,bull$\}$-free graphs.

3.2 Simplicial and Antisimplicial Vertices in $\{P_5, \overline{P_5}, C_5\}$-Free Graphs

In this section we prove the following result:

3.2.1. *All prime $\{P_5, \overline{P_5}, C_5\}$-free graphs have both a simplicial vertex, and an antisimplicial vertex.*

Along the way we establish 3.2.9, a result which is helpful in finding simplicial and antisimplicial vertices in prime $\{P_5, \overline{P_5}\}$-free graphs.

First, we make the following three easy observations:

3.2.2. *If G is a prime graph, then G is connected and anticonnected.*

Proof. Passing to the complement if necessary, we may suppose G is not connected. Since G has at least four vertices, there exists a component C of $V(G)$ such that $|V(G) \setminus C| \geq 2$. However, then $V(G) \setminus C$ is a homogeneous set, a contradiction. This proves 3.2.2. □

We say a vertex $v \in V(G) \setminus X$ *is mixed on an edge of* X, if there exist adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$. Similarly, a vertex $v \in V(G) \setminus X$ *is mixed on a non-edge of* X, if there exist non-adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$.

3.2.3. *Let G be a graph, $X \subseteq V(G)$, and suppose $v \in V(G) \setminus X$ is mixed on X.*

1. *If X is a connected subset of $V(G)$, then v is mixed on an edge of X.*

2. *If X is an anticonnected subset of $V(G)$, then v is mixed on a non-edge of X.*

Proof. Suppose X is a connected subset of $V(G)$. Since v is mixed on X, both $X \cap N(v)$ and $X \setminus N(v)$ are non-empty. As $G[X]$ is connected, there exists an edge given by $x \in X \cap N(v)$ and $y \in X \setminus N(v)$, and v is mixed on $\{x, y\}$. This proves 3.2.3.1. Passing to the complement, we get 3.2.3.2. This proves 3.2.3.

□

3.2.4. *Let G be a graph, $X_1, X_2 \subseteq V(G)$ with $X_1 \cap X_2 = \emptyset$, and $v \in V(G) \setminus (X_1 \cup X_2)$.*

1. *If G is P_5-free, and X_1, X_2 are connected subsets of $V(G)$ anticomplete to each other, then v is not mixed on both X_1 and X_2.*

2. *If G is $\overline{P_5}$-free, and X_1, X_2 are anticonnected subsets of $V(G)$ complete to each other, then v is not mixed on both X_1 and X_2.*

Proof. Suppose G is P_5-free, X_1, X_2 are disjoint connected subsets of $V(G)$ anticomplete to each other, and v is mixed on both X_1 and X_2. By 3.2.3.1, v is mixed on an edge of X_1, given by say $x_1, y_1 \in X_1$ with v adjacent to x_1 and non-adjacent to y_1, and an edge of X_2, given by say $x_2, y_2 \in X_2$ with v adjacent to x_2 and non-adjacent to y_2. However, then

$y_1 - x_1 - v - x_2 - y_2$ is a P_5 in G, a contradiction. This proves 3.2.4.1. Passing to the complement, we get 3.2.4.2. This proves 3.2.4.

□

As a consequence of 3.2.3 and 3.2.4 we obtain the following two useful results:

3.2.5. *Let u and v be non-adjacent vertices in a $\overline{P_5}$-free graph G, and let A be an anticonnected subset of $N(u) \cap N(v)$. Then no vertex $w \in V(G) \setminus (A \cup \{u, v\})$ can be mixed on both A and $\{u, v\}$.*

Proof. Since A and $\{u, v\}$ are disjoint anticonnected subsets of $V(G)$ complete to each other, 3.2.5 follows from 3.2.4.2.

□

3.2.6. *Let u, v and w be three pairwise non-adjacent vertices in a $\{P_5, \overline{P_5}\}$-free graph G such that w is mixed on an anticonnected subset A of $N(u) \cap N(v)$. Then no vertex $z \in N(w) \setminus (A \cup \{u, v\})$ can be mixed on $\{u, v\}$.*

Proof. Suppose there exists a vertex $z \in N(w) \setminus (A \cup \{u, v\})$ which is mixed on $\{u, v\}$, with say z adjacent to v and non-adjacent to u. Since w is mixed on A, by 3.2.3.2, it follows that w is mixed on a non-edge of A, given by say $x, y \in A$ with w adjacent to x and non-adjacent to y. By 3.2.5, z is not mixed on A. However, if z is anticomplete to A, then $y - u - x - w - z$ is a P_5 in G, and if z is complete to A, then $\{x, y, w, u, z\}$ is a $\overline{P_5}$ in G, in both cases a contradiction. This proves 3.2.6.

□

Now, we can start to prove 3.2.1.

3.2.7. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Then G has an antisimplicial vertex, or admits a 1-join.*

Proof. Suppose G does not admit a 1-join. Let W be a maximal subset of vertices that has a partition $A_1 \cup ... \cup A_k$ with $k \geq 2$ such that:

- $A_1, ..., A_k$ are all anticonnected subsets of $V(G)$, and
- $A_1, ..., A_k$ are pairwise complete to each other.

(1) $V(G) \setminus W$ *is non-empty.*

Proof: By 3.2.2, G is anticonnected, which implies that $V(G) \setminus W$ is non-empty. This proves *(1)*.

(2) Every $v \in V(G) \setminus W$ *is either anticomplete to or mixed on* A_i *for each* $i \in \{1, ..., k\}$.

Proof: Suppose $v \in V(G) \setminus W$ is complete to some A_i. Take B to be the union of all the A_j to which v is complete. However, since $\{v\} \cup W \backslash B$ is anticonnected and complete to B, it follows that $W' = B \cup (\{v\} \cup W \backslash B)$ contradicts the maximality of W. This prove *(2)*.

(3) If for some $i \in \{1, ..., k\}$, $v \in V(G) \setminus W$ *is mixed on* A_i*, then* v *is anticomplete to* $W \backslash A_i$.

Proof: By 3.2.4.2, any $v \in V(G) \setminus W$ is mixed on at most one A_i, and so together with *(2)* this proves *(3)*.

(4) Every vertex in $V(G) \setminus W$ *is mixed on exactly one* A_i*, for some* $i \in \{1, ..., k\}$.

Suppose not. Let $X \subseteq V(G) \setminus W$ be the set of vertices anticomplete to W, which is non-empty by *(2)* and *(3)*. By 3.2.2, G is connected, and so there exists an edge given by $v \in X$ and $u \in V(G) \setminus (X \cup W)$. By *(2)*, u is mixed on some A_i, and so, by 3.2.3.2, u is mixed on a non-edge of A_i, given by say $x_i, y_i \in A_i$ with u adjacent to x_i and non-adjacent to y_i. However, by *(3)*, u is anticomplete to $W \setminus A_i$, and so for $j \neq i$ and a vertex $z \in A_j$ we get that $v - u - x_i - z - y_i$ is a P_5 in G, a contradiction. This proves *(4)*.

And so, by *(3)* and *(4)*, we can partition $V(G) = A_1 \cup ... \cup A_k \cup B_1 \cup ... \cup B_k$, where each B_i is the set of vertices mixed on A_i and anticomplete to $(A_1 \cup ... \cup A_k) \setminus A_i$.

(5) $B_1, ..., B_k$ *are pairwise anticomplete.*

Proof: Suppose for $i \neq j$, $b_i \in B_i$ is adjacent to $b_j \in B_j$. By 3.2.3.2, b_i is mixed on a non-edge of A_i, given by say $x_i, y_i \in A_i$ with b_i adjacent to x_i and non-adjacent to y_i. As b_j is mixed on A_j, there exists $x_j \in A_j$ non-adjacent to b_j, however then $b_j - b_i - x_i - x_j - y_i$ is a P_5 in G, a contradiction. This proves *(5)*.

(6) Exactly one B_i is non-empty.

Proof: By *(1)* and *(4)*, at least one B_i is non-empty. Suppose for $i \neq j$, B_i and B_j are both non-empty. Then, by *(5)*, $A = B_i$, $B = A_i$, $C = (A_1 \cup ... \cup A_k) \backslash A_i$ and $D = (B_1 \cup ... \cup B_k) \backslash B_i$ is a 1-join, a contradiction. This proves *(6)*.

Hence, by *(6)*, we may assume B_1 is non-empty while $B_2, \ldots, B_k$ are all empty.

(7) $k = 2$ and $|A_2| = 1$.

Proof: Since $A_2 \cup ... \cup A_k$ is not a homogeneous set, *(6)* implies that $k = 2$ and $|A_2| = 1$. This proves *(7)*.

Let a be the vertex in A_2.

(8) B_1 is a stable set.

Proof: Suppose not. Then there exists a component B of B_1 with $|B| > 1$. Since a is anticomplete to B_1, and B is a component of B_1, as G is prime, it follows that there exist $a_1 \in A_1$ which is mixed on B. Thus, by 3.2.3.1, a_1 is mixed on an edge of B, given by say $b, b' \in B$ with a_1 adjacent to b and non-adjacent to b'. Next, partition $A_1 = C \cup D$ with $C = A_1 \cap (N(b) \setminus N(b'))$ and $D = A_1 \setminus C$, where both C and D are non-empty, as $a_1 \in C$ and b' is mixed on A_1. Since A_1 is anticonnected there exists a non-edge given by $c \in C$ and $d \in D$. However, since $d \in D$, it follows that $\{d, a, c, b, b'\}$ is either a $P_5, \overline{P_5}$ or C_5 in G, a contradiction. This proves *(8)*.

Thus, by *(8)*, a is an antisimplicial vertex. This proves 3.2.7.

□

Next, we observe:

3.2.8. *Let u and v be non-adjacent vertices in a prime $\overline{P_5}$-free graph G. Then either*

- *$N(u) \cap N(v)$ is a clique, or*
- *there exists a vertex $w \in V(G) \setminus (N(u) \cup N(v) \cup \{u, v\})$ which is mixed on an anticonnected subset of $N(u) \cap N(v)$.*

Proof. Suppose $N(u) \cap N(v)$ is a not clique. Then there exists an anticomponent A of $N(u) \cap N(v)$ with $|A| > 1$. Since $\{u, v\}$ is complete to $N(u) \cap N(v)$, and A is a anticomponent of $N(u) \cap N(v)$, as G is prime, it follows that there exists $w \in V(G) \setminus ((N(u) \cap N(v)) \cup \{u, v\})$ which is mixed on A. Thus, by 3.2.5, w is not mixed on $\{u, v\}$, and so w is anticomplete to $\{u, v\}$. This proves 3.2.8.

□

A useful consequence of 3.2.8 is the following:

3.2.9. *Let v be a vertex in a prime $\{P_5, \overline{P_5}\}$-free graph G.*

1. *If v is antisimplicial, and we choose u non-adjacent to v such that $|N(u) \cap N(v)|$ is minimum, then u is a simplicial vertex.*
2. *If v is simplicial, and we choose u adjacent to v such that $|N(u) \cup N(v)|$ is maximum, then u is an antisimplicial vertex.*

Proof. Suppose v is antisimplicial, we choose u non-adjacent to v such that $|N(u) \cap N(v)|$ is minimum, and u is not simplicial. Since v is antisimplicial, it follows that $N(u) \setminus N(v)$ is empty, and thus, as u is not simplicial, $N(u) \cap N(v)$ is not a clique. Hence, by 3.2.8, there exists some w, non-adjacent to both u and v, which is mixed on an anticonnected subset of $N(u) \cap N(v)$. However, then, by our choice of u, there exists a vertex $z \in N(v) \setminus N(u)$ adjacent to w, contradicting 3.2.6. This proves 3.2.9.1. Passing to the complement, we get 3.2.9.2. This proves 3.2.9.

□

3.2.10. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Then G has a simplicial vertex, or an antisimplicial vertex.*

Proof. Suppose G does not have an antisimpicial vertex. Then, by 3.2.7, it admits a 1-join (A, B, C, D).

(1) A and D are stable sets.

Proof: By symmetry, it suffices to argue that A is a stable set. Suppose not. Then there exists a component A' of A with $|A'| > 1$. Since $C \cup D$ is anticomplete to A, and A' is a component of A, as G is prime, it follows that there exists $b \in B$ which is mixed on A'. Thus, by 3.2.3.1, b is mixed on an edge of A', given by say $a, a' \in A'$ with b adjacent to a' and non-adjacent to a. By 3.2.2, G is connected, and so there exists an edge given by $c \in C$ and $d \in D$. However, then $a - a' - b - c - d$ is a P_5 in G, a contradiction. This proves *(1)*.

Next, fix some $c \in C$, and choose a vertex $a \in A$ such that $|N(a) \cap N(c)|$ is minimum.

(2) a is a simplicial vertex.

Proof: Suppose not. Then, by *(1)*, $N(a) \cap N(c) = N(a) \subseteq B$ is not a clique, and so, by 3.2.8, there exists w, non-adjacent to both a and c, which is mixed on an anticonnected subset of $N(a) \cap N(c)$. Since B is complete to C and anticomplete to D, it follows that w belongs to A. However, then, by our choice of a, there exists a vertex $z \in N(c) \backslash N(a)$ adjacent to w, contradicting 3.2.6. This proves *(2)*.

This completes the proof of 3.2.10. □

Putting things together we can now prove 3.2.1.

Proof of 3.2.1. By 3.2.10, passing to the complement if necessary, there exists an antisimplicial vertex a. And so, by 3.2.9.1, if we choose s non-adjacent to a such that $|N(a) \cap N(s)|$ is minimum, then s is simplicial. This proves 3.2.1. □

3.3 The H_6-Conjecture

In this section we give a counterexample to the H_6-conjecture 3.1.4, and prove 3.1.5, a slightly weaker version of the conjecture. We also give a proof of 3.1.2, and provide a counterexample to 3.1.6.

We begin by establishing some properties of prime graphs. Recall the following theorem of Seinsche [33]:

3.3.1. *If G is a P_4-free graph with at least two vertices, then G is either not connected or not anticonnected.*

Together, 3.2.2 and 3.3.1 imply the following:

3.3.2. *Every prime graph contains P_4.*

Next, as first shown by Hoàng and Khouzam [22], we observe that:

3.3.3. *Let G be a prime graph.*

1. *A vertex $v \in V(G)$ is simplicial if and only if v is a degree one vertex in every copy of P_4 in G containing it.*

2. *A vertex $v \in V(G)$ is antisimplicial if and only if v is a degree two vertex in every copy of P_4 in G containing it.*

Proof. Both forward implications are clear. To prove the converse of 3.3.3.1, suppose there exists a vertex v which is not simplicial and yet is a degree one vertex in every copy of P_4 in G containing it. Then there exists an anticomponent A of $N(v)$ with $|A| > 1$. Since v is complete to A, and A is a anticomponent of $N(v)$, as G is prime, it follows that there exists $u \in V(G) \setminus (N(v) \cup \{v\})$ which is mixed on A. Thus, by 3.2.3.2, u is mixed on a non-edge of A, given by say $x, y \in A$ with u adjacent to x and non-adjacent to y. However, then $y - v - x - u$ is a P_4 in G with v having degree two, a contradiction. This proves 3.3.3.1. Passing to the complement, we get 3.3.3.2. This proves 3.3.3.

□

Finally, we observe that:

3.3.4. *Let G be a prime graph.*

1. *The set of antisimplicial vertices in G is a clique.*

2. *The set of simplicial vertices in G is a stable set.*

Proof. Suppose there exist non-adjacent antisimplicial vertices $a, a' \in V(G)$. Since a is antisimplicial, it follows that $N(a') \setminus N(a)$ is empty. Similarly, $N(a) \setminus N(a')$ is also empty. However, this implies that $\{a, a'\}$ is a homogeneous set in G, a contradiction. This proves 3.3.4.1. Passing to the complement, we get 3.3.4.2. This proves 3.3.4.

□

3.3.5. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Let A be the set of antisimplicial vertices in G, and let S be the set of simplicial vertices in G. Then $G[A \cup S]$ is a split graph which is both connected and anticonnected.*

Proof. 3.3.4 implies that $G[A \cup S]$ is a split graph, where A is a clique and S is a stable set. By 3.2.9.1, every vertex in A has a non-neighbor in S, and, by 3.2.9.2, every vertex in S has a neighbor in A. Thus, $G[A \cup S]$ is both connected and anticonnected. This proves 3.3.5.

□

We are finally ready to give a proof of 3.1.2, first shown in [30] by Hayward and Nastos.

3.3.6. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph, then there exists a copy of P_4 in G whose vertices of degree one are simplicial, and whose vertices of degree two are antisimplicial.*

Proof. Let A be the set of antisimplicial vertices in G, and let S be the set of simplicial vertices in G. By 3.2.1, both A and S are non-empty. Hence, $G[A \cup S]$ is a graph with at least two vertices, which, by 3.3.5, is both connected and anticonnected, and so, by 3.3.1, it follows that $G[A \cup S]$ contains P_4. Since 3.3.4 implies that A is a clique and S is a stable set, it follows that every copy of P_4 in $G[A \cup S]$ is of the desired form. This proves 3.3.6.

□

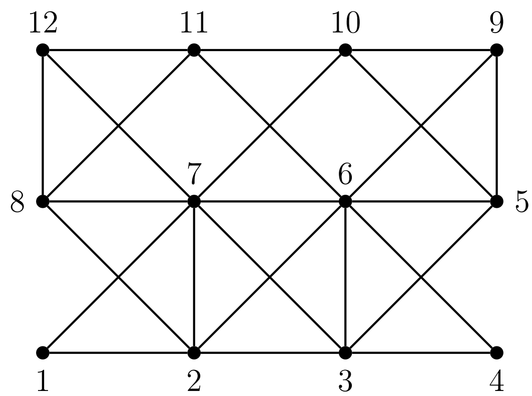


Figure 3.2: Counterexample to the H_6-conjecture, where additionally $\{2, 3\}$ is complete to $\{9, 10, 11, 12\}$.

Next, we turn our attention to the H_6-conjecture. A result of Hoàng and Reed [23] implies the following:

3.3.7. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then G or $\overline{G}$ contains H_6.*

In hopes of saying more along these lines, motivated by 3.3.6 and 3.3.7, Hayward and Nastos posed 3.1.4, which we restate:

3.3.8 (The H_6-Conjecture)**.** *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and whose two vertices of degree three are antisimplicial.*

In Figure 3.2 we give a counterexample to 3.3.8. The graph G in Figure 3.2 contains C_4, and so, by 3.1.3, is not split. The mapping $\phi : V(G) \to V(\overline{G})$

$$\phi := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 1 & 4 & 2 & 7 & 5 & 8 & 6 & 11 & 9 & 12 & 10 \end{pmatrix}$$

is an isomorphism between G and $\overline{G}$. Thus, as G is self-complementary, it suffices to check that G is P_5-free, which is straight forward, as is verifying that G is prime, and we leave the details to the reader. The set of simplicial vertices in G is $\{1, 4\}$, and the set of antisimplicial vertices in G is $\{2,3\}$. However, no copy of C_4 in G contains $\{2,3\}$, and so there does not exist a copy of H_6 of the desired form.

However, all is not lost as we can prove 3.1.5, a slightly weaker version of the H_6-conjecture, which we restate:

3.3.9. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is antisimplicial.*

Proof. By 3.3.6, there exist simplicial vertices s, s', and antisimplicial vertices a, a' such that $s - a - a' - s'$ is a P_4 in G. Now, choose maximal subsets A of antisimplicial vertices in G, and S of simplicial vertices in G such that $a, a' \in A$, $s, s' \in S$, every vertex in A has a neighbor in S, and every vertex in S has a non-neighbor in A.

(1) Any graph containing a vertex which is both simplicial and antisimplicial is split.

Proof: By definition, if a vertex $v \in V(G)$ is both simplicial and antisimplicial, then $N(v)$ is a clique and $V(G) \setminus N(v)$ is a stable set. This proves *(1)*.

(2) There exists no vertex $v \in V(G) \setminus (A \cup S)$ adjacent to a vertex $u \in S$ and non-adjacent to a vertex $w \in A$.

Proof: Suppose not. If u is adjacent to w, then $N(u)$ is not a clique, and if u is non-adjacent to w, then $V(G) \setminus N(w)$ is not a stable set, in both cases a contradiction. This proves *(2)*.

By *(1)* and *(2)*, we can partition $V(G) = A \cup S \cup B \cup C \cup D$, where B is the set of vertices complete to A and anticomplete to S, C is the set of vertices complete to A with a neighbor in S, and D is the set of vertices anticomplete to S with a non-neighbor in A. Recall 3.3.4 implies that A is a clique and S is a stable set.

(3) No vertex of $C \cup D$ is simplicial or antisimplicial.

Proof: Consider a vertex $c \in C$. Then there exists $s_c \in S$ adjacent to c. Hence, c is not antisimplicial, as otherwise we could add c to A contrary to maximality. By construction,

s_c has a non-neighbor $a_c \in A$. Since c is complete to the A, it follows that $N(c)$ is not a clique, and thus c is not simplicial. Hence, C contains no simplicial or antisimplicial vertices. Passing to the complement, we get that no vertex in D is simplicial or antisimplicial. This proves *(3)*.

(4) We may assume that C is a clique, and D is a stable set.

Proof: By symmetry, it is enough to argue that if D is not a stable set, then the theorem holds. Suppose we have an edge given by $x, y \in D$. By definition, any antisimplicial vertex is adjacent to at least one of x and y. And so, as x and y both have non-neighbors in A, there exists $a_x, a_y \in A$ such that a_x is adjacent to x and non-adjacent to y, and a_y is adjacent to y and non-adjacent to x. Since S is anticomplete to D, it follows that a_x and a_y do not have a common neighbor $s'' \in S$, as otherwise $\{a_x, y, s'', x, a_y\}$ is a $\overline{P_5}$ in G, a contradiction. By construction, every vertex in A has a neighbor in S, and so there exists $s_x \in S$ adjacent to a_x and non-adjacent to a_y, and $s_y \in S$ adjacent to a_y and non-adjacent to a_x. However, then $\{s_x, a_x, a_y, s_y, x, y\}$ is a copy of H_6 in G of the desired form. Passing to the complement, we may also assume that C is a clique. This proves *(4)*.

(5) For all $d \in D$ and $u \in A$, $N(d) \subseteq N(u) \cup \{u\}$.

Proof: By *(4)*, $A \cup C$ is a clique and $D \cup S$ is a stable set. Thus, for any $d \in D$, it follows that $N(d) \subseteq A \cup B \cup C$. Since A is complete to B, it follows that any $a \in A$ is complete to $(A \setminus \{a\}) \cup B \cup C$. This proves *(5)*.

(6) We may assume both C and D are empty.

Proof: By symmetry, it is enough to argue that if D is non-empty, then the theorem holds. Suppose D is non-empty, and choose $d \in D$ with $|N(d)|$ minimum. Then there exists $a_d \in A$ non-adjacent to d. By *(3)* and *(5)*, $N(a_d) \cap N(d) = N(d)$ is not a clique, and so, by 3.2.8, there exists a vertex w, non-adjacent to both a_d and d, which is mixed on an anticonnected subset of $N(d)$. Since a_d is complete to $(A \setminus \{a_d\}) \cup B \cup C$, it follows that $w \in D \cup S$. If

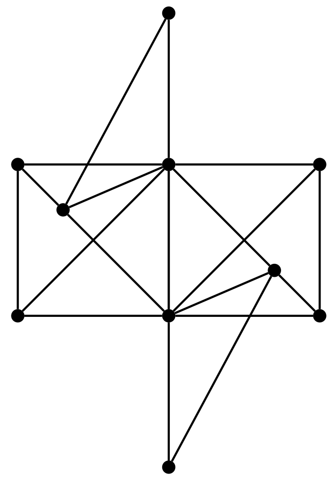


Figure 3.3: Counterexample to Conjecture 3.3.10.

$w \in D$, then, by our choice of d, there exists $z \in N(w) \setminus N(d)$ which, by *(5)*, is adjacent to a_d, contradicting 3.2.6. Hence, $w \in S$. Since w is mixed on an anticonnected subset of $N(d)$, by 3.2.3.2, w is mixed on a non-edge of $N(d)$, given by say $x, y \in N(d)$ with w adjacent to x and non-adjacent to y. Since $A \cup C$ is a clique, and B is complete to A and anticomplete to S, it follows that $x \in C$ and $y \in B$. By construction, every vertex in A has a neighbor in S, and so there exists $s_d \in S$ adjacent to a_d. Since s_d is mixed on $\{a_d, d\}$ and non-adjacent to y, 3.2.5 implies that s_d is anticomplete to $\{x, y\}$. However, then $\{s_d, a_d, x, w, y, d\}$ is a copy of H_6 in G of the desired form. Passing to the complement, we may also assume that C is empty. This proves *(6)*.

By *(6)*, since G is prime, it follows that $|B| \leq 1$, implying that G is a split graph, a contradiction. This proves 3.3.9.

□

Another conjecture which seemed plausible for a while is 3.1.6, which we restate:

3.3.10. *If G is a $\{P_5, \overline{P_5}\}$-free graph, then either*

- *G is isomorphic to C_5, or*

- *G is a split graph, or*
- *G has a homogeneous set, or*
- *G or $\overline{G}$ admits a 1-join.*

However, with Paul Seymour we found the counterexample in Figure 3.3. The graph in Figure 3.3 contains C_4 and $\overline{C_4}$, and so, by 3.1.3, is not split; we leave the rest of the details to the reader.

3.4 $\{P_5, \overline{P_5}$,bull$\}$-Free Graphs

In this section we give a short proof of 3.1.1, and of Fouquet's result 3.4.4 on the structure of $\{P_5, \overline{P_5}$,bull$\}$-free graphs. The following is joint work with Max Ehramn.

The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set $\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}$. Let O_k be the bipartite graph on $2k$ vertices with bipartition $(\{a_1, \dots a_k\}, \{b_1, \dots b_k\})$ in which a_i is adjacent to b_j if and only if $i + j \geq k + 1$. If a graph G is isomorphic to O_k for some k, then we call G a *half graph*. Note that by construction half graphs are prime. In [13] the Chudnovsky and Seymour proved:

3.4.1. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that both G and H are prime, and that both G and $\overline{G}$ are not half graphs. Then there exists an induced subgraph H' of G, isomorphic to H, and a vertex $v \in V(G) \setminus V(H')$, such that $G[V(H') \cup \{v\}]$ is prime.*

Next, we give a proof of Fouquet's result 3.1.1, which we restate:

3.4.2. *If G is a prime $\{P_5, \overline{P_5}\}$-free graph which contains C_5, then G is isomorphic to C_5.*

Proof. Suppose not, and so C_5 is a proper induced subgraph of G. Since C_5 is self-complementary both G and $\overline{G}$ contain an odd cycle, hence are non-bipartite, and thus not half graphs. As C_5 is prime, by 3.4.1, there exists a subgraph H induced by $\{v_1, v_2, v_3, v_4, v_5\}$ isomorphic to

C_5, and a vertex $v \in V(G)\backslash V(H)$ such that the subgraph of G induced by $V(H) \cup \{v\}$ is prime. Considering the complement, we may assume v is adjacent to at most two vertices in $V(H)$. To avoid a homogeneous set in $G[V(H) \cup \{v\}]$, by symmetry, the only possibilities are for $N(v) = \{v_1\}$, in which case $v - v_1 - v_2 - v_3 - v_4$ is a P_5 in G, or for $N(v) = \{v_1, v_2\}$, in which case $v - v_2 - v_3 - v_4 - v_5$ is a P_5 in G, in both cases a contradiction. This proves 3.4.2. □

Thus, to understand prime $\{P_5, \overline{P_5}$,bull$\}$-free graphs it is enough to study prime $\{P_5, \overline{P_5}, C_5$,bull$\}$-free graphs.

3.4.3. *If G is a prime $\{P_5, \overline{P_5}, C_5$,bull$\}$-free graph, then either G or $\overline{G}$ is a half graph.*

Proof. Suppose not. By 3.3.2, G contains P_4, which is isomorphic to O_2. Since G and $\overline{G}$ are not half graphs, it follows that P_4 is a proper induced subgraph of G. As P_4 is prime, by 3.4.1, there exists a subgraph H induced by $\{v_1, v_2, v_3, v_4\}$ isomorphic to P_4, and a vertex $v \in V(G)\backslash V(H)$ such that the subgraph of G induced by $V(H) \cup \{v\}$ is prime. Considering the complement, we may assume v is adjacent to at most two vertices in H. To avoid a homogeneous set in $G[V(H) \cup \{v\}]$, by symmetry, the only possibilities are for $N(v) = \{v_1\}$, in which case $v - v_1 - v_2 - v_3 - v_4$ is a P_5 in G, for $N(v) = \{v_1, v_4\}$, in which case $v - v_1 - v_2 - v_3 - v_4 - v$ is a C_5 in G, or for $N(v) = \{v_2, v_3\}$, in which case $\{v, v_1, v_2, v_3, v_4\}$ is a bull in G, in all cases a contradiction. This proves 3.4.3. □

Putting things together we obtain Fouquet's original structural result [17]:

3.4.4. *If G is a $\{P_5, \overline{P_5}$,bull$\}$-free graph, then either*

- $|V(G)| \leq 2$, *or*
- *G is isomorphic to C_5, or*
- *G has a homogeneous set, or*
- *G or $\overline{G}$ is a half graph.*

Proof. As all graphs on three vertices have a homogeneous set, 3.4.4 immediately follows from 3.4.2 and 3.4.3.

