Risk Preference Smoothness and the APT

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Abstract

Attitudes of risk aversion and diversification preference are closely linked to the degree of smoothness of the underlying preference relation over monetary risks, represented in terms of the first and second variations of the preference functional in the direction of incremental risks, evaluated at an initial portfolio. Under general maintained assumptions, "exact" and "approximate" versions of the Arbitrage Pricing Theory (APT) may be derived respectively from first- and second-degree smoothness of risk preferences among investors in a competitive, finite asset market.

Keywords: Preference Smoothness, Risk Aversion, Diversification Preference, Arbitrage Pricing Theory.
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1. Introduction

In myriad different settings, the theory of choice under uncertainty poses the question: How does a decision maker evaluate marginal risks as increments to an initial wealth endowment? In general, an individual's willingness to accept an incremental monetary risk depends on its joint distribution with stochastic initial wealth, as well as the individual's attitudes of risk aversion and diversification preference.

This paper demonstrates that individual attitudes of risk aversion and diversification preference are closely linked with the degree of smoothness of the underlying preference relation over monetary risks, represented in terms of the first and second variations of the preference functional in the direction of incremental risks, evaluated at an initial portfolio. Preferences are smooth to the first degree if the first variation of the preference functional is a continuous linear form over incremental risks which are mean-independent of initial wealth. They are smooth to the second degree if, in addition, the second variation of the preference functional is a continuous quadratic form over zero-mean, mean-independent risk increases. These definitions of smoothness apply generally to any preference relation over monetary risks, and not merely to expected utility preferences.

Under the maintained assumptions of continuity and monotonicity, first-degree risk preference smoothness implies diversification preference, which specifies that a small share of any mean-independent risk increment having positive expected value is utility enhancing. Diversification preference is equivalent to the Arrow-Lind (1970) property that any small, mean-independent risk increment is valued according to its expectation.

Under the same maintained assumptions, second-degree smoothness of risk preference implies the existence of a generalized Arrow-Pratt index of risk aversion. For risk preferences which exhibit second-degree smoothness, the risk premium associated with any small, independent risk increment is proportional to its variance.

The idea of preference smoothness permits a simple, unified treatment of the Arbitrage Pricing Theory (APT) in a perfectly competitive, finite asset market. With general maintained assumptions about investor preferences, first-degree preference smoothness yields the "exact" version of the APT, and second-degree preference smoothness yields an "approximate" version. Thus, the "exact" APT derives essentially from global diversification preference among investors plus aggregate diversifiability of the market portfolio. The stronger, "approximate" version of the APT applies if investor preferences support an Arrow-Pratt index of risk aversion, which permits a variance-based asymptotic bound for deviations of equilibrium asset prices from those predicted by the "exact" APT in a market that is not aggregately diversified.

2. Preliminaries

Monetary risks are identified with the vector space D of bounded random variables defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$, for which $\mathcal{P}$ is an atomless probability measure. The distribution function for a monetary risk $\tilde{x}:D\to\mathbb{R}$ is $F_{\tilde{x}}(x) = \mathcal{P}(\tilde{x}\le x)$. Among monetary risks, $\tilde{x}$ *strictly dominates* $\tilde{y}$ *stochastically to the first degree*, denoted $\tilde{x} >^1 \tilde{y}$, if $F_{\tilde{x}}(z)\le F_{\tilde{y}}(z)$ for all z and $F_{\tilde{x}}(z_o)<F_{\tilde{y}}(z_o)$ for some z_o. Equivalently, $\tilde{x} >^1 \tilde{y}$ if $E(g(\tilde{x})) > E(g(\tilde{y}))$ for every strictly increasing function $g:\mathbb{R}\to\mathbb{R}$ (Brumelle and Vickson, 1975).

Any preference relation $V{:}D \to \mathbb{R}$ is assumed to be complete and transitive. In subsequent discussion, I will also invoke the following basic properties:

Continuity: $V(\tilde{y}) \to V(\tilde{x})$ whenever $\tilde{y} \to \tilde{x}$ in distribution.

Monotonicity: $V(\tilde{y}) > V(\tilde{x})$ whenever $\tilde{y} >^1 \tilde{x}$.

Convexity: If $V(\tilde{x}) > V(\tilde{x}+\tilde{y})$ then $V(\tilde{x}+\epsilon\tilde{y}) > V(\tilde{x}+\tilde{y})$ for all $\epsilon \in (0,1)$.

Pseudo-Convexity: If $V(\tilde{x}) \geq (>)\ V(\tilde{x}+\tilde{y})$ then $V(\tilde{x}) \geq (>)\ V(\tilde{x}+\epsilon\tilde{y})$ for all $\epsilon > 1$.

