A GL(3) Kuznetsov Trace Formula and the Distribution of Fourier Coefficients of Maass Forms

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ABSTRACT

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We study the problem of the distribution of certain GL(3) Maass forms, namely, we obtain a Weyl's law type result that characterizes the distribution of their eigenvalues, and an orthogonality relation for the Fourier coefficients of these Maass forms. The approach relies on a Kuznetsov trace formula on GL(3) and on the inversion formula for the Lebedev-Whittaker transform. The family of Maass forms being studied has zero density in the set of all GL(3) Maass forms and contains all self-dual forms. The self-dual forms on GL(3) can also be realised as symmetric square lifts of GL(2) Maass forms by the work of Gelbart-Jacquet. Furthermore, we also establish an explicit inversion formula for the Lebedev-Whittaker transform, in the nonarchimedean case, with a view to applications.

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Chapter 1

Introduction

Understanding the spectral theory of the Laplace operator Δ has interested mathematicians and physicists for a long time. This interest derives from the fact that the Laplacian spectrum is strongly connected to the geometry of the underlying space. From an arithmetic perspective this problem is particularly appealing when the underlying space is a quotient of the hyperbolic upper half plane $\mathbb{H}$ by a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, where Γ acts on $\mathbb{H}$ by fractional linear transformations.

When the discrete group Γ is equal to $\mathrm{SL}_2(\mathbb{Z})$ (or a congruence subgroup), the quotient $\Gamma\backslash\mathbb{H}$ has finite volume and is noncompact, as it has a finite number of cusps. These properties are reflected in the spectral decomposition of the Laplace operator, i.e. the spectrum of Δ has both a discrete part and a continuous part. The eigenfunctions of Δ that form the discrete part of the spectrum are the so-called Maass cusp forms. Moreover, an important feature of the Laplacian spectrum in this case is that it can be used to decompose the space of square-integrable functions on the quotient $\Gamma\backslash\mathbb{H}$.

The first result on the distribution of the discrete eigenvalues of Δ on $\Gamma\backslash\mathbb{H}$ is due to Selberg [Selberg, 1956]. He developed the Selberg trace formula which relates the eigenvalues of Maass cusp forms with the conjugacy classes of Γ (up to some contribution of the continuous spectrum). Using this formula, Selberg was able to count the number of eigenvalues with a value less than a given parameter $X > 0$, in analogy to the results already known for compact Riemannian surfaces.

Since the work of Selberg, trace formulas have been a thoroughly researched subject in number theory. In general, these formulas relate the spectral decomposition of a space of functions (spectral side) to information about the structure of an algebraic group (geometric side). The work in this thesis is centered around a trace formula of the type first developed by Kuznetsov. For $\{\phi_j\}$ an orthonormal basis for eigenfunctions for the discrete spectrum of Δ, indexed by increasing eigenvalue $\lambda_j = \frac{1}{4} + x_j^2$, Kuznetsov [Kuznetsov, 1980] showed a formula relating, on one hand, a sum over the spectral information associated to the functions ϕ_j, weighted by their Fourier coefficients $a_j(n)$ and a test function h,

$$\sum_{j=1}^{\infty} \frac{a_j(n)\overline{a_j(m)}}{\cosh \pi x_j} h(x_j)$$

and on the other hand, a sum over Kloosterman sums $S(n, m; c)$, weighted by values of Φ_h, a Bessel transform of h,

$$\sum_{c=1}^{\infty} \frac{1}{c} S(n, m; c) \Phi_h \left(\frac{4\pi\sqrt{nm}}{c} \right)$$

.

Kuznetsov's formula is a powerful tool in analytic number theory with many applications, for example to the Linnik's conjecture [Deshouillers and Iwaniec, 198283] and to problems on the distribution of quadratic roots mod p [Duke *et al.*, 1995].

Versions of the Kuznetsov trace formula have been proved for $\mathrm{SL}_3(\mathbb{Z})\backslash(\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{Z}))$ by Blomer [Blomer, 2013], Buttcane [Buttcane, 2012], Goldfeld and Kontorovich [Goldfeld and Kontorovich, 2013] and Li [Li, 2010] and have been applied to a wide range of topics: moments of L-functions, exceptional eigenvalues, and statistics of low-lying zeros of L-functions. As in the GL(2) Kuznetsov trace formula, the GL(3) formulas give equalities between spectral information about Maass forms on one side and Kloosterman sums on the other side.

In this thesis, we prove two main results; one is an application of a GL(3) Kuznetsov trace formula and the other one deals with the problem of inverting the Lebedev-Whittaker transform, which plays an important role in the study of this type of trace formula.

The first of those results establishes a Weyl's law type theorem for the spectral parameters of a "thin" family $\mathcal{F}$ of Maass forms, whose spectral parameters are approximately like those of self-dual forms, and an orthogonality relation for the Fourier coefficients $A_j(m_1, m_2)$ of that same family.

Theorem 1.0.1. *Let ϕ_j be a set of orthogonal $GL(3)$ Hecke-Maass forms with spectral parameters $\nu^{(j)}$ and Fourier coefficients $A_j(n_1, n_2)$. Let $h_{T,R,\kappa}$ be a family of smooth functions essentially* *supported on the spectral parameters of the Maass forms in $\mathcal{F}$. Then,*

$$\sum_{j\geq 1} \frac{\left|h_{T,R,\kappa}(\nu^{(j)})\right|}{\mathcal{L}_j} = c_{R,\kappa}\frac{T^{4+3R}}{(\log T)^{1/\kappa}} + \mathcal{O}\left(T^{3+3R+\epsilon}\right), \qquad (T\to\infty).$$

Moreover,

$$\sum_{j\geq 1} A_j(m_1,m_2)\overline{A_j(n_1,n_2)}\frac{\left|h_{T,R,\kappa}(\nu^{(j)})\right|}{\mathcal{L}_j} = \begin{cases} \sum\limits_{j\geq 1}\frac{\left|h_{T,R,\kappa}(\nu^{(j)})\right|}{\mathcal{L}_j} + \mathcal{O}\left(T^{3R+3+\epsilon}\right), & \text{if } {m_1=n_1 \atop m_2=n_2}, \\ \mathcal{O}\left(T^{3R+3+\epsilon}\right), & \text{otherwise.} \end{cases}$$

A more precise version of this theorem is stated in Theorem 4.1.1. It is useful to note that the second statement in Theorem 1.0.1 can be thought of as a version of the well known orthogonality relation of Dirichlet characters,

$$\sum_{\chi \bmod q} \chi(n)\overline{\chi(m)} = \begin{cases} \phi(q), & \text{if } n\equiv m \bmod q, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

for integers m,n coprime to q, where the sum on the left is over all characters (mod q). Since Dirichlet characters can be viewed as automorphic representations of $GL(1,\mathbb{A}_{\mathbb{Q}})$, this result can be interpreted as the simplest case of the orthogonality relation conjectured by Zhou [Zhou, 2014] concerning Fourier-Whittaker coefficients of Maass forms on the space $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$, $n\geq 2$. This orthogonality relation conjectured by Zhou was proved by Bruggeman [Bruggeman, 1978] in the case $n=2$ and by Goldfeld-Kontorovich [Goldfeld and Kontorovich, 2013] and Blomer [Blomer, 2013] in the case $n=3$. Versions of this result have applications to the Sato-Tate problem for Hecke operators, both in the holomorphic [Conrey *et al.*, 1997], [Serre, 1997] and non-holomorphic setting [Sarnak, 1987], [Zhou, 2014], as well as to the problem of determining symmetry types of families of L-functions [Goldfeld and Kontorovich, 2013] as introduced in the work of Katz-Sarnak [Katz and Sarnak, 1999].

The version of Weyl's law presented above tells us that the family $\mathcal{F}$ of Maass forms being studied has zero density in the space of all Maass forms, indexed by increasing spectral parameters. This is the first result of this kind obtained using a GL(3) trace formula. After normalizing by the weight factor T^{3R} (which is needed for technical reasons), Theorem 1.0.1 tells us that $\mathcal{F}$ has roughly T^4 Maass forms with eigenvalue up to T^2. For comparison, Weyl's law for all Maass forms on the space $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$ says that there are T^5 Maass forms with eigenvalue up to

T^2, by work of Miller [Miller, 2001]. Miller's result has since been obtained in more general settings in [Lapid and Müller, 2009], [Lindenstrauss and Venkatesh, 2007], [Müller, 2007]. Furthermore, $\mathcal{F}$ contains all self-dual Maass forms, which is composed exactly of the Maass forms that are obtained by the Gelbart-Jacquet lift [Gelbart and Jacquet, 1978], i.e. the ones that arise as symmetric square lifts of GL(2) Maass forms.

The approach used to prove Theorem 1.0.1 is to apply the Kuznetsov trace formula for GL(3) developed by Blomer [Blomer, 2013] and Goldfeld-Kontorovich [Goldfeld and Kontorovich, 2013], which we rederive carefully in Section 2.8, to a suitable family of smooth test functions related to $h_{T,R,\kappa}$.

The other main result is an inversion formula for the Lebedev-Whittaker transform in the nonarchimedean case, which can be described as follows. Let $h : Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p) \to C$ be a smooth function. The p-adic Whittaker transform of h is defined to be

$$h^\sharp(\alpha) := \int\limits_{T(\mathbb{Z}_p)\backslash T(\mathbb{Q}_p)} h(t)W_\alpha(t)d^\times t,$$

where W_α is a Whittaker function on $\mathrm{GL}_n(\mathbb{Q}_p)$.

Whittaker functions on local nonarchimedean fields (such as $\mathbb{Q}_p$) have been studied by several authors since they were introduced in 1967 by Jacquet [Jacquet, 1967]. Shintani [Shintani, 1976] obtained an explicit formula for these Whittaker functions, which was then generalized in different directions in other works (see [Casselman and Shalika, 1980], [Miyauchi, 2014]).

Theorem 1.0.2. *Let $H : \mathcal{S}^n \to \mathbb{C}$ holomorphic and permutation invariant, where $\mathcal{S}$ is an open annulus containing the unit circle. Let*

$$H^\flat(t) = \frac{1}{n!(2\pi i)^{n-1}} \int\limits_{\mathbb{T}} H(\beta)W_{1/\beta}(t) \prod_{\substack{i,j=1\\ i\neq j}}^{n} (\beta_i - \beta_j)\, d\beta^\star.$$

Under certain convergence conditions,

$$(H^\flat)^\sharp(\alpha) = H(\alpha)$$

for $\alpha_1 \cdots \alpha_n = 1$.

At the end of Chapter 3 we use this inversion formula to give an integral representation for a local L-function associated to a symmetric square lift. An example of another possible application would be on the choice of test functions for an adelic Kuznetsov trace formula, similarly to what has been achieved in [Goldfeld and Kontorovich, 2013]. We should remark that the inversion formula was obtained in the setting of p-adic reductive groups by Delorme [Delorme, 2013]. However, the approach presented here provides a more explicit presentation of the result in this particular setting and its simple derivation solely relies on complex analysis.

The thesis is organized as follows. In Chapter 2, we do a quick introduction to the theory of Hecke-Maass forms for $\mathrm{SL}(2,\mathbb{Z})$ and for $\mathrm{SL}(3,\mathbb{Z})$, Eisenstein series and Kloostermann sums. We also rederive a Kuznetsov trace formula due to Blomer [Blomer, 2013], including many of the details and calculations. Chapter 3 covers the Lebedev-Whittaker transform and its inversion formula, in both the archimdean and nonarchimedean cases. The inversion formula is derived for $\mathrm{GL}(3)$ in the archimedean case and for $\mathrm{GL}(n)$ in the nonarchimdean case (Theorem 1.0.2). Finally, Chapter 4 establishes Theorem 1.0.1, as an application of the Kuznetsov trace formula. We do a careful analysis of all terms that arise from the trace formula, in both the spectral and geometric side, obtaining bounds and asymptotic expressions for those terms.

Chapter 2

Preliminaries

In this chapter we lay the foundations for the rest of the thesis, by compiling definitions and results on Maass forms for $\mathrm{SL}(3,\mathbb{Z})$, Eisenstein series and Kloosterman sums. It culminates with the statement of a GL(3) Kuznetsov trace formula following [Blomer, 2013] and [Goldfeld and Kontorovich, 2013]. The GL(3) Kuznetsov trace formula is an equality which relates Fourier coefficients of Hecke-Maass forms and Eisenstein series on one side, to Kloosterman sums on the other side. Sections 2.2 and 2.4 introduce the theory of Hecke-Maass forms, Section 2.5 covers the theory of Eisenstein series, and Section 2.6 that of Kloosterman sums. In the final section of this chapter (Section 2.8) the Kuznetsov trace formula is stated and a sketch of its proof is given.

