Arbitrage, Gains from Trade and Social Diversity:
A Unified Perspective on Resource Allocation

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ARBITRAGE, GAINS FROM TRADE AND SOCIAL DIVERSITY: A UNIFIED PERSPECTIVE ON RESOURCE ALLOCATION

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We trade because we are different. Gains from trade and the scope for mutually advantageous reallocation depend naturally on the diversity of the traders' preferences and endowments. The market owes its existence to the diversity of those who make up the economy.

An excess of diversity could however stretch the ability of economic institutions to operate efficiently: this has recently been a concern in regions experiencing extensive and rapid migration, such as US and the ex-USSR. Are there natural limits on the degree of social diversity with which our institutions can cope? This paper will argue that there are. I shall argue that not only is a certain amount of diversity essential for the functioning of markets, but, at the other extreme, that too much diversity of a society's preferences and endowments may hinder its ability to allocate resources efficiently. This will be examined rigorously in the context of two classical forms of resource allocation: by markets, and by social choice or voting, arguably those most frequently used in modern economies.

Until quite recently, diversity has been an elusive concept. However, a precise measure of *social diversity* will be given here in terms of the preferences and endowments of individuals. This concept is robust to small errors in measurements, and independent of the units of measurement. I shall establish that too much social diversity in this sense can interfere with the efficient performance of markets and with the achievement of social choices.

Shifting the angle of inquiry slightly sheds a different light on the subject. If a society allocates resources efficiently, whether by markets or by collective choices, then this society must exhibit no more than a certain degree of social diversity. There is therefore an implicit prediction here about the characteristics of economies which evolve mechanisms for allocating resources efficiently: they will have only a limited degree of social diversity in my sense. Economies which do not succeed in allocating resource efficiently are not likely to be observed in practice, so that enduring economies are likely to have limited social diversity.

The precise degree of social diversity which is consistent with the market reaching efficient allocations is described here by a condition of *limited arbitrage.* Intuitively, this gages the extent of the gains from trade. This paper defines limited arbitrage precisely from the endowments and preferences, and then defines the degree of social diversity which it implies. It provides examples of economies with and without limited arbitrage. It shows by means of examples why limited arbitrage separates those markets which have a competitive equilibrium from those that do not, and why it simultaneously

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separates those economies which have well defined social choice rules from those which do not.

From this analysis a new unified perspective emerges on the central question of resource allocation. This is the existence of a well-defined connection between two classic forms of resource allocation which have been considered separate and almost antagonistic until now: markets and public choices. The same limitation on social diversity links these two forms of resource allocation. Limited arbitrage is necessary and sufficient for the existence of competitive market allocations. It is also necessary and sufficient for the existence of well-defined social choice rules. One can actually translate one form of resource allocation into the other. The success of both hinges on the same limitation on the social diversity of the economy. The economies which we observe in practice, if successful at either form of resource allocation, will exhibit a limited amount of social diversity. Turning this proposition around it implies that increases in diversity beyond this threshold may call for forms of resource allocation different from both.

1. Limited Arbitrage and Gains from Trade. Intuitively limited arbitrage bounds the extent of gains from trade in the economy. To offer a formal perspective one needs a few definitions. An economy E has $H \geq 2$ traders who trade $N \geq 2$ commodities or assets, so that the trading space is R^N; when short sales are not allowed the trading space is instead R^{N+}. A trader i is described by an initial endowment $\Omega_i \in R^N$, and by a preference represented by a utility function $u_i : R^N \to R$.

One wishes to identify those trading opportunities which could yield unbounded utility increases for the *ith* trader. These are described by net trades in $A_i = \{y \in R^N : \forall \lambda > 0,\ u_i(\Omega_i + \lambda y) > u_i(\Omega_i)$ and $\lim_{\lambda \to \infty} u_i(\Omega_i + \lambda y) = \sup_{x \in R^N} u_i(x)\}$, a concept new to the literature, which contains global information about the trader and is therefore called a *global cone*. The trader's *market cone* is the set of all those prices at which all trading opportunities in A_i are unaffordable, $D_i = \{p \in R^{N+} : < p, (y - \Omega_i) > > 0\}$.[1] The existence of both competitive equilibrium and social choice rules is shown to depend on the relation between the traders' market cones; this relation also provides a framework for measuring social diversity.

DEFINITION 1. *The market economy E has limited arbitrage when all its market cones intersect:* $\bigcap_{i=1}^{H} D_i \neq \phi$.

This means that there exists one price, the same for all traders, at which the trades they can afford only increase their utilities by limited, or bounded, amounts. The concept of limited arbitrage can also be interpreted in terms of gains from trade, defined as the maximum increment in the sum of utilities which the traders can achieve by reallocating the economy's resources:

$$\text{Gains from trade } = G(E) = Sup\left(\sum_{i=1}^{H} u_i(x_i) - u_i(\Omega_i)\right),$$

where for all i $u_i(x_i) \geq u_i(\Omega_i)$ and $\sum_{i=1}^{H}(x_i - \Omega_i) = 0$.

PROPOSITION 2. *An economy E satisfies limited arbitrage if and only if it has bounded gains from trade, namely $G(E) < \infty$.*

For a proof see the Appendix. The geometry of limited arbitrage is simple: it means that the traders' global cones cannot contain net trades which add up to zero: $\sim \exists\; x_i, x_j$ such that $x_i + x_j = 0$, $x_i \in A_i$ and $x_j \in A_j$. In other words: all global cones A_i must lie on one side of a given price hyperplane.

Figure 1 illustrates an economy E_1 with two traders and two assets which has limited arbitrage. Its global cones are A_1 and A_2 and the price line p leaves both cones on one side. Therefore net trades in directions which lead to unbounded utility gains are unaffordable by all traders from their initial endowments at price p. The gains from trade in this economy $G(E_1)$ are bounded.