Continuity in distribution means that the preference relation V is indifferent among identically distributed monetary risks. Thus, continuous (in distribution) preferences are *state-independent* in the sense that they depend only on distributions of final payoffs, but not on the states of nature in which these payoffs occur. Monotonicity is the condition that "more is preferred to less"; while convexity implies that averages are preferred to extremes. Pseudo-convexity (a condition distinct from convexity) simply says that if a given incremental risk is distasteful, then larger portions of the same risk are likewise distasteful.

I next present standard definitions of risk aversion and diversification preference. Hereafter, I adopt the convention that $\tilde{m} \in D$ represents stochastic initial wealth, while $\tilde{x} \in D$ is an incremental monetary risk. $I(\tilde{m}) \subset D$ is the set of monetary risks that are stochastically independent of initial wealth $\tilde{m}$, while $MI(\tilde{m}) \subset D$ is the set of monetary risks that are mean-independent of $\tilde{m}$: $MI(\tilde{m}) = \{\tilde{x} \in D\colon E(\tilde{x}|\tilde{m}) = 0\}$. $ZI(\tilde{m})$ and $ZMI(\tilde{m})$ indicate the subsets of zero-mean monetary risks that are, respectively, independent and mean-independent of $\tilde{m}$: $ZI(\tilde{m}) = \{\tilde{x} \in I(\tilde{m})\colon E(\tilde{x}) = 0\}$ and $ZMI(\tilde{m}) = \{\tilde{x} \in MI(\tilde{m})\colon E(\tilde{x}) = 0\}$. Notice that $MI(\tilde{m})$ and $ZMI(\tilde{m})$ are vector spaces, but that $I(\tilde{m})$ and $ZI(\tilde{m})$ in general are not because stochastic independence is not preserved under addition. However, completion of $I(\tilde{m})$ as a vector space in D with respect to convergence in distribution yields exactly $MI(\tilde{m})$ (Chew and Herk, 1995).

Definition. The preference relation V on D shows *aversion to independent increases in risk* if $V(\tilde{m}+\tilde{x}) \leq V(\tilde{m})$ for all $\tilde{m}\in D$ and $\tilde{x}\in I(\tilde{m})$ with $E(\tilde{x})\leq 0$. Likewise, V shows *aversion to mean-preserving increases in risk* if the same condition applies for all $\tilde{x}\in MI(\tilde{m})$ with $E(\tilde{x})\leq 0$.

Definition. The preference relation V on D shows *diversification preference* if for all $\tilde{m}\in D$ and $\tilde{x}\in MI(\tilde{m})$ with $E(\tilde{x})>0$, there exists $\epsilon^*>0$ such that $V(\tilde{m} + \epsilon\tilde{x})) > V(\tilde{m})$ for all $\epsilon<\epsilon^*$.

Throughout the paper, I will maintain the assumptions that risk preferences are continuous, monotone, and averse to independent risk increases. Hereafter, a *standard* preference relation V on D is one which exhibits these properties. Note that a standard preference relation is not necessarily either convex or pseudo-convex, nor does it necessarily exhibit diversification preference.[1]

Further aspects of individual preference with respect to small risk increments are expressed in terms of the first and second variations of the preference relation V. My usage of this terminology exactly follows standard definitions from the calculus of variations. The first variation of V at initial wealth $\tilde{m}$ in the direction of the incremental risk $\tilde{x}$ is

$$\delta V_{\tilde{m}}(\tilde{x}) = d/d\epsilon \ V(\tilde{m}+\epsilon\tilde{x})|_{0^+}.$$

Similarly, the second variation of V at $\tilde{m}$ in the direction $\tilde{x}$ is

$$\delta^2 V_{\tilde{m}}(\tilde{x}) = d^2/d\epsilon^2 \; V(\tilde{m}+\epsilon\tilde{x})|_{0^+}.$$

The first variation $\delta V_{\tilde{m}}(\cdot)$ can also be referred to naturally as the *marginal utility of incremental risk* at stochastic initial wealth $\tilde{m}$.

Definition. The utility functional V on D is *smooth to the first degree* if for all $\tilde{x}\in D$, $\delta V_{(\cdot)}(\tilde{x})$ is continuous, and for all $\tilde{m}\in D$, $\delta V_{\tilde{m}}(\cdot)$ is a continuous linear functional on $MI(\tilde{m})$.[2]

Definition. The utility functional V on D is *smooth to the second degree* if it is smooth to the first degree; and for all $\tilde{m}\in D$, $\delta^2 V_{\tilde{m}}(\cdot)$ is a continuous quadratic functional on $ZMI(\tilde{m})$.