2.1 Maass Forms for $\mathbf{SL}(2,\mathbb{Z})$

In this section we state the definition of Maass form for $\mathrm{SL}(2,\mathbb{Z})$ and establish some of its properties. This should provide some insight into the theory of Maass forms for $\mathrm{SL}(3,\mathbb{Z})$ in the sections to follow.

Definition 2.1.1 (Upper Half Plane). *Let $\mathfrak{h}^2$ be the upper half plane*

$$\{z = x + iy : x, y \in \mathbb{R}, y > 0\}.$$

Remark 2.1.2. *Alternatively, the upper half plane can be realized as $GL(2,\mathbb{R})/(O(2,\mathbb{R})\cdot\mathbb{R}^\times)$ via the map*

$$z = x + iy \mapsto \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}.$$

There is an action of $\mathrm{GL}(2,\mathbb{R})$ on $\mathfrak{h}^2$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ and $z \in \mathfrak{h}^2$.

The space of complex-valued functions defined on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2$ (with the above action) can be equipped with the following inner product.

Definition 2.1.3 (Petersson Inner Product). *Let $\phi, \phi' : SL(2,\mathbb{Z})\backslash\mathfrak{h}^2 \to \mathbb{C}$. We define their inner product*

$$\langle \phi, \phi' \rangle := \int\limits_{SL(2,\mathbb{Z})\backslash\mathfrak{h}^2} \phi(z)\overline{\phi'(z)}\frac{dx\,dy}{y^2}.$$

Furthermore, define the L^2-norm as $\|\phi\|_2^2 := \langle \phi, \phi \rangle$ and the L^2-space as $\mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2) := \{\phi : SL(2,\mathbb{Z})\backslash\mathfrak{h}^2 \to \mathbb{C} : \|\phi\|_2 < \infty\}$.

Definition 2.1.4 (Maass form for $\mathbf{SL}(2,\mathbb{Z})$). *A Maass form of type $\nu \in \mathbb{C}$ for $SL(2,\mathbb{Z})$ is a non-zero smooth function $\phi \in \mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ satisfying*

- *ϕ is an eigenfunction of the Laplace operator $\Delta := -y^2\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)$ with eigenvalue $\frac{1}{4} - \nu^2$;*
- *ϕ is cuspidal, i.e., $\int\limits_0^1 \phi(z)\,dx = 0$.*

Remark 2.1.5. *The Laplace operator is invariant under the group actions*

$$z \mapsto \frac{az+b}{cz+d},$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$.

Definition 2.1.6. *For $n = 1, 2, \ldots,$ define the Hecke operator T_n acting on $\mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ by*

$$T_n\phi(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ 0\le b<d}} \phi\left(\frac{az+b}{d}\right),$$

where $\phi \in \mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2)$.

Definition 2.1.7 (Hecke-Maass Form). *A Hecke-Maass form for $SL(2,\mathbb{Z})$ is a Maass form ϕ for $SL(2,\mathbb{Z})$ satisfying:*

$$T_n\phi = \lambda_n\phi$$

for every $n = 1, 2, \ldots$. The λ_n are called the Hecke eigenvalues of ϕ.

With the above definitions we can now define an L-function associated to a Hecke-Maass form for $\mathrm{SL}(2,\mathbb{Z})$.

Definition 2.1.8 (L-function). *Let $s \in \mathbb{C}$, $\Re(s) > \frac{3}{2}$, and let ϕ be a Hecke-Maass form for $SL(2,\mathbb{Z})$ with Hecke eigenvalues λ_n. Define the L-function associated to ϕ, L_ϕ, by the series*

$$L_\phi(s) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n^s}.$$

We shall also define the Rankin-Selberg L-function associated to a Hecke-Maass form for $\mathrm{SL}(2,\mathbb{Z})$ as it will prove useful later on.

Definition 2.1.9 (Rankin-Selberg L-function). *Let $s \in \mathbb{C}$, $\Re(s)$ sufficiently large, and let ϕ be a Hecke-Maass form for $SL(2,\mathbb{Z})$ with Hecke eigenvalues λ_n. Define the L-function associated to ϕ, denoted L_ϕ, by the series*

$$L_{\phi\times\tilde{\phi}}(s) := \zeta(2s)\sum_{n=1}^{\infty} \frac{|\lambda_n|^2}{n^s},$$

where $\zeta(s) := \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function.

2.2 Maass Forms for $\mathrm{SL}(3,\mathbb{Z})$

The goal of this section is to establish a definition of a Maass form for $\mathrm{SL}(3,\mathbb{Z})$. Most of the components of the definition of a Maass form for $\mathrm{SL}(2,\mathbb{Z})$ can be generalized in a straightforward manner to $\mathrm{SL}(3,\mathbb{Z})$, with the exception of the Laplace eigenfunction condition which requires a bit of work. The role of the Laplace operator will be taken over by a rank two polynomial algebra of differential operators.

Definition 2.2.1 (Generalized Upper Half Plane). *The generalized upper half plane* $\mathfrak{h}^3$ *is the set of* 3×3 *matrices of the form* $z = x \cdot y$*, where*

$$x = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}, \; y = \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $x_1, x_2, x_3 \in \mathbb{R}$ *and* $y_1, y_2 > 0$.

Remark 2.2.2. *By the Iwasawa decomposition (see Proposition 1.2.6 [Goldfeld, 2006]) we have*

$$GL(3, \mathbb{R}) = \mathfrak{h}^3 \cdot O(3, \mathbb{R}) \cdot \mathbb{R}^\times, \qquad \mathfrak{h}^3 \cong GL(3, \mathbb{R})/(O(3, \mathbb{R}) \cdot \mathbb{R}^\times).$$

From the description of $\mathfrak{h}^3$ in the remark we can see that there is a group action of $\mathrm{SL}(3, \mathbb{Z})$ on $\mathfrak{h}^3$ by left multiplication. This action gives rise to the quotient space $\mathrm{SL}(3, \mathbb{Z})\backslash\mathfrak{h}^3$. We can also define an inner product for complex-valued functions on this space.

Definition 2.2.3 (Petersson Inner Product on $\mathfrak{h}^3$). *Let* $\phi, \phi' : SL(3, \mathbb{Z})\backslash\mathfrak{h}^3 \to \mathbb{C}$*. We define their inner product*

$$\langle \phi, \phi' \rangle := \int_{SL(3,\mathbb{Z})\backslash\mathfrak{h}^3} \phi(z)\overline{\phi'(z)}\, d^*z,$$

where $d^*z := \frac{dx_1\, dx_2\, dx_3\, dy_1\, dy_2}{y_1^3 y_2^3}$.

Furthermore, define the L^2*-norm as* $\|\phi\|_2^2 := \langle \phi, \phi \rangle$*, and the* L^2*-space as*

$$\mathcal{L}^2(SL(3, \mathbb{Z})\backslash\mathfrak{h}^3) := \left\{\phi : SL(3, \mathbb{Z})\backslash\mathfrak{h}^3 \to \mathbb{C} : \|\phi\|_2 < \infty\right\}.$$

Remark 2.2.4. *The measure* $d^*z = \frac{dx_1\, dx_2\, dx_3\, dy_1\, dy_2}{y_1^3 y_2^3}$ *is the left-invariant Haar measure on* $\mathfrak{h}^3$ *(see Proposition 1.5.2 in [Goldfeld, 2006]).*

Definition 2.2.5. *Let* $\alpha \in \mathfrak{gl}(3, \mathbb{R})$*, the Lie algebra of* $GL(3, \mathbb{R})$*, and let* $\phi : GL(3, \mathbb{R}) \to \mathbb{C}$ *be a smooth function, we define*

$$D_\alpha \phi(g) := \frac{\partial}{\partial t} \phi(g \cdot \exp(t\alpha))\bigg|_{t=0},$$

with $g \in GL(3, \mathbb{R})$.

The differential operators D_α with $\alpha \in \mathfrak{gl}(3, \mathbb{R})$ generate an associative algebra $\mathcal{D}^3$ defined over $\mathbb{R}$, where multiplication of two operators is given by their composition. Let $\mathfrak{D}^3$ be the center of $\mathcal{D}^3$.

Lemma 2.2.6. *Every $D \in \mathfrak{D}^3$ is well-defined on smooth function on $SL(3,\mathbb{Z})\backslash\mathfrak{h}^3$. In other words, for*

$$\phi : SL(3,\mathbb{Z})\backslash GL(3,\mathbb{R})/\left(O(3,\mathbb{R})\cdot\mathbb{R}^\times\right) \to \mathbb{C}$$

we have

$$(D\phi)(\gamma\cdot g\cdot k\cdot z) = D\phi(g),$$

where $\gamma \in SL(3,\mathbb{Z})$, $g \in GL(3,\mathbb{R})$, $k \in O(3,\mathbb{R})$, $z \in \mathbb{R}^\times$ is a scalar matrix.

Proof. See Proposition 2.3.1 [Goldfeld, 2006]. □

The subalgebra $\mathfrak{D}^3$ of these differential operators has a rather explicit description.

Proposition 2.2.7. *Let $E_{i,j} \in \mathfrak{gl}(3,\mathbb{R})$ be the matrix with the value 1 at the (i,j)-th entry and zeros elsewhere, for $1 \le i,j \le 3$. Define the differential operators*

$$\Delta_2 := \sum_{i_1=1}^{3}\sum_{i_2=1}^{3} D_{E_{i_1,i_2}} \circ D_{E_{i_2,i_1}}, \Delta_3 := \sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{i_1=1}^{3} D_{E_{i_1,i_2}} \circ D_{E_{i_2,i_3}} \circ D_{E_{i_3,i_1}}.$$

Then, every $D \in \mathfrak{D}^3$ can be written as polynomial (with coefficients in $\mathbb{R}$) in Δ_2 and Δ_3, i.e,

$$\mathfrak{D}^3 = \mathbb{R}[\Delta_2, \Delta_3].$$

Furthermore,

$$\Delta_2 = y_1^2\frac{\partial^2}{\partial y_1^2} + y_2^2\frac{\partial^2}{\partial y_2^2} - y_1y_2\frac{\partial^2}{\partial y_1\partial y_2} + y_1^2(x_2^2+y_2^2)\frac{\partial^2}{\partial x_3^2} + y_1^2\frac{\partial^2}{\partial x_1^2} + y_2^2\frac{\partial^2}{\partial x_2^2} + 2y_1^2x_2\frac{\partial^2}{\partial x_1\partial x_3}$$

and $-\Delta_2$ is the Laplace operator

Proof. See Proposition 2.3.5 and Equation 6.1.1 [Goldfeld, 2006]. □

Now we define a family of functions on $\mathfrak{h}^3$ that are eigenfunctions for all $D \in \mathfrak{D}^3$.

Definition 2.2.8. *For $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$ and $z = x\cdot y \in \mathfrak{h}^3$ (with x and y as in Definition 2.2.1), we define the function $I_s : \mathfrak{h}^3 \to \mathbb{C}$, by the condition:*

$$I_\nu(z) := y_1^{1+\nu_1+2\nu_2}y_2^{1+2\nu_1+\nu_2}.$$

Lemma 2.2.9. *For every $\nu \in \mathbb{C}^2$ and $D \in \mathfrak{D}^3$, the function I_ν is an eigenfunction of D. We denote by $\lambda_\nu(D)$ the corresponding eigenvalue.*

Proof. See Proposition 2.4.3 [Goldfeld, 2006]. Our definition of I_ν differs by the map $\nu_1 \mapsto \nu_1 - \frac{1}{3}$, $\nu_2 \mapsto \nu_2 - \frac{1}{3}$. □

Remark 2.2.10. *The Laplace eigenvalue $\lambda_\nu(-\Delta_2)$ is given by $1 - 3(\nu_1^2 + \nu_1\nu_2 + \nu_2^2)$. This is a straightforward computation using the explicit decription of $-\Delta_2$ given in Proposition 2.2.7. Note that $I_\nu(z)$ does not depend on x_1, x_2, x_3 so*

$$\begin{aligned} -\Delta_2 I_\nu &= -y_1^2 \frac{\partial^2 I_\nu}{\partial y_1^2} - y_2^2 \frac{\partial^2 I_\nu}{\partial y_2^2} + y_1 y_2 \frac{\partial^2 I_\nu}{\partial y_1 \partial y_2} \\ &= (1 + \nu_1 + 2\nu_2)(1 + 2\nu_1 + \nu_2) I_\nu - (1 + \nu_1 + 2\nu_2)(\nu_1 + 2\nu_2) I_\nu - (1 + 2\nu_1 + \nu_2)(2\nu_1 + \nu_2) I_\nu \\ &= \left(1 - 3(\nu_1^2 + \nu_1\nu_2 + \nu_2^2)\right) I_\nu. \end{aligned}$$

We can now define the notion of Maass form for $\mathrm{SL}(3, \mathbb{Z})$.

