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Abstract

This paper investigates the empirical performance of a new class of uninsurable risk models in the context of UK indexed bond market. Using closed form expressions for pricing kernels, we test the ability of three consumption-based models to price indexed bonds in the UK, and find that the standard general equilibrium, complete markets model is soundly rejected in favour of two uninsurable-risk models. Using the estimated bond price equation, impulse response analysis is undertaken to understand the effects of three macroeconomic fundamental shocks on real interest rates. In contrast to the estimates that typically arise in equity markets, the estimated coefficient of relative risk aversion is found to be small in this class of models with uninsurable risk.

- Keywords:uninsurable consumption risk, incomplete market, inflation indexed bond, impulse response, VAR
- JEL Codes:E21, G12, G17

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1 Introduction

The effect of uninsurable risk on financial asset prices has generated increasing interest among financial economists in recent years (Constantinides and Duffie, 1996, Kocherloakota and Pistaferri, 2007, 2009, Basu et al., 2011). While the majority of research in this area has focused on linking macroeconomic factors to stock prices and exchange rates, less attention has been paid to explaining bond prices and the term structure of real and nominal yields until recent years. Recent macro-finance papers on bonds include Piazzesi and Schneider (2007), Eraker (2008) and Rudebusch and Swanson (2012) who explore the implications of Epstein and Zin (1991) preferences for important bond market variables including the term premium.[1]However, all these papers are based on a complete market/ full risk sharing paradigm.

The aim of this paper is to derive the implications of uninsurable risks for the bond pricing and real interest rate behavior. To the best of our knowledge, this is the first paper that attempts to make such a connection between uninsurable risk and the inflation indexed bond market and the underlying real interest rates.[2] While bond prices and the associated real interest rates are arguably the most fundamental bond market aggregates, the difficulty in measuring the expected inflation poses challenge in measuring the *ex ante* real interest rate. A partial solution to this problem is provided by inflation-indexed bonds since the real value of their expected coupons is almost wholly independent of expected inflation. A number of papers have attempted to model real rates using such bonds, including Barr and Campbell (1997) and Piazzesi and Schneider (2007). Of these, Piazzesi

[1] See Rudebusch (2010) for an excellent survey of three types of macro-finance models employed in the recent literature to explain the relation between macro economic variables and the term structure.

[2] A huge literature exists in exploring the implication of incomplete markets for asset pricing puzzles (see for example, Heaton and Lucas (1995), Thelmer (1993) and by others). but they do not explicitly deal with inflation indexed bonds and the underlying term structure of real interest rates. Basu et al. (2011) investigate the implications of uninsurable risk to address various financial market puzzles including the risk premium puzzle and the risk free rate puzzle. However, they do not explore the implications of uninsurable risk for term structure of real interest rates which are the major focus of this paper.

and Schneider (2007) are closest to the present paper in that they link real rates to consumption growth via the usual Euler equation. Our paper differs from Piazzesi and Schneider (2007) in several respects. First, Piazzesi and Schneider (2007) explore the implications of aggregate risk while our endeavour is to understand the role of idiosyncratic uninsurable consumption risk for bond price behavior, which is the central aim of this paper. Second, their paper employs a two-variable VAR in inflation and consumption growth while our VAR includes these variables with an additional variable to reflect uninsurable risk. Third, while Piazzesi and Schneider (2007) calibrate the two parameters of their asset pricing model to the short and long ends of an estimated nominal yield curve, we estimate the parameters by maximum likelihood based on fitting the model to market prices of bond.

As in Basu et al. (2011) the pricing kernels are derived using a lognormal process for the cross-sectional distribution of consumption which has strong empirical validity.[3] The bond price is thus a log linear function of three state variables whose expected future values are constructed from a vector autoregression. The lognormality of the consumption process yields a simple analytical form for bond prices in the tradition of affine yield curve models.[4] Two of the macroeconomic state variables, namely aggregate consumption growth and the cross sectional log variance of consumption (which represents the uninsurable risk) come directly from the underlying equilibrium model. A third factor, inflation, is included because we are fitting the market prices of indexed bonds, which are likely to depend on expected inflation due to

[3] Battistin et al. (2009) establish that the cross-sectional consumption distribution in both the US Consumer Expenditure Survey (CEX) and the UK British Family Expenditure Survey (FES) is approximately log normal within demographically homogeneous groups. This is due to the fact that the Gibrat's law applies to consumption. Brzozowski et al. (2009) provide further empirical evidence that the cross-sectional distribution of consumption within cohort groups in Canada may be approximated by a log-normal distribution. Blundell and Lewbel (1999) also provide powerful empirical evidence of log-normality of the cross-sectional distribution of consumption in a variety of data sets. Attanasio et al. (2004) assume log-normality of the cross-sectional distribution of household consumption when studying the evolution of inequality in consumption in the US both within cohorts and for the all population.

[4] See for example, Campbell et al. (1997, Ch. 11) for a comprehensive exposition of this class of affine yield models.

the imperfect nature of their indexation. The estimated VAR is then used to provide the expectation proxies that allow us to estimate the structural parameters (agent's risk aversion and the time preference) by fitting the closed form price equation to market data.

Our estimated model of bond pricing is consistent with the common finding that the standard complete market model performs poorly against the data while the models with uninsurable risk fare better. We use the estimates of the real yield curve provided by the Bank of England as a check on the plausibility of the yields implied by our estimated bond price equation. Although there are discrepancies between the Bank's rates and ours, particularly at the short end of the curve, the incomplete market models come closer to the Bank estimates than does the standard RA model. This suggests that uninsurable risks matter for the long term real interest rates. However, our estimated coefficient of risk aversion is found to be too small which is indicative of the standard limitation of the expected utility models in replicating the empirical bond premium as pointed out by Rudebusch and Swanson (2012).

Finally, an impulse response analysis based on our three factor model reveals that a rise in inflation lowers real interest rates of nearly all maturities (due to the news it carries about future consumption); a rise in economic growth raises real interest rates, which is consistent with the permanent income hypothesis, and a greater uninsurable consumption risk lowers real rates as one predicts from standard theory of precautionary savings. A similar impulse response analysis for US real rates was performed by Piazzesi and Schneider (2007) although they did not explore the impulse response with respect to cross section consumption variance which is our measure of uninsurable risk.

The paper is organized as follows. The following section lays out the basic setup for the three pricing kernels. Section 3 presents the applications of these pricing kernels to UK indexed-bond prices. Section 4 discusses the estimation methods and the data. Section 5 presents the estimation results. Section 6 concludes and suggests areas for future research.

2 Theoretical models

2.1 Three Pricing Kernels

Our benchmark case is the traditional complete market model with homogeneous agents. With a power utility function (with risk aversion parameter γ), the stochastic discount factor is given by:

$$M_{t+1}^{RA} = \frac{\beta c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \tag{1}$$

where β is the subjective discount factor and c_t is the aggregate consumption at date t.