The economy of Figure 2 does not satisfy limited arbitrage: there are two directions of net trades $w_1' \in A_1$ and $w_1 \in A_2$, yielding unbounded increases in utility and which sum up to zero. Therefore, there is no price p at which all net trades in A_1 and in A_2 are unaffordable from initial endowments. The gains from trade in this economy are unbounded.

Section 3 shows that the boundedness of possible gains from trade, which we now know to be equivalent to limited arbitrage, is fundamental to the existence of a competitive equilibrium: it is necessary and sufficient. Intuitively this is reasonable: an economy such as that in Figure 2, where traders wish to take unboundedly large and opposed trading positions, cannot reach an equilibrium. Desired trades are just too diverse to be accommodated within the same economy.

2. Limited Arbitrage and No-Arbitrage. In financial markets an *arbitrage opportunity* exists when individuals can make unbounded gains at no cost, or, equivalently, by taking no risks. For example, buying an asset in a market where its price is low while simultaneously selling it at another where its price is high can lead to unbounded gains at no risk to the trader. *No-arbitrage* means that such opportunities do not exist, and it provides a standard way of pricing a financial asset: precisely so that no arbitrage opportunities should arise between this and other related assets. Since trading does not cease until all arbitrage opportunities are extinguished, at a market clearing equilibrium there is no-arbitrage.

The simplest illustration of the link between limited arbitrage and no-arbitrage is an economy E where the traders' initial endowments are zero, $\Omega_i = 0$ for $i = 1, 2$. Here *no-arbitrage* at the initial endowments means that there are no trades which could increase the traders' utility at zero cost: gains from trade in E must be zero. By contrast, E has *limited arbitrage* when no trader can increase utility beyond a given bound at zero cost; as seen in Proposition 2 of Section 1, gains from trade are bounded. In summary: no-arbitrage requires that there should be *no* gains from trade at zero cost, while limited arbitrage requires that there should be only bounded or *limited* gains from trade.

The two concepts are related but nonetheless quite different. No-arbitrage is a market clearing condition: it is used to describe an allocation at which there is no further reason to trade. It can be applied at the initial allocations, but then it means that there is no reason for trade: the economy is autarchic and therefore not very interesting. By contrast, limited arbitrage is applied only to the economy's initial data, the traders' endowments and preferences, and it does not imply that the economy is autarchic. Quite to the contrary, it is valuable in predicting whether the economy can ever reach a competitive equilibrium, and allows us to do this simply by examining the economy's initial conditions. This is the subject of the next section.

3. Limited Arbitrage and Market Equilibrium. Limited arbitrage identifies fully those economies which have a competitive market equilibrium.[2] I concentrate here on competitive market allocations because they are Pareto efficient, which makes them desirable from the point of view of resource allocation, while other lesser concepts of equilibrium are not[3]. A competitive equilibrium for the economy E is a price p^* and an allocation $x_1^*...x_H^* \in R^{N \times H}$ such that each trader $i = 1...H$ maximizes utility within a budget, $u_i(x_i^*) = Max(u_i(x))$ for $x \in \{y \in R^N :< p^*, y - \Omega_i >= 0\}$, and all N markets clear, $\sum_{i=1}^{H}(x_i - \Omega_i) = \{0\}$. When the market has bounds on short sales, its trading space is the positive orthant R^{N+} and the market cones are slightly different; they are denoted ∂D_i and defined in the Appendix. The condition of limited arbitrage is however always the same: all market cones intersect, $\cap_{i=1}^{H} \partial D_i \neq \phi$. The following holds in economies E with or without short sales:

PROPOSITION 1. *The economy E has a competitive equilibrium if and only if it satisfies limited arbitrage.*

The Appendix has a proof for economies with short sales; Graciela Chichilnisky [4] has a proof for economies without short sales. Other sufficient conditions for existence of an equilibrium with short sales–in finite or infinite dimensions[4]–are in Chichilnisky and Geoffrey M. Heal [12], but limited arbitrage is the first necessary and sufficient condition for existence of a competitive equilibrium in economies with or without short sales. Sufficiency requires that the Pareto frontier of the economy be *bounded* and *closed*: both can fail in economies with short sales, and both are crucial for existence. The Appendix establishes that limited arbitrage implies both. It is intuitively clear that limited arbitrage is needed for an equilibrium to exist. Otherwise, as seen in Figure 2, traders with very diverse preferences wish to take unboundedly short and long positions against each other. Desired trades are too diverse to be accommodated within the same economy.

At first sight it may seem surprising that limited arbitrage is also necessary for the existence of a competitive equilibrium in economies where no short sales are allowed. However, equilibrium fails here in a similar way: it fails when traders wish to take unboundedly large positions which the bounded resources of the economy cannot accommodate. Paradoxically, this occurs when some of the traders have zero income, i.e. when some prices are zero, and some traders' endowments are in the boundary of R^{N+}. Kenneth Arrow [2] gave an example of a standard economy which has no competitive equilibrium: one with two traders and two goods, without short sales, and where preferences are continuous, concave and increasing. Trader one owns only the first good, which trader two does not like.[5] The second trader has strictly positive endowments of both goods but does not value the first good; therefore at an equilibrium the second trader does not trade, the first good must be free, and the first trader's income zero. Trader one likes the first good, which is free.[6] Therefore there can be no competitive equilibrium: no allocation can maximize the first trader's utility when the first good is free. This example can be extended to economies with any number of traders and of goods, and where some traders with positive income wish and can afford unbounded positions. Arrow and Lionel McKenzie introduced *resource relatedness* and *irreducibility* to solve the non-existence problem [2]: they eliminate traders with zero income by restricting divergences in tastes and in endowments. It should be obvious by now why the failure of existence is the same in economies with or without short sales. The failure originates from some traders wishing to take unbounded positions which that can afford at the going prices. An interesting angle on this problem is that, when no short sales are allowed, the failure of existence occurs when some traders have zero income because what they own is of no market value, while some traders wish, and can afford, to take unboundedly large trading positions.

