Methods for Computing Genus Distribution Using Double-Rooted Graphs

Imran Farid Khan

Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2012

ABSTRACT

Methods for Computing Genus Distribution Using Double-Rooted Graphs

Imran Farid Khan

This thesis develops general methods for computing the *genus distribution* of various types of graph families, using the concept of *double-rooted graphs*, which are defined to be graphs with two vertices designated as roots (the methods developed in this dissertation are limited to the cases where one of the two roots is restricted to be of valence two). I define *partials* and *productions*, and I use these as follows: *(i)* to compute the genus distribution of a graph obtained through the vertex amalgamation of a double-rooted graph with a single-rooted graph, and to show how these can be used to obtain recurrences for the genus distribution of iteratively growing infinite graph families. *(ii)* to compute the genus distribution of a graph obtained *(a)* through the operation of self-vertex-amalgamation on a double-rooted graph, and *(b)* through the operation of edge-addition on a double-rooted graph, and finally *(iii)* to develop a method to compute the recurrences for the genus distribution of the graph family generated by the Cartesian product $P_3 \Box P_n$.

Table of Contents

List of Figures

List of Tables

Acknowledgments

I want to express my deepest gratitude to my PhD advisor, Professor Jonathan L. Gross, not only for his excellent teaching and academic guidance, but for his patience and kindness in helping me throughtout. He has become the archetypical PhD supervisor for me, and a role-model I can only wish I can live up to.

The list of acknowledgements will remain incomplete without Professor Irfan Ullah Chaudhary and Omar Khan, who were my teachers during my undergraduate years at NUCES-FAST. They helped me in innumerable ways, especially in overcoming both personal and systemic shortcomings, and in broadening my perspective.

All thanks to my wife and research partner, Mehvish, for giving me enough energy to go through the slow, tedious and uncertain process of applying to graduate schools and obtaining a scholarship, and for keeping me focussed.

Lastly, I would also like to acknowledge Fulbright/HEC/USAID scholarship program that made my studies here possible.

Dedicated to my parents: Farid-uz-Zaman
and Mujahida Khanum

Chapter 1

Introduction

Graphs are defined combinatorially, and although graphs are often drawn on paper in order to illustrate various concepts intuitively, almost everything to be said about graphs can be said without referring to such drawings. A distinguishing feature of topological graph theory is that it provides a topological structure to a graph, in order to study *embeddings* of a graph on *surfaces*. This structure makes it possible to refer to a drawing of a graph in a mathematically precise way, and it also becomes possible to discuss different drawings of the same graph.

Though much of the topological graph theory employs advanced combinatorial techniques, the topology remains indispensible. Very little knowledge of topology is assumed here, and we will provide the necessary definitions with brief explanations.

1.1 Basic Concepts and Definitions

We take a ***graph*** to be connected; a graph need not be simple, i.e., it may have self-loops and multiple edges between two vertices. We use the words ***degree*** and ***valence*** of a vertex to mean the same thing. Each edge has two ***edge-ends***, in the topological sense, even if it has only one endpoint.

A ***surface*** is defined to be a two-dimensional manifold, which we envision as having a model that is embedded in some Euclidean space; it is ***closed*** if its model is (a) boundaryless, (b) finite (in the sense that there exists a real number such that the maximum distance between any two points of the surface is less than that real number), and (c) the end-points of any open arc in the surface are in the surface itself.

A theorem of topology classifies the closed surfaces into two categories: *orientable* and *non-orientable* (an orientable surface is a surface in which one can define and rely on a global notion of "left" and "right", whereas in a non-orientable surface, one cannot). All of the surfaces within these two categories are completely known. Figure 1.1 illustrates the surfaces: in the top row we give the orientable surfaces, i.e. sphere, torus, double-torus, etc, and in the bottom row we give representations of the non-orientable surfaces, i.e. surface with one crosscap (also known as the projective plane), surface with two crosscaps (also known as the Klein bottle), surface with three crosscaps, etc. A crosscap is constructed by first excising an open disc from the sphere, and then closing off the boundary created with a Möbius band.

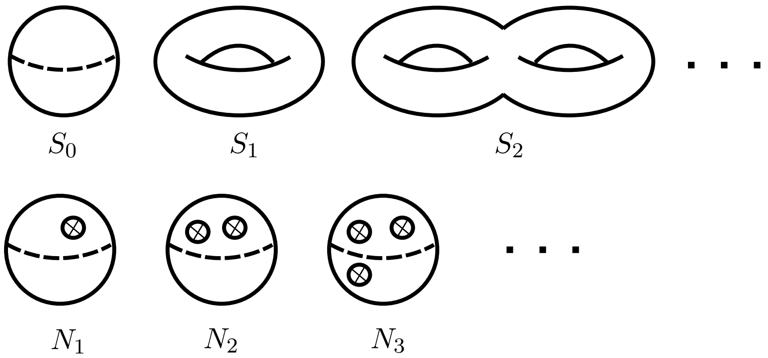


Figure 1.1: Classification of closed surfaces.

All of the surfaces mentioned hereafter in this document are assumed to be both orientable and closed.

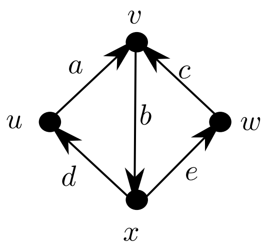
An ***embedding*** is represented by a crossing-free drawing of a graph on a surface. More precisely, an embedding ι_G of a graph G on a surface S_G is constructed as follows: we define a function f_{V_G} that maps each of the vertices of the graph G to distinct points on S_G; then for each edge e with endpoints u and v, we define a continuous, bijective function $f_e : I = [0, 1] \to S_G$, where $f_e(0) := f_{V_G}(u)$ and $f_e(1) := f_{V_G}(v)$ such that *(i)* the interior of $f_e(I)$ does not intersect any of the points in $f_{V_G}(V_G)$, and *(ii)* for any two edges e and d, the intersection of the interiors of $f_e(I)$ and $f_d(I)$ is empty. Every edge e thus has two

edge-ends, i.e. near $f_e(0)$ an edge-end e^+, and near $f_e(1)$ an edge-end e^-. We visualize an edge-end as a short half-open line-segment that contains a little bit of the interior of the edge as well, rather than just the endpoint.

The ***regions*** of an embedding $\iota_G \to S_G$ are defined to be the connected components of $S_G \backslash i_G$. If each region of an embedding is *homeomorphic* to an open disc, then that embedding is said to be a *cellular* embedding. A ***face*** is defined to be a region along with its boundary. All of the embeddings discussed in this document are assumed to be cellular.

The ***genus*** of an embedding of a graph G is defined to be the genus of the surface on which the graph is embedded. To compute the genus g of an embedding, we use ***Euler's polyhedral formula*** for orientable surfaces, i.e. $\#v - \#e + \#f = 2 - 2g$, where $\#v, \#e$ and $\#f$ correspond to the number of vertices, the number of edges and the number of faces of the embedding, respectively.

A ***rotation at a vertex*** v is defined to be a cyclic permutation of the edge-ends incident on vertex v. These edge-ends are then interpreted as being cyclically incident on vertex v in the counter-clockwise direction (the choice of the direction is arbitrary, but it needs to remain consistent within an analysis). A ***rotation system of a graph*** G is an assignment of rotations to all vertices of the graph, as illustrated in Figure 1.2.



v : c^- a^- b^+
u : a^+ d^-
x : e^+ b^- d^+
w : c^+ e^-

Figure 1.2: A spherical embedding and the corresponding rotation system

Since there are $(deg(v) - 1)!$ different rotations that are possible at a vertex, the total number of distinct rotation systems is

$$\prod_{v \in V_G} (deg(v) - 1)!$$

It is easy to see that every embedding induces a unique rotation system of a graph, and with the aid of *Heffter-Edmonds face-tracing algorithm*, it can be shown that every rotation system of a graph induces a unique *equivalence class* of embeddings (two embeddings are said to be ***equivalent*** if they have the same set of face-boundary walks, also abbreviated as ***fb-walks***).

The ***Heffter-Edmonds face-tracing algorithm*** takes as input a rotation system of a graph G. Using this algorithm, we start constructing an fb-walk with any edge-end, say k^+, that has not yet been explored, and check which rotation of a vertex has the other edge-end (i.e. k^-) in it. Then we read the edge-end that follows k^-, say l^σ (where $\sigma \in \{+,-\}$), and write it after k^+ in the fb-walk being constructed. We keep on doing this until we arrive back at k^+, at which point we consider the construction of fb-walk to be complete. Similarly, we continue listing other fb-walks until all edge-ends have been explored. For instance in Figure 1.2, if we start reading the fb-walk from the edge-end e^+, then the next edge-end in the fb-walk would be c^+ since it follows e^- in the rotation system. Continuing this way, we would have $(e^+c^+a^-d^-)$, $(e^-b^-c^-)$ and $(a^+b^+d^+)$ as the three fb-walks. The list of fb-walks can then be used to construct an embedding of the graph on the surface of the genus indicated by the number of fb-walks using Euler's polyhedral equation. For instance, in Figure 1.2, we have $4 - 5 + 3 = 2 - 2g$, correctly giving $g = 0$ as the genus of the graph embedding corresponding to the given rotation system.

The problem of computation of the *genus distribution* of a graph is concerned with the genus of all orientable embeddings of that graph, i.e. the genus distribution is the sequence $g_0(G), g_1(G), ...$ where $g_i(G)$ represents the number of embeddings of the graph G on surface of genus i. Clearly,

$$\sum_{i=0} g_i(G) = \prod_{v \in V_G} (deg(v) - 1)!$$

.

1.2 Related Work

Computation of the genus distribution was first introduced as an invariant for graphs in 1987 by Gross and Furst in [Gross and Furst, 1987]. Explicit formulas for the genus distribution of two families of graphs (closed-end ladders and cobblestone paths) were computed by Furst, Gross and Statman in [Furst *et al.*, 1989]. These were the first explicit computations of genus distributions for infinite families of graphs. Later, McGeoch, in his Phd thesis [McGeoch, 1987], computed the genus distributons for circular and Möbius ladders (the seeming chronological discrepency arises from the accelerated publication date of a thesis). Gross, Robbins and Tucker were able to compute the the genus distribution of bouquets of circles in [Gross *et al.*, 1989], using a combinatorial formula of Jackson [Jackson, 1987]. In 1989, it was shown by Thomassen [Thomassen, 1989] that the minimum genus problem is NP-complete. However, it was still possible to compute the genus distributions for many families of graphs, for instace, those having a high-degree of symmetry, so the interest in this problem continued. In 1990 Rieper, in his Phd thesis [Rieper, 1990], was able to use group characters to enumerate the genus distribution of dipoles. Over the next two decades, there have been numerous investigations of genus distributions. These include [Chen *et al.*, 2011b], [Kwak and Lee, 1993], [Kwak and Lee, 1994], [Mull, 1999], [Stahl, 1990], [Stahl, 1991a], [Stahl, 1991b], [Tesar, 2000], [Visentin and Wieler, 2007], [Wan and Liu, 2006], and [Wan and Liu, 2008].

Researchers have developed several methods to compute the genus distributions, as well as *total embedding distributions* (that include orientable as well as non-orientable embeddings). Most of these methods are very specific to the particular graph families whose genus distribution is being computed, as they rely on the particular symmetries in those graph families. Of note among the methods that also have some general applicability, are methods that employ overlap matrices and edge-attaching surgery.

Overlap matices were first defined and employed by Mohar in [Mohar, 1989]. The main theorem states that if a certain overlap matrix is associated with a graph embedding, then the genus of the embedding is equal to half the rank of the associated overlap matrix if the embedding is orientable, or is equal to it if the embedding is non-orientable. This theorem was later used by Chen, Gross and Rieper in [Chen *et al.*, 1994] to derive recurrences for the rank-distribution polynomial of certain graph families (necklaces, closed-end ladders, and cobblestone paths), that can then be used to compute the total embedding distributions of a graph family. Subsequently, this method has been used to compute genus distributions of numerous graph families. For instance, [Chen *et al.*, 2011d] uses these methods to derive recurrences for rank-distribution polynomial for the Ringel ladders (also see [Chen *et al.*, 2006]). In order to compute total embedding distributions of a graph family using this method, recursively defined rank-distribution polynomials need to be set-up for that graph family (which is nontrivial for all but very few simple graph families). By solving the recurrences, it has been possible to compute closed formulas for genus distributions of a few graph families.

Edge-attaching surgery was developed in [Kwak and Shim, 2002] to compute the total embedding distribution of bouquets B_n (and dipoles D_n) by considering three different ways of attaching an edge to B_{n-1}: *(i)* attaching it in one face with a twist, *(ii)* without a twist, and *(iii)* attaching it so that one edge-end is in one face and the second edge-end in another face (with or without a twist – where by the term *twist* we mean that traversing the edge reverses the rotational sense). In this process, the method recursively generates all embeddings of $B_n = B_{n-1} + e$. In order to use this method, each iteration of the computation needs to be explicitly carried out. That is, the embedding distribution of B_i is computed from that of B_{i-1} for each $i < n - 1$. The embedding distribution polynomial of B_{n-1} contains information about the number of embeddings of each type $t_1 \cdots t_{2(i-1)}$, where $2(i-1)$ is equal to twice the number of edges of the bouquet B_{i-1}, and where t_k represents that there are t_k k-sided regions in the embedding (for instance, the embedding given in Figure 1.2 is of type $t_3^2 t_4$). This method is hard to generalize, and it was used for computing total embedding distributions only of bouquets and dipoles.

Our recent work, starting with [Gross *et al.*, 2010] and including [Gross, 2011c], [Gross, 2011b], [Poshni *et al.*, 2010], [Poshni *et al.*, 2012], [Khan *et al.*, 2010], [Khan *et al.*, 2011], is general in its application, in the sense that no graph-family-specific proofs are needed. Morever, there are infinitely many graph families whose genus distribution can be computed using these methods. The content of [Khan *et al.*, 2010] and [Khan *et al.*, 2011] is the basis for Chapters 2–4 of this thesis.

Most of the graph-families for which genus distribution has been computed are *linear graph-families*. By this we mean that there is one fundamental graph whose iterated amalgamations with itself define the infinite graph family. The earliest such linear graph-families for which the genus distribution was computed were, as mentioned earlier, closed-end ladders and cobblestone paths [Furst *et al.*, 1989]. The method used there was significantly generalized in [Gross *et al.*, 2010], using the concept of *rooted partitioned genus distributions.* The key idea was based on having a partitioned genus distribution of a double-rooted graph (G, u, v), where $deg(u) = deg(v) = 2$, that contained information about the number of fb-walks that are incident on root vertices u and v, and also about the characteristics of these fb-walks. This information was shown to be enough for deriving recurrences for the genus distribution of an infinite family of graphs that is iteratively constructed from the base graph (G, u, v) using repeated vertex amalgamations. Chapter 2 presents a significant generalization of these results.

Using these ideas, [Gross, 2011c] develops methods for computing the genus distribution of the self-vertex-amalgamation of a double-rooted graph (G, u, v), i.e. the genus distribution of the graph obtained via amalgamating the two root vertices with each other. Chapters 3 and 4 develop a generalization of this.

Other important results related to the methods developed in this work are concerned with computations of genus distributions using *edge-amalgamation* [Poshni *et al.*, 2010], *self-edge-amalgamation* [Poshni *et al.*, 2012], and of non-linear graph families like *cubic outerplanar graphs* [Gross, 2011b], *4-regular outerplanar graphs* [Poshni *et al.*, 2011], *3-regular Halin graphs* [Gross, 2011a], and $P_3 \square P_n$ [Khan *et al.*, 2012] – this also makes up the last chapter of this thesis.

1.3 Thesis Statement

As mentioned earlier, my work is concerned with developing methods for computing the genus distributions of graph families. The methods that I have developed are more general than the methods that already exist, and they can be used to compute genus distributions of infinitely many graph families, without proving graph-family specific theorems.

Chapters 2–4 of my thesis develop methods that can be used to compute

- the genus distribution of the graph families obtained through iterative amalgamations of a double-rooted graph (G, u, v) with itself, where $deg(u) \geq 2$ and $deg(v) = 2$. We refer to these as *open chains* (Chapters 2 and 4),
- the genus distribution of *closed chains*, obtained though self-vertex-amalgamation of the two *root vertices* of the open chains with each other, and the genus distributions of the graph family obtained through adding an edge between the root vertices of the open chain (Chapters 3 and 4).

Finally, the last part (Chapter 5) of my research gives a graph-family specific computation, that has the potential of being generalized. I give a surgical method that can be used with some of the structures developed in Chapters 3 and 4 to compute the genus distributions of the graph family generated by the Cartesian product $P_3 \Box P_n$.

Part I

Thesis

Chapter 2

Vertex Amalgamation

This chapter is concerned with counting the embeddings of a graph in a surface. In [Gross *et al.*, 2010], we showed how to calculate the genus distribution of an iterated amalgamation of copies of a graph whose genus distribution is already known and is further analyzed into a *partitioned genus distribution* (which is defined for a *double-rooted graph*). Our methods were restricted there to the case with two 2-valent roots. In this chapter we substantially extend the method in order to allow one of the two roots to have arbitrarily high valence.

2.1 Introduction

By the ***vertex-amalgamation*** of the rooted graphs (G, t) and (H, u), we mean the graph obtained from their disjoint union by merging the roots t and u. We denote the operation of amalgamation by an asterisk, i.e.,

$$(G, t) * (H, u) \;=\; (X, w)$$

where X is the amalgamated graph and w the merged root.

Remark Some of the calculations in this chapter are quite intricate, and it appears that taking the direct approach here to amalgamating two graphs at roots of arbitrarily high degree might be formidable. We observe that a vertex of arbitrary degree can be split (by inverse contraction) into two vertices of smaller degree. Effects on the genus distribution that arise from splitting a vertex are described by [Gross, 2010].

Embeddings induced by an amalgamation of two embedded graphs

We say that the pair of embeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ ***induce*** the set of embeddings of $X = G * H$ whose rotations have the same cyclic orderings as in G and H, and that this set of embeddings of X is the result of ***amalgamating the two embeddings*** $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$.

Proposition 1. *For any two embeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ of graphs into surfaces, the number of embeddings of the amalgamated graph $(X, w) = (G, t) * (H, u)$ whose rotation systems are consistent with the embeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ is*

$$(deg(u) + deg(t) - 1) \cdot \binom{deg(t) + deg(u) - 2}{deg(u) - 1} \tag{2.1}$$

Proof. Formula (2.1) is a symmetrization of Formula (1.1) of [Gross *et al.*, 2010]. □

In the amalgamation $(G, t) * (H, u) = (X, w)$, when one of the roots t and u is 1-valent, the genus distribution of the resulting graph is easily derivable via bar-amalgamations (see [Gross and Furst, 1987]). For the case where

$$deg(t) = deg(u) = 2,$$

methods for calculating the genus distribution are developed in [Gross *et al.*, 2010]. For the purposes of this chapter, we assume that $deg(t) = 2$ and $deg(u) = n \geq 2$. A pair of such embeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ induce, in accordance with Formula (2.1), $n^2 + n$ embeddings of the amalgamated graph X. We observe that for each such embedding $\iota_X : X \to S_X$, we have

$$\gamma(S_X) = \begin{cases} \gamma(S_G) + \gamma(S_H) & \text{or} \\ \gamma(S_G) + \gamma(S_H) + 1 \end{cases}$$

Terminology The difference $\gamma(S_X) - (\gamma(S_G) + \gamma(S_H))$ is called the ***genus increment of the amalgamation***, or more briefly, the ***genus increment*** or ***increment***.

Proposition 2. *In any vertex-amalgamation $(G, t) * (H, u) = (X, w)$ of two graphs, the increment of genus lies within these bounds:*

$$\left\lceil \frac{1 - deg(t) - deg(u)}{2} \right\rceil \leq \gamma(S_X) - (\gamma(S_G) + \gamma(S_H)) \leq \left\lfloor \frac{deg(t) + deg(u) - 2}{2} \right\rfloor$$

Proof. See [Gross *et al.*, 2010]. □

Double-rooted graphs

By a ***double-rooted graph*** (H, u, v) we mean a graph with two vertices designated as roots. Double-rooted graphs were first introduced in [Gross *et al.*, 2010] as they lend themselves natrually to iterated amalgmation. For the purposes of this chapter, root u is assumed to have degree $n \geq 2$, whereas root v is 2-valent. Our focus here, is the graph amalgamation $(G, t) * (H, u, v)$ when $deg(t) = deg(v) = 2$ and $deg(u) = n \geq 2$. This is illustrated in Figure 2.1.

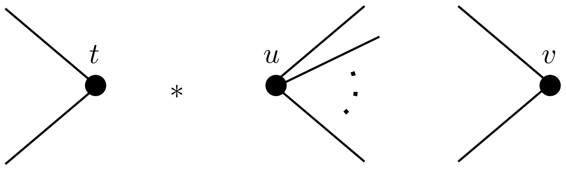


Figure 2.1: $(G, t) * (H, u, v)$ when $deg(t) = deg(v) = 2$ and $deg(u) = n \geq 2$.

When two single-rooted graphs are amalgamated, the amalgamated graph has the merged vertices of amalgamation as its root. If we iteratively amalgamate several single-rooted graphs, we obtain a graph with a root of whose degree is the sum of the degrees of the constituent roots. We use double-rooted graphs when we want to calculate the genus distribution of a chain of copies (as in §3 and §4) of the same graph (or of different graphs).

2.2 Double-root Partials and Productions

The genus distribution of the set of embeddings of $(X, w) = (G, t) * (H, u)$ whose rotation systems are consistent with those of fixed embeddings $G \to S_G$ and $H \to S_H$, depends only on $\gamma(S_G)$, $\gamma(S_H)$, and the respective numbers of faces of the embeddings $G \to S_G$ and $H \to S_h$ in which the two vertices of amalgamation t and u lie. Accordingly, we partition the embeddings of a single-rooted graph (G, t) with $deg(t) = 2$ in a surface of genus i into the subset of ***type-d_i embeddings***, in which root t lies on two distinct fb-walks, and the subset of ***type-s_i embeddings***, in which root t occurs twice on the same fb-walk. Moreover, we define

$$\begin{aligned} d_i(G,t) &= \text{the number of embeddings of type-}d_i\text{, and} \\ s_i(G,t) &= \text{the number of embeddings of type-}s_i. \end{aligned}$$

Thus,

$$g_i(G,t) \;=\; d_i(G,t) + s_i(G,t).$$

The numbers $d_i(G,t)$ and $s_i(G,t)$ are called ***single-root partials***. The sequences $\{d_i(G,t) \mid i \geq 0\}$ and $\{s_i(G,t) \mid i \geq 0\}$ are called ***partial genus distributions***.

Since $deg(u) = n$ in a double-rooted graph (H,u,v), there are n face corners incident at u (i.e., u occurs n times in the fb-walks — we will call them ***u-corners*** from now on), some or all of which might belong to the same face.

Suppose further that the n occurrences of root u in fb-walks of different faces are distributed according to the partition $p_1p_2\cdots p_r$ of n (where r is the number of faces incident at root u). For each such partition $p_1p_2\cdots p_r$, we define the following ***double-root partials*** of the genus distribution of a graph (H,u,v), such that root u is n-valent and root v is 2-valent:

$f_{p_1p_2\cdots p_r}d_i$ = the number of embeddings of (H,u,v) in the surface S_i such that the n occurrences of root u are distributed in r fb-walks, according to the partition $p_1p_2\cdots p_r$, and the two occurrences of v lie on two *different* fb-walks.

$f_{p_1p_2\cdots p_r}s_i$ = the number of embeddings of (H,u,v) in the surface S_i such that the n occurrences of root u are distributed in r fb-walks, according to the partition $p_1p_2\cdots p_r$, and that the two occurrences of v lie on the *same* fb-walk.

NOTATION We write the partition $p_1p_2\cdots p_r$ of an integer in non-ascending order.

A ***production*** for an amalgamation

$$(G,t) * (H,u,v) \;=\; (X,v)$$

of a single-rooted graph (G, t) with a double-rooted graph (H, u, v) (where $deg(t) = deg(v) = 2$, and $deg(u) \geq 2$) is an expression of the form

$$\begin{aligned} p_i(G,t) * q_j(H,u,v) \quad \longrightarrow \quad & \alpha_1\, d_{i+j}(G*H,v) \;+\; \alpha_2\, d_{i+j+1}(G*H,v) \\ & + \alpha_3\, s_{i+j}(G*H,v) \;+\; \alpha_4\, s_{i+j+1}(G*H,v) \end{aligned}$$

where p_i is a single-root partial and q_j is a double-root partial, and where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are integers. It means that amalgamation of a type-p_i embedding of graph (G, u) and a type-q_j embedding of graph (H, u, v) induces a set of α_1 type-d_{i+j}, α_2 type-d_{i+j+1}, α_3 type-s_{i+j}, and α_4 type-s_{i+j+1} embeddings of $G * H$. We often write such a rule in the form

$$p_i * q_j \;\longrightarrow\; \alpha_1\, d_{i+j} \;+\; \alpha_2\, d_{i+j+1} + \alpha_3\, s_{i+j} \;+\; \alpha_4\, s_{i+j+1}$$

Sub-partials of $f_{p_1p_2\cdots p_r}d_i$

In the course of developing productions for amalgamating a single-rooted graph (G, t) to a double-rooted graph (H, u, v), we shall discover that we sometimes need to refine a double-root partial into sub-partials. The following two types of numbers are the ***sub-partials of*** $f_{p_1p_2\cdots p_r}d_i$:

$f_{p_1p_2\cdots p_r}d'_i$ = the number of type-$f_{p_1p_2\cdots p_r}d_i$ embeddings such that at most one of the r fb-walks incident at u is the same as one of the two fb-walks incident at v;

$f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}$ = the number of type-$f_{p_1p_2\cdots p_r}d_i$ embeddings such that the two fb-walks (corresponding to subscripts l and m) incident at v have p_l and p_m occurrences of u, where $l < m$ (so that, in general, $p_l \geq p_m$), and $r > 1$.

Note that the value of the latter sub-partial of a graph (H, u, v) would be the same for any two pairs (p_a, p_b) and (p_l, p_m) such that $(p_l, p_m) = (p_a, p_b)$. Also note that, in general, we have

$$f_{p_1p_2\cdots p_r}d_i \;=\; f_{p_1p_2\cdots p_r}d_i' \;+ \sum_{\substack{\text{over all distinct} \\ \text{pairs } (p_l,p_m) \\ \text{with } l<m}} f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}$$

Example 3. For instance, $f_{112}d_4 = f_{112}d_4' + f_{112}d_4^{(1,1)} + f_{112}d_4^{(1,2)}$, since $(1,1)$ and $(1,2)$ are the distinct pairs.

Lemma 4. *Let x represent a face of an embedded graph (H,u,v) with $p_x > 0$ u-corners. There are $p_x(p_x+1)$ ways to insert two edge-ends into the u-corners of this face.*

Proof. Since there are p_x u-corners, there are p_x choices for the location of the first edge-end. After the first edge-end is inserted, the number of u-corners is p_x+1. Thus, there are p_x+1 choices for the second edge-end. Hence, there are a total of $p_x(p_x+1)$ choices (see Figure 2.2). □

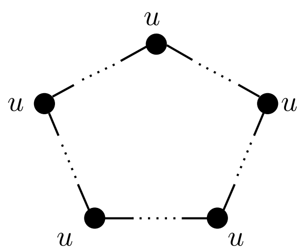


Figure 2.2: Since $p_x = 5$, there are $30 = 5 * 6$ ways to insert two edge-ends into the u-corners of this face.

Lemma 5. *Let x and y be two faces of an embedded graph (H,u,v), with $p_x > 0$ and $p_y > 0$ u-corners, respectively. There are $2p_xp_y$ ways to insert two edge-ends at root u, such that one edge-end is in face x and the other in face y.*

Proof. There are p_x choices for the edge-end that is inserted into face x, and for each such choice, there are p_y choices for the other edge-end (see Figure 2.3). Since either of the two edge-ends can be the one that is inserted into face x, we need to multiply p_xp_y by 2. □

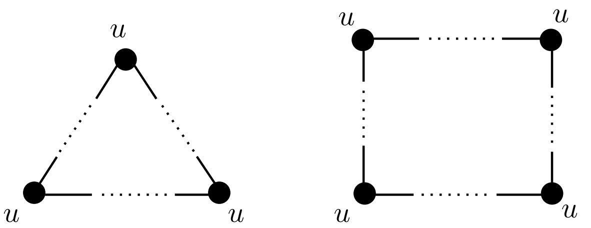


Figure 2.3: Since $p_x = 3$ and $p_y = 4$, there are $24 = 2 \cdot 3 \cdot 4$ ways to insert two edge-ends with one edge-end in each of the two faces.

2.3 Vertex Amalgamation

In theorems 6–8 below, productions are listed and proven for the case where the embedding of the double-rooted graph is of type $f_{p_1 p_2 \cdots p_r} d_j$.