Definition 2.2.11 (Maass Form for SL$(3, \mathbb{Z})$). *A Maass form of type $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ for SL$(3, \mathbb{Z})$ is a non-zero smooth function $\phi \in \mathcal{L}^2(SL(3, \mathbb{Z}) \backslash \mathfrak{h}^3)$ satisfying*

- *for every $D \in \mathfrak{D}^3$, ϕ is an eigenfunction of D with eigenvalue $\lambda_\nu(D)$;*
- *ϕ is cuspidal, i.e., $\int\limits_{(SL(3,\mathbb{Z}) \cap U) \backslash U} \phi(uz)\, du = 0$ for $U = U_{1,2}, U_{2,1}$, where*

$$U_{1,2} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad U_{2,1} = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Remark 2.2.12. *This definition can be easily generalized to define Maass forms for congruence subgroups of SL$(3, \mathbb{Z})$.*

2.3 Fourier-Whittaker Expansion

In this section we define Whittaker functions, establish some of their properties and state the Fourier-Whittaker expansion of a Maass form for $\mathrm{SL}(3, \mathbb{Z})$.

Definition 2.3.1 (Siegel set). *Let $a, b \geq 0$. We define the Siegel set $\Sigma_{a,b} \subset \mathfrak{h}^3$ to be the set of all matrices $z = x \cdot y$ such that*

$$|x_1|, |x_2|, |x_3| \leq b, \qquad y_1, y_2 > a,$$

where x and y are defined as in 2.2.1.

For $m = (m_1, m_2) \in \mathbb{Z}^2$ and $u \in U_3(\mathbb{R})$ of the form

Let $U_3(\mathbb{R})$ be the set of matrices u of the form

$$u = \begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

where $u_1, u_2, u_3 \in \mathbb{R}$.

Their characters can be indexed by a pair of integers $m = (m_1, m_2)$ such that the character ψ_m of $U_3(\mathbb{R})$ is defined on a matrix u as in 2.1 by

$$\psi_m(u) := e^{2\pi i(m_1 u_1 + m_2 u_2)}. \tag{2.2}$$

Definition 2.3.2 (Whittaker Function). *An $SL(3,\mathbb{Z})$-Whittaker function of type $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, associated to a character ψ of $U_3(\mathbb{R})$, is a smooth function $\Psi : \mathfrak{h}^3 \to \mathbb{C}$ satisfying the following conditions:*

- $\Psi(uz) = \psi(u)\Psi(z)$, *for all* $u \in U_3(\mathbb{R})$, $z \in \mathfrak{h}^3$;
- $D\Psi(z) = \lambda_\nu(D)\Psi(z)$, *for all* $D \in \mathfrak{D}^3$, $z \in \mathfrak{h}^3$;
- $\int\limits_{\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}} |\Psi(z)|^2 \, d^*z < \infty$.

Definition 2.3.3 (Jacquet's Whittaker Function). *Fix a character ψ_m of $U_3(\mathbb{R})$ (as in 2.1, 2.2), with non-zero m_1, m_2, and $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. Define Jacquet's Whittaker function to be*

$$W_\nu(z; m) := \int\limits_{U_3\mathbb{R}} I_\nu(w \cdot u \cdot z)\overline{\psi_m(u)} \, d^*u,$$

where $z \in \mathfrak{h}^3$, $d^*u = du_1\, du_2\, du_3$, and $w = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$.

Remark 2.3.4. *This integral representation only makes sense for $\Re(\nu_1), \Re(\nu_2) \gg 1$, where the integral converges (absolutely and uniformly in compact sets). The function can be extended for any $\nu \in \mathbb{C}^2$ analytically.*

Remark 2.3.5. *We can also define the completed Jacquet's Whittaker function to be*

$$W_\nu^*(z;m) := \pi^{-3\nu_1-3\nu_2}\Gamma\left(\frac{1+3\nu_1}{2}\right)\Gamma\left(\frac{1+3\nu_2}{2}\right)\Gamma\left(\frac{1+3\nu_1+3\nu_2}{2}\right)W_\nu(z;m).$$

This normalization will be useful for some of the formulas.

Proposition 2.3.6 (Multiplicity One). *The function $W_\nu(z;m)$ is an $SL(3,\mathbb{Z})$-Whittaker function of type $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$ associated to a character ψ of $U_3(\mathbb{R})$. Furthermore, if Ψ is an $SL(3,\mathbb{Z})$-Whittaker function of type $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$ associated to a character ψ of $U_3(\mathbb{R})$, with sufficient decay in y_1, y_2, so that*

$$\int_0^\infty\int_0^\infty y_1^{\sigma_1}y_2^{\sigma_2}|\Psi(y)|\,dy_1\,dy_2$$

for sufficiently large σ_1, σ_2, then

$$\Psi(z) = c\cdot W_\nu(z;m)$$

for some $c\in\mathbb{C}$.

Proof. See Proposition 5.5.2 and Theorem 6.1.6 [Goldfeld, 2006]. □

Theorem 2.3.7 (Fourier-Whittaker Expansion). *Let ϕ be a Maass form of type ν for $SL(3,\mathbb{Z})$ then*

$$\phi(z) = \sum_{\gamma\in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})}\sum_{m_1=1}^\infty\sum_{m_2\neq 0}\frac{A(m_1,m_2)}{|m_1m_2|}\cdot W_\nu^*\left(\begin{pmatrix}|m_1m_2| & & \\ & m_1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}z;(1,\operatorname{sgn}(m_2))\right),$$

where $A(m_1,m_2)\in\mathbb{C}$ is the (m_1,m_2)-th Fourier coefficient.

Proof. See Theorem 5.3.2 and Equation 6.2.1 [Goldfeld, 2006]. □

Remark 2.3.8. *The Fourier coefficients $A(m_1,m_2)$ can be computed the following way:*

$$\frac{A(m_1,m_2)}{|m_1m_2|}\cdot W_\nu^*\left(\begin{pmatrix}|m_1m_2|y_1y_2 & & \\ & m_1y_1 & \\ & & 1\end{pmatrix};(1,\operatorname{sgn}(m_2))\right)$$
$$=\int_0^1\int_0^1\int_0^1\phi(z)e^{-2\pi i(m_1x_1+m_2x_2)}\,dx_1\,dx_2\,dx_3$$

Note that the Whittaker functions that show up in the Fourier-Whittaker expansion (Theorem 2.3.7) are associated to either the character $\psi_{(1,1)}$ or the character $\psi_{(1,-1)}$.

Define $W_\nu^*(z) := W_\nu(z;(1,1))$ dropping the dependence on the character. If $z = y$ is a diagonal matrix then we have $W_\nu^*(y) = W_\nu^*(y;(1,1)) = W_\nu^*(y;(1,-1))$. We shall also denote

$$W_\nu^*(y_1, y_2) := W_\nu^*\left(\begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right).$$

Define $W_\nu(z)$ and $W_\nu(y_1, y_2)$ similarly.

Proposition 2.3.9 (Double Mellin Inversion). *The Whittaker function W_ν^* has an integral representation as a double Mellin inversion*

$$W_\nu^*(y) = \frac{y_1 y_2 \pi^{3/2}}{(2\pi i)^2} \int_{(C_2)} \int_{(C_1)} \frac{\prod_{j=1}^3 \Gamma\left(\frac{s_1-\alpha_j}{2}\right)\Gamma\left(\frac{s_2+\alpha_j}{2}\right)}{4\pi^{s_1+s_2}\Gamma\left(\frac{s_1+s_2}{2}\right)} y_1^{-s_1} y_2^{-s_2}\, ds_1\, ds_2,$$

where $y_1, y_2 > 0$, $\nu \in \mathbb{C}^2$,

$$\alpha_1 = 2\nu_1 + \nu_2, \quad \alpha_2 = \nu_2 - \nu_1, \quad \alpha_3 = -\nu_1 - 2\nu_2,$$

$C_i > \max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$ and (C_i) denotes the integration path (oriented upward) along the vertical line $\Re(s_i) = C_i$.

Proof. See Equation 10.1 [Bump, 1984]. Notice that [Bump, 1984] has the roles of ν_1, ν_2 interchanged in the definition of I_v and that our completed Whittaker function differs by a factor of $\pi^{3/2}$. □

Remark 2.3.10. *Note that $W_\nu^*(y)$ is invariant under permutations of $(\alpha_1, \alpha_2, \alpha_3)$, and*

$$W^*_{(\nu_1,\nu_2)}(y_1, y_2) = W^*_{(\nu_2,\nu_1)}(y_2, y_1) = \overline{W^*_{(\overline{\nu_1},\overline{\nu_2})}(y_1, y_2)}.$$

Using the double Mellin transform and some estimates on the Gamma function it is possible to obtain some bounds on the Whittaker function.

Proposition 2.3.11. *Let $\theta = \max(|\Re(\alpha_1)|, |\Re(\alpha_2)|, |\Re(\alpha_3)|)$ and assume $\theta \le 1/2$. Let $\theta < \sigma_1 < \sigma_2$ and $\varepsilon > 0$. Then for any $\sigma_1 \le c_1, c_2 \le \sigma_2$ we have*

$$W_\nu^*(y) \ll \frac{y_1 y_2}{(1+|\nu_1|+|\nu_2|)^{1/2-\varepsilon}} \left(\frac{y_1}{1+|\nu_1|+|\nu_2|}\right)^{-c_1} \left(\frac{y_2}{1+|\nu_1|+|\nu_2|}\right)^{-c_2}.$$

Proof. See Proposition 1 [Blomer, 2013]. □

Proposition 2.3.12 (Stade's Formula). *For $s \in \mathbb{C}$, $\nu, \mu \in \mathbb{C}^2$,*

$$\int_0^\infty \int_0^\infty W_\nu^*(y)\overline{W_\mu^*(y)}(y_1^2 y_2)^s \frac{dy_1\, dy_2}{(y_1 y_2)^3} = \frac{\pi^{3(1-s)}}{\Gamma\left(\frac{3s}{2}\right)} \prod_{1 \le j,k \le 3} \Gamma\left(\frac{s+\alpha_j+\overline{\beta_k}}{2}\right),$$

where the α_i are defined as in Proposition 2.3.9 and the β_i are defined analogously in terms of μ_1, μ_2.

Proof. See Theorem 3.1 [Stade, 1993] observing that Stade's definition of the Whittaker function differs by a factor of $\pi^{3/2}$. □

2.4 Hecke-Maass Forms and L-functions

In this section we define Hecke operators and their simultaneous eigenfunctions, the Hecke-Maass forms. We then define L-functions attached to these forms, which have an Euler product.

Definition 2.4.1. *For $n = 1, 2, \ldots$, define the Hecke operator T_n acting on $\mathcal{L}^2(SL(3,\mathbb{Z})\backslash\mathfrak{h}^3)$ by*

$$T_n\phi(z) = \frac{1}{n} \sum_{\substack{abc=n \\ 0\le c_1,c_2<c \\ 0\le b_1<b}} \phi\left(\begin{pmatrix} a & b_1 & c_1 \\ 0 & b & c_2 \\ 0 & 0 & c \end{pmatrix} z\right),$$

where $\phi \in \mathcal{L}^2(SL(3,\mathbb{Z})\backslash\mathfrak{h}^3)$.

The operators T_n commute with the differential operators in $\mathfrak{D}^3$. Therefore, we may simultaneously diagonalize the space $\mathcal{L}^2(\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{h}^3)$ by all these operators.

Definition 2.4.2 (Hecke-Maass form). *A Hecke-Maass form for $SL(3,\mathbb{Z})$ is a Maass form ϕ for $SL(3,\mathbb{Z})$ satisfying:*

$$T_n\phi = \lambda_n\phi$$

for every $n = 1, 2, \ldots$. The λ_n are called the Hecke eigenvalues of ϕ.

Theorem 2.4.3 (Hecke Relations). *Let ϕ be a Hecke-Maass form for $SL(3,\mathbb{Z})$ with Fourier coefficients $A(m_1, m_2)$, normalized such that $A(1,1) = 1$. Then,*

$$T_n\phi = A(n,1)\phi.$$

Furthermore, we have the following relations

$$\begin{aligned}
A(m_1m_1', m_2m_2') &= A(m_1,m_2)A(m_1',m_2'), \ \text{if } (m_1m_2, m_1'm_2') = 1;\\
A(n,1)A(m_1,m_2) &= \sum_{\substack{abc=n\\ a|m_1\\ b|m_2}} A\left(\frac{m_1c}{a}, \frac{m_2a}{b}\right);\\
A(1,n)A(m_1,m_2) &= \sum_{\substack{abc=n\\ a|m_1\\ b|m_2}} A\left(\frac{m_1b}{a}, \frac{m_2c}{b}\right);\\
A(m_1,m_2) &= \sum_{d|(m_1,m_2)} \mu(d)A\left(\frac{m_1}{d},1\right)A\left(1,\frac{m_2}{d}\right),
\end{aligned}$$

where μ is the Möbius function.