It turns out that the existence of a competitive equilibrium is decided within sets of at most $N+1$ traders. In an economy E with H traders[7], each subset of traders $\theta \subset \{1...H\}$ defines a *subeconomy* E_θ of E:

PROPOSITION 2. *The economy E has a competitive equilibrium if and only if every subeconomy E_θ with at most $N+1$ traders does, where N is the dimension of the trading space.*

The proof of this proposition is in Chichilnisky [4], [10]. It is due to the fact that all market cones in E intersect when every subfamily of at most $N+1$ market cones does.

The easiest way to visualize the connection between limited arbitrage and the existence of an equilibrium is in an economy with two traders who have linear utility functions [8] and where short sales are allowed. Such an economy has a competitive equilibrium when, and only when, the two traders' preferences are identical; otherwise it is always possible to find a sequence of affordable trades along which the utility of both traders increases without bound, such as that illustrated in Figure 2. This economy has limited arbitrage precisely when the two linear preferences are identical, and only then: the global cones A_1, A_2 of linear preferences are half-spaces, and the market cones D_1, D_2 are half lines defined by the preferences' gradients. The market cones intersect if and only if the two gradients are identical. Therefore this economy has a competitive equilibrium if and only if it satisfies limited arbitrage.

Simple non-linear examples can also be given: Figure 2 illustrates an economy without limited arbitrage. At any prices there is a sequence of affordable trades $(w_n, w'_n)_{n=1,2...}$ in Figure 2 along which the traders can achieve unbounded utility levels, so that no competitive equilibrium can exist.

There are standard conditions which ensure the existence of a competitive equilibrium, such as that no indifference surface intersects the boundary of R^{N+}, or that all traders have a strictly positive endowment of every single good. These are very strong conditions, and Arrow and Hahn find them "unrealistic" [2], p. 80. In any case all these conditions imply limited arbitrage, because being necessary for existence, limited arbitrage must be satisfied by any economy which has a competitive equilibrium.

4. Social Diversity, Limited Arbitrage and Efficient Markets. What if the economy does not have limited arbitrage? Then it is *socially diverse*:

DEFINITION 1. *The economy E is* socially diverse *when* $\bigcap_{i=1}^{H} D_i = \phi$.

This concept is robust under small errors in measurement and is independent of the units of measurement or choice of numeraire. If E is not socially diverse, all economies sufficiently close in endowments and preferences have the same property: the concept is *structurally stable.* Social diversity admits different "shades"; these can be measured, for example, by the smallest number of market cones which do not intersect:

DEFINITION 2. *The economy E has index of diversity $I(E) = H - K$ if $K + 1$ is the smallest number s.t. $\exists \Upsilon \subset \{1...H\}$ with cardinality of $\Upsilon = K+1$, and $\bigcap_{i\in\Upsilon} D_i = \phi$.*

The index $I(E)$ ranges between 0 and $H - 1$: the larger the index, the larger the social diversity. The index is smallest when all the market cones intersect: then all social diversity disappears, and is replaced by limited arbitrage. Proposition 1 implies:

PROPOSITION 3. *The index of social diversity is $I(E)$ if and only if every subeconomy of E with $H - I(E)$ traders has a competitive equilibrium.*

5. Limited Arbitrage and Social Choice. It turns out that limited arbitrage, or the absence of social diversity, is crucial for achieving resource allocation via social choice. Social choice rules allocate resources by assigning a social preference $\Phi(u_1...u_H)$ to each profile $(u_1...u_H)$ of individual preferences[9] of an economy E, in a way which respects ethical axioms. The social preference ranks allocations in $R^{N\times H}$, and is used to locate an optimal allocation. This procedure requires, of course, that an appropriate social choice rule Φ should exist: the role of limited arbitrage is important because it ensures existence. This was demonstrated rigorously in Chichilnisky [8], and will be illustrated below.

There are two main approaches to social choice. One is Arrow's: his axioms of social choice require that the rules Φ be non-dictatorial, independent of irrelevant alternatives, and satisfy a Pareto condition. A second approach requires, instead, that the rule Φ be continuous, anonymous, and respect unanimity, Chichilnisky [6]. Though the two sets of axioms are quite different, I show below that limited arbitrage is nevertheless closely connected with both.

Arrow's impossibility theorem established that a social choice rule Φ does not exist in general; the problem of social choice has no solution unless individual preferences are restricted. Duncan Black [3] established that "single peakedness" of preferences is a sufficient restriction. Using different axioms, Chichilnisky [6], [8] established that a social choice rule Φ does not generally exist; Chichilnisky and Heal [11] established for the first time a necessary and sufficient restriction for the resolution of the social choice paradox: the *contractibility* of the space of preferences[10], which can be interpreted as a limitation on preference diversity, Heal [13]. In all cases, therefore, the problem of social choice is resolved by restricting the diversity of individual preferences.

I shall show next that the traders' preferences in the economy E satisfy limited arbitrage if and only if they contain no *Condorcet triples* of large utility values. Condorcet triples are building blocks of Arrow's impossibility theorem, and are at the root of the social choice problem. Thus limited arbitrage eliminates the source of Arrow's impossibility theorem for choices of large utility values.

DEFINITION 1. *A Condorcet triple is a collection of three preferences over a choice set X, represented by utilities $u_i : X \to R$, $i = 1,2,3$, and three choices α, β, γ within a feasible set $Y \subset X$ such that $u_1(\alpha) > u_1(\beta) > u_1(\gamma)$, $u_2(\gamma) > u_2(\alpha) > u_2(\beta)$, and $u_3(\beta) > u_3(\gamma) > u_3(\alpha)$.*

Within an economy E, the social choice problem is about the choice of allocations: choices are in $X = R^{N\times H}$. An allocation $(x_1...x_H)$ is *feasible* if $\sum_i (x_i - \Omega_i) = 0$. Preferences over allocations are induced naturally by the traders' preferences over private consumption: $u_i(x_1...x_H) \geq u_i(y_i...y_H) \Leftrightarrow u_i(x_i) \geq u_i(y_i)$.