In two influential papers (2007, 2009) Kocherlakota and Pistaferri (K-P hereafter) introduce consumer heterogeneity and uninsurable risk for two distinct market environments: (i) incomplete market (INC) where private skill shocks are uninsurable, (ii) partial insurance environment where the private skill shocks are partially insured by an insurance company who stipulate long term contracts with agents subject to a truth revelation constraint for eliciting efforts and private skill shocks. The latter environment is constrained Pareto efficient and K-P call it private information Pareto optimal ($PIPO$) environment.

Using the law of large numbers K-P demonstrate that the pricing kernels for these two market environments can be written as:

$$M_{t+1}^{INC} = \frac{\beta \sum_i c_{it+1}^{-\gamma} prob(i)}{\sum_i c_{it}^{-\gamma} prob(i)} \tag{2}$$

$$M_{t+1}^{PIPO} = \frac{\beta \sum_i c_{it}^{\gamma} prob(i))}{\sum_i c_{it+1}^{\gamma} prob(i)} \tag{3}$$

where c_{it} is the consumption of individual i at date t and $prob(i)$ is the cross sectional probability of the occurrence of the $i$the household in the population.

2.2 Lognormal Parameterization of the Consumption Process

In a similar spirit as in Constantinides and Duffie (1996), we consider a lognormal parameterization of the post-trade consumption process. We represent the *post-trade* allocation of consumption as follows.[5] The ith investor's consumption is:

$$c_{i,t} = \delta_{i,t}.c_t \tag{4}$$

where $\delta_{i,t}$ is the ith investor's share in aggregate consumption , c_t. This specification basically means that the log of individual consumption is the sum of the log of aggregate per capita consumption and the uninsurable consumption due to the idiosyncratic uninsurable skill shock. De Santis (2007) also assumes this log-additive specification to estimate the welfare cost of business cycles.

We assume the following lognormal process for $\delta_{i,t}$:

$$\delta_{i,t} = \exp(u_{i,t}\sqrt{x_t} - \frac{x_t}{2}) \tag{5}$$

where $u_{i,t}$ is standard normal i.i.d. shock, and x_t is the cross sectional variance of log consumption.

Note that in Constantinides and Duffie (1996) because of the assumption of an endowment economy, a lognormal process for consumption simply imposes restriction on the individual endowments. Since K-P (2007, 2009) have a production economy, such a lognormal consumption process imposes further restrictions on preference, technology and stochastic processes for skill shocks. These restrictions may not necessarily be unique. In the Appendix A, we have identified one such specification and provided a microfoundation of the posited consumption process (4).

[5] Constantinides and Duffie (1996) write the post trade allocation in terms of a consumption growth rate while we write here in terms of a level of consumption. The motivation for doing this is to apply this post-trade allocation to the Kocherlakota-Pistaferri (2007, 2009) discounting methodology. The Kocherlakota-Pistaferri incomplete market discount factor is based on the cross sectional moments of consumption in level while Sarkissian (2003) and Semenov (2008) use the Constantinides-Duffie (1996) discount factor which is based on the cross sectional average of the intertemporal marginal rates of substitution.

The s^{th} raw moment of the cross sectional distribution of consumption is given by:

$$E_i\left(c_{i,t}^s\right) = c_t^s \exp\left(\frac{(s^2-s)}{2} x_t\right) \tag{6}$$

Note that, by construction, aggregate consumption is the sum of individual consumption, which can be checked by setting $s=1$. Following the same approach as in Basu et al. (2011), we next derive the unique pricing kernel for each environment, namely INC and PIPO.

2.3 Lognormal Pricing Kernels

Substitute (6) into (2), and evaluating at $s=-\gamma$ to obtain the following pricing kernel for the *INC* environment :

$$M_{t+1}^{INC} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \exp\left(\frac{\gamma(\gamma+1)}{2}\left(x_{t+1}-x_t\right)\right) \tag{7}$$

Likewise, substitution of (6) into (3), and evaluating at $s=\gamma$, yields the pricing kernel for *PIPO*:

$$M_{t+1}^{PIPO} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \exp\left(\frac{-\gamma(\gamma-1)}{2}\left(x_{t+1}-x_t\right)\right) \tag{8}$$

In the absence of any information frictions and heterogeneity ($x_t = 0$ for all t), both (7) and (8) reduce to (1).

2.3.1 Difference from Long-run risk models

Pricing kernels (7) and (8) involve a latent variable x_t which gives rise to heterogeneity in consumption streams. The latent variable x_t captures idiosyncratic risk in consumption and it washes out in the aggregate. Thus by construction this latent variable x_t will not show up in the per capita consumption variable.[6] Our pricing kernels are thus fundamentally different from the long run risk model of Bansal and Yaron (2004) and Piazzesi and Schneider (2007). The latter deal with long run aggregate risk which does

[6] To verify this set $s=1$ in (6) and we obtain $E_i(c_{i,t}) = c_t$.

not wash out in the aggregate while our focus is exclusively on idiosyncratic risk which disappears in the aggregate. [7]

3 Application to UK Indexed Bonds

3.1 Pricing pure-real zero-coupon bonds

We start by considering the real price, P_{nt}^R, of a zero-coupon bond with maturity n, and develop this into the nominal price of the imperfectly indexed coupon bonds that are traded in the UK. P_{nt}^R can be written as follows for each of the market environments, $h = RA, INC, PIPO$:

$$P_{nt}^R = E_t\left[P_{n-1,t+1}^R M_{t+1}^h\right] \tag{9}$$

Assuming log normality we get the following expression for the log real price of a perfectly indexed zero-coupon bond of maturity n,

$$p_{nt}^R = E_t[m_{t+1}^h + p_{n-1,t+1}^R] + \frac{1}{2}Var_t[m_{t+1}^h + p_{n-1,t+1}^R] \tag{10}$$

where

$$m_{t+1}^{RA} = ln(\beta) - \gamma g_{t+1} \tag{11}$$

$$m_{t+1}^{INC} = ln(\beta) - \gamma g_{t+1} + \left(\frac{\gamma(\gamma+1)}{2}\right) v_{t+1} \tag{12}$$

$$m_{t+1}^{PIPO} = ln(\beta) - \gamma g_{t+1} - \left(\frac{\gamma(\gamma-1)}{2}\right) v_{t+1} \tag{13}$$

and $g_{t+1} \equiv c_{t+1} - c_t$, $v_{t+1} \equiv x_{t+1} - x_t$.[8]

[7] Bansal and Yaron (2004) employ a non-expected utility (Epstein-Zin (1991)) type preference while we focus on a simple expected utility model. To make a valid comparison between their pricing kernel and ours, we need to reduce their model to an expected utility paradigm. It is easy to check that in Bansal and Yaron (2004), if one sets their parameter $\theta = 1$, their pricing kernel (their equation 1) reduces to a standard RA pricing kernel $\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$ which stands in sharp contrast with our INC and PIPO pricing kernels (7) and (8). This clearly demonstrates the stark difference between the pricing kernel of long run risk model and the KP pricing kernels that we employ in our model.

