Quantum difference equations for quiver varieties

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ABSTRACT

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For an arbitrary Nakajima quiver variety X, we construct an analog of the quantum dynamical Weyl group acting in its equivariant K-theory. The correct generalization of the Weyl group here is the fundamental groupoid of a certain periodic locally finite hyperplane arrangement in $\mathrm{Pic}(X)\otimes\mathbb{C}$. We identify the lattice part of this groupoid with the operators of quantum difference equation for X. The cases of quivers of finite and affine type are illustrated by explicit examples.

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Chapter 1

Introduction

In the present work we explain how to construct the *quantum difference equation* for a large class of symplectic varieties - Nakajima quiver varieties. The Nakajima quiver varieties form a wide class of symplectic resolutions with extremely rich internal symmetry. Special examples of these spaces are known as moduli spaces of instantons (sheaves) on complex surfaces whose study for many decades has brought together physics and geometry. The quiver varieties provide an ideal "laboratory" for the study of enumerative geometry and quantum physics by means of the correspondence between geometry and representation theory. The idea of the correspondence is straightforward: it was shown in [26; 25] that the geometry of a Nakajima variety X_Q is governed by the representation theory of a certain associated quantum loop algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$. Many difficult geometric problems about X_Q can be translated into the language of representation theory where they can have surprisingly simple solutions.

The quantum difference equation is an object playing an important role in different fields, from enumerative geometry to representation theory. On the representation theory side, our construction is supposed to generalize the notion of quantum dynamical Weyl group for quantized loop algebras $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ corresponding to a general quiver Q. On the algebraic geometry side we study quantum deformation of the tensor product in $K_G(X)$ - the quantum K-theory ring of X_Q. The operators that we construct in this work may be understood as operators of quantum multiplication by tautological line bundles. In this introduction we give a short description of our motivations and discuss the structure of this work.

1.1 Quantum differential equation

1.1.1

Let X be a smooth quasiprojective variety. By definition, the quantum cohomology ring of X is a $H^2(X)$-parametric deformation of the classical cup product:

$$a * b = a \cup b + \sum_{d>0} C_d(a,b) z^d$$

where $d \in H_2(X)$. The terms with $d > 0$ are called quantum corrections and are given by three-point Gromow-Witten invariants of X. In other words, quantum corrections count the rational curves meeting three given cycles in $H^\bullet(X)$, see [9] for an introduction.

Given a quantum cohomology ring of X one constructs the *quantum connection* in the trivial $H^\bullet(X)$ bundle over $H^2(X)$:

$$\nabla_\lambda = \frac{d}{d\lambda} - \lambda * \tag{1.1}$$

The properties of the quantum product $*$ are translated into the flatness of the quantum connection:

$$[\nabla_\lambda, \nabla_\lambda] = 0. \tag{1.2}$$

The flat sections of the quantum connection, i.e. the solutions of the *quantum differential equation*

$$\frac{d}{d\lambda}\Psi(\lambda) = \lambda * \Psi(\lambda) \tag{1.3}$$

play an important role in enumerative geometry. They can be thought of as generating functions for rational Gromow-Witten invariants of X [15].

1.1.2

As was explained in [25] the quantum cohomology of the Nakajima variety X_Q, corresponding to a quiver Q is best described in the language of representation theory for a certain associated Yangian $Y_\hbar(\mathfrak{g}_Q)$. It is well known that the Yangians come together with a natural flat Yangian-valued connection on Cartan subalgebra $\mathfrak{h}_Q \subset \mathfrak{g}_Q$, known as *Casimir connection* [39]. This connection can be written explicitly in terms of the roots of $\mathfrak{g}_Q$ as follows:

$$\nabla_\lambda = \frac{d}{d\lambda} + \lambda^{(1)} - \hbar \sum_{\alpha>0} (\lambda, \alpha) \frac{z^\alpha}{1 - z^\alpha} e_\alpha e_\alpha \tag{1.4}$$

Here $\lambda \in \mathfrak{h}_Q$ and $\lambda^{(1)}$ is a degree one "loop" generator in the Yangian. The sum runs over the set of positive roots and e_α are the corresponding generators of $\mathfrak{g}_Q$. The main result of [25] is the identification of the quantum connection (1.1) for a Nakajima variety X_Q with the Casimir connection (1.4) for the associated Yangian $Y_\hbar(\mathfrak{g}_Q)$. In particular, the Cartan subalgebra gets identified with the second cohomology of a Nakajima variety $\mathfrak{h}_Q \simeq H^2(X_Q, \mathbb{C})$. As a vector space it is spanned by first Chern classes of tautological bundles over X_Q, such that $\lambda^{(1)}$ is identified with the operator of classical multiplication by $c_1(\mathcal{V})$ for some tautological bundle $\mathcal{V}$. The non-Cartan terms in (1.4) vanishing at $z = 0$ describe the quantum corrections to these operators in the quantum cohomology ring. This provides a representation theoretic description of the quantum cohomology of Nakajima varieties.

1.1.3

In addition to flatness (1.2), the quantum connection for Nakajima varieties must satisfy much more restrictive conditions [25]. Let $\mathscr{R}(u) \in Y_\hbar^{\otimes 2}(\mathfrak{g}_Q)$ be the universal R-matrix of a Yangian $Y_\hbar(\mathfrak{g}_Q)$ with spectral parameter u. The quantum Knizhnik-Zamolodchikov operator is a difference operator acting in the space of rational functions of u and $\hbar^\lambda$ with values in $Y_\hbar^{\otimes 2}(\mathfrak{g}_Q)$ defined explicitly by

$$\mathcal{K} = \hbar^\lambda \otimes 1 \mathscr{R}(u) T_u$$

where $\lambda \in \mathfrak{h}_Q$ and $T_u f(u) = f(uq)$. The main property of the Casimir connection (1.4) (and therefore of the quantum connection for the Nakajima varieties) is its commutativity with with qKZ operators:

$$[\Delta(\nabla_\lambda), \mathcal{K}] = 0 \tag{1.5}$$

where Δ is the coproduct of $Y_\hbar(\mathfrak{g}_Q)$. Commutativity with qKZ operators, is a very strong restriction on connection ∇_λ which, under some assumptions, fixes it uniquely.

1.1.4

K-theory is often understood as "exponentiation" of cohomology. The similarity between the two theories is close to the relation among trigonometric and rational functions. An example of this principle is best illustrated by the Chern character, which, on the level of line bundles, is given by the exponentiation of Chern classes:

$$\mathrm{ch}(\mathscr{L}) = \exp(c_1(\mathscr{L})) = 1 + c_1(\mathscr{L}) + \cdots$$

In view of this, one can expect that the quantum connection (1.1) is the first term in the expansion of some *quantum difference operator*, which is some "wise exponentiation" of the corresponding differential operator:

$$\mathcal{A}_{\mathscr{L}}(z) =: \exp(\nabla_\lambda) := 1 + \nabla_\lambda + \cdots \tag{1.6}$$

The flatness condition (1.2) is upgraded to commutativity:

$$[\mathcal{A}_{\mathscr{L}_1}(z), \mathcal{A}_{\mathscr{L}_2}(z)] = 0 \tag{1.7}$$

for any two given line bundles $\mathscr{L}_1, \mathscr{L}_2 \in \mathrm{Pic}(X)$.

This exponentiation phenomenon is well known in representation theory and is parallel to the transition from Yangians $Y_\hbar(\mathfrak{g}_Q)$ to quantum loop groups $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$. Thus, in view of the previous sections, it is natural to expect that the difference operator $\mathcal{A}_{\mathscr{L}}(z)$ has a representation theoretic description in terms of $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$, similar to the description (1.4) of the quantum connection in terms of the Yangians.

In the case of Nakajima varieties we have a crucial restrictive condition for the difference operator $\mathcal{A}_{\mathscr{L}}(z)$, generalizing (1.5) which now reads:

$$[\mathcal{A}_{\mathscr{L}}(z), \mathcal{K}] = 0$$

where the qKZ operator $\mathcal{K}$ is same as above with the Yangian R-matrix replaced by the universal R-matrix of $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$. As we will prove in Section 6, for Nakajima varieties this condition fixes the quantum difference operator $\mathcal{A}_{\mathscr{L}}(z)$ up to a scalar multiple. We will use this property to prove the uniqueness of the operators we construct in this thesis.

1.1.5

The right framework to justify the intuitive exposition of the previous sections is the quantum K-theory of Nakajima varieties. The construction of the quantum K-theory for Nakajima varieties follows an established path of quantum cohomology [31]. In this construction we pass from the moduli space of stable maps to a different compactification known as *quasimaps.* We recall this construction in Section 4. Analogously to the case of the virtual fundamental class for the moduli of stable maps, the quasimap moduli space is equipped with a natural virtual fundamental sheaf, which allows one to do the enumerative calculations. As shown in Section 8 of [31], the generating functions of K-theoretic invariants (the so called capping operators) satisfy the *quantum difference equation* generalizing the quantum differential equation (1.3):

$$\Psi(zq^{\mathscr{L}}) = M_{\mathscr{L}}(z)\Psi(z)$$

where q - is the equivariant weight of some torus acting on the quasimap moduli space. In the equivariant setting, the operator $M_{\mathscr{L}}(z)$ belongs to a bigger family of commuting difference operators. If we denote $\mathcal{A}_{\mathscr{L}}(z) = M_{\mathscr{L}}(z)T_{\mathscr{L}}$ where $T_{\mathscr{L}}$ is a shift operator $T_{\mathscr{L}}f(z) = f(zq^{\mathscr{L}})$ then this operator satisfies both (1.7) and (1.5). The role of R-matrices in qKZ operators is played by some geometrically constructed solutions of the Yang-Baxter equation. Their construction is explained in Section 2.

1.2 Quantum dynamical Weyl group

1.2.1

Let M_λ and M_μ be two Verma modules of a quantized Kac-Moody Lie algebra $\mathscr{U}_\hbar(\mathfrak{g})$. Let V be an $\mathfrak{h}$-diagonalizable module. Let us consider the intertwiner $\Phi^v : M_\lambda \to M_\mu \otimes V$ defined uniquely by:

$$\Phi^v(v_\lambda) = v_\mu \otimes v + \quad \textit{lower weight terms},$$

where v_λ and v_μ are normalized highest weight vectors of the Verma modules, and $v \in V$ is a vector with weight $\lambda - \mu$.

Let s be an element of the Weyl group of $\mathfrak{g}$ corresponding to a simple reflection. As was shown in [13] there exists a unique operator $A_s(\lambda) : V \to V$ obtained by restriction of the intertwiner Φ to Verma module $M_{s\lambda}$, i.e. the operator such that:

$$\Phi^v_\lambda\,(v^\lambda_{s\lambda}) = v^{\lambda-\mu}_{s(\lambda-\mu)} \otimes A_s(\lambda)v + \quad \textit{lower weight terms.}$$

The operators $A_s(\lambda)$ are invertible and their matrix elements are rational functions of $\hbar^\lambda$. The set of these operators gives rise to the braid group representation on the space of functions from the dualized Cartan subalgebra $\mathfrak{h}^*$ to V and is called the dynamical Weyl group of V.

The dynamical Weyl group has several beautiful application in representation theory [13; 12]. In particular, the elements corresponding to the lattice part of the Weyl group form a (commutative) set of operators commuting with qKZ. By our discussion from the previous section these operators (up to a multiple) must coincide with quantum difference operators.

1.2.2

The method of Etingof and Varchenko is, hoverer, not available for the quantum loop algebras $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ associated to a general quiver Q. For such algebras both crucial elements of their construction - the Verma modules M_λ and the simple reflections s, are not available. For example, the algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ for the Jordan quiver is isomorphic to the Elliptic hall algebra, all roots of which are imaginary and thus do not generate reflections, see Section 8.

In our construction we start from certain $\mathrm{Pic}(X_Q)$ - periodic hyperplane arrangement in $\mathrm{Pic}(X_Q)\otimes \mathbb{C}$. Using this arrangement we construct a difference operator:

$$\mathcal{A}_{\mathscr{L}}(z) \sim \mathscr{L}\mathbf{B}_{w_n}\cdots\mathbf{B}_{w_1}$$

where $w_1 \cdots w_n$ is a set of the hyperplanes separating the fundamental alcove ∇ and $\nabla + \mathscr{L}$ (see Section 2 for details). In our construction the Etingof-Varchenko operators $A_s(\lambda)$ are substituted by $\mathbf{B}_w$ canonically assigned to each hyperplane w (in the Etingof-Varchenko picture these are the root hyperplanes). The advantage of our method is that the operators $\mathbf{B}_w$ always exist and are constructed from unique solutions of certain qKZ-like equations, see Section 5.

1.2.3

The program of constructing the general Yangians $Y_\hbar(\mathfrak{g}_Q)$ and identifying their Casimir connections with the quantum connection for Nakajima varieties was born out of conjectures made by Nekrasov and Shatashvili on one hand [30; 29], and Bezrukavnikov and his collaborators — on the other.

It was already predicted by Etingof that the correct K-theoretic version of the quantum connection should be identified with a similar generalization of dynamical difference equations studied by Tarasov, Varchenko, Etingof, and others (see e.g. [38; 13]) for finite-dimensional Lie algebras $\mathfrak{g}$. In particular, Balagovic proved [2] that for a finite-dimensional $\mathfrak{g}$, the dynamical equations degenerate to the Casimir connection in the appropriate limit (1.6). While both our methods and objects of study differ significantly from the above cited works, it is fundamentally this vision of Etingof that is realized here.

1.3 Importance and possible applications

1.3.1

For quivers of affine ADE type, Nakajima varieties are moduli of framed coherent sheaves on the corresponding surfaces. In particular, the Hilbert schemes $\mathrm{Hilb}(S, \mathrm{points})$, where S is an ADE surface, are Nakajima varieties. Quantum differential equations for those were determined earlier in [32; 22], and play a key role in enumerative geometry of curves in threefolds. Such enumerative theories exists in different flavors known as the Gromov-Witten and the Donaldson-Thomas theories[1]. A highly nontrivial equivalence between the two was conjectured in [20; 21] and its proof for toric varieties given in [23] rests on reconstructing both from the quantum difference equation for the Hilbert schemes of points in A_n surfaces.

In fact, it may be accurate to say that the GW/DT correspondence in the generality known today, see especially [34] for a state-of-the-art results, is proven by breaking the threefolds in pieces until we get to an ADE surface fibration, for which the computations on both sides can be equated to a computation in quantum cohomology of $\mathrm{Hilb}(S, \mathrm{points})$. It is not surprising that such a connection exists, because a curve

$$C \to \mathrm{Hilb}(S, \mathrm{points})$$

defines a subscheme of $C \times S$. However, it is very important for S to be a symplectic surface for this correspondence to remain precise enumeratively, and not be corrected by contributions of nonmatching strata in different moduli spaces.

As a particular case of our general result, we compute the quantum difference connection in the quantum K-theory of $\mathrm{Hilb}(S, \mathrm{points})$. This has an entirely parallel use in K-theoretic Donaldson-Thomas theory for threefolds, see [31]. There is great interest in this theory, for instance because of its conjectural connection to curve-counting in Calabi-Yau 5-folds, which is expected to be an

[1]Here the threefold need not be Calabi-Yau, to point out a frequent misconception. For example, the equivariant Donaldson-Thomas theory of toric varieties is a very rich subject with many applications in mathematical physics.

algebro-geometric version of computing the contribution of membranes to the index of M-theory, see [28]

1.3.2

Another reason why quantum differential equations are important is because the conjectures of Bezrukavnikov and his collaborators relate them to representation theory of *quantizations* of X, see for example [10] and also e.g. [5; 6] for subsequent developments.

Much technical and conceptual progress in representation theory has been achieved by treating algebras of interest, such as universal enveloping algebras of semisimple Lie algebras, as the quantization of algebraic symplectic varieties, see e.g. [4; 7; 18; 3], especially in prime characteristic. By construction, Nakajima varieties are algebraic symplectic reductions of linear symplectic representations, and hence come with a natural family of quantizations $\widehat{X}_\lambda$. Here λ is a parameter of the quantization, which is of the same nature as commutative deformations of X, e.g. the central character in the case

$$\mathscr{U}(\mathfrak{g})\big/\text{central character} = \text{Quantization of } T^*G/B\,.$$

For example, the Hilbert scheme of n points in the plane yields the spherical subalgebra of Cherednik's double affine Hecke algebra of $\mathfrak{gl}(n)$ — a structure of great depth and importance in applications.

Using quantization in characteristic $p \gg 0$, one constructs an action of the fundamental group of the complement of a certain periodic locally finite arrangement of rational hyperplanes in $H^2(X,\mathbb{C})$ by autoequivalences of $D^b_{\mathsf{T}}(\operatorname{Coh} X)$. It is known in special cases and is conjectured in general that these hyperplanes coincide with those considered in this work and, moreover, one conjectures a precise identification of the resulting action on $K_{\mathsf{T}}(X)$ with the monodromy of the quantum differential equation. This can be verified for the Hilbert schemes of points and other Nakajima varieties with isolated fixed points under a torus action [32; 8] and it is quite possible that similar arguments can be made to work for general Nakajima varieties. There are parallel links between the singularities of (1.4) and representation theory of $\widehat{X}_\lambda$ for special values of λ in characteristic 0, see [10].

1.3.3

An important structure which emerges from the quantization viewpoint is an association of a t-structure on $D^b_{\mathsf{T}}(\operatorname{Coh} X)$ to each alcove of the complement of hyperplanes in $H^2(X,\mathbb{R})$. The abelian hearts of the corresponding t-structures are identified with $\widehat{X}_\lambda$-modules for the corresponding range of parameters λ. In this way, the action of the fundamental group by derived autoequivalences of $\operatorname{Coh} X$ fits into an action of the fundamental groupoid

$$\mathbb{B} = \pi_1 \left(H^2(X,\mathbb{C}) \setminus \text{affine root hyperplanes} \right)$$

by derived equivalences between the categories of $\widehat{X}_\lambda$-modules. In particular, $\mathbb{B}$ acts on the common K-theory $K_{\mathsf{T}}(X)$ of all these categories.

The main object constructed in this work is a *dynamical* extension of the action of $\mathbb{B}$ on $K_{\mathsf{T}}(X)$. By definition, this means that the operators of $\mathbb{B}$ depend on the Kähler variables z and the braid relations are understood accordingly.

To be precise, in this work we construct a dynamical action of $\mathbb{B}$ and we *prove* its relation to the quantum difference equation. The connection with quantization in characteristic $p \gg 0$ is not considered here, see [8]. Similarly, a categorical lift of the dynamical action at this point remains an open problem. It is possible that it easier to categorify the *monodromy* of the quantum difference equation, which can be characterized in terms of an action of an elliptic quantum group on the elliptic cohomology of Nakajima varieties, see [1].

Chapter 2

Equivariant K-theory of Nakajima varieties and R-matrices

2.1 Stable envelopes in K-theory

2.1.1

Let X be a algebraic symplectic variety and G a reductive group acting on X. Since the algebraic symplectic form ω on X is unique up to a multiple, the group G scales ω by a character $\hbar$. Replacing G by its double cover if necessary, we can assume that $\hbar^{1/2}$ exists.

Let $\mathsf{A} \subset G$ be a torus in the center of G and in the kernel of $\hbar$. By definition, K-theoretic stable envelope is a K-theory class on the product

$$\mathrm{Stab} \subset K_G(X \times X^{\mathsf{A}}),$$

with the same support as the cohomological stable envelope, satisfying certain degree conditions for the restriction to $X^{\mathsf{A}} \times X^{\mathsf{A}}$. It defines a wrong way map

$$\mathrm{Stab} : K_G(X^{\mathsf{A}}) \to K_G(X),$$

which we denote by the same symbol.

2.1.2

The construction of stable envelopes requires additional data, namely the choice of:

- a cone $\mathfrak{C} \subset \mathrm{Lie}(\mathsf{A})$, which divides the normal directions to X^{A} into attracting and repelling ones and determines the support of Stab,
- a polarization $T^{1/2} \in K_G(X)$, which is a choice of a half of the tangent bundle $TX \in K_G(X)$, that is, a solution of

$$T^{1/2} + \hbar^{-1} \otimes \left(T^{1/2}\right)^{\vee} = TX \tag{2.1}$$

  in $K_G(X)$,
- a slope $s \in \mathrm{Pic}(X) \otimes_Z \mathbb{Q}$, which should be suitably generic, see below.

Of these pieces of data, the cone $\mathfrak{C}$ is exactly the same as in cohomology. The polarization reduces in cohomology to a certain sign, while the slope parameter is genuinely K-theoretic.

We recall from [25], Section 2.2.7, that a Nakajima variety, like any symplectic reduction of a cotangent bundle, has natural polarizations. For any polarization $T^{1/2}$, there is the opposite polarization

$$T^{1/2}_{\mathrm{opp}} = \hbar^{-1} \otimes \left(T^{1/2}\right)^{\vee} . \tag{2.2}$$

2.1.3

Let $\mathscr{N}$ be the normal bundle to X^{A} in X. The A-weights v appearing in $\mathscr{N}$ define hyperplanes $\{v = 0\}$ in $\mathrm{Lie}\,\mathsf{A}$. By definition, the cone

$$\mathfrak{C} \subset \mathrm{Lie}\,\mathsf{A} \setminus \bigcup_v \{v = 0\}$$

is one of the chambers of the complement. We write $v > 0$ if v is positive on $\mathfrak{C}$. A choice of $\mathfrak{C}$ thus determines the decomposition

$$\mathscr{N} = \mathscr{N}_+ \oplus \mathscr{N}_-$$

into attracting and repelling directions, with the corresponding attracting manifold

$$\mathrm{Attr} = \left\{(x, y), \lim_{a \to 0} a \cdot x = y\right\} \subset X \times X^{\mathsf{A}}$$

where $a \to 0$ means that $v(a) \to 0$ for all $v > 0$.

2.1.4

Let F be a component of X^{A}. By Koszul resolution,

$$\mathscr{O}_{\mathrm{Attr}}\Big|_{F\times F} = \mathscr{O}_{\mathrm{diag}\,F} \otimes \Lambda^{\bullet}_{-}\mathscr{N}_{-}^{\vee}\,,$$

where the subscript in $\Lambda^{\bullet}_{-}$ indicates an alternating sum of exterior powers. We require

$$\mathrm{Stab}\Big|_{F\times F} = \pm \text{ line bundle} \otimes \mathscr{O}_{\mathrm{Attr}}\Big|_{F\times F}$$

where the sign and the line bundle are determined by the choice of polarization.

Concretely, let

$$T^{1/2}\big|_F = T^{1/2}_0 \oplus T^{1/2}_{\neq 0}$$

be the splitting of the polarization into trivial and nontrivial A-characters. We have

$$\mathscr{N}_{-} \ominus T^{1/2}_{\neq 0} = \hbar^{-1}\left(T^{1/2}_{>0}\right)^{\vee} \ominus T^{1/2}_{>0}\,,$$

and therefore the determinant of this virtual vector bundle is a square (recall that we replace G by its double cover if the character $\hbar$ is not a square). We set

$$\mathrm{Stab}\Big|_{F\times F} = (-1)^{\mathrm{rk}\,T^{1/2}_{>0}}\left(\frac{\det \mathscr{N}_{-}}{\det T^{1/2}_{\neq 0}}\right)^{1/2} \otimes \mathscr{O}_{\mathrm{Attr}}\Big|_{F\times F}\,. \tag{2.3}$$

2.1.5

The key property of stable envelopes are degree bounds satisfied by $\mathrm{Stab}\big|_{F_2\times F_1}$, where F_1 and F_2 are two different components of X^{A}. Note that because of the support condition, this restriction vanishes unless $F_2 < F_1$ in the partial ordering defined by the closures of attracting manifolds, that is, by

$$\exists x, \quad \lim_{a\to 0} a^{\pm 1}x \in F_{\pm} \Rightarrow F_{+} > F_{-}\,.$$

Recall that in cohomology the degree bounds reads

$$\deg_{\mathsf{A}} \mathrm{Stab}\Big|_{F_2\times F_1} < \deg_{\mathsf{A}} \mathrm{Stab}\Big|_{F_2\times F_2}\,, \tag{2.4}$$

where $\deg_{\mathsf{A}}$ for an element of

$$H^{\bullet}_{G}(X^{\mathsf{A}},\mathbb{Q}) \cong H^{\bullet}_{G/\mathsf{A}}(X^{\mathsf{A}},\mathbb{Q}) \otimes \mathbb{Q}[\mathrm{Lie}\,\mathsf{A}]$$

is its degree in the variables $\mathrm{Lie}\,\mathsf{A}$.

2.1.6

Now in K-theory the degree $\deg_{\mathsf{A}} f$ of a Laurent polynomial

$$f = \sum_{\mu \in \mathsf{A}^\wedge} f_\mu \, a^\mu \in \mathbb{Z}[\mathsf{A}] = K_{\mathsf{A}}(\mathrm{pt})$$

is its Newton polygon

$$\deg_{\mathsf{A}} f = \text{Convex hull } (\{\mu, f_\mu \neq 0\}) \subset \mathsf{A}^\wedge \otimes_{\mathbb{Z}} \mathbb{Q} \,,$$

with the natural partial ordering on polygons defined by inclusion.

Such definition has a caveat, in that the degree of an invertible function a^μ should really be zero, and so the Newton polygons shoud really be considered up to translation by the lattice $\mathsf{A}^\wedge$. If we want to compare two Newton polygons by inclusion, a possibility of inclusion after a shift appears, and this is where the slope parameter s comes in.

The K-theoretic analog of (2.4) is the following condition

$$\deg_{\mathsf{A}} \mathrm{Stab}_s \Big|_{F_2 \times F_1} \otimes s \Big|_{F_1} \ \subset \ \deg_{\mathsf{A}} \mathrm{Stab}_s \Big|_{F_2 \times F_2} \otimes s \Big|_{F_2} , \tag{2.5}$$

where the weight of a fractional line bundle $s \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a fractional weight, that is, an element of $\mathsf{A}^\wedge \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that (2.5) is independent on the A-linearization of s. The dependence of the stable envelope Stab_s on the slope s is indicated for emphasis in the LHS of (2.5). The degree of $\mathrm{Stab}\Big|_{F_2 \times F_2}$ is given by (2.3) and is indepent of s.

Remark 1. *Observe that for a sufficiently generic s the inclusion in* (2.5) *is necessarily strict, as the inclusion between fractional shifts of integral polytopes.*

2.1.7

To keep track of the weights of the line bundles s restricted to components of the fixed locus, it is convenient to introduce a locally constant map (a form of moment map)

$$\boldsymbol{\mu} : X^{\mathsf{A}} \to H_2(X, \mathbb{Z}) \otimes \mathsf{A}^\wedge \,, \tag{2.6}$$

defined up to an overall translation, such that

$$\boldsymbol{\mu}(F_1) - \boldsymbol{\mu}(F_2) = [C] \otimes v$$

if there is an irreducible A-invariant curve joining F_1 and F_2 with tangent weight v at F_1. For any s, we then have

$$\text{weight } s\big|_{F_1} - \text{weight } s\big|_{F_2} = (s, C)\, v\,.$$

By construction

$$\text{Stab}_{F_2\times F_1} \neq 0 \Rightarrow \boldsymbol{\mu}(F_1) - \boldsymbol{\mu}(F_2) \in H_2(X, \mathbb{Z})_{\text{eff}} \otimes \mathsf{A}^\wedge_{>0}\,, \tag{2.7}$$

where $\mathsf{A}^\wedge_{>0}$ is the cone of weights positive on $\mathfrak{C}$.

2.1.8

By the same argument as in cohomology, it is easy to see that a K-theory class Stab which is supported on the cohomological stable envelope, satisfies the normalization (2.3) and the degree (2.5) condition is necessarily unique. Existence of stable envelopes may be shown [24] under more restrictive geometric hypotheses than in cohomology. These hypotheses are satisfied for Nakajima varieties.

2.1.9

Uniqueness of stable envelopes implies the following transformation law under duality on $X \times X^{\mathsf{A}}$

$$\left(\text{Stab}_{\mathfrak{C}, T^{1/2}, s}\right)^\vee = (-1)^{\frac{\dim X^{\mathsf{A}}}{2}}\, \hbar^{-\frac{\dim X}{2}}\, \text{Stab}_{\mathfrak{C}, T^{1/2}_{\text{opp}}, -s}\,. \tag{2.8}$$

Here $T^{1/2}_{\text{opp}}$ is the opposite polarization (2.2).