In the preceding definitions, smoothness of the preference relation V to either the first or second degree is understood to imply that the associated directional derivatives exist. Notice that first-degree smoothness refers to the behavior of $\delta V_{\tilde{m}}$ over incremental risks which are mean-independent of initial wealth $\tilde{m}$; whereas second-degree smoothness refers to the behavior of $\delta^2 V_{\tilde{m}}$ on the smaller subset of mean-independent incremental risks having zero mean.

3. Preference Smoothness

First-degree smoothness of a standard preference relation V implies that the first variation $\delta V_{\tilde{m}}(\tilde{x})$ is proportional to $E(\tilde{x})$ for any incremental risk $\tilde{x}$ which is mean-independent of initial wealth $\tilde{m}$. This in turn implies diversification preference. The following theorem states these results in global terms.

Theorem 1. *Let* V *be a standard preference relation on* D. *If* V *is smooth to the first degree, then:*

(i) $\delta V_{\tilde{m}}(\tilde{x}) = \delta V_{\tilde{m}}(1)E(\tilde{x})$ *for all* $\tilde{m}\in D$ *and* $\tilde{x}\in MI(\tilde{m})$.

(ii) V *exhibits diversification preference.*

Proof. See Chew and Herk (1995).[3]

In Theorem 1(i), $\delta V_{\tilde{m}}(1)$ is the marginal utility of income evaluated at stochastic initial wealth $\tilde{m}$.

If V is a standard preference relation on D which is pseudo-convex, then first-degree smoothness also implies aversion toward all mean-independent increases in risk (Chew and Herk, 1995).

The next theorem is a counterpart to Theorem 1. Second-degree smoothness of a standard preference relation V on D implies that the second variation $\delta^2 V_{\tilde{m}}(\tilde{x})$ is proportional to $E(\tilde{x}^2)$ for any zero-mean, incremental risk $\tilde{x}$ which is independent of initial wealth $\tilde{m}$.

Theorem 2. *Let* V *be a standard preference relation on* D. *If* V *is smooth to the second degree, then* $\delta^2 V_{\tilde{m}}(\tilde{x}) = \delta^2 V_{\tilde{m}}(\tilde{1}_{\tilde{m}})E(\tilde{x}^2)$ *for all* $\tilde{x}\in ZI(\tilde{m})$, *where* $\tilde{1}_{\tilde{m}}$ *denotes any risk in* $ZI(\tilde{m})$ *for which* $E(\tilde{1}_{\tilde{m}}^{\,2}) = 1$.

Proof: Let $V:D\to\mathbb{R}$ be a standard preference relation that is smooth to the second degree. Notice that D is a subspace of $\mathcal{L}^2(\Omega,\mathcal{F},\mathcal{P})$. By the Riesz representation theorem, there exists a measurable function $g_{\tilde{m}} \in \mathcal{L}^2(\Omega,\mathcal{F},\mathcal{P})$ such that

$$\delta V_{\tilde{m}}(\tilde{x}) = \int g_{\tilde{m}}(\omega)\tilde{x}(\omega)\,d\mathcal{P}(\omega)$$

for all $\tilde{x}\in ZMI(\tilde{m})$. Since V is state-independent, $g_{\tilde{m}}$ admits the representation $g_{\tilde{m}}(\omega) = g(\tilde{m}(\omega))$ for some measurable function $g:\mathbb{R}\to\mathbb{R}$. Consequently,

$\delta^2 V_{\tilde{m}}(\tilde{x}) = E[g(\tilde{m})]E(\tilde{x}^2)$ for $\tilde{x} \in ZI(\tilde{m})$.

Let $\tilde{1}_{\tilde{m}}$ be any monetary risk in $ZI(\tilde{m})$ for which $E(\tilde{1}_{\tilde{m}}^2) = 1$. Substituting $\tilde{1}_{\tilde{m}}$ for $\tilde{x}$ in the expression for $\delta^2 V_{\tilde{m}}(\tilde{x})$ establishes that $E[g(\tilde{m})] = \delta^2 V_{\tilde{m}}(\tilde{1}_{\tilde{m}})$, from which the theorem follows. *Q.E.D.*

The *certainty equivalent* $CE_{\tilde{m}}(\tilde{x}) \in \mathbb{R}$ is the nonstochastic increment to initial wealth $\tilde{m}$ which is indifferent to the incremental risk $\tilde{x}$. It is defined by

$$V(\tilde{m} + \tilde{x}) = V(\tilde{m} + CE_{\tilde{m}}(\tilde{x})) . \tag{1}$$

Equation (1) has a unique solution because monotonicity of V implies nonsatiation. The certainty equivalent $CE_{\tilde{m}}$ is itself a utility function on D which expresses the same preferences as V, subject to the normalization $CE_{\tilde{m}}(c) = c$ for every scalar c. Consequently, $CE_{\tilde{m}}$ inherits all of V's properties with respect to continuity, monotonicity, convexity, pseudo-convexity, risk aversion, and diversification preference. The following elementary lemma shows that $CE_{\tilde{m}}$ also inherits V's properties of smoothness.