[8] In K-P's (2007, 2009) setup, there are both aggregate and individual shocks and the former are completely hedged by a set of aggregate-shock contingent claims. In our

3.2 Pricing imperfectly indexed coupon bonds

UK indexed bonds are indexed to the change in goods prices[9] over a base period starting 8 months before their issue date, and ending 8 months before their redemption date.[10] We approximate this eight-month lag by 3 calendar quarters because we are using quarterly data. Thus the total inflation compensation for an n-period zero-coupon indexed bond is Q_{t+n-3}/Q^*, where Q^* is the goods price for the bond's base period, which leaves the bond's real price exposed to inflation over final 3 periods of its life. Thus equation (9) becomes

$$P_{nt}^{R} = E_t\left[\left(\prod_{s=1}^{n} M_{t+s}^{h}\right)\frac{Q_{t+n-3}}{Q^*}\frac{1}{Q_{t+n}}\right] \tag{14}$$

from which we get the nominal price of the bond as,

$$P_{nt}^{Nom} = \frac{Q_t}{Q^*}E_t\left[\left(\prod_{s=1}^{n} M_{t+s}^{h}\right)\frac{Q_{t+n-3}}{Q_{t+n}}\right] \tag{15}$$

After log-linearizing (15) and denoting the lower cases as the log of upper cases, gives the nominal price as

$$p_{nt}^{Nom} = (q_t - q^*) + E_t[z_{n,t+1}] + \frac{1}{2}Var_t[z_{n,t+1}] \tag{16}$$

where

$$z_{n,t+1} = \sum_{s=1}^{n} m_{t+s} - \sum_{s=0}^{2} \pi_{t+n-s} \tag{17}$$

and $\pi_{t+s} = q_{t+s} - q_{t+s-1}$.

The nominal price, in natural units, of a bond that pays a quarterly

bond economy, if these contingent claims do not exist in addition to bonds, PIPO market environment is not constrained Pareto optimal and the use of the PIPO discount factor is not justified. In order to avoid this problem, we assume that there are both these contingent claims and bonds traded but for brevity we focus on bonds and do not present a fully specified model which is available from the author upon request.

[9] Measured by the Retail Prices Index (RPI).

[10] The indexation method for UK bonds changed in 2005 (after the end of our sample). For bonds issued since that date the indexation lag is 3 months.

coupon[11] C can then be expressed as a linear combination of zero coupon log prices as follows:

$$P_{nt}^{Nom,c} = \sum_{s=1}^{n} exp(p_{st}^{Nom})C + exp(p_{nt}^{Nom}) \tag{18}$$

This price is exposed to changes in current inflation to the extent that it influences expectations of future inflation and the consumption components of the stochastic discount factor.

4 Estimation Method and Data

Our focus is on maximum likelihood estimation of the log-linearized bond pricing models described above. We use a 'panel' of observed prices consisting of a time-series of a selection of about six bonds in each period. The structural parameters can be estimated from a single cross-section, or from a time-series of prices for a single bond. Subject to parameter stability, the simultaneous use of both cross-sectional and time-series data should increase the efficiency of the estimates and provide a sharper test of the model than we get from either cross-section or time-series estimation alone.

4.1 A vector autoregressive model for the state variables

The nominal coupon bond price $P_{nt}^{Nom,c}$ in (18) through (16) depends on expectations of the three state variables; consumption growth (g), the change in the cross-sectional variance of consumption (v), and inflation (π), which we generate from a separately estimated vector autoregression as explained below.

[11] UK indexed coupons are paid 6-monthly. We fit this into our quarterly model by assuming half of the 6-monthly coupon to be paid each quarter. This introduces a small error due to the overvaluation of each coupon that accompanies our assumption that half of it is paid earlier than it is in reality.

Let w_t be a vector of state variables

$$w_t = \begin{pmatrix} g_t \\ v_t \\ \pi_t \end{pmatrix} \tag{19}$$

where all variables are in logs.

We assume the state vector to be autoregressive

$$w_{t+1} = A + Bw_t + \epsilon_{t+1} \tag{20}$$

where

$$\epsilon_{t+1} \sim N(0, \Omega_\epsilon) \qquad \forall t$$

We define a coefficient vector ϕ_R for each of the 3 models, consistent with equations (11) to (13), as follows:

$$\phi_R^{RA} = \begin{pmatrix} -\gamma \\ 0 \\ 0 \end{pmatrix}, \phi_R^{INC} = \begin{pmatrix} -\gamma \\ \frac{\gamma(\gamma+1)}{2} \\ 0 \end{pmatrix}, \phi_R^{PIPO} = \begin{pmatrix} -\gamma \\ \frac{-\gamma(\gamma-1)}{2} \\ 0 \end{pmatrix}$$

along with a second vector ϕ_L that captures the effects of inflation, as

$$\phi_L^{RA} = \begin{pmatrix} -\gamma \\ 0 \\ -1 \end{pmatrix}, \phi_L^{INC} = \begin{pmatrix} -\gamma \\ \frac{\gamma(\gamma+1)}{2} \\ -1 \end{pmatrix}, \phi_L^{PIPO} = \begin{pmatrix} -\gamma \\ \frac{-\gamma(\gamma-1)}{2} \\ -1 \end{pmatrix}$$

The log of the pricing kernels can then be written in the following general form,

$$m_{n,t+1}^h = ln(\beta) + \phi_R' w_{t+1} \tag{21}$$

and, from (17),

$$E_t[z_{n,t+1}] = \sum_{i=1}^{n-3} \phi_R' E_t[w_{t+i}] + \sum_{i=n-2}^{n} \phi_L' E_t[w_{t+i}] + ln(\beta) \quad (22)$$

$$Var_t[z_{n,t+1}] = \sum_{i=1}^{n-3} \phi_R' \Omega_{t+i} \phi_R + \sum_{i=n-2}^{n} \phi_L' \Omega_{t+i} \phi_L \quad (23)$$

where

$$E_t[w_{t+i}] = \tilde{B}_i A + B w_t \quad (24)$$

$$\Omega_{t+i} = \sum_{j=0}^{i-1} B^j \Omega_\epsilon B^{j'} \quad \forall\ t \quad (25)$$

and

$$\tilde{B}_i = \sum_{j=0}^{i-1} B^j$$

After substituting (22) and (23) into (16), the real price of the indexed zero-coupon bond can then be expressed in familiar affine form as:

$$p_{nt}^R = G_n + H_n w_t \quad (26)$$

where

$$\begin{aligned} G_n &= ln(\beta) + \left(\sum_{i=1}^{n-3} \phi_R' \tilde{B}_i A + \sum_{i=n-2}^{n} \phi_L' \tilde{B}_i A \right) + \\ &\quad \frac{1}{2} \sum_{i=1}^{n-3} \sum_{j=0}^{i-1} \left(\phi_R' B^j \Omega_\epsilon B^{j'} \phi_R \right) + \\ &\quad \frac{1}{2} \sum_{i=n-2}^{n} \sum_{j=0}^{i-1} \left(\phi_L' B^j \Omega_{t+i} B^{j'} \phi_L \right) \quad (27) \\ H_n &= \sum_{i=1}^{n-3} \phi_R' B + \sum_{i=n-2}^{n} \phi_L' B \quad (28) \end{aligned}$$

The log nominal price follows as $p_{nt}^{Nom} = p_{nt}^R + q_t$ which we substitute into (18) to obtain our estimation equation.[12]

$$P_{nt}^{Nom,c} = \sum_{s=1}^{n} exp(p_{st}^{R} + q_t)C + exp(p_{nt}^{R} + q_t) \tag{29}$$

We first estimate the vector autoregression for the state variables in order to obtain estimates of A, B and Ω_ϵ, and then use maximum likelihood to estimate the parameters (β and γ) of the asset pricing models by fitting equation (29) to market prices. The pricing errors are assumed to be normally and independently distributed, and homoskedastic across both maturities and time.