DEFINITION 2. *In an economy E a family of preferences $\{u_1...u_H\}$ has a Condorcet triple of size k if there exists three feasible allocations $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k) \in X \subset R^{N\times 3}$; $\beta^k = (\beta_1^k, \beta_2^k, \beta_3^k)$ and $\gamma^k = (\gamma_1^k, \gamma_2^k, \gamma_3^k)$, and three preferences $u_1^k, u_2^k, u_3^k \in \{u_1...u_H\}$ which define a Condorcet triple, and such that each trader achieves at least a utility level k at each choice*: $\min_{i=1,2,3}\{[u_i^k(\alpha_i^k), u_i^k(\beta_i^k), u_i(\gamma_i^k)]\} > k$.

The following shows that limited arbitrage eliminates Condorcet triples on matters of great importance, namely on those with utility level approaching the supremum of utilities:

PROPOSITION 3. *Let E be a market economy E with no bounds on short sales. Then E has social diversity if and only if its traders' preferences have Condorcet triples of every size.*[11] *Equivalently, E has limited arbitrage if and only for some $k > 0$, the traders' preferences have no Condorcet triples of size larger than k.*

A proof is in the Appendix: it relies on the fact that limited arbitrage is equivalent to bounded gains from trade, Proposition 2 of Section 1.

Turning now to the second approach to social choice, Chichilnisky [10] [8], the link connecting markets with social choices is still very close but takes a different form: the contractibility of the space of preferences, which is necessary and sufficient for continuous, anonymous rules which respect unanimity [11], is shown to be equivalent to limited arbitrage [10]. Therefore limited arbitrage, or equivalently the lack of social diversity, is necessary and sufficient for resource allocation via social choice rules. Formally: let P consist of all those preferences which are *similar* to those of the market economy E, in the sense that their gradients are in the intersection of the market cones of the traders, see [8] and [10]. Intuitively a preference is similar to that of trader i when it prefers those allocations which assign i a consumption vector which u_i prefers.

THEOREM 4. *A continuous anonymous social choice rule $\Phi : P^k \to P$ which respects unanimity exists which for every $k \geq 2$ if and only if the economy E has limited arbitrage.*

For a proof see [8].

6. Appendix. Definitions: An *economy* E is defined by its trading space and its traders $E = \{X, \Omega_i \in R^{N+}, u_i : X \to R,\ i = 1...H\}$, where $X = R^N$, or $X = R^{N+}$ when no short sales are allowed. The traders' *preferences* $u_i : X \to R$ are continuous, concave and increasing: $x \geq y \Rightarrow u_i(x) \geq u_i(y)$ and $u_i(0) = 0$. When the trading space $X = R^{N+}$, if an indifference surface of positive utility intersects the boundary of R^{N+} all indifference surfaces of higher utility do too. When the trading space $X = R^N$ preferences are smooth (C^2), $\exists\varepsilon, K > 0 : \forall x \in R^N, \|Du(x)\| > \varepsilon$, and $\|D^2u(x)\| < K$, and the directions of gradients of an indifference surface which is not bounded below form a closed set. This includes Cobb-Douglas, CES, strictly concave and linear preferences, and preferences with indifferences which intersect the axis, and which contain halflines. *Global cones* A_i and *market cones* D_i were defined in Section 1. The *market cone* ∂D_i of an economy E with trading space R^{N+} is $\partial D_i = D_i \cap S(E)$ if $S(E) \subset N$, and $\partial D_i = D_i$ otherwise, where $N = \{v \in R^N : \exists i$ with $< v, \Omega_i >= 0\}$, and where $S(E)$ is the set of supports to individually rational allocations: $S(E) = \{v \in R^N : \exists(x_1...x_H) \in R^{H\times N+}$ with $\sum(x_i - \Omega_i) = 0, u_i(x_i) \geq u_i(\Omega_i)$ for all i, and $\forall z_i \in R^{N+}, u_i(z_i) \geq u_i(x_i) \Rightarrow < v, z_i - x_i > \geq 0\}$. The *utility set* is $U(E) = \{U_1, ...U_H \in R^{H+}+ : \forall i = 1...H, \exists x_1...x_H$ with $U_i = u_i(x_i) \geq u_i(\Omega_i)$ where $\sum_{i=1}^H (x_i - \Omega_i) \leq 0\}$. The *Pareto frontier* $P(E) = \{V = V_1...V_H \in U(E) :\sim \exists W_1...W_H \in U(E)$ with $W_j \geq U_j \ \forall\ j$ and for some $h, W_h > U_h\}$.