Theorem 6. *Let $p_1 p_2 \cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-d_i embedding of a single-rooted graph (G, t) is amalgamated to a type-$f_{p_1 p_2 \cdots p_r} d_j$ embedding of a double-rooted graph (H, u, v), with deg(v) = deg(t) = 2 and deg$(u) = n$. Then the following production holds:*

$$d_i * f_{p_1 p_2 \cdots p_r} d'_j \longrightarrow \left(\sum_{x=1}^{r} p_x(p_x + 1) \right) d_{i+j} + \left(\sum_{x=1}^{r} \sum_{y=x+1}^{r} 2 p_x p_y \right) d_{i+j+1} \tag{2.2}$$

Proof. Since at most one of the r faces incident at root u of H is incident at root v of H, it follows that no matter how the root t of G is amalgamated to u, at most one of the two faces incident at v are affected by this amalgamation. It follows that in the amalgamated graph the two occurrences of v remain on two different faces. There are two cases:

case (i) is when both edge-ends incident at root t of graph G are placed into one of the r faces of graph H incident at u. In this case, no new handle is needed, and thus, the genus increment is 0. The coefficient $\sum_{x=1}^{r} p_x(p_x + 1)$ of d_{i+j} counts the number of ways this can happen. The summation goes from 1 to r, since we can put the two edge-ends incident at t into any of the r faces. The term $p_x(p_x + 1)$ follows from Lemma 4.

case (ii) is when the two edge-ends incident at root t of graph G are placed into two different faces incident at u. This necessitates adding a new handle — resulting in a genus increment of 1. The coefficient $\sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y$ of d_{i+j+1} counts the number of ways this can happen, by Lemma 5. □

Theorem 7. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$, and let (p_l, p_m) be a pair such that $1 \leq l < m \leq r$. Suppose that a type-d_i embedding of a single-rooted graph (G, t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}d_j^{(p_l,p_m)}$ embedding of a double-rooted graph (H, u, v), with $deg(v) = deg(t) = 2$ and $deg(u) = n$. Then the following production holds:*

$$\begin{aligned} d_i * f_{p_1p_2\cdots p_r}d_j^{(p_l,p_m)} \quad &\longrightarrow \quad \left(\sum_{x=1}^{r} p_x(p_x+1)\right) d_{i+j} \\ &+ \quad \left(\left(\sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y\right) - 2p_lp_m\right) d_{i+j+1} \\ &+ \quad 2p_lp_m s_{i+j+1} \end{aligned} \tag{2.3}$$

Proof. Let φ_l and φ_m be the two faces incident at root u that are also incident at v, with u occurring p_l times on fb-walk of face φ_l, and p_m times on fb-walk of face φ_m. We note that unless we place one edge-end incident at root t of graph (G, t) into face φ_l and the other edge-end into face φ_m, at most one of the two faces φ_l and φ_m is affected by this amalgamation. Thus, *case (i)* remains the same as in Theorem 6. The first term of the Production (2.3) reflects this similarity. Moreover, *case (ii)* remains the same as in Theorem 6, unless x and y correspond to the faces φ_l and φ_m, which is why we subtract $2p_lp_m$ from the second sum in Production (2.3). If x and y correspond to the faces φ_l and φ_m, then as a result of the amalgamation, the two faces (φ_l and φ_m) combine to become one face having both occurrences of v in its boundary (see Figure 2.4). The third term of the production reflects this. □

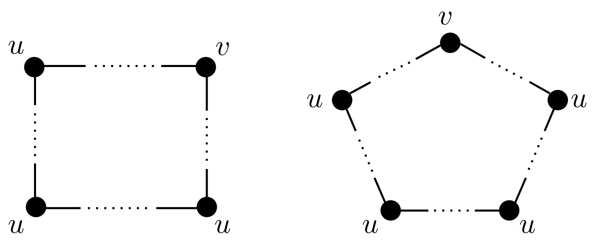


Figure 2.4: Here $p_l = 3$ and $p_m = 4$. Amalgamation combines the two faces, and the resultant face contains both occurrences of v.

NOTATION We sometimes use the shorthand $f_{p_1p_2\cdots p_r}d_j^{\bullet}$ in place of $f_{p_1p_2\cdots p_r}d_j$, to emphasize the absence of any superscript after d_j.

Theorem 8. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-s_i embedding of a single-rooted graph (G,t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}d_j^{\bullet}$ embedding of a double-rooted graph (H,u,v), with deg(v) = deg$(t) = 2$ and deg$(u) = n$. Then the following production holds:*

$$s_i * f_{p_1p_2\cdots p_r}d_j^{\bullet} \quad \longrightarrow \quad (n^2+n)d_{i+j} \tag{2.4}$$

Proof. Suppose that in a type-$f_{p_1p_2\cdots p_r}d_j^{\bullet}$ embedding of graph (H,u,v), the two occurrences of root-vertex v lie on on two different fb-walks W_1 and W_2 that may or may not contain the root-vertex u. Suppose further that the two occurrences of root-vertex t of graph (G,t) lie on fb-walk X. The two occurrences of root-vertex v continue being on two different fb-walks after the operation of vertex amalgamation, unless the fb-walks W_1 and W_2 combine with the fb-walk X under amalgamation into a single fb-walk. But this cannot happen when the embedding of (G,t) is a type-s_i embedding, since a reduction of two faces forces the Euler characteristic to be of odd parity, which is not possible. Thus, there is no genus-increment and all n^2+n resulting embeddings are type-d_{i+j} embeddings. □

Sub-partials of $f_{p_1p_2\cdots p_r}s_i$

To define the sub-partials of $f_{p_1p_2\cdots p_r}s_i$ we need the concept of *strands*, which was introduced and used extensively in [Gross *et al.*, 2010]. When two embeddings are amalgamated, these strands *recombine* with other strands to form new fb-walks.

Definition 9. We define a ***u-strand*** of an fb-walk of a rooted graph (H,u,v) to be a subwalk that starts and ends with the root vertex u, such that u does not appear in the interior of the subwalk.

The following two types of numbers are the relevant sub-partials of the partial $f_{p_1p_2\cdots p_r}s_i$ for graph (H,u,v):

$f_{p_1p_2\cdots p_r}s_i'$ = the number of type-$f_{p_1p_2\cdots p_r}s_i$ embeddings of H such that the two occurrences of v lie in at most one u-strand.

$f_{p_1p_2\cdots p_r}s_i^{(p_l,c)}$ = the number of type-$f_{p_1p_2\cdots p_r}s_i$ embeddings of H such that the two occurrences of v lie in two different u-strands of the fb-walk that is represented by p_l, and such that there are $q \geq 1$ intermediate u-corners between the two occurrences of v. We take c to be equal to $min(q,\, p_l - q)$, i.e., equal to the smaller number of intermediate u-corners between the two occurrences of root-vertex v.

Note that the last sub-partial would be the same for any other pair (p_a, c) such that $p_a = p_l$.

Theorem 10. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-d_i embedding of a single-rooted graph (G,t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}s_j'$ embedding of a double-rooted graph (H,u,v), with $deg(v) = deg(t) = 2$ and $deg(u) = n$. Then the following production holds:*

$$\begin{aligned} d_i * f_{p_1p_2\cdots p_r}s_j' \quad &\longrightarrow \quad \left(\sum_{x=1}^{r} p_x(p_x+1)\right) s_{i+j} \\ &+ \quad \left(\sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y\right) s_{i+j+1} \end{aligned} \tag{2.5}$$

Proof. Since both occurrences of root v of H lie in at most one u-strand of one of the r fb-walks, it follows that regardless of how the u-strands *recombine* in the amalgamation process, these two occurrences remain on that same u-strand; thus, in all of the resultant embeddings, the two occurrences of v are on the same fb-walk. As discussed in the proof of Theorem 6, there are $\sum_{x=1}^{r} p_x(p_x+1)$ embeddings that do not result in any genus-increment (corresponding to both edge-ends at t being inserted into the same face at u), whereas there are $\sum_{y=x+1}^{r} 2p_xp_y$ embeddings that result in a genus increment of 1 (corresponding to inserting both edge-ends at t into the different faces at u). □

Theorem 11. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Then for each distinct p_l, with $l \in \{1, \cdots, r\}$, and for each integer c in the integer interval $[1, \lfloor \frac{p_l}{2} \rfloor]$, when a type-$d_i$ embedding of a single-rooted graph (G,t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}s_j^{(p_l,c)}$ embedding of a double-rooted graph (H,u,v), with $deg(v) = deg(t) = 2$ and $deg(u) = n$, the following production holds:*

$$\begin{aligned} d_i * f_{p_1p_2\cdots p_r}s_j^{(p_l,c)} \quad \longrightarrow & \quad \left(\left(\sum_{x=1}^{r} p_x(p_x+1) \right) - p_l(p_l+1) \right) s_{i+j} \\ + & \quad 2c(p_l - c)\, d_{i+j} \\ + & \quad [c(c+1) + (p_l - c)(p_l - c + 1)]\, s_{i+j} \\ + & \quad \left(\sum_{x=1}^{r} \sum_{y=x+1}^{r} 2p_xp_y \right) s_{i+j+1} \end{aligned} \tag{2.6}$$

Proof. Let φ_l be the face corresponding to p_l, and let w_1 and w_2 be the two (different) u-strands that contain the two occurrences of root v of H (with c intermediate u-corners between the two occurrences of v). It follows that unless the two edge-ends incident at root t of G are both placed into the face φ_l, the two occurrences of root v will lie on the same fb-walk after amalgamation. The first and last terms of the production reflect this.

Now we consider the case when the two edge-ends incident at root t of graph (G,t) are both placed into the face φ_l. Let e_{start_1} and e_{start_2} be the initial edge-ends of u-strands w_1 and w_2, similarly let e_{end_1} and e_{end_2} be the terminal edge-ends of u-strands w_1 and w_2 (we consider that a u-strand starts and ends at root u). This is illustrated in Figure 2.5.

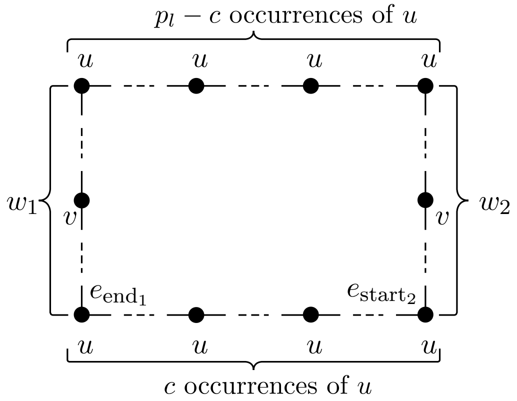


Figure 2.5: Fb-walk of a type-$f_{p_1p_2\cdots p_r}s_j^{(p_l,c)}$ embedding.

.

It is clear that in the fb-walk of the face φ_l, the cyclic order of these four edge-ends is: $e_{start_1}, e_{end_1}, e_{start_2}, e_{end_2}$. If one of the two edge-ends incident at root t is placed between e_{end_1} and e_{start_2} and the other between e_{end_2} and e_{start_1}, then after the strands are recombined, one of the u-strands containing one occurrence of root v clearly recombines with the one t-strand of (G,t) to make a new face (see Figure 2.6, left).

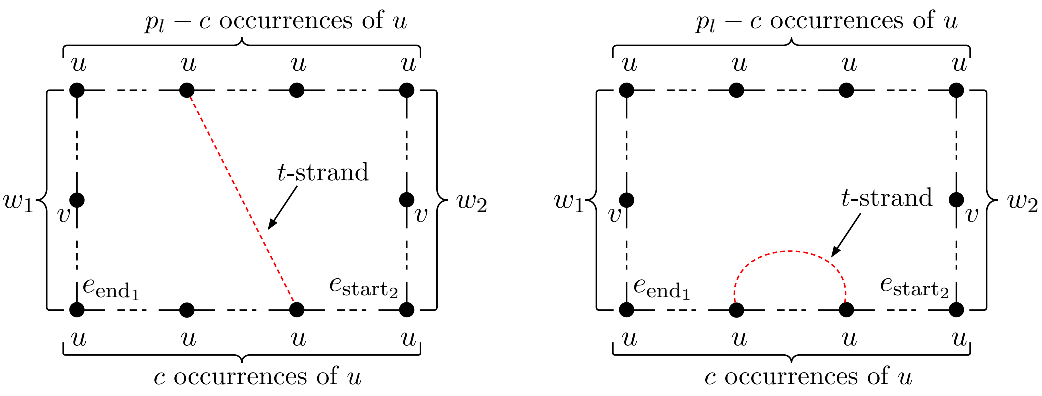


Figure 2.6: The two ways of inserting t-strands.

.

It follows that in this case the two occurrences of root v will lie on two different faces. Since there are a total of p_l u-corners in face φ_l, and there are c intermediate u-corners between the two occurrences of root v of graph (H,u,v), there are $2c(p_l - c)$ ways in all of inserting the two edge-ends incident at root t of graph (G,t) in this way. We multiply by 2 since either of the two edge-ends can be chosen as the first edge-end. The second term of the production reflects this case.

If both of the edge-ends incident at root t are placed between e_{end_1} and e_{start_2}, or between e_{end_2} and e_{start_1}, then the two occurrences of root v lie on the same face after u-strands and t-strands are recombined (see Figure 2.6, right). There are $c(c+1) + (p_l - c)(p_l - c + 1)$ ways this can happen, since there are c and $p_l - c$ intermediate u-corners between w_1 and w_2. □

Notation We sometimes use the shorthand $f_{p_1 p_2 \cdots p_r} s_j^\bullet$ in place of $f_{p_1 p_2 \cdots p_r} s_j$, to emphasize the absence of any superscript after s_j.

Theorem 12. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-s_i embedding of a single-rooted graph (G,t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}s_j$ embedding of a double-rooted graph (H,u,v), with deg(v) = deg$(t) = 2$ and deg$(u) = n$. Then the following production holds:*

$$s_i * f_{p_1p_2\cdots p_r}s_j^{\bullet} \quad\longrightarrow\quad (n^2+n)s_{i+j} \tag{2.7}$$

Proof. Since the two occurrences of root v of H lie on the same fb-walk. One necessary condition for the operation of vertex amalgamation to change this is that both edge-ends at root t of G are inserted into that face. However, since both occurrences of root t are on the same fb-walk, both ends of each t-strand lie in the same u-corner of that face, as illustrated in Figure 2.7. This implies that no new handle is needed as a result of the amalgamation. Thus, there is no genus-increment. □

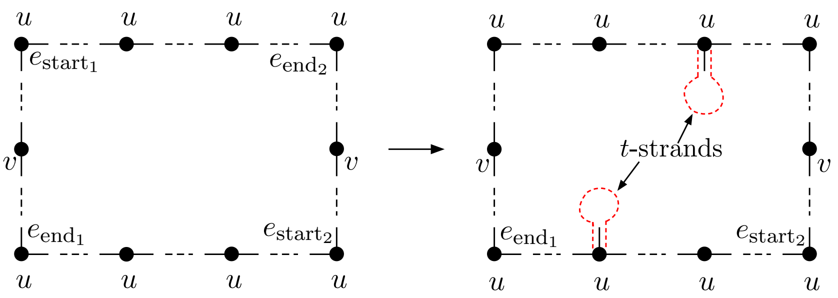


Figure 2.7: Even after the amalgamation, the two occurrences of v remain on the same fb-walk.

.

Corollary 13. *Let $(X,v) = (G,t)*(H,u,v)$, where deg(v) = deg$(t) = 2$ and deg$(u) = n$ for $n \geq 2$. Then for*

$$\alpha_{p_1p_2\cdots p_r} = \sum_{x=1}^{r} p_x(p_x+1) \quad \textit{and} \quad \beta_{p_1p_2\cdots p_r} = \sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y$$

we have

$$d_k(X) = \sum_{\substack{\textit{over all partitions}\\ p_1p_2\cdots p_r \textit{ of } n}} \Bigg[\sum_{i=0}^{k} \alpha_{p_1p_2\cdots p_r}\, d_{k-i}f_{p_1p_2\cdots p_r}d_i'$$

$$+ \sum_{i=0}^{k-1} \beta_{p_1p_2\cdots p_r}\ d_{k-i-1} f_{p_1p_2\cdots p_r} d'_i$$

$$+ \sum_{i=0}^{k} \sum_{\substack{\text{over all} \\ \text{distinct } (p_l,p_m) \\ l<m}} \alpha_{p_1p_2\cdots p_r}\ d_{k-i} f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)}$$

$$+ \sum_{i=0}^{k-1} \sum_{\substack{\text{over all} \\ \text{distinct } (p_l,p_m) \\ l<m}} (\beta_{p_1p_2\cdots p_r} - 2p_lp_m)\ d_{k-i-1} f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)}$$

$$+ \sum_{i=0}^{k} (n^2+n)\ s_{k-i} f_{p_1p_2\cdots p_r} d_i^{\bullet}$$

$$+ \sum_{i=0}^{k} \sum_{\substack{\text{over all} \\ \text{distinct } p_l}} \sum_{c=1}^{\lfloor \frac{p_l}{2} \rfloor} 2c(p_l-c)\ d_{k-i} f_{p_1p_2\cdots p_r} s_i^{(p_l,c)} \Bigg] \tag{2.8}$$

Proof. This equation is derived from Theorems 6, 7, 8 and 11 by a routine transposition of the productions that have the single-root partial d on their right-hand-side. □

Corollary 14. *Let $(X, v) = (G, t) * (H, u, v)$, where $deg(v) = deg(t) = 2$ and $deg(u) = n$ for $n \geq 2$. Then for*

$$\alpha_{p_1p_2\cdots p_r} = \sum_{x=1}^{r} p_x(p_x+1) \quad \text{and} \quad \beta_{p_1p_2\cdots p_r} = \sum_{x=1}^{r} \sum_{y=x+1}^{r} 2p_xp_y$$

we have

$$s_k(X) = \sum_{\substack{\text{over all partitions} \\ p_1p_2\cdots p_r \text{ of } n}} \Bigg[\sum_{i=0}^{k} \alpha_{p_1p_2\cdots p_r}\ d_{k-i} f_{p_1p_2\cdots p_r} s'_i$$

$$+ \sum_{i=0}^{k-1} \beta_{p_1p_2\cdots p_r}\ d_{k-i-1} f_{p_1p_2\cdots p_r} s'_i$$

$$+ \sum_{i=0}^{k-1} \sum_{\substack{\text{over all} \\ \text{distinct } (p_l,p_m) \\ l<m}} 2p_lp_m\ d_{k-i-1} f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)}$$

$$+ \sum_{i=0}^{k} \sum_{\substack{\text{over all} \\ \text{distinct } p_l}} \sum_{c=1}^{\lfloor \frac{p_l}{2} \rfloor} \Big(c(c+1) + (p_l-c)(p_l-c+1)$$

$$+ \alpha_{p_1p_2\cdots p_r} - p_l(p_l+1) \Big)\ d_{k-i} f_{p_1p_2\cdots p_r} s_i^{(p_l,c)}$$

$$+ \sum_{i=0}^{k-1} \sum_{\substack{over\ all \\ distinct\ p_l}} \sum_{c=1}^{\lfloor \frac{p_l}{2} \rfloor} \beta_{p_1 p_2 \cdots p_r}\ d_{k-i-1} f_{p_1 p_2 \cdots p_r} s_i^{(p_l, c)}$$

$$+ \sum_{i=0}^{k} (n^2 + n)\ s_{k-i} f_{p_1 p_2 \cdots p_r} s_i^{\bullet} \Big] \tag{2.9}$$

Proof. This equation is derived from Theorems 7, 10, 11 and 12 by a routine transposition of the productions that have the single-root partial s on their right-hand-side. □

Remark 15. In writing Recursions 2.8 and 2.9, we have suppressed indication of graphs G and H as arguments, in order that they not occupy too many lines. In the examples to follow, we see how restriction of these recursions to particular genus distributions of interest greatly simplifies them. The reason for placing the index variable i of each sum with the second factor, rather than the first, also becomes clear in the applications.

Example 16. We can specify a sequence of ***open chains*** of copies of a double-rooted graph (G, u, v) recursively.

$$(X_1,\, t_1) = (G, v) \qquad \text{(suppressing co-root } u) \tag{2.10}$$

$$(X_m,\, t_m) = (X_{m-1}, t_{m-1}) * (G, u, v) \quad \text{for } m \geq 1 \tag{2.11}$$

For example, consider a chain of copies of the graph $(\dot{K}_4, u, v)$ obtained from the complete graph (K_4, u) by inserting a vertex v as a subdivision point of any edge of the graph (Figure 2.8). We observe that each of the amalgamations results in a vertex of degree 5.

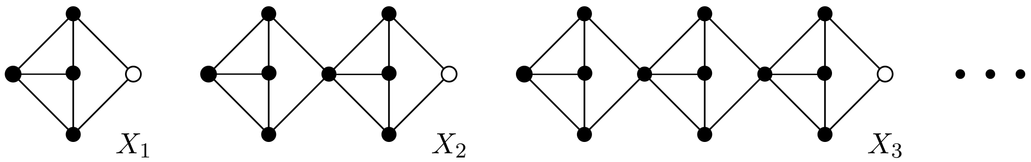


Figure 2.8: X_m is an open chain of m copies of $(\dot{K}_4, u, v)$.

By face-tracing the embeddings of $(\dot{K}_4, u, v)$, we obtain Table 2.1.

Table 2.1: Nonzero partials of $(\dot{K}_4, u, v)$.

k	$f_{111}d'_k$	$f_{21}d_k^{(2,1)}$	$f_{21}s_k^{(2,1)}$	$f_3d'_k$	d_k	s_k	g_k
0	2				2		2
1		6	6	2	8	6	14

By using Recurrences (2.8) and (2.9) for $deg(u) = n = 3$, and the values from Table 2.1, we obtain the following two recurrences, for $m \geq 2, k \geq 0$:

$$d_k(X_m) = 12d_k(X_{m-1}) + 24s_k(X_{m-1}) + 96d_{k-1}(X_{m-1}) + 96s_{k-1}(X_{m-1}) \quad (2.12)$$

$$s_k(X_m) = 48d_{k-2}(X_{m-1}) + 36d_{k-1}(X_{m-1}) + 72s_{k-1}(X_{m-1}) \quad (2.13)$$

Another way of obtaining these recurrences without having to use Recurrences (2.8) and (2.9), is to first list all productions that are relevant for the example at hand (i.e. corresponding to the non-zero double-root partials) using Theorems 6–12; we list the productions for this example in Table 2.2. We can then transpose these productions, and use the values of double-root partials from Table 2.1 on the transposed productions to come up with the desired recurrences.

Table 2.2: The non-zero productions when $deg(u) = 3$.

$d_i * f_{111}d'_j \longrightarrow 6d_{i+j} + 6d_{i+j+1}$
$s_i * f_{111}d^{\bullet}_j \longrightarrow 12d_{i+j}$
$d_i * f_{21}d_j^{(2,1)} \longrightarrow 8d_{i+j} + 4s_{i+j+1}$
$d_i * f_{21}s_j^{(2,1)} \longrightarrow 2d_{i+j} + 6s_{i+j} + 4s_{i+j+1}$
$s_i * f_{21}d^{\bullet}_j \longrightarrow 12d_{i+j}$
$s_i * f_{21}s^{\bullet}_j \longrightarrow 12s_{i+j}$
$d_i * f_3d'_j \longrightarrow 12d_{i+j}$
$s_i * f_3d^{\bullet}_j \longrightarrow 12d_{i+j}$

Using these recurrences and the values of single-root partials in Table 2.1, we obtain the values of single-root partials for X_2, that are listed in Table 2.3. We can then use values of the partials for X_2 to obtain the values of single-root partials for X_3, also listed in Table 2.3. We can iterate this to obtain the genus distribution of X_m for any value of m.

Table 2.3: Single-root partials of X_2 and X_3.

	X_2			X_3		
k	d_k	s_k	g_k	d_k	s_k	g_k
0	24	0	24	288	0	288
1	432	72	504	9216	864	10080
2	1344	816	2160	84096	21888	105984
3		384	384	216576	127872	344448
4				36864	92160	129024

Example 17. As another illustration of the method, we compute the recurrences for the open chains of a graph (G, u, v) in which $deg(u) = n = 6$ (see Figure 2.9). Where, as in previous example, X_1 is the graph G with root s suppressed.

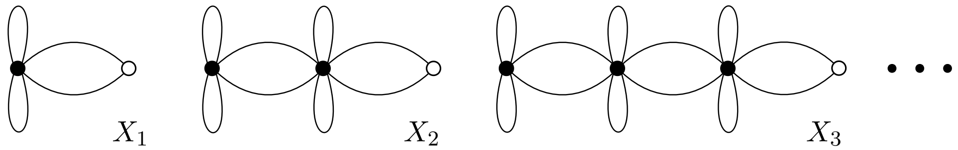


Figure 2.9: X_m is an open chain of m copies of G.

By face-tracing the embeddings of (G, u, v), we obtain Table 2.4.

Table 2.4: Nonzero partials of (G, u, v).

type	$k=0$	$k=1$
$f_{51}d^{(5,1)}$		16
$f_{2211}d^{(2,2)}$	8	
$f_{2211}d^{(2,1)}$	16	
$f_{42}s^{(4,2)}$		8
$f_{42}d^{(4,2)}$		16
$f_{33}d^{(3,3)}$		8
$f_{3111}d^{(3,1)}$	16	
$f_{51}s^{(5,2)}$		32
d_k	40	40
s_k		40
g_k	40	80

Using Recurrences (2.8) and (2.9), we obtain the following two recurrences for $m \geq 2, k \geq 0$:

$$\begin{aligned} d_k(X_m) &= 672d_k(X_{m-1}) + 1680s_k(X_{m-1}) \\ &\quad + 2352d_{k-1}(X_{m-1}) + 1680s_{k-1}(X_{m-1}) \qquad (2.14) \\ s_k(X_m) &= 1008d_{k-2}(X_{m-1}) + 1008d_{k-1}(X_{m-1}) + 1680s_{k-1}(X_{m-1}) \qquad (2.15) \end{aligned}$$

Table 2.5 records the values that these recurrences give us for X_2 and X_3.

Table 2.5: Single-root partials of X_2 and X_3.

	X_2			X_3		
k	d_k	s_k	g_k	d_k	s_k	g_k
0	26880	0	26880	18063360	0	18063360
1	188160	40320	228480	257402880	27095040	284497920
2	161280	147840	309120	867041280	284497920	1151539200
3	0	40320	40320	695439360	600606720	1296046080
4				67737600	230307840	298045440

Remark As illustrated in the preceding examples that when we use the method derived in this chapter for computing the genus distribution of the open chain of a double-rooted graph (G, u, v) (where $deg(u) \geq 2$ and $deg(v) = 2$), once we have the system of recurrences $\langle d_k, s_k : k = 0, 1, 2, \cdots \rangle$, computing the genus distribution of the graph family is a routine task of evaluating those recurrences, that can be done in linear time. Also, the derivation of the system of recurrences is a task that is independent of k (though it does depend on the valence of the root u of the double-rooted graph (G, u, v)). This observation holds true of all methods derived in this dissertation.

Chapter 3

Operations on Double-rooted Graphs

In Chapter 2, we developed a method for computing the genus distribution of open chains of any double-rooted graph (G, u, v) provided that one of the two roots has valence two. In order to do this, we defined double-root partials, and we refined them into sub-partials. The information encoded in those sub-partials can be put to further use. For instance, with a minor refinement in the definition of the double-root sub-partials (Section 3.1), we can derive productions to calculate the effect of amalgamating the two root vertices with each other (Section 3.2), and also the effect of adding an edge between the two root vertices of the double-rooted graph (Section 3.3).

In the next chapter, we will see how we can use these productions to compute the genus distribution of *closed chains* of any double-rooted graph, where one of the two roots is restricted to be of valence two. A closed chain is the graph obtained from an open chain (X_n, u, v) by merging both of its roots.

3.1 Refinement of Sub-partials

We start with the same assumption as at the start of Chapter 2, namely, that we have a double-rooted graph (G, u, v) in which $deg(u) = n \geq 2$ and $deg(v) = 2$. To capture the different ways in which the two roots of a double-rooted graph can occur in shared fb-walks for the purpose of self-vertex-amalgamation, the sub-partials defined in Chapter 2 need to be further refined.

The following three types of numbers are the ***sub-partials of*** $f_{p_1p_2\cdots p_r}d_i(G, u, v)$:

$f_{p_1p_2\cdots p_r}d_i^0$ = the number of type-$f_{p_1p_2\cdots p_r}d_i$ embeddings of (G, u, v) such that none of the fb-walks incident on u is also incident on v;

$f_{p_1p_2\cdots p_r}d_i^{p_k}$ = the number of type-$f_{p_1p_2\cdots p_r}d_i$ embeddings of (G, u, v) such that an fb-walk with p_k occurrences of root u is the same as one of the two fb-walks incident on v;

$f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}$ = the number of type-$f_{p_1p_2\cdots p_r}d_i$ embeddings of (G, u, v) such that the two fb-walks incident on v have p_l and p_m occurrences of u, respectively, where $l < m$ (so that, in general, $p_l \geq p_m$), and $r > 1$.

The following three types of numbers are the relevant sub-partials of the partial $f_{p_1p_2\cdots p_r}s_i$ for graph (G, u, v):

$f_{p_1p_2\cdots p_r}s_i^0$ = the number of type-$f_{p_1p_2\cdots p_r}s_i$ embeddings of (G, u, v) such that none of the fb-walks incident on root u is also incident on root v.

$f_{p_1p_2\cdots p_r}s_i^{p_k}$ = the number of type-$f_{p_1p_2\cdots p_r}s_i$ embeddings of (G, u, v) such that both occurrences of v lie in one u-strand, in the fb-walk represented by p_k.

$f_{p_1p_2\cdots p_r}s_i^{(p_k,c)}$ = the number of type-$f_{p_1p_2\cdots p_r}s_i$ embeddings such that the two occurrences of v lie in two different u-strands of the fb-walk represented by p_k, and such that there are $q \geq 1$ intermediate u-corners between the two occurrences of v. We take c to be equal to $min(q,\, p_k - q)$, i.e., equal to the smaller number of intermediate u-corners between the two occurrences of root-vertex v.

3.2 Self-vertex-amalgamation

Productions for the ***self-vertex-amalgamation*** of the two roots of a double-rooted graph (G, u, v), giving graph G', are of the following form:

$$p_i(G, u, v) \longrightarrow \alpha_1\, g_{i-1}(G') \,+\, \alpha_2\, g_i(G') \,+\, \alpha_3\, g_{i+1}(G')$$

where $p_i(G, u, v)$ is a sub-partial and the coefficients α_k are integers whose sum is n^2+n, i.e. given any embedding ι_G of the double-rooted graph (G, u, v), there are n^2+n embeddings of the graph G' whose rotation systems are consistent with the rotation system corresponding to the embedding ι_G. This essentially follows from Proposition 1, which was, in turn, a symmetrization of Formula (1.1) of [Gross *et al.*, 2010].