Using all of the available data in this way greatly increases the number of degrees of freedom, but does so at the cost of imposing parameter constancy over the sample. Some degree of persistence in the parameter values seems reasonable, so our approach offers a potential efficiency gain over the familiar approach of estimating the yield curve parameters for each period independently. To allow for the possibility that the parameters change with changes in the policy regime we also estimate the model over a number of sub samples, as discussed below.

4.2 Data

We use bond price data from the UK Debt Management Office. Since all indexed bonds with a maturity of 8 months or less, are pure nominal bonds we select only bonds with a residual maturity of 2 years or more. The number of indexed bonds in the market in any quarter is very small, ranging from 7 to 9. We select 6 bonds in each period, aiming for as even a spread as possible across the maturities from 1 to 25 years. When choosing between bonds with similar maturities, we select the one with the largest issue size.

Aggregate real consumption data are from the Office for National Statistics, and the cross sectional variances of the log of real consumption are from the Family Expenditure Survey (FES).[13] Data are quarterly for the

[12] The details of the derivation are presented in Appendix B.

[13] The FES was replaced by the Expenditure and Food Survey, which also covered the

period 1983Q1 to 2004Q4 and are seasonally unadjusted.[14]

4.3 Sub-samples and Monetary Policy Regimes

We estimate the model over the full sample 1983Q2 to 2004Q4, and over the following sub-samples:

Sub sample	Monetary policy regime
1983Q2 to 1992Q3	Monetary-growth and exchange-rate targets.
1992Q4 to 2004Q4	Inflation target: all stages.
1992Q4 to 1997Q2	Inflation target: early stages.
1997Q3 to 2004Q4	Inflation target with Bank of England independence.

From 1983 to 1992 the UK sought to anchor inflation first with control of monetary aggregates, then by using an informal combination of monetary and exchange rate targets, and finally with a 2-year membership of the European Exchange Rate Mechanism (ERM). In the post-ERM period inflation was targeted directly by the Treasury, and then, from 1997, by the newly independent Bank of England. sub-sample estimation provides an informal check of the model's robustness. While it is unlikely that the preference parameters β and γ would change as a result of monetary regime changes, we might expect some instability in their estimates if the model is misspecified.

4.4 Bank of England estimates of real yields

We compare the implied real yields from our models with those estimated by the Bank of England. UK indexed bonds do not provide unambiguous estimates of real rates however, due to the way in which they are indexed: their yields can be calculated only once we have data for expected inflation. The Bank of England calculates zero-coupon real yields conditional on their preferred method of dealing with inflation expectations, and we

National Food Survey, in April 2001.

[14] The details of the computation of the cross sectional variances are presented in Appendix C.

use these yields as a market benchmark for ours. Both sets of yields are based on fitting a model to market prices of indexed bonds. The principal difference between them is that we attempt to model the yield curve using a consumption-based asset pricing model; the Bank data are generated from a curve-fitting exercise that is not based on economic foundations. The Bank's objective is of course different from ours: we seek to estimate and explain real yields while the Bank seeks only to estimate them. Further, the parameters of the Bank model are re-estimated every day, in contrast to ours, which are held constant throughout each sample.

The Bank data are not complete: there are several missing observations at most maturities in our sample period 1983Q3 to 2004Q4, but these gaps do not appear at the same dates for each maturity. Further, the earliest Bank estimates are for 1985Q1, while our implied yields start in 1983Q3. The implications of this are discussed where Bank data are used below.

5 Results

5.1 Maximum Likelihood Estimates of β and γ

Estimates of the coefficients β and γ are presented in Table 1, along with likelihood values, and t-statistics in parentheses. The traditional representative agent model does not perform well. While the estimates of β are reasonable (implying a discount rate of about 1.5%) and highly significant, the estimates of γ are generally imprecise, with small t values, and are sometimes negative.

The estimates for INC and PIPO are substantially better and are statistically significant throughout. The estimates of γ are rather small, however, at about 0.2 and result in estimated bond premia that are also rather small, a familiar problem in DSGE models that employ an expected utility function (see, Rudebusch and Swanson, 2012).[15]

In terms of the likelihood values, the RA model does not perform as well

[15] Estimation using GMM with VAR factors as instruments, produced similar results for γ, with full-sample estimates of $-0.48, 0.18, 0.23$ for RA, INC and PIPO respectively. Basu et al. (2011) also obtained small estimates of γ for the INC model.

as the other two, but the results for the latter are too close to each other to allow us to choose between them. To the extent that a choice can be reached, it seems that the INC specification performs slightly better in the first half of the sample, while the PIPO specification is slightly better in the second. In terms of explaining the prices of indexed bonds however, it is clear that the incomplete-markets models have something to offer over the representative agent model.

We measure the goodness of fit to bond prices using Nagelkerke's (1991) generalized R^2 ,

$$R^2 = 1 - \left(\frac{L(0,0)}{L(\hat{\beta}, \hat{\gamma})} \right)^{\frac{2}{n}}$$

where n is the number of bonds in the sample, and L is the value of the likelihood function.

The results (which are not separately reported) are very similar for each model at around 0.05 for the full sample, and ranging from 0.1 to 0.25 for the sub-samples. The PIPO model has the highest of the three R^2s in each case. The measures suggest that there is a lot of variation in the prices of indexed bonds that is not accounted for by the factors underlying our models. Nevertheless, the models can explain 5% to 25% of the variation in prices, without letting the parameters of the model change from one period to the next, as is standard practice in market applications of no-arbitrage models.

5.2 Testing the plausibility of the model-implied yields

5.2.1 Moments of fitted and actual yields

In this section we ask whether the yields implied by the consumption-based models are consistent with those estimated by the Bank of England. It should be noted that the Bank's estimates are not definitive rates. They are estimates based on a specific methodology, which may or may not be more accurate than ours. Our models' implied yields for period are constructed using equation (29) after adapting the coefficients G_n and H_n to remove the effects of the lagging indexation. Yields are then calculated from the implied prices for a range of maturity values n.