2.2 Slope R-matrices

2.2.1

Following the sign conventions set in Section 3.1.3 of [25], we define the transposition

$$K(X \times Y) \ni \mathscr{E} \mapsto \mathscr{E}^\tau \in K(Y \times X)$$

as a permutation of factors together with a sign $(-1)^{(\dim X - \dim Y)/2}$.

The following is an analog of Theorem 4.4.1 in [25]

Proposition 1.

$$\mathrm{Stab}^{\tau}_{-\mathfrak{C},\,T^{1/2}_{\mathrm{opp}},\,-s} \circ \mathrm{Stab}_{\mathfrak{C},\,T^{1/2},\,s} = 1\,. \tag{2.9}$$

Here we do not distinguish between the structure sheaf of the diagonal and the identity operator by which it acts on the K-theory.

Proof. Since the support of stable envelopes is the same as in cohomology, the convolution (2.9) is an integral K-theory class on $X^{\mathsf{A}} \times X^{\mathsf{A}}$.

Denoting by $\mathscr{S}$ and $\mathscr{S}'$ the two stable envelopes in (2.9), we have

$$\left(\mathscr{S}'^{\tau} \circ \mathscr{S}\right)_{F_3\times F_1} = \sum_{F_1 \geq F_2 \geq F_3} (-1)^{\frac{\operatorname{codim} F_3}{2}} \frac{\mathscr{S}'\big|_{F_2\times F_3} \otimes \mathscr{S}\big|_{F_2\times F_1}}{\Lambda^{\bullet}_{-}\mathscr{N}^{\vee}_{F_2}} \tag{2.10}$$

by equivariant localization and the support condition, where F_i are components of the fixed point locus X^{A}.

Since the convolution (2.10) is integral, its Newton polygon may be estimated directly from (2.10). We denote by

$$\mu = \langle \boldsymbol{\mu}(F_3) - \boldsymbol{\mu}(F_1), s\rangle \in \mathsf{A}^{\wedge} \otimes \mathbb{Q}$$

the difference of weights of s at F_3 and F_1. We have $\mu \notin \mathsf{A}^{\wedge}$ for generic s unless $F_3 = F_1$ because an ample line bundle will pair nonzero with $\boldsymbol{\mu}(F_3) - \boldsymbol{\mu}(F_1)$.

The degree bound (2.5) implies each term is $O(|a|^{\mu})$ as $a \in \mathsf{A}$ goes to infinity in any direction. Since this number is fractional for $F_3 \neq F_1$ while the asymptotics is integral, it follows that terms with $F_1 \neq F_3$ in (2.10) vanish.

The remaining terms with $F_1 = F_2 = F_3$ are easily seen to give the identity operator. □

2.2.2

In the same way, stable envelopes may be defined for real slopes $s \in H^2(X, \mathbb{R})$. They depend on the slope in a locally constant way and change as s crosses certain rational hyperplanes

$$w \overset{\text{def}}{=} \{s \in H^2(X,\mathbb{R}) : (s,\alpha) + n = 0\}\,, \tag{2.11}$$

which we will call *walls*. Here

$$\widehat{\alpha} = (\alpha, n) \in H_2(X,\mathbb{Z}) \oplus \mathbb{Z} \tag{2.12}$$

is an integral affine function on $H^2(X)$, which we call an *affine root* of X. The connected components of the complements to the walls in $H^2(X,\mathbb{R})$ are called *alcoves*.

Below we will see that $\pm\alpha$ is an effective curve class for any affine root $\widehat{\alpha}$. If $n \neq 0$, we set

$$\widehat{\alpha}' = \tfrac{1}{n}\alpha \in H_2(X,\mathbb{Q})\,.$$

This depends only on the wall and not on the particular normalization of its equation.

2.2.3

Let us consider two slopes s and s' separated by a single wall w. To examine the change in stable envelopes across the wall, we define the *wall R-matrix*:

$$R_w^{\mathfrak{C}} = \mathrm{Stab}^{-1}_{\mathfrak{C},T^{1/2},s'} \circ \mathrm{Stab}_{\mathfrak{C},T^{1/2},s}\,. \tag{2.13}$$

To distinguish $R_w^{\mathfrak{C}}$ from its inverse, we assume

$$\langle s'-s,\alpha\rangle > 0\,.$$

for the positive root α defining the corresponding wall. If we crass the wall from s to s' we say that it is *crossed in the positive direction.*

Theorem 1. *We have*

$$R_w^{\mathfrak{C}}\Big|_{F_3\times F_1} = 0$$

unless

$$\boldsymbol{\mu}(F_1) - \boldsymbol{\mu}(F_3) = \widehat{\alpha}' \otimes \mu \tag{2.14}$$

where $\widehat{\alpha}' \in H_2(X,\mathbb{Q})_{\mathrm{eff}}$ *and* μ *is an integral weight of* A *positive on* $\mathfrak{C}$. *In this case*

$$\deg_{\mathsf{A}} R_w^{\mathfrak{C}}\Big|_{F_3\times F_1} = \mu\,.$$

If $n=0$ *the condition* (2.14) *means* $\mu = 0$ *and that* $\boldsymbol{\mu}(F_1)-\boldsymbol{\mu}(F_3)$ *is proportional to* α.

As a corollary of the proof, we will see that

$$R_w^{\mathfrak{C}}\Big|_{F_1\times F_1} = 1\,.$$

Proof. As in the proof of Proposition 1, we see that $R^{\mathfrak{C}}_w$ is an integral K-theory class and we compute its restriction to $F_3 \times F_1$ by localization as in (2.10).

Consider the localization term corresponding to a component F_2 of X^{A}. The slope-dependent part of its degree is

$$\begin{aligned} &\langle \boldsymbol{\mu}(F_2) - \boldsymbol{\mu}(F_1), s\rangle + \langle \boldsymbol{\mu}(F_3) - \boldsymbol{\mu}(F_2), s'\rangle \\ &\quad = \langle \boldsymbol{\mu}(F_3) - \boldsymbol{\mu}(F_1), s'\rangle + \langle \boldsymbol{\mu}(F_2) - \boldsymbol{\mu}(F_1), s' - s\rangle \qquad (2.15) \\ &\quad = \langle \boldsymbol{\mu}(F_3) - \boldsymbol{\mu}(F_1), s\rangle + \langle \boldsymbol{\mu}(F_2) - \boldsymbol{\mu}(F_3), s - s'\rangle\,. \qquad (2.16) \end{aligned}$$

Since the ample cone is open, we may assume that $\pm(s-s')$ is ample. If $s > s'$, the second summand in (2.15) is a negative weight, while the second summand in (2.16) is a positive weight. If $s < s'$, these conclusions are reversed. But in either case,

$$R^{\mathfrak{C}}_w\Big|_{F_3\times F_1} = O(|a|^{\mu})$$

as $a \to 0$ or $a \to \infty$, where

$$\mu = \langle \boldsymbol{\mu}(F_3) - \boldsymbol{\mu}(F_1), x\rangle$$

for $x \in w$ and $a \to 0$ as before means that $v(a) \to 0$ for every positive weight v. Since this is a Laurent polynomial in a, this means vanishing unless μ is an integral weight and $R^{\mathfrak{C}}_w\Big|_{F_3\times F_1}$ is a monomial.

For generic s on the hyperplane (2.11) the weight μ is integral only if

$$\boldsymbol{\mu}(F_1) - \boldsymbol{\mu}(F_3) \in \mathbb{Q}\,\alpha \otimes \mathsf{A}^\wedge\,.$$

From (2.7) and since $(x, \widehat{\alpha}') = -1$ for $n \neq 0$ by construction, we conclude (2.14). If $n = 0$ we have $(x, \alpha) = 0$ and hence $\mu = 0$. □

2.3 Root subalgebras

2.3.1

We recall that Nakajima varieties depend on a quiver with a vertex set I, two dimension vectors $\mathsf{v}, \mathsf{w} \in \mathbb{N}^I$, and a stability parameter $\theta \in \mathbb{R}^I$. The complex deformation parameter $\zeta \in \mathbb{C}^I$, which is the value of the complex moment map in symplectic reduction, will always be set to zero in this paper. We fix θ and denote

$$\mathscr{M}(\mathsf{w}) = \bigsqcup_{\mathsf{v}} \mathscr{M}_\theta(\mathsf{v},\mathsf{w})\,.$$

We take the canonical polarization of Example 3.3.3 in [25] as polarization $T^{1/2}$ of Nakajima varieties.

2.3.2

Let W be a framing space defining a Nakajima variety with dimension w. Let us consider its arbitrary decomposition into a direct sum of subspaces $W = W' \otimes W''$ with dimensions w' and w''. Assume that a torus $\mathsf{A} = \mathbb{C}^\times$ acts on W scaling W' with character a' and W'' with character a''. In this situation we say that A *splits the framing* $\mathsf{w} = a'\mathsf{w}' + a''\mathsf{w}''$.

Such an action induces an action of A corresponding a Nakajima variety $\mathscr{M}(\mathsf{w})$. The basic property of the Nakajima varieties is that the set of the A fixed points is the product of Nakajima varieties for the same quiver but different framings:

$$\mathscr{M}(\mathsf{w})^{\mathsf{A}} = \mathscr{M}(\mathsf{w}') \times \mathscr{M}(\mathsf{w}'')$$

such that

$$K_G(\mathscr{M}(\mathsf{w})^{\mathsf{A}}) = K_G(\mathscr{M}(\mathsf{w}')) \otimes K_G(\mathscr{M}(\mathsf{w}''))$$

As the torus A is one-dimensional, we have only two chambers in its real Lie algebra. These two possible cones correspond to $a \to 0$ and $a \to \infty$. We denote them by $+$ and $-$ respectively. For any slope s these give the stable maps:

$$\mathrm{Stab}_{\pm,s} : K_G(\mathscr{M}(\mathsf{w})) \otimes K_G(\mathscr{M}(\mathsf{w}')) \to K_G(\mathscr{M}(\mathsf{w}+\mathsf{w}'))$$

for any G that commutes with A. To examine the change of the stable map under the change of the chamber we introduce the following *total slope s R-matrix*:

$$\mathscr{R}^s(u) = \mathrm{Stab}^{-1}_{-,s} \circ \mathrm{Stab}_{+,\,s}\,, \tag{2.17}$$

One checks that it depends only on ratio $u = a'/a''$. Just like the cohomological R-matrices, this acts in a localization of $K_G(\mathscr{M}(\mathsf{w})) \otimes K_G(\mathscr{M}(\mathsf{w}'))$. However, the coefficients of the $u \to 0$ or $u \to \infty$

expansion of $\mathscr{R}^s(u)$ are operators in nonlocalized K-theory. The variable u is tradionally called the *spectral parameter*.

The key property of the operators $\mathscr{R}^s(u)$ is the Yang-Baxter equation which they satisfy for any slope s.

2.3.3

We can include the given slope $\mathscr{L}$ into an doubly infinite sequence

$$\dots s_{-2}, s_{-1}, s_0 = s, s_1, s_2, \dots \tag{2.18}$$

such that

$$s_i \to \pm\infty\,, \quad i \to \pm\infty\,,$$

where $s_i \to +\infty$ means that s_i goes to infinity inside the ample cone of X. We can assume that s_i and s_{i+1} are separated by exactly one wall w_i and that the sequence $\{s_i\}$ crosses each wall once. We can write the following obvious identity:

$$\begin{aligned}
&\mathrm{Stab}_{+,\,s} = \\
&\mathrm{Stab}_{+,\,+\infty} \cdots \mathrm{Stab}_{+,\,s_2}\,\mathrm{Stab}^{-1}_{+,\,s_2}\,\mathrm{Stab}_{+,\,s_1}\,\mathrm{Stab}^{-1}_{+,\,s_1}\,\mathrm{Stab}_{+,\,s} = \\
&\mathrm{Stab}_{+,\,+\infty} \cdots R^+_{w_3} R^+_{w_2} R^+_{w_1}.
\end{aligned}$$

Similarly for the negative chamber:

$$\begin{aligned}
&\mathrm{Stab}_{-,\,s} = \\
&\mathrm{Stab}_{-,\,-\infty} \cdots \mathrm{Stab}_{-,\,s_{-1}}\,\mathrm{Stab}^{-1}_{-,\,s_{-1}}\,\mathrm{Stab}_{-,\,s_0}\,\mathrm{Stab}^{-1}_{-,\,s_0}\,\mathrm{Stab}_{-,\,s} = \\
&\mathrm{Stab}_{-,\,-\infty} \cdots (R^-_{w_{-2}})^{-1}(R^-_{w_{-1}})^{-1}(R^-_{w_0})^{-1}.
\end{aligned}$$

In the last case we cross the walls in the negative direction and by our convention from the Section 2.2.3 the corresponding contribution is given by the inverse of the wall R-matrix.

This leads to an infinite factorization of $R^{\mathscr{L}}(u)$ of the following form

$$\mathscr{R}^s(u) \overset{def}{=} \mathrm{Stab}^{-1}_{-,\,s}\,\mathrm{Stab}_{+,\,s} = \overrightarrow{\prod_{i<s}} R^-_{w_i}\; R_\infty \overleftarrow{\prod_{i\geq s}} R^+_{w_i}\,, \tag{2.19}$$

where

$$R_\infty = \mathrm{Stab}^{-1}_{-,-\infty} \circ \mathrm{Stab}_{+,\infty} \ . \tag{2.20}$$

The factorization (2.19) converges and limit operator (2.20) exists in the topology of formal power series around $u=\infty$, as will be explained in the next section.

Similarly, we could first go to negative infinity that would give:

$$\mathrm{Stab}_{+,\,s} =$$

$$\mathrm{Stab}_{+,\,-\infty} \cdots \mathrm{Stab}_{+,\,s_{-1}}\, \mathrm{Stab}^{-1}_{+,\,s_{-1}}\, \mathrm{Stab}_{+,\,s_0}\, \mathrm{Stab}^{-1}_{+,\,s_0}\, \mathrm{Stab}_{+,\,-s} =$$

$$\mathrm{Stab}_{+,\,-\infty} \cdots (R^{+}_{w_{-2}})^{-1} (R^{+}_{w_{-1}})^{-1} (R^{+}_{w_0})^{-1}.$$

and

$$\mathrm{Stab}_{-,\,s} =$$

$$\mathrm{Stab}_{-,\,+\infty} \cdots \mathrm{Stab}_{-,\,s_2}\, \mathrm{Stab}^{-1}_{-,\,s_2}\, \mathrm{Stab}_{-,\,s_1}\, \mathrm{Stab}^{-1}_{-,\,s_1}\, \mathrm{Stab}_{-,\,s} =$$

$$\mathrm{Stab}_{-,\,+\infty} \cdots R^{-}_{w_3} R^{-}_{w_2} R^{-}_{w_1}.$$

This gives another factorization:

$$\mathscr{R}^s(u) \stackrel{def}{=} \mathrm{Stab}^{-1}_{-,\,s}\, \mathrm{Stab}_{+,\,s} = \prod^{\longleftarrow}_{i>0} (R^{-}_{w_i})^{-1} \ R_\infty \prod^{\longrightarrow}_{i\leq 0} (R^{+}_{w_i})^{-1}\,, \tag{2.21}$$

This product converges in topology of power series near $u=0$. We will call this formula Koroshkin-Tolstoy factorization of total R-matrices. As we will see in Section 7.2 in examples of finite type quivers this formula reproduces the results of [19].

2.3.4

Recall that the partial ordering on the fixed point component coincide with "ample partial ordering". If $\theta \in \mathrm{Pic}(X)$ is a choice of ample line bundle, and $\sigma \in \mathfrak{C}$ is a character of A then:

$$F_2 \trianglelefteq F_1 \quad \Leftrightarrow \quad \langle \theta_{F_1}, \sigma \rangle \leq \langle \theta_{F_2}, \sigma \rangle$$

For the Nakajima varieties the ample line bundle corresponds to the choice of stability condition $\theta \in \mathbb{Z}^I$. If the fixed components have the form $F = \mathscr{M}(\mathsf{v},\mathsf{w}) \times \mathscr{M}(\mathsf{v}',\mathsf{w}')$ then, the function defining the ordering takes the following explicit form:

$$\langle \theta_F, \sigma \rangle = \langle \mathsf{v}, \theta \rangle \sigma + \langle \mathsf{v}', \theta \rangle \sigma'$$

All the operators A in K-theory we consider in this paper preserve the weight, i.e., $A = \bigoplus_{\alpha} A_\alpha$ with:

$$A_\alpha : K_G(F_1) \longrightarrow K_G(F_2)$$

and $F_1 = \mathscr{M}(\mathsf{v},\mathsf{w}) \times \mathscr{M}(\mathsf{v}',\mathsf{w}')$, $F_2 = \mathscr{M}(\mathsf{v}+\alpha,\mathsf{w}) \times \mathscr{M}(\mathsf{v}'-\alpha,\mathsf{w}')$. Therefore the difference of ordering function takes the form:

$$\langle \theta_{F_2}, \sigma \rangle - \langle \theta_{F_1}, \sigma \rangle = \langle \alpha, \theta \rangle (\sigma - \sigma') \tag{2.22}$$

In the present text we will always assume that the fixed components are ordered using the positive chamber $\sigma - \sigma' > 0$. Thus the sign of the difference (2.22) is given by a sign of $\langle \alpha, \theta \rangle$.

We will use the following terminology: an operator $A = \bigoplus_{\alpha} A_\alpha$ with A_α as above is *upper-triangular* $\langle \alpha, \theta \rangle > 0$ and *lower-triangular* if $\langle \alpha, \theta \rangle < 0$ for all α. We say that A is *strictly upper-triangular* of *strictly lower-triangular* if in addition $A_0 = 1$. For example, the wall R-matrices R^+_w and R^-_w are strictly upper and lower triangualr respectively. In particular, the Khoroshkin-Tolstoy factorization (2.19) gives a Gauss decomposition of the total R-matrix.

2.3.5

Let $\mathscr{L}_w$ be a line bundle on the wall w. The wall R-matrices $R^{\pm}_w$ are triangular with monomial in spectral parameter u matrix elements:

$$R^{\pm}_w\big|_{F_2 \times F_1} = \begin{cases} 1\,, & F_1 = F_2\,, \\ \propto u^{\langle \boldsymbol{\mu}(F_2) - \boldsymbol{\mu}(F_1), \mathscr{L}_w \rangle}\,, & F_1 \gtrless F_2\,, \\ 0\,, & \text{othewise}\,. \end{cases} \tag{2.23}$$

The condition (2.7) means

$$R^{\pm}_w \to 1\,, \quad w \to \pm\infty\,,$$

in the topology of formal power series.

2.3.6

It means that R_∞ corresponding to the infinite slope is diagonal in the basis of fixed components. The normalization condition (2.3) implies that its diagonal components are given by operator of multiplication by a class of normal bundles in K-theory:

$$R_\infty\big|_{F\times F} = (-1)^{\mathrm{codim}(F)/2}\frac{\prod_{v<0}(v^{1/2}-v^{-1/2})}{\prod_{v>0}(v^{1/2}-v^{-1/2})} \tag{2.24}$$

where v are the Chern roots of $\mathscr{N}_F$. In particular,

$$\lim_{u\to 0} R_\infty = \hbar^{-\Omega} \qquad \lim_{u\to\infty} R_\infty = \hbar^{\Omega} \tag{2.25}$$

where Ω is the codimension function:

$$\Omega(\gamma) = \frac{\mathrm{codim}(F)}{4}\,\gamma \tag{2.26}$$

for a class γ supported on the fixed set component $F \subset \mathscr{M}(\mathsf{w})^{\mathsf{A}}$.

2.3.7

For Nakajima varieties, the codimension function in (2.27) has the following concrete description. For A splitting the framing $\mathsf{w} = a'\mathsf{w}' + a''\mathsf{w}''$. Any component $F \subset \mathscr{M}(\mathsf{v},\mathsf{w})^{\mathsf{A}}$ is of the form

$$F = \mathscr{M}(\mathsf{v}',\mathsf{w}') \times \mathscr{M}(\mathsf{v}'',\mathsf{w}'')$$

for some dimension vectors v',v''. We have, see e.g. Section 2.4.2 in [25],

$$\Omega = \frac{\mathrm{codim}\,F}{4} = \frac{1}{2}(\mathsf{w}',\mathsf{v}'') + \frac{1}{2}(\mathsf{w}'',\mathsf{v}') - \frac{1}{2}(\mathsf{v}',C\mathsf{v}'')\,, \tag{2.27}$$

where C is the Cartan matrix of the quiver, see e.g. Section 2.2.5 of [25]. The map $\boldsymbol{\mu}$ has also a very concrete description, namely

$$\boldsymbol{\mu}(F) = \mathsf{v}' \otimes 1 \tag{2.28}$$

where $1 \in \mathsf{A}^\wedge$ is the weight of u, see e.g. Section 3.2.8 in [25].

2.3.8

Denote by tildas R-matrices conjugated:

$$\widetilde{\mathscr{R}^s(u)} = U^{-1}\,\mathscr{R}^s(u)\,U$$

by diagonal operator U diagonal matrix elements:

$$U\Big|_{F_1\times F_2} = a_1^{(\mathscr{L}_{w_k},\mathsf{v}_1')} a_2^{(\mathscr{L}_{w_k},\mathsf{v}_2')}$$

where $\mathscr{L}_{w_k}$ are the line bundles on walls k with $k \in \{0,-1\}$ and the components F_1, F_2 of the fixed point set are the same as in the previous section. The conjugated R-matrices satisfy the same Yang-Baxter equation as $\mathscr{R}^s(u)$.

From (2.23) and (2.28), we conclude

$$\widetilde{R^{\pm}_{w_i}}\big|_{F_1\times F_2} = \begin{cases} 1\,, & F_1 = F_2\,, \\ \propto u^{\langle \mathsf{v}_2'-\mathsf{v}_1', \mathscr{L}_{w_k}-\mathscr{L}_{w_i}\rangle}\,, & F_1 \gtrless F_2\,, \\ 0\,, & \text{othewise}\,. \end{cases} \tag{2.29}$$

By construction, the sequence (2.18) crosses each affine root hyperplane exactly once and hence

$$(\mathscr{L}_{w_k} - \mathscr{L}_{w_i}, \alpha_i) \gtrless 0\,, \quad i \lessgtr k\,,$$

so that α_i is effective. Therefore

$$\lim_{u\to\infty} R^+_{w_i} = 1\,, \quad i < k\,,$$
$$\lim_{u\to\infty} R^-_{w_i} = 1\,, \quad i > k\,,$$

and hence

$$U\left(\lim_{u\to\infty} \widetilde{\mathscr{R}^s(u)}\right) U^{-1} = \begin{cases} R^-_{w_0}\,\hbar^{\Omega}\,, & k=0\,, \\ \hbar^{\Omega}\, R^+_{w_{-1}}\,, & k=-1\,. \end{cases} \tag{2.30}$$

Similarly, for the factorization (2.21) which converges near $u=0$, we obtain:

$$U\left(\lim_{u\to 0} \widetilde{\mathscr{R}^s(u)}\right) U^{-1} = \begin{cases} \hbar^{-\Omega}(R^+_{w_0})^{-1}\,, & k=0\,, \\ (R^-_{w_{-1}})^{-1}\hbar^{-\Omega}\,, & k=-1\,. \end{cases} \tag{2.31}$$

Since the slope R-matrices $R^-_{w_0}$ and $R^+_{w_{-1}}$ are arbitrary, we conclude the following

Theorem 2. *Slope R-matrices $R^{\pm}_w$ multiplied by $\hbar^{\Omega}$:*

$$\mathsf{R}^{\pm}_w = \hbar^{\Omega} R^{\pm}_w \tag{2.32}$$

satisfy the Yang-Baxter equation.

Chapter 3

Construction of quantum groups

As we explain in Section 2.2 the equivariant K-theory of Nakajima variety provides a set of vector spaces $K_G(\mathscr{M}(\mathsf{w}))$ labeled by a dimension vector $\mathsf{w} \in \mathbb{Z}^n$. For any splitting of the framing $\mathsf{w} = u\mathsf{w}' + \mathsf{w}''$ our construction gives an R-matrix which acts tensor product $K_G(\mathscr{M}(\mathsf{w}')) \otimes K_G(\mathscr{M}(\mathsf{w}''))$ and satisfies the quantum Yang-Baxter equation. This is a well known set up for the Faddeev-Reshetikhin-Takhtadzhyan formalism [36]. Using these data the FRT construction provides a triangular Hopf algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ acting in $K_G(\mathscr{M}(\mathsf{w}))$ for all w.

Similarly, applying the FRT construction to the wall R-matrices $\mathsf{R}^{\pm}_{w}$ one constructs a set of triangular Hopf algebras $\mathscr{U}_\hbar(\mathfrak{g}_w)$ which are, in fact, subalgebras of $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$.

The aim of this section of to review the FRT method and to explain the interaction between Hopf structures of different wall subalgebras $\mathscr{U}_\hbar(\mathfrak{g}_w)$.

3.1 Quiver algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$

3.1.1

For a splitting $\mathsf{w} = u_1\mathsf{w}_1 + \cdots + u_n\mathsf{w}_n$ and a slope $s \subset H^2(\mathscr{M}(\mathsf{w}),\mathbb{R})$ the construction of Sections 2.3.2 provides a set of R-matrices

$$\mathscr{R}^s_{V_i,V_j}(u_i/u_j) \subset \mathrm{End}\Big(V_1 \otimes \cdots \otimes V_n\Big) \otimes \mathbb{C}[u_1^{\pm 1}, ..., u_n^{\pm 1}],$$

with $V_k = K_G(\mathscr{M}(\mathsf{w}_k))$ satisfying the Yang-Baxter equation. We denote

$$V_i(u) \stackrel{def}{=} V_i \otimes \mathbb{C}[u^{\pm 1}]$$

and more generally

$$V_{i_1}(u_1)\otimes\cdots\otimes V_{i_n}(u_n)\stackrel{def}{=}V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\mathbb{C}[u_1^{\pm1},...,u_n^{\pm1}]$$

3.1.2

We have a set of vector spaces $\mathfrak{V}$ such that for any pair $V_i,V_j\in\mathfrak{V}$ we have an R-matrix $\mathscr{R}^s_{V_i,V_j}(u_i/u_j)$.

First, we note that this set is closed with respect to the tensor product. The R-matrix for the tensor products has the following form:

$$\mathscr{R}^s_{\overleftarrow{\bigotimes\limits_{i\in I}}V_i(u_i),\overleftarrow{\bigotimes\limits_{j\in J}}V_i(u_i)}=\overrightarrow{\prod_{i\in I}}\overleftarrow{\prod_{j\in J}}\mathscr{R}^s_{V_i,V_j}(u_i/u_j). \tag{3.1}$$

Second, following [35] we can assume that this set contains dual vector spaces V_i^* with R-matrices defined by the following rules:

$$\mathscr{R}^s_{V_1^*,V_2}=((\mathscr{R}^s_{V_1,V_2})^{-1})^{*_1}$$

$$\mathscr{R}^s_{V_1,V_2^*}=((\mathscr{R}^s_{V_1,V_2})^{-1})^{*_2}$$

$$\mathscr{R}^s_{V_1^*,V_2^*}=((\mathscr{R}^s_{V_1,V_2}))^{*_{12}}$$

where $*_k$ means transpose with respect to the k-th factor. One checks that the R-matrices defined this way satisfy the quantum Yang-Baxter equation in the tensor product of any three spaces from the set $\mathfrak{V}$.

3.1.3

In FRT formalism the quantum algebra $\mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$ is defined as subalgebra

$$\mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)\subset\prod_{V\in\mathfrak{V}}\mathrm{End}(V)$$

generated by matrix elements of

$$\mathscr{R}^s_{V,V_0}(u)\in\mathrm{End}(V)\otimes\mathrm{End}(V_0) \tag{3.2}$$

in the "auxiliary space" V_0 for all choices of $V_0\in\mathfrak{V}$. Such matrix coefficients are rational functions in u with values in $\mathrm{End}(V)$. The algebra $\mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$ by definition is generated by the coefficients of the expansion of these functions as $u \to 0$ and $u \to \infty$. As the degrees of these rational functions are unbounded, there is no universal relations between the coefficients.