PROPOSITION 1. *The global cones A_i of the economy E are open convex sets.*

Proof. A sequence $(v^n)_{n=1,2...} \subset C(A_i) =$ the complement of A_i, defines halflines $(\Gamma^n)_{n=1,2...}$, with $Sup_{\{x:x\in\Gamma^n\}}(u_i(x)) < \infty\ \forall n$. By the assumptions on u_i, $\forall\ n\ \exists y \in \Gamma^n :< Du_i(y), w >= 0$ if $w \in \Gamma^n$. Concavity of u_i implies that $\forall w \in \Gamma^n < Du_i(\lambda y), w > \leq 0$ $\forall\ \lambda > 1$. Assume that on two halflines $\Gamma^n \neq \Gamma^m$ the utility u_i is eventually constant: $\exists y^n \in \Gamma^n$ and $y^m \in \Gamma^m$ such that $\forall \lambda > 1 < Du_i(\lambda y^n), w >= 0\ \forall w \in \Gamma^n$, and $< Du_i(\lambda y^m), w >= 0\ \forall w \in \Gamma^m$, and $u_i(y^n) < u_i(y^m)$. Let Π be a supporting hyperplane for the preferred set of u_i at λy^m; this determines a halfspace Λ of $R^N : \forall q \in \Lambda, u_i(q) < u_i(\lambda y^m)$; note that Π contains an unbounded segment of Γ^m, and Λ an unbounded segment of Γ^n. Therefore $\forall\ K > 0\ \exists\ z^K \in \Gamma^n$ and $w^K \in \Pi : \|z^K - w^K\| > K$ and $\forall K$, $u_i(z^K) = u_i(y^n)$ and $u_i(w^K) = u_i(y^m)$. Since by assumption $\exists\varepsilon > 0 : \forall x$, $\|Du_i(x)\| > \varepsilon$, $\forall K$ the distance between z^K and $\{w \in R^N : u_i(w) = u_i(y^m)\}$ is bounded: $\exists T > 0 : \forall K, \|z^K - w^K\| < T$, a contradiction. The contradiction arises from assuming that u_i is eventually constant on Γ^n and Γ^m with $n \neq m$; therefore $\exists n_o : \forall j \geq n_o$ $\exists y \in \Gamma^j :< Du_i(\lambda y), w > < 0\ \forall w \in \Gamma^j$ and $\forall \lambda > 1$. By concavity of u_i, this implies that along the halfline Γ defined by $v = \lim^n v^n$, u_i is bounded, so that $v \in C(A_i)$. Thus $C(A_i)$ is closed and A_i open. Convexity is immediate.□

THEOREM 2. *Let E be an economy without bounds on short sales. The Pareto frontier of the economy E is bounded if and only if the economy satisfies limited arbitrage. In particular, the economy E has bounded gains from trade, $G(E) < \infty$, if and only if it has limited arbitrage.*

Proof. By contradiction. Assume E has limited arbitrage. If $P(E)$ were *not* bounded there would exist a sequence of net trades $(z_1^j...z_H^j)_{j=1,2...}$ such that $\forall j, \sum_{h=1}^H z_h^j = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for some h.[12] It suffices to consider the case where $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for all h.[13] Consider two exhaustive and exclusive cases: Case 1 and Case 2. *Case 1: For infinitely many j's, $z_h^j \in A_h$ for all h.* Limited arbitrage requires that there exists a hyperplane that leaves all the cones A_h on one side for all h, and this contradicts the fact that $z_h^j \in A_h$ for all h and $\sum_{h=1}^H z_h^j = 0$. Since the contradiction arises from the assumption that $P(E)$ is unbounded, $P(E)$ must be bounded in this case. *Case 2: From some j onwards, $z_h^j \notin A_h$ for some h.* Consider the sequence $\{z_h^j/\|z_h^j\|\}_{j=1,2,...} \subset S^{N-1}$, the $N-1$ sphere in R^N. Since S^{N-1} is compact, it follows that there exists a subsequence, denoted also $\{z_h^j/\|z_h^j\|\}_{j=1,2,...}$ such that $\lim_{j\to\infty} z_h^j/\|z_h^j\| = \alpha_h \in S^{N-1}$ for all $h = 1...H$. Assume first that $\alpha_h \notin A_h$. Note that it suffices to consider utilities with indifference surfaces not bounded below, since when they are bounded below, $P(E)$ is always a bounded set. Then, by assumption, the directions of gradients of each indifference surface define a closed set. Since we assumed that $\alpha_h \notin A_h$, it follows that $Sup_{\lambda \in R+}(u_h(\Omega_h + \lambda\alpha_h) < \infty$. This, together with the assumption on the utilities, implies that if Γ is the halfline defined by the vector α_h, either $\exists w \in \Gamma$ where the gradient $Du_h(z)$ is orthogonal to Γ, or else the utility u_h achieves a maximum at $y \in \Gamma$, and is a constant beyond y. These two alternatives are exhaustive and I will show that in both it is impossible that $\alpha_h = \lim_j z_h^j/\|z_h^j\|$ with $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j) = \infty$. If the gradient $Du_h\,(w)$ is orthogonal to Γ at some point w, and for $\lambda > 1$ $Du_h(\lambda w)$ projected on Γ is negative, then it is also negative in a neighborhood. Therefore for directions β sufficiently close to α_h, $\exists K > 0$ such that $\sup_{x\in\Upsilon}(u_h(x)) < K$, a contradiction. The second alternative is that u_i utility achieves a maximum at w and remains constant thereafter on Γ. Similar reasoning, using convexity, shows that $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) = \infty$ cannot hold either. Since these two alternatives are exhaustive, $\alpha_h \notin A_h$ is impossible, so that $\alpha_h \in A_h$ for all h where $z_h^j \notin A_h$ from some j onwards. Therefore, in Case 2, $\forall h = 1...H$ the vectors $\alpha_h \in A_h^c$, the closure of A_h, and, for some h, $\alpha_h \in A_h$. Since the cones A_h are open by Proposition 1 in the Appendix, there exist nearby vectors $\beta_1...\beta_H$ s.t. $\sum_h \beta_h = 0$ and $\beta_h \in A_h$ for all h, contradicting limited arbitrage. Limited arbitrage thus implies that $P(E)$ is bounded. The reciprocal is immediate.□

THEOREM 3. *Let E be an economy without bounds on short sales. Then E has a competitive equilibrium if and only it has limited arbitrage.*