The results in Table 2 show that the consumption-based models overestimate both the level and volatility of real yields at short maturities but that these errors diminish at longer maturities. The discrepancy in the level of short yields is likely to be due to the fact that the asset pricing models have only two parameters, and the majority of our price data is for bonds of longer maturities. Thus the estimated parameters are generated primarily by the fit to long-bond prices. The Bank data, by contrast, are based on a model that has many more parameters which can fit both long and short ends of the curve. The discrepancy in the volatilities will reflect both this difference in the number of parameters and the fact that the asset pricing models' estimates are held constant throughout the sample, with the result that movements in the factors can create relatively large pricing errors for the poorly fitted short-bond prices. The Bank model on the other hand can alter it's estimated parameters to match the prices of short-dated bonds.[16]

5.3 Real-rate responses to factor shocks

5.3.1 Impulse effects

We examine the impulse responses of real interest rates to shocks to the factors in the form of 1-period ahead expectations of consumption growth, the change in cross-sectional consumption variance and inflation.

The system of equations can be represented as,

$$p^R_{n,t} = G_n + H_n w_t \quad n = 1, 2, ... \tag{30}$$

$$w_t = A + B w_{t-1} + \epsilon_t \tag{31}$$

[16]For each maturity we select only these dates for which Bank of England data are available; these dates are then used for selecting our implied yields. Thus, for example, the 2.5-year full sample starts in 1985Q1 while that for 20-year yields starts in 1986Q3. There are further missing periods within these samples i.e. they do not occur only at the start of the samples.

from which we get the prices as functions of the history of the factor shocks ϵ as

$$p^{R,z}_{n,t} = G_n + H_n(I-B)^{-1}A + H_n(I-BL)^{-1}\epsilon_t \tag{32}$$

Real interest rates at all maturities n follow directly from this equation.

The ϵ_t terms are mutually correlated so we recast these as linear functions of three orthogonal random terms ξ_t and measure the response of real rates to shocks to the latter. Thus we assume that,

$$\begin{aligned} \epsilon_{1t} &= c_{11}\xi_{1t} + c_{12}\xi_{2t} + c_{13}\xi_{3t} \quad (33) \\ \epsilon_{2t} &= c_{21}\xi_{1t} + c_{22}\xi_{2t} + c_{23}\xi_{3t} \quad (34) \\ \epsilon_{3t} &= c_{31}\xi_{1t} + c_{32}\xi_{2t} + c_{33}\xi_{3t} \quad (35) \end{aligned}$$

This leads to the familiar problem that we cannot identify all 9 c_{ij} coefficients from the 6 independent coefficient estimates in $\hat{\Omega}_\epsilon$. We deal with this in the usual way with a Cholesky decomposition of the covariance matrix Ω_ϵ i.e. we impose zero-restrictions on c_{12}, c_{13} and c_{23}. This is equivalent to assuming that the shock ξ_{1t} influences all three variables, ξ_{2t} influences only the latter two, and ξ_{3t} influences only the third. Since the ordering of the ϵs is not unique (we could put the 3 variables in the VAR in any order), and because we have no prior information as to the real-world ordering under these identifying restrictions (if in fact any is correct), we present results for four of the six possible orderings; for the remaining two the reordered Ω is not positive definite and, therefore, there is no Cholesky decomposition.

Thus we define

$$\begin{aligned} \epsilon_t &= C\xi_t \quad (36) \\ E[\xi_t \xi_t'] &= I \quad (37) \end{aligned}$$

where C is lower-triangular

Hence,

$$\Omega_\epsilon = CC' \tag{38}$$

Substituting (36) into the bond price equations we get:

$$p^R_{n,t} = G_n + H_n(I-B)^{-1}A + H_n(I-BL)^{-1}C\xi_t \quad n = 1, 2, ... \tag{39}$$

In order to give the shocks to the ξ a clearer economic meaning we scale them such that they generate a 1 percentage point increase in each of the factors in turn. For example, in Table 3 the first row shows the effects of a shock to ξ_{1t} such that expected consumption growth increases by 1%. In line with the ordering of the VAR, this same ξ_{1t} shock also generates a contemporaneous 26.54% increase in the change of the cross-sectional variance of consumption, and a 0.1183% decline in inflation. The qualitative effects on yields of all of the factor shocks turn out to be robust to changes in the order of the factors.

Table 3 presents the impulse responses of yields at maturities of 2, 5 and 10 years to factor shocks, based on the full-sample estimates. The effect of a shock to inflation that is not accompanied by shocks to the other two factors can be seen from lines 3 and 6 of Table 3. For both the INC and PIPO models, there are small falls in real rates at all maturities. Results for 3-month rates (not reported) show that the short real rate does not respond to changes in expected inflation since agents are assumed to optimize their utility over real magnitudes, and there are no changes in the consumption factors in the utility function. At all longer maturities however, the effect of a current inflation shock on expectations of future consumption growth and variance do have an impact on real rates by altering the utility value of future real returns. The negative response of *ex-ante* real rates is consistent with results found in Barr and Campbell (1997) and others, and provide a possible explanation for their results. This negative impact of expected inflation on real rates arises for all of the VAR orderings, although with inflation placed at position 1 or 2 in the VAR the associated contemporaneous shocks to the other factors also generate negative responses in the 3-month rate.

Increases in expected consumption growth lead to increases in real rates for both models (with the exception of the 2-year rates for the INC model) irrespective of the ordering of the factors: higher consumption growth lowers agent's incentive to save and financial markets respond by offering a higher real yield as the demand for real bonds declines.

Positive shocks to the cross-sectional variance of consumption cause real rates to fall in all cases, which is consistent with the Euler equations (2) and (3) given that the estimated $\gamma < 1$ in both the INC and PIPO models. A higher cross-sectional variance in consumption means that consumers face greater uninsurable risk. In an both INC and PIPO environments with zero or partial insurance, consumers increase their saving for precautionary reasons, and this increase in the supply of loanable funds drives down real interest rates. In a PIPO environment, with partial insurance, an opposing effect on saving will be at work. Greater consumption inequality (i.e. higher variance of consumption) lowers the agency cost because it is cheaper to provide incentives to poor people. This lower agency cost may create a wealth effect that lowers the incentive to save. Thus we expect the effect on the real interest rate to be weaker in the PIPO environment.

Piazzesi and Schneider (2007) present a similar impulse response analysis for US real rates. They obtain a very clean result about real interest rates: growth and inflation surprises move short-maturity real rates in opposite directions but have only small effects on long real rates. In particular, a positive inflation surprise decreases short-maturity real rates (with a half-life of about 5 years), while a growth surprise increases them (with a half life of about 6 months).[17] Both of our incomplete markets models produce impact effects of the same sign as Piazzesi and Schneider (2007) (see lines 3, 6 and 12 of Table 3) and they have only small effects on long rates.

[17] The half-lives are our estimates based on the charts presented in Piazzesi and Schneider (2007) p405.

6 Summary and conclusions

This paper tests three consumption-based asset pricing models applied to indexed bonds in the UK. We employ a three factor model of log normal bond pricing. Our novelty lies in deriving closed form expressions for the pricing kernels of the new class of uninsurable risk models and integrating this with a lognormal affine form bond pricing function. This innovation allows us to derive the price function of indexed coupon bonds in an estimable form with a convenient marriage between VAR based representation of the state variables and the bond price equation. Our central equation is a lognormal bond price equation in which expected values of the state variables are constructed from a parsimonious VAR involving three macroeconomic variables, namely the growth rate of aggregate consumption, cross section variance of consumption and the rate of inflation.