Lemma 1. *Let* V *be a standard preference relation on* D.

(i) *If* V *is smooth to the first degree, then so is* $CE_{\tilde{m}}$ *for any* $\tilde{m} \in D$, *and* $\delta CE_{\tilde{m}}(\tilde{x}) = \delta V_{\tilde{m}}(\tilde{x})/\delta V_{\tilde{m}}(1) = E(\tilde{x})$ *for all* $\tilde{x} \in MI(\tilde{m})$.

(ii) *If* V *is smooth to the second degree, then so is* $CE_{\tilde{m}}$ *for any* $\tilde{m} \in D$, *and* $\delta^2 CE_{\tilde{m}}(\tilde{x}) = \delta^2 V_{\tilde{m}}(\tilde{x})/\delta V_{\tilde{m}}(1) = [\delta^2 V_{\tilde{m}}(\tilde{1}_{\tilde{m}})/\delta V_{\tilde{m}}(1)]E(\tilde{x}^2)$ *for all* $\tilde{x} \in ZI(\tilde{m})$.

Proof: Differentiating both sides of (1) yields

$$\delta CE_{\tilde{m}}(\tilde{x}) = \delta V_{\tilde{m}}(\tilde{x})/\delta V_{\tilde{m}}(1) .$$

Differentiating a second time yields

$$\delta^2 CE_{\tilde{m}}(\tilde{x}) = \frac{\delta^2 V_{\tilde{m}}(\tilde{x}) - \delta^2 V_{\tilde{m}}(1)(\delta CE_{\tilde{m}}(\tilde{x}))^2}{\delta V_{\tilde{m}}(1)} .$$

The lemma is proved by substituting in both expressions according to Theorems 1 and 2. *Q.E.D.*

Taken together, Theorem 1 and Lemma 1(i) show that first-degree smoothness of a standard preference relation implies that for all $\tilde{m} \in D$ and $\tilde{x} \in MI(\tilde{m})$,

$$CE_{\tilde{m}}(\epsilon \tilde{x}) = E(\tilde{x})\epsilon + o(\epsilon), \tag{2}$$

where $o(\epsilon)$ indicates a residual that goes to zero faster than ϵ. Equation (2) is precisely the Arrow-Lind (1970) property, which says that any small, mean-independent risk increment added to initial wealth is valued according to its expectation. The Arrow-Lind property implies diversification preference; in fact, both properties are equivalent for any standard preference relation V for which $\delta V_{\tilde{m}}$ is continuous (see footnote 3).

Theorem 2 with Lemma 1(ii) yields a companion result: Second-degree smoothness of a standard preference relation implies that the certainty equivalent of any independent risk increment has a second-order Taylor expansion in which the first-order term is the expected value of the incremental risk and the second-order term is proportional to its variance. This result is stated formally in the following theorem:

Theorem 3. *Let* V *be a standard preference relation on* D*. If* V *is smooth to the second degree, then for all* $\tilde{m} \in D$ *and* $\tilde{x} \in I(\tilde{m})$,

$$CE_{\tilde{m}}(\epsilon \tilde{x}) = E(\tilde{x})\epsilon - \rho_{\tilde{m}} \operatorname{var}(\tilde{x})(\epsilon^2/2) + o(\epsilon^2) , \tag{3}$$

where $\rho_{\tilde{m}} = -(\delta^2 V_{\tilde{m}}(\bar{1}_{\tilde{m}})/\delta V_{\tilde{m}}(1))$.

Proof. From (1), it follows directly that

$$CE_{\tilde{m}}(\epsilon\tilde{x}) = E(\tilde{x})\epsilon + CE_{\tilde{m}+\epsilon E(\tilde{x})}(\epsilon[\tilde{x}-E(\tilde{x})]).$$

It is necessary to show that the approximation for $CE_{\tilde{m}+\epsilon E(\tilde{x})}(\epsilon[\tilde{x}-E(\tilde{x})])$ as a second-order Taylor series has the desired form. Observe that

$$d/d\epsilon\ CE_{\tilde{m}+\epsilon E(\tilde{x})}(\epsilon[\tilde{x}-E(\tilde{x})])|_{0^+} = d/d\epsilon\ CE_{\tilde{m}}(\epsilon[\tilde{x}-E(\tilde{x})])|_{0^+}$$
$$+\ d/d\epsilon\ CE_{\tilde{m}+\epsilon E(\tilde{x})}(0)|_{0^+}\ .$$