As we will see in the next section, the algebras $\mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$ are isomorphic for all values of slope $s \in H^2(X, \mathbb{R})$. Thus, we denote the resulting algebra by $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$.

3.2 Wall subalgebra $\mathscr{U}_\hbar(\mathfrak{g}_w) \subset \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$

3.2.1

Let $\mathsf{R}^{\pm}_w$ be two R-matrices as in Theorem 2. By definition, these R -matrices are defined for any two vector spaces $V_1, V_2 \in \mathfrak{V}$ and provide solutions of Yang-Baxter equation. Again, we can use the same FRT formalism to define algebras $\mathscr{U}_\hbar(\mathfrak{g}_w)$ generated by matrix elements of $\mathsf{R}^{\pm}_w$. In fact, the wall R-matrices for opposite choices of the chamber of the framing torus are related, and thus it is enough to use only R^{+}_w:

Proposition 2.

$$(R^{\pm}_w)_{21} = R^{\mp}_w \tag{3.3}$$

Proof. Follows from the definition of wall R-matrix and the obvious property $\Omega_{21} = \Omega$. □

3.2.2

Let us define the wall algebra:

$$\mathscr{U}_\hbar(\mathfrak{g}_w) \subset \prod_{V \in \mathfrak{V}} \mathrm{End}(V)$$

as algebra generated by the matrix elements of

$$(\mathsf{R}^{+}_w)_{V,V_0} \in \mathrm{End}(V) \otimes \mathrm{End}(V_0)$$

in the auxiliary vector space V_0 for all $V_0 \in \mathfrak{V}$. By the previous proposition we could define $\mathscr{U}_\hbar(\mathfrak{g}_w)$ using R^{-}_w.

3.2.3

Proposition 3. *The algebra $\mathscr{U}_\hbar(\mathfrak{g}_w)$ is a subalgebra of $\mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$ for every wall.*

Proof. Let us consider the KT factorization of the slope s R-matrix near $u = 0$ (2.19) and $u = \infty$ (2.21). By factorization, the total R-matrix is a product of wall R-matrices in some order $w_1, w_2, w_3, \cdots$. We prove the proposition by induction. In the limit (maybe after some conjugation by a diagonal matrix as in (2.30),(2.31)) we obtain:

$$\lim_{u\to 0} \mathscr{R}^s(u) = \mathsf{R}^+_{w_1}, \quad \lim_{u\to\infty} \mathscr{R}^s(u) = (\mathsf{R}^-_{w_1})^{-1}$$

By definition, $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ is generated by all coefficients in u of matrix elements of $\mathscr{R}^s$ and therefore by matrix elements of $\mathsf{R}^\pm_{w_1}$. We conclude $\mathscr{U}_\hbar(\mathfrak{g}_{w_1}) \subset \mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$.

Now assume that $\mathscr{U}_\hbar(\mathfrak{g}_{w_i}) \subset \mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$ for $i = 1, \cdots, n$. Let us consider the R-matrix $\mathscr{R}^{s'}(u)$ with slope s' between the walls w_n and w_{n+1}. From Khoroshkin-Tolstoy factorization we have:

$$\mathscr{R}^{s'}(u) = T^-_{w_n,w_1} \mathscr{R}^s(u) (T^+_{w_n,w_1})^{-1}$$

where

$$T^+_{w_n,w_1} = \prod^{\leftarrow}_{1\leq i\leq n} R^+_{w_i}, \quad T^-_{w_n,w_1} = \prod^{\rightarrow}_{1\leq i\leq n} R^-_{w}. \tag{3.4}$$

Again, after conjugation by a diagonal matrix we obtain:

$$\lim_{u\to 0} \mathscr{R}^{s'}(u) = \mathsf{R}^+_{w_{n+1}}, \quad \lim_{u\to\infty} \mathscr{R}^{s'}(u) = (\mathsf{R}^-_{w_{n+1}})^{-1}$$

therefore $\mathscr{U}_\hbar(\mathfrak{g}_{w_{n+1}}) \subset \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ and the Proposition follows. □

Corollary 1. *The algebras $\mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$ are isomorphic for all s.*

Proof. By the previous proposition $\mathscr{U}^s_\hbar(\widehat{\mathfrak{g}}_Q)$ is generated by $\mathscr{U}_\hbar(\mathfrak{g}_w)$ for those walls w which contribute to Khoroshkin-Tolstoy factorization of $\mathscr{R}^s(u)$. But, by construction each wall appears in factorization of $\mathscr{R}^s(u)$ exactly one time for all slopes s. □

As all algebras are isomorphic we will denote them simply by $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$.

3.3 Hopf structures

3.3.1

The algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ carries Hopf structures labeled by the slope s. The set $\mathfrak{V}$ is closed with respect to tensor product. It induces the natural projection:

$$\prod_{V\in\mathfrak{V}} \mathrm{End}(V) \to \prod_{V_1,V_2\in\mathfrak{V}} \mathrm{End}(V_1\otimes V_2)$$

which restricts to a coproduct map:

$$\Delta_s : \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q) \to \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)\hat{\otimes}\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$$

Note that this map depends on KT factorization of R-matrix and thus on the slope s.

The set $\mathfrak{V}$ is closed with respect to taking dual $*$ and thus we have an antipode map:

$$S_s : \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q) \to \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$$

which is the restriction of:

$$\mathrm{End}(V) \xrightarrow{*} \mathrm{End}(V^*)$$

The set $\mathfrak{V}$ contains the trivial representation $\mathbb{C}$ which induces a counit map:

$$\epsilon_s : \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q) \to \mathbb{C}$$

One can check that the triple $(\Delta_s, S_s, \epsilon_s)$ satisfies all axioms of Hopf algebra for any slope s. Thus, the algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ becomes triangular Hopf algebra (with triangular structure $\mathscr{R}^s(u)$).

3.3.2

The same procedure applied to R_w^+ in place of $\mathscr{R}^s(u)$ defines a structure of triangular Hopf algebra $(\Delta_w,\, S_w,\, \epsilon_w)$ on $\mathscr{U}_\hbar(\mathfrak{g}_w)$. It should be clear from definitions that $(\Delta_w,\, S_w,\, \epsilon_w)$ does not necessary coincide with restriction of $(\Delta_s,\, S_s,\, \epsilon_s)$ from the ambient algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$. The next proposition explains the relation between these Hopf structures.

3.3.3

Assume that the Khoroshkin-Tolstoy factorization for a slope s R-matirx (2.19) starts with some wall w, i.e. has the form:

$$\mathscr{R}^s(u) = \cdots\, R^+_{w_1}\, R^+_w$$

Proposition 4. *The Hopf structure (Δ_w, S_w, ϵ_w) on $\mathscr{U}_\hbar(\mathfrak{g}_w)$ coincide with restriction of (Δ_s, S_s, ϵ_s) from the ambient algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$.*

Proof. Enough to check this statement for coproducts. We need to show that for any element $x \in \mathscr{U}_\hbar(\mathfrak{g}_w)$:

$$\Delta_s(x) = \Delta_w(x) \in \mathscr{U}_\hbar(\mathfrak{g}_w)^{\otimes 2}$$

Let x be an element of the algebra (3.2) corresponding to some coefficient of series expansion around $u = \infty$ of a matrix coefficient in an auxiliary space V_0. By (3.1) the action of this matrix element in $\Delta_s(x)$ in $V_1 \otimes V_2$ can be written in the form:

$$\Delta_s(x) = \mathrm{Res}_{u=\infty}\Big(\mathrm{tr}_{V_0}(m(u), \mathscr{R}^s_{1,0}(u_1/u)\mathscr{R}^s_{2,0}(u_2/u))\Big)$$

for some finite rank operator $m(u) \in \mathrm{End}(V_0)$.

Let U be the diagonal acting in $V_1 \otimes V_2 \otimes V_0$ with the following elements:

$$U|_{\mathscr{M}(\mathsf{v}_1,\mathsf{w}_1)\times\mathscr{M}(\mathsf{v}_2,\mathsf{w}_2)\times\mathscr{M}(\mathsf{v}_0,\mathsf{w}_0)} = u_1^{\langle \mathsf{v}_1,\mathscr{L}_w\rangle} u_2^{\langle \mathsf{v}_2,\mathscr{L}_w\rangle} u^{\langle \mathsf{v}_0,\mathscr{L}_w\rangle}$$

where $\mathscr{L}_w$ is a line bundle on the wall w.

Let $\widetilde{\mathscr{R}}^s_{1,0}\widetilde{\mathscr{R}}^s_{2,0} = U\mathscr{R}^s_{1,0}\mathscr{R}^s_{2,0}U^{-1}$. By construction and (2.30) the elements $x \in \mathscr{U}_\hbar(g_w)$ correspond to constant in u coefficients with constant matrices $m(u) = m$ of $\widetilde{\mathscr{R}}^s_{1,0}(u_1/u)\widetilde{\mathscr{R}}^s_{2,0}(u_2/u)$, and thus we have:

$$\Delta_s(x) = \mathrm{Const}_u\Big(\mathrm{tr}_{V_0}(m, \widetilde{\mathscr{R}}^s_{1,0}\widetilde{\mathscr{R}}^s_{2,0})\Big) =$$

$$\mathrm{tr}_{V_0}(m, \lim_{u\to\infty} \widetilde{\mathscr{R}}^s_{1,0}\widetilde{\mathscr{R}}^s_{2,0}) = \mathrm{tr}_{V_0}(m, (\mathsf{R}^+_w)_{1,0}(\mathsf{R}^+_w)_{2,0}) \stackrel{def}{=} \Delta_w(x)$$

□

Corollary 2. *If s and w are as above then for $x \in \mathscr{U}_\hbar(\mathfrak{g}_w)$ we have:*

$$\mathscr{R}^s(u)\,\Delta_s(x)\mathscr{R}^s(u)^{-1} = \mathsf{R}^+_w\Delta_w(x)(\mathsf{R}^+_w)^{-1} = (\mathsf{R}^-_w)^{-1}\Delta_w(x)\mathsf{R}^-_w \tag{3.5}$$

with $\mathsf{R}^\pm_w$ as in Theorem 2.

Proof. In any triangular Hopf algebra we have $\mathscr{R}^s(u)\Delta_s(x)\mathscr{R}^s(u)^{-1} = \Delta_s^{op}(x)$. But, for $x \in \mathscr{U}_\hbar(\mathfrak{g}_w)$ we have $\Delta_s^{op}(x) = \Delta_w^{op}(x) = \mathsf{R}_w^+\Delta_w(x)(\mathsf{R}_w^+)^{-1}$. This proves first euqlity. To prove the second line we need to reprove the Proposition 2 using the product formula (2.21) which convergent near $u = \infty$. This gives another triangular structure (2.31) on $\mathscr{U}_\hbar(\mathfrak{g}_w)$ given by R-matrix $(\mathsf{R}_w^-)^{-1}$. Arguing as above, with R_w^+ replaced by $(\mathsf{R}_w^-)^{-1}$ we obtain second equality. □

3.3.4

Let s and s' are two slopes and let Γ be a path in $H^2(X,\mathbb{R})$ connecting them. This path intersects finitely many walls in some order $I_\Gamma = \{w_1, w_2, ..., w_n\}$. We define operators:

$$T^+ = \prod_{w\in I_\Gamma}^{\leftarrow} R_w^+, \quad T^- = \prod_{w\in I_\Gamma}^{\rightarrow} R_w^-$$

Then, from Khoroshkin-Tolstoy factorization we obtain:

$$\mathscr{R}^{s'}(u)T^+ = T^-\mathscr{R}^s(u)$$

which implies that coproducts at different slopes are conjugated:

$$T^+\Delta_{s'} = \Delta_s T^+, \quad T^-\Delta_{s'}^{op} = \Delta_s^{op} T^-. \tag{3.6}$$

3.3.5

As a slope s approaches infinity (in the ample cone) we obtain a special Hopf structure on our algebra with coproduct which we denote by Δ_∞. In this limit the R-matrix coincide with (2.24), and the operators of multiplication by line $\mathscr{L} \in \mathrm{Pic}(X)$ bundles turn to group-like elements:

$$\Delta_\infty(\mathscr{L}) = \mathscr{L} \otimes \mathscr{L} \tag{3.7}$$

3.3.6

Let $\kappa = (\kappa_1, \kappa_2)$ with $\kappa_i \in H^2(X,\mathbb{Z})$. Define an operator $\hbar^\kappa$ acting in $K_G(\mathscr{M}(\mathsf{w}))$ by:

$$\hbar^\kappa(\gamma) = \hbar^{\langle\kappa_1,\mathsf{v}\rangle + \langle\kappa_2,\mathsf{v}\rangle}\gamma$$

for a class γ supported on the component $\mathscr{M}(\mathsf{v},\mathsf{w})$. One can check that $\hbar^\kappa \in \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ with the following properties:

$$\Delta_s(\hbar^\kappa) = \hbar^\kappa \otimes \hbar^\kappa, \quad S_s(\hbar^\kappa) = \hbar^{-\kappa} \tag{3.8}$$

Remind, that the codimension function Ω is quadratic in w, v which gives:

$$S_s \otimes S_s(\Omega) = \Omega \tag{3.9}$$

Finally, in any triangular Hopf algebra we have

$$S_w \otimes S_w(\mathsf{R}^{+}_{w}) = \mathsf{R}^{+}_{w} \tag{3.10}$$

and thus from (3.9) we conclude:

$$S_w \otimes S_w(R^{+}_{w}) = \hbar^{-\Omega}\, R^{+}_{w}\, \hbar^{\Omega} \tag{3.11}$$

We will use some of these simple identities in the following sections.

Chapter 4

Quantum K-theory of Nakajima varieties

4.1 Quaimaps to Nakajima varieties

4.1.1

Let us consider a quiver with set of vertices I and m_{ij} arrows from vertex $i \in I$ to vertex $j \in I$. Let $n = |I|$ be the number of vertices. Recall, that a Nakajima variety $\mathscr{M}(\mathsf{v}, \mathsf{w})$ with dimension vectors $\mathsf{w}, \mathsf{v} \in \mathbb{N}^n$ is defined as the following symplectic reduction:

$$\mathscr{M}(\mathsf{v}, \mathsf{w}) = T^*M /\!\!/\!\!/\!\!/ \mathsf{G} = \mu^{-1}(0) /\!\!/ \mathsf{G}$$

where M is the representation of the quiver:

$$M = \bigoplus_{i,j \in I} \mathrm{Hom}(V_i, V_j) \otimes Q_{ij} \oplus \bigoplus_{i \in I} \mathrm{Hom}(W_i, V_i)$$

by vectors spaces V_i of dimensions v_i and framing spaces W_i of dimensions w_i. We denote by Q_{ij} the linear vector space of dimension m_{ij} (the multiplicity space). The cotangent bundle T^*M is equipped with the canonical action of $\mathsf{G} = \prod_{i \in I} GL(V_i)$ and we denote by

$$\mu : T^*M \to \mathrm{Lie}(\mathsf{G})^*$$

the corresponding moment map.

4.1.2

Let us consider constant maps from a curve $C = \mathbb{P}$ to a Nakajima quiver variety $X = \mathscr{M}(\mathsf{v}, \mathsf{w})$. The moduli space of such maps is of course given by $\mathsf{QM}^0(X) = X$. In this case, we can think of the vector spaces defining the representation of the quiver V_i and W_i as global sections of trivial bundles on the curve C with ranks v_i and w_i respectively.

In general, a quasimap:

$$f : C \dashrightarrow X$$

is defined as a collection of vector bundles $\mathscr{V}_i$, trivial vectors bundles $\mathscr{Q}_{ij}$ and $\mathscr{W}_i$ on the curve C with the same ranks together with a section:

$$f \in H^0\Big(C, \mathscr{M} \otimes \mathscr{M}^*\Big)$$

for

$$\mathscr{M} = \bigoplus_{i,j \in I} \mathscr{H}om(\mathscr{V}_i, \mathscr{V}_j) \otimes \mathscr{Q}_{ij} \oplus \bigoplus_{i \in I} \mathscr{H}om(\mathscr{W}_i, \mathscr{V}_i)$$

The degree of a quasimap is defined as a vector of degrees $d = (\deg(\mathscr{V}_i)) \in \mathbb{Z}^n$.

4.1.3

The moduli space $\mathsf{QM}^d(X)$ parameterizes the degree d quasimaps up to isomorphism which is required to be identity on the curve C, the multiplicity $\mathscr{Q}_{i,j}$ and the framing bundles $\mathscr{W}_i$:

$$\mathsf{QM}^d(X) = \{\text{degree d quasimaps to } \ X\}/\cong$$

This means that moving a point on this moduli space results in varying the bundles $\mathscr{V}_i$ and the section f, while the curve C, bundles $\mathscr{W}_i$ and $\mathscr{Q}_{ij}$ remain fixed.

For a point $p \in C$ we have the evaluation map:

$$\mathsf{QM}^d(X) \xrightarrow{\mathrm{ev}_p} X$$

sending a quasimap to its value at p. This map is well defined on the open set $\mathsf{QM}^d(X)_{\text{nonsing } p} \subset \mathsf{QM}^d(X)$ of the quasimaps which are nonsingular at the point p. The moduli space of relative quasimaps $\mathsf{QM}^d(X)_{\text{relative } p}$ is a resolution of the map ev meaning that we have a commutative diagram:

$$\begin{array}{ccc} & \mathsf{QM}^d(X)_{\text{relative } p} & \\ \nearrow & & \searrow \tilde{\text{ev}} \\ \mathsf{QM}^d(X)_{\text{nonsing } p} & \xrightarrow{\quad \text{ev} \quad} & X \end{array}$$

with *proper* evaluation map $\tilde{\text{ev}}$ from $\mathsf{QM}^d(X)_{\text{relative } p}$ to X. The construction of the moduli space for relative quasimaps is explained in Section 6.5 of [31]. It follows similar construction of relative moduli spaces in Gromow-Witten theory [] and Donaldson-Thomas theory [].

4.2 Difference equations

4.2.1

As explained in [31] the moduli spaces defined in the previous sections carry natural virtual structure sheafs $\widehat{\mathscr{O}}_{\text{vir}}$. Using these virtual sheaves one constructs different enumerative invariants of X. For example, one of the main objects in quantum K-theory is the *capping operator* which is defined as follows: let us consider the moduli space $\mathsf{QM}^d_{\substack{\text{relative} p_1 \\ \text{nonsing} p_2}}(X)$ of quasimaps with relative conditions at $p_1 \in C$ and nonsingular at $p_2 \in C$ (we will assume that $p_1 = 0$ and $p_2 = \infty$ in $C = \mathbb{P}$). These two marked points define the evaluation map:

$$\text{ev} : \mathsf{QM}^d_{\substack{\text{relative} p_1 \\ \text{nonsing} p_2}}(X) \longrightarrow X \times X$$

This moduli space is equipped with an action of $G \times \mathbb{C}^\times$ where action of G comes from its action on the X (see Section **??**) and $\mathbb{C}^\times$ scales the coordinate on $\mathbb{P}$, i.e., the standard torus action which preserves p_1 and p_2. The capping operator is defined as the following $G \times \mathbb{C}^\times$ equivariant push-forward:

$$\mathbf{J} = \sum_{d \in \mathbb{Z}^n} z^d \text{ev}_* \Big(\mathsf{QM}^d_{\substack{\text{relative} p_1 \\ \text{nonsing} p_2}}(X), \widehat{\mathscr{O}}_{\text{vir}} \Big) \in K_G(X)^{\otimes 2}_{\text{localized}} \otimes \mathbb{Q}[[z_1, ..., z_n, q]] \tag{4.1}$$

where q is the equivariant parameter correponding to $\mathbb{C}^\times$ and $z^d = z_1^{d_1} \cdots z_n^{d_n}$.

4.2.2

Assume that we fixed some basis in $K_G(X)$, then the capping operator is represented by a matrix whose entries are certain functions of equivariant parameters u corresponding to G and Kähler parameters z_i. As shown in Section 8.2 of [31] , this matrix is the matrix of fundamental solution of a system of q-difference equations:

$$\begin{aligned} \mathbf{J}(uq,z)\mathsf{E}(u,z) &= \mathsf{S}(u,z)\mathbf{J}(u,z) \\ \mathbf{J}(u,zq^{\mathscr{L}})\mathscr{L} &= \mathbf{M}_{\mathscr{L}}(u,z)\mathbf{J}(u,z) \end{aligned} \tag{4.2}$$

The operators $\mathsf{S}(u,z)$ shifting the equivariant parameters are called *shift operators.* They are constructed using the twisted quasimaps in [31].

The operators $\mathbf{M}_{\mathscr{L}}(u,z)$ corresponding to line bundles $\mathscr{L}\in \mathrm{Pic}(X)$ are called the *quantum difference operators*, they are the main object of study in our paper. Let us explain the notations here. Recall, that the $\mathrm{Pic}(X)$ is generated by the tautological line bundles $\mathscr{L}_i=\det(\mathcal{V}_i)$, $i=1,...,n$.[1] For a bundle $\mathscr{L}=\mathscr{L}_1^{\otimes m_1}\otimes\cdots\otimes\mathscr{L}^{\otimes m_n}$ we use the following notations:

$$zq^{\mathscr{L}}=(z_1q^{m_1},...,z_nq^{m_n})$$

In the right side of the second equation in (4.2) we denote by the same symbol $\mathscr{L}$ the operator of multiplication by a line bundle $\mathscr{L}$ in $K_G(X)$. The operator $\mathsf{E}(u,z)$ is an operator of multiplication by some class in K-theory, in particular it commutes with $\mathscr{L}$.

4.2.3

We can write the system (4.2) in the following equivalent form:

$$\begin{aligned} \mathcal{K}\,\mathbf{J}(u,z) &= \mathbf{J}(u,z)\,\mathcal{K}^{\infty} \\ \mathcal{A}_{\mathscr{L}}\,\mathbf{J}(u,z) &= \mathbf{J}(u,z)\,\mathcal{A}^{\infty}_{\mathscr{L}} \end{aligned} \tag{4.3}$$

with the following q-difference operators:

$$\begin{aligned} \mathcal{K}=T_u^{-1}\mathsf{S}(u,z), &\qquad \mathcal{K}^{\infty}=T_u^{-1}\mathsf{E}(u,z) \\ \mathcal{A}_{\mathscr{L}}=T_{\mathscr{L}}^{-1}\mathbf{M}_{\mathscr{L}}(u,z) &\quad \mathcal{A}^{\infty}_{\mathscr{L}}=T_{\mathscr{L}}^{-1}\mathscr{L} \end{aligned} \tag{4.4}$$

[1] We should comment here that in general, the tautological line bundles generate only a sublattice of $\mathrm{Pic}(X)$

where $T_{\mathscr{L}} f(u,z) = f(u, zq^{\mathscr{L}})$ and $T_u f(u,z) = f(uq, z)$. As $\mathscr{L}$ and $\mathsf{E}(u,z)$ commute, the consistency of this system of difference equations can be presented in the form of "zero curvature" condition:

$$[\mathcal{A}_{\mathscr{L}}, \mathcal{A}_{\mathscr{L}'}] = 0, \quad [\mathcal{A}_{\mathscr{L}}, \mathcal{K}] = 0 \tag{4.5}$$

where by $[A,B] = AB - BA$ we denote the commutators for q-difference operators .

4.2.4

Let $\mathsf{A} = \mathbb{C}^\times$ be a torus splitting the framing as $\mathsf{w} = u\mathsf{w}' + \mathsf{w}''$. This torus acts on the Nakajima variety $X = \mathscr{M}(\mathsf{v}, \mathsf{w})$ with the set of fixed points:

$$X^{\mathsf{A}} = \coprod_{\mathsf{v}' + \mathsf{v}'' = \mathsf{v}} \mathscr{M}(\mathsf{v}', \mathsf{w}') \times \mathscr{M}(\mathsf{v}'', \mathsf{w}'')$$

The stable map defined in the previous section can be used to identify $K_G(X)$ with $K_G(X^{\mathsf{A}})$. The main result of [31] is that after such identification, the shift operator $\mathsf{S}(u,z)$ gets identified with the *quantum Knizhnik-Zamolodzhikov operator* (qKZ).[2]

Theorem 3. *([31]) Let $\nabla \subset H^2(X,\mathbb{R})$ be the alcove uniquely defined by the conditions:*

1) $0 \in H^2(X,\mathbb{R})$ is one of the vertices of ∇

2) $\nabla \subset -C_{ample}$ (opposite of the ample cone)

then for all $s \in \nabla$ we have[3]:

$$Stab_{+,T^{1/2},s}\, \mathsf{S}(u,z)\, Stab^{-1}_{+,T^{1/2},s} = z^{\mathsf{v}'} \mathscr{R}^s(u)$$

Here $\mathscr{R}^s(u)$ is the R-matrix (2.17) corresponding to the slope s. We denote by $z^{\mathsf{v}'}$ an operator diagonal in the basis of A fixed points:

$$z^{\mathsf{v}'}(\gamma) = z_1^{\mathsf{v}'_1} \cdots z_n^{\mathsf{v}'_n} \cdot \gamma \tag{4.6}$$

[2]See Theorem 9.3.1 in [25] for similar statement in the case of equivariant cohomology.

[3]Note, that we use modified quantum parameter z which differs by a sign:

$$z^{\mathsf{v}} \mapsto (-1)^{\mathrm{codim}/2} z^{\mathsf{v}},$$

see Theorem 10.2.8 in [31]. Explicitly, this change of variables amounts in the following substitution of Kähler parameters:

$$z_i \mapsto (-1)^{2\kappa_i} z_i$$

for canonical vector (5.10). To get rid of the minus sign, we will use modified notations in this paper.

for a class γ supported on the component $\mathscr{M}(\mathsf{v}',\mathsf{w}') \times \mathscr{M}(\mathsf{v}'',\mathsf{w}'') \subset X^{\mathsf{A}}$. The same argument can be used to show that

$$Stab_{+,T^{1/2},s}\, \mathsf{E}(u,z)\, Stab^{-1}_{+,T^{1/2},s} = z^{\mathsf{v}'}\mathscr{R}^{\infty}(u)$$

where $\mathscr{R}^{\infty}(u)$ is the R-matrix for the infinite slope (2.24). Therefore, under the above identification the first equation in (4.2) turns to:

$$\mathscr{K}^{s}\mathbf{J}(u,z) = \mathbf{J}(u,z)\,\mathscr{K}^{\infty}.$$

with $\mathscr{K}^{s} = T_u^{-1} z^{\mathsf{v}'}\mathscr{R}^{\infty}(u)$. This is nothing but the well known quantum Knizhnik-Zamolodchikov equation [14].

4.2.5

In Section 5.0.8 we construct a system of difference operators

$$\mathscr{A}^{s}_{\mathscr{L}} = T_{\mathscr{L}}^{-1}\mathbf{B}^{s}_{\mathscr{L}}(u,z), \quad \mathscr{L} \in \mathrm{Pic}(X)$$

with $\mathbf{B}^{s}_{\mathscr{L}}(u,z)$ given explicitly in terms of the algebra $\mathscr{U}_{\hbar}(\widehat{\mathfrak{g}}_{Q})$. These operators commute among themselves and with the qKZ operator $\mathscr{K}^{s}$ for all slopes $s \in H^2(X,\mathbb{R})$:

$$[\mathscr{A}^{s}_{\mathscr{L}}, \mathscr{A}^{s}_{\mathscr{L}'}] = 0, \quad [\mathscr{A}^{s}_{\mathscr{L}}, \mathscr{K}^{s}] = 0 \tag{4.7}$$

We then prove our main result Theorem 5: under identification of theorem Theorem 3 the quantum difference operator $\mathbf{M}_{\mathscr{L}}(u,z)$ is identified with $\mathbf{B}^{s}_{\mathscr{L}}(u,z)$. In particular the compatibility condition (4.5) is identified with (4.7) for the slope s specified in Theorem 3.