□

Chapter 4

Excluding Four-Edge Paths and Their Complements

In this chapter, we prove that a graph G contains no induced four-edge path and no induced complement of a four-edge path if and only if G can be obtained obtained from pentagons and split graphs by repeatedly applying the following operations: substitution, split graph unification, and split graph unification in the complement ("split graph unification" is a new class-preserving operation that is introduced in this chapter). This is a combination of joint work with Maria Chudnovsky and Irena Penev [8], together with later work of Louis Esperet, Laetitia Lemoine and Frederic Maffray [15], and first appeared in [6].

4.1 Introduction

We begin by recalling and rephrasing Fouquet's result 3.1.1 from [17]:

4.1.1. *If G is a $\{P_5, \overline{P_5}\}$-free graph, then either*

- *G contains a homogeneous set,*
- *G is isomorphic to C_5, or*
- *G is C_5-free.*

It follows immediately from 4.1.1 that every $\{P_5, \overline{P_5}\}$-free graph can be obtained by substitution starting from $\{P_5, \overline{P_5}, C_5\}$-free graphs and pentagons. Furthermore, it is easy to check that every graph obtained by substitution starting from pentagons and $\{P_5, \overline{P_5}, C_5\}$-free graphs is $\{P_5, \overline{P_5}\}$-free. We remark that the Strong Perfect Graph Theorem 1.1.2 implies that a $\{P_5, \overline{P_5}\}$-free graph is perfect if and only if it is C_5-free. Thus, every $\{P_5, \overline{P_5}\}$-free graph can be obtained by substitution starting from pentagons and $\{P_5, \overline{P_5}\}$-free perfect graphs. In view of this, the bulk of this chapter focuses on understanding the structure of prime $\{P_5, \overline{P_5}, C_5\}$-free graphs (equivalently: $\{P_5, \overline{P_5}\}$-free perfect graphs).

First, we make the following easy observation:

4.1.2. *Let G be a $\{P_5, \overline{P_5}, C_5\}$-free graph, A and B be non-empty disjoint subsets of $V(G)$, and let t be a vertex in $V(G) \setminus (A \cup B)$ such that:*

- *t is anticomplete to A and complete to B,*
- *every vertex in B has a neighbor in A, and*
- *A is connected.*

Then some vertex of A is complete to B.

Proof. Pick a vertex a in A with the maximum number of neighbors in B. Suppose that a has a non-neighbor y in B. We know that y has a neighbor a' in A. Since A is connected, there is a path $P = a_0$-...-a_k in $G[A]$ with $k \geq 1$, $a_0 = a'$ and $a_k = a$. Choose a' such that k is minimal, and so P is induced and y is anticomplete to $\{a_1, ..., a_k\}$. It follows that $k = 1$, as otherwise $t - y - a_0 - a_1 - a_2$ is a P_5 in G. Since y is adjacent to a' and non-adjacent to a, by our choice of a, it follows that there is a vertex z in B adjacent to a and not to a'. However, if y is non-adjacent to z, then $y - t - z - a - a' - y$ is a C_5 in G, and if y is adjacent to z, then $\{y, a, t, a', z\}$ is a $\overline{P_5}$ in G, in both cases a contradiction. This proves 4.1.2.

□

We say that a set of vertices, or a graph, is *big* if it contains at least two vertices.

4.1.3. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which contains $\overline{C_4}$. Then there exist pairwise disjoint subsets $X_0, X_1, ..., X_m, Y_0, Y_1, ..., Y_m$, with $m \geq 2$, whose union is equal to $V(G)$, such that the following properties hold, where $X = X_0 \cup X_1 \cup ... \cup X_m$ and $Y = Y_0 \cup Y_1 \cup ... \cup Y_m$:*

1. *$X_0, X_1, ..., X_m$ are pairwise anticomplete to each other, where X_0 is a (possibly empty) stable set and $X_1, ..., X_m$ are connected and big.*
2. *For each $i \in \{1, ..., m\}$, $Y_i \neq \emptyset$, every vertex of Y_i is mixed on X_i and complete to $X \setminus (X_i \cup X_0)$, and Y_0 is complete to $X \setminus X_0$.*
3. *$Y_0, Y_1, ..., Y_m$ are pairwise complete to each other. (So each anticomponent of Y is included in some Y_i with $i \in \{0, ..., m\}$.)*
4. *No vertex of $X \setminus X_0$ is mixed on any anticomponent of Y.*
5. *For each $i \in \{1, ..., m\}$, X_i contains a vertex that is complete to Y.*
6. *Every vertex of X_0 is mixed on at most one anticomponent of Y.*
7. *For every big anticomponent Z of Y, the set X_Z of vertices of X_0 that are mixed on Z is not empty. Moreover, if Z and Z' are any two distinct big anticomponents of Y, then $X_Z \cap X_{Z'} = \emptyset$.*
8. *Each big anticomponent Z of Y contains a vertex that is anticomplete to X_Z.*
9. *If Y is not a clique, there is a big anticomponent Z of Y such that X_Z is anticomplete to all big anticomponents of $Y \setminus Z$.*

Proof. Since G contains a $\overline{C_4}$, there is a subset X of $V(G)$ such that $G[X]$ has at least two big components. We choose X maximal with this property. Let $X_1, ..., X_m$ ($m \geq 2$) be the vertex-sets of the big components of $G[X]$, and let $X_0 = X \setminus (X_1 \cup ... \cup X_m)$. Thus, 4.1.3.1 holds. Let $Y = V(G) \setminus X$. We claim that:

(1) For every $y \in Y$ and $i \in \{1, ..., m\}$, y has a neighbor in X_i.

Proof: If y is anticomplete to X_i for some $i \in \{1, ..., m\}$, then $X \cup \{y\}$ induces a subgraph of G with at least two big components (one of which is X_i), which contradicts the maximality of X. This proves *(1)*.

(2) For every vertex $y \in Y$, there is at most one integer $i \in \{1, ..., m\}$ such that y has a non-neighbor in X_i.

Proof: Suppose for distinct $i, j \in \{1, ..., m\}$ that $y \in Y$ has a non-neighbor in both X_i and X_j. By *(1)*, both $N(y) \cap X_i$ and $X_i \setminus N(y)$ are non-empty, and so, since X_i is connected, it follows that there exist adjacent $u_i, v_i \in X_i$ such that y is adjacent to u_i and non-adjacent to v_i. Similarly, there exist adjacent $u_j, v_j \in X_j$ such that y is adjacent to u_j and non-adjacent to v_j. However, then $v_i - u_i - y - u_j - v_j$ is a P_5 in G, a contradiction. This proves *(2)*.

An immediate consequence of *(1)* and *(2)* is the following:

(3) Every vertex $y \in Y$ is either complete to $X \setminus X_0$, or there is a unique integer $i \in \{1, ..., m\}$ such that y is complete to $X \setminus (X_i \cup X_0)$ and mixed on X_i.

For each $i \in \{1, ..., m\}$, let $Y_i = \{y \in Y \mid y \text{ is mixed on } X_i\}$, and let $Y_0 = Y \setminus (Y_1 \cup ... \cup Y_m)$. By *(3)*, the sets $Y_0, Y_1, ..., Y_m$ are pairwise disjoint and their union is Y. For each $i \in \{1, ..., m\}$, since G is prime, X_i is not a homogeneous set, so there exists a vertex in $V(G) \setminus X_i$ that is mixed on X_i; by 4.1.3.1, any such vertex is in Y, and so $Y_i \neq \emptyset$. Thus, 4.1.3.2 holds.

Now, we prove 4.1.3.3. Let Z be an anticomponent of Y, and suppose that $Z \not\subseteq Y_0$. So Z contains a vertex y from Y_i for some $i \in \{1, ..., m\}$; say $y \in Y_1$. Since X_1 is connected, there are adjacent vertices u_1 and v_1 in X_1 such that y is adjacent to u_1 and not to v_1. Consider any non-neighbor z of y in Z. By 4.1.3.2, z has a neighbor x_2 in X_2, and y is complete to X_2. If z is anticomplete to $\{u_1, v_1\}$, then $z - x_2 - y - u_1 - v_1$ is a P_5 in G, a contradiction. If z is complete to $\{u_1, v_1\}$, then $\{z, y, v_1, x_2, u_1\}$ is a $\overline{P_5}$ in G, a contradiction. So z is mixed on X_1, i.e., $z \in Y_1$. Since Z is anticonnected, we can repeat this argument along the edges of a spanning tree of $\overline{G}[Z]$, which implies that $Z \subseteq Y_1$. Thus, 4.1.3.3 holds.

Now, we prove 4.1.3.4. Suppose on the contrary, and up to symmetry, that a vertex x in X_1 is mixed on some anticomponent Z of Y. Since Z is anticonnected, there are non-adjacent vertices $y, z \in Z$ such that x is adjacent to y and not to z. By 4.1.3.2, z has a neighbor u in X_1, so $z \in Y_1$. Since X_1 is connected, there is a path u_0-...-u_k in $G[X_1]$ with $u_0 = u$, $u_k = x$ and $k \geq 1$. Choose u such that k is minimal. By 4.1.3.2, y has a neighbor x_2 in X_2, and since $z \in Y_1$, z is adjacent to x_2. Suppose $k = 1$. If u is non-adjacent to y, then $x - y - x_2 - z - u - x$ is a C_5 in G, and if u is adjacent to y, then $\{y, z, x, x_2, u\}$ is a P_5 in G, in both cases a contradiction. So $k \geq 2$. The minimality of k implies that z is not adjacent to u_1 or u_2, and u is not adjacent to u_2. However, then $x_2 - z - u - u_1 - u_2$ is a P_5 in G, a contradiction.

Now, we prove 4.1.3.5. Observe that any vertex t from a big component of $X \setminus X_i$ is complete to Y_i and anticomplete to X_i, so we can apply Lemma 4.1.2 to X_i, Y_i and t. It follows that some vertex a of X_i is complete to Y_i. By 4.1.3.2, X_i is complete to $Y \setminus Y_i$. Thus, a is complete to Y.

Now, we prove 4.1.3.6. Suppose that a vertex x in X_0 is mixed on two anticompoments Z_1 and Z_2 of Y. For each $j \in \{1, 2\}$, since Z_j is anticonnected, there are non-adjacent vertices y_j and z_j in Z_j such that x is adjacent to y_j and not to z_j. Then $\{y_1, z_1, x, z_2, y_2\}$ is a $\overline{P_5}$ in G, a contradiction.

Now, we prove 4.1.3.7. If Z is any big anticomponent of Y, then, since G is prime, Z is not a homogeneous set, so there exists a vertex of $V(G) \setminus Z$ that is mixed on Z. The definition of Z and 4.1.3.4 imply that any such vertex is in X_0. So $X_Z \neq \emptyset$. The second sentence of 4.1.3.7 follows directly from 4.1.3.6.

Now, we prove 4.1.3.8. Let Z be a big anticomponent of Y. By 4.1.3.3, Z is included in one of $Y_0, Y_1, ..., Y_m$. By 4.1.3.2 and 4.1.3.4, some vertex t of $X \setminus X_0$ is complete to Z, and by 4.1.3.1 t is anticomplete to X_Z. Hence, we can apply Lemma 4.1.2 to Z, X_Z and t in $\overline{G}$, and we obtain that some vertex in Z is complete (in $\overline{G}$) to X_Z.

Finally we prove 4.1.3.9. Suppose that Y is not a clique, and choose a big anticomponent Z of Y that minimizes the number of big anticomponents of Y that are not anticomplete to

X_Z. If this number is 1, then Z satisfies the desired property. So suppose that this number is at least 2, that is, there is a vertex $x \in X_Z$ and a big anticomponent Z' of $Y \setminus Z$ that contains a neighbor of x. There are non-adjacent vertices $y, z \in Z$ such that x is adjacent to y and not to z. By 4.1.3.6, x is complete to Z'. Consider any $t \in X_{Z'}$; there are non-adjacent vertices $y', z' \in Z'$ such that t is adjacent to y' and not to z'. If t has any neighbor in Z, then, by 4.1.3.6, t is complete to Z, and then $\{z, x, t, z', y'\}$ is a $\overline{P_5}$ in G, a contradiction. Since this holds for any $t \in X_{Z'}$, we obtain that $X_{Z'}$ is anticomplete to Z. Now, the choice of Z implies that there is a third big anticomponent Z'' of Y (a big anticomponent of $Y \setminus (Z \cup Z')$) such that some vertex u of $X_{Z'}$ has a neighbor y'' in Z'' and X_Z is anticomplete to Z''. There are non-adjacent vertices $a, b \in Z'$ such that u is adjacent to a and not to b. Then $\{a, b, u, x, y''\}$ is a $\overline{P_5}$ in G, a contradiction. This proves 4.1.3.

□

4.2 The Split Divide

Recall, that a graph G is a *split graph* if there is a partition $V(G) = A \cup B$ such that A is a stable set and B is a clique. Also, recall Földes and Hammer's result 3.1.3 which we restate:

4.2.1. *A graph G is a split graph if and only if G is a $\{C_4, \overline{C_4}, C_5\}$-free graph.*

A *split divide* of a graph G is a partition (A, B, C, L, T) of $V(G)$ such that:

- $|A| \geq 2$, A is complete to B and anticomplete to $C \cup T$, and some vertex of A is complete to L,
- L is a non-empty clique, every vertex of L is mixed on A, and L is complete to $B \cup C$,
- $|C| \geq 2$, some vertex of C is complete to B, and no vertex of C is mixed on any anticomponent of B, and
- T is a (possibly empty) stable set and is anticomplete to C.

Note, that the sets B and T may be empty. The split divide can be thought of as a relaxation of the homogeneous set decomposition: a set $X \subseteq V(G)$ is a homogeneous set in G if no vertex in $V(G) \setminus X$ is mixed on X; in the case of the split divide, the set A is not homogeneous, but all the vertices that are mixed on A lie in the clique L, and adjacency between L and the rest of the graph is heavily restricted.

4.2.2. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Then either G is a split graph, or G or $\overline{G}$ admits a split divide.*

Proof. By 4.2.1 and taking complements, we may assume that G contains a $\overline{C_4}$. Consequently G admits the structure described in 4.1.3, and we use it with the same notation. Suppose that Y is a clique. Let $A = X_1$, $L = Y_1$, $B = Y \setminus Y_1$, $C = X_2 \cup ... \cup X_m$ and $T = X_0$. Then (A, B, C, L, T) is a split divide of G; this follows immediately from the definition of the partition $X_0, X_1, ..., X_m, Y_0, Y_1, ..., Y_m$, the fact that Y is a clique, and items 4.1.3.1–4.1.3.5.

Now, suppose that Y is not a clique. We will show that $\overline{G}$ admits a split divide. By 4.1.3.9, there is a big anticomponent Z of Y such that X_Z is anticomplete to all big anticomponents of $Y \setminus Z$. By 4.1.3.7, $X_Z \neq \emptyset$. By 4.1.3.3, and up to relabeling, we may assume that $Z \subseteq Y_0 \cup Y_1$. Hence, Z is complete to $X_2 \cup ... \cup X_m$, and every vertex of $X_1 \cup (X_0 \setminus X_Z)$ is either complete or anticomplete to Z. Let K be the union of all anticomponents of Y of size 1. So K is a clique and is complete to $Y \setminus K$. Let:

- $A = Z$,
- $L = X_Z$,
- $B = \{x \in X_1 \cup (X_0 \setminus X_Z)$ such that x is anticomplete to $Z\}$,
- $C' = \{x \in X_1 \cup (X_0 \setminus X_Z)$ such that x is complete to $Z\}$,
- $T = \{k \in K$ such that k has a neighbor in $X_Z\}$, and
- $C = X_2 \cup ... \cup X_m \cup (Y \setminus (Z \cup T)) \cup C'$.

We claim that:

(1) No vertex of C is mixed on any component of B.

Proof: For suppose that there is a vertex $c \in C$ and adjacent vertices $u, v \in B$ such that c is adjacent to u and not to v. Since X_0 is a stable set and is anticomplete to X_1, we have $u, v \in \{x \in X_1 \mid x \text{ is anticomplete to } Z\}$. Since c is adjacent to u, we have $c \in (Y \setminus (Z \cup T)) \cup \{x \in X_1 \mid x \text{ is complete to } Z\}$. Pick any $x \in X_Z$ and any vertex $z \in Z$ adjacent to x. However, then $x - z - c - u - v$ is a P_5 in G, a contradiction. This proves *(1)*.

(2) T is complete to C.

Proof: For suppose that there are non-adjacent vertices $t \in T$ and $c \in C$. Since K is complete to $Y \setminus K$ and $T \subseteq K$, we have that $c \notin Y \setminus (Z \cup T)$. Thus, $c \in X_2 \cup ... \cup X_m \cup C'$. By 4.1.3.2, Y_0 and Y_1 are complete to $X_2 \cup ... \cup X_m$; since $Z \subseteq Y_0 \cup Y_1$, it follows that Z is complete to $X_2 \cup ... \cup X_m$. Thus, $X_2 \cup ... \cup X_m \cup C'$ is complete to Z, and so c is complete to Z. Further, since $X_2 \cup ... \cup X_m \cup C' \subseteq X \setminus X_Z$ and X_Z is anticomplete to $X \setminus X_Z$ (because $X_Z \subseteq X_0$), we know that c is anticomplete to X_Z. By the definition of T, t has a neighbor x in X_Z. There are non-adjacent vertices $y, z \in Z$ such that x is adjacent to y and not to z. Since t and c are complete to Z, we see that $\{t, c, x, z, y\}$ is a $\overline{P_5}$ in G, a contradiction. This proves *(2)*.

Next, we observe that:

- $|A| \geq 2$ (because Z is big), A is anticomplete to B (by the definition of B), A is complete to $C \cup T$ (by 4.1.3.2), and some vertex of A is anticomplete to L (by 4.1.3.8).
- L is a non-empty stable set, every vertex of L is mixed on A, and L is anticomplete to $B \cup C$ (by the definition of L, with $L \subseteq X_0$).
- $|C| \geq 2$ (because $X_2 \subseteq C$), some vertex of C is anticomplete to B (every vertex of X_2 has this property), and no vertex of C is mixed on any component of B (by *(1)*).

- T is a clique and is complete to C (by *(2)*).

These observations mean that (A, B, C, L, T) is a split divide in $\overline{G}$. This proves 4.2.2.

□

Let G be a graph that admits a split divide (A, B, C, L, T) as above, where a_0 is a vertex of A that is complete to L, and c_0 is a vertex of C that is complete to B. Consider $G_1 = G[A \cup B \cup \{c_0\} \cup L \cup T]$ and $G_2 = G[\{a_0\} \cup B \cup C \cup L \cup T]$. Then we consider that G is decomposed into the two graphs G_1 and G_2. Note, that G_1 and G_2 are induced subgraphs of G and each of them has strictly fewer vertices than G, since $|A| \geq 2$ and $|C| \geq 2$.

4.3 Split Unification

We now define a composition operation that "reverses" the split divide decomposition. Let A, B, C, L, T be pairwise disjoint vertex sets, and assume that A and C are non-empty. Let a^*, c^* be distinct vertices such that $a^*, c^* \notin A \cup B \cup C \cup L \cup T$.

Let G_1 be a graph with vertex-set $A \cup B \cup L \cup T \cup \{c^*\}$ and adjacency as follows:

- L is a (possibly empty) clique;
- T is a (possibly empty) stable set;
- A is complete to B and anticomplete to T;
- Some vertex a_0 of A is complete to L;
- c^* is complete to $B \cup L$ and anticomplete to $A \cup T$.

Let G_2 be a graph with vertex-set $B \cup C \cup L \cup T \cup \{a^*\}$ and adjacency as follows:

- $G_2[B \cup L \cup T] = G_1[B \cup L \cup T]$;
- T is anticomplete to C;

- L is complete to $B \cup C$;
- a^* is complete to $B \cup L$ and anticomplete to $C \cup T$;
- Some vertex c_0 of C is complete to B, and no vertex of C is mixed on any anticomponent of B.

Under these circumstances, we say that (G_1, G_2) is a *composable pair*. The *split unification* of a composable pair (G_1, G_2) is the graph G with vertex-set $A \cup B \cup C \cup L \cup T$ such that:

- $G[A \cup B \cup L \cup T] = G_1 \setminus c^*$;
- $G[B \cup C \cup L \cup T] = G_2 \setminus a^*$;
- A is anticomplete to C in G.

Thus, to obtain G from G_1 and G_2, we "glue" G_1 and G_2 along their common induced subgraph $G_1[B \cup L \cup T] = G_2[B \cup L \cup T]$, where $L \cup T$ induces a split graph (hence the name of the operation).

We say that a graph G is *obtained by split unification* provided that there exists a composable pair (G_1, G_2) such that G is the split unification of (G_1, G_2). We say that G is *obtained by split unification in the complement* provided that $\overline{G}$ is obtained by split unification. We now prove that every graph which admits a split divide is obtained by split unification from smaller graphs.

4.3.1. *If a graph G admits a split divide, then it is obtained from a composable pair of smaller graphs (each of them isomorphic to an induced subgraph of G) by split unification.*

Proof. Let G be a graph that admits a split divide. Let (A, B, C, L, T) be a split divide of G, let a_0 be a vertex of A that is complete to L, and let c_0 be a vertex of C that is complete to B. Let $G_1 = G[A \cup B \cup L \cup T \cup \{c_0\}]$. Since $|C| \geq 2$, we have $|V(G_1)| < |V(G)|$. Let $G_2 = G[B \cup C \cup L \cup T \cup \{a_0\}]$. Since $|A| \geq 2$, we have $|V(G_2)| < |V(G)|$. Now, (G_1, G_2) is a composable pair, and G is obtained from it by split unification. $\square$

The split unification can be thought of as generalized substitution. Indeed, we obtain the graph G from G_1 and G_2 by first substituting $G_1[A]$ for a^* in G_2, and then reconstructing the adjacency between A and L in G using the adjacency between A and L in G_1. We include B, T and c^* in G_1 in order to ensure that split unification preserves the property of being $\{P_5, \overline{P_5}, C_5\}$-free. In fact, we prove now something stronger than this: split unification preserves the (individual) properties of being P_5-free, $\overline{P_5}$-free, and C_5-free.

4.3.2. *Let (G_1, G_2) be a composable pair and let G be the split unification of (G_1, G_2). Then, for each $H \in \{P_5, \overline{P_5}, C_5\}$, G is H-free if and only if both G_1 and G_2 are H-free.*

Proof. We use the same notation as in the definition of the split unification above. First, suppose that G is H-free. Observe that G_1 is isomorphic to the induced subgraph $G[A \cup B \cup L \cup T \cup \{c_0\}]$, and G_2 is isomorphic to the induced subgraph $G[B \cup C \cup L \cup T \cup \{a_0\}]$. Hence, G_1 and G_2 are H-free. Now, suppose that G_1 and G_2 are H-free and that G contains an induced copy of H. Let W be a five-vertex subset of $V(G)$ such that $G[W]$ is a copy of H in G. We claim that W must contain two non-adjacent vertices b and c with $b \in W \cap B$ and $c \in W \cap C$. For suppose the contrary. Then $W \cap C$ is complete to $W \cap (L \cup B)$ and anticomplete to $W \cap (A \cup T)$. If $|W \cap C| \geq 2$, then either $|W \cap C| \leq 4$, so $W \cap C$ is a proper homogeneous set in $G[W]$ (a contradiction since H is prime), or $W \subseteq C$, so W is isomorphic to an induced subgraph of G_2 (a contradiction since G_2 is H-free). So $|W \cap C| \leq 1$, and then W is isomorphic to an induced subgraph of G_1 (where c^* plays the role of the vertex in $W \cap C$ if there is such a vertex), a contradiction since G_1 is H-free. Therefore, the claim holds. By a similar argument, W must contain two non-adjacent vertices a and ℓ with $a \in W \cap A$ and $\ell \in W \cap L$. Let w be the fifth vertex in W, so that $W = \{a, b, c, \ell, w\}$. By the definition of the split unification, $a - b - \ell - c$ is a P_4 in G. Consequently, we must have one of the following two cases:

(i) W is a P_5 or C_5 in G. So w is anticomplete to $\{b, \ell\}$ and has a neighbor in $\{a, c\}$. Since w is anticomplete to $\{b, \ell\}$, it cannot be in A, B, L or C, so it is in T. But then w should be anticomplete to $\{a, c\}$.

(ii) W is a $\overline{P_5}$ in G. So w is complete to $\{a, c\}$ and has exactly one neighbor in $\{b, \ell\}$. Since

w is adjacent to a, it is not in $C \cup T$, and since it is adjacent to c, it is not in A. Moreover, since w is adjacent to exactly one of b and ℓ, it is not in L. So $w \in B$, and so it is adjacent to ℓ and, consequently, not to b. Hence, b and w lie in the same anticomponent of B, and c is adjacent to exactly one of them, a contradiction (to the last axiom in the definition of a split unification).

□

4.4 The Decomposition Theorem

In this section, we use 4.1.1 and the results of the preceding sections to prove 4.4.1, the main theorem of this chapter.

4.4.1. *A graph G is $\{P_5, \overline{P_5}\}$-free if and only if at least one of the following holds:*

- *G is a split graph,*
- *G is a pentagon,*
- *G is obtained by substitution from smaller $\{P_5, \overline{P_5}\}$-free graphs, or*
- *G or $\overline{G}$ is obtained by split unification from smaller $\{P_5, \overline{P_5}\}$-free graphs.*

Proof. We first prove the "if" part. If G is a split graph or a pentagon, then it is clear that G is $\{P_5, \overline{P_5}\}$-free. Since both P_5 and $\overline{P_5}$ are prime, we know that the class of $\{P_5, \overline{P_5}\}$-free graphs is closed under substitution, and consequently, any graph obtained by substitution from smaller $\{P_5, \overline{P_5}\}$-free graphs is $\{P_5, \overline{P_5}\}$-free. Finally, if G or $\overline{G}$ is obtained by split unification from smaller $\{P_5, \overline{P_5}\}$-free graphs, then the fact that G is $\{P_5, \overline{P_5}\}$-free follows from 4.3.2 and from the fact that the complement of a $\{P_5, \overline{P_5}\}$-free graph is again $\{P_5, \overline{P_5}\}$-free.

For the "only if" part, suppose that G is a $\{P_5, \overline{P_5}\}$-free graph. We may assume that G is prime, for otherwise, G is obtained by substitution from smaller $\{P_5, \overline{P_5}\}$-free graphs, and we are done. If some induced subgraph of G is isomorphic to the pentagon, then by 4.1.1,

G is a pentagon, and again we are done. Thus, we may assume that G is $\{P_5, \overline{P_5}, C_5\}$-free. By 4.2.2, we know that either G is a split graph, or one of G and $\overline{G}$ admits a split divide. In the former case, we are done. In the latter case, 4.3.1 implies that G or $\overline{G}$ is the split unification of a composable pair of smaller $\{P_5, \overline{P_5}, C_5\}$-free graphs, and again we are done. This proves 4.4.1.

□

As an immediate corollary of 4.4.1, we have the following.

4.4.2. *A graph is $\{P_5, \overline{P_5}\}$-free if and only if it is obtained from pentagons and split graphs by repeated substitutions, split unifications, and split unifications in the complement.*

Finally, a proof analogous to the proof of 4.4.1 (but without the use of 4.1.1) yields the following result for $\{P_5, \overline{P_5}, C_5\}$-free graphs.

4.4.3. *A graph G is $\{P_5, \overline{P_5}, C_5\}$-free if and only if at least one of the following holds:*

- *G is a split graph,*
- *G is obtained by substitution from smaller $\{P_5, \overline{P_5}, C_5\}$-free graphs, or*
- *G or $\overline{G}$ is obtained by split unification from smaller $\{P_5, \overline{P_5}, C_5\}$-free graphs.*

Chapter 5

Three-Coloring Triangle-Free Graphs with No Induced Six-Edge Path

In this chapter, we give a polynomial time algorithm which determines if a given triangle-free graph with no induced seven-vertex path is 3-colorable, and gives an explicit coloring if one exists. This is joint work with Maria Chudnovsky and Mingxian Zhong, and first appeared in [10].

5.1 Introduction

The COLORING problem is determining the smallest integer k such that a given graph is k-colorable, and was one of the initial problems Karp [26] showed to be NP-complete. For fixed $k \geq 1$, the k-COLORING problem is deciding whether a given graph is k-colorable. Since Stockmeyer [35] showed that for any $k \geq 3$ the k-COLORING problem is NP-complete, there has been much interest in deciding for which classes of graphs coloring problems can be solved in polynomial time. In this chapter, the general approach that we consider is to fix a graph H and consider the k-COLORING problem restricted to the class of H-free graphs.