The negative-genus increment here might be surprising, but the scenario does arise during self-vertex-amalgamation of an embedding of type $f_{p_1p_2\cdots p_r}s_i^{(p_k,c)}$ (see *case (iii)* in the proof of Production 3.6 below).

Theorem 18. *Let (G, u, v) be a double-rooted graph where deg$(u) \geq 2$ and deg$(v) = 2$, and let G' be the graph obtained by merging the roots u and v. Then for each partition $p_1p_2\cdots p_r$ of deg(u), the following production holds:*

$$\begin{aligned} f_{p_1p_2\cdots p_r}d_i^0(G, u, v) \quad \longrightarrow \quad & \left(\sum_{x=1}^{r} p_x(p_x+1)\right) g_{i+1}(G') \\ & + \left(\sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y\right) g_{i+2}(G') \end{aligned} \tag{3.1}$$

Proof. In an embedding of type $f_{p_1p_2\cdots p_r}d_i^0(G, u, v)$, none of the fb-walks incident on root u are incident on root v. This implies that no matter how the two edge-ends incident at root v are inserted into the rotation system at root vertex u, at least two fb-walks are merged with each other. Figure 3.1 shows an embedding of this type. The fb-walks ψ_x and ψ_y have p_x and p_y u-corners, respectively, though we only show one u-corner for simplicity's sake (see the remark that immediately follows this proof).

Thus, there will be a genus-increment of at least one during self-vertex-amalgamation of an embedding of type $f_{p_1p_2\cdots p_r}d_i^0(G, u, v)$. Let α and β be the two edge-ends incident on root v. There are two cases depending on where these two edge-ends are inserted:

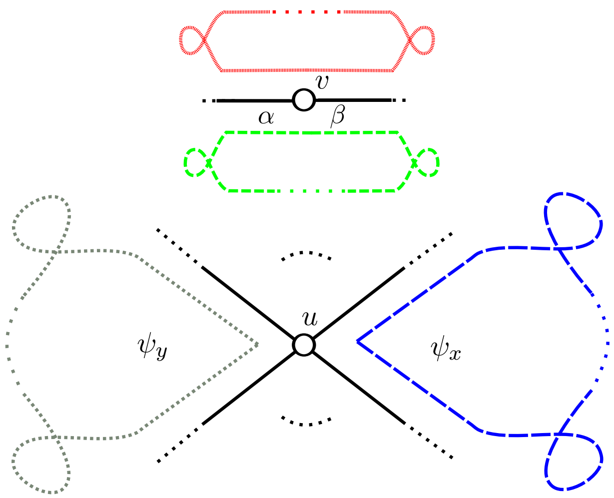


Figure 3.1: A graph embedding of type $f_{p_1 p_2 \cdots p_r} d_i^0(G, u, v)$

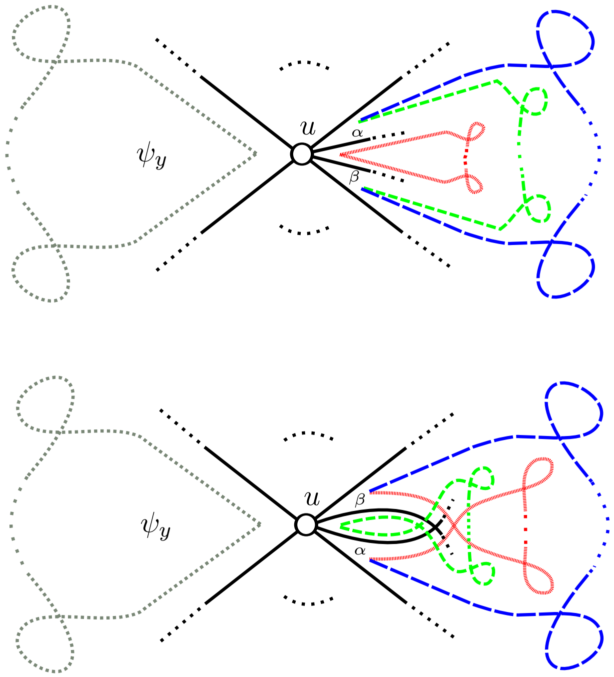


Figure 3.2: *Case (i)* – when both edge-ends are inserted into one fb-walk

case (i) is when the two edge-ends incident at root v are inserted into one fb-walk incident at root u, say in the fb-walk ψ_x containing p_x u-corners. In this case, according to Lemma 4, there will be $p_x(p_x + 1)$ new embeddings of the graph G'. Since both edge-ends are inserted into the the same fb-walk, there is no additional genus-increment. In Figure 3.2 we show the two ways of inserting edge-ends α and β into ψ_x, one with a "twist" (bottom) and one without (top). The first term of the production corresponds to this case.

case (ii) is when α and β are inserted into two different fb-walks incident at root u, say in ψ_x and ψ_y. In this case, since these two fb-walks are merged as a consequence of this operation, there will be another genus-increment (thus the genus rises by two overall) (see Figure 3.3). There are $2p_xp_y$ ways of doing this, according to Lemma 5. The second term of the production corresponds to this case. ♢

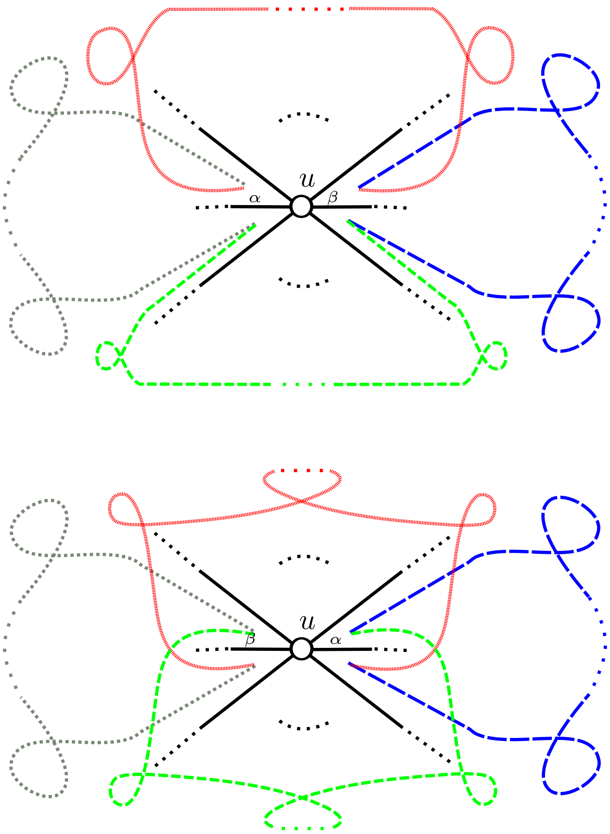


Figure 3.3: *Case (ii)* – when both edge-ends are inserted into two different fb-walks, with a twist (bottom), without a twist (top)

Remark In order to avoid unnecessary clutter, we simplified all of the figures in the previous proof, by showing that each fb-walk contained only one u-corner. We could afford this simplification only because in each of the fb-walks that were involved in the self-vertex-amalgamation, only one strand from each of the involved fb-walks took part in self-vertex-amalgamation. This abstraction works in all of the subsequent proofs as well, except in the case of Production 3.6. The reason why this abstraction doesn't work there is that in an embedding of type $f_{p_1p_2\cdots p_r}s_i^{(p_k,c)}(G,u,v)$, two u-strands of the fb-walk containing p_k u-corners are involved in the amalgamation, instead of only one (as in the embeddings of all other types).

Theorem 19. *Let (G,u,v) be a double-rooted graph where $deg(u) \geq 2$ and $deg(v) = 2$, and let G' be the graph obtained by merging the roots u and v. Then for each partition $p_1p_2\cdots p_r$ of $deg(u)$, the following production holds:*

$$\begin{aligned} f_{p_1p_2\cdots p_r}d_i^{p_k}(G,u,v) \quad \longrightarrow \quad & \left(\sum_{\substack{x=1\\x\neq k}}^{r} p_x(p_x+1)\right) g_{i+1}(G') \\ & + \frac{p_k(p_k+1)}{2}\left(g_i(G')+g_{i+1}(G')\right) \\ & + \left(\sum_{\substack{x=1\\x\neq k}}^{r}\sum_{\substack{y=x+1\\y\neq k}}^{r} 2p_xp_y\right) g_{i+2}(G') \\ & + \left(\sum_{\substack{x=1\\x\neq k}}^{r} 2p_kp_x\right) g_{i+1}(G') \end{aligned} \tag{3.2}$$

Proof. In an embedding of type $f_{p_1p_2\cdots p_r}d_i^{p_k}(G,u,v)$, one of the fb-walks containing p_k u-corners (denoted by ϕ) also contains one v-corner, such that the fb-walk (denoted by φ) containing the other v-corner is not incident on root u. There are four cases:

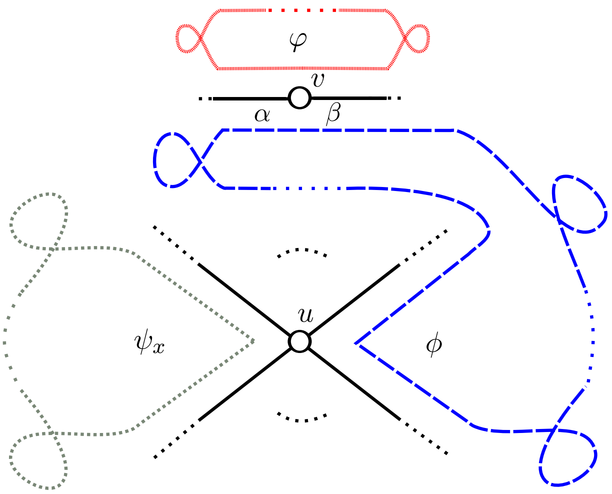


Figure 3.4: A graph embedding of type $f_{p_1p_2\cdots p_r}d_i^{p_k}(G,u,v)$

case (i) and *(ii)* are similar to the cases *(i)* and *(ii)* in the proof of Production 3.1, except that $x \neq k, y \neq k$. The first term of the production corresponds to *case (i)*, and the third term corresponds to *case (ii)*.

case (iii) is when the two edge-ends incident on root v are inserted into the fb-walk ϕ. Since $deg(v) = 2$, there are two sub-cases: *case (iii-a)* when the insertion is done with a twist, and *case (iii-b)* when the insertion is done without a twist. Since ϕ contains only one of the two v-corners, in one of these two cases φ is merged with ϕ (resulting in a genus-increment) – see Figure 3.5 (bottom) – since the number of fb-walks is reduced by one, whereas in the other case the fb-walk ϕ is divided into two fb-walks (with no genus-increment) – see Figure 3.5 (top) – since the number of fb-walk increases by one. The second term of the production corresponds to these two sub-cases.

case (iv) is when the two edge-ends incident on root v are inserted into two different fb-walks, one of which is ϕ. In this case there will be a genus increment of only one (see Figure 3.6 – the number of fb-walks is reduced by one in both cases). The last term of the production corresponds to this case. $\diamondsuit$

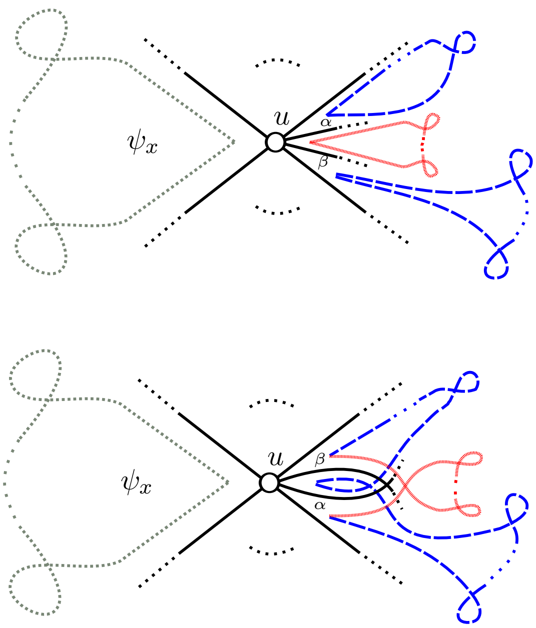


Figure 3.5: *Case (iii)* – when both edge-ends are inserted into ϕ.

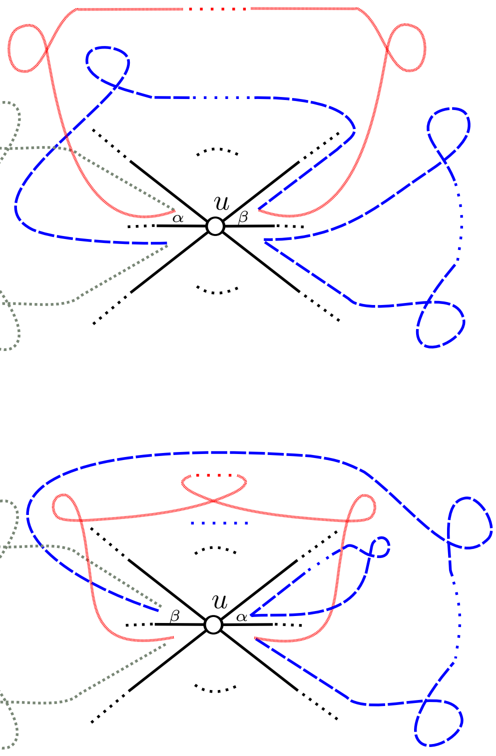


Figure 3.6: *Case (iv)* – when the two edge-ends are inserted into two different fb-walks one of which is ϕ, with a twist (bottom), without a twist (top)

Theorem 20. *Let (G, u, v) be a double-rooted graph where $deg(u) \geq 2$ and $deg(v) = 2$, and let G' be the graph obtained by merging the roots u and v. Then for each partition $p_1 p_2 \cdots p_r$ of $deg(u)$, the following production holds:*

$$
\begin{aligned}
f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)}(G,u,v) \quad \longrightarrow \quad & \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{r} p_x(p_x+1) \right) g_{i+1}(G') \\
& + \frac{p_l(p_l+1)}{2} \left(g_i(G') + g_{i+1}(G') \right) \\
& + \frac{p_m(p_m+1)}{2} \left(g_i(G') + g_{i+1}(G') \right) \\
& + \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{r} \sum_{\substack{y=x+1 \\ y\neq l, y\neq m}}^{r} 2p_x p_y \right) g_{i+2}(G') \\
& + \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{r} (2p_l p_x + 2p_m p_x) \right) g_{i+1}(G') \\
& + 2p_l p_m \, g_i(G')
\end{aligned}
\tag{3.3}
$$

Proof. The proof of this production is similar to the proof of the previous production, except for *case (iii)* and *case (iv)*, since here both of the fb-walks incident on root v are also incident on root u (we call them ϕ and ψ), instead of only one (as in the case of previous production).

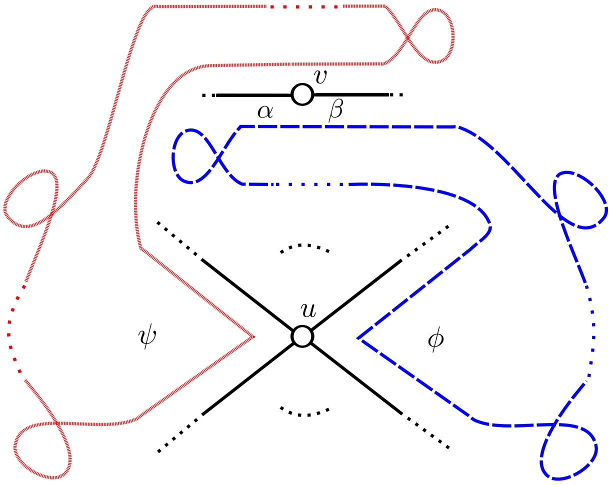


Figure 3.7: A graph embedding of type $f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)}(G,u,v)$

case (iii) is similar to the *case (iii)* of the previous proof, except that there it dealt with the scenario when the two edge-ends incident on root v were inserted into fb-walk ϕ, whereas here it deals with the scenario when the two edge-ends are inserted into the fb-walk ϕ, or into the fb-walk ψ. The analysis remains the same in both cases. The second and third terms of the production correspond to this case.

case (iv) – in an embedding of type $f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}(G,u,v)$, there is one additional sub-case of *case (iv)* of the previous proof: when the two edge-ends incident on root v are inserted into two different fb-walks, one of which is ϕ and the other ψ. In this case, the two fb-walks divide up into three fb-walks, and thus there is no genus increment. Figure 3.8 shows one of the two ways of inserting the two edge-ends in these two fb-walks (the other way gives similar results, we leave its drawing for the reader). The last term of the production corresponds to this case. $\diamondsuit$

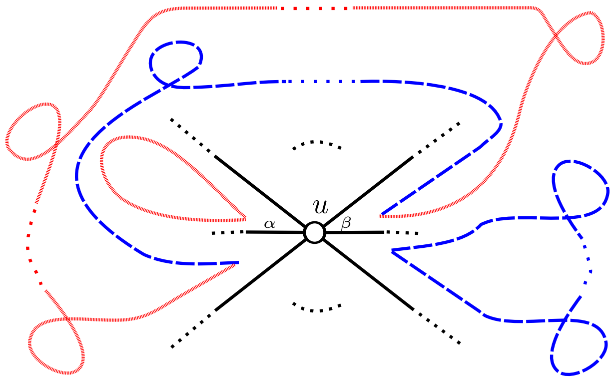


Figure 3.8: *Case (iv)* – when the two edge-ends are inserted into two different fb-walks one of which is ϕ and the other is ψ (we show only one of the two possible ways of doing this).

Theorem 21. *Let (G,u,v) be a double-rooted graph where $deg(u) = n \geq 2$ and $deg(v) = 2$, and let G' be the graph obtained by merging the roots u and v. Then for each partition $p_1p_2\cdots p_r$ of $deg(u)$, the following production holds:*

$$f_{p_1p_2\cdots p_r}s_i^0(G,u,v) \quad \longrightarrow \quad (n^2+n)\,g_{i+1}(G') \tag{3.4}$$

Proof. In an embedding of type $f_{p_1p_2\cdots p_r}s_i^0(G,u,v)$, none of the fb-walks incident on root u are incident on root v. This shows that there is a genus-increment of at least one, also there is no additional genus-increment, since both edge-ends incident on root v lie on the same fb-walk. Thus, all n^2+n embeddings are of type g_{i+1}. $\diamondsuit$

Theorem 22. *Let (G,u,v) be a double-rooted graph where deg$(u)\geq 2$ and deg$(v)=2$, and let G' be the graph obtained by merging the roots u and v. Then for each partition $p_1p_2\cdots p_r$ of deg(u), the following production holds:*

$$\begin{aligned} f_{p_1p_2\cdots p_r}s_i^{p_k}(G,u,v) \quad \longrightarrow \quad & \left(\sum_{\substack{x=1\\x\neq k}}^{r} p_x(p_x+1)\right) g_{i+1}(G') \\ & + p_k(p_k+1)\, g_i(G') \\ & + \left(\sum_{\substack{x=1\\x\neq k}}^{r}\sum_{\substack{y=x+1\\y\neq k}}^{r} 2p_xp_y\right) g_{i+1}(G') \\ & + \left(\sum_{\substack{x=1\\x\neq k}}^{r} p_kp_x\right) g_i(G') + \left(\sum_{\substack{x=1\\x\neq k}}^{r} p_kp_x\right) g_{i+1}(G') \qquad (3.5) \end{aligned}$$

Proof. In an embedding of type $f_{p_1p_2\cdots p_r}s_i^{p_k}(G,u,v)$, of the r fb-walks incident on root u, only one is incident on root v (we refer to this fb-walk as ϕ) such that only one u-strand (denoted by s_1) contains both occurrences of root v. There are four cases:

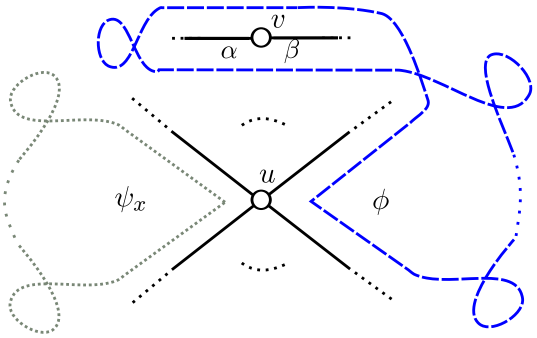


Figure 3.9: A graph embedding of type $f_{p_1p_2\cdots p_r}s_i^{p_k}(G,u,v)$.

case (i) When the two edge-ends incident on root v are inserted into one fb-walk ψ_x such that $\psi_x\neq\phi$. From Lemma 4 we know there are $p_x(p_x+1)$ ways of doing that. There is a genus-increment since the fb-walk ϕ (that contains both occurrences of root v) is merged with ψ_x (see Figure 3.10). The first term of the production represents this.

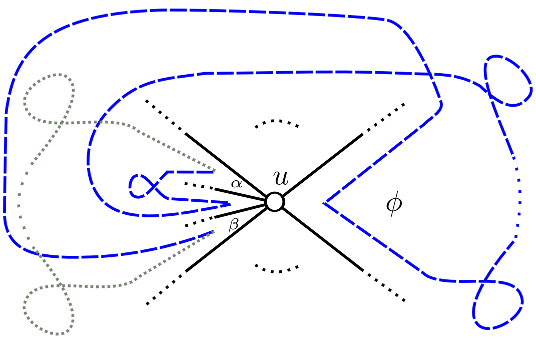


Figure 3.10: *Case (i)* – since the two fb-walks are merged with each other, there is a genus-increment of one (we show only one of the two possible ways of doing this).

case (ii) is when the two edge-ends incident on root v are inserted into fb-walk ϕ. It is clear that there is no genus-increment in this case, since fb-walk ϕ is split into two fb-walks (see Figure 3.11). The second term of the production represents this.

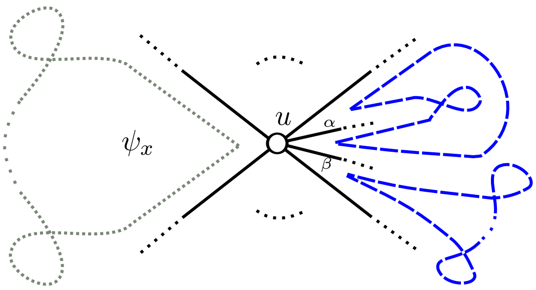


Figure 3.11: *Case (ii)* – since the two fb-walks are merged with each other, there is a genus-increment of one (we show only one of the two possible ways of doing this).

case (iii) When the two edge-ends incident on root v are inserted into two different fb-walks (that we denote with ψ_x and ψ_y), neither of which is ϕ. In this case the fb-walk ϕ is split into two strands, each of which recombines with one of the two fb-walks ψ_x and ψ_y, yielding two fb-walks (which is a reduction in the number of fb-walks by one). Thus there is no genus increment in this case. According to Lemma 5, there are $2p_xp_y$ ways of doing that. The third term of the production represents this.

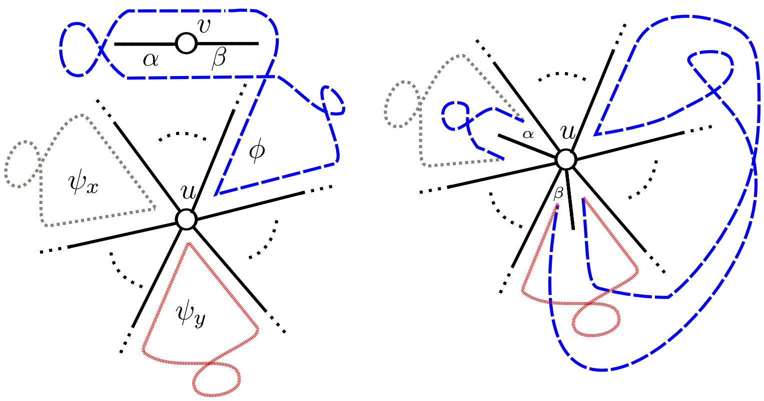


Figure 3.12: *Case (iii)* – when both edge-ends are inserted into two different fb-walks. The embedding before the amalgamation (left), and after the amalgamation (right) (we show only one of the two possible ways of doing this)

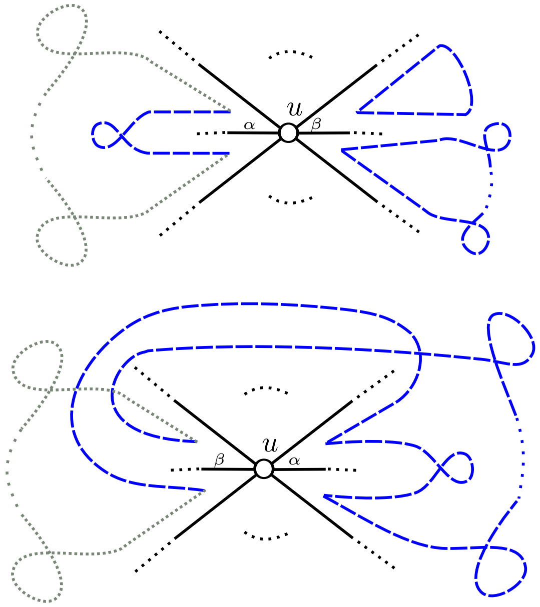


Figure 3.13: *Case (iv)* – when both edge-ends are inserted into two different fb-walks, one of which is ϕ. There are two ways of doing this, one of which results in a genus increment.

case (iv) is similar to *case (iii)*, except that one of the two fb-walks is ϕ. There are two ways of doing this (shown in Figure 3.13). In the first case, the fb-walk ϕ is split into three strands, one of which recombines with fb-walk ϕ_x, and the other two make two new fb-walks (the number of fb-walks rises by one, so the genus remains the same). In the second case, ϕ splits into three strands as well, but these strands recombine with each other, and with fb-walk ϕ_x to give only one fb-walk (the number of fb-walks decreases by one, resulting in a genus-increment). The last term of the production reflects this. $\diamondsuit$

Theorem 23. *Let (G,u,v) be a double-rooted graph where $deg(u) \geq 2$ and $deg(v) = 2$, and let G' be the graph obtained by merging the roots u and v. Then for each partition $p_1p_2\cdots p_r$ of $deg(u)$, the following production holds:*

$$\begin{aligned} f_{p_1p_2\cdots p_r}s_i^{(p_k,c)}(G,u,v) \quad \longrightarrow \quad & \left(\sum_{\substack{x=1\\x\neq k}}^{r} p_x(p_x+1)\right) g_{i+1}(G') \\ & + c(p_k-c)\, g_{i-1}(G') \; + \; [p_k(p_k+1) - c(p_k-c)]\, g_i(G') \\ & + \left(\sum_{\substack{x=1\\x\neq k}}^{r} \sum_{\substack{y=x+1\\y\neq k}}^{r} 2p_xp_y\right) g_{i+1}(G') \\ & + \left(\sum_{\substack{x=1\\x\neq k}}^{r} p_kp_x\right) g_i(G') \; + \; \left(\sum_{\substack{x=1\\x\neq k}}^{r} p_kp_x\right) g_{i+1}(G') \end{aligned} \tag{3.6}$$

Proof. In an embedding of type $f_{p_1p_2\cdots p_r}s_i^{(p_k,c)}(G,u,v)$, of the r fb-walks incident on root u, only one is incident on root v (we refer to this fb-walk as ϕ) such that the two occurrences of root v lie in two different strands (denoted by s_1 and s_2), separated by c u-corners. Let α and β be the two edge-ends incident on root v. Without any loss of generality, we can assume that in u-strand s_1, edge-end β appears before α in the chosen orientation (since $deg(v) = 2$, it is clear that these two edge-ends appear in fb-walks consecutively), whereas in u-strand s_2, edge-end α appears before β. We can further assume that in fb-walk ϕ, the sub-walk w_1 containing c intermediate u-corners starts after s_1 and ends before s_2 (in the chosen orientation). It follows that sub-walk w_2 containing the remaining $p_k - c$ u-corners starts after s_2 and ends before s_1 (in the chosen orientation). Figure 3.14 illustrates these assumptions (we assume that the two ends of the thick dotted line-segments that appear at the start and end of the fb-walk ϕ are connected, though we have not shown this in the figure, to avoid unnecessary clutter).

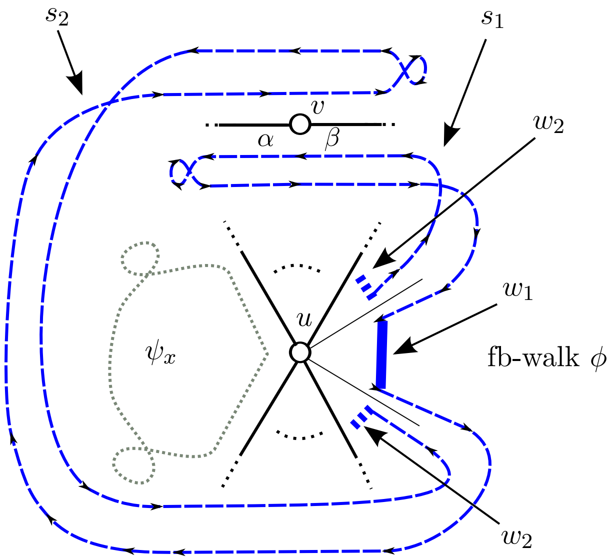


Figure 3.14: A graph embedding of type $f_{p_1p_2\cdots p_r}s_i^{(p_k,c)}(G,u,v)$.

Depending on where edge-ends α and β are inserted during the self-vertex-amalgamation, there are four cases:

case (i) is when the two edge-ends are inserted into an fb-walk, say ψ_x, such that $\psi \neq \phi$. In each of the two ways of doing this, the two fb-walks are merged, reducing the number of fb-walks by one. Thus there is a genus-increment (Figure 3.15 shows one of these two ways, we leave the other drawing for the reader). The first term of the production corresponds to this case.