Chapter 5

Commuting difference operators

5.0.6 Notations and definitions

5.0.6.1

Let $z_i, i = 1, ..., n = |I|$ be formal parameters. Later, they will play a role of Kähler parameters in quantum difference equations. It will be convenient to introduce a formal vector $\lambda = (\mathbf{t}_1, ..., \mathbf{t}_n)$ such that $\hbar^{\mathbf{t}_i} = z_i$. Let us consider a Nakajima variety $X = \mathscr{M}(\mathsf{v}, \mathsf{w})$ and denote by A a subtorus of the framing torus corresponding to decomposition:

$$X^{\mathsf{A}} = \coprod_{\mathsf{v}_1+...+\mathsf{v}_n=\mathsf{v}} \mathscr{M}(\mathsf{v}_1, \mathsf{w}_1) \times \cdots \times \mathscr{M}(\mathsf{v}_n, \mathsf{w}_n) \tag{5.1}$$

In this section we consider rational functions of parameters z_i which takes values in $End(K_{\mathsf{T}}(X))$. Using the above notations we will denote such functions as $f(z_i)$ or $f(\lambda)$. In the last case we understand them as functions of $\lambda \in \mathbb{C}^I$.

In the localized equivariant K-theory of X we can choose a basis consisting of elements supported on the fixed set. The first function we need $\hbar^{\lambda}_{(k)} \in End(K_{\mathsf{T}}(X))$ is defined to by diagonal in the basis supported on the set fixed points:

$$\hbar^{\lambda}_{(k)}(\gamma) = \hbar^{(\lambda, \mathsf{v}_k)}\gamma = z_1^{\mathsf{v}_{k,1}} \cdots z_n^{\mathsf{v}_{k,n}}\gamma$$

for a class γ supported on a component $F = \mathscr{M}(\mathsf{v}_1, \mathsf{w}_1) \times \cdots \times \mathscr{M}(\mathsf{v}_n, \mathsf{w}_n)$.

We will need the so called *dynamical notations* below. Let κ be some fixed linear combination of dimension vectors:

$$\kappa(\mathsf{v}, \mathsf{w}) = A\mathsf{v} + B\mathsf{w} \in \mathbb{C}^I$$

with A and B given by scalar $n \times n$ matrices. Let $f(\lambda) \in End(K_{\mathsf{T}}(X))$ be as above. Then, we define $f(\lambda + \hat{\kappa}_{(i)}) \in End(K_{\mathsf{T}}(X))$ by:

$$f(\lambda + \hat{\kappa}_{(i)})(\gamma) = f(\lambda + \kappa(\mathsf{v}_i, \mathsf{w}_i))(\gamma)$$

for the γ supported on the fixed component $F = \mathscr{M}(\mathsf{v}_1, \mathsf{w}_1) \times \cdots \times \mathscr{M}(\mathsf{v}_n, \mathsf{w}_n)$. We will refer to such a transformation $f(\lambda) \to f(\lambda + \hat{\kappa}_{(i)})$ as *dynamical shift* of f by weight κ in the i-component. In the case of one component we will omit subscript (1) and write $f(\lambda + \hat{\kappa})$.

In the following we will need the q-difference operators which act on the variables z_i by the rule $T_{z_i} f(z_1, ..., z_i, ..., z_n) = f(z_1, ..., z_i q, ..., z_n)$. We extend this to the action of sublattice of Picard group generated by tautological line bundles. Let $\mathcal{V}_i$, $i = 1, ..., n$ be the set of tautological bundles on Nakajima variety and $\mathscr{L}_i = \det \mathcal{V}_i$ the corresponding line bundles. For a line bundle $\mathscr{L} = \mathscr{L}_1^{m_1} \otimes \cdots \otimes \mathscr{L}_n^{m_n}$ define $T_{\mathscr{L}} = T_1^{m_1} \cdots T_n^{m_n}$. Define $\mathbf{s}$ by $\hbar^{\mathbf{s}} = q$, then in λ-notations the action of the difference operators takes the form:

$$T_{\mathscr{L}} f(\lambda) = f(\lambda + \mathbf{s}\mathscr{L}) \tag{5.2}$$

5.0.6.2

Below, we use definitions of triangular operators from Section 2.3.4.

Proposition 5. *There exist unique strictly upper triangular $J^{+}_{w}(\lambda)$ and strictly lower triangular $J^{-}_{w}(\lambda)$ solutions of the following ABRR equations:*

$$J^{+}_{w}(\lambda)\hbar^{-\lambda}_{(1)} \mathsf{R}^{+}_{w} = \hbar^{-\lambda}_{(1)} \hbar^{\Omega} J^{+}_{w}(\lambda), \quad \mathsf{R}^{-}_{w}\hbar^{-\lambda}_{(1)} J^{-}_{w}(\lambda) = J^{-}_{w}(\lambda)\hbar^{\Omega}\hbar^{-\lambda}_{(1)} \tag{5.3}$$

Moreover, $J^{\pm}_{w}(\lambda) \in \mathscr{U}_{\hbar}(\mathfrak{g}_w) \otimes \mathscr{U}_{\hbar}(\mathfrak{g}_w)$ and:

$$S_w \otimes S_w\Big((J^{+}_{w}(\lambda))_{21}\Big) = J^{-}_{w}(\lambda) \tag{5.4}$$

where the subscript (21) stands for the transposition $(a \otimes b)_{(21)} = b \otimes a$ and S_w is the antipode in $\mathscr{U}_{\hbar}(\mathfrak{g}_w)$.

Proof. We write the first ABRR equation in the form:

$$Ad_{\hbar^{\lambda}_{(1)}\hbar^{-\Omega}}\Big(J^{+}_{w}(\lambda)\Big) = J^{+}_{w}(\lambda)(R^{+}_{w})^{-1}$$

(recall that R^+_w and R^+_w are related by Theorem 2). By assumption $J^+_w(\lambda) = \bigoplus_{\langle\alpha,\theta\rangle>0} J^+_w(\lambda)_\alpha$ where θ is the stability parameter of the Nakajima variety. The wall R-matrix R^+_w is upper triangular, thus, it has the same decomposition. In the components the last equation is equivalent to the following system:

$$Ad_{\hbar^\lambda_{(1)}\hbar^{-\Omega}}\Big(J^+_w(\lambda)_\alpha\Big) = J^+_w(\lambda)_\alpha + \cdots$$

where $\cdots$ stands for the lower terms $J^+_w(\lambda)_{\alpha'}$, i.e., the terms with $\langle\alpha,\theta\rangle > \langle\alpha',\theta\rangle$. The operator $Ad_{\hbar^\lambda_{(1)}\hbar^{-\Omega}} - 1$ is invertible for general λ, thus we can solve the last system recursively. Different solutions correspond to different choices of the lower component $J^+_w(\lambda)_0$. By assumption $J^+_w(\lambda)_0 = 1$, thus the solution is unique. By construction of the wall quantum algebra algebra the R-matrix R^+_w is an element of $\mathscr{U}_\hbar(\mathfrak{g}_w)^{\otimes 2}$, thus, same is true for $J^+_w(\lambda)$.

Next, we apply the antipode $S_w \otimes S_w$ and and the transposition to the first ABRR equation and use (3.10)-(3.9) to obtain:

$$\mathsf{R}^-_w \hbar^\lambda_{(2)} S_w \otimes S_w\Big((J^+_w(\lambda))_{21}\Big) = S_w \otimes S_w\Big((J^+_w(\lambda))_{21}\Big)\hbar^\lambda_{(2)}\,\hbar^\Omega$$

It clear that for any upper of lower triangular operator X we have $\hbar^\lambda_{(2)} X \hbar^{-\lambda}_{(2)} = \hbar^{-\lambda}_{(1)} X \hbar^\lambda_{(1)}$, therefore, the last equation takes the form:

$$\mathsf{R}^-_w \hbar^{-\lambda}_{(1)} S_w \otimes S_w\Big((J^+_w(\lambda))_{21}\Big) = S_w \otimes S_w\Big((J^+_w(\lambda))_{21}\Big)\hbar^{-\lambda}_{(1)}\,\hbar^\Omega$$

By uniqueness of the solution we conclude that $S_w \otimes S_w\Big((J^+_w(\lambda))_{21}\Big) = J^-_w(\lambda)$. □

5.0.7 Wall Knizhnik-Zamolodchikov equations

5.0.7.1

Let $F = \mathscr{M}(\mathsf{v}_1,\mathsf{w}_1) \times \mathscr{M}(\mathsf{v}_2,\mathsf{w}_2)$ and $F' = \mathscr{M}(\mathsf{v}'_1,\mathsf{w}'_1) \times \mathscr{M}(\mathsf{v}'_2,\mathsf{w}'_2)$ be two fixed components. As we discussed in Section 2.3.5 the dependence of matrix elements of a wall R-matrix on equivariant parameter u is given by:

$$R^+_w(u)\big|_{F\times F'} \sim u^{\langle \mathsf{v}_1-\mathsf{v}'_1,\mathscr{L}_w\rangle}$$

Let us define $\mathbf{s}$ by $\hbar^{\mathbf{s}} = q$ and $\tau_w = \mathbf{s}\mathscr{L}_w$. Then, obviously, we have

$$\hbar^{\tau_w}_{(1)}\, R^+_w(u)\, \hbar^{-\tau_w}_{(1)} = R^+_w(uq) \tag{5.5}$$

From the previous proposition we obtain:

$$\hbar_{(1)}^{\tau_w} \, J_w^+(u) \, \hbar_{(1)}^{-\tau_w} = J_w^+(uq) \tag{5.6}$$

Shifting $\lambda \to \lambda - \tau_w$ in the ABRR equation (5.3) and using last two identities we find:

$$J_w^+(u, \lambda - \tau_w)\hbar_{(1)}^{-\lambda} \mathsf{R}_w^+(uq) = \hbar_{(1)}^{-\lambda} \hbar^{\Omega} J_w^+(uq, \lambda - \tau_w)$$

and same for J_w^-. Finally, denoting

$$\mathbf{J}_w^{\pm}(\lambda) = J_w^{\pm}(\lambda - \tau_w) \tag{5.7}$$

we rewrite the last relation in the form:

Proposition 6. *There exists unique strictly upper triangular $\boldsymbol{J}_w^+(\lambda) \in \mathscr{U}_\hbar(\mathfrak{g}_w)^{\otimes 2}$ and strictly lower triangular $\boldsymbol{J}_w^-(\lambda) \in \mathscr{U}_\hbar(\mathfrak{g}_w)^{\otimes 2}$ solutions of* ***wall Knizhnik-Zamolodchikov equations****:*

$$\begin{gathered} \boldsymbol{J}_w^+(\lambda)\hbar_{(1)}^{-\lambda} T_u R_w^+ = \hbar_{(1)}^{-\lambda} T_u \hbar^{\Omega} \boldsymbol{J}_w^+(\lambda), \\ R_w^- \hbar_{(1)}^{-\lambda} T_u \boldsymbol{J}_w^-(\lambda) = \boldsymbol{J}_w^-(\lambda)\hbar_{(1)}^{-\lambda} T_u \hbar^{\Omega} \end{gathered} \tag{5.8}$$

where $T_u f(u)$=f(u q)

5.0.8 Dynamical operators $\mathbf{B}^s_{\mathscr{L}}(\lambda)$

5.0.8.1

The following operator is playing a fundamental role in our paper:

$$\boxed{\mathbf{B}_w(\lambda) = \mathbf{m}\Big(1 \otimes S_w(\, \mathbf{J}_w^-(\lambda)^{-1}\,)\Big)\Big|_{\lambda \to \lambda + \kappa}} \tag{5.9}$$

Here S_w is the antipode of the Hopf algebra $\mathscr{U}_\hbar(\mathfrak{g}_w)$ and $m_{21}(a \otimes b) \stackrel{def}{=} ba$. We denote by $\lambda \to \lambda + \kappa$ the shift by the following vector:

$$\kappa = (C\mathsf{v} - \mathsf{w})/2 \tag{5.10}$$

where C is the Cartan matrix of the corresponding quiver. Note that this operator is well defined in evaluation modules of (even infinite dimensional) because the operator $\mathbf{J}_w^-(\lambda)$ is lower triangular and thus $\mathbf{B}_w(\lambda)$ is normally ordered.

5.0.8.2

Let $\mathscr{L} \in \mathrm{Pic}(X)$ be a line bundle. Let us fix a slope $s \in H^2(X, \mathbb{R})$ and choose a path p in $H^2(X, \mathbb{R})$ from s to $s - \mathscr{L}$. We assume that this path positively crosses finitely many walls in the following order $\{w_1, w_2, ..., w_m\}$. For this choice of slope and a line bundle we associate the following operator:

$$\boxed{\mathbf{B}^s_{\mathscr{L}}(\lambda) = \mathscr{L}\mathbf{B}_{w_m}(\lambda) \cdots \mathbf{B}_{w_1}(\lambda) \quad \in \quad \mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)} \tag{5.11}$$

In this formula $\mathscr{L}$ denotes the operator of multiplication by a line bundle in $K_G(X)$. We will also define the q-difference operators:

$$\mathscr{A}^s_{\mathscr{L}} = T^{-1}_{\mathscr{L}}\, \mathbf{B}^s_{\mathscr{L}}(\lambda)$$

where the notations are same as in Section 5.0.6.1.

5.0.8.3

Let $\mathscr{R}^s(u)$ be the slope s R-matrix. We define the quantum Knizhnik-Zamolodchikov operator acting in the space $K_G(X^{\mathsf{A}})$-valued functions explicitly:

$$\mathscr{K}^s = \hbar^{\lambda}_{(1)} T^{-1}_u \mathscr{R}^s(u) \tag{5.12}$$

where $T_u f(u) = f(uq)$ is a q-difference operator.

Theorem 4. *Let s, s' be two slopes separated by a single wall w, then the corresponding q-difference operators are conjugated:*

$$W^{-1}\mathscr{K}^s W = \mathscr{K}^{s'}, \quad W^{-1}\mathscr{A}^s_{\mathscr{L}} W = \mathscr{A}^{s'}_{\mathscr{L}} \tag{5.13}$$

where $W = \boldsymbol{B}_w(\lambda)(R^+_w)^{-1}$ and we assume that passing from s to s' we cross the wall w in the positive direction. .

As a consequence of this theorem we will prove the following result:

Corollary 3. *1) The operator $\boldsymbol{B}^s_{\mathscr{L}}(\lambda)$ does not depend on the choice of path from in (5.11).*
2) The q-difference operators commute for all $\mathscr{L}, \mathscr{L}' \in \mathrm{Pic}(X)$ and $s \in H^2(X, \mathbb{R})$:

$$\mathscr{A}^s_{\mathscr{L}} \mathscr{A}^s_{\mathscr{L}'} = \mathscr{A}^s_{\mathscr{L}'} \mathscr{A}^s_{\mathscr{L}}, \quad \mathscr{A}^s_{\mathscr{L}} \mathscr{K}^s_{\mathscr{L}} = \mathscr{K}^s_{\mathscr{L}} \mathscr{A}^s_{\mathscr{L}}$$

Our final step is to explain the connection between the quantum difference operator $\mathbf{M}_{\mathscr{L}}(\lambda)$ and operator $\mathbf{B}^s_{\mathscr{L}}(\lambda)$. Recall that the quantum difference operators $\mathbf{M}_{\mathscr{L}}(u,\lambda)$ for $\mathscr{L} \in \mathrm{Pic}(X)$ and the shift operator $\mathsf{S}(u,\lambda)$ form a compatible system of difference equations (4.2). The Theorem 3 then identifies the shift operator $\mathsf{S}(u,\lambda)$ with qKZ operator $\mathscr{K}^s$ for some canonical choice of the slope s. We now generalize this theorem to the case of quantum difference operator:

Theorem 5. *Let $\nabla \subset H^2(X,\mathbb{R})$ be the alcove uniquely defined by the conditions:*

1) $0 \in H^2(X,\mathbb{R})$ is one of the vertices of ∇

2) $\nabla \subset -C_{ample}$ (opposite of the ample cone)

then for $s \in \nabla$ we have:

$$Stab_{+,T^{1/2},s}\, \mathsf{S}(u,\lambda)\, Stab^{-1}_{+,T^{1/2},s} = z^{\mathsf{v}'} \mathscr{R}^s(u)$$

$$Stab_{+,T^{1/2},s}\, \boldsymbol{M}_{\mathscr{L}}(u,\lambda)\, Stab^{-1}_{+,T^{1/2},s} = \boldsymbol{B}^s_{\mathscr{L}}(\lambda)$$

Proofs of these results are based on careful analysis of properties of operators $\mathbf{J}^{\pm}_w(\lambda)$, the Hopf structures of algebras $\mathscr{U}_\hbar(\mathfrak{g}_w)$ and interaction of these structures for different slopes. This will be done in the next section.

Chapter 6

Proofs of Theorems 4 and 5

6.1 Cocycle identity

In this section we prove that the solutions of ABRR equations satisfy dynamical cocycle identities. Our presentation follows closely to [11] where this fact was fist proven for simple quantized Lie algebras.

6.1.1

Let $J^{\pm}_w(\lambda)$ be the operators defined by Proposition 5. Let us denote $J^{\pm}(\lambda)^{12} = J^{\pm}_w(\lambda)\otimes 1$, $J^{\pm}(\lambda)^{23} = 1 \otimes J^{\pm}_w(\lambda)$, $J^{\pm}(\lambda)^{12,3} = (\Delta_w \otimes 1)J^{\pm}_w(\lambda)$, $J^{\pm}(\lambda)^{1,23} = (1 \otimes \Delta_w)J^{\pm}_w(\lambda)$ the operators in $\mathscr{U}_\hbar(\mathfrak{g}_w) \otimes \mathscr{U}_\hbar(\mathfrak{g}_w) \otimes \mathscr{U}_\hbar(\mathfrak{g}_w)$. Then we have:

Theorem 6. *The operators $J^{\pm}(\lambda)$ satisfy the dynamical cocycle conditions:*

$$J^{-}(\lambda)^{12,3}J^{-}(\lambda+\hat{\kappa}_{(3)})^{12} = J^{-}(\lambda)^{1,23}J^{-}(\lambda-\hat{\kappa}_{(1)})^{23}$$
$$J^{+}(\lambda+\hat{\kappa}_{(3)})^{12}J^{+}(\lambda)^{12,3} = J^{+}(\lambda-\hat{\kappa}_{(1)})^{23}J^{+}(\lambda)^{1,23} \tag{6.1}$$

with dynamical shift $\kappa = (C\mathsf{v} - \mathsf{w})/2$ where C is the Cartan matrix of the quiver.

We will need the three-component analog of Proposition 5. We start with definition of upper/lower triangular operators acting in a tensor product of three $\mathscr{U}_\hbar(\mathfrak{g}_w)$ modules. Let $X =$

$\mathscr{M}(\mathsf{v},\mathsf{w})$ - be a Nakajima variety, and let A be a torus splitting the framing such that:

$$X^{\mathsf{A}} = \coprod_{\mathsf{v}_1+\mathsf{v}_2+\mathsf{v}_3=\mathsf{v}} \mathscr{M}(\mathsf{v}_1,\mathsf{w}_1)\times \mathscr{M}(\mathsf{v}_2,\mathsf{w}_2)\times \mathscr{M}(\mathsf{v}_3,\mathsf{w}_3). \tag{6.2}$$

We say that the operator $A \in \mathrm{End}(K_{\mathsf{T}}(X^{\mathsf{A}}))$ is *upper triangular* if $A = \bigoplus_{\substack{\langle\alpha,\theta\rangle>0\\ \langle\beta,\theta\rangle<0}} A_{\alpha,\beta}$ where θ is the stability parameter of the Nakajima variety and:

$$A_{\alpha,\beta} : K_{\mathsf{T}}(\mathscr{M}(\mathsf{v}_1,\mathsf{w}_1)\times \mathscr{M}(\mathsf{v}_2,\mathsf{w}_2)\times \mathscr{M}(\mathsf{v}_3,\mathsf{w}_3)) \to$$

$$K_{\mathsf{T}}(\mathscr{M}(\mathsf{v}_1+\alpha,\mathsf{w}_1)\times \mathscr{M}(\mathsf{v}_2+\gamma,\mathsf{w}_2)\times \mathscr{M}(\mathsf{v}_3+\beta,\mathsf{w}_3))$$

Obviously, γ is defined from the condition $\alpha+\beta+\gamma=0$. Similarly, the operator is lower triangular if $A = \bigoplus_{\substack{\langle\alpha,\theta\rangle<0\\ \langle\beta,\theta\rangle>0}} A_{\alpha,\beta}$ with the same $A_{\alpha,\beta}$ as above. Finally, we say that an operator is strictly upper or lower triangular if, in addition, $A_{0,0}=1$. For example, the product of wall R-matrices $R^{+,13}_{w}R^{+,12}_{w}$ or $R^{+,13}_{w}R^{+,23}_{w}$ (where the indices indicate in which components of (6.2) they act), are strictly upper triangular.

In the three-component case we have two types of qKZ operators $\hbar^{\lambda}_{(3)}\mathsf{R}^{+,13}_{w}\mathsf{R}^{+,23}_{w}$ and $\hbar^{-\lambda}_{(1)}\mathsf{R}^{+,13}_{w}\mathsf{R}^{+,12}_{w}$ which correspond to the coproducts of the wall qKZ operators in the first or the second component.

Proposition 7. *If there exists a strictly upper triangular operator $J(\lambda) \in End(K_{\mathsf{T}}(X^{\mathsf{A}}))$ satisfying:*

$$\begin{aligned} J(\lambda)\,\hbar^{\lambda}_{(3)}\mathsf{R}^{+,13}_{w}\mathsf{R}^{+,23}_{w} &= \hbar^{\lambda}_{(3)}\hbar^{\Omega_{13}+\Omega_{23}}\,J(\lambda)\\ J(\lambda)\,\hbar^{-\lambda}_{(1)}\mathsf{R}^{+,13}_{w}\mathsf{R}^{+,12}_{w} &= \hbar^{-\lambda}_{(1)}\hbar^{\Omega_{13}+\Omega_{12}}\,J(\lambda)\end{aligned} \tag{6.3}$$

or strictly lower-triangular operator $J(\lambda) \in End(K_{\mathsf{T}}(X^{\mathsf{A}}))$ satisfying

$$\mathsf{R}^{-,23}_{w}\mathsf{R}^{-,13}_{w}\hbar^{\lambda}_{(3)}J(\lambda) = J(\lambda)\hbar^{\Omega_{23}+\Omega_{13}}\,\hbar^{\lambda}_{(3)}$$

$$\mathsf{R}^{-,12}_{w}\mathsf{R}^{-,13}_{w}\,\hbar^{-\lambda}_{(1)}\,J(\lambda) = J(\lambda)\hbar^{\Omega_{12}+\Omega_{13}}\,\hbar^{-\lambda}_{(1)}$$

then it is unique.

Proof. We prove the upper-triangular case. The lower-triangular case is similar. Following [13] we introduce the operators:

$$A_R(X) = \hbar^{-\Omega_{13}-\Omega_{23}}\hbar_{(3)}^{-\lambda} X \hbar_{(3)}^{\lambda} \mathsf{R}_w^{+,13}\mathsf{R}_w^{+,23},$$

$$A_L(X) = \hbar^{-\Omega_{13}-\Omega_{12}}\hbar_{(1)}^{\lambda} X \hbar_{(1)}^{-\lambda} \mathsf{R}_w^{+,13}\mathsf{R}_w^{+,12}$$

Assume that there exist operator $J(\lambda)$ satisfying conditions of the proposition. Then, obviously $A_R A_L(J(\lambda)) = J(\lambda)$. It is enough to check that the solution for this equation is unique. We are given that $J(\lambda) = \bigoplus_{\substack{\langle\alpha,\theta\rangle>0\\ \langle\beta,\theta\rangle<0}} J_{\alpha,\beta}(\lambda)$, and thus this equation has the following form in components:

$$J_{\alpha,\beta}(\lambda) = Ad_{\hbar_{(1)}^{\lambda}\hbar_{(3)}^{-\lambda}\hbar^{-\bar{\Omega}}}\Big(J_{\alpha,\beta}(\lambda)\Big) + \cdots \tag{6.4}$$

where $\bar{\Omega} = 2\,\Omega_{13} + \Omega_{23} + \Omega_{12}$ and $\cdots$ stands for the lower terms $J_{\alpha',\beta'}(\lambda)$ with

$$\langle\alpha' - \beta', \theta\rangle < \langle\alpha - \beta, \theta\rangle$$

Note that the operator $1 - Ad_{\hbar_{(1)}^{\lambda}\hbar_{(3)}^{-\lambda}\hbar^{-\bar{\Omega}}}$ is invertible for generic λ. This means that all $J_{\alpha,\beta}(\lambda)$ can be expressed trough the lowest term $J_{0,0}(\lambda) = 1$ and therefore they are uniquely determined by (6.4). □

Let $J(\lambda)$ be as in Proposition 5. It is obvious that $J^+(\lambda + \hat{\kappa}_{(3)})^{12}J^+(\lambda)^{12,3}$ is a solution of $A_R(X) = X$. Similarly $J^+(\lambda - \hat{\kappa}_{(1)})^{23}J^+(\lambda)^{1,23}$ is the solution of $A_L(X) = X$. Thus, by the previous proposition, to prove Theorem 6 it is enough to prove the following lemma:

Lemma 1.

$$X = J^+(\lambda + \hat{\kappa}_{(3)})^{12}J^+(\lambda)^{12,3} \quad \textit{is a solution of} \quad A_L(X) = X$$

$$X = J^+(\lambda - \hat{\kappa}_{(1)})^{23}J^+(\lambda)^{1,23} \quad \textit{is a solution of} \quad A_R(X) = X$$

Proof. 1) As noted above the element $X = J^+(\lambda + \hat{\kappa}_{(3)})^{12}J^+(\lambda)^{12,3}$ is a solution of $A_R(X) = X$. Note, that A_R and A_L commute (due to Yang-Baxter equation for R_w^+). Thus, $Y = A_L(X)$ is also a solution of this equation. The solution of $A_R(X) = X$ is uniquely determined by the degree zero part in the third component. Denote this comportment of X by X_0 and similarly for Y by Y_0. Enough to prove that $X_0 = Y_0$. For X_0 we obtain:

$$X_0 = J^+(\lambda + \hat{\kappa}_{(3)})^{12}$$

For Y_0 we have:

$$Y_0 = \hbar^{-\Omega_{13}-\Omega_{12}} \hbar^{\lambda}_{(1)} J^+(\lambda + \hat{\kappa}_{(3)})^{12} \, \hbar^{-\lambda}_{(1)} \hbar^{\Omega_{13}} \mathsf{R}^{+,12}_w \tag{6.5}$$

Denote by $Z = J^+(\lambda+\hat{\kappa}_{(3)})^{12}$. By triangularity of R-matrix and $J(\lambda)$ it factors $Z = \bigoplus\limits_{\alpha\in\mathbb{N}^I} Z_\alpha$ with:

$$Z_\alpha : K_G\Big(\mathscr{M}(\mathsf{v}_1, \mathsf{w}_1) \times \mathscr{M}(\mathsf{v}_2, \mathsf{w}_2) \times \mathscr{M}(\mathsf{v}_3, \mathsf{w}_3)\Big)$$

$$\to K_G\Big(\mathscr{M}(\mathsf{v}_1 + \alpha, \mathsf{w}_1) \times \mathscr{M}(\mathsf{v}_2 - \alpha, \mathsf{w}_2) \times \mathscr{M}(\mathsf{v}_3, \mathsf{w}_3)\Big)$$

Thus, by definition of codimension function (2.27) we have $\hbar^{-\Omega_{13}} Z_\alpha \hbar^{\Omega_{13}} = \hbar^{m_\alpha} Z_\alpha$ where

$$m_\alpha =$$

$$\tfrac{1}{2}\Big(\langle \mathsf{v}_1, \mathsf{w}_3\rangle + \langle \mathsf{v}_3, \mathsf{w}_1\rangle - \langle \mathsf{v}_1, C\mathsf{v}_3\rangle\Big) - \tfrac{1}{2}\Big(\langle \mathsf{v}_1 + \alpha, \mathsf{w}_3\rangle + \langle \mathsf{v}_3, \mathsf{w}_1\rangle - \langle \mathsf{v}_1 + \alpha, C\mathsf{v}_3\rangle\Big)$$

$$= \langle \alpha, \kappa_{(3)}\rangle$$

with $\kappa_{(3)} = (C\mathsf{v}_3 - \mathsf{w}_3)/2$. Therefore, using the dynamical notations we can write the equation (6.5) in the form:

$$Y_0 = \hbar^{-\Omega_{12}} \hbar^{\lambda+\hat{\kappa}_{(3)}}_{(1)} J^+(\lambda + \hat{\kappa}_{(3)})^{12} \, \hbar^{-\lambda-\hat{\kappa}_{(3)}}_{(1)} \mathsf{R}^{+,12}_w$$