Since $CE_{\tilde{m}+\epsilon E(\tilde{x})}(0) = 0$ for all ϵ,

$$d/d\epsilon\ CE_{\tilde{m}+\epsilon E(\tilde{x})}(\epsilon[\tilde{x}-E(\tilde{x})])|_{0^+} = d/d\epsilon\ CE_{\tilde{m}}(\epsilon[\tilde{x}-E(\tilde{x})])|_{0^+} = 0\ ,$$

and also

$$d^2/d\epsilon^2\ CE_{\tilde{m}+\epsilon E(\tilde{x})}(\epsilon[\tilde{x}-E(\tilde{x})])|_{0^+} = d^2/d\epsilon^2\ CE_{\tilde{m}}(\epsilon[\tilde{x}-E(\tilde{x})])|_{0^+}$$
$$= [\delta^2 V_{\tilde{m}}(\tilde{1}_{\tilde{m}})/\delta V_{\tilde{m}}(1)]\mathrm{var}(\tilde{x})\ ,$$

where the second equalities in both instances rely on Lemma 1. *Q.E.D.*

In the Taylor expansion (3) of $CE_{\tilde{m}}(\epsilon\tilde{x})$, the second-order coefficient $\rho_{\tilde{m}} = -(\delta^2 V_{\tilde{m}}(\tilde{1}_{\tilde{m}})/\delta V_{\tilde{m}}(1))$ is simply a generalized Arrow-Pratt index of risk aversion. For example, for expected utility preferences, $V(\tilde{z}) = E(u(\tilde{z}))$ with $u(\cdot) \in \mathbb{C}^2(\mathbb{R})$, $\delta V_{\tilde{m}}(1) = E(u'(\tilde{m}))$ and $\delta^2 V_{\tilde{m}}(\tilde{1}_{\tilde{m}}) = E(u''(\tilde{m}))$. This yields the familiar result that under expected utility, the Arrow-Pratt index of risk aversion at initial wealth $\tilde{m}$ is simply $\rho_{\tilde{m}} = -E(u''(\tilde{m}))/E(u'(\tilde{m}))$. In essence, Theorem 3 shows that there exists an Arrow-Pratt index of risk aversion for any standard preference relation that is smooth to the second degree.

4. Application: Arbitrage Pricing Theory

4.1. *Basic Structure*

In the Arbitrage Pricing Theory (APT), originated by Ross (1976) and generalized by Connor (1984), Chamberlain (1988), and Milne (1988), among others, prices of financial assets are determined by projecting their stochastic end-of-period values onto a limited set of stochastic factors. The residual from this projection is an idiosyncratic risk whose price is either zero in "exact" versions of the APT such as those cited above, or bounded (and asymptotically negligible) in "approximate" versions (Dybvig, 1983; Grinblatt and Titman, 1983). In either its "exact" or "approximate" variants, the power of the APT depends essentially on its ability to estimate closely the equilibrium prices of idiosyncratic risks in perfectly competitive but incomplete asset markets.

In this section, I present a streamlined model of a finite asset exchange economy conforming to standard structural assumptions of the APT. Within this framework, I show that the "exact" APT follows from first-degree preference smoothness (diversification preference) among investors, while the stronger "approximate" APT obtains if investor preferences are smooth to the second degree.

As elsewhere in the paper, financial assets are identified with elements of D, the vector space of bounded monetary risks defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$, where $\mathcal{P}$ is an atomless probability measure.

Consider a pure exchange economy which trades portfolios constituted from a finite collection of elementary assets $\{\tilde{z}_0,\tilde{z}_1,\ldots,\tilde{z}_t\}$ under conditions of perfect competition. The set of tradable portfolios is $T = [\tilde{z}_j]_{j=0}^{t} \cap D$, the linear manifold generated from the elementary assets and restricted to D.

Asset portfolios are traded among a finite number of consumers, each of whom has standard preferences over D which are also convex and pseudo-convex.[4] The $i\underline{th}$ consumer has an initial asset endowment $\tilde{a}^i \in T$, from which the aggregate endowment of the exchange economy is $\sum_i \tilde{a}^i \in T$.

The structure of the asset exchange economy described above closely follows Milne (1988), who proves the following fundamental results:

(1) There exists a competitive equilibrium for the asset exchange economy with equilibrium pricing functional $\pi : T \to \mathbb{R}$.

(2) Letting $\tilde{z}^i \in T$ denote the equilibrium portfolio of consumer i, the competitive equilibrium allocation of asset holdings $\{\tilde{z}^i\}$ over all consumers constitutes a Pareto optimum relative to the aggregate endowment $\sum_i \tilde{a}^i$.

(3) As a consequence of Pareto optimality, the equilibrium pricing functional π is additive on T (there are no arbitrage opportunities).