We call a graph *acyclic* if it is C_n-free for all $n \geq 3$. The *girth* of a graph is the length of its shortest cycle, or infinity if the graph is acyclic. Kamiński and Lozin [31] proved:

5.1.1. *For any fixed $k, g \geq 3$, the k-COLORING problem is NP-complete for the class of graphs with girth at least g.*

As a consequence of 5.1.1, it follows that if the graph H contains a cycle, then for any fixed $k \geq 3$, the k-COLORING problem is NP-complete for the class of H-free graphs. The *claw* is the graph with vertex set $\{a_0, a_1, a_2, a_3\}$ and edge set $\{a_0a_1, a_0a_2, a_0a_3\}$. A theorem of Holyer [24] together with an extension due to Leven and Galil [28] imply the following:

5.1.2. *If a graph H contains the claw, then for every $k \geq 3$, the k-COLORING problem is NP-complete for the class of H-free graphs.*

Hence, the remaining problem of interest is deciding the k-COLORING problem for the class of H-free graphs where H is a fixed acyclic claw-free graph. It is easily observed that every component of an acyclic claw-free graph is a path. And so, we focus on the k-COLORING problem for the class of H-free graphs where H is a connected acyclic claw-free graph, that is, simply a path. Hoàng, Kamiński, Lozin, Sawada, and Shu [21] proved the following:

5.1.3. *For every k, the k-COLORING problem can be solved in polynomial time for the class of P_5-free graphs.*

Additionally, Randerath and Schiermeyer [32] showed that:

5.1.4. *The 3-COLORING problem can be solved in polynomial time for the class of P_6-free graphs.*

While, Huang [25] recently showed that:

5.1.5. *The following problems are NP-complete:*

1. *The 5-COLORING problem is NP-complete for the class of P_6-free graphs.*
2. *The 4-COLORING problem is NP-complete for the class of P_7-free graphs.*

Thus, the remaining open cases of the k-COLORING problem for P_ℓ-free graphs are the following:

1. The 4-COLORING problem for the class of P_6-free graphs.

2. The 3-COLORING problem for the class of P_ℓ-free graphs where $\ell \geq 7$.

Toward extending these polynomial results, it is convenient to consider the following more general coloring problem. A *palette* L of a graph G is a mapping which assigns each vertex $v \in V(G)$ a finite non-empty subset of $\mathbb{N}$, denoted by $L(v)$. A *subpalette* of a palette L of G is a palette L' of G such that $L'(v) \subseteq L(v)$ for all $v \in V(G)$. We say a palette L of the graph G has *order* k if $L(v) \subseteq \{1, ..., k\}$ for all $v \in V(G)$. Notationally, we write (G, L) to represent a graph G and a palette L of G. We say that a k-coloring c of G is a *coloring of* (G, L) provided $c(v) \in L(v)$ for all $v \in V(G)$. We say (G, L) is *colorable*, if there exists a coloring of (G, L). We denote by $(G, \mathcal{L})$ a graph G and a collection $\mathcal{L}$ of palettes of G. We say $(G, \mathcal{L})$ is *colorable* if (G, L) is colorable for some $L \in \mathcal{L}$, and c is a *coloring* of $(G, \mathcal{L})$ if c is a coloring of (G, L) for some $L \in \mathcal{L}$.

Let G be a graph. A subset D of $V(G)$ is called a *dominating set*, if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. Given (G, L), consider a subset $X \subseteq V(G)$ such that $|L(x)| = 1$ for all $x \in X$. For a subset $Y \subseteq V(G) \setminus X$, we say that we *update the palettes of the vertices in* Y *with respect to* X, if for all $y \in Y$ we set

$$L(y) = L(y) \setminus \Big(\bigcup_{u \in N(y) \cap X \text{ with } |L(u)|=1} L(u) \Big).$$

Note that updating can be carried out in time $O(|V(G)|^2)$.

By reducing to an instance of 2-SAT, which Aspvall, Plass and Tarjan [1] showed can be solved in linear time, Edwards [14] proved the following:

5.1.6. *There is an algorithm with the following specifications:*

Input: *A palette L of a graph G such that $|L(v)| \leq 2$ for all $v \in V(G)$.*

Output: *A coloring of (G, L), or a determination that none exists.*

Running time: $O(|V(G)|^2)$.

Let G be a graph. A subset S of $V(G)$ is called *monochromatic* with respect to a given coloring c of G if $c(u) = c(v)$ for all $u, v \in S$. For a palette L, and a set X of subsets of $V(G)$, we say that (G, L, X) is *colorable* if there is a coloring c of (G, L) such that S is monochromatic with respect to c for all $S \in X$. The proof of 5.1.6 is easily modified to obtain the following generalization [34]:

5.1.7. *There is an algorithm with the following specifications:*

Input: *A palette L of a graph G such that $|L(v)| \leq 2$ for all $v \in V(G)$, together with a set X of subsets of $V(G)$.*

Output: *A coloring of (G, L, X), or a determination that none exists.*

Running time: $O(|X||V(G)|^2)$.

Applying 5.1.6 yields the following general approach for 3-coloring a graph. Let G be a graph, and suppose $D \subseteq V(G)$ is a dominating set. Initialize the order 3 palette L of G by setting $L(v) = \{1, 2, 3\}$ for all $v \in V(G)$. Consider a fixed 3-coloring c of $G[D]$, and let L_c be the subpalette of L obtained by updating the palettes of the vertices in $V(G) \setminus D$ with respect to D. By construction, (G, L_c) is colorable if and only if the coloring c of $G[D]$ can be extended to a 3-coloring of G. Since $|L_c(v)| \leq 2$ for all $v \in V(G)$, 5.1.6 allows us to efficiently test if (G, L_c) is colorable. Let $\mathcal{L}$ to be the set of all such palettes L_c where c is a 3-coloring of $G[D]$. It follows that G is 3-colorable if and only if $(G, \mathcal{L})$ is colorable. Assuming we can efficiently produce a dominating set D of bounded size, since there are at most $3^{|D|}$ ways to 3-color $G[D]$, it follows that we can efficiently test if $(G, \mathcal{L})$ is colorable, and so we can decide if G is 3-colorable in polynomial time. This method figures prominently in the polynomial time algorithms for the 3-COLORING problem for the class of P_ℓ-free graphs where $\ell \leq 5$. However, this approach needs to be modified when considering the class of P_ℓ-free graphs when $\ell \geq 6$, since a dominating set of bounded size may not exist. Very roughly, the techniques used in this chapter may be described as such a modification.

In this chapter and [11], we prove that the 3-COLORING problem can be solved in polynomial time for the class of P_7-free graphs. Here we consider the triangle-free case and prove the following:

5.1.8. *There is an algorithm with the following specifications:*

Input: *A $\{P_7, C_3\}$-free graph G.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^{18})$.

Here is a brief outline of the algorithm. Consider a $\{P_7, C_3\}$-free graph G. We begin by establishing two polynomial time procedures 5.2.3 and 5.3.4 which determine if a 3-coloring of a specific induced subgraph of G extends to a coloring of G, and gives an explicit 3-coloring if one exists. More specifically, given an order 3 palette L of G, and a set X of subsets of $V(G)$, 5.2.3 and 5.3.4 allow us to reduce determining if (G, L, X) is colorable to determining if one of polynomially many triples (G', L', X') is colorable, where each of (G, L', X) is "closer" than (G, L, X) to being of the form required by 5.1.7. Next, we introduce a polynomial time "cleaning" procedure 5.4.3, which preprocesses the graph G so that we can apply 5.2.3 and 5.3.4. Next, we use 5.3.4 to show that if G contains a 7-gon, then in polynomial time we can either produce a 3-coloring of G, or determine that none exists. And so, we may assume G is a $\{P_7, C_3, C_7\}$-free graph. A *shell* in a graph G is a pair (C, p), where C is a 6-gon given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p \in V(G) \setminus \{v_0, ..., v_5\}$, such that $N(p) \cap \{v_0, ..., v_5\} = \{v_\ell, v_{\ell+3}\}$ for some $\ell \in \{0, 1, 2\}$. Next, we use 5.3.4 to show that if G contains a shell, then in polynomial time we can either produce a 3-coloring of G, or determine that none exists. And so, we may assume G is a $\{P_7, C_3, C_7, shell\}$-free graph. Finally, we use 5.2.3 to show that if G contains a 5-gon, then in polynomial time we can either produce a 3-coloring of G, or determine that none exists. And so, we may assume G is a $\{P_7, C_3, C_5, C_7\}$-free graph. Since G is P_7-free, it follows that G is C_k-free for all $k > 7$. And so, G is bipartite, and we can easily produce a 2-coloring of G, thus, establishing 5.1.8.

In [11], using different techniques, we prove the following:

5.1.9. *There is a polynomial time algorithm with the following specifications:*

Input: *A P_7-free graph G which contains a triangle.*

Output: *A 3-coloring of G, or a determination that none exists.*

Together, 5.1.8 and 5.1.9, imply the following:

5.1.10. *There is a polynomial time algorithm with the following specifications:*

Input: *A P_7-free graph G.*

Output: *A 3-coloring of G, or a determination that none exists.*

This chapter is organized as follows. In section 2, we prove 5.2.3, and in section 3, we prove 5.3.4. In section 4, we give a preprocessing procedure 5.4.3, so that we can apply 5.2.3 and 5.3.4 to a given $\{P_7, C_3\}$-free graph. In section 5, we prove a lemma that allows us to identify more easily situations where 5.2.3 and 5.3.4 are applicable. In section 6, we prove 5.6.6, which shows that if a $\{P_7, C_3\}$-free graph contains a 7-gon, then 3-COLORING can be solved in polynomial time. In section 7, we prove 5.7.7, which shows that if a $\{P_7, C_3, C_7\}$-free graph contains a shell, then 3-COLORING can be solved in polynomial time. In section 8, we prove 5.8.6, which shows that if a $\{P_7, C_3, C_7, shell\}$-free graph contains a 5-gon, then 3-COLORING can be solved in polynomial time. Finally, in section 9, we tie everything together and give a formal proof of 5.1.8.

5.2 Reducing the Palettes: Part I

In this section, we give a polynomial time procedure 5.2.3 which, given a $\{P_7, C_3\}$-free graph G with palette L, and a set of subsets X of $V(G)$, under certain circumstances, allows us to reduce determining if (G, L, X) is colorable to determining if one of polynomially many triples (G, L', X) is colorable, where each (G, L', X) is "closer" than (G, L, X) to the form required by 5.1.7. More precisely, more vertices have lists of size at most two in the palette L' than in the palette L. We begin with some definitions, and easy lemma and algorithm.

Let G be a graph. A *clique* in G is a set of vertices all pairwise adjacent. A *stable set* in G is a set of vertices all pairwise non-adjacent. The *neighborhood* of a vertex $v \in V(G)$ is the set of all vertices adjacent to v, and is denoted $N(v)$. The *degree* of a vertex $v \in V(G)$ is $|N(v)|$, and is denoted $deg(v)$. A *partition* of a set S is a collection of disjoint subsets of S whose union is S. Let A and B be disjoint subsets of $V(G)$. For a vertex $b \in V(G) \setminus A$, we say that b *is complete to* A if b is adjacent to every vertex of A, and that b *is anticomplete to* A if b is non-adjacent to every vertex of A. If every vertex of A is complete to B, we say A *is complete to* B, and if every vertex of A is anticomplete to B, we say that A *is anticomplete to* B. If $b \in V(G) \setminus A$ is neither complete nor anticomplete to A, we say that b *is mixed on* A. We say G is *connected* if $V(G)$ cannot be partitioned into two disjoint non-empty sets anticomplete to each other. The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. If $\overline{G}$ is connected we say that G is *anticonnected*. For $X \subseteq V(G)$, we say that X is *connected* if $G[X]$ is connected, and that X is anticonnected if $G[X]$ is anticonnected. A *component* of $X \subseteq V(G)$ is a maximal connected subset of X, and an *anticomponent* of X is a maximal anticonnected subset of X.

5.2.1. *Let G be a bipartite $\overline{C_4}$-free graph with bipartition (A, B). If $a, a' \in A$ are such that $deg(a) \leq deg(a')$, then $N(a) \subseteq N(a')$.*

Proof. Suppose not, and so there exists $b \in N(a) \setminus N(a')$. Since $|N(a)| \leq |N(a')|$, it follows that there exists $b' \in N(a') \setminus N(a)$. However, then $\{a, b, a', b'\}$ is a $\overline{C_4}$ in G, a contradiction. This proves 5.2.1.

□

5.2.2. *There is an algorithm with the following specifications:*

Input: *A bipartite $\overline{C_4}$-free graph G together with a bipartition $V(G) = A \cup B$.*

Output: *A partition $A_1 \cup ... \cup A_q$ of A and an ordering $\{b_1, ..., b_{|B|}\}$ of the vertices in B such that for every $i \in \{1, ..., q\}$ and $j \in \{1, ..., |B|\}$ the following hold:*

1. *If $a, a' \in A_i$, then $N(a) = N(a')$, and*
2. *If b_j is complete to A_i, then $A_i \cup ... \cup A_q$ is complete to $\{b_j, ..., b_{|B|}\}$.*

Running time: $O(|V(G)|^2)$.

Proof. In time $O(|V(G)|^2)$ we can compute the degree of each vertex in G, and sort the vertices of A and B by degree, thus obtaining a labeling $a_1, ..., a_{|A|}$ of A such that $deg(a_1) \leq ... \leq deg(a_{|A|})$, and a labeling $b_1, ..., b_{|B|}$ of B such that $deg(b_1) \leq ... \leq deg(b_{|B|})$. Now, let $q = deg(a_{|A|})$, and for each $i \in \{1, ..., q\}$ define $A_i = \{a \in A : deg(a) = i\}$. By applying 5.2.1 twice, it follows that if $a, a' \in A_i$, then $N(a) = N(a')$. Next, suppose b_j is complete to A_i for some $i \in \{1, ..., q\}$ and $j \in \{1, ..., |B|\}$, which implies $A_i \subseteq N(b_j)$ and $b_j \in N(a)$ for all $a \in A_i$. Since $deg(b_j) \leq ... \leq deg(b_{|B|})$, by 5.2.1, it follows that A_i is complete to $\{b_j, ..., b_{|B|}\}$. And, since $deg(a) \geq i$ for all $a \in A_i \cup ... \cup A_q$, by 5.2.1, it follows that $\{b_j, ..., b_{|B|}\}$ is complete to $A_i \cup ... \cup A_q$. This proves 5.2.2.

□

The following is the main result of the section.

5.2.3. *Let G be a $\{P_7, C_3\}$-free graph with $V(G) = \{x\} \cup S \cup \hat{A} \cup \hat{B} \cup Y$, where*

- *x is complete to S and anticomplete to $\hat{A} \cup \hat{B} \cup Y$,*
- *$\hat{A} = \{a_1, ..., a_t\}$ and $\hat{B} = \{b_1, ..., b_t\}$ are stable,*
- *for $i, j \in \{1, ..., t\}$, a_i is adjacent to b_j if and only if $i = j$, and*
- *each vertex of $\hat{A} \cup \hat{B}$ has a neighbor in S.*

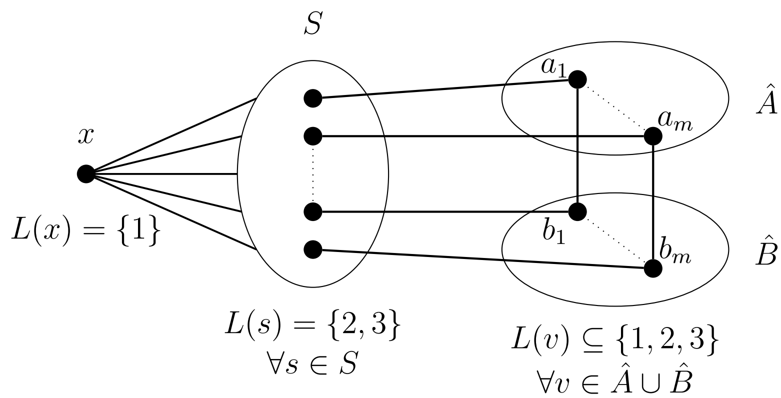


Figure 5.1: By 5.2.3, when we encounter the above situation we can reduce determining if (G, L, X) is colorable to determining if one of the triples (G, L', X) is colorable for some $L' \in \mathcal{L}$, where each of (G, L', X) is "closer" to the form required by 5.1.7 (in particular, $L'(v) \leq 2$ for all $v \in \hat{A} \cup \hat{B}$).

Let L be an order 3 palette of G such that $L(v) \subseteq \{2, 3\}$ for every $v \in S$.

Let X be a set of subsets of $V(G)$.

Then there exists a set $\mathcal{L}$ of $O(|V(G)|^2)$ subpalettes of L such that

(a) For each $L' \in \mathcal{L}$, $L'(v) = L(v)$ for every $v \in \{x\} \cup S \cup Y$, and $|L'(v)| \leq 2$ for every $v \in \hat{A} \cup \hat{B}$, and

(b) (G, L, X) is colorable if and only if (G, L', X) is colorable for at least one $L' \in \mathcal{L}$; and for every $L' \in \mathcal{L}$, every coloring of (G, L', X) is a coloring of (G, L, X).

Moreover, if the partition $\{x\} \cup S \cup \hat{A} \cup \hat{B} \cup Y$ of $V(G)$ is given, then $\mathcal{L}$ can be computed in time $O(|V(G)|^4)$.

Proof. Since G is triangle-free, it follows that S is stable, and that every vertex of S is either anticomplete to or mixed on $\{a_i, b_i\}$ for every $i \in \{1, ..., t\}$. Let H be a bipartite graph with bipartition $V(H) = S \cup \{c_1, ..., c_t\}$, where $s \in S$ is adjacent to c_i in H if and only if s is mixed on $\{a_i, b_i\}$ in G. Note, H can be constructed in time $O(|V(G)|^2)$.

(1) H is a $\overline{C_4}$-free graph.

Proof: Suppose not. Then there exist $s, s' \in S$ such that in G for $i \neq j$, s is mixed on $\{a_i, b_i\}$ and anticomplete to $\{a_j, b_j\}$, and s' is mixed on $\{a_j, b_j\}$ and anticomplete to $\{a_i, b_i\}$. By symmetry, we may assume that s is adjacent to a_i, and s' is adjacent to a_j. However, then $b_i - a_i - s - x - s' - a_j - b_j$ is a P_7 in G, a contradiction. This proves *(1)*.

Write $C = \{c_1, ..., c_t\}$. By *(1)*, applying 5.2.2 in time $O(|V(G)|^2)$ we obtain a partition $S_1 \cup ... \cup S_q$ of S and an ordering $\{d_1, ..., d_t\}$ of the vertices of C. Renumber the vertices of $\hat{A}$ and $\hat{B}$ so that d_k corresponds to the edge $a_k b_k$ for every $k \in \{1, ..., t\}$.

(2) For every $i \in \{1, ..., q\}$ and $j \in \{1, ..., t\}$ the following hold:

(2a) The vertices in S_i are either all anticomplete to or all mixed on $\{a_j, b_j\}$.

(2b) If the vertices in S_i are all mixed on $\{a_j, b_j\}$, then every vertex in $S_i \cup ... \cup S_q$ is mixed on $\{a_k, b_k\}$ for all $k \in \{j, ..., t\}$.

Proof: By 5.2.2.1, it follows that in H every vertex d_j is either complete or anticomplete to S_i. Hence, by the construction of H, in G the vertices in S_i are either all anticomplete to or all mixed on $\{a_j, b_j\}$. This proves *(2a)*. By 5.2.2.2, it follows that in H if d_j is complete to S_i, then $\{d_j, ..., d_t\}$ is complete to $S_i \cup ... \cup S_q$, and *(2b)* follows. This proves *(2)*.

For $j \in \{1, ..., t\}$, we define the *height* of the edge $a_j b_j$ to be the maximum ℓ such that both a_j and b_j have neighbors in $S_\ell \cup ... \cup S_q$. Since every vertex in $\hat{A} \cup \hat{B}$ has a neighbor in S, the height of an edge is well defined. If the height of the edge $\{a_j, b_j\}$ is $\ell < q$, then *(2)* implies that one of the vertices in $\{a_j, b_j\}$ is anticomplete to $S_{\ell+1} \cup ... \cup S_q$, we call this the *small* vertex in $\{a_j, b_j\}$ and denote it by s_j. We call the vertex of $\{a_j, b_j\} \setminus \{s_j\}$ the *large* vertex in $\{a_j, b_j\}$ and denote it by l_j. Then l_j is complete to $S_{\ell+1} \cup ... \cup S_q$. If the edge $a_j b_j$ has height q, then we arbitrary assign $\{l_j, s_j\} = \{a_j, b_j\}$. Next, let N_j be the set of vertices in $S_\ell \cup ... \cup S_q$ adjacent to l_j, and let M_j be the set of vertices in $S_\ell \cup ... \cup S_q$ adjacent to s_j. Clearly, computing the height of a_jb_j, determining the small and large vertices, and computing the N_j and M_j can be done in time $O(|V(G)|^2)$.

(3) For $j \in \{1, ..., t\}$, suppose the edge a_jb_j has height ℓ. Then the following hold:

(3a) $N_j \cup M_j = S_\ell \cup ... \cup S_q$, where N_j, M_j are disjoint, both non-empty, and $M_j \subseteq S_\ell$.

(3b) Let $k \in \{1, ..., t\} \setminus \{j\}$, and let $\{y, z\} = \{a_k, b_k\}$. If y is anticomplete to $S_\ell \cup ... \cup S_q$, then the height of a_kb_k is strictly less than ℓ, $y = s_k$, and both N_j, M_j are proper subsets of N_k .

Proof: Since G is triangle-free, it follows that N_j, M_j are disjoint. By the definition of height, both N_j, M_j are non-empty and, by *(2a)*, it follows that every vertex in S_ℓ is mixed on $\{a_j, b_j\}$. Hence, by *(2b)*, it follows that every vertex in $S_\ell \cup ... \cup S_q$ is mixed on $\{a_j, b_j\}$, and so $N_j \cup M_j = S_\ell \cup ... \cup S_q$. Finally, by our choice of s_j, it follows that $M_j \subseteq S_\ell$. This proves *(3a)*. Next, we prove *(3b)*. Since y is anticomplete to $S_\ell \cup ... \cup S_q$, it follows, by the definition of height, that the height of a_kb_k is strictly less than ℓ, and that $y = s_k$. Hence, by *(3a)*, it follows that l_k is complete to $S_\ell \cup ... \cup S_q$, and so both N_j, M_j are proper subsets of N_k. This proves *(3b)*. This completes the proof of *(3)*.

We say that (G, L, X) has a *type I coloring* if there exists a coloring c of (G, L, X) such that $\{c(a_i), c(b_i)\} = \{2, 3\}$ for some $i \in \{1, ..., t\}$. We now prove the following:

(4) There exists a set $\mathcal{L}_1$ of $O(|V(G)|)$ of subpalettes of L such that

(4a) For each $L_1 \in \mathcal{L}_1$, $L_1(v) = L(v)$ for every $v \in \{x\} \cup S \cup Y$, and $|L_1(v)| \leq 2$ for every $v \in \hat{A} \cup \hat{B}$, and

(4b) (G, L, X) has a type I coloring if and only if (G, L_1, X) is colorable for some $L_1 \in \mathcal{L}_1$; and for every $L_1 \in \mathcal{L}_1$, every coloring of (G, L_1, X) is a type I coloring of (G, L, X).

Moreover, $\mathcal{L}_1$ can be constructed in time $O(|V(G)|^3)$.

Proof: Let $i \in \{1, ..., t\}$ and ℓ be the height of the edge a_ib_i. First, set

- $L_i(l_i) = \{2\}$,
- $L_i(s_i) = \{3\}$, and
- $L_i(v) = L(v)$ for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$.

Next, for each $j \in \{1, ..., t\} \setminus \{i\}$, let $y \in \{a_j, b_j\}$. If $N(y) \cap (S_\ell \cup ... \cup S_q) \neq \emptyset$, then set

$$L_i(y) = \begin{cases} L(y) \setminus \{3\} & , \quad \text{if } N(y) \cap (S_\ell \cup ... \cup S_q) \subseteq N_i \\ L(y) \setminus \{2\} & , \quad \text{if } N(y) \cap (S_\ell \cup ... \cup S_q) \subseteq M_i \\ \{1\} & , \quad \text{otherwise} \end{cases}$$

Otherwise, if y is anticomplete to $S_\ell \cup ... \cup S_q$, then set $L_i(y) = \{2, 3\}$. As above, construct the subpalette L_i' of L, but with the roles of the colors 2 and 3 exchanged.

Clearly, if one of (G, L_i, X) and (G, L_i', X) is colorable, then there exists a type I coloring of (G, L, X). Now, suppose c is a type I coloring of (G, L, X) with $\{c(a_i), c(b_i)\} = \{2, 3\}$. By symmetry, we may assume $c(l_i) = 2$ and $c(s_i) = 3$. We claim that $c(v) \in L_i(v)$ for all $v \in V(G)$. By definition, this is the case for a_i, b_i, and for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$. Since $L_i(v) = L(v) \subseteq \{2, 3\}$ for every $v \in S$, as $c(l_i) = 2$, it follows that every vertex in N_i is colored 3. Similarly, as $c(s_i) = 3$, it follows that every vertex in M_i is colored 2. Hence, by *(3a)*, the colors of all the vertices in $S_\ell \cup ... \cup S_q$ are forced by $c(l_i)$ and $c(s_i)$. Next, consider $j \in \{1, ..., t\} \setminus \{i\}$, and let $\{y, z\} = \{a_j, b_j\}$. If y has a neighbor $y' \in S_\ell \cup ... \cup S_q$, then, as $c(y) \neq c(y')$, it follows that $c(y) \in L_i(y)$ by construction. So we may assume that y is anticomplete to $S_\ell \cup ... \cup S_q$, and so, by construction, $L_i(y) = \{2, 3\}$. By *(3b)*, it follows that the height of $a_j b_j$ is strictly less than ℓ, $y = s_j$, and that both N_i, M_i are proper subsets of N_j. Hence, by construction, $L_i(l_j) = \{1\}$. By *(3a)*, both N_i, M_i are non-empty, and so $c(l_j) = 1$, which implies $c(s_j) \in \{2, 3\}$, and the claim holds.

For every $i \in \{1, ..., t\}$, construct the subpalettes L_i, L_i' of L as above. Then $\mathcal{L}_1 = \{L_1, ..., L_t, L_1', ..., L_t'\}$ satisfies *(4a)* and *(4b)*. For a fixed $i \in \{1, ..., t\}$, the subpalettes L_i, L_i' of L can be constructed in time $O(|V(G)|^2)$, and so $\mathcal{L}_1$ can be constructed in time $O(|V(G)|^3)$. This proves *(4)*.

We say that (G, L, X) has a *type II coloring* if there exist distinct $i, j \in \{1, ..., t\}$, $z \in \{a_j, b_j\}$ and a coloring c of (G, L, X) such that

(II.1) $a_i b_i$ *and* $a_j b_j$ *have the same height* ℓ, *and*

(II.2) N_i *is not a proper subset of* $N(z) \cap (S_\ell \cup ... \cup S_q)$, *and*

(II.3) writing $\{y\} = \{a_j, b_j\} \setminus \{z\}$, *we have* $c(s_i) = c(z) = 1$, *and* $\{c(l_i), c(y)\} = \{2, 3\}$.

We now prove the following:

(5) There exists a set $\mathcal{L}_2$ *of* $O(|V(G)|^2)$ *subpalettes of* L *such that*

(5a) For each $L_2 \in \mathcal{L}_2$, $L_2(v) = L(v)$ *for every* $v \in \{x\} \cup S \cup Y$, *and* $|L_2(v)| \leq 2$ *for every* $v \in \hat{A} \cup \hat{B}$, *and*

(5b) (G, L, X) *has a type II coloring if and only if* (G, L_2, X) *is colorable for some* $L_2 \in \mathcal{L}_2$*; and for every* $L_2 \in \mathcal{L}_2$, *every coloring of* (G, L_2, X) *is a type II coloring of* (G, L, X).

Moreover, $\mathcal{L}_2$ *can be constructed in time* $O(|V(G)|^4)$.

Proof: Let $i, j \in \{1, ..., t\}$ be distinct, $\{y, z\} = \{a_j, b_j\}$, and suppose *(II.1)* and *(II.2)* are satisfied. First, set

- $L_{\{i,j\}}(l_i) = \{2\}$,
- $L_{\{i,j\}}(y) = \{3\}$,
- $L_{\{i,j\}}(s_i) = L_{\{i,j\}}(z) = \{1\}$, and
- $L_{\{i,j\}}(v) = L(v)$ for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$.