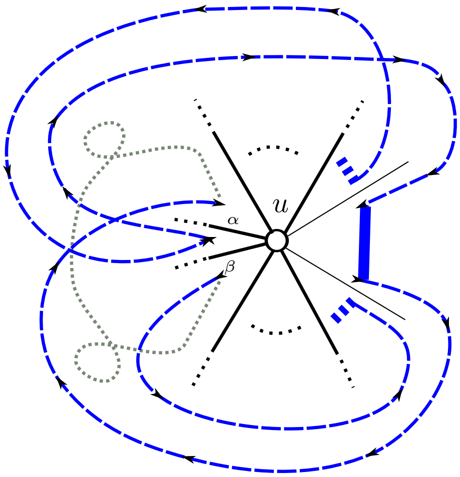


Figure 3.15: *Case (i)* – α and β are inserted into $\psi_x \neq \phi$.

case (ii) is when the two edge-ends incident on root v are inserted into ϕ. There are two sub-cases: *sub-case (ii-a)* when α is inserted into the sub-walk w_1, and α is inserted into the sub-walk w_2, thus splitting each of the two sub-walks w_1, w_2 and the u-strands s_1, s_2 into strands. These strands then recombine to form four different fb-walks as shown in Figure 3.16, thus resulting in a *negative* genus-increment. There are $c(p_k - c)$ ways of doing this, since there are c u-corners in w_1 an $p_k - c$ u-corners in w_2. The second term of the production deals with this scenario.

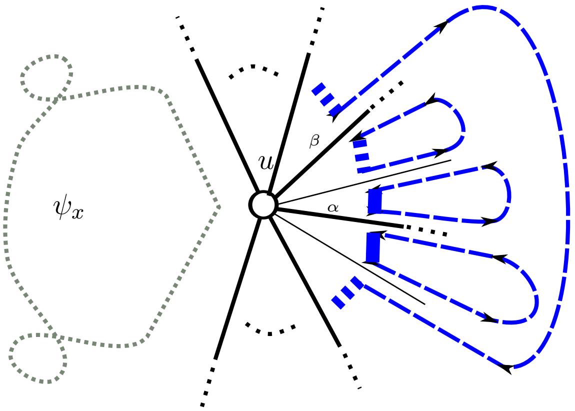


Figure 3.16: *Sub-case (ii-a)*

sub-case (ii-b) corresponds to all of the remaining $p_k(p_k+1) - c(p_k - c)$ ways of inserting the two edge-ends into the fb-walk ϕ, i.e. *(i)* when both edge-ends are inserted into either w_1 or into w_2 (see Figure 3.17 – top), and *(ii)* when α is inserted into w_2 aand β is inserted into w_1 (see Figure 3.17 – bottom). In each of these cases, ϕ is split into two fb-walks (so there is no genus increment). The third term of the production corresponds to this scenario.

case (iii) is when the two edge-ends, α and β are inserted into two different fb-walks, ψ_x and ψ_y, such that $\psi_x \neq \psi_y \neq \phi$. In this case, ϕ is split into two strands, one of which recombines with ψ_x to form a new fb-walk, and the other strand recombines with ψ_y to form another fb-walk. Thus the number of fb-walks reduces by one overall, resulting in a negative genus-increment (see Figure 3.18). The fourth term of the production corresponds to this case.

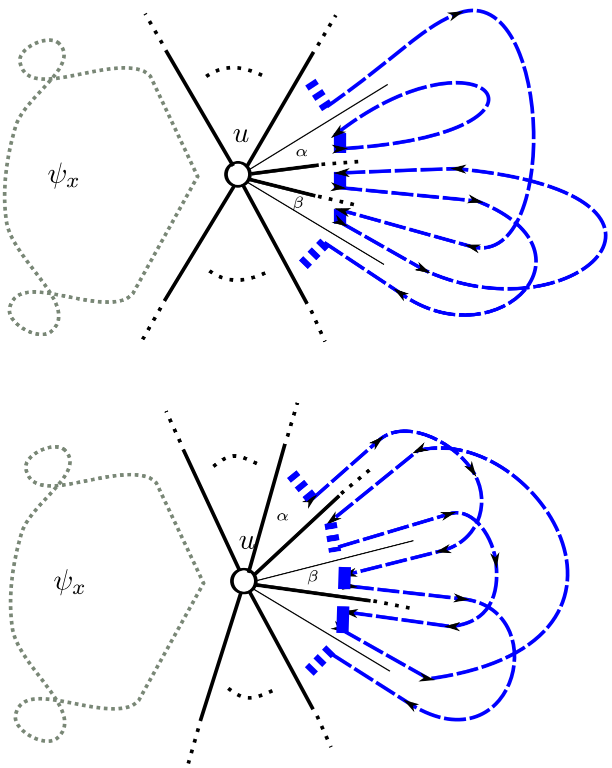


Figure 3.17: *Sub-case (ii-b)*

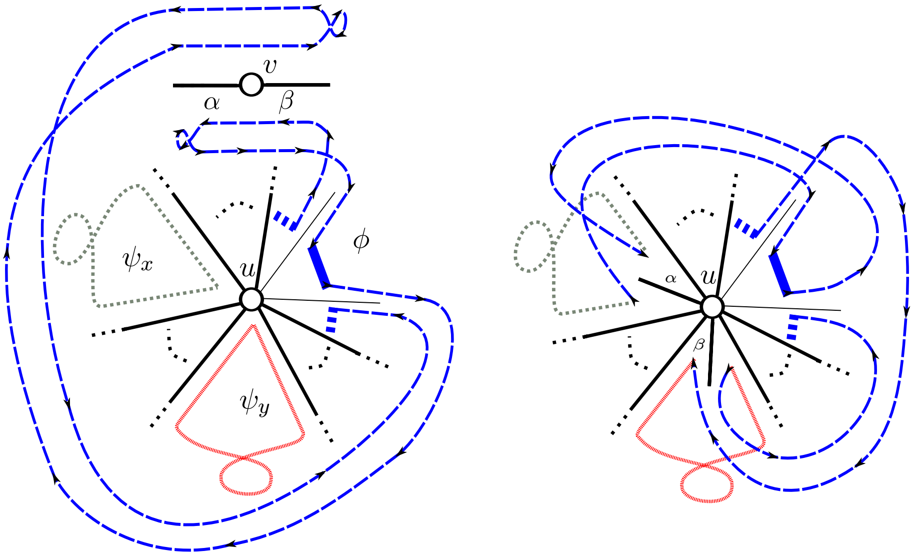


Figure 3.18: *Case (iii)* the two edge-ends are inserted into two different fb-walks, neither of which is ϕ. The embedding before the amalgamation (left), and after the amalgamation (right) (we show only one of the two possible ways of doing this)

case (iv) Corresponds to the scenario when the two edge-ends incident on root v are inserted into two different fb-walks, one of which is ϕ. As with *case (iv)* of the proof of Production 3.5, there are two sub-cases. We illustrate these sub-cases in Figure 3.19. The last two terms of the production correspond to this scenario. ♢

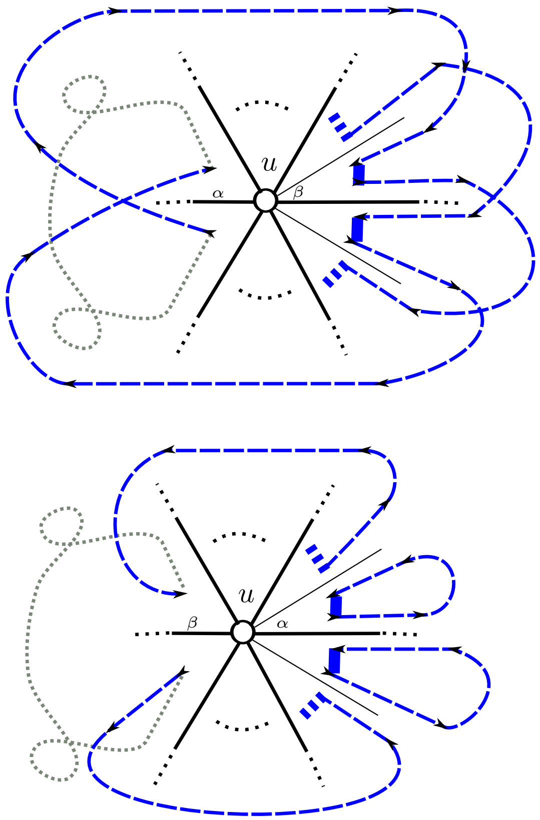


Figure 3.19: *Case (iv)* the two edge-ends are inserted into two different fb-walks, one of which is ϕ.

Example 24. We'll illustrate an application of these productions by computing the genus distribution of the graph obtained through self-vertex-amalgamation of a double-rooted graph (G, u, v), where $deg(u) = 3$ and $deg(v) = 2$.

Since $deg(u) = 3$, the following are the only possible sub-partials for the double-rooted graph:

$f_{111}d_i^0,\ f_{111}d_i^1, f_{111}d_i^{(1,1)},\ f_{21}d_i^0,\ f_{21}d_i^1, f_{21}d_i^2,\ f_{21}d_i^{(2,1)},\ f_3d_i^0,\ f_3d_i^3,\ f_{111}s_i^0,\ f_{111}s_i^1,\ f_{21}s_i^0,$ $f_{21}s_i^1, f_{21}s_i^2,\ f_{21}s_i^{(2,1)},\ f_3s_i^0,\ f_3s_i^3, f_3s_i^{(3,1)}$

Using Theorems 18–23, we obtain the following productions for self-vertex-amalgamation of a double-rooted graph (G, u, v), where $deg(u) = 3$ and $deg(v) = 2$.

$$
\begin{aligned}
f_{111}d_i^0(G,u,v) &\longrightarrow 6\,g_{i+1}(G') \,+\, 6\,g_{i+2}(G') \\
f_{111}d_i^1(G,u,v) &\longrightarrow g_i(G') \,+\, 9\,g_{i+1}(G') \,+\, 2\,g_{i+2}(G') \\
f_{111}d_i^{(1,1)}(G,u,v) &\longrightarrow 4\,g_i(G') \,+\, 8\,g_{i+1}(G') \\
f_{21}d_i^0(G,u,v) &\longrightarrow 8\,g_{i+1}(G') \,+\, 4\,g_{i+2}(G') \\
f_{21}d_i^1(G,u,v) &\longrightarrow g_i(G') \,+\, 11\,g_{i+1}(G') \\
f_{21}d_i^2(G,u,v) &\longrightarrow 3\,g_i(G') \,+\, 9\,g_{i+1}(G') \\
f_{21}d_i^{(2,1)}(G,u,v) &\longrightarrow 8\,g_i(G') \,+\, 4\,g_{i+1}(G') \\
f_3d_i^0(G,u,v) &\longrightarrow 12\,g_{i+1}(G') \\
f_3d_i^3(G,u,v) &\longrightarrow 6\,g_i(G') \,+\, 6\,g_{i+1}(G') \\
f_{111}s_i^0(G,u,v) &\longrightarrow 12\,g_{i+1}(G') \\
f_{111}s_i^1(G,u,v) &\longrightarrow 4\,g_i(G') \,+\, 8\,g_{i+1}(G') \\
f_{21}s_i^0(G,u,v) &\longrightarrow 12\,g_{i+1}(G') \\
f_{21}s_i^1(G,u,v) &\longrightarrow 4\,g_i(G') \,+\, 8\,g_{i+1}(G') \\
f_{21}s_i^2(G,u,v) &\longrightarrow 8\,g_i(G') \,+\, 4\,g_{i+1}(G') \\
f_{21}s_i^{(2,1)}(G,u,v) &\longrightarrow g_{i-1}(G') \,+\, 7\,g_i(G') \,+\, 4\,g_{i+1}(G') \\
f_3s_i^0(G,u,v) &\longrightarrow 12\,g_{i+1}(G') \\
f_3s_i^3(G,u,v) &\longrightarrow 12\,g_i(G') \\
f_3s_i^{(3,1)}(G,u,v) &\longrightarrow 2\,g_{i-1}(G') \,+\, 10\,g_i(G')
\end{aligned}
$$

After transposition, we get the following formula:

$$
\begin{aligned}
g_i(G') \;=\; & 6\,f_{111}d_{i-1}^0 \,+\, 6\,f_{111}d_{i-2}^0 \,+\, f_{111}d_i^1 \,+\, 9\,f_{111}d_{i-1}^1 \,+\, 2\,f_{111}d_{i-2}^1 \\
& +\, 4\,f_{111}d_i^{(1,1)} \,+\, 8\,f_{111}d_{i-1}^{(1,1)} \,+\, 8\,f_{21}d_{i-1}^0 \,+\, 4\,f_{21}d_{i-2}^0 \\
& +\, f_{21}d_i^1 \,+\, 11\,f_{21}d_{i-1}^1 \,+\, 3\,f_{21}d_i^2 \,+\, 9\,f_{21}d_{i-1}^2 \\
& +\, 8\,f_{21}d_i^{(2,1)} \,+\, 4\,f_{21}d_{i-1}^{(2,1)} \,+\, 12\,f_3d_{i-1}^0
\end{aligned}
$$

$$
\begin{aligned}
&+ 6\, f_3 d_i^3 + 6\, f_3 d_{i-1}^3 + 12\, f_{111} s_{i-1}^0 \\
&+ 4\, f_{111} s_i^1 + 8\, f_{111} s_{i-1}^1 + 12\, f_{21} s_{i-1}^0 \\
&+ 4\, f_{21} s_i^1 + 8\, f_{21} s_{i-1}^1 + 8\, f_{21} s_i^2 + 4\, f_{21} s_{i-1}^2 \\
&+ f_{21} s_{i+1}^{(2,1)} + 7\, f_{21} s_i^{(2,1)} + 4\, f_{21} s_{i-1}^{(2,1)} + 12\, f_3 s_{i-1}^0 \\
&+ 12\, f_3 s_i^3 + 2\, f_3 s_{i+1}^{(3,1)} + 10\, f_3 s_i^{(3,1)} \qquad (3.7)
\end{aligned}
$$

This formula can be used to compute the genus distribution of the self-vertex-amalgamation of any double-rooted graph (G, u, v) such that $deg(u) = 3$ and $deg(v) = 2$, provided that we are given the partitioned genus distribution of (G, u, v).

For instance, for the double-rooted graph $(\dot{K}_4, u, v)$ (as defined in Chapter 2 – i.e., a graph obtained from the complete graph (K_4, u) by inserting a vertex v as a subdivision point of any edge of the graph) the partitioned genus distribution is given in Table 3.1.

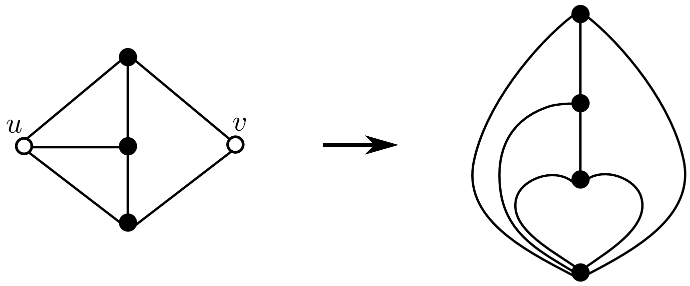


Figure 3.20: Self-vertex-amalgamation of double-rooted graph $(\dot{K}_4, u, v)$.

Table 3.1: Nonzero partials of $(\dot{K}_4, u, v)$.

k	$f_{111}d_k^1$	$f_{21}d_k^{(2,1)}$	$f_3 d_k^3$	$f_{21}s_k^{(2,1)}$	$f_3 s_k^{(3,1)}$	g_k
0	2					2
1		6	2	6		14

We can use the values given in Table 3.1 as base-cases to recursively evaluate Formula 3.7, to compute the genus distribution of the graph $\dot{K}_4'$ obtained through self-vertex-amalgamation of $(\dot{K}_4, u, v)$. Table 3.2 gives the results of this evaluation.

Table 3.2: Genus distribution of the resulting graph

k	g_k
0	8
1	120
2	64

3.3 Edge-addition

In this section, we'll derive productions for calculating the effect of the operation of edge-addition on the double-rooted sub-partials of a double-rooted graph (G, u, v). This enables us to compute the genus distribution of the graph $G+e$ directly from the partitioned genus distribution of (G, u, v). We'll see in the next chapter how these productions can be used to compute the genus distribution of iteratively constructed *chains* of graphs by adding an edge between the two root vertices of open chains.

Theorem 25. *Let (G, u, v) be a double-rooted graph where $deg(u) \geq 2$ and $deg(v) = 2$, and let $G + e$ be the graph obtained by adding an edge e between roots u and v. Then for each partition $p_1 p_2 \cdots p_r$ of $deg(u)$, the following productions hold:*

$$f_{p_1p_2\cdots p_r} d_i^0(G,u,v) \longrightarrow 2n\, g_{i+1}(G+e) \tag{3.8}$$

$$f_{p_1p_2\cdots p_r} d_i^{p_k}(G,u,v) \longrightarrow (2n-p_k)\, g_{i+1}(G+e) + p_k\, g_i(G+e) \tag{3.9}$$

$$f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)}(G,u,v) \longrightarrow (2n-p_l-p_m)\, g_{i+1}(G+e) + (p_l+p_m)\, g_i(G+e) \tag{3.10}$$

$$f_{p_1p_2\cdots p_r} s_i^0(G,u,v) \longrightarrow 2n\, g_{i+1}(G+e) \tag{3.11}$$

$$f_{p_1p_2\cdots p_r} s_i^{p_k}(G,u,v) \longrightarrow (2n-2p_k)\, g_{i+1}(G+e) + 2p_k\, g_i(G+e) \tag{3.12}$$

$$f_{p_1p_2\cdots p_r} s_i^{(p_k,c)}(G,u,v) \longrightarrow (2n-2p_k)\, g_{i+1}(G+e) + 2p_k\, g_i(G+e) \tag{3.13}$$

Proof. Since in any embedding of the double-rooted graph (G, u, v), there are n u-corners and two v-corners, the number of ways of adding an edge between the root u and the root v is $2n$. This explains why the sum of the coefficients in each of these productions is $2n$.

In an embedding of type $f_{p_1p_2\cdots p_r} d_i^0(G,u,v)$ and $f_{p_1p_2\cdots p_r} s_i^0(G,u,v)$, none of the fb-walks incident on root u are also incident on root v. Thus, when an edge is added between u and

v, the corresponding two fb-walks are merged with each other, reducing the number of faces by one. We use Euler's polyhedral equation to compute the genus increment:

$$\#v - (\#e + 1) + (\#f - 1) = (\#v - \#e + \#f) - 2.$$

Thus, there is a genus increment of 1.

In an embedding of type $f_{p_1p_2\cdots p_r}d_i^{p_k}(G,u,v)$, one of the fb-walks incident on root u (containing p_k occurrences of u), is also incident on root v. Let this fb-walk be represented by ϕ. If both of the edge-ends of edge e are inserted in ϕ, there will not be any genus-increment, and in every other case there will be a genus-increment of 1, as argued earlier. Since there are p_k u-corners and only one v-corner in ϕ, there are p_k ways of adding e in this face. The second term in Production 3.9 corresponds to this case, and the first term corresponds to all other cases.

In an embedding of type $f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}(G,u,v)$, two of the fb-walks that are incident on root u are also incident on root v. The second term in Production 3.10 corresponds to the cases where edge e is added in these two fb-walks, whereas the first term corresponds to all other cases.

Productions 3.12 and 3.13 are derived similarly, and we note that the fb-walk containing p_k u-corners also contains *two* v-corners (and thus there are $2p_k$ ways of adding edge e in this face). ♢

Example 26. For the sub-partials of a double-rooted graph (G,u,v), such that $deg(u)=3$ and $deg(v)=2$, these productions become:

$$\begin{aligned}
f_{111}d_i^0(G,u,v) &\longrightarrow 6\,g_{i+1}(G+e)\\
f_{111}d_i^1(G,u,v) &\longrightarrow g_i(G+e) \,+\, 5\,g_{i+1}(G+e)\\
f_{111}d_i^{(1,1)}(G,u,v) &\longrightarrow 2\,g_i(G+e) \,+\, 4\,g_{i+1}(G+e)\\
f_{21}d_i^0(G,u,v) &\longrightarrow 6\,g_{i+1}(G+e)\\
f_{21}d_i^1(G,u,v) &\longrightarrow g_i(G+e) \,+\, 5\,g_{i+1}(G+e)\\
f_{21}d_i^2(G,u,v) &\longrightarrow 2\,g_i(G+e) \,+\, 4\,g_{i+1}(G+e)\\
f_{21}d_i^{(2,1)}(G,u,v) &\longrightarrow 3\,g_i(G+e) \,+\, 3\,g_{i+1}(G+e)\\
f_3d_i^0(G,u,v) &\longrightarrow 6\,g_{i+1}(G+e)
\end{aligned}$$

$$f_3 d_i^3(G,u,v) \longrightarrow 3\,g_i(G+e) + 3\,g_{i+1}(G+e)$$
$$f_{111} s_i^0(G,u,v) \longrightarrow 6\,g_{i+1}(G+e)$$
$$f_{111} s_i^1(G,u,v) \longrightarrow 2\,g_i(G+e) + 4\,g_{i+1}(G+e)$$
$$f_{21} s_i^0(G,u,v) \longrightarrow 6\,g_{i+1}(G+e)$$
$$f_{21} s_i^1(G,u,v) \longrightarrow 2\,g_i(G+e) + 4\,g_{i+1}(G+e)$$
$$f_{21} s_i^2(G,u,v) \longrightarrow 4\,g_i(G+e) + 2\,g_{i+1}(G+e)$$
$$f_{21} s_i^{(2,1)}(G,u,v) \longrightarrow 4\,g_i(G+e) + 2\,g_{i+1}(G+e)$$
$$f_3 s_i^0(G,u,v) \longrightarrow 6\,g_{i+1}(G+e)$$
$$f_3 s_i^3(G,u,v) \longrightarrow 6\,g_i(G+e)$$
$$f_3 s_i^{(3,1)}(G,u,v) \longrightarrow 6\,g_i(G+e)$$

After transposition, we get:

$$\begin{aligned} g_i(G+e) &= 6\,f_{111} d_{i-1}^0 + f_{111} d_i^1 + 5\,f_{111} d_{i-1}^1 \\ &+ 2\,f_{111} d_i^{(1,1)} + 4\,f_{111} d_{i-1}^{(1,1)} + 6\,f_{21} d_{i-1}^0 \\ &+ f_{21} d_i^1 + 5\,f_{21} d_{i-1}^1 + 2\,f_{21} d_i^2 + 4\,f_{21} d_{i-1}^2 \\ &+ 3\,f_{21} d_i^{(2,1)} + 3\,f_{21} d_{i-1}^{(2,1)} + 6\,f_3 d_{i-1}^0 \\ &+ 3\,f_3 d_i^3 + 3\,f_3 d_{i-1}^3 + 6\,f_{111} s_{i-1}^0 \\ &+ 2\,f_{111} s_i^1 + 4\,f_{111} s_{i-1}^1 + 6\,f_{21} s_{i-1}^0 \\ &+ 2\,f_{21} s_i^1 + 4\,f_{21} s_{i-1}^1 + 4\,f_{21} s_i^2 + 2\,f_{21} s_{i-1}^2 \\ &+ 4\,f_{21} s_i^{(2,1)} + 2\,f_{21} s_{i-1}^{(2,1)} + 6\,f_3 s_{i-1}^0 \\ &+ 6\,f_3 s_i^3 + 6\,f_3 s_i^{(3,1)} \end{aligned} \tag{3.14}$$

Using this formula for the graph $(\dot{K}_4, u, v)$ with the values given in given in Table 3.1 as base cases, we obtain the genus distribution of $\dot{K}_4 + e$, given in Table 3.3.

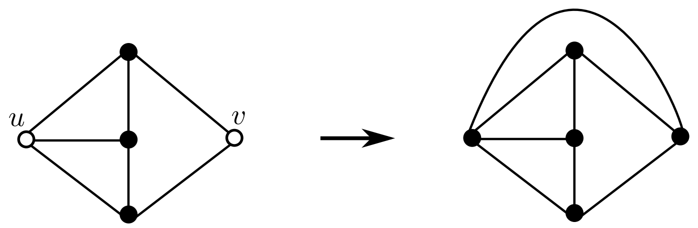


Figure 3.21: Joining the roots of the double-rooted graph $(\dot{K}_4, u, v)$.

Table 3.3: Genus distribution of $\dot{K}_4 + e$.

k	g_k
0	2
1	58
2	36

Remark Using the results of this chapter, we can compute the genus distribution of the self-vertex-amalgamation of a double-rooted graph (G, u, v), and also the genus distribution of the graph $G + e$, obtained by adding an edge between the two root vertices, where one of the two root vertices is restricted to be of valence two. Since at this point we do not have a way of computing double-root partitioned genus distribution of open chains of a double-rooted graph, the applications of the productions derived in this chapter are limited. In the next chapter, we'll develop a method for computing the double-root partitioned genus distributions of open chains of a double-rooted graph (G, u, v), using its double-root partitioned genus distribution only.

Chapter 4

Extended Vertex Amalgamation

In Chapter 3, we developed methods for computing the genus distribution of graphs obtained through the operations of self-vertex-amalgamation and edge-addition, provided that we are given the partitioned genus distribution of a double-rooted graph (G, u, v) with $deg(u) \geq 2$ and $deg(v) = 2$. In order to use these methods iteratively on the sequence of open chains of a double-rooted graph, we need to extend the methods developed in Chapter 2 so that we obtain double-root partitioned genus distributions of open chains.

In this chapter, we derive a method for computing the partitioned genus distribution of the vertex amalgamation of two double rooted graphs, (G, u, v) and (H, a, b), such that $deg(u) \geq 2, deg(a) \geq 2$, and $deg(v) = deg(b) = 2$, in which vertex v is merged with vertex a. The resultant vertex amalgamated graph will be a double-rooted graph (X, u, b). This method is then used to compute the double-root partitioned genus distributions of open chains of any double-rooted graph.

These double-rooted partitioned genus distributions can then be used in coordination with the methods derived in the previous chapter, to compute the genus distributions of closed chains of double-rooted graphs that meet the stated conditions. Also, we use these distributions to compute the genus distributions of the graph families obtained through connecting the two roots of a double-rooted graph with an edge.

4.1 Double-rooted Amalgamations

We extend the definition of a production for vertex amalgamation to the case in which both of the graphs being amalgamated are double-rooted. A production for the double-rooted vertex amalgamation

$$(G, u, v) * (H, a, b) \;=\; (X, u, b)$$

of a double-rooted graph (G, u, v) with a double-rooted graph (H, a, b) (where $deg(v) = deg(b) = 2$, and $deg(u) \geq 2, deg(a) \geq 2$) is an expression of the form

$$p_i(G, u, b) * q_j(H, a, b) \longrightarrow \sum_{\substack{r \text{ ranges over the} \\ \text{six sub-partials}}} (c_{1r} r_{i+j}(X, u, b) + c_{2r} r_{i+j+1}(X, u, b))$$

where p and q are the double-root sub-partials, and where c_{1r}, and c_{2r} are integers. It means that amalgamation of a type-p_i embedding of graph (G, u, v) and a type-q_j embedding of graph (H, a, b) induces a set of c_{1r} type-r_{i+j} and c_{2r} type-r_{i+j+1} embeddings of (X, u, b). Again, it follows from Formula (2.1), that the sum of the integers c_{1r} and c_{2r} (over all r's) in each production is six. Since there are a total of six different double-root sub-partials, there are a total of thirty-six different productions.

In the theorems that follow, we prove a subset of these productions, in which the embedding of the graph (H, a, b) is of type $f_{q_1 q_2 \cdots q_t} d_j^{q_s}$. The two edge-ends incident at root vertex v are denoted by α and β, and the fb-walk containing q_s a-corners that also contains one of the two occurrences of root b, is denoted by φ. When we say that an amalgamation results in a "genus-increment" of m, we mean that the genus of the resultant embedding is $i + j + m$. It should be noted that the genus-increment and the type of the second root b in the vertex-amalgamated graph (X, u, b) has to be consistent with the productions derived in Chapter 2. This is the reason why we do not focus on these details in our proofs below. While stating the theorems below, we omit the triple (G, u, v) from the first amalgamand in the production-head, (H, a, b) from the second amalgamand, and (X, u, b) from each of the terms in the production-body, in order to conserve space.