For simplicity, I assume that asset 0 is riskless, with $\mathcal{P}(\tilde{z}_0 = 1) = 1$. The risk-free asset may be treated as a numeraire by means of the (inessential) normalization $\pi(\tilde{z}_0) = 1$. With this normalization, traded assets can be regarded as "pure" risks, from which the element of time relating to a holding period has been entirely removed.

4.2 *"Exact" and "Approximate" Versions of the APT*

The APT with a finite number of traded assets imposes several additional assumptions concerning the structure of asset values.

Assumption (Factor Structure). Every elementary traded asset $\tilde{z}_j$, $j=1,\ldots,t$ obeys the factor structure

$$\tilde{z}_j = E(\tilde{z}_j) + \beta_{j1}\tilde{\gamma}_1 + \ldots + \beta_{jK}\tilde{\gamma}_K + \tilde{\xi}_j ,$$

where $\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_K\} \subseteq D$ is a collection of common factors, $\beta_{j1},\ldots,\beta_{jK}$ are factor sensitivities of the $j\underline{th}$ elementary asset's end-of-period value, and $\tilde{\xi}_j \in D$ is an idiosyncratic risk specific to the $j\underline{th}$ elementary asset.

By construction, each factor and elementary idiosyncratic risk has zero expected value. In the following, I refer to the linear manifolds generated from the factors and elementary idiosyncratic risks as $F = [\tilde{\gamma}_k]_{k=1}^{K} \cap D$ and $I = [\tilde{\xi}_j]_{j=1}^{t} \cap D$.

Assumption (Independent Idiosyncratic Risks). Elementary idiosyncratic risks $\{\tilde{\xi}_1,\ldots\tilde{\xi}_t\}$ are jointly independent of the factors $\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_K\}$. (Equivalently, every element of the factor space F is independent of every element of the idiosyncratic space I.)[5]

Any tradeable portfolio $\tilde{z} \in T$ may be separated into components from the factor and idiosycratic spaces via $\tilde{z} = \tilde{m} + \tilde{\xi}$, with $\tilde{m} \in F$ and $\tilde{\xi} \in I$. Because of risk aversion, every consumer prefers a *diversified* portfolio $\tilde{m} \in F$ to an *undiversified* portfolio $\tilde{m} + \tilde{\xi}$ which incorporates some nonzero element of idiosyncratic risk $\tilde{\xi} \in I$.

Assumption (Individual Diversifiability). There is a tradeable portfolio corresponding to each factor: $\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_K\} \subseteq T$.

Individual diversifiability implies $F \subseteq T$ and hence $I \subseteq T$. Thus, any element of the factor space F added to any element of the idiosyncratic space I yields a tradeable portfolio, whence $T = F \oplus I$.

The following assumption is optional in the sense that it is imposed in "exact" APT and waived in the "approximate" APT.

Assumption (Aggregate Diversifiability). The aggregate asset endowment is diversified: $\sum_i \tilde{a}^i \in F$.

Theorems 4 and 5 present "exact" and "approximate" versions of the APT. The "exact" APT proceeds from the assumption that the aggregate asset endowment is diversified. If consumers are diversifiers (exhibit first-degree preference smoothness), then in a competitive equilibrium, every consumer holds a diversified portfolio and the price of every idiosyncratic risk is zero. The "approximate" APT assumes that the aggregate asset endowment is not diversified: that is, there exists a positive net supply of (undiversified) idiosyncratic risk. If consumers exhibit second-degree preference smoothness, then in a competitive equilibrium, every consumer holds a positive amount of idiosyncratic risk and the price of any small idiosyncratic risk is bounded away from zero in proportion to its covariance with consumers' equilibrium portfolios.

Theorems 4 and 5 maintain the assumptions that consumers' risk preferences are continuous, monotonic, convex and pseudo-convex, and averse to independent risk increases.

Theorem 4. **("Exact" APT)** *Suppose that the aggregate asset endowment is diversified. If consumers' risk preferences are smooth to the first degree, then in a competitive equilibrium:*

(i) *For every consumer* i, $\tilde{z}^i \in F$.

(ii) *For all* $\tilde{\xi} \in I$, $\pi(\tilde{\xi}) = 0$.

Proof: Observe that for any risk $\tilde{z} \in T$, $(\tilde{z} - \pi(\tilde{z})) \in T$ is self-financing. As in Chen and Ingersoll (1983), (ii) will be true if at least one consumer holds a diversified portfolio. At this consumer's optimal portfolio $\tilde{z}^i \in F$, it must be true that $\delta CE_{\tilde{z}^i}(\tilde{\xi} - \pi(\tilde{\xi})) = 0$; since otherwise this consumer is not satiated at $\tilde{z}^i$ with respect to the incremental risk $(\tilde{\xi} - \pi(\tilde{\xi}))$. Since this incremental risk is mean-independent of $\tilde{z}^i$, first-degree preference smoothness implies $\delta CE_{\tilde{z}^i}[\tilde{\xi} - \pi(\tilde{\xi})] = -\pi(\tilde{\xi})$, whence $\pi(\tilde{\xi}) = 0$.