Proof. See Theorem 6.4.11 [Goldfeld, 2006] for the first statement and the first three relations between the Fourier coefficients. To obtain the last relation, we start by choosing $m_1 = 1$ in the second relation to obtain

$$A(n,1)A(1,m_2) = \sum_{d|(n,m_2)} A\left(\frac{n}{d},1\right)A\left(1,\frac{m_2}{d}\right).$$

By Möbius inversion it follows that

$$\begin{aligned}
\sum_{d|(m_1,m_2)} \mu(d)A\left(\frac{m_1}{d},1\right)A\left(1,\frac{m_2}{d}\right) &= \sum_{d|(m_1,m_2)} \mu(d) \sum_{ed|(m_1,m_2)} A\left(\frac{m_1}{de},\frac{m_2}{de}\right)\\
&= \sum_{e'|(m_1,m_2)} A\left(\frac{m_1}{e'},\frac{m_2}{e'}\right)\sum_{d|e'}\mu(d)\\
&= A(m_1,m_2).
\end{aligned}$$

□

Definition 2.4.4 (Standard L-function). *Let $s \in \mathbb{C}$, $\Re(s) > 2$, and let ϕ be a Hecke-Maass form for $SL(3,\mathbb{Z})$. Define the standard L-function associated to ϕ, denoted L_ϕ, by the series*

$$L_\phi(s) := \sum_{n=1}^{\infty} \frac{A(1,n)}{n^s}.$$

Remark 2.4.5. *The series that defines* $L_\phi(s)$ *is absolutely convergent in the region* $\Re(s) > 2$ *as* $A(1,n) = \mathcal{O}(n)$ *(see Lemma 6.2.2 [Goldfeld, 2006]).*

Remark 2.4.6. *By the first relation in Theorem 2.4.3 one can factor* $L_\phi(s)$ *as*

$$\prod_p L_{\phi,p}(s),$$

where $L_{\phi,p}(s) := \sum_{k=1}^{\infty} \frac{A(1,p^k)}{p^{ks}}$. *By the second and third relations in Theorem 2.4.3 one obtains*

$$A(p,1)A(1,p^k) - A(1,p)A(1,p^{k+1}) = A(1,p^{k-1}) - A(1,p^{k+2}) \text{ for } k \geq 1,$$

$$A(p,1) - A(1,p)A(1,p) = -A(1,p^2).$$

Therefore,

$$A(p,1)L_{\phi,p}(s) - A(1,p)\left(p^s L_{\phi,p}(s) - p^s\right) = p^{-s}L_{\phi,p}(s) - \left(p^{2s}L_{\phi,p}(s) - p^{2s} - p^s A(1,p)\right).$$

Solving the above equation for $L_{\phi,p}(s)$ *yields*

$$L_{\phi,p}(s) = \left(1 - p^{-s}A(1,p) + p^{-2s}A(p,1) - p^{-3s}\right)^{-1}.$$

Definition 2.4.7 (Satake Parameters at p**).** *The roots* $\alpha_{1,p}, \alpha_{2,p}, \alpha_{3,p} \in \mathbb{C}$ *of the polynomial*

$$1 - p^{-s}A(1,p) + p^{-2s}A(p,1) - p^{-3s}$$

are called the Satake parameters of ϕ *at* p.

Theorem 2.4.8 (Functional Equation). *Let* ϕ *be a Hecke-Maass form of type* $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ *for* $SL(3,\mathbb{Z})$, *with associated L-function* $L_\phi(s)$. *Let* $\tilde{L}_\phi(s) := \sum_{n=1}^{\infty} A(n,1)n^{-s}$ *be the dual L-function. Then* $L_\phi(s)$, $\tilde{L}_\phi(s)$, *have holomorphic continuation for* $s \in \mathbb{C}$ *and satisfy the function equation*

$$F_\nu(s)L_\phi(s) = \tilde{F}_\nu(1-s)\tilde{L_\phi}(1-s),$$

where

$$F_\nu(s) = \pi^{-3s/2}\prod_{j=1}^{3}\Gamma\left(\frac{s-\alpha_j}{2}\right),$$

$$\tilde{F}_\nu(s) = \pi^{-3s/2}\prod_{j=1}^{3}\Gamma\left(\frac{s+\alpha_j}{2}\right).$$

Here $\alpha_1 = 2\nu_1 + \nu_2$, $\alpha_2 = \nu_2 - \nu_1$, $\alpha_3 = -\nu_1 - 2\nu_2$ *are the Langlands parameters of* ϕ.

Proof. See Theorem 6.5.15 [Goldfeld, 2006]. Recall that we have normalized $\nu_1 \mapsto \nu_1 + \frac{1}{3}$ and $\nu_2 \mapsto \nu_2 + \frac{1}{3}$. □

2.5 Eisenstein Series and Spectral Decomposition

Definition 2.5.1 (Minimal Parabolic Eisenstein Series). *Define the minimal standard parabolic subgroup for $GL(3,\mathbb{R})$ as*

$$P_{1,1,1} := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

For $z \in \mathfrak{h}^3$ and $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, with $\Re(\nu_1)$, $\Re(\nu_2)$ sufficiently large, define the minimal parabolic Eisenstein series $E_{\min}(z,\nu)$ by the series

$$E_{\min}(z,\nu) := \sum_{\gamma \in (P_{1,1,1} \cap SL(3,\mathbb{Z})) \backslash SL(3,\mathbb{Z})} I_\nu(\gamma z).$$

This function has meromorphic continuation to all $\nu_1, \nu_2 \in \mathbb{C}$.

Proposition 2.5.2. *The minimal parabolic Eisenstein series $E_{\min}(z,\nu)$ has a Fourier-Whittaker expansion (as in 2.3.7) and its Fourier coefficients (for nonzero m_1, m_2) are given by*

$$\frac{A_\nu(m_1,m_2)}{|m_1 m_2|} \frac{W_\nu\left(m_1 y_1, |m_2| y_2\right)}{\zeta(1+3\nu_1)\zeta(1+3\nu_2)\zeta(1+3\nu_1+3\nu_2)} = \int_0^1 \int_0^1 \int_0^1 E_{\min}(z,\nu) e^{-2\pi i (m_1 x_1 + m_2 x_2)}\, dx_1\, dx_2\, dx_3,$$

where $A_\nu(m_1,m_2)$ satisfy the Hecke relations 2.4.3, and

$$A_\nu(m,1) = \sum_{d_1 d_2 d_3 = m} d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3}.$$

Proof. See Theorem 10.8.1 [Goldfeld, 2006]. Due to the normalization in that statement the zeta factors do not show up. See Theorem 7.2 [Bump, 1984] for the computation of those factors. □

Definition 2.5.3 (Maximal Parabolic Eisenstein Series). *Let*

$$P_{2,1} := \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

be a maximal standard parabolic subgroup for $GL(3,\mathbb{R})$. Let $z \in \mathfrak{h}^3$, $s \in \mathbb{C}$, with $\Re(s)$ sufficiently large, and let u be a Hecke-Maass form of type μ for $SL(2,\mathbb{Z})$, normalized such that $\|u\|_2 = 1$. Define the maximal parabolic Eisenstein series associated to u, denoted $E_{\max}(z,s,u)$, by

$$E_{\max}(z,\nu,u) := \sum_{\gamma \in (P_{2,1} \cap SL(3,\mathbb{Z})) \backslash SL(3,\mathbb{Z})} \det(\gamma z)^{1/2+s} u(\tau(\gamma z)),$$

where

$$\tau(z) = \begin{pmatrix} y_2 & x_2 \\ 0 & 1 \end{pmatrix}.$$

Proposition 2.5.4. *The maximal parabolic Eisenstein series $E_{\max}(z,s,u)$ has a Fourier-Whittaker expansion (as in 2.3.7) and its Fourier coefficients (for nonzero m_1, m_2) are given by*

$$c\frac{B_{s,u}(m_1,m_2)}{|m_1 m_2|}\frac{W_{\left(\frac{2\mu}{3},s-\frac{\mu}{3}\right)}(m_1y_1,|m_2|y_2)}{L_u(1+3s)L_{Ad\;u}(1)^{1/2}} = \int_0^1\int_0^1\int_0^1 E_{\max}(z,\nu,u)e^{-2\pi i(m_1x_1+m_2x_2)}\,dx_1\,dx_2\,dx_3,$$

where $B_{s,u}(m_1,m_2)$ satisfy the Hecke relations 2.4.3, and

$$B_{s,u}(m,1) = \sum_{d_1d_2=m}\lambda_u(d_1)d_1^s d_2^{-2s}.$$

Here L_{Adu} is the adjoint L-function defined given by

$$L_{Ad\;u}(s) = \frac{L_{u\times\tilde{u}}(s)}{\zeta(s)},$$

and c is a nonzero absolute constant.

Proof. See page 18 [Blomer, 2013]. □

Proposition 2.5.5. *Let $\nu\in\mathbb{C}^2$ such that $\Re(\nu_1)=\Re(\nu_2)=0$, and $s\in\mathbb{C}$, such that $\Re(s)=0$. Then, the Fourier coefficients $A_\nu(m_1,m_2)$ and $B_{s,u}(m_1,m_2)$ satisfy*

$$A_\nu(m_1,m_2) = \mathcal{O}((m_1m_2)^{\varepsilon}), \qquad B_{s,u}(m_1,m_2) = \mathcal{O}((m_1m_2)^{1/2+\varepsilon}),$$

for every $\varepsilon>0$.

Proof. By Proposition 2.5.2 we have

$$\begin{aligned} A_\nu(m,1) &= \sum_{d_1d_2d_3=m} d_1^{\alpha_1}d_2^{\alpha_2}d_3^{\alpha_3} \\ &\le \sigma(m)^3 \\ &= \mathcal{O}(m^{\varepsilon}). \end{aligned}$$

By Proposition 2.5.4 we have

$$\begin{aligned} B_{s,u}(m,1) &= \sum_{d_1 d_2 = m} \lambda_u(d_1) d_1^s d_2^{-2s} \\ &= \mathcal{O}\left(\sum_{d_1 \mid m} \sqrt{d_1} \right) \\ &= \mathcal{O}(m^{1/2+\varepsilon}), \end{aligned}$$

where the first inequality follows from $\lambda_u(n) = \mathcal{O}(\sqrt{n})$ (see Proposition 3.6.3 [Goldfeld, 2006]). Analogously one can obtain the same bounds for $A_\nu(1,m)$, $B_{s,u}(1,m)$ (by considering the dual objects). The result then follows from the Hecke relations for $A_\nu(m_1,m_2)$, $B_{s,u}(m_1,m_2)$. □

A degenerate case of a maximal parabolic Eisenstein series is given by $E_{\max}(z,s,u)$, where u is a constant function. For $z \in \mathfrak{h}^3$, $s \in \mathbb{C}$, with $\Re(s)$ sufficiently large, define

$$E_{\max}(z,s,\mathbf{1}) := \sum_{\gamma \in (P_{2,1} \cap \mathrm{SL}(3,\mathbb{Z})) \backslash \mathrm{SL}(3,\mathbb{Z})} \det(\gamma z)^{1/2+s}.$$

This Eisenstein series only has degenerate terms (corresponding to $m_1 m_2 = 0$) in its Fourier expansion (see Theorem 2.4 [Friedberg, 1987]).

Define the notation $\int\limits_{(C)}$ to denote the integral along the complex vertical line whose points have real part equal to C.

Theorem 2.5.6 (Langlands Spectral Decomposition). *Let $\phi \in \mathcal{L}^2(SL(3,\mathbb{Z})\backslash\mathfrak{h}^3)$ with sufficient decay such that $\langle \phi, E \rangle$ converges for all Eisenstein series defined in this section. Assume that ϕ is orthogonal to all residues of all such Eisenstein series. Then the function*

$$\phi(z) - \frac{1}{(4\pi i)^2} \int\limits_{(0)} \int\limits_{(0)} \langle \phi, E_{\min}(*,\nu) \rangle E_{\min}(z,\nu)\, d\nu_1\, d\nu_2 - \frac{1}{2\pi i} \sum_{j=0}^{\infty} \int\limits_{(0)} \langle \phi, E_{\max}(*,s,u_j) \rangle E_{\max}(z,s,u_j)\, ds$$

is a Maass form for $SL(3,\mathbb{Z})$, where $\{u_j\}$ for $j = 1, 2, \ldots$ is a basis of Maass forms for $SL(2,\mathbb{Z})$ normalized such that $\|u_j\|_2 = 1$.