As $J^+(\lambda)$ is satisfies condition of Proposition 5 we obtain $Y_0 = J^+(\lambda + \hat{\kappa}_{(3)})^{12}$. Therefore $Y = X$. The second equality is proved analogously. □

6.2 Coproduct of $\mathbf{B}_w(\lambda)$

6.2.1

Let us consider the operators:

$$\tilde{B}'_w(\lambda) = \mathbf{m}\Big(1 \otimes S_w(J^-_w(\lambda)^{-1})\Big), \quad B'_w(\lambda) = \mathbf{m}_{21}\Big(S^{-1}_w \otimes 1(J^-_w(\lambda)^{-1})\Big)$$

where S_w is the antipode of $\mathscr{U}_\hbar(\mathfrak{g}_w)$ and $\mathbf{m}(a\otimes b) \overset{def}{=} ab$, $\mathbf{m}_{21}(a \otimes b) \overset{def}{=} ba$. We define:

$$\tilde{B}_w(\lambda) = \tilde{B}'_w(\lambda + \hat{\kappa}), \quad B_w(\lambda) = B'_w(\lambda - \hat{\kappa}) \tag{6.6}$$

with κ as in the Theorem 6. Then, we have:

Theorem 7.

$$1)\quad \Delta_w \tilde{B}_w(\lambda) = J_w^-(\lambda)\left(\tilde{B}_w(\lambda+\hat{\kappa}_{(2)})\otimes \tilde{B}_w(\lambda-\hat{\kappa}_{(1)})\right)J_w^+(\lambda)$$

$$2)\quad \Delta_w B(\lambda) = J_w^-(\lambda)\left(B_w(\lambda+\hat{\kappa}_{(2)})\otimes B_w(\lambda-\hat{\kappa}_{(1)})\right)J_w^+(\lambda)$$

Proof. Let $X(\lambda) = J_w^-(\lambda)^{-1}$. By Theorem 6:

$$X^{12}(\lambda+\hat{\kappa}_{(3)})X^{12,3}(\lambda) = X^{23}(\lambda-\hat{\kappa}_{(1)})X^{1,23}(\lambda) \tag{6.7}$$

By our conventions from Section 5.0.6.1, the operators $J(\lambda)$ and $X(\lambda)$ are rational functions of $\hbar^{(\lambda,m)} = z^m = z_1^{m_1}\cdots z_1^{m_n}$ where $m=(m_1,\cdots,m_n)$ is multi-index. We write them as power series convergent near $z_i = 0$:

$$X(\lambda) = \sum_{i,m} a_{i,m}\otimes b_{i,m}z^m, \quad J_w^-(\lambda) = X^{-1}(\lambda) = \sum_{i,m}\bar{a}_{i,m}\otimes \bar{b}_{i,m}z^m$$

then

$$\tilde{B}'_w(\lambda) = \mathbf{m}\Big(1\otimes S_w(X(\lambda))\Big) = \sum_{i,m} a_{i,m}S_w(b_{i,m})z^m$$

and in the sumless Sweedler notation we have:

$$\Delta_w \tilde{B}'_w(\lambda) = \sum_{i,m} a_{i,m}^{(1)}S_w(b_{i,m}^{(2)})\otimes a_{i,m}^{(2)}S_w(b_{i,m}^{(1)})z^m$$

We denote by $\hat{A}$ the following contraction:

$$\hat{A}(a_1\otimes a_2\otimes a_3\otimes a_4) = a_1S_w(a_4)\otimes a_2S_w(a_3),$$

then, obviously $\Delta_w\tilde{B}'_w(\lambda) = \hat{A}(\Delta_w\otimes\Delta_w(X))$. From (6.7) we have:

$$\Delta_w\otimes 1(X(\lambda)) = X^{12}(\lambda+\hat{\kappa}_{(3)})^{-1}X^{23}(\lambda-\hat{\kappa}_{(1)})X^{1,23}(\lambda)$$

or in the components:

$$\Delta_w\otimes 1(X(\lambda)) = \sum\left(\bar{a}_{i,m}\otimes\bar{b}_{i,m}\otimes K^m\right)\left(K^{-s}\otimes a_{j,s}\otimes b_{j,s}\right)\left(a_{k,l}\otimes b_{k,l}^{(1)}\otimes b_{k,l}^{(2)}\right)z^{m+s+l} =$$

$$= \sum\left(\bar{a}_{i,m}K^{-s}a_{k,l}\otimes\bar{b}_{i,m}a_{j,s}b_{k,l}^{(1)}\otimes K^m b_{j,s}b_{k,l}^{(2)}\right)z^{m+s+l}$$

where we denoted by $K = \hbar^\kappa$. Now, $\Delta_w\otimes\Delta_w = (1\otimes 1\otimes\Delta_w)(\Delta_w\otimes 1)$ and therefore:

$$\Delta_w\otimes\Delta_w X(z) = \sum\left(\bar{a}_{i,m}K^{-s}a_{k,l}\otimes\bar{b}_{i,m}a_{j,s}b_{k,l}^{(1)}\otimes K^m b_{j,s}^{(1)}b_{k,l}^{(2),(1)}\otimes K^m b_{j,s}^{(2)}b_{k,l}^{(2),(2)}\right)z^{m+s+l}$$

Applying contraction $\hat{A}$, taking into account that antipode S_w is antihomomorphism and $S_w(K) = K^{-1}$ by (3.8) we obtain:

$$\hat{A}(\Delta_w \otimes \Delta_w X) =$$

$$\sum \bar{a}_{i,m} K^{-s} a_{k,l} S_w(b_{k,l}^{(2),(2)}) S_w(b_{j,s}^{(2)}) K^{-m} \otimes \bar{b}_{i,m} a_{j,s} b_{k,l}^{(1)} S_w(b_{k,l}^{(2),(1)}) S_w(b_{j,s}^{(1)}) K^{-m}\, z^{m+s+l}$$

$$= J_w^-(\lambda - \hat{\kappa}_{(1)} - \hat{\kappa}_{(2)}) \sum K^{-s} a_{k,l} S_w(b_{k,l}^{(2),(2)}) S_w(b_{j,s}^{(2)}) \otimes a_{j,s} b_{k,l}^{(1)} S_w(b_{k,l}^{(2),(1)}) S_w(b_{j,s}^{(1)})\, z^{s+l}$$

where $J_w^-(\lambda - \hat{\kappa}_{(1)} - \hat{\kappa}_{(2)}) = \sum \bar{a}_{i,m} K^{-m} \otimes \bar{b}_{i,m} K^{-m} z^m$ and in the last step we used that the whole operator is weight zero and therefore commutes with $K \otimes K$. From the simple Lemma (2) we obtain:

$$\hat{A}(\Delta_w \otimes \Delta_w X(\lambda)) =$$

$$J_w^-(\lambda - \hat{\kappa}_{(1)} - \hat{\kappa}_{(2)}) \sum K^{-s} a_{k,l} S_w(b_{k,l}) S_w(b_{j,s}^{(2)}) \otimes a_{j,s} S_w(b_{j,s}^{(1)})\, z^{s+l} =$$

$$J_w^-(\lambda - \hat{\kappa}_{(1)} - \hat{\kappa}_{(2)}) \tilde{B}'_w(\lambda) \otimes 1 \cdot \Big(\sum K^{-s} S_w(b_{j,s}^{(2)}) \otimes a_{j,s} S_w(b_{j,s}^{(1)})\, z^s \Big)$$

Let us consider the contraction defined by $\hat{P}(a_1 \otimes a_2 \otimes a_3) = S_w(a_3) \otimes a_1 S_w(a_2)$. Then for the expression in the brackets in the last formula we have:

$$\sum K^{-s} S_w(b_{j,s}^{(2)}) \otimes a_{j,s} S_w(b_{j,s}^{(1)})\, z^s = \hat{P}\Big(X_{1,23}(\lambda + \hat{\kappa}_{(3)}) \Big)$$

Again, by from (6.7) we have:

$$X^{1,23}(\lambda + \kappa_{(3)}) = X^{23}(\lambda - \hat{\kappa}_{(1)} + \hat{\kappa}_{(3)})^{-1} X^{12}(\lambda + 2\hat{\kappa}_{(3)}) X^{12,3}(\lambda + \hat{\kappa}_{(3)})$$

$$= \sum K^{-m} a_{j,s} a_{k,l}^{(1)} \otimes \bar{a}_{i,m} b_{j,s} a_{k,l}^{(2)} \otimes \bar{b}_{i,m} K^{m+2s} b_{k,l} K^l z^{s+m+l}.$$

Thus

$$\hat{P}\Big(X^{1,23}(\lambda + \hat{\kappa}_{(3)}) \Big) =$$

$$\sum K^{-l} S_w(b_{k,l}) K^{-m-2s} S_w(\bar{b}_{i,m}) \otimes K^{-m} a_{j,s} a_{k,l}^{(1)} S(a_{k,l}^{(2)}) S_w(b_{j,s}) S_w(\bar{a}_{i,m}) z^{s+m+l}$$

Note, that $a^{(1)}_{k,l}S_w(a^{(2)}_{k,l}) = \epsilon_w(a_{k,l})$, and using the unipotence of $J^-_w(\lambda)$ we find:

$$\hat{P}\Big(X_{1,23}(\lambda+\hat{\kappa}_{(3)})\Big) = \sum K^{-m-2s}S_w(\bar{b}_{i,m})\otimes K^{-m}a_{j,s}S_w(b_{j,s})S_w(\bar{a}_{i,m})z^{s+m}$$

$$= \Big(\sum K^{-2s}\otimes a_{j,s}S_w(b_{j,s})z^s\Big)\Big(\sum K^{-m}S_w(\bar{b}_{i,m})\otimes K^{-m}S_w(\bar{a}_{i,m})z^m\Big)$$

$$= \Big(1\otimes \tilde{B}'_w(\lambda-2\hat{\kappa}_{(1)})\Big)S_w\otimes S_w((J^+_w)_{21})(\lambda-\hat{\kappa}_{(1)}-\hat{\kappa}_{(2)})$$

Overall we obtain the identity:

$$\Delta_w\tilde{B}'(\lambda) = J^-_w(\lambda-\hat{\kappa}_{(1)}-\hat{\kappa}_{(2)})\left(\tilde{B}'_w(\lambda)\otimes\tilde{B}'_w(\lambda-2\hat{\kappa}_{(1)})\right)S_w\otimes S_w((J^-_w)_{21})(\lambda-\hat{\kappa}_{(1)}-\hat{\kappa}_{(3)})$$

Finally, after shifting $\lambda\to\lambda+\hat{\kappa}_{(1)}+\hat{\kappa}_{(2)}$ and using (5.4) we obtain 1). The equation 2) is obtained similarly. □

Lemma 2.

$$\sum S(x^{(2),(2)})\otimes x^{(1)}S(x^{(2),(1)}) = S(x)\otimes 1 \tag{6.8}$$

Proof. Consider the contraction $\hat{C}(a_1\otimes a_2\otimes a_3) = S(a_3)\otimes a_1S(a_2)$ then, obviously

$$\sum S(x^{(2),(2)})\otimes x^{(1)}S(x^{(2),(1)}) = \hat{C}\Big(1\otimes\Delta(\Delta x)\Big) = \hat{C}\Big(\Delta\otimes 1(\Delta x)\Big)$$

$$= S(x^{(2)})\otimes x^{(1)(1)}S(x^{(1)(2)}) = S(x^{(2)})\otimes\epsilon(x^{(1)}) = S(x)\otimes 1$$

□

6.2.2

Corollary 4. *The coproduct of the operator $\boldsymbol{B}_w(\lambda)$ defined by (5.9), has the following form:*

$$\Delta_w(\boldsymbol{B}_w(\lambda)) = \boldsymbol{J}^-_w(\lambda)\left(\boldsymbol{B}_w(\lambda+\hat{\kappa}_{(2)})\otimes\boldsymbol{B}_w(\lambda-\hat{\kappa}_{(1)})\right)\boldsymbol{J}^+_w(\lambda) \tag{6.9}$$

Proof. Shift $\lambda\to\lambda-\tau_w$ and use definitions (5.7) and (5.9). □

6.3 Other properties of $\mathbf{B}_w(\lambda)$

6.3.1

Let us consider the wall qKZ operators as in the Proposition 6:

$$\mathscr{K}_w^+ = T_u \hbar_{(1)}^{-\lambda} \mathsf{R}_w^+, \quad \mathscr{K}_w^- = \mathsf{R}_w^- T_u \hbar_{(1)}^{-\lambda} \tag{6.10}$$

acting in the tensor product of two evaluation modules of $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$.

Proposition 8.

$$\mathscr{K}_w^- \, \Delta_w(\boldsymbol{B}_w(\lambda)) = \Delta_w(\boldsymbol{B}_w(\lambda)) \, \mathscr{K}_w^+ \tag{6.11}$$

Proof. We have

$$\mathscr{K}_w^- \Delta_w(\mathbf{B}_w(\lambda)) \overset{(6.9)}{=}$$

$$\mathsf{R}_w^- T_u \hbar_{(1)}^{-\lambda} \mathbf{J}_w^-(\lambda) \Big(\mathbf{B}_w(\lambda + \hat{\kappa}_{(2)}) \otimes \mathbf{B}_w(\lambda - \hat{\kappa}_{(1)}) \Big) \, \mathbf{J}_w^+(\lambda) \overset{(5.8)}{=}$$

$$\mathbf{J}_w^-(\lambda) T_u \hbar_{(1)}^{-\lambda} \hbar^{\Omega} \Big(\mathbf{B}_w(\lambda + \hat{\kappa}_{(2)}) \otimes \mathbf{B}_w(\lambda - \hat{\kappa}_{(1)}) \Big) \, \mathbf{J}_w^+(\lambda) =$$

$$\mathbf{J}_w^-(\lambda) \Big(\mathbf{B}_w(\lambda + \hat{\kappa}_{(2)}) \otimes \mathbf{B}_w(\lambda - \hat{\kappa}_{(1)}) \Big) \, T_u \hbar_{(1)}^{-\lambda} \hbar^{\Omega} \mathbf{J}_w^+(\lambda) \overset{(5.8)}{=}$$

$$\mathbf{J}_w^-(\lambda) \Big(\mathbf{B}_w(\lambda + \hat{\kappa}_{(2)}) \otimes \mathbf{B}_w(\lambda - \hat{\kappa}_{(1)}) \Big) \mathbf{J}_w^+(\lambda) \hbar_{(1)}^{-\lambda} T_u \mathsf{R}_w^+ =$$

$$\Delta_w(\mathbf{B}_w(\lambda)) \, \mathscr{K}_w^+$$

□

6.3.2

Proposition 9. *For $\mathscr{L} \in \mathrm{Pic}(X)$ the operators $\boldsymbol{B}_w(\lambda)$ satisfy:*

$$\mathscr{L} T_{\mathscr{L}}^{-1} \, \boldsymbol{B}_w(\lambda) = \boldsymbol{B}_{w+\mathscr{L}}(\lambda) \, \mathscr{L} T_{\mathscr{L}}^{-1} \tag{6.12}$$

with shift operators $T_{\mathscr{L}}$ defined by (5.2).

Proof. Let A be a torus splitting the framing $\mathsf{w} = u'\mathsf{w}' + u''\mathsf{w}''$. We a consider Nakajima variety $X = \mathscr{M}(\mathsf{v}, \mathsf{w})$. The components of X^{A} are of the form $F_i = \mathscr{M}(\mathsf{v}'_i, \mathsf{w}') \times \mathscr{M}(\mathsf{v}''_i, \mathsf{w}'')$. Let us consider the operators:

$$S_{\mathfrak{C},s} = i^*_{X^{\mathsf{A}}} \circ \mathrm{Stab}_{\mathfrak{C},T^{1/2},s} : K_G(X^{\mathsf{A}}) \longrightarrow K_G(X^{\mathsf{A}})$$

where $i_{X^{\mathsf{A}}}$ is the inclusion map. Let $\mathscr{L} \in \mathrm{Pic}(X)$ be a line bundle. We denote by $U(\mathscr{L})$ a block diagonal operator acting in $K_G(X^{\mathsf{A}})$ with the following matrix elements:

$$U(\mathscr{L})|_{F_i \times F_i} = \mathscr{L}|_{F_i}$$

Let us consider an operator $\bar{S}_{\mathfrak{C},s} = U(\mathscr{L}) S_{\mathfrak{C},s} U(\mathscr{L})^{-1}$. A conjugation by a diagonal matrix does not change the diagonal elements, thus:

$$\bar{S}_{\mathfrak{C},s}\big|_{F_i \times F_i} = S_{\mathfrak{C},s}|_{F_i \times F_i} \tag{6.13}$$

For the non-diagonal elements we have:

$$\begin{aligned} \deg_{\mathsf{A}} \Big(\bar{S}_{\mathfrak{C},s}\big|_{F_2 \times F_1} \Big) &= \deg_{\mathsf{A}} \Big(S_{\mathfrak{C},s}|_{F_2 \times F_1} \frac{\mathscr{L}|_{F_2}}{\mathscr{L}|_{F_1}} \Big) \\ &\overset{(2.5)}{\subset} \deg_{\mathsf{A}} \Big(S_{\mathfrak{C},s}|_{F_2 \times F_2} \frac{s \otimes \mathscr{L}|_{F_2}}{s \otimes \mathscr{L}|_{F_1}} \Big) \overset{(6.13)}{=} \deg_{\mathsf{A}} \Big(\bar{S}_{\mathfrak{C},s}\big|_{F_2 \times F_2} \frac{s \otimes \mathscr{L}|_{F_2}}{s \otimes \mathscr{L}|_{F_1}} \Big) \end{aligned} \tag{6.14}$$

Note, that the stable map is defined uniquely by these restrictions and thus we conclude: $\bar{S}_{\mathfrak{C},s} = S_{\mathfrak{C},s+\mathscr{L}}$.

Recall that the wall R-matrices are defined by $R^{\pm}_w = S^{-1}_{\pm,s_2} S_{\pm,s_1}$ for two slopes s_1 and s_2 separated by a single wall w. Therefore:

$$U(\mathscr{L}) R^{\pm}_w U(\mathscr{L})^{-1} = R^{\pm}_{w+\mathscr{L}}.$$

Conjugating both sides of ABRR equation (5.3) by $U(\mathscr{L})$ we get:

$$\mathsf{R}^-_{w+\mathscr{L}} \hbar^{-\lambda}_{(1)} U(\mathscr{L}) J^-_w(\lambda) U(\mathscr{L})^{-1} = U(\mathscr{L}) J^-_w(\lambda) U(\mathscr{L})^{-1} \hbar^{-\lambda}_{(1)} \hbar^{\Omega}$$

Thus, by uniqueness of the solution of this equation:

$$U(\mathscr{L}) J^-_w(\lambda) U(\mathscr{L})^{-1} = J^-_{w+\mathscr{L}}(\lambda) \tag{6.15}$$

Without a loss of generality we can assume that $\mathscr{L}=\det(\mathcal{V}_k)$ is the k-th tautological line bundle. Then, we have:

$$U(\mathscr{L})=\tilde{\mathscr{L}}\otimes\tilde{\mathscr{L}}$$

Where $\tilde{\mathscr{L}}$ is the same tautological bundle twisted by some powers of trivial line bundles u' and u'': explicitly for the component $F=\mathscr{M}(\mathsf{v}',\mathsf{w}')\times\mathscr{M}(\mathsf{v}'',\mathsf{w}'')$ we have: $\mathscr{L}|_F=(u')^{\mathsf{v}'_k}\mathscr{L}\otimes(u'')^{\mathsf{v}''_k}\mathscr{L}$.

Let $(J^-_w(\lambda))^{-1}=\sum_i a_i\otimes b_i$, $(J^-_{w+\mathscr{L}}(\lambda))^{-1}=\sum_i a'_i\otimes b'_i$. and denote by $B_w(\lambda)=\sum_i a_iS_w(b_i)$. Then we have:

$$\mathscr{L}B_w(\lambda)\mathscr{L}^{-1}=\tilde{\mathscr{L}}B_w(\lambda)\tilde{\mathscr{L}}^{-1}=\textstyle\sum_i\tilde{\mathscr{L}}a_iS_w(\tilde{\mathscr{L}}a_i)=$$

$$\mathbf{m}\Big(1\otimes S_w(\textstyle\sum_i\tilde{\mathscr{L}}a_i\otimes\tilde{\mathscr{L}}b_i)\Big)\overset{(6.15)}{=}\mathbf{m}\Big(1\otimes S_w(\textstyle\sum_i a'_i\tilde{\mathscr{L}}\otimes b'_i\tilde{\mathscr{L}})\Big)$$

$$=\textstyle\sum_i a'_iS_w(b'_i)=B_{w+\mathscr{L}}(\lambda).$$

In the first equality we substituted $\mathscr{L}$ by $\tilde{\mathscr{L}}$ because for one component case the u-factors cancels. Thus we proved that:

$$\mathscr{L}B_w(\lambda)=B_{w+\mathscr{L}}(\lambda)\mathscr{L}$$

Finally, note that $B_w(\lambda)=\mathbf{B}_w(\lambda+\tau_w)$ and thus:

$$\mathscr{L}\mathbf{B}_w(\lambda+\tau_w)=\mathbf{B}_{w+\mathscr{L}}(\lambda+\tau_{w+\mathscr{L}})\mathscr{L}$$

By definition $\tau_{w+\mathscr{L}}-\tau_w=\mathbf{s}\mathscr{L}$ thus, after substitution $\lambda\to\lambda-\tau_w-\mathbf{s}\mathscr{L}$ we obtain:

$$\mathscr{L}\mathbf{B}_w(\lambda-\mathbf{s}\mathscr{L})=\mathbf{B}_{w+\mathscr{L}}(\lambda)\mathscr{L}$$

which gives (6.12). □

6.4 Proof of Theorem 4

6.4.1

We have

$$\mathscr{K}^s=\hbar^{\lambda}_{(1)}T_u^{-1}\mathscr{R}^s(u),\quad \mathscr{K}^{s'}=\hbar^{\lambda}_{(1)}T_u^{-1}\mathscr{R}^{s'}(u),\quad W=\Delta_w(\mathbf{B}_w(\lambda))(R^+_w)^{-1}.$$

We need to check that $\mathscr{K}^s W = W \mathscr{K}^{s'}$. We have:

$$\mathscr{K}^s W = \hbar^{\lambda}_{(1)} T_u^{-1} \mathscr{R}^s(u) \Delta_w(\mathbf{B}_w(\lambda))(R_w^+)^{-1} =$$

$$\hbar^{\lambda}_{(1)} T_u^{-1} \Delta_w^{op}(\mathbf{B}_w(\lambda)) \mathscr{R}^s(u)(R_w^+)^{-1} =$$

$$\hbar^{\lambda}_{(1)} T_u^{-1} (\mathsf{R}^-)^{-1} \mathsf{R}^- \Delta_w^{op}(\mathbf{B}_w(\lambda)) \mathscr{R}^s(u)(R_w^+)^{-1} =$$

$$\hbar^{\lambda}_{(1)} T_u^{-1} (\mathsf{R}^-)^{-1} \Delta_w(\mathbf{B}_w(\lambda)) \hbar^{\Omega} R_w^- \mathscr{R}^s(u)(R_w^+)^{-1} \overset{(6.11)}{=}$$

$$\Delta_w(\mathbf{B}_w(\lambda))(R_w^+)^{-1} \hbar^{\lambda}_{(1)} T_u^{-1} R_w^- \mathscr{R}^s(u)(R_w^+)^{-1} = W \mathscr{K}^{s'}$$

where the last equality uses $\mathscr{R}^{s'}(u) = R_w^- \mathscr{R}^s(u)(R_w^+)^{-1}$ because by assumption s and s' are separated by a single wall w.

6.4.2

The following proposition describes the action of the difference operator $\mathscr{A}^s_{\mathscr{L}}$ in the tensor product of two $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ modules.

Proposition 10.

$$\Delta_s(\mathscr{A}^s_{\mathscr{L}}) = W_{w_0}(\lambda) W_{w_1}(\lambda) \cdots W_{w_{m-1}}(\lambda) \Delta_\infty(\mathscr{L}) T_{\mathscr{L}}^{-1}$$

where $w_0, \cdots, w_{m-1}$ is the ordered set of walls separating line bundles s and $s + \mathscr{L}$, $W_w(\lambda) = \Delta_w(\boldsymbol{B}_w(\lambda))(R_w^+)^{-1}$ and Δ_∞ is the infinite slope coproduct from Section 3.3.5.

Proof. First, by definition (5.11) we have:

$$\mathscr{A}^s_{\mathscr{L}} = T_{\mathscr{L}}^{-1} \mathscr{L}\, \mathbf{B}_{w_{-m}}(\lambda) \mathbf{B}_{w_{-2}}(\lambda) \cdots \mathbf{B}_{w_{-1}}(\lambda)$$

Where, we denote by $w_{-1}, \cdots w_{-m}$ the ordered set of walls between the slope s and $s - \mathscr{L}$. By Proposition 9 we know that $T_{\mathscr{L}}^{-1} \mathscr{L} \mathbf{B}_{w_k}(\lambda) = \mathbf{B}_{w_{k+m}}(\lambda) T_{\mathscr{L}}^{-1} \mathscr{L}$ and thus we obtain:

$$\mathscr{A}^s_{\mathscr{L}} = \mathbf{B}_{w_0}(\lambda) \mathbf{B}_{w_1}(\lambda) \cdots \mathbf{B}_{w_m}(\lambda)\, \mathscr{L}\, T_{\mathscr{L}}^{-1}$$

with walls w_i as in the proposition.

Next, for coproduct we have:

$$\Delta_s(\mathscr{A}^s_{\mathscr{L}}) = \Delta_s(\mathbf{B}_{w_0}(\lambda)\mathbf{B}_{w_1}(\lambda)\cdots\mathbf{B}_{w_m}(\lambda)\mathscr{L})T_{\mathscr{L}}^{-1}$$

and by (3.6) the coproducts at different slopes are related as follows

$$\Delta_s(\mathbf{B}_{w_k}(\lambda)) = (R^+_{w_0})^{-1}\cdots(R^+_{w_{k-1}})^{-1}\Delta_{w_k}(\mathbf{B}_{w_k})R^+_{w_{k-1}}\cdots R^+_{w_0}.$$

Thus we obtain:

$$\mathscr{A}^s_{\mathscr{L}} = \Delta_{w_0}(\mathbf{B}_{w_0}(\lambda))(R^+_{w_0})^{-1}\cdots\Delta_{w_m}(\mathbf{B}_{w_m}(\lambda))(R^+_{w_m})^{-1}R^+_{w_m}\cdots R^+_{w_0}\Delta_s(\mathscr{L})T_{\mathscr{L}}^{-1}$$

The proposition follows from next Lemma. □

Lemma 3. *Let $w_0,\cdots,w_m$ be the ordered set of walls between s and $s+\mathscr{L}$. Then we have:*

$$\Delta_\infty(\mathscr{L}) = R^+_{w_m}\cdots R^+_{w_0}\Delta_s(\mathscr{L}) \tag{6.16}$$

Proof. By (3.6) the coproducts are related as follows:

$$\Delta_s(\mathscr{L}) = (R^+_{w_0})^{-1}\cdots(R^+_{\infty})^{-1}\Delta_\infty(\mathscr{L})R^+_{\infty}\cdots R^+_{w_0}$$

By definition $\Delta_\infty(\mathscr{L}) = \mathscr{L}\otimes\mathscr{L}$. In particular,

$$\Delta_\infty(\mathscr{L})R^+_{w_k}\Delta_\infty(\mathscr{L})^{-1} = R^+_{w_k+\mathscr{L}} = R^+_{w_{k+m}}$$

We use this identity to cancel all but finitely many factors in the previous expression. The lemma is proven. □

6.4.3

Let s and s' be two slopes separated by a single wall w_0. We choose a path from slope s to $s+\mathscr{L}$ crossing some sequence of walls $w_0, w_1..., w_m$. Similarly, for the path from s' to $s'-\mathscr{L}$ crosses the walls $w_1, w_2..., w_{m+1}$ with $w_m = w_0+\mathscr{L}$. By Proposition 10 we have:

$$\Delta_s(\mathscr{A}^s_{\mathscr{L}}) = W_{w_0}(\lambda)\cdots W_{w_m}(\lambda)\,\Delta_\infty(\mathscr{L})\,T_{\mathscr{L}}^{-1}$$

$$\Delta_{s'}(\mathscr{A}^{s'}_{\mathscr{L}}) = W_{w_1}(\lambda)\cdots W_{w_{m+1}}(\lambda)\,\Delta_\infty(\mathscr{L})\,T_{\mathscr{L}}^{-1}$$

To finish the proof of the Theorem 4 we need to check that

$$W_{w_0}(\lambda)^{-1}\Delta_s(\mathscr{A}^s_{\mathscr{L}})W_{w_0}(\lambda) = \Delta_{s'}(\mathscr{A}^{s'}_{\mathscr{L}}).$$

But this is obvious.