It remains, therefore, to prove (i). Recall that aversion to indepedent risk increases together with first-degree preference smoothness imply aversion to mean-independent risk increases. Suppose to the contrary that a certain consumer i holds an undiversified equilibrium portfolio $\tilde{z}^i = \tilde{m}^i + \tilde{\xi}^i$ for which $\tilde{m}^i \in F$, $\tilde{\xi}^i \in I$, and $\tilde{\xi}^i \neq 0$. Since aggregate diversification of the market portfolio implies $\sum_i \tilde{\xi}^i = 0$, consumer i's undiversified portfolio is inconsistent with Pareto optimality. (See Milne (1988), Lemma 7). *Q.E.D.*

Theorem 5. ("Approximate" APT) *Suppose that the aggregate asset endowment is not diversified. If consumers' risk preferences are smooth to the second degree, then in a competitive equilibrium:*

(i) For every consumer i, $\tilde{z}^i = \tilde{m}^i + \epsilon^i \tilde{\xi}^i$, *with* $\tilde{m}^i \in F$, $\tilde{\xi}^i \in I \setminus \{0\}$ *and* $\epsilon^i > 0$.

(ii) For all $\tilde{\xi} \in I$, $\pi(\tilde{\xi}) = -\rho_{\tilde{m}^i} \operatorname{cov}(\tilde{\xi}^i, \tilde{\xi}) + \hat{r}(\epsilon^i)$, *where* $\rho_{\tilde{m}^i}$ *is the Arrow-Pratt index of risk aversion for consumer* i *evaluated at* $\tilde{m}^i$, *and* $\hat{r}(\epsilon^i)$ *is a remainder for which* $\hat{r}(\epsilon^i) \to 0$ *as* $\epsilon^i \to 0$.

Proof: Part (i) follows from the same arguments used to prove Theorem 4. If to the contrary some consumer holds a diversified portfolio in equilibrium, then $\pi(\tilde{\xi}) = 0$ for all $\tilde{\xi} \in I$, and hence every consumer holds a diversified equilibrium portfolio. But this is impossible because the aggregate endowment is not diversified.

Part (ii) may be proved as follows. In a competitive equilibrium, suppose that consumer i with preference relation V chooses the optimal portfolio $\tilde{z}^i = \tilde{m}^i + \epsilon^i\tilde{\xi}^i$, where $\tilde{m}^i \in F$, $\tilde{\xi}^i \in I\backslash\{0\}$ and $\epsilon^i > 0$. For any $\tilde{\xi} \in I$ and $\vartheta > 0$, $\epsilon^i\tilde{\xi}^i + \vartheta\tilde{\xi} \in I$. According to Theorem 3,

$$\begin{aligned} CE_{\tilde{m}^i}(\epsilon^i\tilde{\xi}^i + \vartheta\tilde{\xi}) \quad = \quad & E(\epsilon^i\tilde{\xi}^i + \vartheta\tilde{\xi}) \\ & - (\rho_{\tilde{m}^i}/2)\left[\operatorname{var}(\tilde{\xi}^i)(\epsilon^i)^2 + \operatorname{var}(\tilde{\xi})\vartheta^2 + 2\operatorname{cov}(\tilde{\xi}^i,\tilde{\xi})\epsilon^i\vartheta\right] \\ & + r(\epsilon^i,\vartheta) \ , \end{aligned}$$

where $\rho_{\tilde{m}^i} = -\delta^2 V_{\tilde{m}^i}(\tilde{1}_{\tilde{m}^i})/\delta V_{\tilde{m}^i}(1)$ is the Arrow-Pratt index of risk aversion for consumer i evaluated at the diversified component $\tilde{m}^i$ of the consumer's equilibrium portfolio $\tilde{z}^i$, and $r(\epsilon^i,\vartheta)$ is a remainder. Differentiating both sides of this expression with respect to ϑ at zero gives

$$\partial/\partial\vartheta \ CE_{\tilde{m}^i}(\epsilon^i\tilde{\xi}^i + \vartheta\tilde{\xi})|_{0^+} \quad = \quad E(\tilde{\xi}) - \rho_{\tilde{m}^i}\operatorname{cov}(\tilde{\xi}^i,\tilde{\xi})\epsilon^i + \hat{r}(\epsilon^i) \ , \tag{4}$$

with $\hat{r}(\epsilon^i) = \partial/\partial\vartheta \ r(\epsilon^i,0)$. Because $\tilde{z}^i$ is an optimal portfolio, (4) equals zero for every $\tilde{\xi} \in I$ for which $\pi(\tilde{\xi}) = 0$. Therefore, replacing $\tilde{\xi} \in I$ by $(\tilde{\xi} - \pi(\tilde{\xi}))$ in (4) and rearranging terms yields

$$\pi(\tilde{\xi}) = -\rho_{\tilde{m}^i}\operatorname{cov}(\tilde{\xi}^i,\tilde{\xi}) + \hat{r}(\epsilon^i) \ .$$