Proof. See Theorem 10.13.1 [Goldfeld, 2006]. □

2.6 Kloosterman Sums

The standard Kloosterman sum $S(m,n,c)$ is defined by

$$S(m,n,c) := \sum_{d \pmod{c}}^{*} e\left(\frac{md+n\overline{d}}{c}\right),$$

where $\overline{d}$ denote the inverse of $d \pmod{c}$ and $e(x) := e^{2\pi i x}$. Here $*$ means that the sum is over invertible classes $\pmod{c}$.

Using geometric arguments Weil [Weil, 1948] showed the following estimate for these sums

$$|S(m,n,c)| \ll c^{1/2+\varepsilon}(m,n,c)^{\varepsilon}.$$

In order to define Kloosterman sums for $\mathrm{SL}(3,\mathbb{Z})$ we will need a few facts about the structure of $\mathrm{GL}(3,\mathbb{R})$.

Definition 2.6.1 (Weyl Group). *Let T be the set of diagonal matrices of $GL(3,\mathbb{R})$. Define the Weyl group W of $GL(3,\mathbb{R})$ to be the quotient N/T, where N is the normalizer of T in $GL(3,\mathbb{R})$, i.e, every element $n \in N$ commutes with every element of T.*

We can choose the following representatives for W:

$$w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$w_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad w_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad w_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proposition 2.6.2 (Bruhat Decomposition). *The group $GL(3,\mathbb{R})$ decomposes as*

$$GL(3,\mathbb{R}) = \bigcup_{w \in W} G_w,$$

where $G_w = U_3(\mathbb{R})\, w\, T\, U_3(\mathbb{R})$.

Proof. See Proposition 10.3.2 [Goldfeld, 2006]. □

Let $\Gamma := \mathrm{SL}(3,\mathbb{Z})$ and $\Gamma_w := (w^{-1} \cdot U_3(\mathbb{Z})^t \cdot w) \cap U_3(\mathbb{Z})$, where t denotes matrix transposition.

Definition 2.6.3 (SL$(3,\mathbb{Z})$ Kloosterman Sum). *Let $w \in W$, $d \in T$, and ψ, ψ' be characters of U_3. Define the $SL(3,\mathbb{Z})$ Kloosterman sum S_w by*

$$S_w(\psi,\psi',d) := \sum_{\substack{\gamma \in U_3(\mathbb{Z})\backslash(\Gamma\cap G_w)/\Gamma_w \\ \gamma = u_1 w d u_2}} \psi(u_1)\psi'(u_2),$$

provided the sum is well-defined, i.e., it does not depend on the Bruhat decomposition of γ. Otherwise, define the Kloosterman sum as zero.

A necessary and sufficient condition for these Kloosterman sums to be well-defined is given in the following lemma.

Lemma 2.6.4. *The Kloosterman sum $S_w(\psi,\psi',d)$ is well-defined if and only if*

$$\psi(dwuw^{-1}) = \psi'(u)$$

for all $u \in (w^{-1} \cdot U_3(\mathbb{R}) \cdot w) \cap U_3(\mathbb{R})$. The characters ψ is extended to $z = uy \in \mathfrak{h}^3$ by $\psi(uy) = \psi(u)$ with the matrix decomposition as in Definition 2.2.1.

Proof. See Lemma 10.6.3 and Proposition 11.2.10 [Goldfeld, 2006]. □

Let

$$d = \begin{pmatrix} d_1 & & \\ & -d_2/d_1 & \\ & & 1/d_2 \end{pmatrix},$$

with d_1, d_2 positive integers. Let $\psi(u) = \psi_{n_1,n_2}(u) = e^{2\pi i(n_1u_1+n_2u_2)}$, where $u = \left(\begin{smallmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Let $\psi' = \psi_{m_2,m_1}$. Assume both ψ,ψ' are non-degenerate, i.e, $n_1n_2 \neq 0$, $m_1m_2 \neq 0$.

These Kloosterman sums have been computed in an explicit form by Bump, Friendberg and Golfeld [Bump *et al.*, 1988]. We collect this information in the following proposition.

Proposition 2.6.5. *Let $d \in T$ and ψ, ψ' be non-degenerate characters of $U_3(\mathbb{R})$, as above. Then,*

$$\begin{aligned}
S_{w_1}(\psi, \psi', d) &= \delta_{d_1,1}\delta_{d_2,1}, \\
S_{w_2}(\psi, \psi', d) &= S_{w_3}(\psi, \psi', d) = 0, \\
S_{w_4}(\psi, \psi', d) &= \tilde{S}(m_1, n_1, n_2, d_1, d_2) \text{ if } d_1|d_2, \\
S_{w_5}(\psi, \psi', d) &= \tilde{S}(m_2, n_2, n_1, d_2, d_1) \text{ if } d_2|d_2, \\
S_{w_6}(\psi, \psi', d) &= S(m_1, m_2, n_1, n_2, d_1, d_2),
\end{aligned}$$

where δ is the Kronecker delta,

$$\tilde{S}(m_1, n_1, n_2, d_1, d_2) := \sum_{\substack{c_1(\bmod d_1),\, c_2(\bmod d_2) \\ (c_1,d_1)=1 \\ (c_2,d_2/d_1)=1}} e\left(\frac{m_1c_1 + n_1\overline{c_1}c_2}{d_1}\right) e\left(\frac{n_2\overline{c_2}}{d_2/d_1}\right),$$

$$\begin{aligned}
S(m_1, m_2, n_1, n_2, d_1, d_2) := \sum_{\substack{b_1,c_1(\bmod d_1),\, b_2,c_2(\bmod d_2) \\ (b_1,c_1,d_1)=(b_2,c_2,d_2)=1 \\ b_1b_2+c_1d_2+c_2d_1\equiv 0(\bmod d_1d_2)}} & e\left(\frac{m_1b_1 + n_1(y_1d_2 - z_1b_2)}{d_1}\right) \\
& \times e\left(\frac{m_2b_2 + n_2(y_2d_1 - z_2b_1)}{d_2}\right),
\end{aligned}$$

and y_1, y_2, z_1, z_2 are such that

$$y_1b_1 + z_1c_1 \equiv 1 \pmod{d_1} \qquad \textit{and} \qquad y_2b_2 + z_2x_2 \equiv 1 \pmod{d_2}.$$

Proof. See Table 5.4 [Bump *et al.*, 1988]. □

Furthermore, we have sharp bounds for these Kloosterman sums. The following bound is due to Larsen in the Appendix [Bump *et al.*, 1988].

Proposition 2.6.6. *For any $\varepsilon > 0$,*

$$\tilde{S}(m_1, n_1, n_2, d_1, d_2) \ll \min\left(\gcd(n_2d_1^2, d_1d_2), \gcd(m_1d_2, n_1d_2, d_1d_2)\right)(d_1d_2)^{\varepsilon}.$$

The bound for the Kloosterman sum associated to the long Weyl element w_6 is due to [Stevens, 1987].

Proposition 2.6.7. *For any $\varepsilon > 0$,*

$$S(m_1, m_2, n_1, n_2, d_1, d_2) \ll (m_1 m_2 n_1 n_2)^{1/2} (d_1 d_2)^{1/2+\varepsilon} (d_1, d_2).$$

Proof. See Theorem 5.1 [Stevens, 1987]. Note that the theorem does not make the dependence on m_1, m_2, n_1, n_2 as these are assumed to be fixed. This dependence can be found by looking into the proof of the theorem, in equation 5.9. □

2.7 Poincaré Series

Poincaré series for $\mathrm{SL}(3, \mathbb{Z})$ were first introduced in [Bump *et al.*, 1988]. These series relate the spectral theory of $\mathrm{SL}(3, \mathbb{Z})\backslash\mathfrak{h}^3$ to $\mathrm{SL}(3, \mathbb{Z})$ Kloosterman sums.

Definition 2.7.1 (Poincaré Series). *Let $z \in \mathfrak{h}^3$, with $z = xy$ as in Definition 2.2.1. Let $F : \mathbb{R}_+^2 \to \mathbb{C}$, m_1, m_2 positive integers and define*

$$F_{m_1,m_2}(z) := e(m_1 x_1 + m_2 x_2) F(m_1 y_1, m_2 y_2).$$

The Poincaré series P_{m_1,m_2} is given by the series

$$P_{m_1,m_2}(z) := \sum_{\gamma \in U_3(\mathbb{Z})\backslash SL(3,\mathbb{Z})} F_{m_1,m_2}(\gamma z).$$

Proposition 2.7.2. *Assume F is a bounded function with the following decay*

$$|F(y_1, y_2)| \ll (y_1 y_2)^{2+\varepsilon},$$

for an $\varepsilon > 0$ as $y_1 \to 0$ or $y_2 \to 0$. Then the Poincaré series $P_{m_1,m_2}(z)$ converges absolutely and uniformly on compact subsets of $\mathfrak{h}^3$. Furthermore, $P_{m_1,m_2} \in \mathcal{L}^2(SL(3, \mathbb{Z})\backslash\mathfrak{h}^3)$.

Proof. First note that P_{m_1,m_2} is invariant under $\mathrm{SL}(3, \mathbb{Z})$, by construction so is a function on $\mathrm{SL}(3, \mathbb{Z})\backslash\mathfrak{h}^3$. To show the convergence properties we follow the idea of Godement in the proof of Theorem 9.1 [Borel, 1966]. For each $z_0 \in \mathrm{SL}(3, \mathbb{Z})\backslash\mathfrak{h}^3$ it suffices to show that

$$\int_{C_{z_0}} |P_{m_1,m_2}(z)| \, d^*z < \infty,$$

where C_{z_0} is a compact set of $\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{h}^3$ containing z_0. Then,

$$\int\limits_{C_{z_0}} |P_{m_1,m_2}(z)|\, d^*z \ \le \int\limits_{(U_3(\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{Z}))C_{z_0}} |F_{m_1,m_2}(z)|\, d^*z\,.$$

It follows from Proposition 1.3.2 [Goldfeld, 2006] there are only finitely many $\gamma \in U_3(\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{Z})$ such that $\gamma z_0 \in \Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$. This means that there exists an $a \ge \frac{\sqrt{3}}{2}$ such that $\gamma z_0 \notin \Sigma_{a,\frac{1}{2}}$ for every $\gamma \in U_3(\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{Z})$. In addition, by taking C_{z_0} sufficiently small, we also conclude that $\gamma z \notin \Sigma_{a,\frac{1}{2}}$ for every $\gamma \in U_3(\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{Z})$, $z \in C_{z_0}$. This implies

$$\int\limits_{(U_3(\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{Z}))C_{z_0}} |F_{m_1,m_2}(z)|\, d^*z \ \le \int_0^1\int_0^1\int_0^1\int_0^a\int_0^a |F(m_1y_1, m_2y_2)|\, dx_1\, dx_2\, dx_3\, \frac{dy_1\, dy_2}{(y_1y_2)^3},$$

and the latter integral converges for F under the decay conditions in the statement. To show the second part of the proposition note that

$$\begin{aligned}\int\limits_{\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{h}^3} |P_{m_1,m_2}(z)|^2\, d^*z &\le \int\limits_{U_3(\mathbb{Z})\backslash\mathfrak{h}^3} |F_{m_1,m_2}(z)|^2\, d^*z \\ &\le \int_0^1\int_0^1\int_0^1\int_0^\infty\int_0^\infty |F_{m_1,m_2}(z)|^2\, dx_1\, dx_2\, dx_3\, \frac{dy_1\, dy_2}{(y_1y_2)^3}.\end{aligned}$$

The last integral converges for bounded F satisfying $|F(y_1,y_2)| \ll (y_1y_2)^{2+\varepsilon}$. □

2.8 Kuznetsov Trace Formula

In this section we present a GL(3) Kuznetsov trace formula following [Blomer, 2013] and [Goldfeld and Kontorovich, 2013]. It is obtained by computing the Petersson inner product of two Poincaré series in different ways. The first way is to use the Langlands spectral decomposition (Theorem 2.5.6). The second way is to use the geometric structure $\mathrm{GL}(3,\mathbb{R})$, as given by the Bruhat decomposition.

To apply the spectral decomposition to a Poincaré series P_{m_1,m_2} we first need to make sure that such a function satisfies the conditions of Theorem 2.5.6. For the remainder of the section, assume that $F : \mathbb{R}_+^2 \to \mathbb{C}$ in the definition of the Poincaré series (Definition 2.7.1) is a bounded function with the following decay

$$|F(y_1,y_2)| \ll (y_1y_2)^{2+\varepsilon},$$

for an $\varepsilon > 0$ as $y_1 \to 0$ or $y_2 \to 0$. In addition, define $F^*(y_1, y_2) := F(y_2, y_1)$ which satisfies the same decay condition.