6.5 Proof of Corollary 3

In fact, the statement of the Corollary 3 follows immediately from Proposition 10. Indeed, we obtain

$$\mathscr{K}^s \Delta_s(\mathscr{A}^s_{\mathscr{L}}) = \mathscr{K}^s W_{w_0}(\lambda) W_{w_1}(\lambda) \cdots W_{w_n}(\lambda) \Delta_\infty(\mathscr{L}) T^{-1}_{\mathscr{L}} \overset{(5.13)}{=}$$

$$W_{w_0}(\lambda) W_{w_1}(\lambda) \cdots W_{w_n}(\lambda) \mathscr{K}^{s+\mathscr{L}} \Delta_\infty(\mathscr{L}) T^{-1}_{\mathscr{L}} = \Delta_s(\mathscr{A}^s_{\mathscr{L}}) \mathscr{K}^s$$

However, in this section we will consider alternative proof. The Theorem 4 says that the difference operators for different choices of slopes are conjugated. Thus, to prove the commutativity it is enough to check it for some particular slope. The natural choice is the "infinite" slope. The aim of this section is to illustrate the general principle: at the infinite slope the difference operators become trivial and the commutativity is obvious.

6.5.1

First, by Theorem 4 we have $X \mathscr{A}^s_{\mathscr{L}} X^{-1} = \mathscr{A}^\infty_{\mathscr{L}}$ where X is a product of wall operators W. As $\mathscr{A}^\infty_{\mathscr{L}}$ does not depend on the choice of a path (its exact form is established in the next section) we conclude that $\mathscr{A}^s_{\mathscr{L}}$ does not depend on this choice.

6.5.2

To compute $\mathscr{A}^\infty_{\mathscr{L}}$ we use the following proposition.

Proposition 11. *If $J^-_w(\lambda)$ is the strictly lower-triangular solution of ABRR equation then:*

$$\lim_{\lambda \to -\infty} J^-_w(\lambda) = 1$$

Proof. By definition, the limit $\lambda \to -\infty$ means that λ goes to infinity in the opposite ample cone. For a Nakajima variety the line bundle corresponding to the stability parameter $\theta \in \mathbb{Z}^I$ is ample. Thus we need to prove that for $\lambda = x\theta$ and $x \in \mathbb{R}$ we have:

$$\lim_{x \to -\infty} J^-_w(x\theta) = 1$$

As the operator J^-_w is strictly lower-triangular, this is equivalent to:

$$\lim_{x \to -\infty} J^-_w(x\theta)_\alpha = 0 \quad \text{for} \quad \alpha \neq 0$$

We prove it by induction on the value of $\langle\alpha,\theta\rangle$. The ABRR equation takes the form:

$$\hbar_{(1)}^{-x\theta} J_w^-(x\theta)\hbar_{(1)}^{x\theta} = (\mathsf{R}_w^-)^{-1} J_w^-(x\theta)\hbar^{\Omega}$$

or, in the components:

$$J_w^-(x\theta)_\alpha \hbar^{-x\langle\alpha,\theta\rangle} = J_w^-(x\theta)_\alpha + \quad \text{lower terms}$$

where the lower terms are $J_w^-(x\theta)_{\alpha'}$ with $\langle\alpha',\theta\rangle > \langle\alpha,\theta\rangle$. Therefore:

$$\lim_{x\to-\infty} J_w^-(x\theta)_\alpha = \lim_{x\to-\infty} \frac{1}{\hbar^{-x\langle\alpha,\theta\rangle}-1}\Big(\text{lower terms}\Big)$$

By induction, the lower terms vanish in this limit. By lower-trianguality of $J_w^-(\lambda)$ we have $\langle\alpha,\theta\rangle < 0$, thus the prefactor vanishes as well. □

Corollary 5.

$$\mathscr{A}_{\mathscr{L}}^{\infty} = \mathscr{L} T_{\mathscr{L}}^{-1} \tag{6.17}$$

Proof. To compute the operator $\mathbf{B}_{\mathscr{L}}^s(\lambda)$ at the infinite slope $s=\infty$ we note that all walls that contribute to the product (5.11) correspond to $\tau_w \to \infty$. By definition (5.7) we have:

$$\lim_{\tau_w\to\infty} \mathbf{J}_w^-(\lambda) = \lim_{\tau_w\to\infty} J_w^-(\lambda-\tau_w) = 1$$

where the second equality is by previous proposition. By (5.9) we obtain $\lim\limits_{\tau_w\to\infty} \mathbf{B}_w(\lambda) = 1$. □

Now, it is obvious that the operators $\mathscr{A}_{\mathscr{L}}^{\infty}$ commute among themselves. But, by Theorem 4 it means that they commute for arbitrary slope

$$[\mathscr{A}_{\mathscr{L}}^s, \mathscr{A}_{\mathscr{L}'}^s] = 0.$$

6.5.3

Let A be a torus splitting the framing $\mathsf{w} = u\mathsf{w}' + \mathsf{w}''$. Let $\mathscr{K}^{\infty} = \hbar_{(1)}^{\lambda} T_u^{-1} \mathscr{R}^{\infty}(u)$ be the corresponding qKZ operator with infinite slope acting in $K_G(\mathscr{M}(\mathsf{w}')) \otimes K_G(\mathscr{M}(\mathsf{w}''))$. By the previous section the operator $\mathscr{A}_{\mathscr{L}}^{\infty}$ acts on this space by $\mathscr{A}_{\mathscr{L}}^{\infty} = \Delta_\infty(\mathscr{L}) T_{\mathscr{L}}^{-1}$. Without a loss of generality we assume that $\mathscr{L} = \det(\mathcal{V}_k)$ is the k-th tautological line bundle. The coproduct at the infinite slope splits, meaning that $\Delta_\infty(\mathscr{L}) = \mathscr{L} \otimes \mathscr{L}$.

The R-matrix $\mathscr{R}^{\infty}(u)$ is the operator of multiplication by class (2.24) in the equivariant K-theory. Therefore, $\mathscr{R}^{\infty}(u)$ commutes with $\hbar^{\lambda}_{(i)}$ and the operators of multiplication by $\Delta_{\infty}(\mathscr{L})$. We need to check that $\mathscr{K}^{\infty}\mathscr{A}^{\infty}_{\mathscr{L}}=\mathscr{A}^{\infty}_{\mathscr{L}}\mathscr{K}^{\infty}$ which therefore means:

$$\Delta_{\infty}(\mathscr{L})\, T^{-1}_{z_k}(\hbar^{\lambda}_{(1)}) = \hbar^{\lambda}_{(1)}\, T^{-1}_{u}(\Delta_{\infty}(\mathscr{L})) \tag{6.18}$$

and follows from definition of $\hbar^{\lambda}_{(1)}$.

6.6 Proof of Theorem 5

6.6.1

Let $\mathsf{A}=\mathbb{C}^{\times}$ be a torus splitting the framing space as $\mathsf{w}=u\mathsf{w}'+\mathsf{w}''$ and acting on a Nakajima variety $X=\mathscr{M}(\mathsf{v},\mathsf{w})$. We denote the components of X^{A} by $F_{\mathsf{v}'}=\mathscr{M}(\mathsf{v}',\mathsf{w}')\times\mathscr{M}(\mathsf{v}'',\mathsf{w}'')$. Note, that we label them by the weight in the first component. For a line bundle $\mathscr{L}$ we have two difference operators acting in $K_G(\mathsf{w}')\otimes K_G(\mathsf{w}'')$ and commuting with the qKZ operator $\mathscr{K}^{s}$ for a slope s as in the Theorem 5. First, by Theorem 3:

$$\mathcal{A}_{\mathscr{L}} = T^{-1}_{\mathscr{L}}\mathbf{N}^{s}_{\mathscr{L}}(u,\lambda)$$

for $\mathbf{N}^{s}_{\mathscr{L}}(u,\lambda)=\mathrm{Stab}_{+,T^{1/2},s}\,\mathbf{M}_{\mathscr{L}}(u,\lambda)\,\mathrm{Stab}^{-1}_{+,T^{1/2},s}$ commutes with qKZ operator at the slope s. Second, by the corollary of Theorem 4, the operator:

$$\mathscr{A}_{\mathscr{L}} = T^{-1}_{\mathscr{L}}\mathbf{B}^{s}_{\mathscr{L}}(u,\lambda)$$

commutes with the same qKZ operator. We prove that they coincide up to a constant multiple:

$$\mathbf{B}^{s}_{\mathscr{L}}(u,\lambda)=\mathbf{N}^{s}_{\mathscr{L}}(u,z)Const$$

6.6.2

Both $\mathbf{N}_{\mathscr{L}}(u,z)$ and $\mathbf{B}^{s}_{\mathscr{L}}(u,z)$ are defined in integral K-theory, in particular they and their inverses are Laurent polynomials in u. It follows that the operator:

$$U(u)=\mathbf{B}^{s}_{\mathscr{L}}(u,z)\mathbf{N}^{-1}_{\mathscr{L}}(u,z)$$

is a Laurent polynomial in u. By construction, this operator commutes with qKZ at a slope s which means that:

$$U(uq) = \hbar^{\lambda}_{(1)}\mathscr{R}^s(u)U(u)\left(\hbar^{\lambda}_{(1)}\mathscr{R}^s(u)\right)^{-1}$$

From Khoroshkin-Tolstoy factorization for the slope s R-matrix we obtain:

$$\mathscr{R}^s(0) = \hbar^{-\Omega}\prod_{0\in w}^{\leftarrow} R^+_w \quad \mathscr{R}^s(\infty) = \prod_{0\in w}^{\rightarrow}(R^-_w)^{-1}\,\hbar^{\Omega}$$

where R^+_w and R^-_w are strictly upper and lower triangular wall R-matrixes. The products run over walls passing through $0 \in H^2(X,\mathbb{R})$. Therefore, the eigenvalues of conjugation by $\hbar^{\lambda}_{(1)}\mathscr{R}^s(u)$ at $u = 0, \infty$ are either 1 or $z^m\hbar^{m'}$ with $m \neq 0$. Solutions in Laurent series in u thus necessarily correspond to eigenvalue 1. In particular, they are regular at $u = 0$ and $u = \infty$. It follows that U is a constant matrix in u.

6.6.3

The constant matrix U commutes with $\hbar^{\lambda}_{(1)}\mathscr{R}^s(u)$. Diagonalizing the matrix $\hbar^{\lambda}_{(1)}\mathscr{R}^s(0)$ we find that U is upper triangular. Similarly diagonalizing $\hbar^{\lambda}_{(1)}\mathscr{R}^s(\infty)$ we find that U is block lower triangular. We conclude that U is block diagonal matrix.

Let us consider the diagonal block $U_{0,0}$ of the matrix U corresponding to the lowest component of the fixed point set:

$$U_{0,0} : K_G(F_0) \to K_G(F_0)$$

Since U commutes with qKZ, the block $U_{0,0}$ commutes with the corresponding block of R-matrix $\mathscr{R}^s_{0,0}(u)$. From definition of R-matrix the matrix element $\mathscr{R}^s_{0,0}(u)$ is the generating function for operators of classical multiplication by tautological classes on F_0. Thus, the operator $U_{0,0}$ is itself an operator of multiplication by some class in $K_G(F_0)$. To finish the proof it remain to note that:

$$U_{0,0} = U_{F_0} \tag{6.19}$$

Where U_{F_0} denotes the same operator U for quiver variety F_0. Indeed, applying (6.19) to X in place of F_0 we conclude, that U is an operator of multiplication in $K_G(X)$. However, so such nonscalar operator can be diagonal in the stable basis. We conclude that $U = Const$.

6.7 $u = 0$ limit

To finish the proof of the theorem we need to prove (6.19). It follows from Propositions 12 and 13 below.

6.7.1

Proposition 12. *The matrix of quantum difference operator* $\boldsymbol{M}_{\mathscr{L}}(0,\lambda)$ *has the following form:*

$$\boldsymbol{M}_{\mathscr{L}}(0,\lambda)_{\mathsf{v}_2,\mathsf{v}_1} = 0 \quad \textit{for} \quad \mathsf{v}_1 \neq 0, \quad \boldsymbol{M}_{\mathscr{L}}(0,\lambda)_{0,0} = \boldsymbol{M}_{\mathscr{L}}(\lambda-\kappa)|_{F_0} \tag{6.20}$$

where $\kappa = (\mathsf{w}+\mathsf{w}' - C(\mathsf{v}+\mathsf{v}'))/2$ *and* C *is the Cartan matrix.*

Proof. First, let us consider the limit $u \to 0$ in the quantum difference equation (4.2):

$$\mathbf{M}_{\mathscr{L}}(u,\lambda)\mathbf{J}(u,z) = \mathbf{J}(u,\lambda)\mathscr{L}$$

First, we have $\mathscr{L}_{\mathsf{v}_2,\mathsf{v}_1} \sim u^{\langle \mathscr{L},\mathsf{v}_2\rangle}$. Second, the matrix of fundamental solution $\mathbf{J}(0,\lambda)$ is block upper triangular, moreover, the "vacuum matrix element" has the form

$$\mathbf{J}(0,\lambda)_{0,0} = \mathbf{J}|_{F_0}(\lambda-\kappa)$$

Thus, we conclude that the operator $\mathbf{M}_{\mathscr{L}}(u,\lambda)$ has the form (6.21).

The existence of the limit $\mathbf{J}(0,\lambda)$ in the stable basis is shown in Section 9.2 of [31]. The upper-triangularity statement follows by inspection of the braking nodes. Every one of them has the weight of the form $(1-q^m a^k)$ and it has to be the case that $k > 0$ for all of them for the limit to be non-vanishing. In particular, the curves which contribute to $\mathbf{J}(0,\lambda)_{0,0}$ never break, therefore, stay entirely within the component F_0. Thus $\mathbf{J}(0,\lambda)_{0,0} = \mathbf{J}|_{F_0}(\lambda + ...)$. The exact form of the shift indicated by dots can be computed as the index limit computation for the vertex Section 7.4 in [31] and gives exactly κ. □

6.7.2

Let us denote $\mathbf{B}(u) = \mathbf{B}^s_{\mathscr{L}}(\lambda)$ for the slope s as in the Theorem 5 and tautological line bundle $\mathscr{L}$.

Proposition 13.

$$\boldsymbol{B}(0)_{\mathsf{v}_2,\mathsf{v}_1} = 0 \quad \textit{for} \quad \mathsf{v}_1 \neq 0, \quad \boldsymbol{B}(0)_{0,0} = \boldsymbol{B}(\lambda-\kappa)|_{F_0} \tag{6.21}$$

Proof. **6.7.3**

First by Proposition 10, in the tensor product of two $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ modules we have:

$$\mathbf{B}(u) = W_{w_0}(\lambda) W_{w_1}(\lambda) \cdots W_{w_{n-1}}(\lambda) \Delta_\infty(\mathscr{L}) \tag{6.22}$$

where $W_w(\lambda) = \Delta_w(\mathbf{B}_w)(R_w^+)^{-1}$ and $w_0, \cdots w_{n-1}$ is the ordered set of walls crossed by an interval from s to $s + \mathscr{L}$.

6.7.4

By Corollary 6.9 we have:

$$\Delta_w(\mathbf{B}_w(\lambda)) = \mathbf{J}_w^-(\lambda) \left(\mathbf{B}_w(\lambda + \hat{\kappa}_{(2)}) \otimes \mathbf{B}_w(\lambda - \hat{\kappa}_{(1)}) \right) \mathbf{J}_w^+(\lambda)$$

Recall, that the operators $\mathbf{J}_w^-(\lambda)$ and R_w^+ are triangular with the following matrix elements:

$$\mathbf{J}_w^\pm(\lambda) = \bigoplus_{\substack{s=0, \\ \pm\langle \alpha, \theta \rangle > 0}}^{\infty} J_{s\alpha}, \quad R_w^\pm(\lambda) = \bigoplus_{\substack{s=0, \\ \pm\langle \alpha, \theta \rangle > 0}}^{\infty} R_{s\alpha}$$

where θ is the stability parameter of the quiver and α is the root defining the wall w:

$$w = \{x \in H^2(X, \mathbb{R}) | \langle x, \alpha \rangle = m\}.$$

The matrix elements are of the form:

$$J_{s\alpha}, R_{s\alpha} : K_G(F_{\mathsf{v}}) \longrightarrow K_G(F_{\mathsf{v}+s\alpha})$$

and by Theorem 1 they have the following dependence on the equivariant parameter u:

$$J_{s\alpha}, R_{s\alpha} \sim u^{s\langle \alpha, \mathscr{L}_w \rangle}.$$

where $\mathscr{L}_w$ is a line bundle on the wall w. We conclude that the matrix elements of $W_w(\lambda)$ have the following form:

$$W_w(\lambda)_{\mathsf{v}_2, \mathsf{v}_1} \sim u^{\langle s\alpha, \mathscr{L}_w \rangle}, \quad \text{if} \quad \mathsf{v}_2 = \mathsf{v}_1 + s\alpha. \tag{6.23}$$

6.7.5

From (6.22) we see that the matrix element $\mathbf{B}^{(1)}_{\mathsf{v}_2,\mathsf{v}_1}$ has takes the form:

$$\mathbf{B}_{\mathsf{v}_2,\mathsf{v}_1} = \sum_{s_0,\cdots,s_{n-1}=0}^{\infty} \mathbf{B}_{\mathsf{v}_2,\mathsf{v}_1}(s_0,\cdots,s_{n-1})$$

where $\mathbf{B}_{\mathsf{v}_2,\mathsf{v}_1}(s_0,\cdots,s_{n-1})$ is the contribution of the following combination of matrix elements:

$$\mathbf{B}_{\mathsf{v}_2,\mathsf{v}_1}(s_0,\cdots,s_{n-1}):$$

$$K_G(F_{\mathsf{v}_1}) \stackrel{W_{w_{n-1}}(\lambda)}{\longrightarrow} K_G(F_{\mathsf{v}_1+s_{n-1}\alpha_{n-1}}) \stackrel{W_{w_{n-2}}(\lambda)}{\longrightarrow} K_G(F_{\mathsf{v}_1+s_{n-1}\alpha_{n-1}+s_{n-2}\alpha_{n-2}}) \stackrel{W_{w_{n-3}}(\lambda)}{\longrightarrow}$$

$$\cdots \stackrel{W_{w_0}(\lambda)}{\longrightarrow} K_G(F_{\mathsf{v}_2})$$

such that

$$s_0\alpha_0 + \cdots + s_{n-1}\alpha_{n-1} = \mathsf{v}_2 - \mathsf{v}_1 \tag{6.24}$$

From (6.23) we see that this matrix element has the following dependence on the spectral parameter: $\mathbf{B}_{\mathsf{v}_2,\mathsf{v}_1}(s_0,\cdots,s_{n-1}) \sim u^{d_{\mathsf{v}_2,\mathsf{v}_1}((s_0,\cdots,s_{n-1}))}$, with exponent:

$$d_{\mathsf{v}_2,\mathsf{v}_1}(s_0,...,s_{n-1}) = s_0\langle\alpha_0,\mathscr{L}_0\rangle + ... + s_{n-1}\langle\alpha_{n-1},\mathscr{L}_{n-1}\rangle + \langle\mathsf{v}_1,\mathscr{L}_n\rangle \tag{6.25}$$

where we denote by $\mathscr{L}_i$ the point at which the interval $(s, s+\mathscr{L})$ intersects the wall w_i and $\mathscr{L}_n = \mathscr{L}$. The last term $\langle\mathsf{v}_1,\mathscr{L}_n\rangle$ comes from $\Delta_\infty(\mathscr{L})$ which is diagonal operator with diagonal matrix lements $\Delta_\infty(\mathscr{L})_{\mathsf{v}_1,\mathsf{v}_1} \sim u^{\langle\mathsf{v}_1,\mathscr{L}\rangle}$.

6.7.6

Note, by our choice of the alcove we can assume that the slope s lies in the arbitrary small neighborhood of $0 \in H^2(X,\mathbb{R})$. Thus we can assume that $\mathscr{L}_0 = 0$ and write:

$$\begin{aligned} &d_{\mathsf{v}_2,\mathsf{v}_1}(s_0,...,s_{n-1}) = \\ &s_1\langle\alpha_1,\mathscr{L}_1-\mathscr{L}_0\rangle + ... + s_{n-1}\langle\alpha_{n-1},\mathscr{L}_{n-1}-\mathscr{L}_0\rangle + \langle\mathsf{v}_1,\mathscr{L}_n-\mathscr{L}_0\rangle \end{aligned} \tag{6.26}$$

The crucial step is to rewrite this equality in the following form:

$$\begin{aligned} & d_{\mathsf{v}_2,\mathsf{v}_1}(s_1,...,s_n) = \\ & \langle \mathsf{v}_1, \mathscr{L}_n - \mathscr{L}_{n-1}\rangle + \\ & \langle \mathsf{v}_1 + s_{n-1}\alpha_{n-1}, \mathscr{L}_{n-1} - \mathscr{L}_{n-2}\rangle + \\ & + \cdots + \\ & \langle \mathsf{v}_1 + s_{n-1}\alpha_{n-1} + \cdots + s_1\alpha_1, \mathscr{L}_1 - \mathscr{L}_0\rangle \end{aligned} \tag{6.27}$$

Now, we have the set of inequalities:

$$\begin{aligned} & \mathsf{v}_1 + s_{n-1}\alpha_{n-1} \geq 0 \\ & \mathsf{v}_1 + s_{n-1}\alpha_{n-1} + s_{n-2}\alpha_{n-2} \geq 0 \\ & \cdots \\ & \mathsf{v}_1 + s_{n-1}\alpha_{n-1} + \cdots + s_1\alpha_1 \geq 0 \end{aligned} \tag{6.28}$$

where $\mathsf{v} \geq 0$ means that the inequality holds for all components of the dimension vector: $\mathsf{v}_i > 0$. If they not satisfied, the matrix element $\mathbf{B}_{\mathsf{v}_2,\mathsf{v}_1}(s_0, s_2, ..., s_{n-1})$ obviously vanishes as the corresponding operator annihilate any class supported on component F_{v_1}.

If $\mathscr{L}$ is one of tautological line bundles, then $\langle \mathsf{v}, \mathscr{L}\rangle > 0$ for $\mathsf{v} > 0$ and therefore $\langle \mathsf{v}\mathscr{L}_i - \mathscr{L}_{i+1}\rangle > 0$. We conclude that:

$$d_{\mathsf{v}_2,\mathsf{v}_1}(s_0,...,s_{n-1}) \geq \langle \mathsf{v}_1, \mathscr{L}\rangle$$

and thus:

$$\lim_{u\to 0} \mathbf{B}_{\mathsf{v}_2,\mathsf{v}_1} = 0 \quad \text{for} \quad \mathsf{v}_1 \neq 0,$$

6.7.7

Now, let us analyze the case $\mathsf{v}_2 = \mathsf{v}_1 = 0$. Substituting $\mathsf{v}_1 = 0$ into (6.27) we see that $d_{\mathsf{v}_2,\mathsf{v}_1}(s_0, ..., s_{n-1}) = 0$ only when $s_1 = s_2 = \cdots = s_{n-1} = 0$. Thus, from (6.24) we conclude: $s_0\alpha_0 = \mathsf{v}_2 = 0$, so that

$s_0 = 0$. It means that only the diagonal matrix elements (all $s_i = 0$) of $W_{w_k}(\lambda)$ contribute to vacuum matrix element $\mathbf{B}(u)_{0,0}$. From (6.22) we

$$\mathbf{B}_{0,0}(0) = \mathbf{B}_{w_0}(\lambda - \kappa) \cdots \mathbf{B}_{w_{n-1}}(\lambda - \kappa)\mathscr{L} = \mathbf{B}^{s}_{\mathscr{L}^{-1}}\big|_{F_0}(\lambda - \kappa)$$

The proposition is proven. □

Chapter 7

Example 1: Cotangent bundles to Grassmannians

The simplest example of quiver is the quiver consisting of one vertex and no edges. In this case the dimension vectors are given by a couple of natural numbers $(\mathsf{v}, \mathsf{w}) = (k, n) \in \mathbb{N}^2$, and the corresponding varieties are isomorphic to cotangent bundles to Grassmannians of k -dimensional subspaces in n-dimensional space:

$$\mathscr{M}(\mathsf{v}, \mathsf{w}) = T^*Gr(k, n) \tag{7.1}$$

The framing torus $\mathsf{A} \simeq (\mathbb{C}^\times)^n$ acts on $W = \mathbb{C}^n$ in a standard way. This induces an action of A on $T^*Gr(k, n)$. Note, that this action preserves the symplectic form on $T^*Gr(k, n)$. Let us denote by $G = \mathsf{A} \times \mathbb{C}^\times$ where the extra factor acts by scaling the fibers of the cotangent bundle. This torus scales the symplectic form with character which we denote $\hbar$.

7.1 Algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ and wall subalgebras $\mathscr{U}_\hbar(\mathfrak{g}_w)$

7.1.1

Let us denote

$$X = \mathscr{M}(\mathsf{w}) = \coprod_{\mathsf{v}} \mathscr{M}(\mathsf{v}, \mathsf{w}) = \coprod_{k=0}^{n} T^*Gr(k, n) \tag{7.2}$$

Note that $\mathscr{M}(1)$ is a variety consisting of two points, thus $K_G(\mathscr{M}(1))$ is two dimensional over $K_G(pt)$. Therefore, if the torus A splits the framing as $\mathsf{w} = u_1 + \cdots + u_n$ then we have:

$$K_G(X) = \mathbb{C}^2(u_1) \otimes \cdots \otimes \mathbb{C}^2(u_n) \tag{7.3}$$

such that the total dimension is 2^n. Note, that $T^*Gr(k,n)^{\mathsf{A}}$ consists of $n!/k!/(n-k)!$ points, such that X^{A} is a set of 2^n points p_i. The fixed point basis of (localized) $K_G(X)$ consists of sheaves $\mathcal{O}_{p_i}$.

7.1.2

We start from the case $n = 2$. We have:

$$X = \text{pt} \cup T^*\mathbb{P}^1 \cup \text{pt}$$

Where pt stands for Nakajima variety consisting of one point. Therefore, the only nontrivial block of R-matrix corresponds to $T^*\mathbb{P}^1$. The action of torus $G = \mathsf{A} \times \mathbb{C}^\times$ is represented in Fig. 7.1. In this picture p_1 and p_2 are two fixed points, corresponding to the points $z = 0$ and $z = \infty$ of the base $\mathbb{P}^1 \subset T^*\mathbb{P}^1$. We also specify explicitly the characters of the tangent spaces to $T^*\mathbb{P}^1$ at the fixed points. For example the tangent space at p_1 is spanned by tangent space to the base with character u_1/u_2 and tangent space to cotangent fiber with character $u_2/(u_1\hbar)$.

To compute the stable envelopes of the fixed points we need to choose a polarization $T^{1/2}$ and a chamber $\mathfrak{C}$. We choose the positive chamber $\mathfrak{C}$ such that $u_1/u_1 \to 0$. The arrows in Fig. 7.1 represent the attracting and repealing directions with respect to this chamber. We choose a polarization $T^{1/2}$ given by cotangent directions.

In our case $H^2(T^*\mathbb{P}^1, \mathbb{R}) = \mathbb{R}$, thus we identify the set of slopes with real numbers $s \in \mathbb{R}$.