Comparisons of the implied real yield curves from our incomplete markets models with those estimated by the Bank of England suggest that our models generate real rates that are about 1.6% greater at the short (2.5-year) end of the curve and about 30 basis points greater at the long (20-year) end. Our models also generate greater volatility in short rates but the volatility of long rates is broadly the same as the Bank's estimates. The differences between the two sets of estimates are likely to lie in: the fact that the Bank re-estimates the parameters of its curve every day, while we impose parameter constancy within each sample period; differences in the way in which inflation expectations are generated (we use a backward-looking VAR while the Bank uses forward-looking break-even rates), and the Bank estimates are not constrained to be functions of consumption and inflation. The last of these points is crucial in the sense that while our estimated models suggest that consumption growth and incomplete markets explain part of the behavior of real rates they clearly do not tell the full story if we regard the Bank's estimates as more representative of actual real rates than ours.

Impulse responses based on the estimated bond price equation for the incomplete markets models suggest that a rise in inflation lowers the real interest rates of almost all maturities while a rise in aggregate consumption growth raises real interest rates. An increase in uninsurable risk, on the other hand, lowers real interest rates.

All of our models give rise to a low estimated coefficient of relative risk aversion, which implies low ex ante bond risk premia. This simply indicates that incomplete market models alone cannot resolve the extant bond premium puzzle. A natural step forward would be to adopt Epstein-Zin preferences. To the best of our knowledge however, there is as yet no theory that integrates these incomplete market models with non-expected utility maximization although this is likely to be an interesting avenue for further research.

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A Microfoundation of the Lognormal Consumption Process (4)

Consider a simple production economy. Each agent i derives instantaneous utility defined over consumption, c_t^i, and labour supply l_t^i . The agent's output, y_t^i, is subject to private and aggregate shocks as follows,

$$y_t^i = \phi(\theta^{i,t}, z^t) l_t^i \tag{A.1}$$

where $\theta^{i,t}$ is the history of private shocks to the productivity of agent i until date t, z^t is the history of shocks to total factor productivity, and $\phi(.)$ is some function of these histories.[18] Although individuals may differ in terms of private skill shock history, each such history $\theta^{i,t} = (\theta_1^i, \theta_2^i, ..., \theta_t^i)$ is drawn from the same probability space with the measure $\pi(\theta^{i,t})$. The public shock history $z^t = (z_1, z_2, ..., z_t)$ has the probability measure $\psi(z^t)$. We assume that the skill shock and the aggregate shock are drawn independently, so that by observing the aggregate shock one cannot infer anything about the idiosyncratic skill shock.

The timing of decisions is as follows. In period t, the agents' first action is to produce their output, they then visit the asset market to trade real bonds of various maturities, then visit the goods markets. The flow budget constraint facing the household is:

$$Q_t c_t^i + \sum_{s=1}^{m} P_{s,t}^{N,z} b_{s,t}^i = Q_t y_t^i + \sum_{s=1}^{m} P_{s-1,t}^{N,z} b_{s-1,t}^i \tag{A.2}$$

where

$P_{s,t}^{N,z} =$ Date t nominal price of a zero coupon bond with maturity s
$c_t^i =$ Consumption of the agent i
$b_{s,t}^i =$ Number of bonds of maturity s held by agent i
$Q_t =$ Nominal price of the good.

[18] For example, suppose the private and aggregate shocks u_t and v_t follow random walks, $u_t = u_{t-1} + \theta_t$ and $v_t = v_{t-1} + z_t$. Then $\phi^i(\theta^t, z^t) = \phi^i(\theta_1 + \theta_2 + ... + \theta_t,\ z_1 + z_2 + ...z_t)$.

The household is assumed to maximize a standard intertemporally separable utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t [U(c_t^i) - V(l_t^i)]$$

where $U(c_t^i)$ is the per period utility from consumption and $V(l_t^i)$ is the instantaneous disutility from labour supply with the property that $U'(c_t^i) > 0$, $U''(c_t^i) < 0$ and $V'(l_t^i) > 0$.

It is straightforward to verify that in both INC and PIPO settings of K-P (2007, 2009) the following intratemporal efficiency condition holds.

$$U'(c_t^i)\phi^i(\theta^{i,t}, z^t) = V^{'}(l_t^i) \tag{A.3}$$

We will now construct an environment which will mimic the lognormal consumption process (4).

Proposition: If $U(c_t^i) = \frac{c_t^{i1-\gamma}}{1-\gamma}$, $V(l_t^i) = \omega_t l_t^i$, $y_t^i = \phi(\theta^{i,t}, z^t) l_t^i = \delta_{i,t}^{\gamma} l_t^i$ where $\delta_{i,t} = \exp(u_{i,t}\sqrt{x_t} - \frac{x_t}{2})$, and $\{\omega_t\}$ is a stochastic process uncorrelated with $\delta_{i,t}$ then the *post-trade* allocation of consumption follows (4).

Proof: Eq (A.3) can be rewritten as:

$$\delta_{i,t}^{\gamma} c_{it}^{-\gamma} = \omega_t$$

=>

$$c_{it} = \omega_t^{-1/\gamma} \delta_{i,t} \tag{A.4}$$

Verify that

$$E(c_{it}) = c_t = \omega_t^{-1/\gamma}$$

Thus eq (A.4) becomes

$$c_{it} = \delta_{i,t} c_t$$

End of proof.

In other words, if there is an aggregate preference shock $\{\omega_t\}$ (which could be persistent), the equilibrium process for individual consumption will look like (4). One needs to be cautious that this is not the only environment which mimics (4). Alternative environment could be constructed as well. The purpose of this appendix is just to illustrate that (4) could be microfounded.

B Derivation of the estimated price equations

B.1 The stochastic discount factors

For the stochastic discount factors (12) and (13),

$$\begin{aligned} M_{t+i} &= \beta G_{t+i}^{-\gamma} \exp\left[\left(\frac{\gamma(\gamma \pm 1)}{2}\right)(x_{t+i} - x_{t+i-1})\right] && \text{(B.5)} \\ \Rightarrow m_{t+i} &= \ln\beta - \gamma g_{t+i} \pm \left[\left(\frac{\gamma(\gamma \pm 1)}{2}\right)(x_{t+i} - x_{t+i-1})\right] && \text{(B.6)} \\ &= \ln\beta - \gamma g_{t+i} \pm \left[\left(\frac{\gamma(\gamma \pm 1)}{2}\right) v_{t+i})\right] && \text{(B.7)} \\ &= \ln\beta + \phi_R' w_{t+i} && \text{(B.8)} \end{aligned}$$

Now substitute this expression for m into the price equation (16) to get,

$$\begin{aligned} p_{nt}^n - (q_t - q^*) = \; & E_t \phi_R' w_{t+1} + \ldots + \phi_R' w_{t+n-3}) + \\ & E_t(\phi_L' w_{t+n-2} + \phi_L' w_{t+n-1} + \phi_L' w_{t+n}) + \\ & \frac{1}{2} Var_t(\ldots) + n \ln\beta && \text{(B.9)} \end{aligned}$$

The terms $\phi_R' w_{t+1} + \ldots + \phi_R' w_{t+n-3}$ come directly from the equation for m above. The others, $\phi_L' w_{t+n-2} + \phi_L' w_{t+n-1} + \phi_L' w_{t+n}$, are a combination of the m terms and inflation, for the last 3 months of the bond's life i.e. the period after the indexation ends, and the bond's real value is exposed to inflation.