Theorem 27. *Let (G, u, v) and (H, a, b) be two double-rooted graphs such that $deg(u) \geq 2, deg(a) \geq 2$ and $deg(v) = deg(b) = 2$, and let (X, u, b) be the graph obtained by amalgamating these two graphs on root vertices v and a. Then for each pair of partitions $p_1 p_2 \cdots p_r$*

of deg(u) and $q_1q_2\cdots q_r$ of deg(a), the following productions hold:

$$f_{p_1p_2\cdots p_r}d_i^0 * f_{q_1q_2\cdots q_t}d_j^{q_s} \longrightarrow \left(\sum_{x=1}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r}d_{i+j}^0 + \left(\sum_{x=1}^{t}\sum_{y=x+1}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r}d_{i+j+1}^0 \tag{4.1}$$

Proof In an embedding of type $f_{p_1p_2\cdots p_r}d_i^0(G,u,v)$, none of the fb-walks that are incident on root u contain root v. Note that in a vertex-amalgamation, the only fb-walks that are affected are those that are incident either on root v or on root a. This implies that for an embedding of type $f_{p_1p_2\cdots p_r}d_i^0(G,u,v)$, the r fb-walks that are incident on root u before vertex-amalgamation, remain incident on it after the amalgamation. This explains why the integer partition $p_1\cdots p_r$ remains unchanged in all of the terms in the production-body of Production 4.1. Also, in an embedding of type $f_{q_1q_2\cdots q_t}d_j^{q_s}(H,a,b)$ only one of the t fb-walks incident on root a is incident on root b, thus after the vertex-amalgamation, the two occurrences of root b continue to be on two different fb-walks. This explains why the second root is of type d in all of the sub-partials of the production-body. It follows further that no matter how root v is amalgamated with root a of the double-rooted graph (H,a,b), none of the fb-walks that are incident on root u in the amalgamated graph (X,u,b) become incident on root b. This explains the superscript of 0 in each of the terms in production body. The coefficients and genus increments follow from Lemmas 4 and 5 and Euler's polyhedral equation. We have seen similar coefficients and genus-increments in earlier chapters, so I do not elaborate here. ♢

Theorem 28. *Let (G,u,v) and (H,a,b) be two double-rooted graphs such that deg(u) ≥ 2, deg(a) ≥ 2 and deg(v) $=$ deg(b) $= 2$, and let (X,u,b) be the graph obtained by amalgamating these two graphs on root vertices v and a. Then for each pair of partitions $p_1p_2\cdots p_r$ of deg(u) and $q_1q_2\cdots q_r$ of deg(a), the following productions hold:*

$$f_{p_1p_2\cdots p_r}d_i^{p_k} * f_{q_1q_2\cdots q_t}d_j^{q_s} \longrightarrow \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r}d_{i+j}^0 + \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r}d_{i+j}^0$$

$$+ \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r} d_{i+j}^{p_k}$$
$$+ \left(\sum_{\substack{x=1 \\ x\neq s}}^{t} \sum_{\substack{y=x+1 \\ y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r} d_{i+j+1}^{0}$$
$$+ \left(\sum_{\substack{x=1 \\ x\neq s}}^{t} 2q_xq_s\right) f_{p_1p_2\cdots p_r} d_{i+j+1}^{p_k} \quad (4.2)$$

Proof In this amalgamation, we need to consider only those cases in which both of the fb-walks ϕ (containing p_k u-corners) and the fb-walk containing φ (containing p_s a-corners) are involved – since in all other cases, what was said in the proof of Production 4.1 suffices (as is also evidenced by the first and the second-last terms in the production-body). There are two cases:

case (i) is when both edge-ends α and β are inserted into the fb-walk φ. For any choice of two a-corners for this operation, there are two ways of doing this: the "first" edge-end is inserted either into the "first" a-corner or into the "second" a-corner. For both of these choices, the fb-walk φ is split into two strands, one of which recombines with the fb-walk ϕ, and the other with the other fb-walk incident on root v. Since only one of these two strands contains the occurrence of root b, one of the embeddings produced is of type $f_{p_1p_2\cdots p_r} d_{i+j}^{0}(X,u,b)$ and the other embedding is of type $f_{p_1p_2\cdots p_r} d_{i+j}^{p_k}(X,u,b)$. The second and third terms in the production-body represent these two cases. The total number of embeddings produced in this scenario is $q_s(q_s+1)$ (from Lemma 4), half of which are of the first type and half of the second type. This explains the coefficients in these two terms.

case (ii) When α and β are inserted in two different fb-walks, one of which is φ (denote the other by ψ_x). After the amalgamation, the two fb-walks are merged into one fb-walk. Thus, ϕ, φ and ψ_x all become one fb-walk that also contains an occurrence of root b. The last term in the production body represents this scenario. Again, the coefficient comes from Lemma 5. ♢

Theorem 29. *Let (G,u,v) and (H,a,b) be two double-rooted graphs such that $deg(u) \geq 2, deg(a) \geq 2$ and $deg(v) = deg(b) = 2$, and let (X,u,b) be the graph obtained by amalgamating these two graphs on root vertices v and a. Then for each pair of partitions $p_1p_2\cdots p_r$*

of deg(u) and $q_1q_2\cdots q_r$ *of deg(a), the following productions hold:*

$$\begin{aligned}
f_{p_1p_2\cdots p_r}d_i^{(p_g,p_h)} * f_{q_1q_2\cdots q_t}d_j^{q_s} \longrightarrow & \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r}d_{i+j}^{0} \\
& + \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r}d_{i+j}^{p_g} \\
& + \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r}d_{i+j}^{p_h} \\
& + \left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r}d_{i+j+1}^{0} \\
& + \left(\sum_{\substack{x=1\\x\neq s}}^{t} 2q_xq_s\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r}d_{i+j+1}^{p_g+p_h} \qquad (4.3)
\end{aligned}$$

Proof This is similar to the proof of Production 4.2 with two key differences.

The first difference is that in *case (i)*, since both of the fb-walks incident on root v are also incident on root u (as opposed to the *case (i)* of the previous proof, where only one fb-walk is incident on root v), when the two strands are recombined, one of the embeddings produced is of type $f_{p_1p_2\cdots p_r}d_{i+j}^{p_g}(X,u,b)$ and the other of type $f_{p_1p_2\cdots p_r}d_{i+j}^{p_h}(X,u,b)$. The second and third terms of the production-body reflect this.

The second difference affects the last two terms of the production-body, i.e., it affects the cases when α and β are inserted into two different fb-walks. As a consequence of this operation, the two fb-walks incident on root v are merged together. In an embedding of type $f_{p_1p_2\cdots p_r}d_i^{(p_g,p_h)}$, both of the fb-walks incident on root v contain u-corners. Since the total number of u-corners in these two fb-walks is p_g+p_h, and since these two are merged, it follows that after the amalgamation, the partition representing the distribution of u corners is $p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r$ instead of $p_1p_2\cdots p_r$ as before. The hat ˆ signs on p_g and p_h represent that the fb-walks corresponding to these two numbers are no longer there, but are merged with each other, represented by the addition of a new part p_g+p_h in the partition. ◇

Theorem 30. *Let* (G,u,v) *and* (H,a,b) *be two double-rooted graphs such that* $deg(u)\geq 2, deg(a)\geq 2$ *and* $deg(v)=deg(b)=2$*, and let* (X,u,b) *be the graph obtained by amalga-*

mating these two graphs on root vertices v and a. Then for each pair of partitions $p_1p_2\cdots p_r$ of deg(u) and $q_1q_2\cdots q_r$ of deg(a), the following productions hold:

$$f_{p_1p_2\cdots p_r}s_i^0 * f_{q_1q_2\cdots q_t}d_j^{q_s} \longrightarrow (n^2+n)\, f_{p_1p_2\cdots p_r}d_{i+j}^0 \tag{4.4}$$

Proof The proof of this production is easy, and is essentially follows from the proof of the Production 8 in Chapter 2. ♢

Theorem 31. *Let (G,u,v) and (H,a,b) be two double-rooted graphs such that deg(u) ≥ 2, deg(a) ≥ 2 and deg(v) $=$ deg(b) $= 2$, and let (X,u,b) be the graph obtained by amalgamating these two graphs on root vertices v and a. Then for each pair of partitions $p_1p_2\cdots p_r$ of deg(u) and $q_1q_2\cdots q_r$ of deg(a), the following productions hold:*

$$\begin{aligned} f_{p_1p_2\cdots p_r}s_i^{p_k} * f_{q_1q_2\cdots q_t}d_j^{q_s} \longrightarrow & \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r}d_{i+j}^0 \\ & + q_s(q_s+1)\, f_{p_1p_2\cdots p_r}d_{i+j}^{p_k} \\ & + \left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r}d_{i+j}^0 \\ & + \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_xq_s\right) f_{p_1p_2\cdots p_r}d_{i+j}^0 \\ & + \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_xq_s\right) f_{p_1p_2\cdots p_r}d_{i+j}^{p_k} \end{aligned} \tag{4.5}$$

Proof In an embedding of type $f_{p_1p_2\cdots p_r}s_i^{p_k}$, the edge-ends α and β lie on the same u-strand. Thus, there is no genus increment in this case (also see the relevant production in Chapter 2). As in the case of the proof of Production 4.2, we need to specifically consider only two cases, as the remaining cases are essentially covered in the proof of Production 4.4.

case (i) is when α and β are inserted into φ. Since both α and β lie on the same u-strand, the partition representing the distribution of the u-corners in fb-walks remains the same after vertex-amalgamation. As opposed to the *case (i)* of the proof of Production

4.2, the fb-walks ϕ and φ are completely merged with each other since both α and β lie on the same u-strand. This implies that in all of the resultant embeddings, the newly merged fb-walk contains one occurrence of root b (the occurrence that was previously on φ). The second term of the production-body reflects this.

case (ii) is when α and β are inserted into two different fb-walks, one of which is φ (we denote the other fb-walk by ψ_x). In this case, the u-strand containing both occurrences of v is split into two fragments. One fragment recombines with ψ_x, and the other fragment recombines with φ. The fragment that recombines with φ now contains an occurrence of root b. According to Lemma 5, there are a total of $2p_xp_k$ embeddings that are yielded, half of these are of each type. The last two terms of the production-body represent this. $\diamondsuit$

Theorem 32. *Let (G,u,v) and (H,a,b) be two double-rooted graphs such that $deg(u) \geq 2, deg(a) \geq 2$ and $deg(v) = deg(b) = 2$, and let (X,u,b) be the graph obtained by amalgamating these two graphs on root vertices v and a. Then for each pair of partitions $p_1p_2\cdots p_r$ of $deg(u)$ and $q_1q_2\cdots q_r$ of $deg(a)$, the following productions hold:*

$$
\begin{aligned}
f_{p_1p_2\cdots p_r}s_i^{(p_k,d)} * f_{q_1q_2\cdots q_t}d_j^{q_s} \longrightarrow & \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r}d_{i+j}^{0} \\
& + q_s(q_s+1)\, f_{p_1p_2\cdots p_r}d_{i+j}^{p_k} \\
& + \left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots\hat{p}_k\cdots p_rd(p_k-d)}d_{i+j}^{0} \\
& + \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_xq_s\right) f_{p_1p_2\cdots\hat{p}_k\cdots p_rd(p_k-d)}d_{i+j}^{d} \\
& + \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_xq_s\right) f_{p_1p_2\cdots\hat{p}_k\cdots p_rd(p_k-d)}d_{i+j}^{p_k-d}
\end{aligned}
\qquad (4.6)
$$

Proof The proof for this production is essentially similar to the proof of Production 4.5, with one important difference: in an embedding of type $f_{p_1p_2\cdots p_r}s_i^{(p_k,d)}$, the edge-ends α and β lie in two different u-strands (denoted by s_1 and s_2). So during the amalgamation, if α and β are inserted into two different fb-walks, say into ψ_x and ψ_y, then one of the two strands recombines with ψ_x and the other with ψ_y, thus yielding two fb-walks, both of which now contain u-corners. Since there are d intermediate u-corners between the u-strands s_1 and s_2, one of the two fb-walks now contains d u-corners and the other fb-walk contains $p_k - d$ u-corners. This implies that the partition that represents the distribution of u-corners after the amalgamation in this case looks like this: $p_1 p_2 \cdots \hat{p}_k \cdots p_r d(p_k - d)$, where the hat sign $\hat{}$ over p_k implies that the fb-walk corresponding to this part is no longer there, but is now divided into two parts, as represented by the addition of two new parts in the partition d and $p_k - d$ at the end. This affects last three terms of the production. ♢

Of the thirty-six productions that are possible, we have given the proof of only six here. The proofs of the remaining productions can be derived using similar techniques. We have listed them in Appendix A as Theorem 43.

Genus Distribution of the Double-Rooted Open Chain of $(\dot{K}_4, u, v)$

Example 33. We extend the definition of the open chain given in Example 16 to so that it has two roots instead of only one. As before, we can specify a sequence of open chains of copies of a double-rooted graph (G, u, v) recursively.

$$(X_1, s_1, t_1) = (G, u, v) \tag{4.7}$$

$$(X_m, s_m, t_m) = (X_{m-1}, s_{m-1}, t_{m-1}) * (G, u, v) \quad \text{for } m \geq 1 \tag{4.8}$$

In order to compute the partitioned genus distribution of the sequence of open chains of the double-rooted graph $(\dot{K}_4, u, v)$, we note that since $deg(u) = 3$ and $deg(v) = 2$, 111, 21 and 3 are the only partitions we need to consider in this case. Table 4.1 lists the non-zero partials of $(\dot{K}_4, u, v)$.

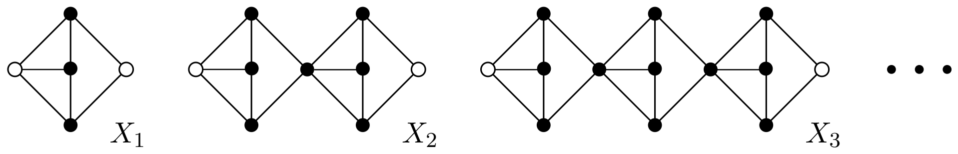


Figure 4.1: Sequence of open-chains of the double-rooted graph $(\dot{K}_4, u, v)$.

Table 4.1: Nonzero partials of $(\dot{K}_4, u, v)$.

j	$f_{111}d_j^1$	$f_{21}d_j^{(2,1)}$	$f_3d_j^3$	$f_{21}s_j^{(2,1)}$	g_j
0	2				2
1		6	2	6	14

Plugging these values into the second amalgamand of the relevant productions of Theorem 43 in Appendix A, and transposing the consequent productions, we obtain the following recurrences (for the intermediate steps, see Appendix B):

$$
\begin{aligned}
f_{111}d_i^0 &= 12\,f_{111}d_i^0 + 96\,f_{111}d_{i-1}^0 + 10\,f_{111}d_i^1 + 40\,f_{111}d_{i-1}^1 + 8\,f_{111}d_i^{(1,1)} + 24\,f_{111}s_i^0\\
&\quad 16\,f_{111}s_i^1 + 96\,f_{111}s_{i-1}^0 + 4\,f_{21}s_i^{(2,1)}\\
f_{111}d_i^1 &= 2\,f_{111}d_i^1 + 8\,f_{111}s_i^1 + 4\,f_{111}d_i^{(1,1)} + 56\,f_{111}d_{i-1}^1 + 72\,f_{111}d_{i-1}^{(1,1)} + 96\,f_{111}s_{i-1}^1 +\\
&\quad 8\,f_{21}s_i^{(2,1)}\\
f_{111}d_i^{(1,1)} &= 12\,f_{111}d_{i-1}^{(1,1)} + 24\,f_{21}s_{i-1}^{(2,1)}\\
f_{21}d_i^0 &= 4\,f_{111}d_{i-1}^{(1,1)} + 12\,f_{21}d_i^0 + 96\,f_{21}d_{i-1}^0 + 10\,f_{21}d_i^2 + 40\,f_{21}d_{i-1}^2 + 10\,f_{21}d_i^1 +\\
&\quad 40\,f_{21}d_{i-1}^1 + 8\,f_{21}d_i^{(2,1)} + 24\,f_{21}s_i^0 + 16\,f_{21}s_i^2 + 16\,f_{21}s_i^1 + 8\,f_{21}s_i^{(2,1)} +\\
&\quad 96\,f_{21}s_{i-1}^0 + 4\,f_3s_i^{(3,1)}\\
f_{21}d_i^1 &= 2\,f_{21}d_i^1 + 2\,f_{21}d_i^{(2,1)} + 8\,f_{21}s_i^1 + 56\,f_{21}d_{i-1}^1 + 36\,f_{21}d_{i-1}^{(2,1)} + 96\,f_{21}s_{i-1}^1 + 4\,f_3s_i^{(3,1)}\\
f_{21}d_i^2 &= 8\,f_{111}d_{i-1}^{(1,1)} + 2\,f_{21}d_i^2 + 56\,f_{21}d_{i-1}^2 + 2\,f_{21}d_i^{(2,1)} + 36\,f_{21}d_{i-1}^{(2,1)} +\\
&\quad 8\,f_{21}s_i^2 + 4\,f_{21}s_i^{(2,1)} + 96\,f_{21}s_{i-1}^2 + 72\,f_{21}s_{i-1}^{(2,1)} + 4\,f_3s_i^{(3,1)}\\
f_{21}d_i^{(2,1)} &= 12\,f_{21}d_{i-1}^{(2,1)} + 24\,f_3s_{i-1}^{(3,1)}\\
f_3d_i^0 &= 4\,f_{21}d_{i-1}^{(2,1)} + 12\,f_3d_i^0 + 96\,f_3d_{i-1}^0 + 10\,f_3d_i^3 + 40\,f_3d_{i-1}^3 + 24\,f_3s_i^0 +\\
&\quad 16\,f_3s_i^3 + 8\,f_3s_i^{(3,1)} + 96\,f_3s_{i-1}^0\\
f_3d_i^3 &= 8\,f_{21}d_{i-1}^{(2,1)} + 2\,f_3d_i^3 + 56\,f_3d_{i-1}^3 + 8\,f_3s_i^3 + 4\,f_3s_i^{(3,1)} + 96\,f_3s_{i-1}^3 +\\
&\quad 72\,f_3s_{i-1}^{(3,1)}\\
f_{111}s_i^0 &= 36\,f_{111}d_{i-1}^0 + 24\,f_{111}d_{i-2}^0 + 48\,f_{111}d_{i-1}^1 + 12\,f_{111}d_{i-1}^{(1,1)} + 72\,f_{111}s_{i-1}^0 + 24\,f_{111}s_{i-1}^1\\
f_{111}s_i^1 &= 48\,f_{111}d_{i-2}^1 + 12\,f_{111}d_{i-1}^1 + 24\,f_{111}d_{i-1}^{(1,1)} + 48\,f_{111}s_{i-1}^1 + 24\,f_{21}s_{i-1}^{(2,1)}
\end{aligned}
$$

$$
\begin{aligned}
f_{21}s_i^0 &= 36\, f_{21}d_{i-1}^0 + 48\, f_{21}d_{i-2}^0 + 24\, f_{21}d_{i-1}^2 + 24\, f_{21}d_{i-1}^1 + 12\, f_{21}d_{i-1}^{(2,1)} + 72\, f_{21}s_{i-1}^0 + \\
&\quad 24\, f_{21}s_{i-1}^2 + 24\, f_{21}s_{i-1}^1 + 12\, f_{21}s_{i-1}^{(2,1)} \\
f_{21}s_i^1 &= 48\, f_{21}d_{i-2}^1 + 12\, f_{21}d_{i-1}^1 + 12\, f_{21}d_{i-1}^{(2,1)} + 48\, f_{21}s_{i-1}^1 + 12\, f_3s_{i-1}^{(3,1)} \\
f_{21}s_i^2 &= 24\, f_{111}d_{i-2}^{(1,1)} + 48\, f_{21}d_{i-2}^2 + 12\, f_{21}d_{i-1}^2 + 12\, f_{21}d_{i-1}^{(2,1)} + 48\, f_{21}s_{i-1}^2 + 24\, f_{21}s_{i-1}^{(2,1)} + 12\, f_3s_{i-1}^{(3,1)} \\
f_{21}s_i^{(2,1)} &= 24\, f_{111}d_{i-2}^{(1,1)} + 12\, f_{21}s_{i-1}^{(2,1)} \\
f_3s_i^0 &= 36\, f_3d_{i-1}^0 + 48\, f_3d_{i-2}^0 + 24\, f_3d_{i-1}^3 + 72\, f_3s_{i-1}^0 + 24\, f_3s_{i-1}^3 + 12\, f_3s_{i-1}^{(3,1)} \\
f_3s_i^3 &= 24\, f_{21}d_{i-2}^{(2,1)} + 48\, f_3d_{i-2}^3 + 12\, f_3d_{i-1}^3 + 48\, f_3s_{i-1}^3 + 24\, f_3s_{i-1}^{(3,1)} \\
f_3s_i^{(3,1)} &= 24\, f_{21}d_{i-2}^{(2,1)} + 12\, f_3s_{i-1}^{(3,1)}
\end{aligned}
$$

Table 4.2 lists the evaluation of these sub-partials for X_2, X_3.

Table 4.2: Partitioned Genus Distribution of X_1, X_2 and X_3.

	X_1		X_2				X_3				
i	0	1	0	1	2	3	0	1	2	3	4
$f_{111}d_i^0$			20	104			280	6464	26272		
$f_{111}d_i^1$	2		4	160			8	736	14336	33408	
$f_{111}d_i^{(1,1)}$					144					3456	
$f_{21}d_i^0$				96				1632	28992	49536	
$f_{21}d_i^1$				12	216			24	1824	22176	
$f_{21}d_i^2$				36	648			72	5472	66528	
$f_{21}d_i^{(2,1)}$		6			72					864	3456
b $f_3d_i^0$				20	104			280	6464	26272	
$f_3d_i^3$		2		4	160			8	736	14336	33408
$f_{111}s_i^0$				48				816	12576	12480	
$f_{111}s_i^1$				24	240			48	3264	24384	
$f_{21}s_i^0$					144				4608	44352	
$f_{21}s_i^1$					72				144	7488	12096
$f_{21}s_i^2$					216				432	22464	36288
$f_{21}s_i^{(2,1)}$		6			72					864	3456
$f_3s_i^0$					48				816	12576	12480
$f_3s_i^3$					24	240			48	3264	24384
$f_3s_i^{(3,1)}$						144					3456

Genus Distributions of X'_m and $X_m + e$

As we mentioned towards the end of Chapter 3, that the Formulas (3.7) and (3.14) are valid for any double-rooted graph (G, u, v) where $deg(u) = 3$ and $deg(v) = 2$. Since this condition is met in all of the terms of the open chain X_m, evaluating these formulas by substituting the values given in Table 4.2, we obtain the genus distribution of both the sequence of closed chains of the complete graph (K_4, u, v) and the sequence of graphs obtained through the addition of an edge between the two roots of the open chain (X_m, s_m, t_m). Tables 4.3 and 4.4 list these evaluations.

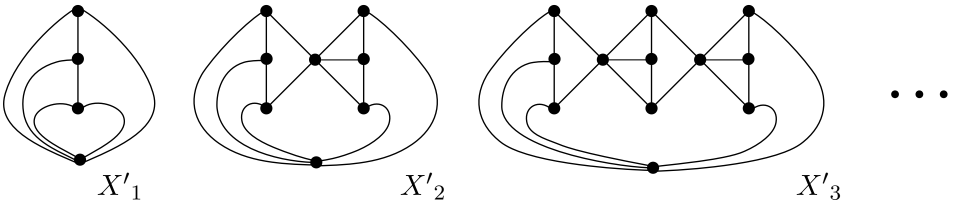


Figure 4.2: The sequence X'_m.

Table 4.3: Genus distribution of X'_m.

k	$g_k(X'_1)$	$g_k(X'_2)$	$g_k(X'_3)$
0	8	4	8
1	120	628	2968
2	64	12776	130176
3		23456	1686368
4			4373632
5			884736

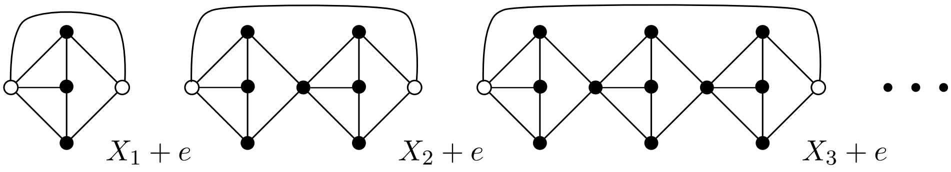


Figure 4.3: The sequence $X_m + e$.

Table 4.4: Genus distribution of $X_m + e$.

k	$g_k(X_1 + e)$	$g_k(X_2 + e)$	$g_k(X_3 + e)$
0	2	4	8
1	58	444	2744
2	36	7136	97600
3		10848	1015552
4			2109696
5			313344

Chapter 5

Genus Distribution of $P_3\square P_n$

In this chapter, we derive a recursion for the genus distribution of the graph family $P_3\square P_n$ with the aid of a modified form of *double-root partials*, and also of a new kind of *production*, which corresponds to a surgical operation more complicated than the vertex- or edge-amalgamation operations used in our earlier work.

5.1 Introduction

A common feature of the graph families whose genus distributions can be computed using the methods that have been developed in earlier chapters (and in our other recent collaborative and individual work), is that they grow linearly and that the iteration step used to obtain the graph family is either a vertex-amalgamation or an edge-amalgamation. An important step beyond amalgamation occurs in the computation of genus distribution of 3-regular Halin graphs [Gross, 2011a]. In this chapter, I present a method to compute the genus distribution of the linear graph family $P_3\square P_n$ (see Figure 5.1), for which the iteration step is once again more complicated than a vertex- or an edge-amalgamation.

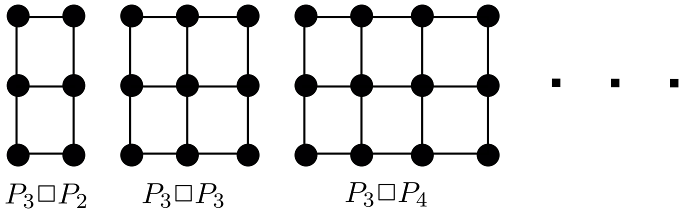


Figure 5.1: $P_3\square P_n$

After computing the genus distributions for various *ladder-type* graph families, a next natural step is computing the genus distributions of *grid-type* graph families. This is the first result in this direction, and it further illustrates the power of the methods that we have been developing in the recent years.

Proposition 34. *For a double-rooted graph (G, u, v) such that deg$(u) = 3$ and deg$(u) = 2$, the possible sub-partials are* $f_{111}d_i^0$, $f_{111}d_i^1$, $f_{111}d_i^{(1,1)}$, $f_{21}d_i^0$, $f_{21}d_i^1$, $f_{21}d_i^2$, $f_{21}d_i^{(2,1)}$, $f_3d_i^0$, $f_3d_i^3$, $f_{111}s_i^0$, $f_{111}s_i^1$, $f_{21}s_i^0$, $f_{21}s_i^1$, $f_{21}s_i^2$, $f_{21}s_i^{(2,1)}$, $f_3s_i^0$, $f_3s_i^3$, $f_3s_i^{(3,1)}$. ◇

The distribution of all of these six sub-partials over all surfaces is called the ***partitioned genus distribution*** of the double-rooted graph (G, u, v).

Let (X_n, u, v) be the graph family depicted by Figure 5.2. Since the root vertex u is 3-valent and the root vertex v is 2-valent, it follows that in any embedding of X_n, root u occurs a total of three times in the fb-walks and root v a total of two times.

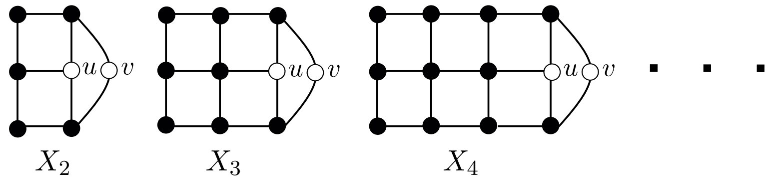


Figure 5.2: The graph sequence X_n

Proposition 35. *In any embedding of (X_n, u, v), at least one of the fb-walks incident on root u is also incident on root v*

Proof Let $c = vx$, $d = ux$ and $e = zx$ be the three edges incident on the vertex x, as shown in Figure 5.3. Thus, the six oriented edges incident on vertex x are $c^+, c^-, d^+, d^-, e^+, e^-$.

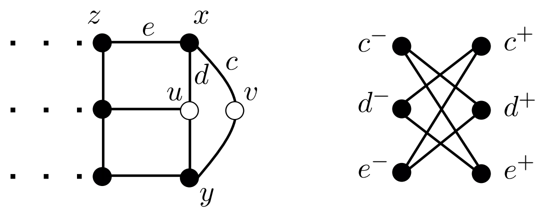


Figure 5.3: The graph X_n (left), and the oriented edges at vertex x (right)

In the bipartite graph at the right, each of the oriented edges leading into vertex x is joined to the two oriented edges that can follow it in an fb-walk of an embedding. Thus, each embedding of X_n induces one of two possible perfect matchings. One of these perfect matchings contains $c^- d^+$, which goes from vertex v to u, and the other matching contains $d^- c^+$, which goes from vertex u to v. Thus, whichever matching is induced, the corresponding embedding has an fb-walk that contains both u and v. $\diamondsuit$

Proposition 36. *An embedding of (X_n, u, v) cannot be any of the following types: (i) $f_{p_1 \cdots p_r} d_i^0$, (ii) $f_{p_1 \cdots p_r} s_i^0$, (iii) $f_{p_1 \cdots p_r} s^{p_k}$*

Proof Proposition 35 rules out types (i) and (ii).

To rule out type (iii), suppose that there exists such an embedding. Then there exists an fb-walk with a u-strand that contains both occurrences of root v. Using the vertex-labels shown in Figure 5.3, it would look like:

$$u \cdots yvx \cdots xvy \cdots u.$$

Here, since the two occurrences of vertex x cannot be consecutive in this u-strand, there must be some intermediate vertices between the two occurrences of x. Root u cannot be one of these intermediate vertices, as otherwise this u-strand would break-up into two u-strands, contradicting the assumption. Root v cannot be one of these intermediate vertices, as v can occur only twice in the fb-walks of an embedding. Thus, vertex z must be one of these intermediate vertices (since there are only three neighbors of vertex x). It also follows that z must be the vertex that immediately follows the first occurrence of x, and also the vertex that immediately precedes the second occurrence of x. Thus, the walk looks like:

$$u \cdots yvxz \cdots zxvy \cdots u$$

where the vertices u and v do not appear as intermediate vertices between the two occurrences of z. But this is impossible because then whatever fb-walk traverses the edge ux would be forced to have the sequence uxu. $\diamondsuit$

Corollary 37. *The only possible non-zero sub-partials for the graph family (X_n, u, v) are in this list:*

$$f_{111} d_i^1, f_{111} d_i^{(1,1)}, f_{21} d_i^1, f_{21} d_i^2, f_{21} d_i^{(2,1)}, f_3 d_i^3, f_{21} s_i^{(2,1)}, f_3 s_i^{(3,1)}$$

Proof List all possible sub-partials for (X_n, u, v), and then eliminate the sub-partials excluded by Proposition 36. $\diamondsuit$

Grid-growth Operation

The details of the ***grid-growth operation*** are illustrated by Figure 5.4.