7.1.3

First we consider the restrictions of stable envelopes to the fixed components. By (2.3) we have:

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_1)\Big|_{p_1} = (-1)^{\mathrm{rk}T^{1/2}_{>0}}\Big(\frac{\det \mathscr{N}_-}{\det T^{1/2}_{\neq 0}}\Big)^{\frac{1}{2}}\Lambda^{\bullet}_{-}\mathscr{N}_-^{\vee}$$

Here, by definition $\mathscr{N}_-$ is the repealing part of the normal bundle to p_1, $T^{1/2}_{>0}$ - attracting part of polarization and $T^{1/2}_{\neq 0}$ non-stationary part of polarization. From the Fig. 7.1 around p_1 we obtain: $\mathscr{N}_- = u_2/u_1/\hbar$, $\mathrm{rk}T^{1/2}_{>0} = 0$, $T^{1/2}_{\neq 0} = \mathscr{N}_- = u_2/u_1/\hbar$, and thus we find:

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_1)\Big|_{p_1} = 1 - \hbar u_1/u_2$$

Same formula with around p_2 with $\mathscr{N}_- = u_2/u_1$, $\mathrm{rk}T^{1/2}_{>0} = 1$, $T^{1/2}_{\neq 0} = u_1/u_2/\hbar$ gives:

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_2)\Big|_{p_2} = (1 - u_2/u_1)\hbar^{1/2}$$

The support condition for stable envelopes means that $\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_2)\Big|_{p_2} = 0$. We conclude, that the stable envelope of the fixed points has the following form:

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_1) = (1 - \hbar\mathcal{O}(1)/u_2)(\mathcal{O}(1)/u_1)^{n(s)}$$

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_2) = (1 - \mathcal{O}(1)/u_1)\sqrt{\hbar}(\mathcal{O}(1)/u_2)^{m(s)}$$

Here $\mathcal{O}(1)$ is the tautological bundle restricting to the fixed points by the rule $\mathcal{O}(1)|_{p_i} = u_i$ for $i = 1, 2$. The exponents $n(s)$ and $m(s)$ are some integer numbers which are to be found from the restriction condition (2.5). Note that this is the only part of the construction which depends on the slope parameter s.

7.1.4

The fractional line bundle corresponding to slope s is $\mathcal{O}(1)^s$. The degree condition (2.5) for the point p_1 therefore means:

$$\deg_{\mathsf{A}}\Big(\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_1)\Big|_{p_2}\Big) \subset \deg_{\mathsf{A}}\Big(\mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_2)\Big|_{p_2} \times \frac{\mathcal{O}(1)^s|_{p_2}}{\mathcal{O}(1)^s|_{p_1}}\Big)$$

or, by above formulas:

$$\deg_{\mathsf{A}}\Big((1-\hbar)\,(u_2/u_1)^{n(s)}\Big) \subset \deg_{\mathsf{A}}\Big((1 - u_2/u_1)\sqrt{h}\,(u_2/u_1)^{s}\Big)$$

which is equivalent to condition:

$$n(s) \in (s, s+1)$$

as the $n(s)$ must be integer we conclude that $n(s) = \lfloor 1+s \rfloor$, i.e., the "floor" integral part of $1+s$. For the second fixed point p_2 computation is same. We conclude:

$$\boxed{\begin{aligned} \mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_1) &= (1-\hbar\mathcal{O}(1)/u_2)\left(\frac{\mathcal{O}(1)}{u_1}\right)^{\lfloor 1+s \rfloor} \\ \mathrm{Stab}_{\mathfrak{C},T^{1/2},s}(p_2) &= (1-\mathcal{O}(1)/u_1)\sqrt{\hbar}\left(\frac{\mathcal{O}(1)}{u_2}\right)^{\lfloor 1+s \rfloor} \end{aligned}} \tag{7.4}$$

For the opposite chamber $-\mathfrak{C}$ the we have $u_1/u_2 \to \infty$. It means that in Fig. 7.1 all arrows are reversed. In particular the stable envelope for $-\mathfrak{C}$ is obtain from the last formula by permuting the fixed points:

$$\boxed{\begin{aligned} \mathrm{Stab}_{-\mathfrak{C},T^{1/2},s}(p_1) &= (1-\mathcal{O}(1)/u_2)\sqrt{\hbar}\left(\frac{\mathcal{O}(1)}{u_1}\right)^{\lfloor 1+s \rfloor} \\ \mathrm{Stab}_{-\mathfrak{C},T^{1/2},s}(p_2) &= (1-\hbar\mathcal{O}(1)/u_1)\left(\frac{\mathcal{O}(1)}{u_2}\right)^{\lfloor 1+s \rfloor} \end{aligned}} \tag{7.5}$$

7.1.5

In agreement with our general theory we see that the stable envelopes are the locally constant functions of the parameter s. From the last set of formulas we see that it changes only when s crosses an integer point. We conclude that the set of walls can be identified with $\mathbb{Z} \subset \mathbb{R}$ and thus alcoves are of the form $(w, w+1) \subset \mathbb{R}$.

The alcove specified by Theorem 5 has the form $\nabla = (-1, 0)$. To compute the R-matrix corresponding to this alcove we choose $s \in \nabla$, then in the basis of fixed points ordered as $[p_2, p_1]$ from above formulas we compute:

$$i^*\mathrm{Stab}_{\mathfrak{C},T^{1/2},s} = \begin{bmatrix} \left(1-u^{-1}\right)\sqrt{\hbar} & 1-\hbar \\ 0 & 1-\hbar u \end{bmatrix}$$

$$i^{*}\mathrm{Stab}_{-\mathfrak{C},T^{1/2},s} = \begin{bmatrix} 1-\hbar u^{-1} & 0 \\ 1-\hbar & (1-u)\sqrt{\hbar} \end{bmatrix}$$

where we denote $u = u_1/u_2$ and i^* is the operation of restriction to fixed points. The total R-matrix for slope s is defined as follows:

$$\mathscr{R}^{s}(u) = \mathrm{Stab}^{-1}_{-\mathfrak{C},T^{1/2},s}\mathrm{Stab}_{\mathfrak{C},T^{1/2},s} = (i^{*}\mathrm{Stab}_{-\mathfrak{C},T^{1/2},s})^{-1}\ (i^{*}\mathrm{Stab}_{\mathfrak{C},T^{1/2},s})$$

and we obtain:

$$\boxed{\mathscr{R}^{s}(u) = \begin{bmatrix} \dfrac{(1-u)\,\hbar^{\frac{1}{2}}}{\hbar-u} & \dfrac{u\,(\hbar-1)}{\hbar-u} \\ \dfrac{\hbar-1}{\hbar-u} & \dfrac{(1-u)\,\hbar^{\frac{1}{2}}}{\hbar-u} \end{bmatrix}} \tag{7.6}$$

7.1.6

The wall R-matrices are defined by (2.13) and similarly to what we have above:

$$R^{\pm}_{w} = (i^{*}\mathrm{Stab}_{\pm\mathfrak{C},T^{1/2},s'})^{-1}\ (i^{*}\mathrm{Stab}_{\pm\mathfrak{C},T^{1/2},s})$$

where s and s' are two slopes separated by a wall w. Let w be an integer representing the wall and $s = w-\epsilon$, $s = w+\epsilon$ for sufficiently small ϵ (obviously enough to take $0<\epsilon<1$). Then from above formulas we obtain:

$$\boxed{R^{+}_{w} = \begin{bmatrix} 1 & \dfrac{1-\hbar}{u^{w}\sqrt{\hbar}} \\ 0 & 1 \end{bmatrix} \quad R^{-}_{w} = \begin{bmatrix} 1 & 0 \\ \dfrac{u^{w}\,(1-\hbar)}{\sqrt{\hbar}} & 1 \end{bmatrix}} \tag{7.7}$$

7.1.7

The KT factorization of R-matrix $s \in \nabla$ has the form (2.19):

$$\mathscr{R}^{s}(u) = \prod^{\rightarrow}_{w<0} R^{-}_{w} R_{\infty} \prod^{\leftarrow}_{w\geq 0} R^{+}_{w} \tag{7.8}$$

This infinite product is convergent in the topology of power series in u^{-1}. From (7.7) we obtain:

$$U = \prod_{w\geq 0}^{\leftarrow} R_w^+ = \cdots R_1^+ R_0^+ = \begin{bmatrix} 1 & \dfrac{1-\hbar}{\sqrt{\hbar}}(1+u^{-1}+\cdots) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{(1-\hbar)u}{\sqrt{\hbar}\,(u-1)} \\ 0 & 1 \end{bmatrix}$$

$$L = \prod_{w<0}^{\rightarrow} R_w^- = R_{-1}^- R_{-2}^- \cdots = \begin{bmatrix} 1 & 0 \\ \dfrac{(1-\hbar)}{\sqrt{\hbar}}(u^{-1}+\cdots) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{(1-\hbar)}{\sqrt{\hbar}\,(u-1)} & 1 \end{bmatrix}$$

Finally, the infinity slope R-matrix is given by (2.24). The attracting and repealing directions are obvious from Fig. 7.1 and we obtain:

$$R_\infty = -\begin{bmatrix} \dfrac{u^{-\frac{1}{2}} - u^{\frac{1}{2}}}{u^{\frac{1}{2}}\hbar^{-\frac{1}{2}} - u^{-\frac{1}{2}}\hbar^{\frac{1}{2}}} & 0 \\ 0 & \dfrac{u^{-\frac{1}{2}}\hbar^{-\frac{1}{2}} - u^{\frac{1}{2}}\hbar^{\frac{1}{2}}}{u^{\frac{1}{2}} - u^{-\frac{1}{2}}} \end{bmatrix}$$

One easily checks that in agreement with (7.6) we have $\mathscr{R}^s(u) = L\,R_\infty\,U$. This gives canonical Gauss decomposition of the R-matrix.

7.1.8

The R-matrix for the whole Nakajima variety X given by (7.2) is of the form:

$$\mathscr{R}^s(u) = \begin{bmatrix} 1 & & \\ & \mathscr{R}^s_{T^*\mathbb{P}} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{(1-u)\hbar^{\frac{1}{2}}}{\hbar - u} & \dfrac{u(\hbar-1)}{\hbar-u} & 0 \\ 0 & \dfrac{\hbar-1}{\hbar-u} & \dfrac{(1-u)\hbar^{\frac{1}{2}}}{\hbar-u} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Up to a scalar multiple one recognizes the standard R-matrix for $\mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_2)$ acting in the tensor product of two fundamental evaluation modules $\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)$. We conclude, that the quiver algebra corresponding to cotangent bundles to Grassmannians is $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q) = \mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_2)$.

7.1.9

The codimension function (2.27) for X is given, obviously, by the following diagonal matrix:

$$\hbar^\Omega = \operatorname{diag}(1, \hbar^{\frac{1}{2}}, \hbar^{\frac{1}{2}}, 1)$$

We obtain that the wall R-matrices defined by the Theorem 2 have the following explicit form:

$$\mathsf{R}_w^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \hbar^{\frac{1}{2}} & (1-\hbar)u^{-w} & 0 \\ 0 & 0 & \hbar^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In particular all wall R-matrices are conjugated to the zeroth one by line bundle:

$$\mathsf{R}_w^+ = \mathcal{O}(w)\mathsf{R}_0^+\mathcal{O}(w)^{-1} \tag{7.9}$$

with $\mathscr{O}(w) = \mathrm{diag}(1, u_2^w, u_1^w, 1)$. One recognizes, that up to a multiple R_0^+ coincides with the standard R-matrix for $\mathscr{U}_\hbar(\mathfrak{sl}_2)$ in the tensor product of two fundamental representations. Thus, the wall subalgebra, which is build by FRT procedure from this R-matrix is $\mathscr{U}_\hbar(\mathfrak{g}_0) \simeq \mathscr{U}_\hbar(\mathfrak{sl}_2)$. As the R-matrices for other wall conjugated, we conclude that $\mathscr{U}_\hbar(\mathfrak{g}_w) \simeq \mathscr{U}_\hbar(\mathfrak{sl}_2)$ for arbitrary wall w.

7.1.10

To summarize, we have an algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q) = \mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_2)$ and a set of subalgebras $\mathscr{U}_\hbar(\mathfrak{g}_w) \simeq \mathscr{U}_\hbar(\mathfrak{sl}_2)$ indexed by walls $w \in \mathbb{Z}$. It is convenient to organize this information as follows: let E, F and K be the standard generators of $\mathscr{U}_\hbar(\mathfrak{sl}_2)$ which we understand as $\mathscr{U}_\hbar(\mathfrak{g}_0)$. Then by (7.9) the wall subalgebra $\mathscr{U}_\hbar(\mathfrak{g}_w)$ is generated by E_w, F_w and K:

$$E_w = \mathcal{O}(w)E\mathcal{O}(w)^{-1}, \quad F_w = \mathcal{O}(w)F\mathcal{O}(w)^{-1}. \tag{7.10}$$

Let us denote $x^+(w) = E_w, \quad x^-(w) = F_{-w}$. One can check that the relations among these generators can be summarized as the Drinfeld's realization of $\mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_2)$: the algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_2)$ is an associative algebra with 1 generated over $\mathbb{C}(\hbar)$ by the elements $x^\pm(k), a(l), K^{\pm 1}$ $(k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\})$

with the following relations:

$$
\begin{aligned}
&KK^{-1} = K^{-1}K = 1 \\
&[a(k), a(m)] = 0, [a(k), K^{\pm}] = 0 \\
&Kx^{\pm}(k)K^{-1} = \hbar^{\pm 2}x^{\pm}(k) \\
&[x^{+}(k), x^{-}(l)] = \frac{1}{\hbar - \hbar^{-1}}\Big(\psi(k+l) - \varphi(k+l)\Big) \\
&[a(k), x^{\pm}(l)] = \pm\frac{[2k]_{\hbar}}{k}x^{\pm}(l+k)
\end{aligned}
\tag{7.11}
$$

with

$$\sum_{m=0}^{\infty} \psi(m)z^{-m} = K\exp\left((\hbar - \hbar^{-1})\sum_{k=1}^{\infty} a(k)z^{-k}\right)$$

$$\sum_{m=0}^{\infty} \varphi(-m)z^{m} = K^{-1}\exp\left(-(\hbar - \hbar^{-1})\sum_{k=1}^{\infty} a(-k)z^{-k}\right)$$

It may be convenient to visualize $\mathscr{U}_{\hbar}(\widehat{\mathfrak{gl}}_2)$ and its subalgebras as in the Figure 7.2 : the wall $\mathscr{U}_{\hbar}(\mathfrak{g}_w)$ corresponds to a line with integer slope w.

F_{-2}

$a(2)$

Figure 7.2: The structure of $\mathscr{U}_{\hbar}(\widehat{\mathfrak{gl}}_2)$. The line through zero corresponds to slope 2 subalgebra $U_q(\mathfrak{sl}_2) \subset \mathscr{U}_{\hbar}(\widehat{\mathfrak{gl}}_2)$ generated by E_2, F_2, K.

F_{-1} E_1 F_2

$a(-2)$ $a(1)$ E_{-2}

7.2 R -matrices

7.2.1

To write the formulas for R-matrices for general variety (7.3) it is enough to substitute all formulas from the previous section by their "universal" versions.

The universal R-matrix for $\mathscr{U}_\hbar(\mathfrak{sl}_2)$ is well known:

$$\mathsf{R} = \hbar^{-H\otimes H/2} \sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k \hbar^{-k(k-1)/2}}{[k]_\hbar!} F^k \otimes E^k \tag{7.12}$$

with H related to K as $K = \hbar^H$. Up to a scalar multiple the codimension function is given by $\hbar^\Omega = \hbar^{-H\otimes H/2}$ thus, we conclude that there is the following universal formula for the wall R-matrices:

$$R_w^+ = \sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k \hbar^{-k(k-1)/2}}{[k]_\hbar!} F_w^k \otimes E_w^k, \tag{7.13}$$

The lower triangular wall R-matrix is obtained by transposition $R_w^- = (R_w^+)_{21}$.

7.2.2

The KT factorization (2.19) provides the following universal formula for the total R-matrix:

$$\mathscr{R}^s(u) = \prod_{w<s}^{\rightarrow} R_w^- \, R_\infty \prod_{w\geq s}^{\leftarrow} R_w^+ \tag{7.14}$$

with $R_w^\pm$ with given explicitly by (7.13). The R -matrix R_∞ is the operator of multiplication by the class of normal bundles (2.24). It can be conveniently expressed in terms of generators $a(m)$ corresponding to the infinite slope in the Fig7.2:

$$R_\infty = c\,\hbar^{H\otimes H/2} \exp\Big((\hbar-\hbar^{-1}) \sum_{n=1}^{\infty} \frac{m}{[2m]_\hbar}\, a(-n)\otimes a(n)\Big)$$

where c is some scalar multiple depending on normalization.

7.3 The quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$

7.3.1

By definition the function $\hbar^\lambda$ on K-theory of $\mathscr{M}(1) = \mathscr{M}(1,1)\coprod \mathscr{M}(0,1)$ as:

$$\hbar^\lambda = \begin{cases} z & \text{on } \mathscr{M}(1,1) \\ 1 & \text{on } \mathscr{M}(0,1) \end{cases} \quad \Leftrightarrow \quad \hbar^\lambda = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} = z^{\frac{1}{2}}\, z^{H/2}$$

From this and (7.13) we see that the the ABRR equation for $\mathscr{U}_\hbar(\mathfrak{sl}_2)$ takes the following form:

$$J^+(z) z^{-H\otimes 1/2}\,\mathsf{R} \;=\; z^{-H\otimes 1/2}\,\hbar^{-H\otimes H/2} J^+(z)$$

with R given by (7.12). This is an equation for strictly upper triangular operator $J(z)$, which means that:

$$J^+(z) = 1 + \sum_{k=1}^{\infty} J_k^+(z)\, F^k \otimes E^k$$

The Proposition 5 says that ABRR equation determines the coefficients $J_k(z)$ uniquely. Computation gives:

$$J^+(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \hbar^{-k(k-1)/2} (\hbar-\hbar^{-1})^k}{[k]_\hbar!\, \prod\limits_{i=1}^{k} (1 - z^{-1} K\otimes K^{-1} \hbar^{2i})}\, F^k \otimes E^k$$

7.3.2

By definition (5.7) we have $\mathbf{J}_w^+(\lambda) = J_w^+(\lambda - \tau_w)$. In out case $\tau_w = w$ and this corresponds to a shift $z \to z\hbar^{-\mathbf{s}w} = zq^{-w}$ for integer wall w. We conclude that:

$$\mathbf{J}_w^+(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \hbar^{-k(k-1)/2} (\hbar-\hbar^{-1})^k}{[k]_\hbar!\, \prod\limits_{i=1}^{k} (1 - z^{-1} q^w K\otimes K^{-1} \hbar^{2i})}\, F_w^k \otimes E_w^k \tag{7.15}$$

7.3.3

The operator $\mathbf{B}_w(z)$ is given by (5.9). To compute it, we need the formulas for antipode S_w of $\mathscr{U}_\hbar(\mathfrak{g}_w)$. They can be obtained directly form the wall R-matrix (7.13). First, from $1\otimes\Delta(\mathsf{R}) = \mathsf{R}_{13}\mathsf{R}_{12}$ and $\Delta\otimes 1(\mathsf{R}) = \mathsf{R}_{13}\mathsf{R}_{23}$ we obtain:

$$\Delta(E) = K^{-1}\otimes E + E\otimes 1, \quad \Delta(F) = 1\otimes F + F\otimes K, \quad \Delta(K) = K\otimes K$$

and thus the antipode corresponding to this coproduct has the form:

$$S(E) = -KE, \quad S(F) = -FK^{-1}, \quad S(K) = K^{-1}$$

7.3.4

The lower triangular solutions of ABRR equation can be computed from (7.15) by $\mathbf{J}^-_w(z) = S_w \otimes S_w(\mathbf{J}^+_w(z)_{21})$. We obtain:

$$\mathbf{m}\Big(1 \otimes S_w(\mathbf{J}^-_w(z)^{-1}))\Big) = \sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k \hbar^{-k(k+3)/2}}{[k]_\hbar! \prod\limits_{i=0}^{k}(1 - z^{-1}q^w K^2 \hbar^{-2i})} K^k E^k_w F^k_w$$

7.3.5

To compute the operator $\mathbf{B}_w(z)$ we need to shift parameter z by κ. By definition, $\kappa = (C\mathsf{v} - \mathsf{w})/2$. Enough to compute the action of κ in one evaluation module $\mathbb{C}^2(u)$ of $\mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_2)$. This module corresponds to $\mathsf{w} = 1$. The Cartan matrix corresponding to our case $C = 2$. We therefore find:

$$\kappa = \begin{cases} 1/2 & \text{on } \mathscr{M}(1,1) \\ -1/2 & \text{on } \mathscr{M}(0,1) \end{cases} \Leftrightarrow \kappa = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} = H/2$$

Thus, we conclude that the shift $\lambda \to \lambda + \hat{\kappa}$ is given by $z \to zK$ and from the definition (5.9) we obtain:

$$\mathbf{B}_w(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k \hbar^{-k(k+3)/2}}{[k]_\hbar! \prod\limits_{i=0}^{k}(1 - z^{-1}q^w K \hbar^{-2i})} K^k E^k_w F^k_w$$

7.3.6

The alcove specified by Theorem 5 corresponds to the interval $\nabla = (-1, 0)$. Let $s \in \nabla$ and $\mathscr{L} = \mathcal{O}(1)$. There is only one wall $w = -1$ between s and $s - 1$. Thus, the definition (5.11) and Theorem 5 give the following explicit formula for the quantum difference operator:

$$\boxed{\mathbf{M}_{\mathcal{O}(1)}(z) = \mathcal{O}(1) \sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k \hbar^{-k(k+3)/2}}{[k]_\hbar! \prod\limits_{i=0}^{k}(1 - z^{-1}q^{-1} K \hbar^{-2i})} K^k E^k_{-1} F^k_{-1}} \tag{7.16}$$

7.3.7

Using (5.11) we can also rewrite this operator as:

$$\mathbf{M}_{\mathcal{O}(1)}(z) = \Big(\sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k\hbar^{-k(k+3)/2}}{[k]_\hbar!\prod_{i=0}^{k}(1-z^{-1}q^{-1}K\hbar^{-2i})} K^k E^k F^k\Big)\mathcal{O}(1).$$

This form is particularly convenient for explicit computations as it expresses the difference operator through the standard $\mathscr{U}_\hbar(\mathfrak{sl}_2)$.

7.3.8

In the limits we obtain:

$$\lim_{z\to 0}\mathbf{M}_{\mathcal{O}(1)}(z) = \mathcal{O}(1) \quad \lim_{q\to 0}\mathbf{M}_{\mathcal{O}(1)}(z) = \mathcal{O}(1)$$

Moreover:

$$\lim_{z\to\infty}\mathbf{M}_{\mathcal{O}(1)}(zq^{-1}) = \Big(\sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k\hbar^{-k(k+3)/2}}{[k]_\hbar!\prod_{i=0}^{k}(1-z^{-1}K\hbar^{-2i})} K^k E^k F^k\Big)\mathcal{O}(1)$$

And therefore we can find the explicit form of glueing matrix, which plays an important role in quantum K-theory [31]:

$$\mathbf{G} = \sum_{k=0}^{\infty} \frac{(-1)^k(\hbar-\hbar^{-1})^k\hbar^{-k(k+3)/2}}{[k]_\hbar!\prod_{i=0}^{k}(1-z^{-1}K\hbar^{-2i})} K^k E^k F^k$$

7.3.9

The fixed point set $T^*Gr(k,n)^{\mathsf{A}}$ consists of $n!/(n-k)!/k!$ points labeled by k-subsets of the set $\{1,2,\cdots,n\}$. In the basis of stable envelopes of the fixed points on $X = T^*\mathbb{P}^1$ ordered as $[[1],[2]]$ the difference operator takes the following explicit form:

$$\mathbf{M}_{\mathcal{O}(1)}(z/q) = \begin{bmatrix} \dfrac{(z-1)\,u_1}{z\hbar^2-1} & \dfrac{z\left(\hbar-\hbar^{-1}\right)u_2}{(1-z\hbar^2)} \\ \dfrac{\left(\hbar-\hbar^{-1}\right)u_1}{(1-z\hbar^2)} & \dfrac{(z-1)\,u_2}{z\hbar^2-1} \end{bmatrix}$$

For $X = T^*\mathbb{P}^2$ in the stable basis $[[1],[2],[3]]$ we obtain:

$$\mathbf{M}_{\mathcal{O}(1)}(z/q) = \begin{bmatrix} \frac{(-1+z\hbar)\,u_1}{z\hbar^3-1} & -\frac{z\,(\hbar+1)\,(\hbar-1)\,u_2}{z\hbar^3-1} & -\frac{u_3 z\,(\hbar+1)\,(\hbar-1)}{\hbar\,(z\hbar^3-1)} \\ -\frac{(\hbar-1)\,(\hbar+1)\,u_1}{\hbar\,(z\hbar^3-1)} & \frac{u_2\,(-1+z\hbar)}{z\hbar^3-1} & -\frac{u_3 z\,(\hbar+1)\,(\hbar-1)}{z\hbar^3-1} \\ -\frac{(\hbar-1)\,(\hbar+1)\,u_1}{z\hbar^3-1} & -\frac{u_2\,(\hbar+1)\,(\hbar-1)}{\hbar\,(z\hbar^3-1)} & \frac{(-1+z\hbar)\,u_3}{z\hbar^3-1} \end{bmatrix}$$

Chapter 8

Example 2: Instanton moduli spaces

In this section we consider the example of Jordan quiver: the quiver consisting of one vertex and a single loop. The dimension vectors are given by two non-negative integer numbers $\mathsf{v} = n$, $\mathsf{w} = r$. The corresponding variety $\mathscr{M}(n, r)$ is the moduli space of framed rank r torsion-free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ with fixed second Chern class $c_2(\mathcal{F}) = n$. A framing of a sheaf $\mathcal{F}$ is a choice of of an isomorphism:

$$\phi : \mathcal{F}|_{L_\infty} \to \mathscr{O}_{L_\infty}^{\oplus r} \tag{8.1}$$

where L_∞ is the line at infinity of $\mathbb{C}^2 \subset \mathbb{P}^2$. This moduli space is usually referred to as instanton moduli space.

Let $\mathsf{A} \simeq (\mathbb{C}^\times)^r$ be the framing torus acting on $\mathscr{M}(n, r)$ by changing the isomorphism (8.1). This torus acts on the instanton moduli space preserving the symplectic form.

Let us denote by $G = \mathsf{A} \times (\mathbb{C}^\times)^2$ where the second factor acts on $\mathbb{C}^2 \subset \mathbb{P}^2$ by scaling the coordinates. This induces an action of G on $\mathscr{M}(n, r)$. The action of this torus scales the symplectic form with a character which we denote by $\hbar$.

We denote the equivariant parameters corresponding to A by $u_1, \cdots, u_r$, and to torus G/A by t_1, t_2 such that the weight of the symplectic form is:

$$\hbar = t_1 t_2$$

8.1 Algebra $\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ and wall subalgebras $\mathscr{U}_\hbar(\mathfrak{g}_w)$

8.1.1

In the special case $r = 1$ the the instanton moduli space coincide with the Hilbert scheme of n points on the complex plane $\mathscr{M}(n,1) = \mathrm{Hilb}^n(\mathbb{C}^2)$. As a vector space, the K-theory of Hilbert schemes can be identified with polynomials on infinite number of variables.

$$\bigoplus_{n=0}^{\infty} K_G(\mathrm{Hilb}^n(\mathbb{C}^2)) = \mathsf{F}(u_1) \overset{def}{=} \mathbb{Q}[p_1, p_2, \cdots] \otimes \mathbb{Q}[u_1^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}] \tag{8.2}$$

If we introduce a grading in the polynomial ring $\mathbb{Q}[p_1, p_2, \cdots]$ by $\deg(p_k) = k$. Then the n-th term on the left side of (8.2) corresponds to degree n.