So a convenient alternative way to write z is,

$$
\begin{aligned}
z_{n,t+1} &= \phi_R' w_{t+1} + ... + \phi_R' w_{t+n-3} + \\
&\quad \phi_L' w_{t+n-2} + \phi_L' w_{t+n-1} + \phi_L' w_{t+n} \qquad \text{(B.10)}
\end{aligned}
$$

B.2 Time series projections for the factors

For the case of a VAR(1) we have,

$$
\begin{aligned}
w_{t+1} &= A + Bw_t + \epsilon_{t+1} \\
&= A_n + B_n w_t + \eta_{t+n} \qquad \text{(B.11)}
\end{aligned}
$$

where

$$A_n = (I + B + ... + B^{n-1})A \qquad \text{(B.12)}$$

$$B_n = B^n \qquad \text{(B.13)}$$

$$\eta_{t+n} = \epsilon_{t+n} + B\epsilon_{t+n-1} + ... + B^{n-1}\epsilon_{t+1} \qquad \text{(B.14)}$$

It follows that, introducing $\Omega_{t+n} \equiv Var_t(\eta_{t+n})$, which we assume to be constant w.r.t t,

$$E_t(w_{t+n}) = A_n + B_n w_t \qquad \text{(B.15)}$$

$$
\begin{aligned}
Var_t(w_{t+n}) &= Var_t(\eta_{t+n}) \\
&= \Omega + B_1 \Omega B_1' + ... + B_{n-1} \Omega B_{n-1}' \\
&= \Omega_{t+n} \qquad \text{(B.16)}
\end{aligned}
$$

More compactly,

$$\Omega_{t+i} = \sum_{j=0}^{i-1} B_j \Omega_\epsilon B_j' \quad \forall t, \text{ and } i = 1...n \qquad \text{(B.17)}$$

B.3 Derivations of $E_t(z)$ and $Var_t(z)$

Given that

$$\begin{aligned} z_{n,t+1} &= \phi_R' w_{t+1} + ... + \phi_R' w_{t+n-3} + \\ &\quad \phi_L' w_{t+n-2} + \phi_L' w_{t+n-1} + \phi_L' w_{t+n} \end{aligned} \tag{B.18}$$

we get,

$$\begin{aligned} E_t(z_{n,t+1}) &= E_t\left[\phi_R' \left(w_{t+1} + ... + w_{t+n-3}\right) + \phi_L' \left(w_{t+n-2} + w_{t+n-1} + w_{t+n}\right)\right] \\ &= \phi_R' \left((A_1 + B_1 w_t) + ... + (A_{n-3} + B_{n-3} w_t)\right) + \\ &\quad \phi_L' \left((A_{n-2} + B_{n-2} w_t) + (A_{n-1} + B_{n-1} w_t) + (A_n + B_n w_t)\right) \\ &= \phi_R' \left(A_1 + ... + A_{n-3}\right) + \phi_L' \left(A_{n-2} + A_{n-1} + A_n\right) + \\ &\quad \phi_R' \left(B_1 w_t + ... + B_{n-3} w_t\right) + \phi_L' \left(B_{n-2} w_t + B_{n-1} w_t + B_n w_t\right) \\ &= \phi_R' \left(A_1 + ... + A_{n-3}\right) + \phi_L' \left(A_{n-2} + A_{n-1} + A_n\right) + \\ &\quad \phi_R' \left(B_1 + ... + B_{n-3}\right) w_t + \phi_L' \left(B_{n-2} + B_{n-1} + B_n\right) w_t \\ &= \left[\phi_R' \left(A_1 + ... + A_{n-3}\right) + \phi_L' \left(A_{n-2} + A_{n-1} + A_n\right)\right] + \\ &\quad \left[\phi_R' \left(B_1 + ... + B_{n-3}\right) + \phi_L' \left(B_{n-2} + B_{n-1} + B_n\right)\right] w_t \\ &= \left(\sum_{i=1}^{n-3} \phi_R' A_i + \sum_{i=n-2}^{n} \phi_L' A_i\right) + \\ &\qquad \left(\sum_{i=1}^{n-3} \phi_R' B_i + \sum_{i=n-2}^{n} \phi_L' B_i\right) w_t \end{aligned} \tag{B.19}$$

$$\begin{aligned}
Var_t(z_{n,t+1}) &= \phi_R' \left(\Omega_{t+1} + ... + \Omega_{t+n-3}\right) \phi_R + \phi_L' \left(\Omega_{t+n-2} + ...\Omega_{t+n}\right) \phi_L \\
&= \phi_R' \left(\sum_{j=0}^{0} B_j \Omega_\epsilon B_j' + ... + \sum_{j=0}^{n-3-1} B_j \Omega_\epsilon B_j' \right) \phi_R + \\
&\quad \phi_L' \left(\sum_{j=0}^{n-2-1} B_j \Omega_\epsilon B_j' + ... \sum_{j=0}^{n-1} B_j \Omega_\epsilon B_j' \right) \phi_L \qquad \text{(B.20)}
\end{aligned}$$

B.4 The final equation for a zero-coupon indexed bond

Recall,

$$p_{nt}^{Nom} - (q_t - q^*) = n \ln \beta + E_t(z_{n,t+1}) + \frac{1}{2} Var_t(z_{n,t+1}) \qquad \text{(B.21)}$$

we can substitute for the conditional expectations and variances of w that appear in z. The expectations introduce a series of terms in the constant A, which when added to the constant conditional variance, gives us the constant term in the price equation.

The time-varying elements, i.e. the terms in the factors w_{t+i}, are all functions of w_t. Hence, the real price, $p_{n,t}^R \equiv p_{nt}^{Nom} - (q_t - q^*)$, is

$$p_{n,t}^R = G_n + H_n w_t \qquad \text{(B.22)}$$

where

$$\begin{aligned}
G_n &= ln(\beta) + \left(\sum_{i=1}^{n-3} \phi_R' A_i + \sum_{i=n-2}^{n} \phi_L' A_i \right) \\
&\quad + \frac{1}{2} \sum_{i=1}^{n-3} \sum_{j=0}^{i-1} \left(\phi_R' B_j \Omega_\epsilon B_j' \phi_R \right) \\
&\quad + \frac{1}{2} \sum_{i=n-2}^{n} \sum_{j=0}^{i-1} \left(\phi_L' B_j \Omega_\epsilon B_j' \phi_L \right) \qquad \text{(B.23)} \\
H_n &= \sum_{i=1}^{n-3} \phi_R' B_i + \sum_{i=n-2}^{n} \phi_L' B_i \qquad \text{(B.24)}
\end{aligned}$$

C Construction of the Cross Sectional Distribution of Consumption

We construct the cross sectional variance of real consumption using the records of daily expenditure from the Family Expenditure Survey (FES) conducted by the Office for National Statistics (ONS). The data we use are based on the expenditure of approximately 6,500 households for a period of 2 weeks in every quarter.