Next, for every $k \in \{1, ..., t\} \setminus \{i, j\}$, consider each $w \in \{a_k, b_k\}$. If $N(w) \cap (S_\ell \cup ... \cup S_q) \neq \emptyset$, then set

$$L_{\{i,j\}}(w) = \begin{cases} L(w) \setminus \{3\} & , \quad \text{if } N(w) \cap (S_\ell \cup ... \cup S_q) \subseteq N_i \\ L(w) \setminus \{2\} & , \quad \text{if } N(w) \cap (S_\ell \cup ... \cup S_q) \subseteq M_i \\ \{1\} & , \quad \text{otherwise} \end{cases}$$

Otherwise, if w is anticomplete to $S_\ell \cup ... \cup S_q$, then set $L_{\{i,j\}}(w) = \{2,3\}$. As above, construct the subpalette $L'_{\{i,j\}}$ of L, but with the roles of the colors 2 and 3 exchanged.

Clearly, if one of $(G, L_{\{i,j\}}, X)$ and $(G, L'_{\{i,j\}}, X)$ is colorable, then there exists a type II coloring of (G, L, X). Now, suppose c is a type II coloring of (G, L, X) with $c(s_i) = c(z) = 1$, and $\{c(l_i), c(y)\} = \{2, 3\}$. By symmetry, we may assume $c(l_i) = 2$ and $c(y) = 3$. We claim that $c(v) \in L_{\{i,j\}}(v)$ for all $v \in V(G)$. By definition, this is the case for a_i, b_i, a_j, b_j, and for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$. Since $L_{\{i,j\}}(v) = L(v) \subseteq \{2, 3\}$ for every $v \in S$, as $c(l_i) = 2$, it follows, that every vertex in N_i is colored 3. Let $M' = N(y) \cap (S_\ell \cup ... \cup S_q)$ and $N' = N(z) \cap (S_\ell \cup ... \cup S_q)$. Similarly, as $c(y) = 3$, it follows that every vertex in M' is colored 2. Hence, $N_i \cap M' = \emptyset$. By *(3a)*, $N_i \cup M_i = N' \cup M'$. Since N_i is not a proper subset of N', it follows that $N_i = N'$ and $M_i = M'$. And so, it follows that the colors of all the vertices in $S_\ell \cup ... \cup S_q$ are forced; namely $c(v) = 3$ for every $v \in N_i$, and $c(v) = 2$ for every $v \in M_i$. Next, consider $k \in \{1, ..., t\} \setminus \{i, j\}$, and let $u \in \{a_k, b_k\}$. If u has a neighbor $u' \in S_\ell \cup ... \cup S_q$, then, as $c(u) \neq c(u')$, it follows that $c(u) \in L_{\{i,j\}}(u)$ by construction. So we may assume that u is anticomplete to $S_\ell \cup ... \cup S_q$, and so, by construction, $L_{\{i,j\}}(u) = \{2, 3\}$. By *(3b)*, it follows that the height of $a_k b_k$ is strictly less than ℓ, $u = s_k$, and that both N_i, M_i are proper subsets of N_k. Hence, by construction, $L_i(l_k) = \{1\}$. By *(3a)*, both N_i, M_i are non-empty, and so $c(l_k) = 1$, which implies $c(s_k) \in \{2, 3\}$, and the claim holds.

For every distinct pair $i, j \in \{1, ..., t\}$ satisfying *(II.1)* and *(II.2)*, construct the subpalettes $L_{\{i,j\}}, L'_{\{i,j\}}$ of L as above. Let $\mathcal{L}_2$ be the set of all the subpalettes thus constructed, and observe that $|\mathcal{L}_2| \leq 4n^2$. Then $\mathcal{L}_2$ satisfies *(5a)* and *(5b)*. For distinct $i, j \in \{1, ..., t\}$, the corresponding subpalettes can be constructed in time $O(|V(G)|^2)$, and so $\mathcal{L}_2$ can be constructed in time $O(|V(G)|^4)$. This proves *(5)*.

We say that (G, L, X) has a *type III coloring* if for some $i \in \{1, ..., t\}$ where $a_i b_i$ has height ℓ, there exists a coloring c of (G, L, X) such that

(III.1) $c(l_i) \in \{2, 3\}$ *and* $c(s_i) = 1$,

(III.2) let $j \in \{1, ..., t\} \setminus \{i\}$ *such that the height of* a_jb_j *is at most* ℓ*; write* $\{y, z\} = \{a_j, b_j\}$. *If* N_i *is a proper subset of* $N(z) \cap (S_\ell \cup ... \cup S_q)$*, then* $c(z) = 1$.

We now prove the following:

(6) Suppose (G, L, X) *has no type I coloring, and no type II coloring. Then there exists a set* $\mathcal{L}_3$ *of* $O(|V(G)|)$ *subpalettes of* L *such that*

(6a) For each $L_3 \in \mathcal{L}_3$, $L_3(v) = L(v)$ *for every* $v \in \{x\} \cup S \cup Y$*, and* $|L_3(v)| \leq 2$ *for every* $v \in \hat{A} \cup \hat{B}$*, and*

(6b) (G, L, X) *has a type III coloring if and only if* (G, L_3, X) *is colorable for some* $L_3 \in \mathcal{L}_3$*; and for every* $L_3 \in \mathcal{L}_3$*, every coloring of* (G, L_3, X) *is a type III coloring of* (G, L, X).

Moreover, $\mathcal{L}_3$ *can be constructed in time* $O(|V(G)|^3)$.

Proof: Let $i \in \{1, ..., t\}$ and ℓ be the height of the edge a_ib_i. First, set

- $L_i(l_i) = \{2\}$,
- $L_i(s_i) = \{1\}$, and
- $L_i(v) = L(v)$ for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$.

Next, for every $j \in \{1, ..., t\} \setminus \{i\}$, consider each $y \in \{a_j, b_j\}$. If $N(y) \cap N_i \neq \emptyset$, then set $L_i(y) = L(y) \setminus \{3\}$. Otherwise, if y is anticomplete to N_i, then, taking $z \in \{a_j, b_j\} \setminus \{y\}$, set

$$L_i(y) = \begin{cases} L(y) \setminus \{1\} & , \quad \text{if } N_i \text{ is a proper subset of } N(z) \cap (S_\ell \cup ... \cup S_q) \\ L(y) \setminus \{3\} & , \quad \text{if } N_i \text{ is not a proper subset of } N(z) \cap (S_\ell \cup ... \cup S_q) \end{cases}$$

As above, construct the subpalette L_i' of L, but with the roles of the colors 2 and 3 exchanged.

First, we argue that every coloring of (G, L_i, X) and (G, L_i', X) is a type III coloring of (G, L, X). Suppose that one of (G, L_i, X) and (G, L_i', X) is colorable. By symmetry, we may assume that c is a coloring of (G, L_i, X), and so $c(l_i) = 2$, $c(s_i) = 1$, and *(III.1)* holds. Now, we show *(III.2)* holds. Suppose for $j \in \{1, ..., t\} \setminus \{i\}$ the height ℓ' of $a_j b_j$ is at most ℓ. Fix $\{y, z\} = \{a_j, b_j\}$. Since $c(l_i) = 2$, every vertex in N_i is colored 3. Suppose N_i is a proper subset of $N(z) \cap (S_\ell \cup ... \cup S_q)$, and so $c(z) \neq 3$. By *(3a)*, it follows that y is anticomplete to N_i, and so, by construction, $c(y) \neq 1$. Thus, since (G, L, X) does not have a type I coloring, it follows that $c(z) = 1$, and so *(III.2)* holds. Hence, c is a type III coloring of (G, L, X).

Next, we argue that if (G, L, X) has a type III coloring, then (G, L_3, X) is colorable for some $L_3 \in \mathcal{L}_3$. Suppose that c is a type III coloring of (G, L, X) with $c(l_i) \in \{2, 3\}$ and $c(s_i) = 1$. By symmetry, we may assume $c(l_i) = 2$. We claim that $c(v) \in L_i(v)$ for all $v \in V(G)$. By definition, this is the case for a_i, b_i, and for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$. Since $L_i(v) = L(v) \subseteq \{2, 3\}$ for every $v \in S$, as $c(l_i) = 2$, it follows that every vertex in N_i is colored 3. Next, consider $j \in \{1, ..., t\} \setminus \{i\}$, and let $\{y, z\} = \{a_j, b_j\}$. If y has a neighbor in N_i, then $c(y) \neq 3$, and it follows that $c(y) \in L_i(y)$ by construction. So we may assume that y is anticomplete to N_i. By *(3a)*, it follows that the height of $a_j b_j$ is at most ℓ, and that z is complete to N_i. Hence, $c(z) \in \{1, 2\}$. If N_i is a proper subset of $N(z) \cap (S_\ell \cup .... \cup S_q)$, then $c(z) = 1$, by *(III.2)*, and so $c(y) \neq 1$. Thus, we may assume that N_i is not a proper subset of $N(z) \cap (S_\ell \cup .... \cup S_q)$. Since z is complete to N_i, this implies that $N_i = N(z) \cap (S_\ell \cup .... \cup S_q)$. Hence, by *(3a)*, it follows that $M_i = N(y) \cap (S_\ell \cup .... \cup S_q)$, and so $a_j b_j$ also has height ℓ. Thus, since (G, L, X) does not admit a type II coloring, it follows that $c(y) \neq 3$, and the claim holds.

For every $i \in \{1, ..., t\}$, construct the subpalettes L_i, L_i' of L as above. Then $\mathcal{L}_3 = \{L_1, ..., L_t, L_1', ..., L_t'\}$ satisfies *(6a)* and *(6b)*. For a fixed $i \in \{1, ..., t\}$, the subpalettes L_i, L_i' of L can be constructed in time $O(|V(G)|^2)$, and so $\mathcal{L}_3$ can be constructed in time $O(|V(G)|^3)$. This proves *(6)*.

Finally, define the additional subpalette $\hat{L}$ of L such that for $v \in V(G)$

$$\hat{L}(v) = \begin{cases} L(v) & , \quad \text{if } v \in V(G) \setminus (\hat{A} \cup \hat{B}) \\ \{1\} & , \quad \text{if } v = l_i \text{ for some } i \in \{1, ..., t\} \\ \{2,3\} & , \quad \text{if } v = s_i \text{ for some } i \in \{1, ..., t\} \end{cases}$$

Define $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \{\hat{L}\}$. Note, $\mathcal{L}$ has size $O(|V(G)|^2)$, as it is dominated by $\mathcal{L}_2$, and can be constructed in time $O(|V(G)|^4)$, by *(4)*,*(5)* and *(6)*. By *(4a)*, *(5a)*, and *(6a)*, it follows that $\mathcal{L}$ satisfies *(a)*. We now argue that $\mathcal{L}$ satisfies *(b)*. Since $\mathcal{L}$ contains only subpalettes of L, it clearly follows that if (G, L', X) is colorable for some $L' \in \mathcal{L}$, then (G, L, X) is colorable; and for every $L' \in \mathcal{L}$, every coloring of (G, L', X) is a coloring of (G, L, X). Now, suppose that c is a coloring of (G, L, X). If c is a type I or II coloring of (G, L, X), then, by *(4b)* and *(5b)*, it follows that (G, L', X) is colorable for some $L' \in \mathcal{L}_1 \cup \mathcal{L}_2$. Hence, we may assume that (G, L, X) admits no coloring of type I or II. If $c(l_i) = 1$ for every $i \in \{1, ..., t\}$, then $c(s_i) \in \{2, 3\}$ for every $i \in \{1, ..., t\}$, and it follows that c is a coloring of $(G, \hat{L}, X)$, so we may assume not.

We claim that c is a type III coloring of (G, L, X). Let $a_i b_i$ be an edge with minimum height ℓ_{min} such that $c(l_i) \in \{2, 3\}$, and subject to that with N_i maximal. By symmetry, we may assume $c(l_i) = 2$. Since c is not a type I coloring of (G, L, X), it follows that $c(s_i) = 1$, and thus *(III.1)* is satisfied. By our choice of ℓ_{min} and i, for all $j \in \{1, ..., t\} \setminus \{i\}$ if the height of $a_j b_j$ is at most ℓ_{min} and N_i is a proper subset of $N(z) \cap (S_\ell \cup ... \cup S_q)$ for some $z \in \{a_j, b_j\}$, then $c(z) = 1$, and so *(III.2)* is satisfied. This proves that c is a type III coloring of (G, L, X). Now, by *(6b)*, it follows that (G, L', X) is colorable for some $L' \in \mathcal{L}_3$. This proves 5.2.3.

□

5.3 Reducing the Palettes: Part II

Let G be a graph, L a palette of G, and X a set of subsets of $V(G)$. Let $\mathcal{P}$ be a collection of triples (G', L', X'), where G' is an induced subgraph of G, L' is a subpalette of $L|_{V(G')}$, and X' is a set of subsets of $V(G')$. We say that $\mathcal{P}$ is *colorable* if at least one element of $\mathcal{P}$ is colorable. We say that $\mathcal{P}$ is a *restriction* of (G, L, X) if $\mathcal{P}$ being colorable implies that (G, L, X) is colorable, and we can extend a coloring of a colorable element of $\mathcal{P}$ to a coloring of (G, L, X) in time $O(|V(G)|)$.

First, we make the following easy observation:

5.3.1. *Let G be a graph, and let $v \in V(G)$. Let L be an order k palette of a graph G, and let X be a set of subsets of $V(G) \setminus \{v\}$. If $|L(v)| = k$ and in every coloring of $(G \setminus v, L|_{V(G)\setminus\{v\}}, X)$ at most $k-1$ colors are used to color $N(v)$, then (G, L, X) is colorable if and only if $(G \setminus v, L|_{V(G)\setminus\{v\}}, X)$ is colorable. Furthermore, we can extend a coloring of $(G \setminus v, L|_{V(G)\setminus\{v\}}, X)$ to a coloring of (G, L, X) in constant time.*

Next, we prove the following:

5.3.2. *Let G be a triangle-free graph, and let $u, v \in V(G)$ be adjacent. Let X be a set of subsets of $V(G) \setminus \{u, v\}$. Let L be an order 3 palette of G. Assume that:*

- *$|L(v)| = 3$, and in every coloring of $(G \setminus \{u, v\}, L|_{V(G)\setminus\{u,v\}}, X)$, at most two colors are used to color $N(v) \setminus \{u\}$, and*
- *$|L(u)| = 2$ with $L(y) \subseteq \{1, 2, 3\} \setminus L(u)$ for all $y \in N(u) \setminus \{v\}$.*

Then (G, L, X) is colorable if and only if $(G \setminus \{u, v\}, L|_{V(G)\setminus\{u,v\}}, X)$ is colorable. Furthermore, we can extend a coloring of $(G \setminus \{u, v\}, L|_{V(G)\setminus\{u,v\}}, X)$ to a coloring of (G, L, X) in constant time.

Proof. Clearly, if (G, L, X) is colorable, then $(G \setminus \{u, v\}, L|_{V(G)\setminus\{u,v\}}, X)$ is colorable. Now, suppose c is a coloring of $(G \setminus \{u, v\}, L|_{V(G)\setminus\{u,v\}}, X)$. We may assume that only colors 1 and 2 are used on $N(v) \setminus \{u\}$. Assigning $c(v) = 3$ and $c(u) \in L(u) \setminus \{3\}$, we obtain a coloring of (G, L, X). This proves 5.3.2.

□

Let Z be a set of subsets of $V(G)$. Given a coloring c of (G, L, Z) and a subset $X \subseteq V(G)$ define the subpalette L_c^X of L as follows: for every $v \in V(G)$, set

$$L_c^X(v) = \begin{cases} c(v) & , \quad \text{if } v \in X \\ L(v) & , \quad \text{otherwise} \end{cases}.$$

For a subset $Y \subseteq V(G) \setminus X$, let $L_c^{X,Y}$ be the subpalette of L_c^X obtained by updating the palettes of the vertices in Y with respect to X.

5.3.3. *If c is a coloring of (G, L, Z), then c is a coloring of $(G, L_c^{X,Y}, Z)$.*

Proof. Clearly, c is a coloring of $(G, L_c^{X,Y}, Z)$, since, by the definition of updating, $c(v) \in L_c^{X,Y}(v)$ for all $v \in V(G)$. This proves 5.3.3.

□

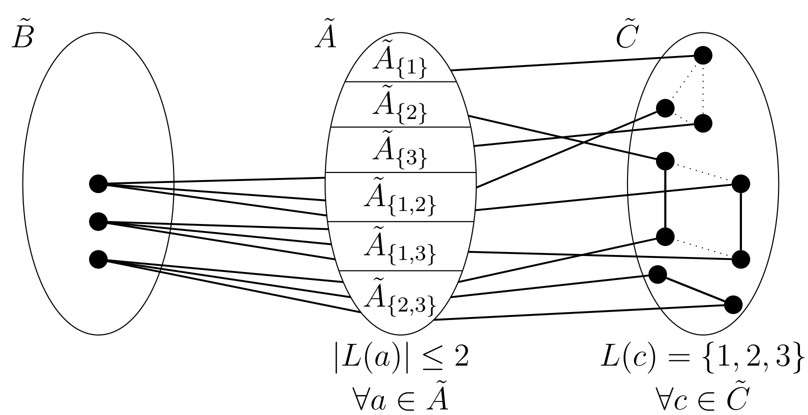


Figure 5.2: By 5.3.4, when we encounter the above situation we can reduce determining if (G, L, Z) is colorable to determining if a restriction $\mathcal{P}$ of (G, L, Z) is colorable. The elements of $\mathcal{P}$ are "closer" to being of the form required by 5.1.7 than (G, L, Z) is.

A vertex u in a graph G is *dominated* if there is $v \in V(G) \setminus \{u\}$ such that u is non-adjacent to v, and $N(u) \subseteq N(v)$. In this case we say that u is *dominated by v*. The following is the main result of this section:

5.3.4. *Let L be an order 3 palette of a $\{P_7, C_3\}$-free graph G, such that $V(G) = \tilde{A} \cup \tilde{B} \cup \tilde{C}$, where*

- *$\tilde{B}$ is anticomplete to $\tilde{C}$,*
- *every component of $\tilde{C}$ has at most 2 vertices,*
- *every vertex of $\tilde{C}$ has a neighbor in $\tilde{A}$,*
- *every vertex of $\tilde{A}$ has a neighbor in $\tilde{C}$,*
- *$|L(a)| \leq 2$ for every $a \in \tilde{A}$,*
- *$|L(c)| = 3$ for every $c \in \tilde{C}$, and*
- *G contains no dominated vertices.*

For every non-empty subset $X \subseteq \{1,2,3\}$, let $\tilde{A}_X = \{a \in \tilde{A}$ with $L(a) = X\}$. For every $c \in \tilde{C}$ and distinct $i, j \in \{1,2,3\}$, let $N_{\{i,j\}}(c) = N(c) \cap \tilde{A}_{\{i,j\}}$, and $M_{\{i,j\}}(c) = \tilde{A}_{\{i,j\}} \setminus N(c)$.

Assume also that:

- *For every $c_1, c_2 \in \tilde{C}$ and $\{i,j,k\} = \{1,2,3\}$, $N_{\{i,j\}}(c_1) \cap M_{\{i,j\}}(c_2)$ is complete to $M_{\{i,k\}}(c_1) \cap N_{\{i,k\}}(c_2)$.*
- *For distinct $i, j \in \{1,2,3\}$, there exists a vertex in $\tilde{B}$ complete to $\tilde{A}_{\{i,j\}}$.*

Let Z be a set of subsets of A. Then there exists a restriction $\mathcal{P}$ of (G, L, Z), of size $O(|V(G)|^7)$, such that

(a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ for every $(G', L', X') \in \mathcal{P}$, and

(b) Every $(G', L', X') \in \mathcal{P}$ is such that $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and $|X'|$ has size $O(|Z| + |V(G)|)$, and

(c) If (G, L, Z) is colorable, then $\mathcal{P}$ is colorable.

Moreover, if the partition $\tilde{A} \cup \tilde{B} \cup \tilde{C}$ *of* $V(G)$ *is given, then* $\mathcal{P}$ *can be computed in time* $O(|V(G)|^7)$.

Proof. Since $|L(a)| \leq 2$ for all $a \in \tilde{A}$, we obtain the partition $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup \tilde{A}_{\{1,2\}} \cup \tilde{A}_{\{1,3\}} \cup \tilde{A}_{\{2,3\}}$ of $\tilde{A}$. Let $i, j \in \{1, 2, 3\}$ be distinct. Since G is triangle-free and there exists a vertex in $\tilde{B}$ complete to $\tilde{A}_{\{i,j\}}$, it follows that $\tilde{A}_{\{i,j\}}$ is stable. Further, there is no vertex $c \in \tilde{C}$ such that $N(c) \subseteq \tilde{A}_{\{i,j\}}$, as such a vertex would be dominated by any vertex in $\tilde{B}$ complete to $\tilde{A}_{\{i,j\}}$. Thus, it follows that:

(1) Every vertex $c \in \tilde{C}$ *such that* $N(c) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$ *for some distinct* $i, j \in \{1, 2, 3\}$ *belongs to a component of* $\tilde{C}$ *of size* 2.

A coloring c of (G, L, Z) is a *type I coloring* if for distinct $i, j \in \{1, 2, 3\}$ there exists $z \in \tilde{C}$ and $x, y \in N_{\{i,j\}}(z)$ with $c(x) = i$ and $c(y) = j$.

(2) There exists a restriction $\mathcal{P}_1$ *of* (G, L, Z)*, of size* $O(|V(G)|^5)$*, such that:*

(2a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ *and* $X' = Z$ *for every* $(G', L', X') \in \mathcal{P}_1$*,*

(2b) Every $(G', L', X') \in \mathcal{P}_1$ *is such that* $L'(v) = L(v)$ *for every* $v \in \tilde{A} \cup \tilde{B}$*, and* $|L'(v)| \leq 2$ *for every* $v \in V(G') \cap \tilde{C}$*, and*

(2c) If (G, L, Z) *has a type I coloring, then* $\mathcal{P}_1$ *is colorable.*

Moreover, $\mathcal{P}_1$ *can be computed in time* $O(|V(G)|^7)$.

Proof: Let $z \in \tilde{C}$, $\{i, j, k\} = \{1, 2, 3\}$, and $x, y \in N_{\{i,j\}}(z)$. Let $U_{(x,y,z)}$ be the set of all vertices $u \in M_{\{i,k\}}(z) \cup M_{\{j,k\}}(z)$ for which there exists $c \in \tilde{C}$ such that c is adjacent to u and anticomplete to $\{x, y\}$. Since $x, y \in N_{\{i,j\}}(z)$, by assumption, it follows that $\{x, y\}$ is complete to $U_{(x,y,z)}$. Set

- $L'(v) = \{i\}$, for all $v \in \{x\} \cup N_{\{i,k\}}(z)$,
- $L'(v) = \{j\}$, for all $v \in \{y\} \cup N_{\{j,k\}}(z)$,

- $L'(v) = \{k\}$ for $v \in \{z\} \cup U_{(x,y,z)}$, and
- $L'(v) = L(v)$, for every $v \in V(G) \setminus (N_{\{i,k\}}(z) \cup N_{\{j,k\}}(z) \cup U_{(x,y,z)} \cup \{x, y, z\})$.

Let $A = \{x, y\} \cup (\tilde{A} \setminus \tilde{A}_{\{i,j\}})$. Next, update the palettes of the vertices in $\tilde{C}$ with respect to A. If $c \in \tilde{C}$ has a neighbor in $(M_{\{i,k\}}(z) \cup M_{\{j,k\}}(z)) \setminus U_{(x,y,z)}$, then it follows, by the definition of $U_{(x,y,z)}$, that c is not anticomplete to $\{x, y\}$. And so after updating, for every $c \in \tilde{C}$ if $N(c) \cap (\tilde{A} \setminus \tilde{A}_{\{i,j\}})$ is non-empty, then $|L'(c)| \leq 2$. Let F be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,j\}}$. Initialize $D^{ij}_{(x,y,z)} = \emptyset$. Consider a vertex $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$. Then $|L'(c_1)| = 3$, and, by *(1)*, c_1 is adjacent to some $c_2 \in \tilde{C}$ which is anticomplete to $\tilde{A}_{\{i,j\}}$. Since every vertex of $\tilde{C}$ has a neighbor in $\tilde{A}$, it follows that $|L'(c_2)| \leq 2$. If $|L'(c_2)| = 1$, set $L'(c_1) = L'(c_1) \setminus L'(c_2)$. Otherwise, $|L'(c_2)| = 2$. Since c_2 is anticomplete to $\tilde{A}_{\{i,j\}}$, it follows that that

$$N(c_2) \cap \tilde{A} \subseteq \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,k\}}(z) \cup N_{\{j,k\}}(z) \cup U_{(x,y,z)}.$$

In particular $|L'(v)| = 1$ for every $v \in N(c_2) \cap \tilde{A}$, and so, by the definition of updating, $L'(v) \subseteq \{1, 2, 3\} \setminus L(c_2)$ for all $v \in N(c_2) \setminus \{c_1\}$. Therefore, by 5.3.2, it follows that $(G \setminus D^{ij}_{(x,y,z)}, L'|_{V(G) \setminus D^{ij}_{(x,y,z)}}, Z)$ is colorable if and only if $(G \setminus (D^{ij}_{(x,y,z)} \cup \{c_1, c_2\}), L'|_{V(G) \setminus (D^{ij}_{(x,y,z)} \cup \{c_1,c_2\})}, Z)$ is colorable. In this case, set $D^{ij}_{(x,y,z)} = D^{ij}_{(x,y,z)} \cup \{c_1, c_2\}$. Carry out this procedure for every $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$. Let $G^{ij}_{(x,y,z)} = G \setminus D^{ij}_{(x,y,z)}$. Repeatedly applying the previous argument, it follows that (G, L', Z) is colorable if and only if $(G^{ij}_{(x,y,z)}, L'|_{V(G^{ij}_{(x,y,z)})}, Z)$ is colorable. By assumption, there exists $b \in \tilde{B}$ complete to $\tilde{A}_{\{i,j\}}$. Let $\mathcal{L}'$ be the set of $O(|V(G)|^2)$ subpalettes of $L'|_{V(G^{ij}_{(x,y,z)})}$ obtained from 5.2.3 applied with

- $x = b$,
- $S = \tilde{A}_{\{i,j\}}$,
- $\hat{A} \cup \hat{B} = F$,
- $Y = V(G^{ij}_{(x,y,z)}) \setminus (\{b\} \cup \tilde{A}_{\{i,j\}} \cup F)$, and

- $X = Z$.

For each $\hat{L} \in \mathcal{L}'$, define the subpalette $L^{ij}_{(x,y,z)}(\hat{L})$ of $L|_{V(G^{ij}_{(x,y,z)})}$ as follows: For $v \in V(G^{ij}_{(x,y,z)})$ set

$$L^{ij}_{(x,y,z)}(\hat{L})(v) = \begin{cases} \hat{L}(v) & , \quad \text{if } v \in \tilde{C} \setminus D^{ij}_{(x,y,z)} \\ L'(v) & , \quad \text{otherwise} \end{cases}$$

Let

$$\mathcal{P}^{ij}_{(x,y,z)} = \{(G^{ij}_{(x,y,z)}, L^{ij}_{(x,y,z)}(\hat{L}), Z) : \hat{L} \in \mathcal{L}'\}.$$

Let $\mathcal{P}_1$ be the union of all $\mathcal{P}^{ij}_{(x,y,z)}$. Since there are at most $3|V(G)|^3$ choices of x, y, z and $\{i, j\}$, and each set F can be found in time $O(|V(G)|^2)$, it follows that building $\mathcal{P}_1$ requires $O(|V(G)|^3)$ applications of 5.2.3, and so $\mathcal{P}_1$ can be constructed in time $O(|V(G)|^7)$. Now, we argue that $\mathcal{P}_1$ is indeed a restriction (G, L, Z). Suppose $\mathcal{P}^{ij}_{(x,y,z)}$ is colorable. By 5.2.3, it follows that $(G^{ij}_{(x,y,z)}, L'|_{V(G^{ij}_{(x,y,z)})}, Z)$ is colorable, and so, as observed above, by construction and 5.3.2, it follows that (G, L', Z) is colorable. Since L' is a subpalette of L, we deduce that (G, L, Z) is colorable. This proves that $\mathcal{P}_1$ is indeed a restriction (G, L, Z).