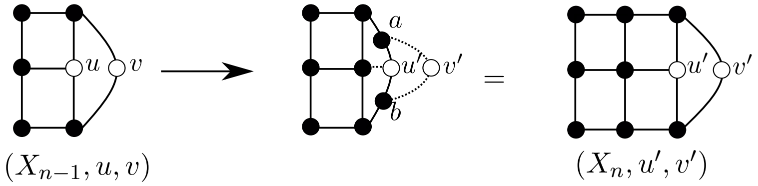


Figure 5.4: The grid-growth operation

The operation consists of the following steps:

- Subdivide the two edges incident on root v, and call the two new vertices a and b, as shown; there is only one way to do this in the given embedding of X_{n-1}.
- Add an edge joining a and b, sub-divide it, and call the new vertex v'; there are four ways to insert the new edge into the embedding, since vertices a and b are both 2-valent.
- Add an edge joining roots u and v; there are 6 ways of adding an edge between roots u and v since $deg(u) = 3$ and $deg(v) = 2$.

Thus, there are 24 ways of performing this operation on an embedding of X_{n-1}.

Productions

Our key idea is to represent the partitioned genus distribution of X_n in terms of the partitioned genus distribution of X_{n-1}. To do so, we define an operation that is applied to each of the types of embeddings of X_{n-1} in order to obtain embeddings of X_n. We once again represent this operation as a ***production***:

$$p_i(X_{n-1}, u, v) \longrightarrow \sum_{\substack{j \text{ ranges over all} \\ \text{sub-partials resulting} \\ \text{from sub-partial } p_i}} c_j q_j(X_n, u', v')$$

where p_i is a sub-partial for X_{n-1} and j is a function that determines the genus of the resulting sub-partial, and where c_j is the corresponding coefficient. Note that these coefficients sum to 24, as argued earlier. The left-hand-side of the production is referred to as the ***production-head***, and the right-hand-side is the ***production-body***.

Our strategy is first to compute the partitioned genus distribution of the double-rooted graph (X_n, u, v) in the next section, and then to use the productions for the operation of edge-addition to a double-rooted graph (as derived in [Khan *et al.*, 2011]) to compute the genus distribution of $P_3 \Box P_n$ in Section 5.3.

5.2 Genus Distribution of (X_n, u, v)

To calculate the distribution of the embeddings of X_n from the distribution of the embeddings of X_{n-1}, we derive the productions listed in Theorem 38.

Theorem 38. *The following are valid productions. (We omit the triple (X_{n-1}, u, v) from the production-head and (X_n, u', v') from the production-body in order to conserve space.)*

$$\begin{aligned}
f_{111}d_i^1 &\longrightarrow f_{111}d_i^1 + 4\,f_{21}d_{i+1}^1 + 4\,f_{21}d_{i+1}^2 + 3\,f_{21}d_{i+1}^{(2,1)} + f_3 d_{i+1}^3 \\
&\quad + 3\,f_{21}s_{i+1}^{(2,1)} + 8\,f_3 s_{i+2}^{(3,1)} \\
f_{111}d_i^{(1,1)} &\longrightarrow 2\,f_{111}d_i^1 + 2\,f_{21}d_{i+1}^1 + 2\,f_{21}d_{i+1}^2 + 6\,f_{21}d_{i+1}^{(2,1)} + 2\,f_3 d_{i+1}^3 \\
&\quad + 6\,f_{21}s_{i+1}^{(2,1)} + 4\,f_3 s_{i+2}^{(3,1)} \\
f_{21}d_i^1 &\longrightarrow f_{111}d_i^1 + 4\,f_{21}d_{i+1}^1 + 4\,f_{21}d_{i+1}^2 + 3\,f_{21}d_{i+1}^{(2,1)} + f_3 d_{i+1}^3 \\
&\quad + 3\,f_{21}s_{i+1}^{(2,1)} + 8\,f_3 s_{i+2}^{(3,1)} \\
f_{21}d_i^2 &\longrightarrow 2\,f_{111}d_i^1 + 2\,f_{21}d_{i+1}^1 + 2\,f_{21}d_{i+1}^2 + 6\,f_{21}d_{i+1}^{(2,1)} + 2\,f_3 d_{i+1}^3 \\
&\quad + 6\,f_{21}s_{i+1}^{(2,1)} + 4\,f_3 s_{i+2}^{(3,1)} \\
f_{21}d_i^{(2,1)} &\longrightarrow 3\,f_{111}d_i^1 + 9\,f_{21}d_{i+1}^{(2,1)} + 3\,f_3 d_{i+1}^3 + 9\,f_{21}s_{i+1}^{(2,1)} \\
f_3 d_i^3 &\longrightarrow 3\,f_{111}d_i^1 + 9\,f_{21}d_{i+1}^{(2,1)} + 3\,f_3 d_{i+1}^3 + 9\,f_{21}s_{i+1}^{(2,1)} \\
f_{21}s_i^{(2,1)} &\longrightarrow 8\,f_{111}d_i^{(1,1)} + 8\,f_{21}d_{i+1}^{(2,1)} + 8\,f_3 s_{i+1}^{(3,1)} \\
f_3 s_i^{(3,1)} &\longrightarrow 12\,f_{111}d_i^{(1,1)} + 12\,f_3 s_{i+1}^{(3,1)}
\end{aligned}$$

Proof We can represent an embedding of the type $f_{111}d_i^1$ using a drawing of the type shown in Figure 5.5.

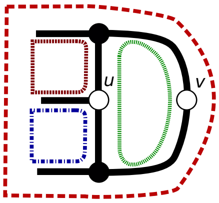


Figure 5.5: $f_{111}d_i^1$

For each of the four ways of inserting an edge between the vertices used to subdivide the two edges incident on root v in X_{n-1}, we show in Figures 5.6–5.9 the six ways in which an edge between the roots u and v can be added, along-with its effect on the fb-walks. This information is then sufficient to derive the production for $f_{111}d_i^1$. In the captions of each of the Figures 5.6–5.9, we give the types of each of the six embeddings in left-to-right, top-to-bottom order.

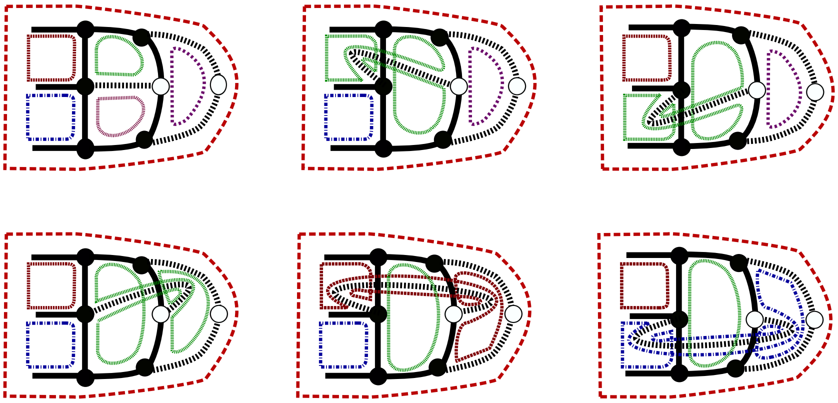


Figure 5.6: $f_{111}d_i^1, f_{21}d_{i+1}^1, f_{21}d_{i+1}^1, f_3d_{i+1}^3, f_{21}d_{i+1}^2, f_{21}d_{i+1}^2$

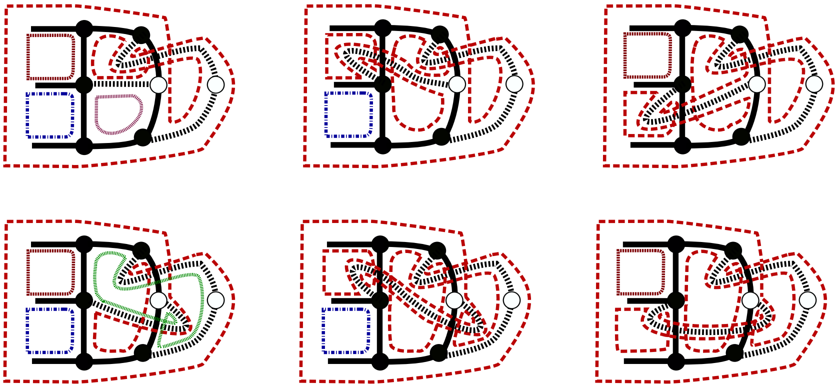


Figure 5.7: $f_{21}s^{(2,1)}_{i+1}, f_3 s^{(3,1)}_{i+2}, f_3 s^{(3,1)}_{i+2}, f_{21}d^{(2,1)}_{i+1}, f_3 s^{(3,1)}_{i+2}, f_3 s^{(3,1)}_{i+2}$

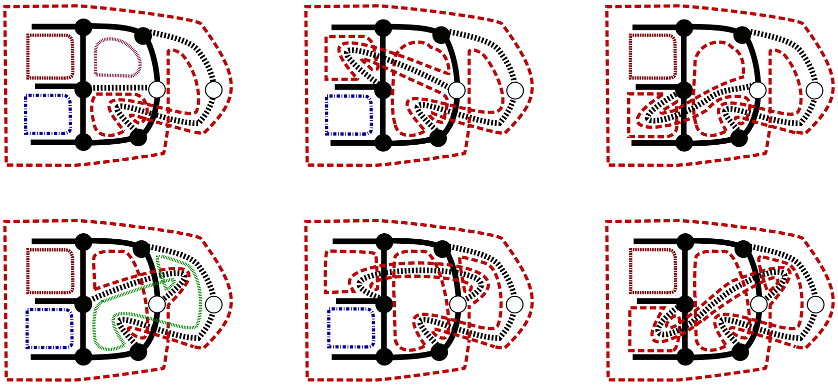


Figure 5.8: $f_{21}s^{(2,1)}_{i+1}, f_3 s^{(3,1)}_{i+2}, f_3 s^{(3,1)}_{i+2}, f_{21}d^{(2,1)}_{i+1}, f_3 s^{(3,1)}_{i+2}, f_3 s^{(3,1)}_{i+2}$

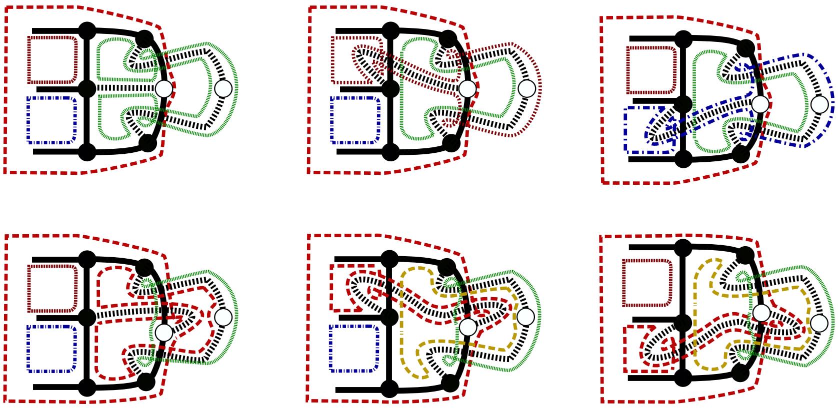


Figure 5.9: $f_{21}s^{(2,1)}_{i+1}, f_{21}d^2_{i+1}, f_{21}d^2_{i+1}, f_{21}d^{(2,1)}_{i+1}, f_{21}d^1_{i+1}, f_{21}d^1_{i+1}$

Collecting the terms given in these captions, we get the production for the sub-partial $f_{111}d^1_i$. The remaining productions are derived similarly, by using similar models for the embedding-types, as given in Figures 5.10–5.16. ♢

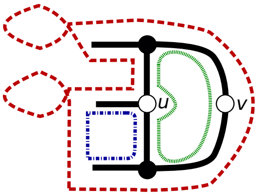


Figure 5.10: $f_{111}d_i^{(1,1)}$

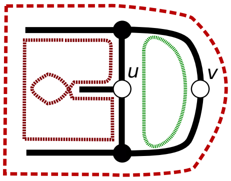


Figure 5.11: $f_{21}d_i^1$

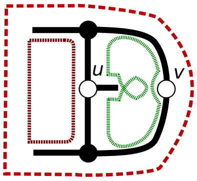


Figure 5.12: $f_{21}d_i^2$

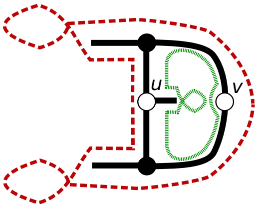


Figure 5.13: $f_{21}d_i^{(2,1)}$

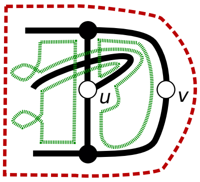


Figure 5.14: $f_3d_i^3$

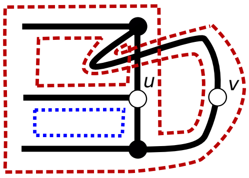


Figure 5.15: $f_{21}s_i^{(2,1)}$

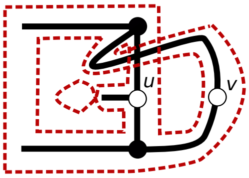


Figure 5.16: $f_3s_i^{(3,1)}$

Theorem 39. *After transposing the productions given in Theorem 38, i.e. by moving terms on the right-hand-side to the left-hand-side, we obtain the following recurrences. (We here omit the triple* (X_n, u, v) *from the term on the left-hand-side of the equation and the triple* (X_{n-1}, u, v) *from the terms on the right-hand-side.)*

$$\begin{aligned}
f_{111}d_i^1 &= f_{111}d_i^1 + 2\,f_{111}d_i^{(1,1)} + f_{21}d_i^1 + 2\,f_{21}d_i^2 + 3\,f_{21}d_i^{(2,1)} + 3\,f_3d_i^3 \\
f_{111}d_i^{(1,1)} &= 8\,f_{21}s_i^{(2,1)} + 12\,f_3s_i^{(3,1)} \\
f_{21}d_i^1 &= 4\,f_{111}d_{i-1}^1 + 2\,f_{111}d_{i-1}^{(1,1)} + 4\,f_{21}d_{i-1}^1 + 2\,f_{21}d_{i-1}^2 \\
f_{21}d_i^2 &= 4\,f_{111}d_{i-1}^1 + 2\,f_{111}d_{i-1}^{(1,1)} + 4\,f_{21}d_{i-1}^1 + 2\,f_{21}d_{i-1}^2 \\
f_{21}d_i^{(2,1)} &= 3\,f_{111}d_{i-1}^1 + 6\,f_{111}d_{i-1}^{(1,1)} + 3\,f_{21}d_{i-1}^1 + 6\,f_{21}d_{i-1}^2 + 9\,f_{21}d_{i-1}^{(2,1)}
\end{aligned}$$

$$
\begin{aligned}
& + 9\, f_3 d^3_{i-1} \;+\; 8\, f_{21} s^{(2,1)}_{i-1} \\
f_3 d^3_i \;&=\; f_{111} d^1_{i-1} \;+\; 2\, f_{111} d^{(1,1)}_{i-1} \;+\; f_{21} d^1_{i-1} \;+\; 2\, f_{21} d^2_{i-1} \;+\; 3\, f_{21} d^{(2,1)}_{i-1} \\
& + 3\, f_3 d^3_{i-1} \\
f_{21} s^{(2,1)}_i \;&=\; 3\, f_{111} d^1_{i-1} \;+\; 6\, f_{111} d^{(1,1)}_{i-1} \;+\; 3\, f_{21} d^1_{i-1} \;+\; 6\, f_{21} d^2_{i-1} \;+\; 9\, f_{21} d^{(2,1)}_{i-1} \\
& + 9\, f_3 d^3_{i-1} \\
f_3 s^{(3,1)}_i \;&=\; 8\, f_{111} d^1_{i-2} \;+\; 4\, f_{111} d^{(1,1)}_{i-2} \;+\; 8\, f_{21} d^1_{i-2} \;+\; 4\, f_{21} d^2_{i-2} \;+\; 8\, f_{21} s^{(2,1)}_{i-1} \\
& + 12\, f_3 s^{(3,1)}_{i-1} \qquad \diamondsuit
\end{aligned}
$$

Computations

By face-tracing, we obtain the double-root partials for (X_2, u, v) given in Table 5.1.

Table 5.1: Nonzero partials of (X_2, u, v).

k	$f_{111}d^1_k$	$f_{111}d^{(1,1)}_k$	$f_{21}d^1_k$	$f_{21}d^2_k$	$f_{21}d^{(2,1)}_k$	$f_3d^3_k$	$f_{21}s^{(2,1)}_k$	$f_3s^{(3,1)}_k$	g_k
0	2								2
1					6	2	6		14

Plugging these values into the recurrences of Theorem 39, yields the values for the sub-partials of (X_3, u, v) given in Table 5.2.

Table 5.2: Nonzero partials of (X_3, u, v).

k	$f_{111}d^1_k$	$f_{111}d^{(1,1)}_k$	$f_{21}d^1_k$	$f_{21}d^2_k$	$f_{21}d^{(2,1)}_k$	$f_3d^3_k$	$f_{21}s^{(2,1)}_k$	$f_3s^{(3,1)}_k$	g_k
0	2								2
1	24	48	8	8	6	2	6		102
2					120	24	72	64	280

By substitution of these values into the recurrences, we next obtain the values for the sub-partials of (X_4, u, v) given in Table 5.3.

By a similar substitution we obtain the following values for the sub-partials of (X_5, u, v):

Table 5.3: Nonzero partials of (X_4, u, v).

k	$f_{111}d_k^1$	$f_{111}d_k^{(1,1)}$	$f_{21}d_k^1$	$f_{21}d_k^2$	$f_{21}d_k^{(2,1)}$	$f_3d_k^3$	$f_{21}s_k^{(2,1)}$	$f_3s_k^{(3,1)}$	g_k
0	2								2
1	168	48	8	8	6	2	6		246
2	432	1344	240	240	552	168	504	64	3544
3					1872	432	1296	1824	5424

Table 5.4: Nonzero partials of (X_5, u, v).

k	$f_{111}d_k^1$	$f_{111}d_k^{(1,1)}$	$f_{21}d_k^1$	$f_{21}d_k^2$	$f_{21}d_k^{(2,1)}$	$f_3d_k^3$	$f_{21}s_k^{(2,1)}$	$f_3s_k^{(3,1)}$	g_k
0	2								2
1	312	48	8	8	6	2	6		390
2	6000	4800	816	816	984	312	936	64	14728
3	6912	32256	5856	5856	22032	6000	18000	6432	103344
4					31104	6912	20736	43968	102724

5.3 Genus Distribution of $P_3\Box P_n$

In this section, we use the productions for *edge-addition* to a double-rooted graphs that were derived in Chapter 3, to obtain the genus distribution of $P_3\Box P_n$. The relevant productions for (X_m, u, v) are:

$$
\begin{aligned}
f_{p_1p_2\cdots p_r}d_i^{p_k}(G,u,v) &\longrightarrow (2n-p_k)\,g_{i+1}(G') \,+\, p_k\,g_i(G') \\
f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}(G,u,v) &\longrightarrow (2n-p_l-p_m)\,g_{i+1}(G') \,+\, (p_l+p_m)\,g_i(G') \\
f_{p_1p_2\cdots p_r}s_i^{(p_k,c)}(G,u,v) &\longrightarrow (2n-2p_k)\,g_{i+1}(G') \,+\, 2p_k\,g_i(G')
\end{aligned}
$$

Theorem 40. *The following productions hold:*

$$\begin{aligned}
f_{111}d_i^1(X_{n-1},u,v) &\longrightarrow 5\,g_{i+1}(P_3\Box P_n) + g_i(P_3\Box P_n)\\
f_{111}d_i^{(1,1)}(X_{n-1},u,v) &\longrightarrow 4\,g_{i+1}(P_3\Box P_n) + 2\,g_i(P_3\Box P_n)\\
f_{21}d_i^1(X_{n-1},u,v) &\longrightarrow 5\,g_{i+1}(P_3\Box P_n) + g_i(P_3\Box P_n)\\
f_{21}d_i^2(X_{n-1},u,v) &\longrightarrow 4\,g_{i+1}(P_3\Box P_n) + 2\,g_i(P_3\Box P_n)\\
f_{21}d_i^{(2,1)}(X_{n-1},u,v) &\longrightarrow 3\,g_{i+1}(P_3\Box P_n) + 3\,g_i(P_3\Box P_n)\\
f_3d_i^3(X_{n-1},u,v) &\longrightarrow 3\,g_{i+1}(P_3\Box P_n) + 3\,g_i(P_3\Box P_n)\\
f_{21}s_i^{(2,1)}(X_{n-1},u,v) &\longrightarrow 2\,g_{i+1}(P_3\Box P_n) + 4\,g_i(P_3\Box P_n)\\
f_3s_i^{(3,1)}(X_{n-1},u,v) &\longrightarrow 6\,g_i(P_3\Box P_n)
\end{aligned}$$

Proof Rewrite the relevant productions given above for $deg(u) = 3$ (using the fact that the number 3 has these three partitions: $111, 21$ and 3). ♢

Theorem 41. *The genus distribution of $P_3\Box P_n$ is as follows:*

$$\begin{aligned}
g_i(P_3\Box P_n) &= 5\,f_{111}d_{i-1}^1(X_{n-1},u,v) + f_{111}d_i^1(X_{n-1},u,v)\\
&+ 4\,f_{111}d_{i-1}^{(1,1)}(X_{n-1},u,v) + 2\,f_{111}d_i^{(1,1)}(X_{n-1},u,v)\\
&+ 5\,f_{21}d_{i-1}^1(X_{n-1},u,v) + f_{21}d_i^1(X_{n-1},u,v)\\
&+ 4\,f_{21}d_{i-1}^2(X_{n-1},u,v) + 2\,f_{21}d_i^2(X_{n-1},u,v)\\
&+ 3\,f_{21}d_{i-1}^{(2,1)}(X_{n-1},u,v) + 3\,f_{21}d_i^{(2,1)}(X_{n-1},u,v)\\
&+ 3\,f_3d_{i-1}^3(X_{n-1},u,v) + 3\,f_3d_i^3(X_{n-1},u,v)\\
&+ 2\,f_{21}s_{i-1}^{(2,1)}(X_{n-1},u,v) + 4\,f_{21}s_i^{(2,1)}(X_{n-1},u,v)\\
&+ 6\,f_3s_i^{(3,1)}(X_{n-1},u,v)
\end{aligned}$$

Proof Simple transposition of the productions given in Theorem 40. ♢

Using Theorem 41, we compute the values in Table 5.5.

Table 5.5: Genus distribution of $P_3 \square P_n$.

k	$P_3 \square P_3$	$P_3 \square P_4$	$P_3 \square P_5$	$P_3 \square P_6$
0	2	2	2	2
1	58	202	346	490
2	36	1524	9540	27924
3		576	35904	345984
4			9504	797184
5				155520

Part II

Conclusions

Chapter 6

Conclusions

6.1 Conclusions

My research contributes to the study of the computations of genus distributions of graphs in general, as well as in specific ways. The general part of my research is summarized in the following points.

- If one is given the single-root partial genus distribution of a single-rooted graph (G, t), and single-root partitioned genus distribution of a single-rooted graph (H, u), when $deg(t) = 2$ and $deg(u) \geq 2$, then one can compute the genus distribution of the graph $G * H$ obtained through their vertex-amalgamation. One simply adds up all terms on the right-hand-sides of Recurrences (2.8) and (2.9) after converting double-root partials to single-root partials (by ignoring the second root).

- If one is given the double-root partitioned genus distribution of a double-rooted graph (G, u, v) where $deg(v) = 2$ and $deg(u)$ can be arbitrarily large, then one can use results of Chapter 2 to compute the recurrences for the single-root partitial genus distribution of the sequence of open chains of the double-rooted graph.

- If one is given the double-root partitioned genus distribution of two double-rooted graphs (G, u, v) and (H, a, b), when $deg(v) = deg(b) = 2$, $deg(u) \geq 2$ and $deg(a) \geq 2$, then one can use results of Chapter 4 to compute the double-root partitioned genus distribution of the graph obtained through the vertex-amalgamation of the two graphs, amalgamated at root vertices u and a.

- If one is given the double-root partitioned genus distribution of a double-rooted graph (G, u, v), when $deg(v) = 2$ and $deg(u) \geq 2$, then one can use results of Chapter 4 to compute the recurrences for the double-root partitioned genus distribution of the sequence of open chains of the double-rooted graph.

- If one is given the double-root partitioned genus distribution of a double-rooted graph (G, u, v), when $deg(v) = 2$ and $deg(u) \geq 2$, then one can use results of Chapters 3 and 4 to compute the genus distribution of the sequence of closed chains obtained through amalgamation of the two root vertices in the open chains, and also to compute the genus distribution of the sequence of graph chains obtained through the addition of an edge between the two root vertices in the open chains.

It is interesting to note that extending the methods developed in Chapters 2–4 to the cases where neither root vertex has valence 2 may present complications. As illustrated in Example 42, the genus-increment can sometimes be negative (see Research Problem 1 below).

Example 42. Figure 6.1 shows two toroidal embeddings of the single-rooted dipole (D_3, u).

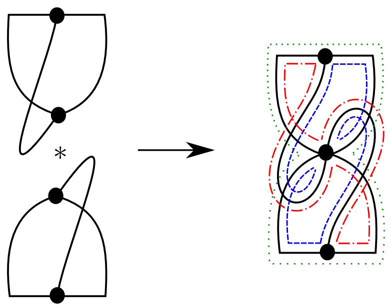


Figure 6.1: A consistent embedding of $D_3 * D_3$ with negative genus-increment.

One of the 40 embeddings of the amalgamated graph $X = (D_3, u) * (D_3, u)$ that is consistent with those two embeddings is also shown in the figure. Note that this is also a toroidal embedding, since

$$V - E + F = 3 - 6 + 3 = 0.$$

Thus, the genus-increment in this case is -1.

Research Problem 1. Develop methods for computing the genus distributions when both amalgamated vertices may have arbitrarily large degrees. For instance, one might augment the present approach with other surgical operations, such as splitting a vertex.

Research Problem 2. Develop methods to solve simultaneous recurrences like (2.12), (2.13) and (2.14), (2.15).

Research Problem 3. The genus distributions that we have computed using our methods are all unimodal. This appears to support the conjecture that all graphs have unimodal genus distributions. A natural question to ask is whether the vertex-amalgamation of two graphs with unimodal distributions has a unimodal genus distribution? Also, whether the operations of self-vertex-amalgamation and edge-addition on a graph with unimodal genus distribution produced graphs with unimodal distributions?

Research Problem 4. Results of [Poshni *et al.*, 2010] have been successfully used to compute the genus distributions of cubic outerplanar graphs in [Gross, 2011b]. Can the results of this thesis be similarly used to compute the genus distributions of other non-linear families of graph?

The specific part of my research enables one to compute the genus distribution of the graph family $P_3 \Box P_n$. Theorem 39 is a set of recurrences for calculating the genus distribution of the mesh graphs $P_3 \Box P_n$. This calculation is based on a complex graph surgery operation by which the auxiliary graph X_n is obtained from X_{n-1}, after which an edge is added to construct $P_3 \Box P_n$ by joining the two roots of X_n.

As indicated by Bodlaender [Bodlaender, 1998], the mesh graphs $P_3 \Box P_n$ have treewidth 3, as do the Halin graphs. This is in contrast to the families of graphs whose genus distribution were more easily calculated and were either of treewidth 2, or derivable by simple surgery on a graph of treewidth 2. A recent result by Gross [Gross, 2012] has developed a quadratic-time algorithm for computing the genus distribution of graph families of fixed treewidth and bounded degree, based on newly general forms of partials and productions.

Research Problem 5 There is no consistent and simple notation for the partials of a k-rooted graph. Nonetheless, there may be a straightforward (albeit tedious) extension of the method given here for computing the genus distribution of $P_4 \Box P_n$, or more generally of $P_{k+1} \Box P_n$.

6.2 Future Directions

We have already specified a few of the research problems that are of immediate interest in the previous section, but more broadly speaking there may be other potential uses of the ideas developed in my work.

My work (and the work of my coauthors) is a foundation for a variety of problems. The most important problem in this area is definitely the computation of the genus distributions of the complete graphs K_n, because of the importance of this problem in the history of topological graph theory. Computation of the minimum genus of complete graphs K_n was the main hurdle in the proof of Heawood's conjecture on coloring maps on surfaces of higher genus. The eventual solution by Ringel and Youngs in 1968 had to break down the problem into twelve different cases, each of which was solved separately. The proof consisted of 300 pages, but it was arguably the methods employed in that proof that helped established the discipline of topological graph theory. If our work leads to the computation of genus distribution of complete graphs, it might significantly simplify the proof.

Additionally, since our results are giving us genus distributions of graph families like $P_3 \Box P_n$, cubic and 4-regular outerplanar graphs, cubic Halin graphs, one might hope that in the near future genus distributions of many familiar graph families would be known (graph families like wheels, Peterson graphs, circulant graphs, bipartite graphs, planar graphs, graphs with known minimum genus etc.), as opposed to largely artificially constructed graph families like closed-end ladders and cobblestone paths.

It is known that the shadow graphs of knots are all 4-regular planar graphs. Since we have been computing genus distributions of graph families like 4-regular outerplanar graphs, it might be interesting to investigate if there is any relationship between the knot polynomials and the genus distribution polynomials of the shadow graphs.

The methods we have developed compute a property (genus distributions) of graphs inductively through amalgamations and other similar operations. Are there other properties of graphs that can be similarly computed using productions and/or analogically defined partials?

Part III

Bibliography

Bibliography

[Beineke *et al.*, 2009] L. W. Beineke, R. J. Wilson, J. L. Gross, and T. W. Tucker (editors). *Topics in Topological Graph Theory.* Cambridge University Press, 2009.

[Bodlaender, 1998] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical Comp. Sci.*, 209:1–45, 1998.

[Bonnington *et al.*, 2000] C. P. Bonnington, M. J. Grannell, T. S. Griggs, and J. Širáň. Exponential families of non-isomorphic triangulations of complete graphs. *J. Combin. Theory (B)*, 78:169–184, 2000.

[Chen *et al.*, 1994] J. Chen, J. L. Gross, and R. G. Rieper. Overlap matrices and total imbedding distributions. *Discrete Math.*, 128:73–94, 1994.