Our procedure mimics Kocherlakota and Pistaferri (2009, 2007). First, the household level consumption of nondurables and services is calculated by adding the nominal consumption of nondurables and services for each household. We follow the definition of nondurable and services of Attanasio and Weber (1995). Second, since the household consumption data are two week durations only, we multiply them by 6.5 to obtain quarterly consumption. Third, we divide this quarterly consumption of each household by the number of people in each household in that quarter to derive the quarterly nominal, per capita consumption of nondurables and services for each household unit. Fourth, by dividing the quarterly data by the quarterly CPI for all items (not seasonally adjusted) (the CPI is from the OECD main economic indicators) with the basis of 2005:Q1, we get the quarterly real per capita consumption for all the relevant households.

C.1 Measurement errors

KP (2009) alert us to measurement errors from the use of cross section expenditure data. In our context, if these measurement errors appear multiplicatively they do not impact the pricing kernels. To see this, define the measured consumption as:

$$\hat{c}_{i,t} = c_{i,t} \exp(\xi_{i,t})$$

where the measurement error $\xi_{i,t}$ is stationary, i.i.d. across households, and uncorrected with z_t. Then we get:

$$\hat{x}_t - \hat{x}_{t-1} = x_t - x_{t-1}$$

Since we work with the first difference of the variance of log consumption, the measurement error is not an issue.

Table 1: Estimation results.

	RA			INC			PIPO		
	β	γ	LF	β	γ	LF	β	γ	LF
1983-2004	0.9950	-0.0144		0.9958	0.1328		0.9962	0.1727	
	(11.55)	(-0.3115)	5.099	(23.59)	(11.16)	5.792	(17.41)	(8.012)	5.767
1983-1992	0.9948	-0.0071		0.9955	0.1319		0.9957	0.1607	
	(11.59)	(-0.1666)	4.983	(18.63)	(7.613)	5.468	(13.69)	(5.423)	5.434
1992-2004	0.9959	0.0501		0.9962	0.1336		0.9967	0.1897	
	(3.05)	(0.3478)	5.237	(14.61)	(8.043)	6.154	(10.04)	(5.356)	6.171
1992-1997	0.9964	0.3363		0.9941	0.1371		0.9947	0.2085	
	(2.82)	(2.635)	6.517	(20.60)	(12.59)	8.160	(15.20)	(7.320)	8.308
1997-2004	0.9975	0.0609		0.9977	0.1353		0.9983	0.2004	
	(1.91)	(0.4200)	6.583	(9.24)	(4.580)	7.692	(3.66)	(3.094)	7.737

LF is the value of the log-likelihood function. Figures in parentheses are t-statistics for $H_0(\beta) = 1$ and $H_0(\gamma) = 0$.

Table 2: Moments of estimate yields.

		2.5 year			
		BoE	RA	INC	PIPO
1985-2004	Mean	2.99	4.62	4.56	4.56
	Var	0.796	2.13	1.62	1.54
1985-1992	Mean	3.40	5.49	5.48	5.47
	Var	0.651	2.54	2.10	2.01
1992-2004	Mean	2.69	3.97	3.88	3.89
	Var	0.686	1.09	0.553	0.529
1992-1997	Mean	3.11	4.88	4.74	4.74
	Var	0.251	1.36	0.658	0.591
1997-2004	Mean	2.45	3.33	3.27	3.27
	Var	0.771	0.852	0.492	0.487
		10 year			
		BoE	RA	INC	PIPO
1985-2004	Mean	3.22	3.77	3.71	3.71
	Var	0.667	1.71	1.32	1.26
1985-1992	Mean	3.93	4.57	4.55	4.55
	Var	0.0980	2.04	1.75	1.70
1992-2004	Mean	2.76	3.23	3.14	3.13
	Var	0.495	0.983	0.527	0.505
1992-1997	Mean	3.49	4.14	4.03	4.02
	Var	0.101	1.30	0.668	0.608
1997-2004	Mean	2.28	2.56	2.50	2.50
	Var	0.174	0.686	0.432	0.429
		20 year			
		BoE	RA	INC	PIPO
1985-2004	Mean	3.02	3.36	3.32	3.34
	Var	0.770	1.12	0.765	0.751
1985-1992	Mean	4.02	4.16	4.23	4.28
	Var	0.0726	1.13	1.04	0.982
1992-2004	Mean	2.76	3.09	3.00	2.99
	Var	0.628	0.957	0.513	0.492
1992-1997	Mean	3.65	3.99	3.89	3.88
	Var	0.0574	1.27	0.650	0.593
1997-2004	Mean	2.18	2.42	2.37	2.36
	Var	0.146	0.667	0.421	0.418

Table 3: Impulse responses to factor shocks.

Cholesky order.			Factor shocks.			Yields: INC			Yields: PIPO		
			Growth	Variance.	Inflation	2 year	5 year	10 year	2 year	5 year	10 year
1.000	2.000	3.000	1.000	26.54	-0.1183	0.0131	0.0053	0.0027	0.0222	0.0091	0.0045
1.000	2.000	3.000	0.0000	1.000	7.971e-005	-0.0005	-0.0002	-9.300e-005	-0.0004	-0.0002	-9.004e-005
1.000	2.000	3.000	0.0000	0.0000	1.000	-0.0016	-0.0007	-0.0004	-0.0041	-0.0018	-0.00087
2.000	1.000	3.000	0.0037	1.000	-0.0002	-0.0011	-0.0005	-0.0002	-0.0011	-0.0004	-0.0002
2.000	1.000	3.000	1.000	0.0000	-0.1204	0.0254	0.0103	0.0051	0.0342	0.0139	0.0069
2.000	1.000	3.000	0.0000	0.0000	1.000	-0.0016	-0.0007	-0.0004	-0.0041	-0.0018	-0.0009
1.000	3.000	2.000	1.000	7.812	-0.1183	0.0131	0.0053	0.0027	0.0223	0.0091	0.0045
1.000	3.000	2.000	0.0000	0.3635	1.000	-0.0022	-0.0002	-0.0005	-0.0046	-0.0020	-0.0010
1.000	3.000	2.000	0.0000	1.000	0.0000	-0.0016	-0.0007	-0.0003	-0.0015	-0.0006	-0.0003
3.000	2.000	1.000	-0.1000	-1.424	1.000	-0.0004	-0.0005	-0.0007	-0.0068	-0.0028	-0.0014
3.000	2.000	1.000	0.0014	1.000	0.0000	-0.0004	0.0103	-8.586e-005	-0.0004	-0.0002	-8.043e-005
3.000	2.000	1.000	1.000	0.0000	0.0000	0.0252	-0.0007	0.0051	0.0337	0.0136	0.0068