By construction and 5.2.3, *(2a)* and *(2b)* hold. Next, we show that *(2c)* holds. We need to prove that if (G, L, Z) admits a type I coloring, then $\mathcal{P}_1$ is colorable. Suppose c is a type I coloring of (G, L, Z). Let $z \in \tilde{C}$ with $c(z) = k$ and $x, y \in N_{\{i,j\}}(z)$ with $c(x) = i$ and $c(y) = j$. We claim that the restriction $\mathcal{P}^{ij}_{(x,y,z)}$ is colorable. Since z is complete to $N_{\{i,k\}}(z) \cup N_{\{j,k\}}(z)$, it follows that $c(v) = i$ for every $v \in N_{\{i,k\}}(z)$ and $c(v) = j$ for every $v \in N_{\{j,k\}}(z)$. By assumption, for every $c_1, c_2 \in \tilde{C}$ we have that $N_{\{i,j\}}(c_1) \cap M_{\{i,j\}}(c_2)$ is complete to $M_{\{i,k\}}(c_1) \cap N_{\{i,k\}}(c_2)$. Taking $c_1 = z$, it follows that $\{x, y\}$ is complete to every $v \in M_{\{i,k\}}(z) \cup M_{\{j,k\}}(z)$ for which there exists $c \in \tilde{C}$ such that c is adjacent to v and anticomplete to $\{x, y\}$, that is, $\{x, y\}$ is complete to $U_{(x,y,z)}$. Consequently, $c(v) = k$ for all $v \in U_{(x,y,z)}$. Also $c(v) = k$ for every $v \in \tilde{B}$ that is complete to $\tilde{A}_{\{i,j\}}$. By 5.3.3, it follows that c is a coloring of $(G, L^{A,\tilde{C}}_c, Z)$. Let F be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,j\}}$. Let C' be the set of vertices $v \in \tilde{C} \setminus F$ with $N(v) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$. By construction, $c(v) \in L''(v)$ for all $L'' \in \mathcal{L}^{ij}_{(x,y,z)}$ and $v \in V(G^{ij}_{(x,y,z)}) \setminus (C' \cup F)$. Next, we show that the claim holds for every vertex in $C' \setminus D^{ij}_{(x,y,z)}$.

By *(1)* and construction, every vertex $c_1 \in C' \setminus D^{ij}_{(x,y,z)}$ is adjacent to some $c_2 \in \tilde{C} \setminus D^{ij}_{(x,y,z)}$, and $|L^{A,\tilde{C}}_c(c_2)| = 1$. Since $c(c_1) \neq c(c_2)$, it follows that $c(c_1) \in L''(c_1)$ for all $L'' \in \mathcal{L}^{ij}_{(x,y,z)}$. Now $\mathcal{P}^{ij}_{(x,y,z)}$ is colorable by 5.2.3 and *(2c)* follows. This proves *(2)*.

Let $\mathcal{X} = \{N(v) \cap \tilde{A}_{\{i,j\}} : v \in \tilde{C}, 1 \leq i < j \leq 3\}$, and let $\mathcal{Z} = Z \cup \mathcal{X}$.

From *(2)* it follows that:

(3) If (G, L, Z) does not have a type I coloring, then (G, L, Z) is colorable if and only if $(G, L, \mathcal{Z})$ is colorable.

(4) Assume that (G, L, Z) does not have a type I coloring. If there exists a vertex in $\tilde{C}$ with neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, then there exists a restriction $\mathcal{P}_2$ of size $O(|V(G)|^2)$ of (G, L, Z) such that:

(4a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ and $X' = \mathcal{Z}$ for every $(G', L', X') \in \mathcal{P}_2$,

(4b) Every $(G', L', X') \in \mathcal{P}_2$ is such that $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and

(4c) If (G, L, Z) is colorable, then $\mathcal{P}_2$ is colorable.

Moreover, $\mathcal{P}_2$ can be computed in time $O(|V(G)|^4)$.

If there does not exist a vertex in $\tilde{C}$ with neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, then $\mathcal{P}_2 = \emptyset$.

Proof: Let $w \in \tilde{C}$. For $\{i, j, k\} = \{1, 2, 3\}$, let $P_{\{j,k\}}(w)$ be the set of vertices $v \in M_{\{j,k\}}(w)$ for which there exists $c \in \tilde{C}$ adjacent to v and anticomplete to $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$. By assumption, it follows that $P_{\{j,k\}}(w)$ is complete to $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$. Thus, if $v \in M_{\{j,k\}}(w) \setminus P_{\{j,k\}}(w)$, then every vertex in $N(v) \cap \tilde{C}$ has a neighbor in $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$.

If there does not exist a vertex in $\tilde{C}$ with neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, set $\mathcal{P}_2 = \emptyset$, and halt. Otherwise, suppose $w \in \tilde{C}$ has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$

and $\tilde{A}_{\{2,3\}}$. Let $\{i,j,k\} = \{1,2,3\}$ be such that $P_{\{j,k\}}(w) = \emptyset$ (we will show later that if (G, L, Z) is colorable, then such $P_{\{j,k\}}(w)$ exists); if no such $\{i,j,k\}$ exists, set $\mathcal{P}_2 = \emptyset$, and halt. Set

- $L'(v) = \{i\}$, for all $v \in \{w\} \cup P_{\{i,j\}}(w)$,
- $L'(v) = \{j\}$, for all $v \in N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w)$,
- $L'(v) = \{k\}$, for all $v \in N_{\{i,k\}}(w)$, and
- $L'(v) = L(v)$, for every $v \in V(G) \setminus (N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w) \cup P_{\{i,j\}}(w) \cup \{w\})$.

Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w) \cup P_{\{i,j\}}(w)$. Next, update the palettes of all the vertices in $\tilde{C}$ with respect to A. If $c \in \tilde{C}$ has a neighbor in $(M_{\{i,j\}}(w) \cup M_{\{j,k\}}(w)) \setminus P_{\{i,j\}}$, then c is not anticomplete to $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w)$, and so after updating, for every $c \in \tilde{C}$, if $N(c) \cap (\tilde{A} \setminus M_{\{i,k\}}(w)) \neq \emptyset$, then $|L'(v)| \leq 2$. Let F to be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $M_{\{i,k\}}(w)$. Initialize $D_{(i,j,k)} = \emptyset$. Consider a vertex $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w)$. Then $|L'(c_1)| = 3$, and, by *(1)*, c_1 is adjacent to some $c_2 \in \tilde{C}$. Since in every coloring of $(G \setminus c_1, L', \mathcal{Z})$ at most two colors appear in $N(c_1) \subseteq \{c_2\} \cup M_{\{i,k\}}(w)$, by 5.3.1, it follows that $(G \setminus D_{(i,j,k)}, L'|_{V(G)\setminus D_{(i,j,k)}}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D_{(i,j,k)} \cup \{c_1\}), L'|_{V(G)\setminus(D_{(i,j,k)}\cup\{c_1\})}, \mathcal{Z})$ is colorable. In this case, set $D_{(i,j,k)} = D_{(i,j,k)} \cup \{c_1\}$. Carry out this procedure for every $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w)$. Let $G_{(i,j,k)} = G \setminus D_{(i,j,k)}$. Repeatedly applying the previous argument, it follows that $(G, L', \mathcal{Z})$ is colorable if and only if $(G_{(i,j,k)}, L'|_{V(G_{(i,j,k)})}, \mathcal{Z})$ is colorable. By assumption, there exists $b \in \tilde{B}$ complete to $\tilde{A}_{\{i,k\}}$; therefore b is complete to $M_{\{i,k\}}(w)$. Let $\mathcal{L}'$ be the set of $O(|V(G)|^2)$ subpalettes of $L'|_{V(G_{(i,j,k)})}$ obtained from 5.2.3 applied with

- $x = b$,
- $S = M_{\{i,k\}}(w)$,
- $\hat{A} \cup \hat{B} = F$,

- $Y = V(G_{(i,j,k)}) \setminus (\{b\} \cup M_{\{i,k\}}(w) \cup F)$, and
- $X = \mathcal{Z}$.

For each $\hat{L} \in \mathcal{L}'$, define the subpalette $L_{(i,j,k)}(\hat{L})$ of $L|_{V(G_{(i,j,k)})}$ as follows: For $v \in V(G_{(i,j,k)})$ set

$$L_{(i,j,k)}(\hat{L})(v) = \begin{cases} \hat{L}(v) & , \quad \text{if } v \in \tilde{C} \setminus D_{(i,j,k)} \\ L'(v) & , \quad \text{otherwise} \end{cases}$$

Let

$$\mathcal{P}_{(i,j,k)} = \{(G_{(i,j,k)}, L_{(i,j,k)}(\hat{L}), \mathcal{Z}) : \hat{L} \in \mathcal{L}'\}.$$

Let $\mathcal{P}_2$ be union of all $\mathcal{P}_{(i,j,k)}$. Since w can be found in time $O(|V(G)|^2)$, there are 6 choices of (i,j,k), and each set F can be found in time $O(|V(G)|^2)$, it follows that building $\mathcal{P}_2$ requires 6 applications of 5.2.3, and so $\mathcal{P}_2$ can be constructed in time $O(|V(G)|^4)$. Now, we argue that $\mathcal{P}_2$ is indeed a restriction of (G,L,Z). Suppose $\mathcal{P}_{(i,j,k)}$ is colorable. By 5.2.3, it follows that $(G_{(i,j,k)}, L'|_{V(G_{(i,j,k)})}, \mathcal{Z})$ is colorable, and so, as observed above, by construction and 5.3.1, it follows that $(G, L', \mathcal{Z})$ is colorable. Since L' is a subpalette of L, and $Z \subseteq \mathcal{Z}$, we deduce that (G,L,Z) is colorable.

By construction and 5.2.3, *(4a)* and *(4b)* hold. Next, we show that *(4c)* holds. Suppose c is a coloring of (G,L,Z) with $c(w) = i$ where $w \in \tilde{C}$ has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$. By *(3)*, c is a coloring of $(G,L,\mathcal{Z})$, and by symmetry, we may assume that $j,k \in \{1,2,3\} \setminus \{i\}$ are such that $c(v) = j$ for all $v \in N_{\{j,k\}}(w)$. We claim that the restriction $\mathcal{P}_{(i,j,k)}$ is colorable. Clearly, every set in $\mathcal{Z}$ is monochromatic in c. It follows that $c(v) = j$ for every $v \in N_{\{i,j\}}(w)$, and $c(v) = k$ for every $v \in N_{\{i,k\}}(w)$. Since $N_{\{j,k\}}(w)$ is complete to $P_{\{i,j\}}(w)$, it follows that $c(v) = i$ for every $v \in P_{\{i,j\}}(w)$. Since $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$ is complete to $P_{\{j,k\}}(w)$, it follows that $P_{\{j,k\}}(w)$ is empty. Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w) \cup P_{\{i,j\}}(w)$.

By 5.3.3, it follows that c is a coloring of $(G, L_c^{A,\tilde{C}})$. Let F to be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $M_{\{i,k\}}(w)$. Let C' be the set of vertices $v \in \tilde{C} \setminus F$ with $N(v) \cap \tilde{A} \subseteq M_{\{i,k\}}(w)$. By construction, $c(v) \in L''(v)$ for all $L'' \in \mathcal{L}_{(i,j,k)}$ and $v \in V(G_{(i,j,k)}) \setminus (C' \cup F)$. Next, we show that the claim holds for every vertex in $C' \setminus D_{(i,j,k)}$. By *(1)* and construction, every vertex $c_1 \in C' \setminus D_{(i,j,k)}$ is adjacent to some vertex $c_2 \in \tilde{C} \setminus D_{(i,j,k)}$. By construction, both c_1 and c_2 have a neighbor in $M_{\{i,k\}}(w)$, hence belong to F, and so *(4c)* follows from 5.2.3. This proves *(4)*.

A coloring c of (G, L, Z) is a *type II coloring* if there exists $w \in \tilde{C}$ with neighbors $x, y \in \tilde{A}_{\{1,2\}} \cup \tilde{A}_{\{1,3\}} \cup \tilde{A}_{\{2,3\}}$ such that $c(x) \neq c(y)$.

(5) Assume that (G, L, Z) does not have a type I coloring, and that no vertex of $\tilde{C}$ has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$. If G has a type II coloring, then there exists a restriction $\mathcal{P}_3$ of (G, L, Z) of size $O(|V(G)|^7)$ such that:

(5a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ and $X' = \mathcal{Z}$ for every $(G', L', X') \in \mathcal{P}_3$,

(5b) Every $(G', L', X') \in \mathcal{P}_3$ is such that $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and

(5c) $\mathcal{P}_3$ is colorable.

Moreover, $\mathcal{P}_3$ can be computed in time $O(|V(G)|^7)$.

If G does not have a type II coloring, then $\mathcal{P}_3 = \emptyset$.

Proof: Let $\{i, j, k\} = \{1, 2, 3\}$. If there does not exist a vertex $w \in \tilde{C}$ anticomplete to $\tilde{A}_{\{k\}}$ with $N_{\{i,k\}}(w) \neq \emptyset$ and $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \neq \emptyset$, set $\mathcal{P}_3 = \emptyset$ and halt. Otherwise, let $w \in \tilde{C}$ be anticomplete to $\tilde{A}_{\{k\}}$ with $N_{\{i,k\}}(w) \neq \emptyset$ and $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \neq \emptyset$. Since no vertex of $\tilde{C}$ has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, it follows that either $N_{\{i,j\}}(w) = \emptyset$ or $N_{\{j,k\}}(w) = \emptyset$. Set

- $L'(w) = \{k\}$,

- $L'(v) = \{i\}$, for every $v \in N_{\{i,k\}}(w)$,
- $L'(v) = \{j\}$, for every $v \in N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w)$, and
- $L'(v) = L(v)$, for every $v \in V(G) \setminus (N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w) \cup N_{\{j,k\}}(w) \cup \{w\})$.

Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w) \cup N_{\{j,k\}}(w) \cup \{w\}$. First, update the palettes of the vertices in $\tilde{A}$ with respect to A. Then, update the palettes of all the vertices in $\tilde{C}$ with respect to $\tilde{A}$. And so after updating, for every $c \in \tilde{C}$ if $N(c) \cap A$ is non-empty, then $|L'(c)| \leq 2$. Let $v \in \tilde{C}$ with $|L'(v)| = 3$. Then, by the definition of updating, $N(v) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$. Let $F_{\{i,k\}}$ be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,k\}}$, and let $F_{\{j,k\}}$ be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $\tilde{A}_{\{j,k\}}$. Initialize $D^w_{(i,j,k)} = \emptyset$. Consider a vertex $c_1 \in \tilde{C} \setminus (F_{\{i,k\}} \cup F_{\{j,k\}})$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$ and $|L'(c_1)| = 3$. Recall that in every coloring of $(G \setminus c_1, L', \mathcal{Z})$ at most two colors appear in $N(c_1) \cap (M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w))$. Therefore, if c_1 is anticomplete to $\tilde{C} \setminus \{c_1\}$, then, by 5.3.1, it follows that $(G \setminus D^w_{(i,j,k)}, L'|_{V(G)\setminus D^w_{(i,j,k)}}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D^w_{(i,j,k)} \cup \{c_1\}), L'|_{V(G)\setminus(D^w_{(i,j,k)}\cup\{c_1\})}, \mathcal{Z})$ is colorable. In this case, set $D^w_{(i,j,k)} = D^w_{(i,j,k)} \cup \{c_1\}$. So we may assume c_1 is adjacent to some $c_2 \in \tilde{C}$. Suppose that c_1 is anticomplete to at least one of $M_{\{i,k\}}(w)$ and $M_{\{j,k\}}(w)$. Then in every coloring of $(G \setminus c_1, L', \mathcal{Z})$ at most one color appears in $N(c_1) \cap (M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w))$, and so, since $N(c_1) \subseteq \{c_2\} \cup M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$, we deduce that at most two colors appear in $N(c_1)$. Thus, again by 5.3.1, $(G \setminus D^w_{(i,j,k)}, L'|_{V(G)\setminus D^w_{(i,j,k)}}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D^w_{(i,j,k)} \cup \{c_1\}), L'|_{V(G)\setminus(D^w_{(i,j,k)}\cup\{c_1\})}, \mathcal{Z})$ is colorable. In this case, set $D^w_{(i,j,k)} = D^w_{(i,j,k)} \cup \{c_1\}$. Therefore we may assume that c_1 has both a neighbor in $M_{\{i,k\}}(w)$ and a neighbor in $M_{\{j,k\}}(w)$. Since $c_1 \notin F_{\{i,k\}} \cup F_{\{j,k\}}$, it follows that c_2 is anticomplete to $\tilde{A}_{\{i,k\}} \cup \tilde{A}_{\{j,k\}}$. This implies that every neighbor of c_2 in $\tilde{A}_{\{i,j\}}$ either belongs to $N_{\{i,j\}}(w)$ or is complete to $N_{\{i,k\}}(w)$. This implies that $|L'(y)| = 1$ for every $y \in N(c_2) \setminus \{c_1\}$, and so, by the definition of updating, $L'(y) \subseteq \{1,2,3\} \setminus L(c_2)$ for all $y \in N(c_2) \setminus \{c_1\}$. If $|L'(c_2)| = 1$, set $L'(c_1) = L'(c_1) \setminus L'(c_2)$. Otherwise, $|L'(c_2)| = 2$ and so, by 5.3.2, it follows that $(G \setminus D^w_{(i,j,k)}, L'|_{V(G)\setminus D^w_{(i,j,k)}}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D^w_{(i,j,k)} \cup \{c_1, c_2\}), L'|_{V(G) \setminus (D^w_{(i,j,k)} \cup \{c_1,c_2\})}, \mathcal{Z})$ is colorable. In this case, set $D^w_{(i,j,k)} = D^w_{(i,j,k)} \cup \{c_1, c_2\}$. Carry out this procedure for every $c_1 \in \tilde{C} \setminus (F_{\{i,k\}} \cup F_{\{j,k\}})$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$ and $|L'(c_1)| = 3$. Let $G^w_{(i,j,k)} = G \setminus D^w_{(i,j,k)}$. Repeatedly applying the previous argument, it follows that $(G, L', \mathcal{Z})$ is colorable if and only if $(G^w_{(i,j,k)}, L'|_{V(G^w_{(i,j,k)})}, \mathcal{Z})$ is colorable. By assumption, there exists $b \in \tilde{B}$ complete to $\tilde{A}_{\{i,k\}}$. Let $\mathcal{L}'$ be the set of $O(|V(G)|^2)$ subpalettes of $L'|_{V(G^w_{(i,j,k)})}$ obtained from 5.2.3 applied with

- $x = b$,
- $S = \tilde{A}_{\{i,k\}}$,
- $\hat{A} \cup \hat{B} = F_{\{i,k\}}$,
- $Y = V(G^w_{\{i,j,k\}}) \setminus (\{b\} \cup \tilde{A}_{\{i,k\}}(w) \cup F_{\{i,k\}})$, and
- $X = \mathcal{Z}$.

For each $\hat{L} \in \mathcal{L}'$, define the subpalette $L^w_{(i,j,k)}(\hat{L})$ of $L|_{V(G_{(i,j,k)})}$ as follows: For $v \in V(G_{(i,j,k)})$ set

$$L^w_{(i,j,k)}(\hat{L})(v) = \begin{cases} \hat{L}(v) & , \quad \text{if } v \in \tilde{C} \setminus D^w_{(i,j,k)} \\ L'(v) & , \quad \text{otherwise} \end{cases}$$

$$\mathcal{P}^w_{(i,j,k)} = \{(G^w_{(i,j,k)}, L^w_{(i,j,k)}(\hat{L}), \mathcal{Z}) : \hat{L} \in \mathcal{L}'\}.$$

By assumption, there exists $b' \in \tilde{B}$ complete to $\tilde{A}_{\{j,k\}}$. For each $\hat{L} \in \mathcal{L}'$, let $\mathcal{L}''(\hat{L})$ be the set of $O(|V(G)|^2)$ subpalettes of $\hat{L}$ obtained from 5.2.3 applied with

- $x = b'$,
- $S = A_{\{j,k\}}$,
- $\hat{A} \cup \hat{B} = F_{\{j,k\}}$,
- $Y = V(G^w_{\{i,j,k\}}) \setminus (\{b'\} \cup A_{\{j,k\}}(w) \cup F_{\{j,k\}})$, and

- $X = \mathcal{Z}$.

Finally, for every $\hat{L} \in \mathcal{L}'$ and $L'' \in \mathcal{L}''(\hat{L})$, define the subpalette $L^w_{(i,j,k)}(\hat{L}, L'')$ of $L|_{V(G_{(i,j,k)})}$ as follows: For every $v \in V(G^w_{(i,j,k)})$ set

$$L^w_{\{i,j,k\}}(\hat{L}, L'')(v) = \begin{cases} L''(v) & , \quad \text{if } v \in \tilde{C} \setminus D^w_{(i,j,k)} \\ \hat{L}(v) & , \quad \text{otherwise} \end{cases}$$

Let

$$\mathcal{P}^w_{(i,j,k)} = \{(G^w_{(i,j,k)}, L^w_{(i,j,k)}(\hat{L}, L''), \mathcal{Z}) : \hat{L} \in \mathcal{L}', L'' \in \mathcal{L}''(\hat{L})\}.$$

Let $\mathcal{P}_3$ be the union of all $\mathcal{P}^w_{(i,j,k)}$. Since there are at most $6n$ choices of w and $\{i, j, k\}$, and $F_{\{i,k\}}$, $F_{\{j,k\}}$ can be found in time $O(|V(G)|^2)$, it follows that building each $\mathcal{P}^w_{(i,j,k)}$ requires $O(|V(G)|^2)$ applications of 5.2.3, and so $\mathcal{P}_3$ can be constructed in time $O(|V(G)|^7)$. Now, we argue that $\mathcal{P}_3$ is indeed a restriction of (G, L, Z). Suppose some $\mathcal{P}^w_{(i,j,k)}$ is colorable. By 5.2.3, it follows that $(G^w_{(i,j,k)}, L'|_{V(G^w_{(i,j,k)})}, \mathcal{Z})$ is colorable, and so, as argued above, by 5.3.1 and 5.3.2, it follows that $(G, L', \mathcal{Z})$ is colorable. Since L' is a subpalette of L, and $Z \subseteq \mathcal{Z}$, we deduce that (G, L, Z) is colorable, and a coloring of (G, L, Z) can be reconstructed in linear time.

Suppose c is a type II coloring of (G, L, Z). By (3), c is a coloring of $(G, L, \mathcal{Z})$. By construction and 5.2.3, *(5a)* and *(5b)* hold. Next, we show that *(5c)* holds. Let $w \in \tilde{C}$ have neighbors of two different colors (under c) in $\tilde{A}_{\{1,2\}} \cup \tilde{A}_{\{1,3\}} \cup \tilde{A}_{\{2,3\}}$. By symmetry, we may assume $c(w) = k$ with $N_{\{i,k\}}(w) \neq \emptyset$. Then w is anticomplete to $\tilde{A}_{\{k\}}$. Since G admits a type II coloring and no type I coloring, we deduce that $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \neq \emptyset$. We claim that $\mathcal{P}^w_{(i,j,k)}$ is colorable. It follows that $c(v) = i$ for every $v \in N_{\{i,k\}}(w)$ and that $c(u) = j$ for every $u \in N_{\{j,k\}}(w)$. We claim that $c(v) = j$ for every $v \in N_{\{i,j\}}(w)$. Suppose not. Then $N_{\{i,j\}}(w) \neq \emptyset$ and $N_{\{j,k\}}(w) = \emptyset$. Since c is a type II coloring, it follows that there exists $y \in N_{\{i,j\}}(w)$ with $c(y) = j$. But since (G, L, Z) has no type I coloring, it follows that $c(u) = j$ for every $u \in N_{\{j,k\}}(w)$. This proves the claim.

Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w) \cup N_{\{j,k\}}(w) \cup \{w\}$. Let M be the palette of G obtained from L by first updating the palettes of the vertices in $\tilde{A}$ with respect to A, and then updating the palettes of the vertices of $\tilde{C}$ with respect to $\tilde{A}$. It follows that c is a coloring of $(G, M, \mathcal{Z})$.

Let $F_{\{i,k\}}$ be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,k\}}$, and let $F_{\{j,k\}}$ be the vertex set of the 2-vertex components of $\tilde{C}$ such that both vertices of the component have a neighbor in $\tilde{A}_{\{j,k\}}$. Let C' be the set of vertices $v \in \tilde{C} \setminus (F_{\{i,k\}} \cup F_{\{j,k\}})$ with $N(v) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$. By construction, $c(v) \in L''(v)$ for all $L'' \in \mathcal{L}^w_{\{i,j,k\}}$ and $v \in V(G^w_{\{i,j,k\}}) \setminus (C' \cup F_{\{i,k\}} \cup F_{\{j,k\}})$. Next, we consider vertices in $C' \setminus D^w_{(i,j,k)}$. By construction, every vertex $c_1 \in C' \setminus D^w_{(i,j,k)}$ has both a neighbor in $M_{\{i,k\}}(w)$, and a neighbor in $M_{\{j,k\}}(w)$, is adjacent to some vertex $c_2 \in \tilde{C}$, and c_2 has a neighbor in at least one of $\tilde{A}_{\{i,k\}}$ and $\tilde{A}_{\{j,k\}}$. Moreover, $|L_c^{\tilde{A},\tilde{C}}(c_2)| = 1$, and so, since $c(c_1) \neq c(c_2)$, it follows that $c(c_1) \in L''(c_1)$ for all $L'' \in \mathcal{L}^w_{(i,j,k)}$. Now $\mathcal{P}^w_{(i,j,k)}$ is colorable by 5.2.3, and *(5c)* follows. This proves *(5)*.

(6) Assume (G, L, Z) does not have a type I or a type II coloring. Then there exists a restriction $\{(G', L', Z')\}$ of (G, L, Z) such that the following hold:

(6a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$,

(6b) $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and

(6c) If (G, L, Z) is colorable, then (G', L', Z') is colorable.

Moreover, (G', L', Z') can be computed in time $O(|V(G)|^2)$.

Proof: Let C' be the set of vertices in $\tilde{C}$ anticomplete to $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}}$. Let $G' = G \setminus C'$, let $W = \{N(v) \cap \tilde{A} : v \in C'\}$, and let $Z' = Z \cup W$. Update the palettes of all the vertices in $\tilde{C} \setminus C'$ with respect to $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}}$, and let L' be the palette $L|_{V(G')}$. Clearly, *(6a)* and *(6b)* hold. Suppose that (G, L, Z) is colorable. Since (G, L, Z) has no type I and no type II coloring, it follows that every set in Z' is monochromatic in every coloring of (G, L, Z), and therefore (G', L', Z') is colorable. Thus *(6c)* holds.

Now, we argue that $\{(G', L', Z')\}$ is a restriction of (G, L, Z). Let c be a coloring of (G', L', Z'); we need to show that c can be extended to a coloring of (G, L, Z) in time $O(|V(G)|)$. Let $c_1 \in C'$. Then c_1 is anticomplete to $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}}$, and $|L(c_1)| = 3$. It follows that in every coloring of (G, L, Z) only one color appears in $N(c_1) \setminus \tilde{C}$. Recall also that c_1 has at most one neighbor in C. Now, repeatedly applying 5.3.1 to vertices of C', it follows that c can be extended to a coloring of (G, L, Z) in time $O(|V(G)|)$. This prove *(6)*.

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \{(G', L', Z')\}$ be the union of the sets of restrictions from *(2)*,*(4)*, *(5)* and *(6)*. Then $\mathcal{P}$ is a restriction of (G, L, Z). We claim that $\mathcal{P}$ satisfies the conclusion of the theorem. By *(2)*,*(4)*, *(5)* and *(6)*, it follows that $\mathcal{P}$ has size $O(|V(G)|^7)$, and that *(a)* and *(b)* hold. Next, we show that *(c)* holds. Since $\mathcal{P}$ is a restriction of (G, L, Z), by definition, it follows that if $\mathcal{P}$ is colorable, then (G, L, Z) is colorable. By *(2c)*, if (G, L, Z) has a type I coloring, then $\mathcal{P}_1$ is colorable. So we may assume (G, L, Z) does not have a type I coloring. By *(4c)*, if some vertex of $\tilde{C}$ has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, then (G, L, Z) is colorable if and only if $\mathcal{P}_2$ is colorable. So we may assume no such vertex exists. By *(5c)*, if (G, L, Z) has a type II coloring, then $\mathcal{P}_3$ is colorable. So we may assume (G, L, Z) does not have a type II coloring. By *(6)*, it follows that (G, L, Z) is colorable if and only if (G', L', Z') is colorable. This proves 5.3.4.

□

5.4 Cleaning

In this section, we identify two configurations whose presence in a graph G allows us to delete some vertices and obtain an induced subgraph G' of G, such that G' is k-colorable if and only G is k-colorable. Furthermore, these two configurations can be efficiently recognized, and so at the expense of carrying out a polynomial time procedure we may assume a given graph does not contain either configuration.