It remains to show that $\hat{r}(\epsilon^i) \to 0$ as $\epsilon^i \to 0$. First, observe that for any $\tilde{m},\tilde{x},\tilde{z} \in D$ and $\epsilon,\vartheta > 0$, $V(\tilde{m} + CE_{\tilde{m}}(\epsilon\tilde{x} + \vartheta\tilde{z})) = V(\tilde{m} + \epsilon\tilde{x} + CE_{\tilde{m}+\epsilon\tilde{x}}(\vartheta\tilde{z}))$. Consequently,

$$\delta V_{\tilde{m}}(1)\ \partial/\partial\vartheta \ CE_{\tilde{m}}(\epsilon\tilde{x} + \vartheta\tilde{z})|_{0^+} \quad = \quad \delta V_{\tilde{m}+\epsilon\tilde{x}}(1)\ \delta CE_{\tilde{m}+\epsilon\tilde{x}}(\tilde{z}),$$

and

$$\begin{aligned} \lim_{\epsilon\downarrow 0} \ \partial/\partial\vartheta \ CE_{\tilde{m}}(\epsilon\tilde{x} + \vartheta\tilde{z})|_{0^+} \quad & = \quad \lim_{\epsilon\downarrow 0} \delta CE_{\tilde{m}+\epsilon\tilde{x}}(\tilde{z}) \\ & = \quad \delta CE_{\tilde{m}}(\tilde{z}) \ . \end{aligned} \tag{5}$$

The last pair of equalities relies on the continuity of $\delta V_{(\cdot)}(\tilde{y})$ (and hence also of $\delta CE_{(\cdot)}(\tilde{y})$) for any given $\tilde{y} \in D$.

Applying (5) to (4) establishes that

$$\lim_{\epsilon \downarrow 0} \hat{r}(\epsilon^{i}) = \lim_{\epsilon \downarrow 0} \delta CE_{\tilde{m}^{i}}(\tilde{\xi}) - E(\tilde{\xi}) = 0$$

for all $\tilde{\xi} \in I$. *Q.E.D.*

Remarks: (1) The proof of "exact" APT (Theorem 4) remains valid even if the elementary idiosyncratic risks $\tilde{\xi}_j$, j=1,...,t, are *individually* independent (or even mean-independent) of the factors $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_K\}$. This weaker assumption is enough to insure that every idiosyncratic risk is mean-independent of every diversified portfolio. Geanokoplos and Oh (1991) prove the "exact" APT with mean-independent idiosyncratic risks. However, the "approximate" APT (Theorem 5) requires the stronger assumption that elementary idiosyncratic risks are *jointly* independent of the factors, which implies that idiosyncratic risks are independent of diversified portfolios.

(2) The pricing equation in the "approximate" APT is reminiscent of the security market line in the Capital Asset Pricing Model (CAPM). In the "approximate" APT, the equilibrium price of an arbitrary idiosyncratic risk is approximated according to its correlation with idiosyncratic risks actually held in consumers' equilibrium portfolios.

(3) Grinblatt and Titman (1983) prove a version of the "approximate" APT like that in Theorem 5, but under more restrictive assumptions. In particular, they assume that consumers have expected utility preferences, and that elementary idiosyncratic risks are jointly independent *of each other* as well as the factors.

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Notes

1. Convexity and psuedo-convexity play no role in the theorems of this section concerning preference smoothness, and hence do not appear in my definition of standard preferences. I have enumerated these properties together with other basic preference assumptions because they will be important in the derivation of the APT in Section 4.

2. First-degree preference smoothness imposes stronger conditions than those in Chew and Herk (1995), where $\delta V_{\tilde{m}}(\tilde{x})$ is assumed to be continuous with respect to the incremental risk $\tilde{x}$, but not necessarily with respect to initial wealth $\tilde{m}$.

3. Chew and Herk prove that if $\delta V_{\tilde{m}}(\cdot)$ exists and is continuous on D, then conditions (i) and (ii) are equivalent to each other, and also to first-degree smoothness of V.

4. Thus, each consumer's risk preferences are continuous, monotonic, convex and pseudo-convex, and averse to independent risk increases.

5. On the significance of joint independence of the factors and elementary idiosyncratic risks, see Remark (1) following the proof of Theorem 5.

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