Proof. Necessity first. Assume that E has a competitive equilibrium consisting of a price p^*, and an allocation $x_1^*...x_H^*$. If limited arbitrage is not satisfied, then $\exists i$ and $z_i \in A_i$ such that $< p^*, z_i > \leq 0$. Therefore $\forall \lambda > 0$, $\Omega_i + \lambda z_i$ is in trader $i's$ budget set and for some $\lambda > 0$, $u_i(\lambda z_i) > u_i(x_i^*)$, a contradiction: the equilibrium cannot exist. Sufficiency next. I shall use Negishi's theorem to prove existence of a quasiequilibrium, namely a price and an allocation at which markets clear and traders minimize cost subject to their utility levels [14]; this suffices since a quasiequilibrium is a competitive equilibrium when allocations with strictly lower utility are feasible ([2]) as they are here. Negishi's proof applies Kakutani's fixed point theorem on the Pareto frontier and requires that the utility set $U(E)$ be convex, bounded and closed. This is immediate when the indifference surfaces are bounded below, Arrow and Hahn [2]. Therefore I consider only utilities which do not satisfy this condition. Convexity is immediate: see e.g. [2], and boundedness follows from Theorem 2 in the Appendix: therefore I need only establish that $P(E)$ is closed when limited arbitrage is satisfied. Let $\Upsilon = \{(w_1...w_H) \in R^{N\times H} : \sum_{i=1}^H (w_i - \Omega_i) = 0\}$. For any $r \in R^{H+}$ let $v = (v_1...v_H) \in R^{H+} = Sup(U(E) \cap r)$, which exists because $U(E)$ is bounded by Theorem 2 above. Consider a sequence of Pareto efficient utility levels $U^n = (u_1(z_1^n), ...u_H(z_H^n))$ in $P(E)$ converging to v, $(z_1^n...z_H^n) \in \Upsilon$. I shall prove that

$v \in P(E)$. Since $U(E)$ is bounded and utilities are increasing, there exists an allocation $(y_1 \ldots y_H) \in R^{N \times H}$, which may or not be feasible : $v = (u_1(y_1) \ldots u_H(y_H)) = \lim_n (u_1(z_1^n), \ldots u_H(z_H^n))$. By standard arguments, the directions of all the gradients must draw close to each other, i.e. $\forall h = 1 \ldots H$, $\lim_{n,m \to \infty} [(Du_h(z_h^n)/\|Du_h(z_h^n)\|) - (Du_h(z_h^m)/\|Du_h(z_h^m)\|)] = 0$. Let $s_h^n = Du_h(z_h^n)/\|Du_h(z_h^n)\| \in S^{N-1}$, the unit sphere in R^N; by compactness there exist a point of accumulation $s \in R^N$ for $(s_h^n)_{n=1,2\ldots}$ which is common to all h. Since $\forall h$, $u_h(z_h^n) \to v_h$ then $\forall \varepsilon > 0$ $\exists T$ and $w_h^n \in R^N$ such that $u_h(w_h^n) = v_h$ and $\|(Du_h(z_h^n)/\|Du_h(z_h^n)\| - Du_h(y_h^m)/\|Du_h(y_h^m)\|)\| < \varepsilon$ for $n > T$; without loss choose the sequence $\{z^n\}_{n=1,2\ldots} = \{z_1^n \ldots z_H^n\}_{1,2,\ldots}$ so that $\forall n$ and $\forall h$, $z_h^n = u_h^{-1}(v_h)$. By compactness of S^{N-1} there exists a subsequence, denoted also $\{Du_1(z_1^n)/\|Du_1(z_1^n)\|, \ldots, Du_H(z_H^n)/\|Du_H(z_H^n)\|\}_{n=1,2\ldots}$ converging to the common direction $(s \ldots s) \in R^{N \times H}$. By the assumptions on preferences, there exists $z = (z_1 \ldots z_H) \in R^{N \times H}$ such that $z_h \in u_h^{-1}(v_h)$ and $Du_h(z_h) = \lambda_h s$ for some $\lambda_h > 0$. It is now standard to show that $\sum_h (z_h - \Omega_h) = 0$; if the sequence of allocations $(z_1^n \ldots z_H^n)_{n=1,2\ldots}$ is bounded or has a bounded subsequence this is immediate. If not, define the set Q of allocations (which may or not be feasible) attaining the utility level $v_1 \ldots v_H$ and having gradients colinear with s : $Q = \{(q_1 \ldots q_H) \in R^{N \times H} : Du_h(q_h) = \lambda s \text{ for some } \lambda, \text{ and } v_h = u_h(q_h)\}$. Q is not empty, since $z = (z_1 \ldots z_H) \in Q$; Q is a closed, convex, affine subset of R^N. But the sequence of allocations $(z_1^n \ldots z_H^n)$ approaches the set Q as closely as desired, and it is feasible. Therefore Q and Υ are at zero distance, so they intersect, i.e. there exists a feasible allocation $(y_1 \ldots y_H) \in R^{N \times H}$ such that $Du_h(y_h) = \lambda_h s$ and $v_h = u_h(y_h)$, so $P(E)$ is closed.□

PROPOSITION 4. *Let E be an economy without bounds on short sales and $H \geq 3$ traders. E has Condorcet triple of all sizes if and only if it is socially diverse, i.e. if and only if it does not satisfy limited arbitrage.*

Proof. Let E have limited arbitrage. For each $k > 0$, let $(\alpha^k, \beta^k, \gamma^k) \in R^{3 \times N \times H}$ and $u_1^k, u_2^k, u_3^k \subset \{u_1 \ldots u_H\}$ be a Condorcet triple of size k. Without loss assume that $\forall i$, $\Omega_i = 0$, and choose a utility representation : $\forall i$, $Sup_{\{x : x \in R^N\}}(u_i(x)) = \infty$. The three allocations are feasible $\forall k$, e.g. $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k) \in R^{N \times 3}$, $\sum_{i=1}^3 (\alpha_i^k) = 0$, and $\lim_{k \to \infty} (\min_{i=1,2,3} (u_i(\alpha_1^k), u_i(\beta_i^k), u_i(\gamma_i^k)) = \infty$. There exist therefore three traders called $1, 2$, and 3 and a corresponding sequence of allocations $(\theta^k)_{k=1,2\ldots} = (\theta_1^k, \theta_2^k, \theta_3^k)_{k=1,2\ldots}$: $\forall k$, $\sum_i \theta_i^k = 0$ and $\forall i = 1, 2, 3$, $\sup_{k \to \infty} u_i(\theta_i^k) = \infty$. This implies that E has unbounded gains from trade, which contradicts Theorem 6. Therefore E cannot have Condorcet triples of every size.