Let G be a graph. Recall that a vertex $v \in V(G)$ is *dominated* by a vertex $u \in V(G) \setminus \{v\}$ if u is non-adjacent to v and $N(v) \subseteq N(u)$. In terms of coloring, dominated vertices are useful because of the following:

5.4.1. *Let $\mathcal{F}$ be a set of graphs and G be an $\mathcal{F}$-free graph. If $v \in V(G)$ is dominated by $u \in V(G) \setminus \{v\}$, then $G \setminus v$ is an $\mathcal{F}$-free graph which is k-colorable if and only if G is k-colorable. Furthermore, we can extend any k-coloring of $G \setminus v$ to a k-coloring of G in constant time.*

Proof. Clearly, $G \setminus v$ is $\mathcal{F}$-free, and if G is k-colorable, then $G \setminus v$ is k-colorable. Now, suppose c is a k-coloring of $G \setminus v$. Since v is non-adjacent to u and $N(v) \subseteq N(u)$, it follows, assigning $c(v) = c(u)$, that c extends to a coloring of G. This proves 5.4.1. □

Let G be a graph. Recall that for a subset $X \subseteq V(G)$, we say that a vertex $v \in V(G) \setminus X$ is *mixed* on X, if v is not complete and not anticomplete to X. We say that a pair of disjoint non-empty stable sets (A, B) form a *non-trivial homogeneous pair of stable sets* if $2 < |A| + |B| < |V(G)|$ and no vertex $v \in V(G) \setminus (A \cup B)$ is mixed on A or mixed on B. In terms of coloring, non-trivial homogeneous pairs of stable sets are useful because of the following:

5.4.2. *Let $\mathcal{F}$ be a set of graphs and let G be an $\mathcal{F}$-free graph. Let (A, B) be a non-trivial homogeneous pair of stable sets and choose $a \in A$ and $b \in B$, adjacent if possible. Then $G \setminus ((A \cup B) \setminus \{a, b\})$ is an $\mathcal{F}$-free graph which is k-colorable if and only if G is k-colorable. Furthermore, we can extend any k-coloring of $G \setminus ((A \cup B) \setminus \{a, b\})$ to a k-coloring of G in linear time.*

Proof. Clearly, $G \setminus ((A \cup B) \setminus \{a, b\})$ is $\mathcal{F}$-free, and if G is k-colorable, then $G \setminus ((A \cup B) \setminus \{a, b\})$ is k-colorable. Now, suppose c is a k-coloring of $G \setminus ((A \cup B) \setminus \{a, b\})$. By our choice of $a \in A$ and $b \in B$, if A is not anticomplete to B, then $c(a) \neq c(b)$. Since both A and B are stable and no vertex $v \in V(G) \setminus (A \cup B)$ is mixed on A or mixed on B, it follows, assigning $c(a') = c(a)$ for all $a' \in A$ and $c(b') = c(b)$ for all $b' \in B$, that c extends to a coloring of G. This proves 5.4.2. □

We say a graph G is *clean* if G has no dominated vertices and no non-trivial homogeneous pair of stable sets.

5.4.3. *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *A clean induced subgraph G' of G such that, for every integer k, G' is k-colorable if and only if G is k-colorable.*

Running time: $O(|V(G)|^5)$.

Furthermore, we can extend any k-coloring of G' to a k-coloring of G in linear time.

Proof. Since there are $O(|V(G)|^2)$ potential pairs of non-adjacent vertices $u, v \in V(G)$ and we can verify in time $O(|V(G)|)$ if $N(v) \subseteq N(u)$, it follows that in time $O(|V(G)|^3)$ we can find a dominated vertex in G, if one exists. In [27], King and Reed give an algorithm, that runs is time $O(|V(G)|^4)$, which is easily modified to produce a non-trivial homogeneous pair of stable sets, if one exists. And so, we can find a dominated vertex or a non-trivial homogeneous pair of stable sets in time $O(|V(G)|^4)$, if one exists. Hence, iteratively, 5.4.3 follows from the observations made in 5.4.1 and 5.4.2.

□

Thus, 5.4.3 implies that given a graph G, we may assume G is clean at the expense of carrying out a time $O(|V(G)|^5)$ procedure. It is also clear from 5.4.1 and 5.4.2, that given a graph G we may extend a k-coloring of the resulting clean induced subgraph G' of G produced by 5.4.3 to a k-coloring of G in time $O(|V(G)|)$.

5.5 A Useful Lemma

In this section we prove a general lemma which will be of great use when trying to apply 5.3.4 to clean graphs.

5.5.1. *Let G be a clean, connected $\{P_7, C_3\}$-free graph with $V(G) = P \cup Q \cup R \cup S \cup T$ such that:*

- *$P \cup T$ is anticomplete to $R \cup S$,*
- *S is anticomplete to $P \cup Q$,*
- *every vertex in R has a neighbor in Q, and*
- *there exist $q_0 \in Q$ and $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q_0$ is an induced path.*
- *for every $q \in Q$, there exist $p_2, p_3 \in P$ and $p_1 \in P \cup \{q_0\}$ such that $p_1 - p_2 - p_3 - q$ is an induced path.*

Then the following hold:

1. *S is empty*
2. *If for every $q \in Q$, there exist $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q$ is an induced path, then every component of R has size at most two.*
3. *Every component of R is bipartite, and if some component X of R has more than two vertices, then q_0 is complete to one side of the bipartition of $G[X]$.*

Proof. *(1) Let $q \in Q$. There is no induced path $q - r_1 - r_2 - r_3$ such that $r_1 \in R$ and $r_2, r_3 \in S$. Moreover, if there exist $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q$ is an induced path, then there is no induced path $q - r_1 - r_2 - r_3$ such that $r_1, r_2, r_3 \in R \cup S$.*

Proof: Suppose there exist $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q$ is an induced path, and $q - r_1 - r_2 - r_3$ is an induced path with $r_1, r_2, r_3 \in R \cup S$. Then, since P is anticomplete to $R \cup S$, it follows that $p_1 - p_2 - p_3 - q - r_1 - r_2 - r_3$ is a P_7 in G, a contradiction. Next suppose that $r_1 \in R$ and $r_2, r_3 \in S$. There exist $p_1, p_2 \in P$ such that $q - p_1 - p_1 - q_0$ is an induced path. Since $q_0 - r_1 - r_2 - r_3$ is not an induced path in G by the previous argument, it follows that q_0 is anticomplete to $\{r_1, r_2, r_3\}$. But now $q_0 - p_1 - p_1 - q - r_1 - r_2 - r_3$ is an induced path in G, a contradiction. This proves *(1)*.

(2) Every vertex in S has a neighbor in R.

Proof: Partition $S = S' \cup S''$, where S' is the set of vertices in S with a neighbor in R and $S'' = S \setminus S'$. Suppose S'' is non-empty. Since G is connected and S'' is anticomplete to $V(G) \setminus S$, it follows that there exists $s'' \in S''$ adjacent to $s' \in S'$. By definition, there exists $r \in R$ adjacent to s' and non-adjacent to s''. Since every vertex in R has a neighbor in Q, there exists $q \in Q$ adjacent to r. Now $q - r - s' - s''$ is an induced path, contrary to *(1)*. This proves *(2)*.

(3) S is stable.

Proof: Suppose $s, s' \in S$ are adjacent. By *(2)*, there exists $r \in R$ adjacent to s. Since G is triangle-free, s' is non-adjacent to r. Since every vertex in R has a neighbor in Q, there exists $q \in Q$ adjacent to r. However, then $q - r - s - s'$ is an induced path, contradicting *(1)*. This proves *(3)*.

Now, we prove 5.5.1.1. Consider a vertex $s \in S$. Since S is anticomplete to $P \cup Q$ and, by *(3)*, S is stable, it follows that S is anticomplete to $V(G) \setminus R$. By *(2)*, there exists $r \in R$ adjacent to s. Since every vertex in R has a neighbor in Q, there exists $q \in Q$ adjacent to r; if possible, choose q and r such that there exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. Since s is not dominated by q, there exists $r' \in N(s) \setminus N(q)$. Since G is triangle-free, r' is non-adjacent to r, and it follows that $q - r - s - r'$ is an induced path. It follows from *(1)* there do not exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path, and in particular $q \neq q_0$. This implies that q_0 is anticomplete to $N(s)$, and that there exist $p_1, p_2 \in P$ such that $q - p_1 - p_2 - q_0$ is an induced path. But now $q_0 - p_2 - p_1 - q - r - s - r'$ is a P_7 in G, a contradiction. This proves 5.5.1.1.

(4) Every component of R is bipartite.

Proof: Suppose X is a component of R which is not bipartite. Since G is $\{P_7, C_3\}$-free, it follows that $G[X]$ contains either a C_5 or a C_7. Let $x_1 - x_2 - ... - x_{2k+1} - x_1$ be either a 5-gon or a 7-gon given by $x_1, ..., x_{2k+1} \in R$ with $k \in \{2, 3\}$. Let $q \in Q$ be a vertex with a neighbor in $\{x_1, ..., x_{2k+1}\}$. Since $2k + 1$ is odd and G is triangle-free, we may assume that q

is adjacent to x_1 and non-adjacent to x_2, x_3. But now $q - x_1 - x_2 - x_3$ is an induced path, and so *(1)* implies that there do not exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. This implies that q_0 is anticomplete to $\{x_1, ..., x_{2k+1}\}$, and that there exist $p_1, p_2 \in P$ such that $q - p_1 - p_2 - q_0$ is an induced path. But now $q_0 - p_2 - p_1 - q - x_1 - x_2 - x_3$ is a P_7 in G, a contradiction. This proves *(4)*.

(5) Let X be a component of R, and (X_1, X_2) be a bipartition of $G[X]$. Let $q \in Q$ be such that there exist $p_1, p_2, p_3 \in P$ where $q - p_1 - p_2 - p_3$ is an induced path. Then q is not mixed on either X_1 or X_2.

Proof: Suppose there exists a vertex $q \in Q$ adjacent to x and non-adjacent to x', where $x, x' \in X_1$. Since X is connected, by choosing x and x' at minimum distance from each other in $G[X]$, we may assume that there exists $x_2 \in X_2$ adjacent to both x and x'. Since G is triangle-free, it follows that q is non-adjacent to x_2. Now $q - x - x_2 - x'$ is an induced path, and so *(1)* implies that there do not exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. This proves *(5)*.

Let X be a component of R, and (X_1, X_2) be a bipartition of $G[X]$. First we prove 5.5.1.2. Suppose that for every vertex of Q there exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. By *(5)*, no vertex of Q is mixed on X_1, and similarly, no vertex of Q is mixed on X_2. Since $P \cup T$ is anticomplete to R, it follows that $V(G) \setminus Q$ is anticomplete to R, and in particular to X. Hence, (X_1, X_2) is a homogeneous pair of stable sets, and so, since G is clean, it follows that $|X| \leq 2$. This proves 5.5.1.2.

Finally, we prove 5.5.1.3. Suppose that $|X| > 2$. Since (X_1, X_2) is not a homogeneous pair of stable sets, it follows that there exists a vertex $q \in Q$ adjacent to x and non-adjacent to x', where $x, x' \in X_1$, say. Since X is connected, by choosing x and x' at minimum distance from each other in $G[X]$, we may assume that there exists $x_2 \in X_2$ adjacent to both x and x'. Since G is triangle-free, it follows that q is non-adjacent to x_2. By *(*5*)*, $q \neq q_0$, and there exist $p_1, p_2 \in P$ such that $q - p_1 - p_2 - q_0$ is a path. Since $q_0 - p_2 - p_1 - q - x - x_2 - x'$ is not a P_7 in G, it follows that q_0 has a neighbor in $\{x, x', x_2\}$, and 5.5.1.3 follows from *(*5*)*. □

5.6 7-gons

In this section we show that if a $\{P_7, C_3\}$-free graph contains a 7-gon, then in polynomial time we can decide if the graph is 3-colorable, and give a coloring if one exists. Let C be an n-gon in a graph G. For a vertex $v \in V(G) \setminus V(C)$, we call the neighbors of v in $V(C)$ the *anchors of* v *in* C.

We begin with some definitions. Let C be a 7-gon in a graph G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. We say that a vertex $v \in V(G) \setminus V(C)$ is:

- a *clone at* i, if $N(v) \cap V(C) = \{v_{i-1}, v_{i+1}\}$ for some $i \in \{0, 1, ..., 6\}$, where all indices are mod 7,
- a *propeller at* $\{i, i+3\}$, if $N(v) \cap V(C) = \{v_i, v_{i+3}\}$ for some $i \in \{0, 1, ..., 6\}$, where all indices are mod 7,
- a *star at* i, if $N(v) \cap V(C) = \{v_{i-2}, v_i, v_{i+2}\}$for some $i \in \{0, 1, ..., 6\}$, where all indices are mod 7.

The following shows how we can partition the vertices of G based on their anchors in C.

5.6.1. *Let G be a $\{P_7, C_3\}$-free graph, and suppose C is a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. If $v \in V(G) \setminus V(C)$, then for some $i \in \{0, 1, ..., 6\}$ either:*

1. *v is a clone at i,*
2. *v is a propeller at $\{i, i+3\}$,*
3. *v is a star at i, or*
4. *v is anticomplete to $V(C)$.*

Proof. Consider a vertex $v \in V(G) \setminus V(C)$. If v is anticomplete to $V(C)$, then 5.6.1.4 holds. Thus, we may assume $N(v) \cap V(C) \neq \emptyset$. Since G is triangle-free, and 7 is odd, we may assume that v is adjacent to v_0, and anticomplete to $\{v_6, v_1, v_2\}$. If v is adjacent to v_4, then, since G is triangle-free, 5.6.1.2 holds, so we may assume not. Since $v - v_0 - v_1 - v_2 - v_3 - v_4 - v_5$ is not a P_7 in G, it follows that v has a neighbor in $\{v_3, v_5\}$. If v is complete to $\{v_3, v_5\}$, then 5.6.1.3 holds; if v is adjacent to v_3 and not to v_5, then 5.6.1.2 holds; and if v is adjacent to v_6 and not to v_3, then 5.6.1.1 holds. This proves 5.6.1. □

Let G be a triangle-free graph, and let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. Using 5.6.1 we partition $V(G) \setminus V(C)$ as follows:

- Let $CL^C(i)$ be the set of clones at i, and define $CL^C = \bigcup_{i=0}^{6} CL^C(i)$.
- $P^C(i)$ be the set of propellers at $\{i, i+3\}$, and define $P^C = \bigcup_{i=0}^{6} P^C(i)$.
- Let $S^C(i)$ be the set of stars at i and define $S^C = \bigcup_{i=0}^{6} S^C(i)$.
- Let A^C be the set of vertices anticomplete to $V(C)$.

By 5.6.1, it follows that $V(G) = V(C) \cup CL^C \cup P^C \cup S^C \cup A^C$. Furthermore, we partition $A^C = X^C \cup Y^C \cup Z^C$, where

- X^C is the set of vertices in A^C with a neighbor in P^C,
- Y^C is the set of vertices in $A^C \setminus X^C$ with a neighbor in S^C, and
- $Z^C = A^C \setminus (X^C \cup Y^C)$.

And so, for a given 7-gon C in time $O(|V(G)|^2)$ we obtain the partition $V(C) \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C \cup Z^C$ of $V(G)$. Now, we establish several properties of this partition.

5.6.2. *If G is a $\{P_7, C_3\}$-free graph, then A^C is anticomplete to CL^C for every 7-gon C in G.*

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. Suppose there exists a vertex $u \in A^C$ adjacent to $v \in CL^C$. By symmetry, we may assume $v \in CL^C(0)$. However, then $u - v - v_1 - v_2 - v_3 - v_4 - v_5$ is a P_7 in G, a contradiction. This proves 5.6.2. □

By definition and 5.6.2 it follows that:

5.6.3. *If G is a $\{P_7, C_3\}$-free graph, then for every 7-gon C in G the following hold:*

1. *A^C is anticomplete to $V(C) \cup CL^C$.*
2. *Every vertex in X^C has a neighbor in P^C (and possibly S^C).*
3. *$Y^C \cup Z^C$ is anticomplete to P^C.*
4. *Every vertex in Y^C has a neighbor in S^C.*
5. *Z^C is anticomplete to S^C.*

Recall, for a fixed subset X of $V(G)$, we say a vertex $v \in V(G) \setminus X$ *is mixed on an edge of* X, if there exist adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$. We need the following two facts:

5.6.4. *If G is a $\{P_7, C_3\}$-free graph, then for every 7-gon C in G the following hold:*

1. *No vertex in P^C is mixed on an edge of A^C.*
2. *X^C is stable and anticomplete to $Y^C \cup Z^C$.*

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. Suppose for adjacent $a, a' \in A^C$, there exists $p \in P^C$ which is adjacent to a' and non-adjacent to a. By symmetry, we may assume $p \in P^C(0)$. However, then $a - a' - p - v_3 - v_4 - v_5 - v_6$ is a P_7 in G, a contradiction. This proves 5.6.4.1.

Consider a vertex $x \in X^C$. By 5.6.3.2, x has a neighbor $p \in P^C$. If there exists $x' \in N(x) \cap A^C$, then, since G is triangle-free, it follows that p is non-adjacent to x', and so

p is mixed on an edge of A^C, contradicting 5.6.4.1. This proves 5.6.4.2, and completes the proof of 5.6.4.

□

5.6.5. *Let G be a clean, connected $\{P_7, C_3\}$-free graph. Then for every 7-gon C in G the following hold:*

1. *Z^C is empty.*

2. *Every component of Y^C is a singleton or an edge.*

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$.

(1) For every $s \in S^C$, there exists $t_1, t_2, t_3 \in V(C)$ such that $t_1 - t_2 - t_3 - s$ is an induced path.

Proof: Consider a vertex $s \in S^C$. By symmetry, we may assume $s \in S^C(0)$. And so, $v_4 - v_3 - v_2 - s$ is the desired induced path. This proves *(1)*.

By 5.6.3, 5.6.4.2 and *(1)*, we may apply 5.5.1 letting $P = V(C)$, $Q = S^C$, $R = Y^C$, $S = Z^C$, and $T = CL^C \cup P^C \cup X^C$. Then 5.5.1.1 and 5.5.1.2 follow immediately from 5.5.1. This proves 5.6.5.

□

Now, we prove the main result of the section.

5.6.6. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3\}$-free graph G which contains a 7-gon.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^{10})$.

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$, and observe that C can be found in time $O(|V(G)|^7)$. Since G is clean, by 5.6.5.1, it follows that Z^C is empty, and so we may partition $V(G) = V(C) \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C$ as usual. Next, fix a 3-coloring c of $G[V(C)]$. Define the order 3 palette L_c^C of G as follows:

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \\ \{1,2,3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $CL^C \cup P^C \cup S^C$ with respect to $V(C)$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$, while $|L_c^C(v)| = 3$ for all $v \in X^C \cup Y^C$. Observe that, by construction, (G, L_c^C) is colorable if and only the 3-coloring c of $G[V(C)]$ extends to a 3-coloring of G.

Let $A' = P^C \cup S^C$, and for every non-empty subset $X \subseteq \{1,2,3\}$, define $A'_X = \{a \in A'$ with $L_c^C(a) = X\}$.

(1) For every $u \in X^C \cup Y^C$ and $\{i,j,k\} = \{1,2,3\}$, $N(u) \cap A'_{\{i,j\}}$ is complete to $A'_{\{i,k\}} \setminus N(u)$.

Proof: It is enough to show that for every $x \in N(u) \cap A'_{\{i,j\}}$ and $y \in A'_{\{i,k\}} \setminus N(u)$, such that x is non-adjacent to y, there exists an induced 6-vertex path $x - p_1 - ... - p_5$ with $p_1, ..., p_5 \in V(C) \cup CL^C \cup \{y\}$. For if such a path exists, then, since, by 5.6.3.1, u is anticomplete to $V(C) \cup CL^C \cup \{y\}$, it follows that $u - x - p_1 - ... - p_5$ is a P_7 in G, a contradiction.

Since $x \in A'_{\{i,j\}}$ and $y \in A'_{\{i,k\}}$, it follows from the definition of updating that all the anchors of x are colored k, and all the anchors of y are colored j. In particular, this implies that x and y have no anchors in common.

Let $\{a, b\} = \{x, y\}$. First, suppose a is a star. By symmetry, we may assume $a \in S^C(0)$. Since a and b have no anchors in common, it follows that b is anticomplete to $\{v_0, v_2, v_5\}$. Since $b \in P^C \cup S^C$, it follows that $|N(b) \cap (V(C) \setminus \{v_0, v_2, v_5\})| \geq 2$ and so, by symmetry and 5.6.1, we may assume that v_1 is an anchor of b, that is, that $b \in P^C(1) \cup S^C(1) \cup S^C(6)$. Further by symmetry, we may assume that $b \in P^C(1) \cup S^C(1)$. Suppose $a = x$. If $b \in P^C(1)$, then $x - v_0 - v_1 - y - v_4 - v_3$ is the desired path, and if $b \in S^C(1)$, then $x - v_0 - v_1 - y - v_3 - v_4$

is the desired path. Thus, we may assume $b = x$. If $b \in P^C(1)$, then $x - v_4 - v_3 - v_2 - y - v_0$ is the desired path, and if $b \in S^C(1)$, then $x - v_3 - v_4 - v_5 - y - v_0$ is the desired path. Hence, we may assume neither of x, y is a star, that is, that both x and y are propellers. By symmetry, we may assume $x \in P^C(0)$, and therefore y is anticomplete to $\{v_0, v_3\}$. Since $y \in P^C$, it follows that $|N(y) \cap (V(C) \setminus \{v_0, v_3\})| = 2$ and so, by symmetry, we may assume that v_1 is an anchor of y, that is, that $y \in P^C(1) \cup P^C(5)$. If $y \in P^C(1)$, then $x - v_0 - v_1 - y - v_4 - v_5$ is the desired path, and if $y \in P^C(5)$, then $x - v_3 - v_2 - v_1 - y - v_5$ is the desired path. By our initial observation, this proves *(1)*.

(2) For all distinct $i, j \in \{1, 2, 3\}$ some vertex of $V(C) \cup CL^C$ is complete to $A'_{\{i,j\}}$.

Proof: If $A'_{\{i,j\}} = \emptyset$, then *(2)* trivially holds. Thus, we may assume $A'_{\{i,j\}} \neq \emptyset$. Let $\{i, j, k\} = \{1, 2, 3\}$ and define K to be the set of vertices of $V(C)$ with a neighbor in $A'_{\{i,j\}}$. Since we have updated, it follows that $c(v) = k$ for every $v \in K$. Since $G[V(C)]$ has no stable set of size 4, it follows that $|K| \leq 3$. Since $A'_{\{i,j\}} \subseteq P^C \cup S^C$, it follows that $|K| \geq 2$. If $|K| = 2$, then $A_{\{i,j\}} \subseteq P^C$, and it follows that K is complete to $A'_{\{i,j\}}$. Hence, we may assume $|K| = 3$. By symmetry, we may assume that $K = \{v_0, v_3, v_5\}$. Since G is triangle-free, it follows that $A'_{\{i,j\}} \subseteq P^C(0) \cup S^C(5)$, and so v_3 is complete to $A_{\{i,j\}}$. This proves *(2)*.

By 5.6.4.2 and 5.6.5, it follows that every component of $X^C \cup Y^C$ has at most two vertices. And so, by 5.6.3, *(1)* and *(2)*, we can apply 5.3.4 with

- $\tilde{A} = A'$,
- $\tilde{B} = V(C) \cup CL^C$,
- $\tilde{C} = X^C \cup Y^C$, and
- $Z = \emptyset$.

Let $\mathcal{P}_c^C$ be the restriction of $(G, L_c^C, \emptyset)$, of size $O(|V(G)|^7)$, thus obtained. By 5.3.4, $\mathcal{P}_c^C$ can be computed in time $O(|V(G)|^7)$. By 5.3.4*(c)*, we have that $(G, L_c^C, \emptyset)$ (and, equivalently,

(G, L_c^C)) is colorable if and only if $\mathcal{P}_c^C$ is colorable. Consider $(G', L', X') \in \mathcal{P}_c^C$. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$, by 5.3.4*(b)*, it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Thus, since $|X'|$ has size $O(|V(G)|)$, applying 5.1.7, we can test in time $O(|V(G)|^3)$ if (G', L', X') is colorable, and extend the coloring to (G, L_c^C). Consequently, via $O(|V(G)|^7)$ applications of 5.1.7, we can determine if $\mathcal{P}_c^C$ is colorable and extend any coloring of a colorable $(G', L', X') \in \mathcal{P}_c^C$ to a coloring of G. That is, in time $O(|V(G)|^{10})$ we can determine if the 3-coloring c of $G[V(C)]$ extends to a 3-coloring of G, and give an explicit 3-coloring c' of G such that $c'(v) = c(v)$ for all $v \in V(C)$, if one exists. Finally, let $\mathcal{P}$ be the union of $\mathcal{P}_c^C$ taken over all 3-colorings c of $G[V(C)]$. Since there are at most 7^3 3-colorings of $G[V(C)]$, it follows that we can test in time $O(|V(G)|^{10})$ if $\mathcal{P}$ is colorable. Since every 3-coloring of G restricts to a 3-coloring of $G[V(C)]$, it follows that G is 3-colorable if and only if $\mathcal{P}$ is colorable. This proves 5.6.6.

□

5.7 Shells

We remind the reader, that a shell in a graph G is a pair (C, p), where C is a 6-gon given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p \in V(G) \setminus \{v_0, ..., v_5\}$, such that $N(p) \cap \{v_0, ..., v_5\} = \{v_\ell, v_{\ell+3}\}$ for some $\ell \in \{0, 1, 2\}$. In this section we show that if a $\{P_7, C_3, C_7\}$-free graph contains a shell, then in polynomial time we can decide if the graph is 3-colorable, and give a coloring if one exists.

We begin with some definitions. Let C be a 6-gon in a graph G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We say that a vertex $v \in V(G) \setminus V(C)$ is:

- a *leaf at* i, if $N(v) \cap V(C) = \{v_i\}$ for some $i \in \{0, 1, ..., 5\}$,
- a *clone at* i, if $N(v) \cap V(C) = \{v_{i-1}, v_{i+1}\}$ for some $i \in \{0, 1, ..., 5\}$, where all indices are mod 6,
- a *propeller at* $\{i, i+3\}$, if $N(v) \cap V(C) = \{v_i, v_{i+3}\}$ for some $i \in \{0, 1, ..., 5\}$, where all

indices are mod 6,

- an *even star*, if $N(v) \cap V(C) = \{v_0, v_2, v_4\}$,
- an *odd star*, if $N(v) \cap V(C) = \{v_1, v_3, v_5\}$.

The following shows how we can partition the vertices of G based on their anchors in C.

5.7.1. *Let G be a triangle-free graph, and suppose C is a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. If $v \in V(G) \setminus V(C)$, then for some $i \in \{0, 1, ..., 5\}$ either:*

1. *v is a leaf at i,*
2. *v is a clone at i,*
3. *v is a propeller at $\{i, i+3\}$, where all indices are mod 6,*
4. *v is an even star,*
5. *v is an odd star, or*
6. *v is anticomplete to $V(C)$.*

Proof. Consider a vertex $v \in V(G) \setminus V(C)$. If v is anticomplete to $V(C)$, then 5.7.1.6 holds. Thus, we may assume $N(v) \cap V(C) \neq \emptyset$. By symmetry, suppose that $v_0 \in N(v) \cap V(C)$. Since G is triangle-free, it follows that v is anticomplete to $\{v_1, v_5\}$. Suppose v is non-adjacent to v_3. If v is anticomplete to $\{v_2, v_4\}$, then 5.7.1.1 holds. If v is mixed on $\{v_2, v_4\}$, then 5.7.1.2 holds. If v is complete to $\{v_2, v_4\}$, then 5.7.1.4 holds. Thus, we may assume v is adjacent to v_3. Since G is triangle-free, it follows that v is anticomplete to $\{v_2, v_4\}$, and so 5.7.1.3 holds. This proves 5.7.1.

□

Let G be a triangle-free graph and C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We partition $V(G) \setminus V(C)$ as follows:

- Let $M^C(i)$ be the set of leaves at i and define $M^C = \bigcup_{i=0}^{5} M^C(i)$.
- Let $CL^C(i)$ be the set of clones at i and define $CL^C = \bigcup_{i=0}^{5} CL^C(i)$.
- Let $P^C(\{i, i+3\})$ be the set of propellers at $\{i, i+3\}$ and define $P^C = \bigcup_{i=0}^{5} P^C(\{i, i+3\})$.
- Let S_0^C be the set of even stars.
- Let S_1^C be the set of odd stars.
- Let $S^C = S_0^C \cup S_1^C$.
- Let A^C be the set of vertices anticomplete to $V(C)$.

By 5.7.1, it follows that $V(G) = V(C) \cup M^C \cup CL^C \cup P^C \cup S^C \cup A^C$. Furthermore, we partition $A^C = X^C \cup Y^C \cup Z^C$, where

- X^C is the set of vertices in A^C with a neighbor in CL^C,
- Y^C is the set of vertices in $A^C \setminus X^C$ with a neighbor in P^C,
- $Z^C = A^C \setminus (X^C \cup Y^C)$.