Lemma 2.8.1. *Let $E_{\min}(z,\nu)$ be the minimal parabolic Eisenstein series and assume $\Re(\nu_1) = \Re(\nu_2) = 0$. For u be a Hecke-Maass form of type μ, let $E_{\max}(z,s,u)$ be the maximal parabolic Eisenstein series associated to u. Assume $\Re(s) = \Re(\mu) = 0$. Then the Petersson inner products $\left\langle E_{\min}(*,\nu), P_{m_1,m_2}\right\rangle$, $\left\langle E_{\max}(*,s,u), P_{m_1,m_2}\right\rangle$ converge and satisfy*

$$\left\langle E_{\min}(*,\nu), P_{m_1,m_2}\right\rangle = m_1 m_2 \int_0^\infty\int_0^\infty A_\nu(m_1,m_2)\frac{W_\nu\left(y_1,y_2\right)\overline{F(y_1,y_2)}}{\zeta(1+3\nu_1)\zeta(1+3\nu_2)\zeta(1+3\nu_1+3\nu_2)}\frac{dy_1dy_2}{(y_1y_2)^3},$$

$$\left\langle E_{\max}(*,s,u), P_{m_1,m_2}\right\rangle = c m_1 m_2 \int_0^\infty\int_0^\infty B_{s,u}(m_1,m_2)\frac{W_{\left(\frac{2\mu}{3},s-\frac{\mu}{3}\right)}\left(y_1,y_2\right)\overline{F(y_1,y_2)}}{L_u(1+3s)L_{Ad\ u}(1)^{1/2}}\frac{dy_1dy_2}{(y_1y_2)^3},$$

for some absolute constant $c > 0$.

Proof. Replace $P_{m_1,m_2}(z)$ by its definition as a series to obtain

$$\left\langle E_{\min}(*,\nu), P_{m_1,m_2}\right\rangle = \int\limits_{\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{h}^3} E_{\min}(z,\nu) \sum_{\gamma\in U_3(\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{Z})} \overline{F_{m_1,m_2}(\gamma z)}\, d^*z.$$

As $E_{\min}(z,\nu)$ is invariant under the action of $\mathrm{SL}(3,\mathbb{Z})$ (on the variable z), one obtains

$$\begin{aligned}\left\langle E_{\min}(*,\nu), P_{m_1,m_2}\right\rangle &= \int\limits_{U_3(\mathbb{Z})\backslash\mathfrak{h}^3} E_{\min}(z,\nu)\overline{F_{m_1,m_2}(z)}d^*z \\ &= \int_0^\infty\int_0^\infty \overline{F(m_1y_1,m_2y_2)}\int_0^1\int_0^1\int_0^1 E_{\min}(z,\nu)e^{-2\pi i(m_1x_1+m_2x_2)}\,dx_1\,dx_2\,dx_3\,\frac{dy_1dy_2}{(y_1y_2)^3} \\ &= \int_0^\infty\int_0^\infty \frac{A_\nu(m_1,m_2)}{m_1m_2}\frac{W_\nu\left(m_1y_1,m_2y_2\right)\overline{F(m_1y_1,m_2y_2)}}{\zeta(1+3\nu_1)\zeta(1+3\nu_2)\zeta(1+3\nu_1+3\nu_2)}\frac{dy_1dy_2}{(y_1y_2)^3},\end{aligned}$$

where the last equality follows from Proposition 2.5.2. The latter integral converges absolutely due to the decay properties of F (2.8) and the bounds on the Whittaker functions (2.3.11). To finish the proof do the change of variables $(y_1,y_2) \mapsto \left(\frac{y_1}{m_1}, \frac{y_2}{m_2}\right)$.

For the second inner product we follow a similar strategy:

$$\left\langle E_{\max}(z,s,u), P_{m_1,m_2}\right\rangle = \int\limits_{\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{h}^3} E_{\max}(z,s,u) \sum_{\gamma\in U_3(\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{Z})} \overline{F_{m_1,m_2}(\gamma z)}d^*z.$$

As $E_{\min}(z,\nu)$ is invariant under the action of $\mathrm{SL}(3,\mathbb{Z})$ (on the variable z), one obtains

$$\left\langle E_{\max}(z,s,u), P_{m_1,m_2}\right\rangle = \int\limits_{U_3(\mathbb{Z})\backslash\mathfrak{h}^3} E_{\min}(z,\nu)\overline{F_{m_1,m_2}(z)}\, d^*z$$

which can be written as

$$\int\limits_0^\infty\int\limits_0^\infty \overline{F(m_1y_1,m_2y_2)}\int\limits_0^1\int\limits_0^1\int\limits_0^1 E_{\max}(z,s,u)e^{-2\pi i(m_1x_1+m_2x_2)}\,dx_1\,dx_2\,dx_3\,\frac{dy_1dy_2}{(y_1y_2)^3}$$
$$= c\int\limits_0^\infty\int\limits_0^\infty \frac{B_{s,u}(m_1,m_2)}{m_1m_2}\frac{W_{\left(\frac{2\mu}{3},s-\frac{\mu}{3}\right)}(m_1y_1,m_2y_2)\,\overline{F(m_1y_1,m_2y_2)}}{L_u(1+3s)L_{\mathrm{Ad}\ u}(1)^{1/2}}\frac{dy_1dy_2}{(y_1y_2)^3},$$

where the last equality follows from Proposition 2.5.4. The latter integral again converges absolutely due to the decay properties of F (2.8) and the bounds on the Whittaker functions (2.3.11). Perform a change of variables as before to obtain the equality in the theorem. □

Remark 2.8.2. *Note that P_{m_1,m_2} is orthogonal to the degenerate Eisenstein series $E_{\max}(z,s,u)$ with u a constant function. This follows from $E_{\max}(z,s,u)$ only having degenerate terms in its Fourier-Whittaker expansion. The same is true for all residues of Eisenstein series.*

The second way to compute the Petersson inner product of two Poincaré series requires an explicit Fourier expansion for the Poincaré series as obtained in Theorem 5.1 [Bump *et al.*, 1988].

Theorem 2.8.3. *Let m_1,m_2,n_1,n_2 be positive integers. Then*

$$\int\limits_0^1\int\limits_0^1\int\limits_0^1 P_{n_1,n_2}(z)e^{-2\pi i(m_1x_1+m_2x_2)}\,dx_1\,dx_2\,dx_3 = S_1+S_{2a}+S_{2b}+S_3,$$

where

$$S_1 = \delta_{m_1,n_1}\delta_{m_2,n_2}F(n_1y_1,n_2y_2),$$
$$S_{2a} = \sum_{\varepsilon=\pm1}\sum_{\substack{d_1|d_2\\ m_2d_1^2=n_1d_2}}\tilde{S}(\varepsilon m_1,n_1,n_2,d_1,d_2)\tilde{J}_F(y_1,y_2,\varepsilon m_1,n_1,n_2,d_1,d_2),$$
$$S_{2b} = \sum_{\varepsilon=\pm1}\sum_{\substack{d_2|d_1\\ m_1d_2^2=n_2d_1}}\tilde{S}(\varepsilon m_1,n_1,n_2,d_1,d_2)\tilde{J}_{F^*}(y_1,y_2,\varepsilon m_2,n_2,n_1,d_2,d_1),$$
$$S_3 = \sum_{\varepsilon_1,\varepsilon_2=\pm1}\sum_{d_1,d_2=1}^\infty S(\varepsilon_1m_1,\varepsilon_2m_2,n_1,n_2,d_1,d_2)J_F(y_1,y_2,\varepsilon_1m_1,\varepsilon_2m_2,n_1,n_2,d_1,d_2).$$

The Kloosterman sums S and $\tilde{S}$ have been defined in Section 2.6 and the weight functions J and $\tilde{J}$ are given by

$$\tilde{J}_F(y_1,y_2,\varepsilon m_1,n_1,n_2,d_1,d_2) = y_1^2 y_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e\left(\frac{n_1 y_2 d_2}{d_1^2}\cdot\frac{x_1 x_2}{x_1^2+1}\right) e\left(\frac{n_2 d_1}{y_1 y_2 d_2^2}\cdot\frac{x_2}{x_1^2+x_2^2+1}\right)$$
$$\times\, e\left(-m_1 x_1 y_1\right) F\left(\frac{n_1 y_2 d_2}{d_1^2}\cdot\frac{\sqrt{x_1^2+x_2^2+1}}{x_1^2+1}, \frac{n_2 d_1}{y_1 y_2 d_2^2}\cdot\frac{\sqrt{x_1^2+1}}{x_1^2+x_2^2+1}\right) dx_1\, dx_2,$$

$$J_F(y_1,y_2,m_1,m_2,n_1,n_2,d_1,d_2) = (y_1 y_2)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e\left(-m_1 x_1 y_1 - m_2 x_2 y_2\right)$$
$$\times\, e\left(-\frac{n_1 d_2}{y_2 d_1^2}\cdot\frac{x_1 x_3 + x_2}{x_2^2+x_3^2+1}\right) e\left(-\frac{n_2 d_1}{y_2 d_2^2}\cdot\frac{x_2(x_1 x_2 - x_3)+x_1}{(x_1 x_2 - x_3)^2 + x_1^2+1}\right)$$
$$\times\, F\left(\frac{n_1 d_2}{y_2 d_1^2}\cdot\frac{\sqrt{(x_1 x_2 - x_3)^2+x_1^2+1}}{x_2^2+x_3^2+1}, \frac{n_2 d_1}{y_2 d_2^2}\cdot\frac{\sqrt{x_2^2+x_3^2+1}}{(x_1 x_2 - x_3)^2+x_1^2+1}\right) dx_1\, dx_2\, dx_3\,.$$

Proof. Apply Theorem 5.1 [Bump *et al.*, 1988] replacing $I_{\nu_1,\nu_2}(\tau)E_{n_1,n_2}(\tau)$ by F_{n_1,n_2} and setting x_1, x_2 equal to zero. Note that the conditions $\Re(\nu_1), \Re(\nu_2) > 2/3$ is equivalent to the decay condition 2.8. To obtain the weight functions J and $\tilde{J}$ do the change of variables that maps $(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ to $(y_1\zeta_1, y_2\zeta_2, y_1 y_2\zeta_3, y_1 y_2 \zeta_4)$ on the integrals in Table 5.4 [Bump *et al.*, 1988]. □

As the Fourier coefficients of Maass forms and Eisenstein series are Petersson inner products involving a Whittaker function we make the following definition.

Definition 2.8.4 (Lebedev-Whittaker Transform). *Let $F : \mathbb{R}_+^2 \to \mathbb{C}$ be a function satisfying 2.8. Define its Lebedev-Whittaker transform $F^\# : D \times D \to \mathbb{C}$ as*

$$F^\#(\nu) = \int_0^\infty\int_0^\infty \overline{W_\nu^*(y)} F(y) \frac{dy_1 dy_2}{(y_1 y_2)^3}, \tag{2.3}$$

where $W_\nu(y)$ is the Whittaker function and $D \subset \mathbb{C}$ is of the form $(-\delta,\delta)\times i\mathbb{R}$ for some $\delta > 0$.

We can finally state the GL(3) Kuznetsov trace formula.