Conversely, if E has no limited arbitrage, there exist three traders, called $1, 2, 3$, with preferences u_1, u_2, u_3 and three vectors in R^N, $a \in A_1, b \in A_2$, $c \in A_3$, which add up to zero. For any integer $k > 0$, and small $\varepsilon > 0$ consider the vector $\Delta = (\varepsilon, \ldots, \varepsilon) \in R^{N+}$ and the following three allocations: $\alpha^k = (ka, kb - 2\Delta, kc + 2\Delta)$, $\beta^k = (ka - \Delta, kb, kc + \Delta)$ and $\gamma^k = (ka - 2\Delta, kb - \Delta, kc + 3\Delta)$; each allocation is feasible, e.g. $ka + kb - 2\Delta + kc + 2\Delta = k(a + b + c) = 0$. For each $k > 0$ the three allocations $\alpha^k, \beta^k, \gamma^k$ and the three utilities u_1, u_2, u_3 define a Condorcet triple of size $m(k)$, with $\lim_{k \to \infty} m(k) = \infty$.□

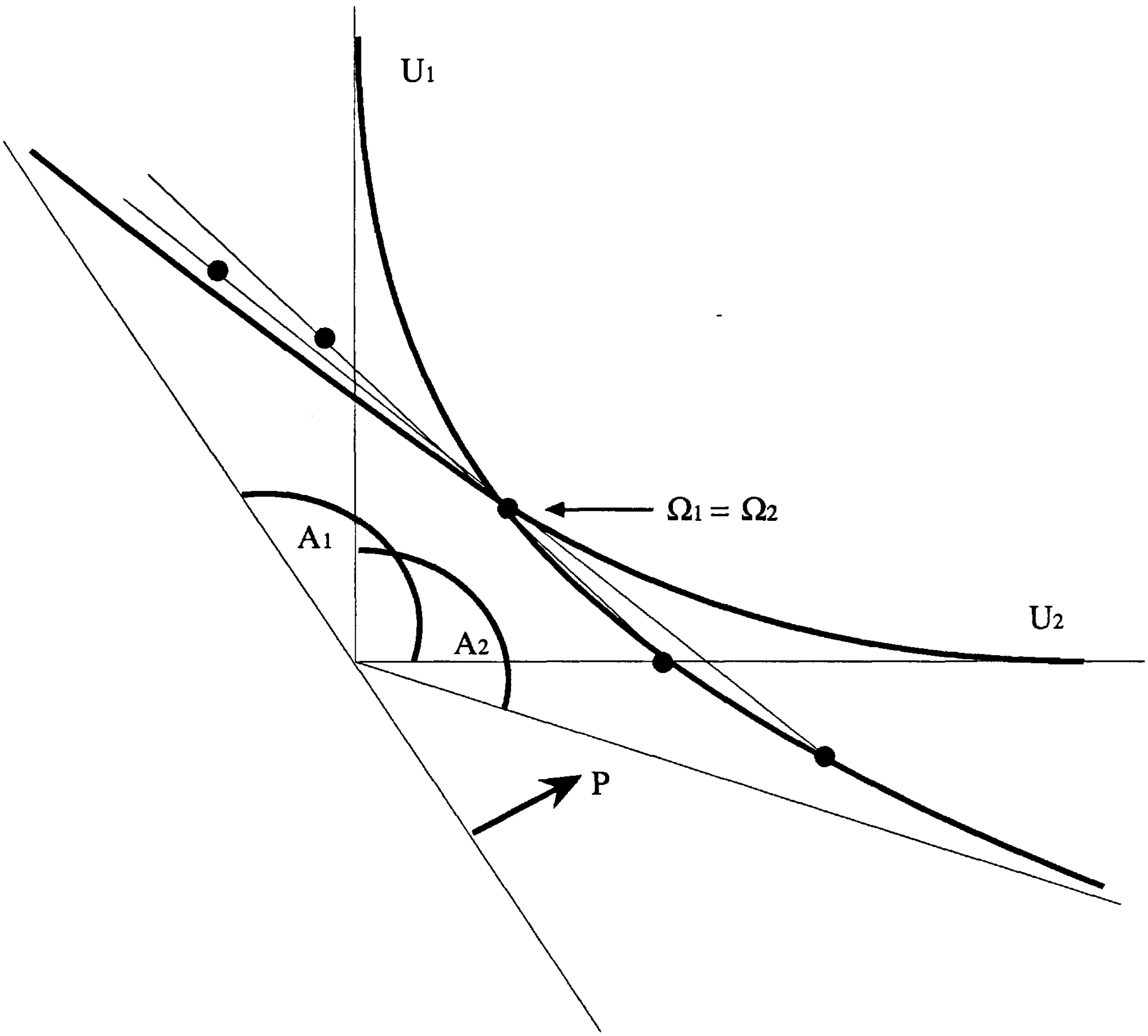

Figure1: limited arbitrage is satisfied. The two global cones lie in the halfspace defined by P. There are no feasible trades that increase utilities without limit: these would consist of pairs of points symmetrically placed about the common initial endowment, and as shown such pairs of points lead to utility values below those of the endowments at a bounded distance from the initial endowments.