And so, given a 6-gon C in G in time $O(|V(G)|^2)$ we obtain the partition $V(C) \cup M^C \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C \cup Z^C$ of $V(G)$. Now, we establish several properties of this partition. The following is immediate:

5.7.2. *If G is a triangle-free graph, then for every 6-gon C in G the following hold:*

1. *Every vertex in X^C has a neighbor in CL^C.*
2. *Every vertex in Y^C has a neighbor in P^C.*
3. *CL^C is anticomplete to $Y^C \cup Z^C$.*

Next, we show:

5.7.3. *If G is a $\{P_7, C_3, C_7\}$-free graph, then for every 6-gon C in G the following hold:*

1. *M^C is anticomplete to A^C.*
2. *For every $q \in M^C \cup CL^C \cup P^C$, there exists $p_1, p_2, p_3 \in V(C)$ such that $p_1 - p_2 - p_3 - q$ is an induced path.*
3. *X^C is stable and anticomplete to $Y^C \cup Z^C$.*
4. *Z^C is anticomplete to $V(G) \setminus (Y^C \cup S^C)$.*
5. *For every $i \in \{0, ..., 5\}$, if $M^C(i)$ is non-empty, then $M^C(i+2) \cup M^C(i-2)$ is empty, where all indices are mod 6.*
6. *No vertex in A^C has a neighbor in $CL^C(i)$ and $CL^C(j)$ for $i \neq j$.*

Proof. Let C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. Suppose there exists $a \in A^C$ adjacent to $m \in M^C$. By symmetry, we may assume $m \in M^C(0)$. However, then $a - m - v_0 - v_1 - v_2 - v_3 - v_4$ is a P_7 in G, a contradiction. This proves 5.7.3.1.

Consider a vertex $q \in M^C \cup CL^C \cup P^C$. By symmetry, we may assume q is adjacent to v_0 and non-adjacent to v_1, v_2, and so $v_2 - v_1 - v_0 - q$ is an induced path. This proves 5.7.3.2.

Consider a vertex $x \in X^C$. By 5.7.2.1, there exists $c \in CL^C$ adjacent to x. By symmetry, we may assume $c \in CL^C(0)$. Let C' be the 6-gon given by $c - v_1 - v_2 - v_3 - v_4 - v_5 - c$. Suppose there exists $x' \in A^C$ adjacent to x. Since G is triangle-free, it follows that c is non-adjacent to x'. However, then $x \in M^{C'}$ is adjacent to $x' \in A^{C'}$, contrary to 5.7.3.1. This prove 5.7.3.3.

By 5.7.2.3 and 5.7.3.1, it follows that 5.7.3.4 holds.

To prove 5.7.2.5, suppose there exists $m \in M^C(0)$ and $m' \in M^C(2)$. If m is non-adjacent to m', then $m' - v_2 - v_3 - v_4 - v_5 - v_0 - m$ is a P_7 in G, and if m is adjacent to m', then $m - m' - v_2 - v_3 - v_4 - v_5 - v_0 - m$ is a C_7 in G, in both cases a contradiction. This proves 5.7.3.5.

We now prove 5.7.3.6. Assume $a \in A^C$ is adjacent to $c \in CL^C$. By symmetry, we may assume $c \in CL^C(0)$. Suppose there exists $c' \in CL^C \setminus CL^C(0)$ adjacent to a. Since G is triangle-free, it follows that c is non-adjacent to c'. By symmetry, it suffices to consider $c' \in CL^C(1) \cup CL^C(2) \cup CL^C(3)$. If $c' \in CL^C(1)$, then $a - c' - v_2 - v_3 - v_4 - v_5 - c - a$ is a C_7 in G, if $c' \in CL^C(2)$, then $v_2 - v_3 - c' - a - c - v_5 - v_0$ is a P_7 in G, and if $c' \in CL^C(3)$, then $v_0 - v_1 - c - a - c' - v_4 - v_3$ is a P_7 in G, in all three cases a contradiction. This proves 5.7.3.6. This proves 5.7.3.

□

Let C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We say that a graph G has a *type I coloring with respect to C* if there exists a 3-coloring c of G such that $c(v_i) = c(v_{i+3})$ for every $i \in \{0, 1, 2\}$.

5.7.4. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3, C_7\}$-free graph G.*

Output: *A type I coloring of G with respect to some 6-gon in G, or a determination that none exists.*

Running time: $O(|V(G)|^{16})$.

Proof. In time $O(|V(G)|^6)$, we can enumerate all 6-gons in G. If G is C_6-free, then clearly G does not have a type I coloring and we may halt. Hence, we may assume there exists a 6-gon C in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. In time $O(|V(G)|^2)$, we can partition $V(G) = V(C) \cup M^C \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C \cup Z^C$ as usual. If S^C is non-empty, then G does not have a type I coloring with respect to C, since, by definition, the anchors of any star in C receive three distinct colors in any type I coloring. Hence, we may assume S^C is empty. Next, fix a 3-coloring c of $G[V(C)]$ such that $c(v_i) = c(v_{i+3})$ for every $i \in \{1, 2, 3\}$, where all indices are mod 6. Define the order 3 palette L_c^C of G as follows: For every $v \in V(G)$, set

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \\ \{1,2,3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $M^C \cup CL^C \cup P^C$ with respect to $V(C)$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C \cup Z^C)$, while $|L_c^C(v)| = 3$ for all $v \in X^C \cup Y^C \cup Z^C$. Additionally, $|L_c^C(v)| = 2$ if and only if $v \in M^C \cup P^C$. Observe that, by construction, (G, L_c^C) is colorable if and only the 3-coloring c of $G[V(C)]$ extends to a type I coloring of G.

By 5.7.2, 5.7.3.1, 5.7.3.2 and 5.7.3.4, we may apply 5.5.1 letting $P = V(C) \cup M^C$, $Q = CL^C \cup P^C$, $R = X^C \cup Y^C$, $S = Z^C$, and $T = \emptyset$. It follows that Z^C is empty and that every component of $X^C \cup Y^C$ has size at most two. Let $A' = CL^C \cup P^C$, and for every non-empty subset $X \subseteq \{1,2,3\}$, define $A'_X = \{a \in A' \text{ with } L_c^C(a) = X\}$. Since for $v \in A'$, $|L_c^C(v)| = 2$ if and only if $v \in P^C$, it follows that if $|X| = 2$, then $A_X = P^C(\{i, i+3\})$ for some $i \in \{0,1,2\}$.

(1) For every $c_1, c_2 \in X^C \cup Y^C$ and $\{i,j,k\} = \{1,2,3\}$, $(N(c_1) \cap A'_{\{i,j\}}) \setminus N(c_2)$ is complete to $(N(c_2) \cap A'_{\{i,k\}}) \setminus N(c_1)$.

Proof: We may assume $A'_{\{i,j\}} = P^C(\{0,3\})$ and $A'_{\{i,k\}} = P^C(\{1,4\})$. Suppose there exists $p_1 \in P^C(\{0,3\}) \setminus N(c_2)$ adjacent to c_1 and $p_2 \in P^C(\{1,4\}) \setminus N(c_1)$ adjacent to c_2 such that p_1 is non-adjacent to p_2. If c_1 is non-adjacent to c_2, then $c_2 - p_2 - v_1 - v_2 - v_3 - p_1 - c_1$ is a P_7 in G, and if c_1 is adjacent to c_2, then $c_1 - c_2 - p_2 - v_1 - v_2 - v_3 - p_1 - c_1$ is a C_7 in G, in both cases a contradiction. Hence, it follows that p_1 is adjacent to p_2. This proves *(1)*.

(2) For all distinct $i,j \in \{1,2,3\}$ some vertex of $V(C) \cup M^C$ is complete to $A'_{\{i,j\}}$.

Proof: After updating, it follows that $|L_c^C(v)| = 2$ if and only if $v \in M^C \cup P^C$. By symmetry, we may assume $A'_{\{i,j\}} = P^C(\{0,3\})$. Hence, $\{v_0, v_3\}$ is complete to $\tilde{A}_{\{i,j\}}$. This proves *(2)*.

By *(1)* and *(2)*, we can apply 5.3.4 with

- $\tilde{A} = CL^C \cup P^C$,

- $\tilde{B} = V(C) \cup M^C$,
- $\tilde{C} = X^C \cup Y^C$, and
- $Z = \emptyset$.

Let $\mathcal{P}_c^C$ be the restriction of $(G, L_c^C, \emptyset)$ of size $O(|V(G)|^7)$ thus obtained. By 5.3.4, $\mathcal{P}_c^C$ can be computed in time $O(|V(G)|^7)$. By 5.3.4*(c)*, we have that $(G, L_c^C, \emptyset)$ (and equivalently (G, L_c^C)) is colorable if and only if $\mathcal{P}_c^C$ is colorable. Consider $(G', L', X') \in \mathcal{P}_c^C$. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$, by 5.3.4*(b)*, it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Since $|X'|$ has size $O(|V(G)|)$, applying 5.1.7, it follows that we can determine in time $O(|V(G)|^3)$ if (G', L', X') is colorable, and if it is, extend the coloring to a coloring of G. Therefore, via $O(|V(G)|^7)$ applications of 5.1.7, we can determine if $\mathcal{P}_c^C$ is colorable. That is, in time $O(|V(G)|^{10})$ we can determine if the 3-coloring c of $G[V(C)]$ extends to a type I coloring of G, and give an explicit type I coloring c' of G such that $c'(v) = c(v)$ for all $v \in V(C)$, if one exists. Finally, let $\mathcal{P}$ be the union of $\mathcal{P}_c^C$ taken over all 6-gons C in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$ with S^C empty and all 3-colorings c of $G[V(C)]$ such that $c(v_i) = c(v_{i+3})$ for every $i \in \{1, 2, 3\}$, where all indices are mod 6. Since every type I coloring of G restricts to such a coloring of $G[V(C)]$, it follows that G has a type I coloring if and only if $\mathcal{P}$ is colorable. Since there are $O(|V(G)|^6)$ 6-gons in G and 3! such colorings of $G[V(C)]$, it follows that $\mathcal{P}$ consists of $O(|V(G)|^6)$ restrictions $\mathcal{P}_c^C$, and so by the previous argument, we can determine in time $O(|V(G)|^{16})$ if G admits a type I coloring, and construct such a coloring if one exists. This proves 5.7.4. □

Let C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We say that a graph G has a *type II coloring with respect to C* (or just a *type II coloring* when the details are not important) if there exist a 3-coloring c of G such that $c(p_1) \neq c(p_2)$ for some $p_1, p_2 \in P^C(0, 3)$.

5.7.5. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3, C_7\}$-free graph G, that does not admit a type I coloring.*

Output: *A type II coloring of G with respect to some 6-gon in G, or a determination that none exists.*

Running time: $O(|V(G)|^{18})$.

Proof. In time $O(|V(G)|^8)$, we can enumerate all triples (C, p_1, p_2) in G, where C is a 6-gon given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p_1, p_2 \in P^C(0,3)$. If G has no such triple, then clearly G does not have a type II coloring and we may halt. Hence, we may assume there exists a 6-gon C in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p_1, p_2 \in P^C(0,3)$. In time $O(|V(G)|^2)$, we can partition $V(G) = V(C) \cup M^C \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C \cup Z^C$ as usual. Write $D = V(C) \cup \{p_1, p_2\}$. Fix a 3-coloring c of $G[D]$ such that $c(p_1) \neq c(p_2)$. Since G does not admit a type I coloring, we may assume that $c(v_0) = c(v_3) = 1, c(v_1) = c(v_5) = 2, c(v_2) = c(v_4) = 3, c(p_1) = 2$, and $c(p_2) = 3$. Define the order 3 palette L_c^C of G as follows: For every $v \in V(G)$, set

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in D \\ \{1,2,3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $M^C \cup CL^C \cup (P^C \setminus \{p_1, p_2\}) \cup S^C$ with respect to D. And so, $|L_c^C(v)| \leq 2$ if and only if $v \in V(G) \setminus (X^C \cup Y^C \cup Z^C)$. Moreover, for $v \in V(G) \setminus (X^C \cup Y^C \cup Z^C)$, $|L(v)| = 2$ only if $v \in M^C \cup CL^C(0) \cup CL^C(3) \cup P^C(0,3)$. By construction, (G, L_c^C) is colorable if and only the 3-coloring c of $G[D]$ extends to a type II coloring of G.

Observe that for every $v \in S^C(0)$ and $i \in \{1,2\}$, $v - v_2 - v_3 - p_i$ is an induced path in G, and for every $v \in S^C(1)$ and $i \in \{1,2\}$, $v - v_1 - v_0 - p_i$ is an induced path in G. Let W^C be the vertices of Z^C with a neighbor in S^C. Now by 5.7.3.1 and 5.7.3.2, we may apply 5.5.1 letting $P = V(C)$, $Q = M^C \cup CL^C \cup P^C \cup S^C$, $R = X^C \cup Y^C \cup W^C$, $S = Z^C \setminus W^C$, $T = \emptyset$, and $q_0 = p_1$. It follows that $Z^C \setminus W^C = \emptyset$, that every component of R is bipartite, and if some component of R has more than two vertices, then p_1 is complete to at least one side of the bipartition. Symmetrically, if some component of R has more than two vertices, then p_2 is complete to at least one side of the bipartition.

Let R_1 be the union of the components of R that contain a vertex complete to $\{p_1, p_2\}$. For every $v \in R_1$ that is complete to $\{p_1, p_2\}$, set $L_c^C(v) = \{1\}$. For every component X of R_1 with $|X| > 1$, proceed as follows. Let (A, B) be a bipartition of $G[X]$. By 5.5.1 and the definition of R_1, it follows that one of A, B is complete to $\{p_1, p_2\}$. We may assume that A is complete to $\{p_1, p_2\}$. Therefore $L_c^C(a) = \{1\}$ for every $a \in A$. Now set $L_c^C(b) = \{2, 3\}$ for every $b \in B$. Note that this does not change the colorability of (G, L_c^C). Observe that at this stage $|L_c^C(v)| < 3$ for every $v \in R_1$, and $|L_c^C(v)| = 3$ for every $v \in R \setminus R_1$.

Let R_2 be the union of all components Y of $R \setminus R_1$ such that $Y = \{y\}$ and y has a neighbor in $\{p_1, p_2\}$. For every $v \in R_2$, update the list of v with respect to $\{p_1, p_2\}$.

(1) Let $v \in CL^C(0) \cup CL^C(3)$ be adjacent to $y \in R$. Then each of p_1, p_2 has a neighbor in $\{v, y\}$.

Proof: Suppose not. We may assume that $v \in CL^C(0)$. If p_1 is anticomplete to $\{v, y\}$, then $y - v - v_1 - v_0 - p_1 - v_3 - v_4$ is a P_7 in G. This proves that either v or y is adjacent to p_1. Similarly, either v or y is adjacent to p_2. This proves *(1)*.

Let $C' = R \setminus (R_1 \cup R_2)$. Then $|L_c^C(y)| = 3$ for every $y \in C'$. Moreover, no vertex of C' is complete to $\{p_1, p_2\}$, and if Y is a component of C' with $|Y| = 1$, then Y is anticomplete to $\{p_1, p_2\}$. Let A' be the set of vertices of $M^C \cup CL^c \cup P^C \cup S^C$ with a neighbor in C'. For every non-empty subset $X \subseteq \{1, 2, 3\}$, define $A'_X = \{a \in A' \text{ with } L_c^C(a) = X\}$.

Suppose that $v \in A' \cap (CL^C(0) \cup CL^C(3))$ has a neighbor y in C'. By 5.7.3.3, $\{y\}$ is a component of R. It follows from the definition of C' that y is anticomplete to $\{p_1, p_2\}$. Now (1) implies that v is complete to $\{p_1, p_2\}$, and, in particular, $L_c^C(v) = \{1\}$. Consequently, for $v \in A'$, $|L_c^C(v)| = 2$ if and only if $v \in P^C(0, 3)$ and $L_c^C(v) = \{2, 3\}$. Thus $A'_{\{1,2\}} = A'_{\{1,2\}} = \emptyset$, and v_0 is complete to $A_{\{2,3\}}$.

We apply 5.3.4 with

- $\tilde{A} = A'$,
- $\tilde{B} = V(G) \setminus (A' \cup C')$,

- $\tilde{C} = C'$, and
- $Z = \emptyset$.

Let $\mathcal{P}_c^{C,p_1,p_2}$ be the restriction of $(G, L_c^C, \emptyset)$ of size $O(|V(G)|^7)$ thus obtained. By 5.3.4, $\mathcal{P}_c^{C,p_1,p_2}$ can be computed in time $O(|V(G)|^7)$. By 5.3.4*(c)*, we have that $(G, L_c^C, \emptyset)$ (and equivalently (G, L_c^C)) is colorable if and only if $\mathcal{P}_c^{C,p_1,p_2}$ is colorable. Consider $(G', L', X') \in \mathcal{P}_c^{C,p_1,p_2}$. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus V(C')$, by 5.3.4*(b)*, it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Since X' has size $O(|V(G)|)$, applying 5.1.7, it follows that we can test in time $O(|V(G)|^3)$ if (G', L', X') is colorable, and extend the coloring to (G, L_c^C) if it is. Therefore, via $O(|V(G)|^7)$ applications of 5.1.7, we can determine in time $O(|V(G)|^{10})$ if the 3-coloring c of $G[D]$ extends to a type II coloring of G, and give an explicit type II coloring c' of G such that $c'(v) = c(v)$ for all $v \in D$, if one exists. Finally, let $\mathcal{P}$ be the union of $\mathcal{P}_c^{C,p_1,p_2}$ taken over all triples (C, p_1, p_2) where C is a 6-gon given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p_1, p_2 \in P^C(0,3)$, and all 3-colorings c of $G[V(C) \cup \{p_1, p_2\}]$ such that $c(p_1) \neq c(p_2)$. Since every type II coloring of G restricts to such a coloring of $G[V(C) \cup \{p_1, p_2\}]$ for some C, p_1, p_2 , it follows that G has a type II coloring if and only if $\mathcal{P}$ is colorable. Since there are $O(|V(G)|^8)$ such triples (C, p_1, p_2) in G and 2 such colorings of $G[V(C) \cup \{p_1, p_2\}]$ for each (C, p_1, p_2), the restriction $\mathcal{P}$ is the union of $O(|V(G)|^8)$ restrictions $\mathcal{P}_c^{C,p_1,p_2}$. Therefore by the previous argument, we can determine in time $O(|V(G)|^{18})$ if G admits a type II coloring, and construct such a coloring if one exists. This proves 5.7.5. □

Now, suppose (C, p) is a shell in G. We partition $V(G) \setminus (V(C) \cup \{p\})$ as follows:

- Let Q_p^C be the set of vertices in $V(G) \setminus (V(C) \cup \{p\})$ with a neighbor in $V(C) \cup \{p\}$.
- Let R_p^C be the set of vertices in $V(G) \setminus (V(C) \cup \{p\} \cup Q_p^C)$ with a neighbor in Q_p^C.
- Let $S_p^C = V(G) \setminus (V(C) \cup \{p\} \cup Q_p^C \cup R_p^C)$.
- Let PL_p^C be the set of vertices in $V(G) \setminus V(C)$ adjacent to p and anticomplete to $V(C)$. Note, that PL_p^C is a subset of Q_p^C.

5.7.6. *Let G be a clean, connected $\{P_7, C_3, C_7\}$-free graph. Then for every shell (C, p) in G the following hold:*

1. *$M^C \cup CL^C \cup P^C \cup S^C \cup PL_p^C$ gives a partition of Q_p^C.*
2. *Every vertex of Q_p^C with a neighbor in R_p^C either belongs to PL_p^C or has at least two neighbors in $V(C)$.*
3. *S_p^C is empty.*
4. *Every component of R_p^C has size at most two.*

Proof. Let (C, p) be a shell in G, where C is the 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$ and $p \in P^C$. By 5.7.1, it follows that 5.7.6.1 holds. Since R_p^C is a subset of A^C, by 5.7.3.1, it follows that M^C is anticomplete to R_p^C. And so, by definition, 5.7.1, and 5.7.6.1, it follows that 5.7.6.2 holds.

(1) For every $s \in S^C$, there exists $p_1, p_2, p_3 \in V(C) \cup \{p\}$ such that $p_1 - p_2 - p_3 - s$ is an induced path.

Proof: Consider a vertex $s \in S^C$. By symmetry, we may assume both s and p are adjacent to v_0, that is, that $s \in S_0^C$ and $p \in P^C(\{0, 3\})$. Since G is triangle-free, it follows that s is non-adjacent to p. Then $v_3 - p - v_0 - s$ is the desired induced path. This proves *(1)*.

By definition, S_p^C is anticomplete to $V(C) \cup \{p\} \cup Q_p^C$. Since G is clean and connected, by 5.7.3.2 and *(1)*, we may apply 5.5.1 letting $P = V(C) \cup \{p\}$, $Q = Q_p^C$, $R = R_p^C$, $S = S_p^C$ and $T = \emptyset$. It follows that 5.7.6.3 and 5.7.6.4 hold. This proves 5.7.6.

□

5.7.7. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3, C_7\}$-free graph G which contains a shell.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^{18})$.

Proof. By 5.7.4 and 5.7.5 in time $O(|V(G)|^{18})$ we can produce a type I or a type II coloring of G, if one exists. Hence, we may assume there does not exists a type I or a type II coloring of G. Let C be the 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and suppose (C, p) is a shell in G. Observe that such an induced subgraph can be found in time $O(|V(G)|^7)$. Since G is clean, by 5.7.6.3, it follows that S_p^C is empty, and so we may partition $V(G) = V(C) \cup \{p\} \cup Q_p^C \cup R_p^C$ as usual. Next, fix a 3-coloring c of $G[V(C) \cup \{p\}]$, that is not a type I coloring with respect to C. Define the order 3 palette L_c^C of G as follows: For every $v \in V(G)$, set

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \cup \{p\} \\ \{c(p)\} & , \quad \text{if } v \notin V(C) \cup \{p\}, \text{ and } v \text{ has the same anchors as } p \text{ in } C \\ \{1,2,3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in Q_p^C with respect to $V(C) \cup \{p\}$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus R_p^C$, while $|L_c^C(v)| = 3$ for all $v \in R_p^C$. Observe that, since G does not have a type II coloring, (G, L_c^C) is colorable if and only if the 3-coloring c of $G[V(C) \cup \{p\}]$ extends to a 3-coloring of G.

Let A' be the set of vertices in Q_p^C with a neighbor in R_p^C, and for every non-empty subset $X \subseteq \{1, 2, 3\}$, define $A'_X = \{a \in A' \text{ with } L_c^C(a) = X\}$.

(1) Let $\{i, j, k\} = \{1, 2, 3\}$. If $q_1 \in A'_{\{i,j\}}$, $q_2 \in A'_{\{j,k\}}$, and $r_1 \in R_p^C \cap (N(q_1) \setminus N(q_2))$ and $r_2 \in R_p^C \cap (N(q_2) \setminus N(q_1))$, then q_1 is adjacent to q_2.

Proof: Suppose q_1 is non-adjacent to q_2. Since $L(q_1) = \{i, j\}$ and $L(q_2) = \{j, k\}$, it follows that $N(q_1) \cap N(q_2) \cap (V(C) \cup \{p\})$ is empty. First, assume $q_1 \in S^C$. By symmetry, we may assume $q_1 \in S_0^C$ and $p \in P^C(\{0, 3\})$. Since G is triangle-free, it follows that q_1 is non-adjacent to p. Suppose $q_2 \in PL_p^C$. However, if r_1 is non-adjacent to r_2, then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2$ is a P_7 in G, and if r_1 is adjacent to r_2, then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2 - r_1$ is a C_7 in G, in both case a contradiction. Hence, $q_2 \notin PL_p^C$. By 5.7.6.2, it follows that $|N(q_2) \cap \{v_1, v_3, v_5\}| \geq 2$.

Suppose q_2 is adjacent to v_3. Since G is triangle-free, it follows that p is non-adjacent to q_2. However, if r_1 is non-adjacent to r_2, then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2$ is a P_7 in G, and if r_1 is adjacent to r_2, then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2 - r_1$ is a C_7 in G, in both case a contradiction. Hence, q_2 is non-adjacent to v_3, and so q_2 is complete to $\{v_1, v_5\}$. If p is non-adjacent to q_2, then $v_4 - v_3 - p - v_0 - v_1 - q_2 - r_2$ is a P_7 in G, a contradiction. Hence, p is adjacent to q_2. However, if r_1 is non-adjacent to r_2, then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2$ is a P_7 in G, and if r_1 is adjacent to r_2, then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2 - r_1$ is a C_7 in G, in both case a contradiction. By symmetry, this proves that neither q_1 nor q_2 belongs to S^C.

Next, suppose $q_1 \in P^C$. By symmetry, we may assume $q_1 \in P^C(\{0, 3\})$. Suppose first that q_2 is adjacent to v_1. Since G is triangle-free, it follows that q_2 is non-adjacent to v_2. However, if r_1 is non-adjacent to r_2, then $r_1 - q_1 - v_3 - v_2 - v_1 - q_2 - r_2$ is a P_7 in G, and if r_1 is adjacent to r_2, then $r_1 - q_1 - v_3 - v_2 - v_1 - q_2 - r_2 - r_1$ is a C_7 in G, in both case a contradiction. By symmetry, it follows that q_2 is anticomplete to $\{v_1, v_2, v_4, v_5\}$. Since $q_1 \in A'_{\{i,j\}}$ and $q_2 \in A'_{\{j,k\}}$, it follows that $q_2 \in PL^C_p$, $c(p) \neq c(v_0)$, and p is non-adjacent to q_1. Since $|L^C_c(q_1)| = 2$, it follows that $p \notin P^C(\{0, 3\})$, and so we may assume that $p \in P^C(\{1, 4\})$. Now, if r_1 is non-adjacent to r_2, then $r_2 - q_2 - p - v_1 - v_0 - q_1 - r_1$ is a P_7 in G, and if if r_1 is adjacent to r_2, then $r_2 - q_2 - p - v_1 - v_0 - q_1 - r_1 - r_2$ is a C_7 in G, in both cases a contradiction. By symmetry, this proves that neither q_1 nor q_2 belongs to P^C.

Since not both q_1 and q_2 are adjacent to p, by 5.7.6.1 and 5.7.6.2, we may assume $q_1 \in CL^C$ is non-adjacent to p. By symmetry, we may assume $q_1 \in CL^C(1)$. Since $r_1 - q_1 - v_0 - v_1 - p - v_4 - v_3$ is not a P_7 in G, it follows that $p \notin P^C(\{1, 4\})$. And so, we may assume $p \in P^C(\{0, 3\})$. Suppose $q_2 \in CL^C$ also. Since $N(q_1) \cap N(q_2) \cap V(C)$ is empty, we may assume that $q_2 \in CL^C(0) \cup CL^C(4)$. Suppose $q_2 \in CL^C(0)$. Let C' be the 6-gon in G given by $q_2 - v_1 - v_2 - v_3 - v_4 - v_5 - q_2$. Then $r_2 \in M^{C'}(0)$ and $q_1 \in M^{C'}(2)$, contrary to 5.7.6.2. Hence, $q_2 \in CL^C(4)$. However, if r_1 is non-adjacent to r_2, then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2$ is a P_7 in G, and if r_1 is adjacent to r_2, then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2 - r_1$ is a C_7 in G, in both case a contradiction. Hence, $q_2 \notin CL^C$. By 5.7.6.1 and 5.7.6.2, it follows $q_2 \in PL^C_p$. However, if r_1 is non-adjacent to r_2, then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2$ is a P_7 in G, and if r_1 is adjacent to r_2, then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2 - r_1$ is a C_7 in G, in both case a contradiction. This proves *(1)*.

(2) For all distinct $i, j \in \{1, 2, 3\}$ some vertex of $V(C) \cup \{p\}$ is complete to $A'_{\{i,j\}}$.

Proof: If $A'_{\{i,j\}} = \emptyset$, then *(2)* trivially holds. Thus, we may assume $A'_{\{i,j\}} \neq \emptyset$. Let $\{i, j, k\} = \{1, 2, 3\}$ and define K to be the set of vertices of $V(C) \cup \{p\}$ with a neighbor in $A'_{\{i,j\}}$. Since we have updated, it follows that $c(v) = k$ for every $v \in K$. Since $G[V(C) \cup \{p\}]$ has no stable set of size 4, it follows that $|K| \leq 3$. If $|K| = 1$, then, by definition, the unique vertex of K is complete to $A'_{\{i,j\}}$. Hence, we may assume that $|K| \geq 2$. If $|K| = 2$, then, by 5.7.6.2, either $p \in K$ is complete to $A'_{\{i,j\}}$, or $p \notin K$ and then K is complete to $A'_{\{i,j\}}$. In either case, *(2)* holds. And so we may assume $|K| = 3$. By 5.7.6.1 and 5.7.6.2, it follows that $A'_{\{i,j\}} \subseteq CL^C \cup P^C \cup S^C \cup PL^C_p$. First, suppose $p \notin K$. It follows that $PL^C_p \cap A'_{\{i,j\}}$ is empty. By symmetry, we may assume $K = \{v_0, v_2, v_4\}$. Suppose further that v_0 and v_2 are not complete to $A'_{\{i,j\}}$. By *(1)*, it follows that there exists $c_3 \in A'_{\{i,j\}} \cap CL^C(3)$, and $c_5 \in A'_{\{i,j\}} \cap CL^C(5)$. Since G is triangle-free, it follows that c_3 is non-adjacent to c_5. By definition, there exists $r_3, r_5 \in R^C_p$ such that c_3 is adjacent to r_3, and c_5 is adjacent to r_5. By 5.7.3.6, it follows that c_3 is non-adjacent to r_5, and c_5 is non-adjacent to r_3. Let C' be the 6-gon in G given by $v_0 - v_1 - v_2 - c_3 - v_4 - c_5 - v_0$. Then $r_3 \in M^{C'}(3)$ and $r_5 \in M^{C'}(5)$, contrary to 5.7.3.5. And so, it follows that $p \in K$.