Theorem 2.8.5 (Kuznetsov Trace Formula). *Let m_1, m_2, n_1, n_2 be positive integers. Let $F : \mathbb{R}_+^2 \to \mathbb{C}$ be a function satisfying 2.8. Let $\{\phi_j\}_{j=0}^\infty$ be an orthonormal basis of Hecke-Maass forms for $SL(3,\mathbb{Z})$, ordered by Laplacian eigenvalue and normalized such that the first Fourier-Whittaker* *coefficient $A_j(1,1)$ is equal to 1. Denote by $\nu^{(j)} = \left(\nu_1^{(j)}, \nu_2^{(j)}\right)$ the type of the Maass form ϕ_j. Let $\{u_j\}$ be a basis of Hecke-Maass form for $SL(2,\mathbb{Z})$, ordered by Laplacian eigenvalue and normalized to have norm 1. Denote by μ_j the type of of the Maass form u_j. Then for some absolute constant $c > 0$ the following equality holds:*

$$\mathcal{C} + \mathcal{E}_{min} + \mathcal{E}_{max} = \Sigma_1 + \Sigma_{2a} + \Sigma_{2b} + \Sigma_3, \tag{2.4}$$

where

$$\mathcal{C} = \sum_{j=0}^{\infty} A_j(m_1, m_2)\overline{A_j(n_1, n_2)} \frac{\left|F^{\#}\left(\nu_1^{(j)}, \nu_2^{(j)}\right)\right|^2}{6L_{Ad\phi_j}(1) \prod\limits_{k=1}^{3} \Gamma\left(\frac{1+3\nu_k^{(j)}}{2}\right)\Gamma\left(\frac{1-3\nu_k^{(j)}}{2}\right)}, \tag{2.5}$$

with $\nu_3^{(j)} := \nu_1^{(j)} + \nu_2^{(j)}$;

$$\mathcal{E}_{min} = \frac{1}{(4\pi i)^2} \int\limits_{-i\infty}^{i\infty} \int\limits_{-i\infty}^{i\infty} \frac{A_\nu(m_1, m_2)\overline{A_\nu(n_1, n_2)} \left|F^{\#}(\nu_1, \nu_2)\right|^2}{\prod\limits_{k=1}^{3} \left|\zeta(1+3\nu_k)\Gamma\left(\frac{1+3\nu_k}{2}\right)\right|^2} d\nu_1\, d\nu_2, \tag{2.6}$$

with $\nu_3 := \nu_1 + \nu_2$;

$$\mathcal{E}_{max} = \frac{c}{2\pi i} \sum_{j=0}^{\infty} \int\limits_{-i\infty}^{i\infty} \frac{B_{(s,u_j)}(m_1, m_2)\overline{B_{(s,u_j)}(n_1, n_2)} \left|F^{\#}\left(\frac{2\mu_j}{3}, s - \frac{\mu_j}{3}\right)\right|^2}{L_{Ad\,u_j}(1)\left|L_{u_j}(1+3s)\right|^2 \prod\limits_{\substack{k,l=1 \\ k\neq l}}^{3} \Gamma\left(\frac{1+\delta_k-\delta_l}{2}\right)} \, ds, \tag{2.7}$$

with $\sigma_1 = \mu_j + s$, $\sigma_2 = s - \mu_j$, $\sigma_3 = -2s$;

$$\Sigma_1 = \mathbf{1}_{\left\{\substack{m_1=n_1 \\ m_2=n_2}\right\}} \int\limits_0^\infty \int\limits_0^\infty |F(y_1, y_2)|^2 \frac{dy_1 dy_2}{(y_1 y_2)^3} = \mathbf{1}_{\left\{\substack{m_1=n_1 \\ m_2=n_2}\right\}} \left\langle F, F \right\rangle, \tag{2.8}$$

$$\Sigma_{2a} = \sum_{\varepsilon=\pm1} \sum_{\substack{d_1|d_2 \\ m_2 d_1^2 = n_1 d_2}} \frac{\tilde{S}(\varepsilon m_1, n_1, n_2, d_1, d_2)}{d_1 d_2} \tilde{\mathcal{J}}_{\varepsilon,F}\left(\sqrt{\frac{m_1 n_1 n_2}{d_1 d_2}}\right), \tag{2.9}$$

$$\Sigma_{2b} = \sum_{\varepsilon=\pm1} \sum_{\substack{d_2|d_1 \\ m_1 d_2^2 = n_2 d_1}} \frac{\tilde{S}(\varepsilon m_2, n_2, n_1, d_2, d_1)}{d_1 d_2} \tilde{\mathcal{J}}_{\varepsilon,F^*}\left(\sqrt{\frac{m_2 n_1 n_2}{d_1 d_2}}\right), \tag{2.10}$$

$$\Sigma_3 = \sum_{\varepsilon_1,\varepsilon_2=\pm1} \sum_{d_1,d_2=1}^{\infty} \frac{S(\varepsilon_1 m_1, \varepsilon_2 m_2, n_1, n_2, d_1, d_2)}{d_1 d_2} \mathcal{J}_{\varepsilon_1,\varepsilon_2,F}\left(\frac{\sqrt{m_1 n_2 d_1}}{d_2}, \frac{\sqrt{m_2 n_1 d_2}}{d_1}\right). \tag{2.11}$$

Furthermore, the weight functions $\mathcal{J}$, $\tilde{\mathcal{J}}$ given by

$$\tilde{\mathcal{J}}_{\varepsilon,F}(A) = A^{-2} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \overline{F(Ay_1, y_2)} e(-\varepsilon A x_1 y_1) F\left(y_2 \frac{\sqrt{x_1^2+x_2^2+1}}{x_1^2+1}, \frac{A}{y_1 y_2} \frac{\sqrt{x_1^2+1}}{x_1^2+x_2^2+1} \right)$$
$$\times e\left(y_2 \frac{x_1 x_2}{x_1^2+1} + \frac{A}{y_1 y_2} \frac{x_2}{x_1^2+x_2^2+1} \right) \frac{dx_1 dx_2 dy_1 dy_2}{y_1 y_2^2}, \quad (2.12)$$

$$\mathcal{J}_{\varepsilon_1,\varepsilon_2,F}(A_1, A_2) = (A_1 A_2)^{-2} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \overline{F(A_1 y_1, A_2 y_2)} e(-\varepsilon_1 A_1 x_1 y_1 - \varepsilon_2 A_2 x_2 y_2)$$
$$\times F\left(\frac{A_2}{y_2} \frac{\sqrt{(x_1 x_2 - x_3)^2 + x_1^2 + 1}}{x_2^2+x_3^2+1}, \frac{A_1}{y_1} \frac{\sqrt{x_2^2+x_3^2+1}}{(x_1 x_2 - x_3)^2 + x_1^2 + 1} \right)$$
$$\times e\left(-\frac{A_2}{y_2} \frac{x_1 x_3 + x_2}{x_2^2+x_3^2+1} - \frac{A_1}{y_1} \frac{x_2(x_1 x_2 - x_3) + x_1}{(x_1 x_2 - x_3)^2 + x_1^2 + 1} \right) \frac{dx_1 dx_2 dx_3 dy_1 dy_2}{y_1 y_2}. \quad (2.13)$$

Proof. We shall compute $\frac{\langle P_{n_1,n_2}, P_{m_1,m_2} \rangle}{n_1 n_2 m_1 m_2}$ in two different ways to obtain the two sides of the equality. To obtain the left hand side of 2.4 apply the spectral decomposition (Theorem 2.5.6) to both Poincaré series and also decompose the space of Maass form via a basis of Hecke-Maass form as in the statement of the theorem. Note that these Poincaré series satisfy the conditions of Theorem 2.5.6 according to Lemma 2.8.1 and Remark 2.8.2. It follows that

$$\frac{\left\langle P_{n_1,n_2}, P_{m_1,m_2} \right\rangle}{n_1 n_2 m_1 m_2} = \sum_{j=0}^{\infty} \frac{\left\langle \phi_j, P_{m_1,m_2} \right\rangle \overline{\left\langle \phi_j, P_{n_1,n_2} \right\rangle}}{\|\phi_j\| (n_1 n_2 m_1 m_2)}$$
$$+ \frac{1}{(4\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\left\langle E_{\min}(*,\nu), P_{m_1,m_2} \right\rangle \overline{\left\langle E_{\min}(*,\nu), P_{n_1,n_2} \right\rangle}}{n_1 n_2 m_1 m_2} \, d\nu_1 \, d\nu_2$$
$$+ \frac{1}{2\pi i} \sum_{j=0}^{\infty} \int_{-i\infty}^{i\infty} \frac{\left\langle E_{\max}(*,s,u_j), P_{m_1,m_2} \right\rangle \overline{\left\langle E_{\max}(*,s,u_j), P_{n_1,n_2} \right\rangle}}{n_1 n_2 m_1 m_2} \, ds \, .$$

Applying Lemma 2.8.1 to compute the inner products involving the Eisenstein series one obtains the terms $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$. Note that the Hecke-Maass forms for $\mathrm{SL}(2,\mathbb{Z})$ satisfy the Ramanujan conjecture at infinity so their type is purely imaginary. The Gamma factors in the integrands of $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ appear as the quotient between W_ν^* and W_ν.

To obtain the term associated to the Hecke-Maass forms for $\mathrm{SL}(3,\mathbb{Z})$ it is necessary to compute the inner products $\left\langle \phi_j, P_{m_1,m_2} \right\rangle$. Start by expanding P_{m_1,m_2} using its definition as a series to obtain

$$\left\langle \phi_j, P_{m_1,m_2} \right\rangle = \int\limits_{\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{h}^3} \phi_j(z) \sum_{\gamma\in U_3(\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{Z})} \overline{F_{m_1,m_2}(\gamma z)} d^*z.$$

As ϕ_j is invariant under the action of $\mathrm{SL}(3,\mathbb{Z})$, one obtains

$$\begin{aligned}
\left\langle \phi_j, P_{m_1,m_2} \right\rangle &= \int\limits_{U_3(\mathbb{Z})\backslash\mathfrak{h}^3} \phi_j(z)\overline{F_{m_1,m_2}(z)} d^*z \\
&= \int_0^\infty\int_0^\infty \overline{F(m_1y_1, m_2y_2)} \int_0^1\int_0^1\int_0^1 \phi_j(z)e^{-2\pi i(m_1x_1+m_2x_2)}\, dx_1\, dx_2\, dx_3\, \frac{dy_1dy_2}{(y_1y_2)^3} \\
&= \int_0^\infty\int_0^\infty \frac{A_\nu(m_1,m_2)}{m_1m_2} W_\nu\left(m_1y_1, m_2y_2\right) F(m_1y_1, m_2y_2) \frac{dy_1dy_2}{(y_1y_2)^3},
\end{aligned}$$

where the last equality follows from Remark 2.3.8. To obtain the cuspidal term $\mathcal{C}$ do a change of variables $(y_1, y_2) \mapsto \left(\frac{y_1}{m_1}, \frac{y_2}{m_2}\right)$ and refer to Equation 2.12 [Goldfeld and Kontorovich, 2013] for an explicit expression for the L^2-norm of ϕ_j,

$$\|\phi_j\| = 6L_{\mathrm{Ad}\phi_j}(1)\prod_{k=1}^{3}\Gamma\left(\frac{1+3\nu_k^{(j)}}{2}\right)\Gamma\left(\frac{1-3\nu_k^{(j)}}{2}\right).$$

Collecting the results obtained so far we get

$$\frac{\left\langle P_{n_1,n_2}, P_{m_1,m_2} \right\rangle}{n_1n_2m_1m_2} = \mathcal{C} + \mathcal{E}_{min} + \mathcal{E}_{max}.$$

We will now compute the inner product $\left\langle P_{n_1,n_2}, P_{m_1,m_2} \right\rangle$ directly using the Fourier expansion of Poincaré series as in Theorem 2.8.3.

$$\begin{aligned}
\left\langle P_{n_1,n_2}, P_{m_1,m_2} \right\rangle &= \int\limits_{U_3(\mathbb{Z})\backslash\mathfrak{h}^3} P_{n_1,n_2}(z)\overline{F_{m_1,m_2}(z)} d^*z \\
&= \int_0^\infty\int_0^\infty \overline{F(m_1y_1, m_2y_2)} \int_0^1\int_0^1\int_0^1 P_{n_1,n_2}(z)e^{-2\pi i(m_1x_1+m_2x_2)}\, dx_1\, dx_2\, dx_3\, \frac{dy_1dy_2}{(y_1y_2)^3} \\
&= \int_0^\infty\int_0^\infty \overline{F(m_1y_1, m_2y_2)} \left(S_1 + S_{2a} + S_{2b} + S_3\right) \frac{dy_1dy_2}{(y_1y_2)^3},
\end{aligned}$$

where S_1, S_{2a}, S_{2b}, S_3 are as in Theorem 2.8.3. We then proceed to compute the integrals $\int_0^\infty \int_0^\infty \overline{F(m_1y_1, m_2y_2)} S_\tau \frac{dy_1dy_2}{(y_1y_2)^3}$ for $\tau \in \{1, 2a, 2b, 3\}$. Recall the definitions of $\Sigma_1, \Sigma_{2a}, \Sigma_{2b}$ and Σ_3 as given by equations 2.8, 2.9, 2.10 and 2.11, respectively. We get

$$\int_0^\infty \int_0^\infty \overline{F(m_1y_1, m_2y_2)} S_1 \frac{dy_1dy_2}{(y_1y_2)^3} = n_1n_2m_1m_2\Sigma_1,$$

after changing variables $(y_1, y_2) \mapsto \left(\frac{y_1}{m_1}, \frac{y_2}{m_2}\right)$. Furthermore,

$$\begin{aligned}\int_0^\infty \int_0^\infty \overline{F(m_1y_1, m_2y_2)} S_{2a} \frac{dy_1dy_2}{(y_1y_2)^3} &= \sum_{\varepsilon=\pm 1} \sum_{\substack{d_1|d_2 \\ m_2d_1^2=n_1d_2}} \tilde{S}(\varepsilon m_1, n_1, n_2, d_1, d_2) \\ &\quad \times \int_0^\infty \int_0^\infty \overline{F(m_1y_1, m_2y_2)} \tilde{J}_F(y_1, y_2, \varepsilon m_1, n_1, n_2, d_1, d_2) \frac{dy_1dy_2}{(y_1y_2)^3} \\ &= \sum_{\varepsilon=\pm 1} \sum_{\substack{d_1|d_2 \\ m_2d_1^2=n_1d_2}} \tilde{S}(\varepsilon m_1, n_1, n_2, d_1, d_2) \frac{n_1n_2m_1m_2}{d_1d_2} \tilde{\mathcal{J}}_\varepsilon\left(\sqrt{\frac{m_1n_1n_2}{d_1d_2}}\right) \\ &= n_1n_2m_1m_2\Sigma_{2a},\end{aligned}$$

where the penultimate equality follows from the change of variables

$$(y_1, y_2) \mapsto \left(\sqrt{\frac{n_1n_2}{m_1d_1d_2}}y_1, \frac{d_1^2}{n_1d_2}y_2\right).$$