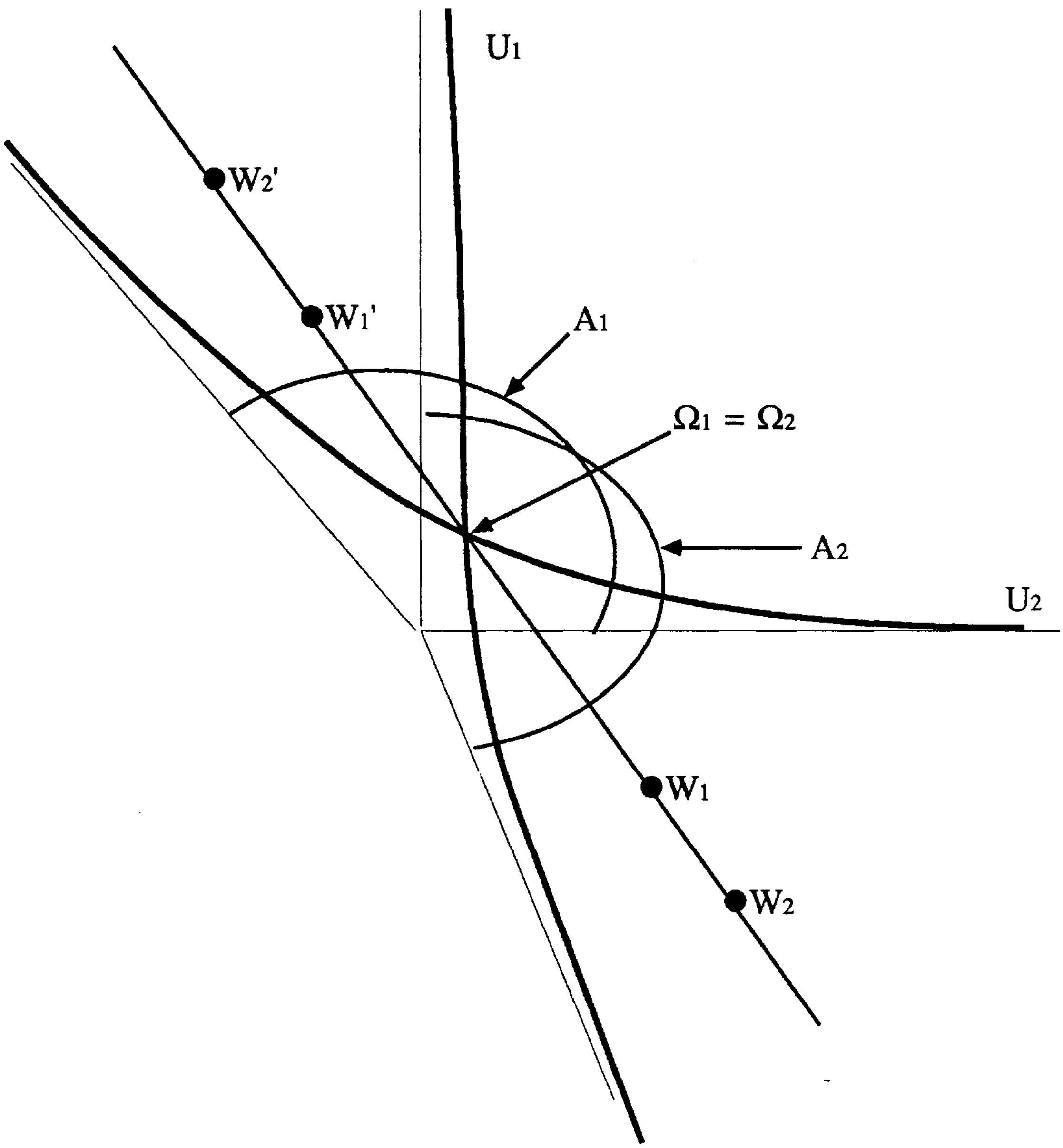

Figure 2: Limited arbitrage does not hold. The global cones are not contained in a half space, and there are sequences of feasible allocations such as W_1 and W_1', W_2 and W_2', which produce unbounded utilities.

Notes.

[1]Global cones and market cones were introduced in Chichilnisky [4], as was the concept of limited arbitage

[2]Utilities are continuous, concave, non-decreasing and satisfy mild regularily conditions; they include Cobb-Douglas and CES utilities, utilities with indifferences which intersect the orthants, linear and partially linear utilities, see the Appendix.

[3]For example, neither a quasiequilibrium nor a compensated equilibrium is generally Pareto efficient, see Arrow and Hahn [2].

[4]Chichilnisky and Heal [12] prove existence in Sobolev spaces, which are Hilbert spaces made of either measurable, continuous or smooth functions.

[5]I.e. $\forall a > 0$ and $\forall x, y \in R^2$, $u_2(x + a, y) = u_2(x, y)$.

[6]I.e. $\forall x, y \in R^2$ and $\forall a > 0, u_1(x + a, y) > u_1(x, y)$.

[7]Each trader has a preference $u_i : R^N \to R$ satisfying the stated conditions, and an endowment $\Omega_i \in R^N$.

[8]Utilities are linear when $\forall \alpha, \beta \in R$, $u_i(\alpha x + \beta y) = \alpha u_i(x) + \beta u_i(y)$.

[9]In the economy E the traders' preferences are defined over private consumption $u_i : R^N \to R$, but they define automatically preferences over allocations in $R^{N\times H}$: $u_i(x_1...x_H) \geq u_i(y_1...y_H) \Leftrightarrow u_i(x_i) \geq u_i(y_i)$.

[10]A space X is contractible when there exists a continuous map $f : X \times [0,1] \to X$ and $x_o \in X$ such that $\forall x$, $f(x, 0) = x$ and $f(x, 1) = x_o$.

[11]Without loss of generality assume that for all i, $Sup_{\{x; x \in R^N\}} u_i(x) = \infty$

[12]Without loss of generality, we normalized utilities so that $\sup_{x \in R^N}(u_h(x)) = \infty$.

[13]By the assumptions on preferences if $\exists (u_1^j...u_H^j)_{j=1,2...}$: $\forall j$, $\sum_{h=1}^H u_h^j = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + u_h^j)) \to \infty$ for some h, $u_h(\Omega_h + u_h^j) \geq u_h(\Omega_h)$ $\forall h$, then $\exists (z_1^j...z_H^j)_{j=1,2...}$: $\forall j$, $\sum_{h=1}^H z_h^j = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for all h.

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