[Chen *et al.*, 2006] Y. C. Chen, Y. P. Liu, and T. Wang. The total embedding distributions of cacti and necklaces. *Acta Math. Sinica — English Series*, 22:1583–1590, 2006.

[Chen *et al.*, 2011a] Y. Chen, T. Mansour, and Q. Zou. Embedding distributions and Chebyshev polynomials. *Graph Combinator.*, 2011.

[Chen *et al.*, 2011b] Y. Chen, T. Mansour, and Q. Zou. Embedding distributions of generalized fan graphs. *Canad. Math. Bull.*, 2011.

[Chen *et al.*, 2011c] Y. Chen, T. Mansour, and Q. Zou. On the total embeddings for some types of graphs. not yet published, 2011.

[Chen *et al.*, 2011d] Y. Chen, L. Ou, and Q. Zou. Total embedding distributions of Ringel ladders. *Discrete Math.*, 311:2463–2474, 2011.

[Chen, 2010] Y. Chen. Lower bounds for the average genus of a CF-graph. *Electronic J. Combin.*, 17:#R150, 2010.

[Conder and Dobcsányi, 2001] M. Conder and P. Dobcsányi. Determination of all regular maps of small genus. *J. Combin. Theory (B)*, 81:224–242, 2001.

[Furst *et al.*, 1989] M. L. Furst, J. L. Gross, and R. Statman. Genus distribution for two classes of graphs. *J. Combin. Theory (B)*, 46:22–36, 1989.

[Goddyn *et al.*, 2007] L. Goddyn, R. B. Richter, and J. Širáň. Triangular embeddings of complete graphs from graceful labellings of paths. *J. Combin. Theory (B)*, 97:964–970, 2007.

[Grannell and Griggs, 2008] M. J. Grannell and T. S. Griggs. A lower bound for the number of triangular embeddings of some complete graphs and complete regular tripartite graphs. *J. Combin. Theory (B)*, 98:637–650, 2008.

[Gross and Furst, 1987] J. L. Gross and M. L. Furst. Hierarchy for imbedding-distribution invariants of a graph. *J. Graph Theory*, 11:205–220, 1987.

[Gross and Tucker, 1987] J. L. Gross and T. W. Tucker. *Topological Graph Theory*. Dover, 2001; (original edn. Wiley, 1987), 1987.

[Gross *et al.*, 1989] J. L. Gross, D. P. Robbins, and T. W. Tucker. Genus distributions for bouquets of circles. *J. Combin. Theory (B)*, 47:292–306, 1989.

[Gross *et al.*, 2010] J. L. Gross, I. F. Khan, and M. I. Poshni. Genus distribution of graph amalgamations: Pasting at root-vertices. *Ars Combinatoria*, 94:33–53, 2010.

[Gross, 2009] J. L. Gross. *Distribution of embeddings*, Chapter 3 in *Topics in Topological Graph Theory (ed. Beineke, Wilson, Gross, Tucker)*, pages 45–61. Cambridge University Press, 2009.

[Gross, 2010] J. L. Gross. Genus distribution of graphs under surgery: Adding edges and splitting vertices. *New York J. Math.*, 16:116168, 2010.

[Gross, 2011a] J. L. Gross. Embeddings of cubic Halin graphs: a surface-by-surface inventory. available at: http://www.cs.columbia.edu/ gross/research/Halin(Bled)-opt.pdf, 20 pages, 2011.

[Gross, 2011b] J. L. Gross. Genus distributions of cubic outerplanar graphs. *J. Graph Algorithms Appl.*, 23:295–316, 2011.

[Gross, 2011c] J. L. Gross. Genus distributions of graph amalgamations: Pasting at root-vertices. *Australasian J. Combin.*, 49:19–38, 2011.

[Gross, 2012] J. L. Gross. Embeddings of graphs of fixed treewidth and bounded degree. manuscript, 28 pages, 2012.

[Jackson and Visentin, 1990] D. M. Jackson and T. I. Visentin. A character-theoretic approach to embeddings of rooted maps in an orientable surface of given genus. *Trans. Amer. Math. Soc.*, 322:343–363, 1990.

[Jackson and Visentin, 2001] D. M. Jackson and T. I. Visentin. *An atlas of the smaller maps in orientable and nonorientable surfaces.* Chapman & Hall/CRC Press, 2001.

[Jackson, 1987] D. M. Jackson. Counting cycles in permutations by group characters with an application to a topological problem. *Trans. Amer. Math. Soc.*, 299:785–801, 1987.

[Khan *et al.*, 2010] I. F. Khan, M. I. Poshni, and J. L. Gross. Genus distribution of graph amalgamations: Pasting when one roots has arbitrary degree. *Ars Math. Contemporanea*, 3:121–138, 2010.

[Khan *et al.*, 2011] I. F. Khan, M. I. Poshni, and J. L. Gross. Genus distribution of graphs: Self-pasting and edge-addition when one root has arbitrary degree. In Preparation, 2011.

[Khan *et al.*, 2012] I. F. Khan, M. I. Poshni, and J. L. Gross. Genus distribution of $P_3 \square P_n$. Accepted by Discrete Math., 2012.

[Korzhik and Voss, 2002] V. P. Korzhik and H-J Voss. Exponential families of non-isomorphic non-triangular orientable genus embeddings of complete graphs. *J. Combin. Theory (B)*, 86:86–211, 2002.

[Kwak and Lee, 1993] J. H. Kwak and J. Lee. Genus polynomials of dipoles. *Kyungpook Math. J.*, 33:115–125, 1993.

[Kwak and Lee, 1994] J. H. Kwak and J. Lee. Enumeration of graph embeddings. *Discrete Math.*, 135:129–151, 1994.

[Kwak and Shim, 2002] J. H. Kwak and S. H. Shim. Total embedding distributions for bouquets of circles. *Discrete Math.*, 248:93–108, 2002.

[McGeoch, 1987] L. A. McGeoch. *Algorithms for two graph problems: computing maximum-genus imbedding and the two-server problem.* PhD thesis, Carnegie-Mellon University, 1987.

[Mohar and Thomassen, 2001] B. Mohar and C. Thomassen. *Graphs on Surfaces.* Johns Hopkins Press, 2001.

[Mohar, 1989] B. Mohar. An obstruction to embedding graphs in surfaces. *Discrete Math.*, 78:135–142, 1989.

[Mull *et al.*, 1988] B. G. Mull, R. G. Rieper, and A. T. White. Enumerating 2-cell imbeddings of complete graphs. *Proc. Amer. Math. Soc.*, 103:321–330, 1988.

[Mull, 1999] B. P. Mull. Enumerating the orientable 2-cell imbeddings of complete bipartite graphs. *J. Graph Theory*, 30:77–90, 1999.

[Poshni *et al.*, 2010] M. I. Poshni, I. F. Khan, and J. L. Gross. Genus distribution of graphs under edge-amalgamations. *Ars Mathematica Contemporanea*, 3:69–86, 2010.

[Poshni *et al.*, 2011] M. I. Poshni, I. F. Khan, and J. L. Gross. Genus distribution of 4-regular outerplanar graphs. 2011.

[Poshni *et al.*, 2012] M. I. Poshni, I. F. Khan, and J. L. Gross. Genus distribution of graphs under self-edge-amalgamations. *Ars Mathematica Contemporanea*, 5:127–148, 2012.

[Rieper, 1990] R. G. Rieper. *The enumeration of graph embeddings.* PhD thesis, Western Michigan University, 1990.

[Riskin, 1995] A. Riskin. On the enumeration of polyhedral embeddings of Cartesian products of cycles. *Ars Combin.*, 41:193–198, 1995.

[Stahl, 1990] S. Stahl. Region distributions of graph embeddings and Stirling numbers. *Discrete Math.*, 82:57–78, 1990.

[Stahl, 1991a] S. Stahl. Permutation-partition pairs iii: Embedding distributions of linear families of graphs. *J. Combin. Theory (B)*, 52:191–218, 1991.

[Stahl, 1991b] S. Stahl. Region distributions of some small diameter graphs. *Discrete Math.*, 89:281–299, 1991.

[Stahl, 1995a] S. Stahl. Bounds for the average genus of the vertex amalgamation of graphs. *Discrete Math.*, 142:235–245, 1995.

[Stahl, 1995b] S. Stahl. On the average genus of the random graph. *J. Graph Theory*, 20:1–18, 1995.

[Stahl, 1997] S. Stahl. On the zeroes of some genus polynomials. *Canad. J. Math.*, 49:617–640, 1997.

[Tesar, 2000] E. H. Tesar. Genus distribution of Ringel ladders. *Discrete Math.*, 216:235–252, 2000.

[Thomassen, 1989] C. Thomassen. The graph genus problem is NP-complete. *J. Algorithms*, 10:568–576, 1989.

[Visentin and Wieler, 2007] T. I. Visentin and S. W. Wieler. On the genus distribution of (p, q, n)-dipoles. *Electronic J. of Combin.*, 14:#R12, 2007.

[Wan and Liu, 2006] L. X. Wan and Y. P. Liu. Orientable embedding distributions by genus for certain types of graphs. *Ars Combin.*, 79:97–105, 2006.

[Wan and Liu, 2008] L. X. Wan and Y. P. Liu. Orientable embedding genus distribution for certain types of graphs. *J. Combin. Theory (B)*, 47:19–32, 2008.

[White, 2001] A. T. White. *Graphs of Groups on Surfaces.* North-Holland, 2001.

Part IV

Appendices

Appendix A

Complete Set of Productions for Extended Vertex Amalgamation

Theorem 43. *Let (G, u, v) and (H, a, b) be two double-rooted graphs such that $deg(u) \geq 2, deg(a) \geq 2$ and $deg(v) = deg(b) = 2$, and let (X, u, b) be the graph obtained by amalgamating these two graphs on root vertices v and a. Then for each pair of partitions $p_1 p_2 \cdots p_r$ of $deg(u)$ and $q_1 q_2 \cdots q_r$ of $deg(a)$, the following productions hold (we omit the triple (G, u, v) from the first amalgamand in the production-head, (H, a, b) from the second amalgamand, and (X, u, b) from each of the terms in the production-body) :*

$$f_{p_1p_2\cdots p_r} d_i^0 * f_{q_1q_2\cdots q_t} d_j^0 \longrightarrow \left(\sum_{x=1}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} d_{i+j}^0 + \left(\sum_{x=1}^{t}\sum_{y=x+1}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r} d_{i+j+1}^0 \tag{A.1}$$

$$f_{p_1p_2\cdots p_r} d_i^{p_k} * f_{q_1q_2\cdots q_t} d_j^0 \longrightarrow \left(\sum_{x=1}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} d_{i+j}^0 + \left(\sum_{x=1}^{t}\sum_{y=x+1}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r} d_{i+j+1}^0 \tag{A.2}$$

$$f_{p_1p_2\cdots p_r} d_i^{(p_g,p_h)} * f_{q_1q_2\cdots q_t} d_j^0 \longrightarrow \left(\sum_{x=1}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} d_{i+j}^0$$

$$+ \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2q_x q_y \right) f_{p_1 p_2 \cdots (p_g + p_h) \cdots \hat{p}_g \cdots \hat{p}_h \cdots p_r} d^0_{i+j+1} \quad \text{(A.3)}$$

$$f_{p_1 p_2 \cdots p_r} s^0_i * f_{q_1 q_2 \cdots q_t} d^0_j \longrightarrow (n^2 + n)\, f_{p_1 p_2 \cdots p_r} d^0_{i+j} \quad \text{(A.4)}$$

$$f_{p_1 p_2 \cdots p_r} s^{p_k}_i * f_{q_1 q_2 \cdots q_t} d^0_j \longrightarrow (n^2 + n)\, f_{p_1 p_2 \cdots p_r} d^0_{i+j} \quad \text{(A.5)}$$

$$f_{p_1 p_2 \cdots p_r} s^{(p_k, d)}_i * f_{q_1 q_2 \cdots q_t} d^0_j \longrightarrow \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2q_x q_y \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} d^0_{i+j} \quad \text{(A.6)}$$

$$f_{p_1 p_2 \cdots p_r} d^0_i * f_{q_1 q_2 \cdots q_t} d^{q_s}_j \longrightarrow \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2q_x q_y \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j+1} \quad \text{(A.7)}$$

$$f_{p_1 p_2 \cdots p_r} d^{p_k}_i * f_{q_1 q_2 \cdots q_t} d^{q_s}_j \longrightarrow \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \left(\frac{q_s (q_s + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \left(\frac{q_s (q_s + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} d^{p_k}_{i+j}$$

$$+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} \sum_{\substack{y=x+1 \\ y \neq s}}^{t} 2q_x q_y \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j+1}$$

$$+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} 2q_x q_s \right) f_{p_1 p_2 \cdots p_r} d^{p_k}_{i+j+1} \quad \text{(A.8)}$$

$$f_{p_1 p_2 \cdots p_r} d^{(p_g, p_h)}_i * f_{q_1 q_2 \cdots q_t} d^{q_s}_j \longrightarrow \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \left(\frac{q_s (q_s + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} d^{p_g}_{i+j}$$

$$+ \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r} d_{i+j}^{p_h}$$
$$+ \left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r} d_{i+j+1}^{0}$$
$$+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} 2q_xq_s\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r} d_{i+j+1}^{p_g+p_h} \quad \text{(A.9)}$$

$$f_{p_1p_2\cdots p_r} s_i^0 * f_{q_1q_2\cdots q_t} d_j^{q_s} \longrightarrow (n^2+n)\, f_{p_1p_2\cdots p_r} d_{i+j}^0 \quad \text{(A.10)}$$

$$f_{p_1p_2\cdots p_r} s_i^{p_k} * f_{q_1q_2\cdots q_t} d_j^{q_s} \longrightarrow \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} d_{i+j}^0$$
$$+\ q_s(q_s+1)\, f_{p_1p_2\cdots p_r} d_{i+j}^{p_k}$$
$$+ \left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r} d_{i+j}^{0}$$
$$+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_xq_s\right) f_{p_1p_2\cdots p_r} d_{i+j}^{0}$$
$$+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_xq_s\right) f_{p_1p_2\cdots p_r} d_{i+j}^{p_k} \quad \text{(A.11)}$$

$$f_{p_1p_2\cdots p_r} s_i^{(p_k,d)} * f_{q_1q_2\cdots q_t} d_j^{q_s} \longrightarrow \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} d_{i+j}^0$$
$$+\ q_s(q_s+1)\, f_{p_1p_2\cdots p_r} d_{i+j}^{p_k}$$
$$+ \left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots\hat{p}_k\cdots p_r d(p_k-d)} d_{i+j}^{0}$$
$$+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} q_xq_s\right) f_{p_1p_2\cdots\hat{p}_k\cdots p_r d(p_k-d)} d_{i+j}^{d}$$

$$+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d(p_k - d)} d_{i+j}^{p_k - d} \tag{A.12}$$

$$\begin{aligned}
f_{p_1 p_2 \cdots p_r} d_i^0 * f_{q_1 q_2 \cdots q_t} d_j^{(q_l, q_m)} \longrightarrow & \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} d_{i+j}^0 \\
& + \left(\sum_{\substack{x=1 \\ xnel, x \neq m}}^{t} \sum_{\substack{y=x+1 \\ y \neq l, y \neq m}}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots p_r} d_{i+j+1}^0 \\
& + 2 q_l q_m \, f_{p_1 p_2 \cdots p_r} s_{i+j+1}^0
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
f_{p_1 p_2 \cdots p_r} d_i^{p_k} * f_{q_1 q_2 \cdots q_t} d_j^{(q_l, q_m)} \longrightarrow & \left(\sum_{\substack{x=1 \\ x \neq l, x \neq m}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} d_{i+j}^0 \\
& + \left(\frac{q_l (q_l + 1)}{2} + \frac{q_m (q_m + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} d_{i+j}^0 \\
& + \left(\frac{q_l (q_l + 1)}{2} + \frac{q_m (q_m + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} d_{i+j}^{p_k} \\
& + \left(\sum_{\substack{x=1 \\ x \neq l, x \neq m}}^{t} \sum_{\substack{y=x+1 \\ y \neq l, y \neq m}}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots p_r} d_{i+j+1}^0 \\
& + \left(\sum_{\substack{x=1 \\ x \neq l, x \neq m}}^{t} (2 q_x q_l + 2 q_x q_m) \right) f_{p_1 p_2 \cdots p_r} d_{i+j+1}^{p_k} \\
& + 2 q_l q_m \, f_{p_1 p_2 \cdots p_r} s_{i+j+1}^{p_k}
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
f_{p_1 p_2 \cdots p_r} d_i^{(p_g, p_h)} * f_{q_1 q_2 \cdots q_t} d_j^{(q_l, q_m)} \longrightarrow & \left(\sum_{\substack{x=1 \\ x \neq l, x \neq m}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} d_{i+j}^0 \\
& + \left(\frac{q_l (q_l + 1)}{2} + \frac{q_m (q_m + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} d_{i+j}^{p_g} \\
& + \left(\frac{q_l (q_l + 1)}{2} + \frac{q_m (q_m + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} d_{i+j}^{p_h}
\end{aligned}$$

$$+ \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} \sum_{\substack{y=x+1 \\ y\neq l, y\neq m}}^{t} 2q_x q_y \right) f_{p_1 p_2 \cdots (p_g+p_h) \cdots \hat{p}_g \cdots \hat{p}_h \cdots p_r} d^0_{i+j+1}$$

$$+ \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} (2q_x q_l + 2q_x q_m) \right) f_{p_1 p_2 \cdots (p_g+p_h) \cdots \hat{p}_g \cdots \hat{p}_h \cdots p_r} d^{p_g+p_h}_{i+j+1}$$

$$+ \ 2q_l q_m \, f_{p_1 p_2 \cdots (p_g+p_h) \cdots \hat{p}_g \cdots \hat{p}_h \cdots p_r} s^{(p_g+p_h, min(p_g,p_h))}_{i+j+1} \quad \text{(A.15)}$$

$$f_{p_1 p_2 \cdots p_r} s^0_i * f_{q_1 q_2 \cdots q_t} d^{(q_l,q_m)}_j \longrightarrow (n^2+n) \, f_{p_1 p_2 \cdots p_r} d^0_{i+j} \quad \text{(A.16)}$$

$$f_{p_1 p_2 \cdots p_r} s^{p_k}_i * f_{q_1 q_2 \cdots q_t} d^{(q_l,q_m)}_j \longrightarrow \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} q_x(q_x+1) \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \ (q_l(q_l+1) + q_m(q_m+1)) \, f_{p_1 p_2 \cdots p_r} d^{p_k}_{i+j}$$

$$+ \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} \sum_{\substack{y=x+1 \\ y\neq l, y\neq m}}^{t} 2q_x q_y \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} (q_x q_l + q_x q_m) \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} (q_x q_l + q_x q_m) \right) f_{p_1 p_2 \cdots p_r} d^{p_k}_{i+j}$$

$$+ \ 2q_l q_m \, f_{p_1 p_2 \cdots p_r} d^{p_k}_{i+j} \quad \text{(A.17)}$$

$$f_{p_1 p_2 \cdots p_r} s^{(p_k,d)}_i * f_{q_1 q_2 \cdots q_t} d^{(q_l,q_m)}_j \longrightarrow \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} q_x(q_x+1) \right) f_{p_1 p_2 \cdots p_r} d^0_{i+j}$$

$$+ \ (q_l(q_l+1) + q_m(q_m+1)) \, f_{p_1 p_2 \cdots p_r} d^{p_k}_{i+j}$$

$$+ \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} \sum_{\substack{y=x+1 \\ y\neq l, y\neq m}}^{t} 2q_x q_y \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} d^0_{i+j}$$

$$+ \left(\sum_{\substack{x=1 \\ x\neq l, x\neq m}}^{t} (q_x q_l + q_x q_m) \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} d^d_{i+j}$$

$$+ \left(\sum_{\substack{x=1 \\ x \neq l, x \neq m}}^{t} (q_x q_l + q_x q_m) \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d(p_k - d)} d_{i+j}^{p_k - d}$$

$$+ \; 2 q_l q_m \, f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d(p_k - d)} d_{i+j}^{(d, p_k - d)} \tag{A.18}$$

$$f_{p_1 p_2 \cdots p_r} d_i^0 * f_{q_1 q_2 \cdots q_t} s_j^0 \longrightarrow \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^0 + \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots p_r} s_{i+j+1}^0 \tag{A.19}$$

$$f_{p_1 p_2 \cdots p_r} d_i^{p_k} * f_{q_1 q_2 \cdots q_t} s_j^0 \longrightarrow \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^0 + \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots p_r} s_{i+j+1}^0 \tag{A.20}$$

$$f_{p_1 p_2 \cdots p_r} d_i^{(p_g, p_h)} * f_{q_1 q_2 \cdots q_t} s_j^0 \longrightarrow \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^0 + \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots (p_g + p_h) \cdots \hat{p}_g \cdots \hat{p}_h \cdots p_r} s_{i+j+1}^0 \tag{A.21}$$

$$f_{p_1 p_2 \cdots p_r} s_i^0 * f_{q_1 q_2 \cdots q_t} s_j^0 \longrightarrow (n^2 + n) \, f_{p_1 p_2 \cdots p_r} s_{i+j}^0 \tag{A.22}$$

$$f_{p_1 p_2 \cdots p_r} s_i^{p_k} * f_{q_1 q_2 \cdots q_t} s_j^0 \longrightarrow (n^2 + n) \, f_{p_1 p_2 \cdots p_r} s_{i+j}^0 \tag{A.23}$$

$$f_{p_1 p_2 \cdots p_r} s_i^{(p_k, d)} * f_{q_1 q_2 \cdots q_t} s_j^0 \longrightarrow \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^0 + \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d(p_k - d)} s_{i+j}^0 \tag{A.24}$$

$$f_{p_1 p_2 \cdots p_r} d_i^0 * f_{q_1 q_2 \cdots q_t} s_j^{q_s} \longrightarrow \left(\sum_{x=1}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^0 + \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots p_r} s_{i+j+1}^0 \tag{A.25}$$

$$
\begin{aligned}
f_{p_1p_2\cdots p_r} d_i^{p_k} * f_{q_1q_2\cdots q_t} s_j^{q_s} \longrightarrow &\left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} s_{i+j}^{0} \\
&+ \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r} s_{i+j}^{0} \\
&+ \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r} s_{i+j}^{p_k} \\
&+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} \sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r} s_{i+j+1}^{0} \\
&+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} 2q_xq_s\right) f_{p_1p_2\cdots p_r} s_{i+j+1}^{p_k}
\end{aligned} \tag{A.26}
$$

$$
\begin{aligned}
f_{p_1p_2\cdots p_r} d_i^{(p_g,p_h)} * f_{q_1q_2\cdots q_t} s_j^{q_s} \longrightarrow &\left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} s_{i+j}^{0} \\
&+ \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r} s_{i+j}^{p_g} \\
&+ \left(\frac{q_s(q_s+1)}{2}\right) f_{p_1p_2\cdots p_r} s_{i+j}^{p_h} \\
&+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} \sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r} s_{i+j+1}^{0} \\
&+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} 2q_xq_s\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r} s_{i+j+1}^{p_g+p_h}
\end{aligned} \tag{A.27}
$$

$$
f_{p_1p_2\cdots p_r} s_i^{0} * f_{q_1q_2\cdots q_t} s_j^{q_s} \longrightarrow (n^2+n)\, f_{p_1p_2\cdots p_r} s_{i+j}^{0} \tag{A.28}
$$

$$
\begin{aligned}
f_{p_1p_2\cdots p_r} s_i^{p_k} * f_{q_1q_2\cdots q_t} s_j^{q_s} \longrightarrow &\left(\sum_{\substack{x=1\\x\neq s}}^{t} q_x(q_x+1)\right) f_{p_1p_2\cdots p_r} s_{i+j}^{0} \\
&+ q_s(q_s+1)\, f_{p_1p_2\cdots p_r} s_{i+j}^{p_k} \\
&+ \left(\sum_{\substack{x=1\\x\neq s}}^{t} \sum_{\substack{y=x+1\\y\neq s}}^{t} 2q_xq_y\right) f_{p_1p_2\cdots p_r} s_{i+j}^{0}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{0} \\
&+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{p_k} \qquad \text{(A.29)}
\end{aligned}
$$

$$
\begin{aligned}
f_{p_1 p_2 \cdots p_r} s_i^{(p_k,d)} * f_{q_1 q_2 \cdots q_t} s_j^{q_s} \longrightarrow\ & \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{0} \\
&+ q_s (q_s + 1) f_{p_1 p_2 \cdots p_r} s_{i+j}^{p_k} \\
&+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} \sum_{\substack{y=x+1 \\ y \neq s}}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} s_{i+j}^{0} \\
&+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} s_{i+j}^{d} \\
&+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} s_{i+j}^{p_k - d} \qquad \text{(A.30)}
\end{aligned}
$$

$$
\begin{aligned}
f_{p_1 p_2 \cdots p_r} d_i^{0} * f_{q_1 q_2 \cdots q_t} s_j^{(q_s,c)} \longrightarrow\ & \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{0} \\
&+ [c(c+1) + (q_s - c)(q_s - c + 1)] f_{p_1 p_2 \cdots p_r} s_{i+j}^{0} \\
&+ 2c(q_s - c) f_{p_1 p_2 \cdots p_r} d_{i+j}^{0} \\
&+ \left(\sum_{x=1}^{t} \sum_{y=x+1}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots p_r} s_{i+j+1}^{0} \qquad \text{(A.31)}
\end{aligned}
$$

$$
\begin{aligned}
f_{p_1 p_2 \cdots p_r} d_i^{p_k} * f_{q_1 q_2 \cdots q_t} s_j^{(q_s,c)} \longrightarrow\ & \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{0} \\
&+ \left(\frac{c(c+1)}{2} + \frac{(q_s - c)(q_s - c + 1)}{2} \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{0}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{c(c+1)}{2}+\frac{(q_s-c)(q_s-c+1)}{2}\right) f_{p_1p_2\cdots p_r}s^{p_k}_{i+j}\\
&+\ 2c(q_s-c)\, f_{p_1p_2\cdots p_r}d^{p_k}_{i+j}\\
&+\left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t}2q_xq_y\right) f_{p_1p_2\cdots p_r}s^{0}_{i+j+1}\\
&+\left(\sum_{\substack{x=1\\x\neq s}}^{t}2q_xq_s\right) f_{p_1p_2\cdots p_r}s^{p_k}_{i+j+1} \qquad \text{(A.32)}
\end{aligned}
$$

$$
\begin{aligned}
f_{p_1p_2\cdots p_r}d_i^{(p_g,p_h)} * f_{q_1q_2\cdots q_t}s_j^{(q_s,c)} \longrightarrow &\left(\sum_{\substack{x=1\\x\neq s}}^{t}q_x(q_x+1)\right) f_{p_1p_2\cdots p_r}s^{0}_{i+j}\\
&+\left(\frac{c(c+1)}{2}+\frac{(q_s-c)(q_s-c+1)}{2}\right) f_{p_1p_2\cdots p_r}s^{p_g}_{i+j}\\
&+\left(\frac{c(c+1)}{2}+\frac{(q_s-c)(q_s-c+1)}{2}\right) f_{p_1p_2\cdots p_r}s^{p_h}_{i+j}\\
&+\ 2c(q_s-c)\, f_{p_1p_2\cdots p_r}d(^{(p_g,p_h)}_{i+j}\\
&+\left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t}2q_xq_y\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r}s^{0}_{i+j+1}\\
&+\left(\sum_{\substack{x=1\\x\neq s}}^{t}2q_xq_s\right) f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r}s^{p_g+p_h}_{i+j+1} \qquad \text{(A.33)}
\end{aligned}
$$

$$
f_{p_1p_2\cdots p_r}s_i^{0} * f_{q_1q_2\cdots q_t}s_j^{(q_s,c)} \longrightarrow (n^2+n)\, f_{p_1p_2\cdots p_r}s^{0}_{i+j} \qquad \text{(A.34)}
$$

$$
\begin{aligned}
f_{p_1p_2\cdots p_r}s_i^{p_k} * f_{q_1q_2\cdots q_t}s_j^{(q_s,c)} \longrightarrow &\left(\sum_{\substack{x=1\\x\neq s}}^{t}q_x(q_x+1)\right) f_{p_1p_2\cdots p_r}s^{0}_{i+j}\\
&+\ q_s(q_s+1)\, f_{p_1p_2\cdots p_r}s^{p_k}_{i+j}\\
&+\left(\sum_{\substack{x=1\\x\neq s}}^{t}\sum_{\substack{y=x+1\\y\neq s}}^{t}2q_xq_y\right) f_{p_1p_2\cdots p_r}s^{0}_{i+j}\\
&+\left(\sum_{\substack{x=1\\x\neq s}}^{t}q_xq_s\right) f_{p_1p_2\cdots p_r}s^{0}_{i+j}
\end{aligned}
$$

$$+ \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{p_k} \tag{A.35}$$

$$\begin{aligned}
f_{p_1 p_2 \cdots p_r} s_i^{(p_k,d)} * f_{q_1 q_2 \cdots q_t} s_j^{(q_s,c)} \longrightarrow & \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x (q_x + 1) \right) f_{p_1 p_2 \cdots p_r} s_{i+j}^{0} \\
& + (c(c+1) + (q_s - c)(q_s - c + 1))\, f_{p_1 p_2 \cdots p_r} s_{i+j}^{p_k} \\
& + 2c(q_s - c)\, f_{p_1 p_2 \cdots p_r} s_{i+j}^{(p_k,d)} \\
& + \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} \sum_{\substack{y=x+1 \\ y \neq s}}^{t} 2 q_x q_y \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} s_{i+j}^{0} \\
& + \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} s_{i+j}^{d} \\
& + \left(\sum_{\substack{x=1 \\ x \neq s}}^{t} q_x q_s \right) f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d (p_k - d)} s_{i+j}^{p_k - d}
\end{aligned} \tag{A.36}$$

Proof The proofs of these productions can be obtained by considering the various cases in which the fb-walks split into strands and then recombine. These are in spirit similar to the proofs already given. ♢

Appendix B

Derivation of the Recurrences for The Open Chain of (K_4, u, v)

Table B.1 lists the non-zero partials of $(\dot{K}_4, u, v)$.