Without loss of generality, we may assume $p \in P^C(\{0, 3\})$. Let $\{a, b\} = K \cap V(C)$. By symmetry, we may assume $a = v_1$ and $b \in \{v_4, v_5\}$. We may also assume p is not complete to $A'_{\{i,j\}}$, as otherwise *(2)* holds immediately. By 5.7.6.1 and 5.7.6.2, it follows that there exists $q \in A'_{\{i,j\}}$ complete to $\{a, b\}$ and non-adjacent to p. We may also assume that a is not complete to $A'_{\{i,j\}}$, as otherwise *(2)* holds immediately. And so, by 5.7.6.2, there exists $q' \in PL^C_p \cap A'_{\{i,j\}}$. By definition, there exists $r, r' \in R^C_p$ such that r is adjacent to q, and r' is adjacent to q'. Suppose r is adjacent to q'. Since G is triangle-free, it follows that q is non-adjacent to q'. But now $v_2 - v_1 - q - r - q' - p - v_3 - v_2$ is a C_7 in G, a contradiction. Hence, it follows that r is non-adjacent to q', and, by symmetry, that r' is non-adjacent to

q. Suppose q is non-adjacent to q'. If r is non-adjacent to r', then $r-q-v_1-v_0-p-q'-r'$ is a P_7 in G, and if r is adjacent to r', then $r-q-v_1-v_0-p-q'-r'-r$ is a C_7 in G, in both cases a contradiction. Hence, q is adjacent to q'. Let C'''' be the 6-gon in G given by $v_1-v_2-v_3-p-q'-q-v_1$. If q is adjacent to v_5 (and therefore not to v_4), then $r'-q'-q-v_1-v_2-v_3-v_4$ is a P_7 in G, a contradiction. Hence, it follows that q is non-adjacent to v_5, that is, that $b=v_4$. Since c is not a type I coloring with respect to C, and since $c(v_1)=c(v_4)=c(p)$, it follows that $c(v_0)=c(v_2)$, and $c(v_3)=c(v_5)$. Applying the fact that G admits no type I coloring to the 6-gon $v_1-v_2-v_3-p-q'-q-v_1$, we deduce that in every coloring c' of (G,L_c^C), $c'(q)=c(v_2)$ and $c'(q')=c(v_3)$. However, applying the fact that G admits no type I coloring with respect to the 6-gon $v_4-v_5-v_0-p-q'-q-v_4$, we deduce that in every coloring c' of (G,L_c^C), $c'(q)=c(v_5)$ and $c'(q')=c(v_0)$. But this implies that $c(v_0)=c(v_2)=c(v_5)$, a contradiction. This proves *(2)*.

By 5.7.6.4, it follows that every component of R_p^C has at most two vertices. And so, by 5.7.6.1, 5.7.6.2, *(1)* and *(2)*, we can apply 5.3.4 with

- $\tilde{A}=A'$
- $\tilde{B}=V(C)\cup\{p\}\cup(Q_p^C\setminus A')$,
- $\tilde{C}=R_p^C$, and
- $Z=\emptyset$.

Let $\mathcal{P}_c$ be the restriction of (G,L_c^C) of size $O(|V(G)|^7)$ thus obtained, and let $\mathcal{P}$ be the union of $\mathcal{P}_c$ taken over all 3-colorings c of $G[V(C)\cup\{p\}]$ that are not type I colorings. By 5.3.4, and since there are at most 7^3 3-colorings of $G[V(C)\cup\{p\}]$, it follows that $\mathcal{P}$ can be computed in time $O(|V(G)|^7)$. By 5.3.4*(c)*, we have that (G,L_c^C) is colorable if and only if $\mathcal{P}_c$ is colorable. Since every 3-coloring of G restricts to a 3-coloring of $G[V(C)\cup\{p\}]$, it follows that G is 3-colorable if and only if $\mathcal{P}$ is colorable.

Consider $(G',L',X')\in\mathcal{P}$. Then $(G',L',X')\in\mathcal{P}_c$ for some coloring c of $G[V(C)\cup\{p\}]$, and c is not a type I or a type II coloring. Since $|L_c^C(v)|\le 2$ for all $v\in V(G)\setminus R_p^C$, by 5.3.4*(b)*, it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Thus, since by 5.3.4 X' has size $O(|V(G)|)$, applying 5.1.7, we can test in time $O(|V(G)|^3)$ if (G', L', X') is colorable. Therefore, via $O(|V(G)|^7)$ applications of 5.1.7, we can determine if $\mathcal{P}$ is colorable and extend any coloring of a colorable $(G', L', X') \in \mathcal{P}$ to a coloring of G in linear time. Consequently, in time $O(|V(G)|^{10})$ we can determine if $\mathcal{P}$ is colorable. This proves 5.7.7.

□

5.8 5-gons

In this section we show that if a $\{P_7, C_3, C_7, shell\}$-free graph contains a 5-gon, then in polynomial time we can decide if the graph is 3-colorable, and give a coloring if one exists.

Let C be a 5-gon in a graph G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. We say that a vertex $v \in V(G) \setminus V(C)$ is:

- a *leaf at* i, if $N(v) \cap V(C) = \{v_i\}$ for some $i \in \{0, 1, ..., 4\}$,
- a *clone at* i, if $N(v) \cap V(C) = \{v_{i-1}, v_{i+1}\}$ for some $i \in \{0, 1, ..., 4\}$, where all indices are mod 5.

The following shows how we can partition the vertices of G based on their anchors in C.

5.8.1. *Let G be a triangle-free graph, and suppose C is a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. If $v \in V(G) \setminus V(C)$, then for some $i \in \{0, 1, ..., 4\}$, either:*

1. *v is a leaf at i,*
2. *v is a clone at i, or*
3. *v is anticomplete to $V(C)$.*

Proof. Consider a vertex $v \in V(G) \setminus V(C)$. If v is anticomplete to $V(C)$, then 5.8.1.3 holds. Thus, we may assume $N(v) \cap V(C) \neq \emptyset$, and, by symmetry, suppose that $v_0 \in N(v) \cap V(C)$. If $|N(v) \cap V(C)| = 1$, then 5.8.1.1 holds, and so we may assume $|N(v) \cap V(C)| \geq 2$. Since

G is triangle-free, it follows that v is anticomplete to $\{v_1, v_4\}$. Since G is triangle-free, it follows that v is mixed on $\{v_2, v_3\}$ and so 5.8.1.2 holds. This proves 5.8.1. □

Let G be a triangle-free graph. Suppose C is a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. Using 5.8.1 we partition $V(G) \setminus V(C)$ as follows:

- Let $M^C(i)$ be the set of leaves at i and define $M^C = \bigcup_{i=0}^{4} M^C(i)$.
- Let $CL^C(i)$ be the set of clones at i and define $CL^C = \bigcup_{i=0}^{4} CL^C(i)$.
- Let A^C be the set of vertices anticomplete to $V(C)$.

By 5.8.1, it follows that $V(G) = V(C) \cup M^C \cup CL^C \cup A^C$. Furthermore, we partition $A^C = X^C \cup Y^C \cup Z^C$, where

- X^C is the set of vertices in A^C with a neighbor in M^C,
- Y^C is the set of vertices in $A^C \setminus X^C$ with a neighbor in CL^C, and
- $Z^C = A^C \setminus (X^C \cup Y^C)$.

Finally, we define subsets of X^C, Y^C and M^C, for every $i \in \{0, ..., 4\}$ as follows:

- Let $X^C(i)$ be the set of vertices of X^C with a neighbor in $M^C(i)$.
- Let $Y^C(i)$ be the set of vertices of Y^C with a neighbor in $CL^C(i)$.
- Let M^C_i be the set of vertices of M^C with a neighbor in $X^C(i)$.

And so, for a given 5-gon C in time $O(|V(G)|^2)$ we obtain the partition $V(C) \cup M^C \cup CL^C \cup X^C \cup Y^C \cup Z^C$ of $V(G)$. Now, we establish several properties of this partition. By definition and 5.8.1, it follows that:

5.8.2. *If G is a triangle-free graph, then for every 5-gon C in G the following hold:*

1. *Every vertex in X^C has a neighbor in M^C.*

2. *Every vertex in Y^C has a neighbor in CL^C.*

3. *Y^C is anticomplete to M^C.*

Recall that for a fixed subset X of $V(G)$, we say a vertex $v \in V(G) \setminus X$ *is mixed on an edge of X*, if there exist adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$.

5.8.3. *If G is a $\{P_7, C_3\}$-free graph, then for every 5-gon C in G the following hold:*

1. *No vertex in M^C is mixed on an edge of A^C.*

2. *X^C is stable and anticomplete to $Y^C \cup Z^C$.*

3. *Both $M^C(i)$ and $CL^C(i)$ are stable for every $i \in \{0, ..., 4\}$.*

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. Suppose for adjacent $a, a' \in A^C$, there exists $m \in M^C$ which is adjacent to a' and non-adjacent to a. By symmetry, we may assume $m \in M^C(0)$. However, then $a - a' - m - v_0 - v_1 - v_2 - v_3$ is a P_7 in G, a contradiction. This proves 5.8.3.1.

Consider a vertex $x \in X^C$. By 5.8.2.1, there exists $m \in M^C$ adjacent to x. If there exists $x' \in N(x) \cap A^C$, then, since G is triangle-free m is non-adjacent to x', and it follows that m is mixed on an edge of A^C, contradicting 5.8.3.1. This proves 5.8.3.2.

For every $i \in \{0, ..., 4\}$, by definition v_i is complete to $M^C(i)$ and v_{i+1} is complete to $CL^C(i)$, where all indices are mod 5. Since G is triangle-free, it follows that 5.8.3.3 holds. This proves 5.8.3.

□

5.8.4. *Let G be a clean, connected $\{P_7, C_3, C_7, shell\}$-free graph. Then for every 5-gon C in G and $i \in \{0, ..., 4\}$ the following hold:*

1. *$X^C(i)$ is anticomplete to $M^C \setminus M^C(i)$; in other words, $M_i^C \subseteq M^C(i)$.*

2. *$X^C(0) \cup ... \cup X^C(4)$ gives a partition of X^C.*

3. *Every vertex in $X^C(i)$ has a neighbor in $CL^C(i)$.*

4. *$X^C(i)$ is anticomplete to $CL^C \setminus CL^C(i)$.*

5. *$Y^C(i)$ is anticomplete to $CL^C(i+1) \cup CL^C(i-1)$, where all indices are mod 5.*

6. *M_i^C is anticomplete to $V(G) \setminus (M_i^C \cup CL^C(i) \cup X^C(i))$.*

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. It is enough to prove this statement for $i = 0$. Let $x \in X^C(0)$, and let $m \in M^C(0)$ be adjacent to x. Then $x \in M_0^C$. By 5.8.3.2, X^C is stable and anticomplete to $Y^C \cup Z^C$, and so it follows that $N(x) \subseteq M^C \cup CL^C$. Suppose there exists $m' \in N(x) \cap (M^C \setminus M_0^C)$. Since G is triangle-free, m is non-adjacent to m'. By symmetry, we may assume $m' \in M^C(3) \cup M^C(4)$. However, if $m' \in M^C(4)$, then $m' - x - m - v_0 - v_1 - v_2 - v_3$ is a P_7 in G, and if $m' \in M^C(3)$, then $m - x - m' - v_3 - v_2 - v_1 - v_0 - m$ is a C_7 in G, in both cases, a contradiction. Hence, x is anticomplete to $M^C \setminus M_0^C$. This proves 5.8.4.1, which, by 5.8.2.1, immediately implies 5.8.4.2. Since v_0 is complete to $M^C(0)$ and G has no dominated vertices, it follows that there exists $c \in CL^C \setminus (CL^C(1) \cup CL^C(4))$ adjacent to x. Since G is triangle-free, c is non-adjacent to m. Suppose $c \notin CL^C(0)$. By symmetry, we may assume $c \in CL^C(2)$. However, then $v_0 - m - x - c - v_3 - v_4 - v_0$ with v_1 is a shell in G, a contradiction. Hence, x has a neighbor in $CL^C(0)$. This proves 5.8.4.3. Now, we prove 5.8.4.4 and 5.8.4.5. We have already shown that $X^C(0)$ is anticomplete to $CL^C(2) \cup CL^C(3)$. Let $c \in CL^C(0)$ be adjacent to $z \in X^C(0) \cup Y^C(0)$. Suppose there exists $c' \in CL^C(1) \cup CL^C(4)$ adjacent to z. By symmetry, we may assume $c' \in CL^C(1)$. Since G is triangle-free, c' is non-adjacent to c. However, then $c - z - c' - v_2 - v_3 - v_4 - c$ with v_1 is a shell in G, a contradiction. Hence, $X^C(0)$ is anticomplete to $CL^C \setminus CL^C(0)$, and $Y^C(0)$ is anticomplete to $CL^C(1) \cup CL^C(4)$. This proves 5.8.4.4 and 5.8.4.5.

Next we prove 5.8.4.6. Recall m is an arbitrary vertex of $M^C(0)$, and that $x \in X^C(0)$ is adjacent to m and to $c \in CL^C(0)$. By definition, 5.8.2.3 and 5.8.4.1, it follows that M_0^C

anticomplete to $(X^C \setminus X^C(0)) \cup Y^C \cup Z^C$. Suppose there exists $m' \in M^C \setminus M_0^C$ adjacent to m. By 5.8.3.3, it follows that $M^C(0)$, and thus M_0^C, is stable, and so it follows that $m' \in M^C \setminus M^C(0)$. Since G is triangle-free, x is non-adjacent to m'. By symmetry, we may assume $m' \in M^C(1) \cup M^C(2)$. However, if $m' \in M^C(1)$, then $x - m - m' - v_1 - v_2 - v_3 - v_4$ is a P_7 in G, and if $m' \in M^C(2)$, then $m - m' - v_2 - v_3 - v_4 - v_0 - m$ with v_1 is a shell in G, in both cases, a contradiction. Hence, M_0^C is anticomplete to $M^C \setminus M_0^C$. Finally, we show that M_0^C is anticomplete to $CL^C \setminus CL^C(0)$. Since v_0 is complete to $M^C(0)$ and G is triangle-free, it follows that M_0^C is anticomplete to $CL^C(1) \cup CL^C(4)$. Suppose there exists $c'' \in CL^C(2) \cup CL^C(3)$ adjacent to m. Since G is triangle-free, c'' is anticomplete to $\{c, x\}$. By symmetry, we may assume $c'' \in CL^C(2)$. However, then $x - m - c'' - v_3 - v_4 - c - x$ with v_0 is a shell in G, a contradiction. Hence, M_0^C is anticomplete to $CL^C \setminus CL^C(0)$. This proves 5.8.4.6.

□

5.8.5. *Let G be a clean, connected $\{P_7, C_3, C_7, S_7\}$-free graph. Then for every 5-gon C in G the following hold:*

1. *Z^C is empty.*
2. *Every component of Y^C has size two.*
3. *$Y^C(0), ..., Y^C(4)$ are pairwise disjoint and anticomplete to each other.*
4. *Every component of $M_i^C \cup X^C(i) \cup Y^C(i)$ has size two for every $i \in \{0, ..., 4\}$.*

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$.

(1) For every $c \in CL^C$, there exists $b_1, b_2, b_3 \in V(C)$ such that $b_1 - b_2 - b_3 - c$ is an induced path.

Proof: Consider a vertex $c \in CL^C$. By symmetry, we may assume $c \in CL^C(0)$. And so, $v_3 - v_2 - v_1 - c$ is the desired induced path. This proves *(1)*.

By 5.8.2, 5.8.3.2 and *(1)*, we may apply 5.5.1 letting $P = V(C)$, $Q = CL^C$, $R = Y^C$, $S = Z^C$, and $T = M^C \cup X^C$. It thus follows that 5.8.5.1 holds and that every component of Y^C has size at most two. Now, suppose $y \in Y^C$ is a singleton component of Y^C. By 5.8.2.2, there exists $c \in CL^C$ adjacent to y. By symmetry, we may assume $c \in CL^C(0)$, and so $y \in Y^C(0)$. By 5.8.4.5, y is anticomplete to $CL^C(1) \cup CL^C(4)$, and so, by 5.8.2.3, it follows that y is anticomplete to $V(G) \setminus (CL^C(0) \cup CL^C(2) \cup CL^C(3))$. Since v_1 does not dominate y, it follows that y has a neighbor in $CL^C(3)$. Since v_4 does not dominate y, it follows that y has a neighbor in $CL^C(2)$. However, then y has a neighbor in $CL^C(2)$ and in $CL^C(3)$, contrary to 5.8.4.5. This proves 5.8.5.2.

(2) If $\{y, y'\}$ is the vertex set of a component of Y^C, then there exists a unique $i \in \{0, ..., 4\}$ such that vertices of $CL^C(i)$ have neighbors in $\{y, y'\}$.

Proof: Suppose $\{y, y'\}$ is the vertex set of a component of Y^C. By 5.8.2.2, there exists $c \in CL^C$ adjacent to y. Since G is triangle-free, y' is non-adjacent to c. By symmetry, we may assume $c \in CL^C(0)$. Let C' be the 5-gon given by $c - v_1 - v_2 - v_3 - v_4 - c$. It follows that $y \in M^{C'}(0)$, $y' \in X^{C'}(0)$ and, since G is triangle-free, $CL^C(j) = CL^{C'}(j)$ for $j = 0, 2, 3$. And so, by 5.8.4.4 applied to C', it follows that y' has a neighbor in $CL^{C'}(0) = CL^C(0)$ and is anticomplete to $CL^C(2) \cup CL^C(3)$. In particular, $y' \in Y^C(0)$. By 5.8.4.5 applied to C, it follows that y' is anticomplete to $CL^C(1) \cup CL^C(4)$. And so, it follows that y' is anticomplete to $CL^C \setminus CL^C(0)$. But now reversing the roles of y and y', it follows that y is anticomplete to $CL^C \setminus CL^C(0)$. This proves *(2)*.

By 5.8.2.2, every vertex $y \in Y^C$ has a neighbor in CL^C and so *(2)* implies that $Y^C(0), ..., Y^C(4)$ are pairwise disjoint and anticomplete to each other. This proves 5.8.5.3.

(3) Every component of $M_i^C \cup X^C(i)$ has size two for every $i \in \{0, ..., 4\}$.

Proof: We may assume $i = 0$. By 5.8.3.3, it follows that $M^C(0)$, and thus M_0^C, is stable. By definition, 5.8.4.3 and 5.8.4.4, it follows that every vertex in $M_0^C \cup X^C(0)$ has a neighbor in $CL^C(0) \cup \{v_0\}$. By 5.8.3.2, 5.8.4.1, 5.8.4.2 and 5.8.4.4, it follows that $X^C(0)$ is anticomplete to $V(G) \setminus (X^C(0) \cup M_0^C \cup CL^C(0))$. By 5.8.4.6, it follows that M_0^C is anticomplete to $V(G) \setminus (M_0^C \cup CL^C(0) \cup X^C(0))$. Since $v_0 - v_1 - v_2 - v_3$ is an induced path, by *(1)*, we may apply 5.5.1 letting $P = V(C) \setminus \{v_0\}$, $Q = CL^C(0) \cup \{v_0\}$, $R = M_0^C \cup X^C(0)$, $S = \emptyset$, and $T = V(G) \setminus (V(C) \cup M_0^C \cup CL^C(0) \cup X^C(0))$. Hence, every component of $M_0^C \cup X^C$ has size at most two. However, since every vertex in M_0^C has a neighbor in $X^C(0)$ and every vertex in $X^C(0)$ has a neighbor in M_0^C, it follows that *(3)* holds.

By 5.8.2.3 and 5.8.3.2, it follows that $M^C \cup X^C$ is anticomplete to Y^C. Hence, together 5.8.5.2 and *(3)* imply that 5.8.5.4 holds. This proves 5.8.5.

□

Now, we prove the main result of the section.

5.8.6. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3, C_7, shell\}$-free graph G which contains a 5-gon.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^6)$.

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$; clearly C can be found in time $O(|V(G)|^5)$. In time $O(|V(G)|^2)$, we can partition $V(G) = V(C) \cup CL^C \cup M^C \cup X^C \cup Y^C \cup Z^C$ as well as determine $X^C(i), Y^C(i)$ and M_i^C for every $i \in \{0, ..., 4\}$. Since G is clean, by 5.8.5.1, it follows that Z^C is empty and, by 5.8.4.2 and 5.8.5.3, we obtain the partitions $X^C(0) \cup ... \cup X^C(4)$ of X^C and $Y^C(0) \cup ... \cup Y^C(4)$ of Y^C. Next, fix a 3-coloring c of $G[V(C)]$. By symmetry, we may assume $c(v_1) = c(v_3)$ and $c(v_2) = c(v_4)$. Define the order 3 palette L_c^C of G as follows:

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \\ \{1, 2, 3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $CL^C \cup M^C$ with respect to $V(C)$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$. Furthermore, $|L_c^C(v)| = 2$ if and only if $v \in M^C \cup CL^C(2) \cup CL^C(3)$. Now, update the vertices in $X^C \cup Y^C$ with respect to $CL^C \cup M^C$. By 5.8.4.3, it follows that $|L_c^C(v)| = 3$ if and only if $v \in X^C(2) \cup Y^C(2) \cup X^C(3) \cup Y^C(3)$. For every $j \in \{2, 3\}$, by 5.8.4.3, every vertex of $M_j \cup X^C(j) \cup Y^C(j)$ has a neighbor in $CL^C(j) \cup \{v_j\}$ and, by 5.8.5.4, every component of $M_j \cup X^C(j) \cup Y^C(j)$ has size 2. Let $\mathcal{L}_1$ be the set of $O(|V(G)|^2)$ subpalettes of L_c^C obtained from 5.2.3 applied with

- $x = v_1$,
- $S = CL^C(2) \cup \{v_2\}$,
- $\hat{A} \cup \hat{B} = M_2^C \cup X^C(2) \cup Y^C(2)$,
- $Y = V(G) \setminus (\{v_1, v_2\} \cup CL^C(2) \cup M_2 \cup X^C(2) \cup Y^C(2))$, and
- $X = \emptyset$.

Next, for every $L \in \mathcal{L}_1$, let $\mathcal{L}(L)$ be the set of $O(|V(G)|^2)$ subpalettes of L obtained from 5.2.3 applied with

- $x = v_4$,
- $S = CL^C(3) \cup \{v_3\}$,
- $\hat{A} \cup \hat{B} = M_3^C \cup X^C(3) \cup Y^C(3)$, and
- $Y = V(G) \setminus (\{v_3, v_4\} \cup CL^C(3) \cup M_3 \cup X^C(3) \cup Y^C(3))$, and
- $X = \emptyset$.

Finally, let $\mathcal{L}_c = \{\mathcal{L}(L) : L \in \mathcal{L}_1\}$ be the set of $O(|V(G)|^4)$ subpalettes of L_c^C thus obtained. By 5.2.3, $\mathcal{L}_c$ can be computed in time $O(|V(G)|^6)$. Since $X = \emptyset$, by 5.2.3*(b)*, we have that (G, L_c^C) is colorable if and only if $(G, \mathcal{L}_c)$ is colorable. Let $\mathcal{L}$ be the union of the sets $\mathcal{L}_c$ taken over all 3-colorings c of $G[V(C)]$. Then G is 3-colorable if and only if $(G, \mathcal{L})$ is 3-colorable.

Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C(2) \cup Y^C(2) \cup X^C(3) \cup Y^C(3))$, by 5.2.3*(a)*, it follows that $|L(v)| \leq 2$ for all $L \in \mathcal{L}$ and $v \in V(G)$. Thus, by 5.1.6 we can test in time $O(|V(G)|^2)$ if (G, L) is colorable for every $L \in \mathcal{L}$. Since there are at most 5^3 3-colorings of $G[V(C)]$, it follows that $\mathcal{L}$ consists of $O(|V(G)|^4)$ subpalettes of L_c^C, and so, via $O(|V(G)|^4)$ applications of 5.1.6, we can determine if $(G, \mathcal{L})$ is colorable. That is, in time $O(|V(G)|^6)$ we can determine if there exists a 3-coloring c of $G[V(C)]$ that extends to a 3-coloring of G, and give an explicit 3-coloring c' of G such that $c'(v) = c(v)$ for all $v \in V(C)$, if one exists. Since every 3-coloring of G restricts to a 3-coloring of $G[V(C)]$, this proves 5.8.6.

□

5.9 Main Algorithm

In this section we prove the main result of this chapter 5.1.8, which we restate:

5.9.1. *There is an algorithm with the following specifications:*

Input: *A $\{P_7, C_3\}$-free graph G.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^{18})$.

Proof. By 5.4.3, at the expense of carrying out a time $O(|V(G)|^5)$ procedure we may assume G is clean. Via breadth-first search in time $O(|V(G)|^2)$ we can determine the components of G, and so we may also assume G is connected. By 5.6.6, if G contains a 7-gon, then in time $O(|V(G)|^{10})$ we can either produce a 3-coloring of G, or determine that none exists. Hence, we may assume G is a $\{P_7, C_3, C_7\}$-free graph. By 5.7.7, if G contains a shell, then in time $O(|V(G)|^{18})$ we can either produce a 3-coloring of G, or determine that none exists. Hence, we may assume G is a $\{P_7, C_3, C_7, \mathit{shell}\}$-free graph. By 5.8.6, if G contains a 5-gon, then in time $O(|V(G)|^6)$ we can either produce a 3-coloring of G, or determine that none exists. Hence, we may assume G is a $\{P_7, C_3, C_5, C_7, \mathit{shell}\}$-free graph. Since G is P_7-free, it follows that G is C_k-free for all $k > 7$. And so, G is bipartite and so via breadth-first search in time $O(|V(G)|^2)$ we can produce a 2-coloring of G. This proves 5.9.1.

□

Chapter 6

Open Problems

In this chapter, we pose several open questions related to the work of this thesis.

6.1 Excluding Four-Edge Paths and Antipaths

A general question of interest in structural graph theory is to understand which families of graphs characterized by excluding induced subgraphs can be explicitly constructed. In chapter 4, we consider the class of graphs with no induced four-edge path and no induced complement of a four-edge path, and prove a decomposition theorem 4.4.1 for all such graphs. Determining if our result can be extended to a complete structure theorem for this class, rests on being able to answer the following question:

Question. *Given* $\{P_5, \overline{P_5}, C_5\}$*-free graphs* G_1 *and* G_2*, when is* (G_1, G_2) *is a composable pair?*

6.2 Coloring Graphs with Forbidden Induced Subgraphs

The problem of determining the chromatic number of an arbitrary graph is one of the initial problems Karp [26] showed to be NP-complete. We focus on the restricted problem of determining the complexity of coloring graphs which do not contain some fixed induced subgraph. In Chapter 5 and [11], specifically 5.1.8 together with 5.1.9, we show that the

3-COLORING problem can be solved in polynomial time for the class of P_7-free graphs. An obvious next question is the following:

Question. *What is the complexity of* 3*-COLORING* P_ℓ*-free graphs where* $\ell \geq 8$*?*

Another question is whether our algorithmic approach from 5.1.8 and 5.1.9 can be extended and adapted to solve the following more general coloring problems:

Question. *Let* L *be an order 3 palette of a* $\{P_7, C_3\}$*-free graph* G*, what is the complexity of determining whether* (G, L) *is colorable?*

Question. *Let* L *be an order 3 palette of a* P_7*-free graph* G *which contains a triangle, what is the complexity of determining whether* (G, L) *is colorable?*

Additionally, motivated by the potential algorithmic implications, we also ask the following structural question:

Question. *Which* P_7*-free graphs* G *have a dominating set of size at most* $\log |V(G)|$*? What about a constant size dominating set?*

Finally, by 5.1.3 and 5.1.5, it follows that that the 4-COLORING problem can be solved in polynomial time for the class of P_5-free graphs, yet is NP-complete for the class of P_7-free graphs. Thus, we ask about the remaining case:

Question. *What is the complexity of* 4*-COLORING* P_6*-free graphs?*

We note that recently some progress has been made on this question. Huang [25] showed that:

6.2.1. *The 4-coloring problem can be solved in polynomial time for the class of* $\{P_6, C_4\}$*-free graphs.*

While in joint work with Maria Chudnovsky, Juraj Stacho and Mingxian Zhong [9], we proved the following:

6.2.2. *The 4-coloring problem can be solved in polynomial time for the class of* $\{P_6, C_5\}$*-free graphs.*

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