8.1.2

The fixed point set $\mathrm{Hilb}^n(\mathbb{C}^2)^G$ is discrete. Its elements are labeled by partitions ν with $|\nu| = n$. The structure sheaves of this points $\mathcal{O}_\nu$ with form a basis of the localized K-theory. The polynomials representing the elements of this basis under isomorphism (8.2) are the Macdonald polynomials P_ν in Haiman normalization [17]. To fix the norms we write first several Macdonald polynomials here:

$$P_{[1]} = p_1, \quad P_{[2]} = \frac{1+t_1}{2}\, p_1^2 + \frac{1-t_1}{2}\, p_2, \quad P_{[1,1]} = \frac{1+t_2}{2}\, p_1^2 + \frac{1-t_2}{2}\, p_2$$

$$P_{[3]} = \frac{(1+t_1)(1+t_1+t_1^2)}{6}\, p_1^3 + \frac{(1-t_1)(1+t_1+t_1^2)}{2}\, p_1 p_2 + \frac{(1+t_1)(1-t_1)^2}{3}\, p_3$$

$$P_{[1,1,1]} = \frac{(1+t_2)(1+t_2+t_2^2)}{6}\, p_1^3 + \frac{(1-t_2)(1+t_2+t_2^2)}{2}\, p_1 p_2 + \frac{(1+t_2)(1-t_2)^2}{3}\, p_3$$

$$P_{[2,1]} = \frac{1+t_1 t_2 + 2t_1 + 2t_2}{6}\, p_1^2 + \frac{1-t_1 t_2}{2}\, p_2 p_1 + \frac{(1-t_1)(1-t_2)}{3}\, p_3$$

8.1.3

Assume, that the torus A splits the framing by $\mathsf{w} = u_1 + \cdots + u_r$ then in the notations of Section 2.3.2 we obtain:

$$\bigoplus_{n=0}^{\infty} K_G(\mathscr{M}(n,r)) = \mathsf{F}(u_1) \otimes \cdots \otimes \mathsf{F}(u_r) \tag{8.3}$$

8.1.4

Let us set $\mathbf{Z} = \mathbb{Z}^2$, $\mathbf{Z}^* = \mathbf{Z}\ \{(0,0)\}$ and:

$$\mathbf{Z}^+ = \{(i,j) \in \mathbf{Z}; i > 0 \ \text{ or } \ i = 0, \ \ j > 0\}, \quad \mathbf{Z}^- = -\mathbf{Z}^+$$

Set

$$n_k = \frac{(t_1^{\frac{k}{2}} - t_1^{-\frac{k}{2}})(t_2^{\frac{k}{2}} - t_2^{-\frac{k}{2}})(\hbar^{-\frac{k}{2}} - \hbar^{\frac{k}{2}})}{k}$$

and for vector $\mathbf{a} = (a_1, a_2) \in \mathbf{Z}$ denote by $\deg(\mathbf{a})$ the greatest common divisor of a_1 and a_2. We set $\epsilon_{\mathbf{a}} = \pm 1$ for $\mathbf{a} \in \mathbf{Z}^{\pm}$. For a pair non-collinear vectors we set $\epsilon_{\mathbf{a},\mathbf{b}} = \mathrm{sign}(\det(\mathbf{a},\mathbf{b}))$.

The "toroidal" algebra $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ is an associative algebra with 1 generated by elements $e_{\mathbf{a}}$ and $K_{\mathbf{a}}$ with $\mathbf{a} \in \mathbf{Z}$, subject to the following relations [37]:

- elements $K_{\mathbf{a}}$ are central and

$$K_0 = 1, \quad K_{\mathbf{a}} K_{\mathbf{b}} = K_{\mathbf{a}+\mathbf{b}}$$

- if $\mathbf{a}$, $\mathbf{b}$ are two collinear vectors then:

$$[e_{\mathbf{a}}, e_{\mathbf{b}}] = \delta_{\mathbf{a}+\mathbf{b}} \frac{K_{\mathbf{a}}^{-1} - K_{\mathbf{a}}}{n_{\deg(\mathbf{a})}} \tag{8.4}$$

- if $\mathbf{a}$ and $\mathbf{b}$ are such that $\deg(\mathbf{a}) = 1$ and the triangle $\{(0,0), \mathbf{a}, \mathbf{b}\}$ has no interior lattice points then

$$[e_{\mathbf{a}}, e_{\mathbf{b}}] = \epsilon_{\mathbf{a},\mathbf{b}} K_{\alpha(\mathbf{a},\mathbf{b})} \frac{\Psi_{\mathbf{a}+\mathbf{b}}}{n_1}$$

  where

$$\alpha(\mathbf{a},\mathbf{b}) = \begin{cases} \epsilon_{\mathbf{a}}(\epsilon_{\mathbf{a}}\mathbf{a} + \epsilon_{\mathbf{b}}\mathbf{b} - \epsilon_{\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}))/2 & \text{if } \ \epsilon_{\mathbf{a},\mathbf{b}} = 1 \\ \epsilon_{\mathbf{b}}(\epsilon_{\mathbf{b}}\mathbf{b} + \epsilon_{\mathbf{b}}\mathbf{b} - \epsilon_{\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}))/2 & \text{if } \ \epsilon_{\mathbf{a},\mathbf{b}} = -1 \end{cases}$$

  and elements $\Psi_{\mathbf{a}}$ are defined by:

$$\sum_{k=0}^{\infty} \Psi_{k\mathbf{a}} z^k = \exp\Big(\sum_{m=1}^{\infty} n_m \, e_{m\,\mathbf{a}} z^m\Big)$$

  for $\mathbf{a} \in \mathbf{Z}$ such that $\deg(\mathbf{a}) = 1$.

It is convenient to visualize the algebra $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ as in the Figure 8.1. Heisenberg subalgebras of $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ are labeled by $w \in Q$ and correspond to lines with slope w in this picture.

$e_{(-2,2)}$

Figure 8.1: The line with slope 2 corresponds to Heisenberg subalgebra generated by $e_{k,2k}$ for $k \in \mathbb{Z}$.

8.1.5

The action of $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ on the K-theory (8.2) was constructed in [37]. The central elements act in this representation by: $e_{(-2,1)}$

$$K_{(1,0)} = t_1^{-\frac{1}{2}} t_2^{-\frac{1}{2}}, \quad K_{(0,1)} = 1 \tag{8.5}$$

$e_{(-2,0)}$ $e_{(2,-2)}$

In particular, the "vertical" generators commute commute in (8.2):

$$[e_{(0,m)}, e_{(0,n)}] = 0$$

$e_{(2,-1)}$

and "horizontal" form a Heisenberg subalgebra:

$$[e_{(m,0)}, e_{(n,0)}] = \frac{-m}{(t_1^{m/2} - t_1^{-m/2})(t_2^{m/2} - t_2^{-m/2})} \delta_{n+m}$$

Explicitly, these generators act as follows:

$$e_{(m,0)} = \begin{cases} \dfrac{1}{(t_1^{m/2} - t_1^{-m/2})(t_2^{m/2} - t_2^{-m/2})} p_{-m} & m < 0 \\ \\ -m \dfrac{\partial}{\partial p_m} & m > 0 \end{cases} \tag{8.6}$$

and

$$e_{(0,m)}(P_\nu) = u_1^{-m} \mathrm{sign}(k) \left(\frac{1}{1 - t_1^m} \sum_{i=1}^{\infty} t_1^{m(\lambda_i - 1)} t_2^{m(i-1)} \right) P_\nu \tag{8.7}$$

It is clear that $e_{(0,m)}$ and $e_{(m,0)}$ generate the whole $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$. Thus the last two formulas give explicit presentation of $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ action on the Fock space.

8.1.6

The geometric R -matrix for the instanton moduli space was computed in [27]. It was shown that this R -matrix coincide with R -matrix for $U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$. In particular, this implies that $\mathscr{U}_\hbar(\mathfrak{g}_Q) \simeq U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$.

8.2 R-matrices

8.2.1

Recall, that the quantum Heisenberg algebra is an algebra generated by elements e, f and a cental element K modulo the following relations:

$$[e,e]=[f,f]=0, \quad [e,f]=\frac{K-K^{-1}}{c-c^{-1}} \tag{8.8}$$

The Fock space $\mathsf{F}=\mathbb{Q}[x]\otimes\mathbb{Q}[c^{\pm 1}]$ is a natural module over the Heisenberg algebra with the following cation:

$$e(p)=xp, \quad f(p)=-\frac{dp}{dx}, \quad K(p)=cp$$

so that c is a formal parameter fixing value of cental element K in F. The Heisenberg algebra is a Hopf algebra with the following coproduct:

$$\Delta(e)=e\otimes 1+K^{-1}\otimes e$$

$$\Delta(f)=f\otimes K+1\otimes f$$

$$\Delta(K)=K\otimes K$$

antipode:

$$S(e)=-Ke, \quad S(f)=-K^{-1}f, \quad S(K)=K^{-1}$$

and counit:

$$\varepsilon(e)=\varepsilon(f)=0, \quad \varepsilon(K)=1$$

We consider the tensor product $\mathsf{F}\otimes\mathsf{F} = \mathbb{Q}[x,y]\otimes\mathbb{Q}[c^{\pm 1}]$, and define codimension function by $c^{\Omega}(x^i y^j) = c^{i+j}x^i y^j$. We consider the following upper and lower triangular R-matrices.

$$\mathsf{R}^{+} = c^{-\Omega}\exp(-(c-c^{-1})\, f\otimes e), \quad \mathsf{R}^{-} = c^{-\Omega}\exp(-(c-c^{-1})\, e\otimes f)$$

Proposition 14. *The R-matrices satisfy the QYBE in* $\mathsf{F}^{\otimes 3}$*:*

$$\mathsf{R}^{\pm}_{23}\mathsf{R}^{\pm}_{13}\mathsf{R}^{\pm}_{12} = \mathsf{R}^{\pm}_{12}\mathsf{R}^{\pm}_{13}\mathsf{R}^{\pm}_{23}$$

and have the following properties:

$$\mathsf{R}^{+}\Delta = \Delta_{21}\mathsf{R}^{+}, \quad \mathsf{R}^{-}\Delta_{21} = \Delta\mathsf{R}^{-}$$

where Δ_{21} *is the opposite coproduct, and*

$$1\otimes\Delta(\mathsf{R}^{+}) = \mathsf{R}^{+}_{13}\mathsf{R}^{+}_{12}, \quad \Delta\otimes 1(\mathsf{R}^{+}) = \mathsf{R}^{+}_{13}\mathsf{R}^{+}_{23}$$

$$1\otimes\Delta(\mathsf{R}^{-}) = \mathsf{R}^{-}_{12}\mathsf{R}^{-}_{13}, \quad \Delta\otimes 1(\mathsf{R}^{-}) = \mathsf{R}^{-}_{23}\mathsf{R}^{-}_{13}$$

8.2.2

The Picard group $\mathrm{Pic}(X) = \mathbb{Z}$ is generated by $\mathcal{O}(1)$. It acts on $H^2(X,\mathbb{R}) = \mathbb{R}$ by shifts. The explicit computation of stable map [16] for a slope $s\in H^2(X,\mathbb{R})$ for $\mathscr{M}(n,r)$ shows that Stab^s is a locally constant function which changes only at the walls:

$$\text{walls} = \{w = \frac{a}{b}\in\mathbb{R} \,:\, a\in\mathbb{Z}, \quad b\in\{1,2,...,n\}\}$$

Therefore, the set of walls for $\mathscr{M}(r) = \coprod_{n=0}^{\infty}\mathscr{M}(n,r)$ is identified with rational numbers $\mathbb{Q}\subset\mathbb{R}$.

8.2.3

For $w\in\mathbb{Q}\cup\{\infty\}$ we denote by $d(w)$ and $n(w)$ the denominator and numerator of rational number. We set $d(\infty)=0$ and $n(\infty)=1$. From (8.4) we see that

$$\alpha^w_k = e_{(d(w)k,n(w)k)}, \quad k\in\mathbb{Z}$$

generate a Heisenberg subalgebra of $H_w\subset U_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ with the following relations:

$$[\alpha^w_{-k},\alpha^w_k] = \frac{K^{kd(w)}_{(1,0)} - K^{-kd(w)}_{(1,0)}}{n_k}$$

As shown in [27] the wall subalgebra $\mathscr{U}_\hbar(\mathfrak{g}_w)\subset\mathscr{U}_\hbar(\widehat{\mathfrak{g}}_Q)$ gets identified with this Heisenberg subalgebra $\mathscr{U}_\hbar(\mathfrak{g}_w) = H_w$.

8.2.4

We conclude, that the R-matrix R^+_w for the wall $w \in \mathbb{Q}$ corresponding to the Heisenberg subalgebra $\mathscr{U}_\hbar(\mathfrak{g}_w)$ takes the form:

$$R^+_w = \prod_{k=0}^{\infty} \exp(-n_k\, \alpha^w_k \otimes \alpha^w_{-k}) = \exp\Big(- \sum_{k=0}^{\infty} n_k\, \alpha^w_k \otimes \alpha^w_{-k} \Big)$$

The lower triangular R-matrix is obtained by the transposition:

$$R^-_w = \prod_{k=0}^{\infty} \exp(-n_k\, \alpha^w_{-k} \otimes \alpha^w_{k}) = \exp\Big(- \sum_{k=0}^{\infty} n_k\, \alpha^w_{-k} \otimes \alpha^w_{k} \Big)$$

As the cental element of the elliptic Hall algebra acts in the Fock K-theory by $K_{(1,0)} = \hbar^{-1/2}$ cental parameter c of the quantum Heisenber algebra generated by $e = \alpha^w_{-k}$ and $f = \alpha^w_k$ is given by $c = \hbar^{-kd(w)/2} = (t_1 t_2)^{-kd(w)/2}$.

8.2.5

Let us fix a slope $s \in H^2(X, \mathbb{R}) = \mathbb{R}$. The Khoroshkin-Tolstoy factorization (2.19) provides the following universal formula for the total R-matrix:

$$\mathscr{R}^s(u) = \prod_{\substack{w\in\mathbb{Q}\\ w<s}}^{\rightarrow} R^-_w\, R_\infty \prod_{\substack{w\in\mathbb{Q}\\ w>s}}^{\leftarrow} R^+_w \tag{8.9}$$

The infinite slope R -matrix R_∞ is the operator of multiplication by normal bundles (2.24). From explicit formula for action of α^∞_k (8.7) we can obtain:

$$R_\infty = \exp\Big(- \sum_{k=0}^{\infty} n_k\, \alpha^\infty_{-k} \otimes \alpha^\infty_{k} \Big)$$

This, together with formulas from the previous section give the following beautiful universal expression for a slope s R-matrix:

$$\mathscr{R}^s(u) = \prod_{w\in\mathbb{Q}\cup\{\infty\}}^{\leftarrow s} \exp\Big(- \sum_{k=0}^{\infty} n_k\, \alpha^w_{-k} \otimes \alpha^w_{k} \Big)$$

8.3 The quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$

8.3.1

Assume that $r = 2$ such that $K_G(\mathscr{M}(r)^{\mathsf{A}}) = \mathsf{F}(u_1) \otimes \mathsf{F}(u_2)$. In this situation

$$\mathscr{M}(n,2)^{\mathsf{A}} = \coprod_{n_1+n_2=n} \mathrm{Hilb}^{n_1}(\mathbb{C}^2) \times \mathrm{Hilb}^{n_2}(\mathbb{C}^2)$$

The Cartan matrix for the Jordan quiver $C = 0$ and therefore the codimension function (2.27) takes the form:

$$\Omega = \frac{\operatorname{codim} F}{4} = \frac{n_1}{2} + \frac{n_2}{2} = \frac{n}{2}$$

In particular it acts as multiplication by a scalar on $K_G(\mathscr{M}(n,2)^{\mathsf{A}})$. Therefore the ABRR equation (5.3) for a wall $w \in \mathbb{Q}$ takes the form:

$$R_w^- \hbar^{-\lambda} J_w^-(z) = J_w^-(z) \hbar^{-\lambda} \tag{8.10}$$

We are looking for a strictly lower-triangular solution $J_w^-(z) \in \mathscr{U}_\hbar(\mathfrak{g}_w)^{\otimes 2}$ which means that $J_w^-(z)$ is of the form:

$$J_w^-(z) = \exp\Big(\sum_{k=0}^{\infty} J_k(z)\, \alpha_{-k}^w \otimes \alpha_k^w \Big)$$

Simple computation gives the following solution:

$$J_w^-(z) = \exp\Big(- \sum_{k=0}^{\infty} \frac{n_k\, K_{(1,0)} \otimes K_{(1,0)}^{-1}}{1 - z^{-kd(w)} K_{(1,0)} \otimes K_{(1,0)}^{-1}}\, \alpha_{-k}^w \otimes \alpha_k^w \Big)$$

8.3.2

The shift $\lambda \to \lambda - \tau_w$ corresponds to substitution $z \to z q^{-w}$. Thus by definition (5.7) we obtain:

$$\mathbf{J}_w^-(z) = \exp\Big(- \sum_{k=0}^{\infty} \frac{n_k\, K_{(1,0)} \otimes K_{(1,0)}^{-1}}{1 - z^{-kd(w)} q^{kn(w)} K_{(1,0)} \otimes K_{(1,0)}^{-1}}\, \alpha_{-k}^w \otimes \alpha_k^w \Big) \tag{8.11}$$

8.3.3

From Section 8.2.1 is is clear that the antipode in $\mathscr{U}_\hbar(\mathfrak{g}_w)$ has the following form:

$$S_w(\alpha_k^w) = -K_{(1,0)}^{-kd(w)} \alpha_k^w$$

From this we obtain that:

$$\mathbf{m}\big(1 \otimes S_w(\mathbf{J}_w^-(z)^{-1})\big) =: \exp\Big(- \sum_{k=0}^{\infty} \frac{n_k\, K_{(1,0)}^{kd(w)}}{1 - z^{-kd(w)} q^{kn(w)} K_{(1,0)}^{2kd(w)}}\, \alpha_{-k}^w \alpha_k^w \Big) :$$

The symbol :: stands for the normal ordering meaning that all "annihilation" operators α_k^w with $k > 0$ act first.

8.3.4

The Cartan matrix of the Jordan quiver is trivial $C = 0$ and therefore $\kappa = (C\mathsf{v} - \mathsf{w})/2 = -r/2$. The shift $\lambda \to \lambda + \kappa$ thus corresponds to $z \to z\hbar^{-r/2}$, of for all r at once $z \to z\, K_{(1,0)}$. From (5.9) we obtain:

$$\mathbf{B}_w(z) =: \exp\Big(\sum_{k=0}^{\infty} \frac{n_k\, K_{(1,0)}^{kd(w)}}{1 - z^{-kd(w)} q^{kn(w)} K_{(1,0)}^{kd(w)}}\, \alpha_{-k}^{w} \alpha_{k}^{w}\Big):$$

8.3.5

Let $\mathscr{L} = \mathcal{O}(1)$ be the generator of Picard group. Let $\nabla \subset \mathbb{R}$ be the alcove specified by Theorem 5. If $s \in \nabla$, then the interval $(s, s - \mathscr{L})$ contains all walls $w \in \mathbb{Q}$ such that $-1 \leq w < 0$. Therefore, by definition (5.11) we obtain the following explicit formula for quantum difference operator:

$$\boxed{\mathbf{M}_{\mathcal{O}(1)}(z) = \mathcal{O}(1) \prod_{\substack{w\in\mathbb{Q} \\ -1\leq w<0}}^{\leftarrow} :\exp\Big(\sum_{k=0}^{\infty} \frac{n_k\, \hbar^{-krd(w)/2}}{1 - z^{-kd(w)} q^{kn(w)} \hbar^{-krd(w)/2}}\, \alpha_{-k}^{w} \alpha_{k}^{w}\Big):} \tag{8.12}$$

where we used that in the K-theory of instanton moduli space $\mathscr{M}(n, r)$ the central element acts by the scalar $K_{(1,0)} = \hbar^{-r/2}$.

8.3.6

Let us consider some limits of the difference operator. First, for all terms in in the previous formula $d(w) > 0$ and $n(w) < 0$. Thus we have:

$$\lim_{q\to 0} \mathbf{M}_{\mathcal{O}(1)}(z) = \lim_{z\to 0} \mathbf{M}_{\mathcal{O}(1)}(z) = \mathcal{O}(1)$$

Second, to compute the limit of $\mathbf{M}_{\mathcal{O}(1)}(zq^{-1})$ as $q \to \infty$ we note that for all terms in (8.12) $d(w) + n(w) \geq 0$. Moreover $d(w) + n(w) = 0$ only for $w = -1$. We conclude that:

$$\lim_{q\to\infty} \mathbf{M}_{\mathcal{O}(1)}(zq^{-1}) = \mathcal{O}(1) : \exp\Big(\sum_{k=0}^{\infty} \frac{n_k\, \hbar^{-kr/2}}{1 - z^{-k} \hbar^{-kr/2}}\, \alpha_{-k}^{-1} \alpha_{k}^{-1}\Big):$$

In the quantum K-theory the glueing matrix can be defined as the limit:

$$\mathbf{G} = \lim_{q\to\infty} \mathbf{M}_{\mathscr{L}}(zq^{-1}) \mathscr{L}^{-1}$$

for ample $\mathscr{L}$. We note, that $\mathcal{O}(1)\alpha_k^w\mathcal{O}(1)^{-1} = \alpha_k^{w+1}$, which gives the following formula for $\mathbf{G}$ matrix:

$$\mathbf{G} =: \exp\Big(\sum_{k=0}^{\infty} \frac{n_k\,\hbar^{-kr/2}}{1-z^{-k}\hbar^{-kr/2}}\,\alpha^0_{-k}\alpha^0_k\Big):$$

The action of "horizontal" Heisenberg algebra α_k^0 on the K-theory is given by (8.6). Using these formula gluing matrix can be easily computed explicitly.

8.3.7

Let us consider the example of $X = \mathrm{Hilb}^2(\mathbb{C}^2)$. The walls which contribute to (8.12) are $w = -1$ and $w = -1/2$. The quantum difference operator takes the form:

$$\mathcal{A}_{\mathcal{O}(1)} = T_z^{-1}\mathcal{O}(1)\mathbf{B}_{-1}(z)\mathbf{B}_{-\frac{1}{2}}(z)$$

Using the identity (6.12) we can also write it in the form:

$$\mathcal{A}_{\mathcal{O}(1)} = \mathbf{B}_0(z)\mathbf{B}_{\frac{1}{2}}(z)\mathcal{O}(1)T_z^{-1}$$

which means that:

$$\mathbf{M}_{\mathcal{O}(1)}(zq^{-1}) = \mathbf{B}_0(z)\mathbf{B}_{\frac{1}{2}}(z)\mathcal{O}(1)$$

We consider the basis of stable envelopes $s_{[1,1]}$ $s_{[2]}$. This basis in the Fock space is given by the plethystic Schur polynomials [16].In this basis we obtain explicitly:

$$\mathbf{B}_0(z\hbar^{1/2}) = \frac{z-1}{\left(z^2{t_1}^2{t_2}^2-1\right)\left(zt_1t_2-1\right)}\begin{bmatrix} z^2t_1t_2-1 & -\left(t_1t_2-1\right)z \\ -\left(t_1t_2-1\right)z & z^2t_1t_2-1 \end{bmatrix}$$

$$\mathbf{B}_{\frac{1}{2}}(z\hbar^{1/2}) = 1 + \frac{z^2\left(t_1t_2-1\right)}{z^2{t_1}^2{t_2}^2-q}\begin{bmatrix} -1 & t_2 \\ t_1 & -t_1t_2 \end{bmatrix}$$

$$\mathcal{O}(1) = \begin{bmatrix} t_2 & 0 \\ -t_1t_2+1 & t_1 \end{bmatrix}$$

8.3.8

Let us consider the example of $X = \mathrm{Hilb}^2(\mathbb{C}^3)$. Similarly, in this case the quantum difference operator takes the form:

$$\mathbf{M}_{\mathcal{O}(1)}(zq^{-1}) = \mathbf{B}_0(z)\mathbf{B}_{\frac{1}{3}}(z)\mathbf{B}_{\frac{1}{2}}(z)\mathbf{B}_{\frac{2}{3}}(z)\mathcal{O}(1)$$

In the stable basis ordered as $s_{[1,1,1]}, s_{[2,1]}, s_{[3]}$ we obtain explicitly:

$$\mathbf{B}_0(z\hbar^{1/2}) = \frac{(z-1)(zt_1t_2+1)(t_1t_2-1)}{(z^3{t_1}^3{t_2}^3-1)(z^2{t_1}^2{t_2}^2-1)(zt_1t_2-1)}\times$$

$$\begin{bmatrix} \frac{\left(z^2t_1t_2-1\right)\left(z^3{t_1}^2{t_2}^2-1\right)}{(t_1t_2-1)(zt_1t_2+1)} & -z\left(z^2t_1t_2-1\right) & \frac{z^2\left(z{t_1}^2{t_2}^2-1\right)}{zt_1t_2+1} \\ -z\left(z^2t_1t_2-1\right) & \frac{z^4{t_1}^2{t_2}^2-z^3{t_1}^2{t_2}^2+z^2{t_1}^2{t_2}^2-2\,z^2t_1t_2+z^2-z+1}{t_1t_2-1} & -z\left(z^2t_1t_2-1\right) \\ \frac{z^2\left(z{t_1}^2{t_2}^2-1\right)}{zt_1t_2+1} & -z\left(z^2t_1t_2-1\right) & \frac{\left(z^2t_1t_2-1\right)\left(z^3{t_1}^2{t_2}^2-1\right)}{(t_1t_2-1)(zt_1t_2+1)} \end{bmatrix}$$

$$\mathbf{B}_{1/3}(z\hbar^{1/2}) = 1 + \frac{z^3(t_1t_2-1)}{z^3{t_1}^3{t_2}^3-q}\begin{bmatrix} -1 & t_2 & -{t_2}^2 \\ t_1 & -t_1t_2 & t_1{t_2}^2 \\ -{t_1}^2 & {t_1}^2t_2 & -{t_2}^2{t_1}^2 \end{bmatrix}$$

$$\mathbf{B}_{1/2}(z\hbar^{1/2}) = 1 + \frac{z^2t_1t_2(t_1t_2-1)}{z^2{t_1}^2{t_2}^2-q}\times$$

$$\begin{bmatrix} -\frac{t_1t_2+t_1-1}{{t_1}^2t_2} & \frac{t_1t_2-1}{{t_1}^2} & \frac{t_2}{{t_1}^2} \\ \frac{(t_1t_2-1)(t_1t_2+t_1-1)}{{t_2}^2{t_1}^2} & -\frac{(t_1t_2-1)^2}{{t_1}^2t_2} & -\frac{t_1t_2-1}{{t_1}^2} \\ -\frac{(t_1t_2+t_1-1)(t_1t_2-t_1-1)}{t_1{t_2}^2} & \frac{(t_1t_2-1)(t_1t_2-t_1-1)}{t_1t_2} & \frac{t_1t_2-t_1-1}{t_1} \end{bmatrix}$$

$$\mathbf{B}_{2/3}(z\hbar^{1/2}) = 1 + \frac{z^3(t_1t_2-1)t_1t_2}{z^3{t_1}^3{t_2}^3-q^2}\times$$

$$\begin{bmatrix} \frac{t_1{t_2}^2-t_1-t_2}{{t_1}^2t_2} & -\frac{t_2(t_1t_2-t_1-1)}{{t_1}^2} & -\frac{{t_2}^2}{{t_1}^2} \\ -\frac{(t_1t_2+t_1-1)\left(t_1{t_2}^2-t_1-t_2\right)}{{t_2}^2{t_1}^2} & \frac{(t_1t_2+t_1-1)(t_1t_2-t_1-1)}{{t_1}^2} & \frac{t_2(t_1t_2+t_1-1)}{{t_1}^2} \\ \frac{\left({t_1}^2+t_1t_2-1\right)\left(t_1{t_2}^2-t_1-t_2\right)}{t_1{t_2}^2} & -\frac{(t_1t_2-t_1-1)\left({t_1}^2+t_1t_2-1\right)}{t_1} & -\frac{t_2\left({t_1}^2+t_1t_2-1\right)}{t_1} \end{bmatrix}$$

$$\mathcal{O}(1) = \begin{bmatrix} {t_2}^3 & 0 & 0 \\ -\left(t_1t_2-1\right)\left(t_2+1\right)t_2 & t_1t_2 & 0 \\ \left(t_1t_2-1\right)\left(t_2{t_1}^2+{t_2}^2t_1-1\right) & -\left(t_1t_2-1\right)\left(t_1+1\right)t_1 & {t_1}^3 \end{bmatrix}$$

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