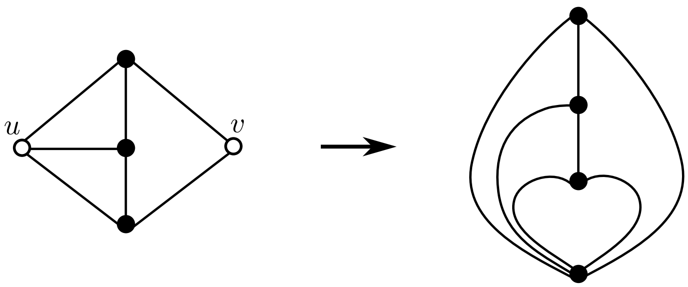


Figure B.1: Self-vertex-amalgamation of double-rooted graph $(\dot{K}_4, u, v)$.

Table B.1: Nonzero partials of $(\dot{K}_4, u, v)$.

j	$f_{111}d^1_j$	$f_{21}d^{(2,1)}_j$	$f_3d^3_j$	$f_{21}s^{(2,1)}_j$	g_j
0	2	0	0	0	2
1	0	6	2	6	14

Plugging in these values for the second amalgamant in the relevant productions of Theorem 43, we obtain the following 24 productions.

For $f_{111}d_0^1 = 2$, we obtain the following six productions:

$$\begin{aligned}
f_{p_1p_2\cdots p_r}d_i^0 *, 2 &\longrightarrow 6\, f_{p_1p_2\cdots p_r}d_i^0 \;+\; 6\, f_{p_1p_2\cdots p_r}d_{i+1}^0 \\
f_{p_1p_2\cdots p_r}d_i^{p_k} *, 2 &\longrightarrow 5\, f_{p_1p_2\cdots p_r}d_i^0 \;+\; f_{p_1p_2\cdots p_r}d_i^{p_k} \\
&\quad +\; 2\, f_{p_1p_2\cdots p_r}d_{i+1}^0 \;+\; 4\, f_{p_1p_2\cdots p_r}d_{i+1}^{p_k} \\
f_{p_1p_2\cdots p_r}d_i^{(p_g,p_h)} *, 2 &\longrightarrow 4\, f_{p_1p_2\cdots p_r}d_i^0 \;+\; f_{p_1p_2\cdots p_r}d_i^{p_g} \\
&\quad +\; f_{p_1p_2\cdots p_r}d_i^{p_h} \;+\; 2\, f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r}d_{i+1}^0 \\
&\quad +\; 4\, f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r}d_{i+1}^{p_g+p_h} \\
f_{p_1p_2\cdots p_r}s_i^0 *, 2 &\longrightarrow 12\, f_{p_1p_2\cdots p_r}d_i^0 \\
f_{p_1p_2\cdots p_r}s_i^{p_k} *, 2 &\longrightarrow 8\, f_{p_1p_2\cdots p_r}d_i^0 \;+\; 4\, f_{p_1p_2\cdots p_r}d_i^{p_k} \\
f_{p_1p_2\cdots p_r}s_i^{(p_k,d)} *, 2 &\longrightarrow 4\, f_{p_1p_2\cdots p_r}d_i^0 \;+\; 2\, f_{p_1p_2\cdots p_r}d_i^{p_k} \\
&\quad +\; 2\, f_{p_1p_2\cdots\hat{p}_k\cdots p_r d(p_k-d)}d_i^0 \;+\; 2\, f_{p_1p_2\cdots\hat{p}_k\cdots p_r d(p_k-d)}d_i^d \\
&\quad +\; 2\, f_{p_1p_2\cdots\hat{p}_k\cdots p_r d(p_k-d)}d_i^{p_k-d}
\end{aligned}$$

For $f_3d_1^3 = 2$, we obtain the following six productions:

$$\begin{aligned}
f_{p_1p_2\cdots p_r}d_i^0 *, 2 &\longrightarrow 12\, f_{p_1p_2\cdots p_r}d_{i+1}^0 \\
f_{p_1p_2\cdots p_r}d_i^{p_k} *, 2 &\longrightarrow 6\, f_{p_1p_2\cdots p_r}d_{i+1}^0 \;+\; 6\, f_{p_1p_2\cdots p_r}d_{i+1}^{p_k} \\
f_{p_1p_2\cdots p_r}d_i^{(p_g,p_h)} *, 2 &\longrightarrow 6\, f_{p_1p_2\cdots p_r}d_{i+1}^{p_g} \;+\; 6\, f_{p_1p_2\cdots p_r}d_{i+1}^{p_h} \\
f_{p_1p_2\cdots p_r}s_i^0 *, 2 &\longrightarrow 12\, f_{p_1p_2\cdots p_r}d_{i+1}^0 \\
f_{p_1p_2\cdots p_r}s_i^{p_k} *, 2 &\longrightarrow 12\, f_{p_1p_2\cdots p_r}d_{i+1}^{p_k} \\
f_{p_1p_2\cdots p_r}s_i^{(p_k,d)} *, 2 &\longrightarrow 12\, f_{p_1p_2\cdots p_r}d_{i+1}^{p_k}
\end{aligned}$$

For $f_{21}d_1^{(2,1)} = 6$, we obtain the following six productions:

$$
\begin{aligned}
f_{p_1p_2\cdots p_r} d_i^0 *, 6 &\longrightarrow 8\, f_{p_1p_2\cdots p_r} d_{i+1}^0 \;+\; 4\, f_{p_1p_2\cdots p_r} s_{i+2}^0 \\
f_{p_1p_2\cdots p_r} d_i^{p_k} *, 6 &\longrightarrow 4\, f_{p_1p_2\cdots p_r} d_{i+1}^0 \;+\; 4\, f_{p_1p_2\cdots p_r} d_{i+1}^{p_k} \\
&\quad +\; 4\, f_{p_1p_2\cdots p_r} s_{i+2}^{p_k} \\
f_{p_1p_2\cdots p_r} d_i^{(p_g,p_h)} *, 6 &\longrightarrow 4\, f_{p_1p_2\cdots p_r} d_{i+1}^{p_g} \;+\; 4\, f_{p_1p_2\cdots p_r} d_{i+1}^{p_h} \\
&\quad +\; 4\, f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r} s_{i+2}^{(p_g+p_h,min(p_g,p_h))} \\
f_{p_1p_2\cdots p_r} s_i^0 *, 6 &\longrightarrow 12\, f_{p_1p_2\cdots p_r} d_{i+1}^0 \\
f_{p_1p_2\cdots p_r} s_i^{p_k} *, 6 &\longrightarrow 8\, f_{p_1p_2\cdots p_r} d_{i+1}^{p_k} \;+\; 4\, f_{p_1p_2\cdots p_r} d_{i+1}^{p_k} \\
f_{p_1p_2\cdots p_r} s_i^{(p_k,d)} *, 6 &\longrightarrow 8\, f_{p_1p_2\cdots p_r} d_{i+1}^{p_k} \;+\; 4\, f_{p_1p_2\cdots\hat{p}_k\cdots p_r d(p_k-d)} d_{i+1}^{(d,p_k-d)}
\end{aligned}
$$

And, finally, for $f_{21}s_1^{(2,1)} = 6$, we obtain the following six productions:

$$
\begin{aligned}
f_{p_1p_2\cdots p_r} d_i^0 *, 6 &\longrightarrow 2\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \;+\; 4\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \\
&\quad +\; 2\, f_{p_1p_2\cdots p_r} d_{i+1}^0 \;+\; 4\, f_{p_1p_2\cdots p_r} s_{i+2}^0 \\
f_{p_1p_2\cdots p_r} d_i^{p_k} *, 6 &\longrightarrow 2\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \;+\; 2\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \\
&\quad +\; 2\, f_{p_1p_2\cdots p_r} s_{i+1}^{p_k} \;+\; 2\, f_{p_1p_2\cdots p_r} d_{i+1}^{p_k} \\
&\quad +\; 4\, f_{p_1p_2\cdots p_r} s_{i+2}^{p_k} \\
f_{p_1p_2\cdots p_r} d_i^{(p_g,p_h)} *, 6 &\longrightarrow 2\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \;+\; 2\, f_{p_1p_2\cdots p_r} s_{i+1}^{p_g} \\
&\quad +\; 2\, f_{p_1p_2\cdots p_r} s_{i+1}^{p_h} \;+\; 2\, f_{p_1p_2\cdots p_r} d_{i+1}^{(p_g,p_h)} \\
&\quad +\; 4\, f_{p_1p_2\cdots(p_g+p_h)\cdots\hat{p}_g\cdots\hat{p}_h\cdots p_r} s_{i+2}^{p_g+p_h} \\
f_{p_1p_2\cdots p_r} s_i^0 *, 6 &\longrightarrow 12\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \\
f_{p_1p_2\cdots p_r} s_i^{p_k} *, 6 &\longrightarrow 2\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \;+\; 6\, f_{p_1p_2\cdots p_r} s_{i+1}^{p_k} \\
&\quad +\; 2\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \;+\; 2\, f_{p_1p_2\cdots p_r} s_{i+1}^{p_k} \\
f_{p_1p_2\cdots p_r} s_i^{(p_k,d)} *, 6 &\longrightarrow 2\, f_{p_1p_2\cdots p_r} s_{i+1}^0 \;+\; 4\, f_{p_1p_2\cdots p_r} s_{i+1}^{p_k} \\
&\quad +\; 2\, f_{p_1p_2\cdots p_r} s_{i+1}^{(p_k,d)} \;+\; 2\, f_{p_1p_2\cdots\hat{p}_k\cdots p_r d(p_k-d)} s_{i+1}^{d}
\end{aligned}
$$

$$+\ 2\, f_{p_1 p_2 \cdots \hat{p}_k \cdots p_r d(p_k-d)} s_{i+1}^{p_k-d}$$

For $p_1 \cdots p_r = 111$, these twenty-four productions become:

$$
\begin{aligned}
2\, f_{111} d_i^0 &\longrightarrow 6\, f_{111} d_i^0 \,+\, 6\, f_{111} d_{i+1}^0 \\
2\, f_{111} d_i^1 &\longrightarrow 5\, f_{111} d_i^0 \,+\, f_{111} d_i^1 \,+\, 2\, f_{111} d_{i+1}^0 \,+\, 4\, f_{111} d_{i+1}^1 \\
2\, f_{111} d_i^{(1,1)} &\longrightarrow 4\, f_{111} d_i^0 \,+\, 2\, f_{111} d_i^1 \,+\, 2\, f_{21} d_{i+1}^0 \,+\, 4\, f_{21} d_{i+1}^2 \\
2\, f_{111} s_i^0 &\longrightarrow 12\, f_{111} d_i^0 \\
2\, f_{111} s_i^1 &\longrightarrow 8\, f_{111} d_i^0 \,+\, 4\, f_{111} d_i^1 \\
2\, f_{111} d_i^0 &\longrightarrow 12\, f_{111} d_{i+1}^0 \\
2\, f_{111} d_i^1 &\longrightarrow 6\, f_{111} d_{i+1}^0 \,+\, 6\, f_{111} d_{i+1}^1 \\
2\, f_{111} d_i^{(1,1)} &\longrightarrow 12\, f_{111} d_{i+1}^1 \\
2\, f_{111} s_i^0 &\longrightarrow 12\, f_{111} d_{i+1}^0 \\
2\, f_{111} s_i^1 &\longrightarrow 12\, f_{111} d_{i+1}^1 \\
6\, f_{111} d_i^0 &\longrightarrow 8\, f_{111} d_{i+1}^0 \,+\, 4\, f_{111} s_{i+2}^0 \\
6\, f_{111} d_i^1 &\longrightarrow 4\, f_{111} d_{i+1}^0 \,+\, 4\, f_{111} d_{i+1}^1 \,+\, 4\, f_{111} s_{i+2}^1 \\
6\, f_{111} d_i^{(1,1)} &\longrightarrow 8\, f_{111} d_{i+1}^1 \,+\, 4\, f_{21} s_{i+2}^{(2,1)} \\
6\, f_{111} s_i^0 &\longrightarrow 12\, f_{111} d_{i+1}^0 \\
6\, f_{111} s_i^1 &\longrightarrow 12\, f_{111} d_{i+1}^1 \\
6\, f_{111} d_i^0 &\longrightarrow 6\, f_{111} s_{i+1}^0 \,+\, 2\, f_{111} d_{i+1}^0 \,+\, 4\, f_{111} s_{i+2}^0 \\
6\, f_{111} d_i^1 &\longrightarrow 4\, f_{111} s_{i+1}^0 \,+\, 2\, f_{111} s_{i+1}^1 \,+\, 2\, f_{111} d_{i+1}^1 \,+\, 4\, f_{111} s_{i+2}^1 \\
6\, f_{111} d_i^{(1,1)} &\longrightarrow 2\, f_{111} s_{i+1}^0 \,+\, 4\, f_{111} s_{i+1}^1 \,+\, 2\, f_{111} d_{i+1}^{(1,1)} \,+\, 4\, f_{21} s_{i+2}^2 \\
6\, f_{111} s_i^0 &\longrightarrow 12\, f_{111} s_{i+1}^0 \\
6\, f_{111} s_i^1 &\longrightarrow 4\, f_{111} s_{i+1}^0 \,+\, 8\, f_{111} s_{i+1}^1
\end{aligned}
$$

For $p_1 \cdots p_r = 21, p_g = 2, p_h = 1$ and for $p_k = 1$ or 2, these are:

$$2\, f_{21} d_i^0 \longrightarrow 6\, f_{21} d_i^0 \,+\, 6\, f_{21} d_{i+1}^0$$

$$2\, f_{21} d_i^2 \longrightarrow 5\, f_{21} d_i^0 \,+\, f_{21} d_i^2 \,+\, 2\, f_{21} d_{i+1}^0 \,+\, 4\, f_{21} d_{i+1}^2$$

$$2\, f_{21} d_i^1 \longrightarrow 5\, f_{21} d_i^0 \,+\, f_{21} d_i^1 \,+\, 2\, f_{21} d_{i+1}^0 \,+\, 4\, f_{21} d_{i+1}^1$$

$$2\, f_{21} d_i^{(2,1)} \longrightarrow 4\, f_{21} d_i^0 \,+\, f_{21} d_i^2 \,+\, f_{21} d_i^1 \,+\, 2\, f_3 d_{i+1}^0 \,+\, 4\, f_3 d_{i+1}^3$$

$$2\, f_{21} s_i^0 \longrightarrow 12\, f_{21} d_i^0$$

$$2\, f_{21} s_i^2 \longrightarrow 8\, f_{21} d_i^0 \,+\, 4\, f_{21} d_i^2$$

$$2\, f_{21} s_i^1 \longrightarrow 8\, f_{21} d_i^0 \,+\, 4\, f_{21} d_i^1$$

$$2\, f_{21} s_i^{(2,1)} \longrightarrow 4\, f_{21} d_i^0 \,+\, 2\, f_{21} d_i^2 \,+\, 2\, f_{111} d_i^0 \,+\, 2\, f_{111} d_i^1 \,+\, 2\, f_{111} d_i^1$$

$$2\, f_{21} d_i^0 \longrightarrow 12\, f_{21} d_{i+1}^0$$

$$2\, f_{21} d_i^2 \longrightarrow 6\, f_{21} d_{i+1}^0 \,+\, 6\, f_{21} d_{i+1}^2$$

$$2\, f_{21} d_i^1 \longrightarrow 6\, f_{21} d_{i+1}^0 \,+\, 6\, f_{21} d_{i+1}^1$$

$$2\, f_{21} d_i^{(2,1)} \longrightarrow 6\, f_{21} d_{i+1}^2 \,+\, 6\, f_{21} d_{i+1}^1$$

$$2\, f_{21} s_i^0 \longrightarrow 12\, f_{21} d_{i+1}^0$$

$$2\, f_{21} s_i^2 \longrightarrow 12\, f_{21} d_{i+1}^2$$

$$2\, f_{21} s_i^1 \longrightarrow 12\, f_{21} d_{i+1}^1$$

$$2\, f_{21} s_i^{(2,1)} \longrightarrow 12\, f_{21} d_{i+1}^2$$

$$6\, f_{21} d_i^0 \longrightarrow 8\, f_{21} d_{i+1}^0 \,+\, 4\, f_{21} s_{i+2}^0$$

$$6\, f_{21} d_i^2 \longrightarrow 4\, f_{21} d_{i+1}^0 \,+\, 4\, f_{21} d_{i+1}^2 \,+\, 4\, f_{21} s_{i+2}^2$$

$$6\, f_{21} d_i^1 \longrightarrow 4\, f_{21} d_{i+1}^0 \,+\, 4\, f_{21} d_{i+1}^1 \,+\, 4\, f_{21} s_{i+2}^1$$

$$6\, f_{21} d_i^{(2,1)} \longrightarrow 4\, f_{21} d_{i+1}^2 \,+\, 4\, f_{21} d_{i+1}^1 \,+\, 4\, f_3 s_{i+2}^{(3,1)}$$

$$6\, f_{21} s_i^0 \longrightarrow 12\, f_{21} d_{i+1}^0$$

$$6\, f_{21} s_i^2 \longrightarrow 8\, f_{21} d_{i+1}^2 \,+\, 4\, f_{21} d_{i+1}^2$$

$$6\, f_{21} s_i^1 \longrightarrow 8\, f_{21} d_{i+1}^1 \,+\, 4\, f_{21} d_{i+1}^1$$

$$6\, f_{21} s_i^{(2,1)} \longrightarrow 8\, f_{21} d_{i+1}^2 \,+\, 4\, f_{111} d_{i+1}^{(1,1)}$$

$$6\, f_{21} d_i^0 \longrightarrow 6\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} d_{i+1}^0 \,+\, 4\, f_{21} s_{i+2}^0$$

$$6\, f_{21} d_i^2 \longrightarrow 2\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} s_{i+1}^2 \,+\, 2\, f_{21} d_{i+1}^2 \,+\, 4\, f_{21} s_{i+2}^2$$

$$6\, f_{21} d_i^1 \longrightarrow 2\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} s_{i+1}^1 \,+\, 2\, f_{21} d_{i+1}^1 \,+\, 4\, f_{21} s_{i+2}^1$$

$$6\, f_{21} d_i^{(2,1)} \longrightarrow 2\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} s_{i+1}^2 \,+\, 2\, f_{21} s_{i+1}^1 \,+\, 2\, f_{21} d_{i+1}^{(2,1)} \,+\, 4\, f_3 s_{i+2}^3$$

$$6\, f_{21} s_i^0 \longrightarrow 12\, f_{21} s_{i+1}^0$$

$$6\, f_{21} s_i^2 \longrightarrow 2\, f_{21} s_{i+1}^0 \,+\, 6\, f_{21} s_{i+1}^2 \,+\, 2\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} s_{i+1}^2$$

$$6\, f_{21} s_i^1 \longrightarrow 2\, f_{21} s_{i+1}^0 \,+\, 6\, f_{21} s_{i+1}^1 \,+\, 2\, f_{21} s_{i+1}^0 \,+\, 2\, f_{21} s_{i+1}^1$$

$$6\, f_{21} s_i^{(2,1)} \longrightarrow 2\, f_{21} s_{i+1}^0 \,+\, 4\, f_{21} s_{i+1}^2 \,+\, 2\, f_{21} s_{i+1}^{(2,1)} \,+\, 2\, f_{111} s_{i+1}^1 \,+\, 2\, f_{111} s_{i+1}^1$$

For $p_1 \cdots p_r = 3, p_k = 3, d = 1$, these are:

$$2\, f_3 d_i^0 \longrightarrow 6\, f_3 d_i^0 \,+\, 6\, f_3 d_{i+1}^0$$

$$2\, f_3 d_i^3 \longrightarrow 5\, f_3 d_i^0 \,+\, f_3 d_i^3 \,+\, 2\, f_3 d_{i+1}^0 \,+\, 4\, f_3 d_{i+1}^3$$

$$2\, f_3 s_i^0 \longrightarrow 12\, f_3 d_i^0$$

$$2\, f_3 s_i^3 \longrightarrow 8\, f_3 d_i^0 \,+\, 4\, f_3 d_i^3$$

$$2\, f_3 s_i^{(3,1)} \longrightarrow 4\, f_3 d_i^0 \,+\, 2\, f_3 d_i^3 \,+\, 2\, f_{21} d_i^0 \,+\, 2\, f_{21} d_i^1 \,+\, 2\, f_{21} d_i^2$$

$$2\, f_3 d_i^0 \longrightarrow 12\, f_3 d_{i+1}^0$$

$$2\, f_3 d_i^3 \longrightarrow 6\, f_3 d_{i+1}^0 \,+\, 6\, f_3 d_{i+1}^3$$

$$2\, f_3 s_i^0 \longrightarrow 12\, f_3 d_{i+1}^0$$

$$2\, f_3 s_i^3 \longrightarrow 12\, f_3 d_{i+1}^3$$

$$2\, f_3 s_i^{(3,1)} \longrightarrow 12\, f_3 d_{i+1}^3$$

$$6\, f_3 d_i^0 \longrightarrow 8\, f_3 d_{i+1}^0 \,+\, 4\, f_3 s_{i+2}^0$$

$$6\, f_3 d_i^3 \longrightarrow 4\, f_3 d_{i+1}^0 \,+\, 4\, f_3 d_{i+1}^3 \,+\, 4\, f_3 s_{i+2}^3$$

$$6\, f_3 s_i^0 \longrightarrow 12\, f_3 d_{i+1}^0$$

$$6\, f_3 s_i^3 \longrightarrow 8\, f_3 d_{i+1}^3 \;+\; 4\, f_3 d_{i+1}^3$$

$$6\, f_3 s_i^{(3,1)} \longrightarrow 8\, f_3 d_{i+1}^3 \;+\; 4\, f_{21} d_{i+1}^{(2,1)}$$

$$6\, f_3 d_i^0 \longrightarrow 6\, f_3 s_{i+1}^0 \;+\; 2\, f_3 d_{i+1}^0 \;+\; 4\, f_3 s_{i+2}^0$$

$$6\, f_3 d_i^3 \longrightarrow 2\, f_3 s_{i+1}^0 \;+\; 2\, f_3 s_{i+1}^0 \;+\; 2\, f_3 s_{i+1}^3 \;+\; 2\, f_3 d_{i+1}^3 \;+\; 4\, f_3 s_{i+2}^3$$

$$6\, f_3 s_i^0 \longrightarrow 12\, f_3 s_{i+1}^0$$

$$6\, f_3 s_i^3 \longrightarrow 2\, f_3 s_{i+1}^0 \;+\; 6\, f_3 s_{i+1}^3 \;+\; 2\, f_3 s_{i+1}^0 \;+\; 2\, f_3 s_{i+1}^3$$

$$6\, f_3 s_i^{(3,1)} \longrightarrow 2\, f_3 s_{i+1}^0 \;+\; 4\, f_3 s_{i+1}^3 \;+\; 2\, f_3 s_{i+1}^{(3,1)} \;+\; 2\, f_{21} s_{i+1}^1 \;+\; 2\, f_{21} s_{i+1}^2$$

After transposition, we get the following recurrences:

$$\begin{aligned}
f_{111} d_i^0 \;&=\; 12\, f_{111} d_i^0 + 96\, f_{111} d_{i-1}^0 + 10\, f_{111} d_i^1 + 40\, f_{111} d_{i-1}^1 + 8\, f_{111} d_i^{(1,1)} + 24\, f_{111} s_i^0 \\
&\quad 16\, f_{111} s_i^1 + 96\, f_{111} s_{i-1}^0 + 4\, f_{21} s_i^{(2,1)} \\
f_{111} d_i^1 \;&=\; 2\, f_{111} d_i^1 + 8\, f_{111} s_i^1 + 4\, f_{111} d_i^{(1,1)} + 56\, f_{111} d_{i-1}^1 + 72\, f_{111} d_{i-1}^{(1,1)} + 96\, f_{111} s_{i-1}^1 + \\
&\quad 8\, f_{21} s_i^{(2,1)} \\
f_{111} d_i^{(1,1)} \;&=\; 12\, f_{111} d_{i-1}^{(1,1)} + 24\, f_{21} s_{i-1}^{(2,1)} \\
f_{21} d_i^0 \;&=\; 4\, f_{111} d_{i-1}^{(1,1)} + 12\, f_{21} d_i^0 + 96\, f_{21} d_{i-1}^0 + 10\, f_{21} d_i^2 + 40\, f_{21} d_{i-1}^2 + 10\, f_{21} d_i^1 + \\
&\quad 40\, f_{21} d_{i-1}^1 + 8\, f_{21} d_i^{(2,1)} + 24\, f_{21} s_i^0 + 16\, f_{21} s_i^2 + 16\, f_{21} s_i^1 + 8\, f_{21} s_i^{(2,1)} + \\
&\quad 96\, f_{21} s_{i-1}^0 + 4\, f_3 s_i^{(3,1)} \\
f_{21} d_i^1 \;&=\; 2\, f_{21} d_i^1 + 2\, f_{21} d_i^{(2,1)} + 8\, f_{21} s_i^1 + 56\, f_{21} d_{i-1}^1 + 36\, f_{21} d_{i-1}^{(2,1)} + 96\, f_{21} s_{i-1}^1 + 4\, f_3 s_i^{(3,1)} \\
f_{21} d_i^2 \;&=\; 8\, f_{111} d_{i-1}^{(1,1)} + 2\, f_{21} d_i^2 + 56\, f_{21} d_{i-1}^2 + 2\, f_{21} d_i^{(2,1)} + 36\, f_{21} d_{i-1}^{(2,1)} + \\
&\quad 8\, f_{21} s_i^2 + 4\, f_{21} s_i^{(2,1)} + 96\, f_{21} s_{i-1}^2 + 72\, f_{21} s_{i-1}^{(2,1)} + 4\, f_3 s_i^{(3,1)} \\
f_{21} d_i^{(2,1)} \;&=\; 12\, f_{21} d_{i-1}^{(2,1)} + 24\, f_3 s_{i-1}^{(3,1)} \\
f_3 d_i^0 \;&=\; 4\, f_{21} d_{i-1}^{(2,1)} + 12\, f_3 d_i^0 + 96\, f_3 d_{i-1}^0 + 10\, f_3 d_i^3 + 40\, f_3 d_{i-1}^3 + 24\, f_3 s_i^0 + \\
&\quad 16\, f_3 s_i^3 + 8\, f_3 s_i^{(3,1)} + 96\, f_3 s_{i-1}^0 \\
f_3 d_i^3 \;&=\; 8\, f_{21} d_{i-1}^{(2,1)} + 2\, f_3 d_i^3 + 56\, f_3 d_{i-1}^3 + 8\, f_3 s_i^3 + 4\, f_3 s_i^{(3,1)} + 96\, f_3 s_{i-1}^3 +
\end{aligned}$$

$$72\, f_3 s_{i-1}^{(3,1)}$$

$$f_{111} s_i^0 = 36\, f_{111} d_{i-1}^0 + 24\, f_{111} d_{i-2}^0 + 48\, f_{111} d_{i-1}^1 + 12\, f_{111} d_{i-1}^{(1,1)} + 72\, f_{111} s_{i-1}^0 + 24\, f_{111} s_{i-1}^1$$

$$f_{111} s_i^1 = 48\, f_{111} d_{i-2}^1 + 12\, f_{111} d_{i-1}^1 + 24\, f_{111} d_{i-1}^{(1,1)} + 48\, f_{111} s_{i-1}^1 + 24\, f_{21} s_{i-1}^{(2,1)}$$

$$f_{21} s_i^0 = 36\, f_{21} d_{i-1}^0 + 48\, f_{21} d_{i-2}^0 + 24\, f_{21} d_{i-1}^2 + 24\, f_{21} d_{i-1}^1 + 12\, f_{21} d_{i-1}^{(2,1)} + 72\, f_{21} s_{i-1}^0 +$$

$$24\, f_{21} s_{i-1}^2 + 24\, f_{21} s_{i-1}^1 + 12\, f_{21} s_{i-1}^{(2,1)}$$

$$f_{21} s_i^1 = 48\, f_{21} d_{i-2}^1 + 12\, f_{21} d_{i-1}^1 + 12\, f_{21} d_{i-1}^{(2,1)} + 48\, f_{21} s_{i-1}^1 + 12\, f_3 s_{i-1}^{(3,1)}$$

$$f_{21} s_i^2 = 24\, f_{111} d_{i-2}^{(1,1)} + 48\, f_{21} d_{i-2}^2 + 12\, f_{21} d_{i-1}^2 + 12\, f_{21} d_{i-1}^{(2,1)} + 48\, f_{21} s_{i-1}^2 + 24\, f_{21} s_{i-1}^{(2,1)} + 12\, f_3 s_{i-1}^{(3,1)}$$

$$f_{21} s_i^{(2,1)} = 24\, f_{111} d_{i-2}^{(1,1)} + 12\, f_{21} s_{i-1}^{(2,1)}$$

$$f_3 s_i^0 = 36\, f_3 d_{i-1}^0 + 48\, f_3 d_{i-2}^0 + 24\, f_3 d_{i-1}^3 + 72\, f_3 s_{i-1}^0 + 24\, f_3 s_{i-1}^3 + 12\, f_3 s_{i-1}^{(3,1)}$$

$$f_3 s_i^3 = 24\, f_{21} d_{i-2}^{(2,1)} + 48\, f_3 d_{i-2}^3 + 12\, f_3 d_{i-1}^3 + 48\, f_3 s_{i-1}^3 + 24\, f_3 s_{i-1}^{(3,1)}$$

$$f_3 s_i^{(3,1)} = 24\, f_{21} d_{i-2}^{(2,1)} + 12\, f_3 s_{i-1}^{(3,1)}$$