A similar calculation gives

$$\int_0^\infty \int_0^\infty \overline{F(m_1y_1, m_2y_2)} S_{2b} \frac{dy_1dy_2}{(y_1y_2)^3} = n_1n_2m_1m_2\Sigma_{2b}.$$

Finally,

$$\begin{aligned}\int_0^\infty\int_0^\infty \overline{F(m_1y_1,m_2y_2)}S_3\frac{dy_1dy_2}{(y_1y_2)^3} &= \sum_{\varepsilon_1,\varepsilon_2=\pm1}\sum_{d_1,d_2=1}^\infty S(\varepsilon_1m_1,\varepsilon_2m_2,n_1,n_2,d_1,d_2)\\ &\times\int_0^\infty\int_0^\infty \overline{F(m_1y_1,m_2y_2)}J_F(y_1,y_2,\varepsilon_1m_1,\varepsilon_2m_2,n_1,n_2,d_1,d_2)\frac{dy_1dy_2}{(y_1y_2)^3}\\ &= \sum_{\varepsilon_1,\varepsilon_2=\pm1}\sum_{d_1,d_2=1}^\infty S(\varepsilon_1m_1,\varepsilon_2m_2,n_1,n_2,d_1,d_2)\frac{n_1n_2m_1m_2}{d_1d_2}\\ &\qquad\times\mathcal{J}_{\varepsilon_1,\varepsilon_2}\left(\frac{\sqrt{m_1n_2d_1}}{d_2},\frac{\sqrt{m_2n_1d_2}}{d_1}\right)\\ &= n_1n_2m_1m_2\Sigma_3,\end{aligned}$$

where the penultimate equality follows from the change of variables $(y_1,y_2)\mapsto\left(\sqrt{\frac{n_2d_1}{m_1^2d_2}}y_1,\frac{n_1d_2}{m_2^2d_1}y_2\right)$. It follows from the above computations that

$$\mathcal{C}+\mathcal{E}_{min}+\mathcal{E}_{max}=\frac{\langle P_{n_1,n_2},P_{m_1,m_2}\rangle}{n_1n_2m_1m_2}=\Sigma_1+\Sigma_{2a}+\Sigma_{2b}+\Sigma_3,$$

as desired. □

Chapter 3

Inverse Lebedev-Whittaker Transform

This chapter is devoted to the Lebedev-Whittaker transform on $\mathrm{GL}(n)$, in particular to its inverse transform. We will analyse this transform in both the archimedean (for the real group $\mathrm{GL}(3,\mathbb{R})$) and nonarchimedean (for p-adic groups $\mathrm{GL}(n,\mathbb{Q}_p)$) cases. The inverse transform for $\mathrm{GL}(3,\mathbb{R})$ will be crucial to the application of GL(3) Kuznetsov trace formula that shall be described in the following chapter.

3.1 Archimedean Case

On archimedean local fields Whittaker functions have been thoroughly studied and an inversion formula for the Lebedev-Whittaker transform in known due to the works of Wallach [Wallach, 1992] and Goldfeld-Kontorovich [Goldfeld and Kontorovich, 2012]. We will follow in this section the presentation of this inversion formula in the latter. We shall restrict ourselves to the inverse transform for $\mathrm{GL}(n,\mathbb{R})$ for $n = 3$ as it will be the only case we will need to use.

Definition 3.1.1 (Lebedev-Whittaker Inverse). *Let $g : i\mathbb{R}^2 \to \mathbb{C}$. Define its Lebedev-Whittaker inverse $g^\flat$ by*

$$g^\flat(y) := \frac{1}{(\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} g(\nu) W_\nu^*(y) \frac{d\nu_1 \nu_2}{\prod_{j=1}^{3} \Gamma\left(\frac{3\nu_j}{2}\right) \Gamma\left(-\frac{3\nu_j}{2}\right)},$$

where $\nu_3 = \nu_1 + \nu_2$, provided the integral converges.

Remark 3.1.2. *A sufficient condition for the convergence of the integral is that g extends holo-*

morphically to a function defined on D^2, with $D = (-\delta, \delta) \times i\mathbb{R}$, and satisfying

$$|g(\nu)| \ll \exp\left(-\frac{3\pi}{4}\sum_{j=1}^{3}|\nu_j|\right)\prod_{j=1}^{3}(1+|\nu_j|)^{-10} \tag{3.1}$$

See Section 2.2 [Goldfeld and Kontorovich, 2013].

Theorem 3.1.3 (Lebedev-Whittaker Inversion). *Let $\delta > 0$ and $g : D^2 \to \mathbb{C}$ holomorphic, with $D = (-\delta, \delta) \times i\mathbb{R}$. Assume that $g(\nu_1, \nu_2)$ is invariant under permutations of $(2\nu_1 + \nu_2, \nu_2 - \nu_1, -\nu_1 - 2\nu_2)$, and satisfies 3.1. Then*

$$(g^\sharp)^\flat(\nu) = g(\nu)$$

for $\nu \in D^2$. Furthermore,

$$\left\langle g^\#, g^\# \right\rangle := \frac{1}{(\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\left|g^\#(\nu)\right|^2 d\nu_1 d\nu_2}{\prod_{j=1}^{3}\Gamma\left(\frac{3\nu_j}{2}\right)\Gamma\left(\frac{-3\nu_j}{2}\right)} = \langle g, g\rangle\,. \tag{3.2}$$

Proof. See Theorem 1.6, Corollary 1.9 and Theorem 3.4 [Goldfeld and Kontorovich, 2012]. Note that our definitions of $f \mapsto f^\sharp$ and $g \mapsto g^\flat$ differ from the ones in [Goldfeld and Kontorovich, 2012] by conjugation of the Whittaker functions. This does not affect the result as $\overline{W_\nu^*} = W_{\overline{\nu}}^*$. The proof in Theorem 3.4 shows the equality of $(g^\sharp)^\flat$ and g on $(i\mathbb{R})^2$ and this equality can be extended to D^2 as both functions are holomorphic. □

3.2 Nonarchimedean Case

Let $G = \mathrm{GL}_n(\mathbb{Q}_p)$ for which we have the Iwasawa decomposition $G = UTK$, where $U = U(\mathbb{Q}_p)$ is the unipotent radical of the standard Borel subgroup, $T = T(\mathbb{Q}_p)$ is the torus of diagonal matrices and $K = \mathrm{GL}_n(\mathbb{Z}_p)$ is the maximal compact subgroup. Let Z be the center of G. Let ψ be a character on U induced from a character ψ' on $\mathbb{Q}_p$ via

$$\psi(u) = \psi'\left(\sum_{i=1}^{n-1} u_{i,i+1}\right), \qquad (\text{where } u = (u_{i,j}) \in U)\,.$$

Definition 3.2.1 (Spherical Hecke Algebra). *Define the spherical Hecke algebra ${}^K\mathcal{H}^K$ to be the set of locally constant, compactly supported function $f : GL_n(\mathbb{Q}_p) \to \mathbb{C}$ satisfying*

$$f(k_1 g k_2) = f(g)$$

for $k_1, k_2 \in K$*,* $g \in GL_n(\mathbb{Q}_p)$*.*

One can now define Whittaker functions on $\mathrm{GL}_n(\mathbb{Q}_p)$.

Definition 3.2.2 (Whittaker Function on $\mathbf{GL}_n(\mathbb{Q}_p)$). *A Whittaker function on* $GL_n(\mathbb{Q}_p)$ *is a* K*-finite smooth function of moderate growth* $W : GL_n(\mathbb{Q}_p) \to \mathbb{C}$ *that satisfies*

$$W(ug) = \psi(u)W(g)$$

for some fixed character ψ *as defined above and any* $u \in U$*, and*

$$\int_G W(gx)\phi(x)\,dx = \lambda(\phi)W(g)$$

for each algebra homomorphism $\lambda : {}^K\mathcal{H}^K \to \mathbb{C}$*,* $\phi \in {}^K\mathcal{H}^K$ *and* $g \in GL_n(\mathbb{Q}_p)$*.*

Let $h : Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p) \to C$ be a smooth function. The p-adic Whittaker transform of h is defined to be

$$\int_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} h(t)W(t)d^\times t,$$

where W is a Whittaker function on $\mathrm{GL}_n(\mathbb{Q}_p)$.

The main goal of the present section is to obtain an inversion formula for the p-adic Whittaker transform. The precise inversion formula for the Whittaker transform is given in Theorem 3.2.10. The work in this section is inspired by the one of Goldfeld-Kontorovich [Goldfeld and Kontorovich, 2012] for the archimedean case. A crucial step in our approach is the computation of an integral of a product of two p-adic Whittaker functions. One can view this computation as a nonarchimedean analogue of Stade's formula [Stade, 2001].

3.2.1 Inversion Formula

The goal of this subsection is to provide all the necessary definitions and results that allow us to state the main theorem of this section 3.2.10, which gives an inverse formula for the p-adic Whittaker transform.

We start by noting Whittaker functions on $\mathrm{GL}_n(\mathbb{Q}_p)$ arise naturally from certain irreducible representation of this group.

Definition 3.2.3 (Whittaker Function associated to a Representation π). *Let V be a complex vector space and let (π, V) be an irreducible generic representation of G. Then the space*

$$Hom_U(\pi, \psi) = \{f : V \to \mathbb{C} \text{ linear} : f(\pi(u)v) = \psi(u)f(v), \forall v \in V, u \in U\}$$

is one-dimensional generated by an element λ. For $v \in V$ a newform, define a Whittaker function W associated to π as

$$W(g) = \lambda(\pi(g)v).$$

Remark 3.2.4. *As the space of newforms in V is one-dimensional then the Whittaker function associated to a representation π is well-defined up to a constant. Furthermore, a Whittaker function associated to an irreducible generic representation π of $GL_n(\mathbb{Q}_p)$ is a nonzero Whittaker function on $GL_n(\mathbb{Q}_p)$. For proofs of these statements see [Jacquet* et al.*, 1981] and [Gelfand and Kazhdan, 1975].*

In order to explicitly describe Whittaker functions associated to certain representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ we proceed to quote a theorem of Shintani [Shintani, 1976]. For $g = zutk \in \mathrm{GL}_n(\mathbb{Q}_p)$ define the Iwasawa coordinates

$$g = zu \begin{pmatrix} t_1 \cdots t_{n-1} & & & \\ & \ddots & & \\ & & t_1 & \\ & & & 1 \end{pmatrix} k,$$

where $z \in Z$, $u \in U$, $t \in T$ and $k \in K$.

Theorem 3.2.5 (Shintani). *Let π be an irreducible unramified generic representation of $GL_n(\mathbb{Q}_p)$ and W_π the Whittaker function associated to a representation π (normalized such that $W_\pi(\mathrm{id}) = 1$, where* id *denotes the $n \times n$ identity matrix) as in Definition 3.2.3. We can write the L-function associated to π as*

$$L(s, \pi) = \prod_{i=1}^{n} \left(1 - \alpha_i p^{-s}\right)^{-1},$$

where $\alpha_i \in \mathbb{C}$ are all nonzero and $\alpha_1 \cdots \alpha_n = 1$. Then, for

$$t = \begin{pmatrix} t_1 \cdots t_{n-1} & & & \\ & \ddots & & \\ & & t_1 & \\ & & & 1 \end{pmatrix}$$

we have

$$W_\pi(t) = \begin{cases} \delta^{1/2}(t) s_{\text{-}\log|t|_p}(\alpha), & \text{if } t_i \in \mathbb{Z}_p \text{ for } i = 1, \dots, n-1, \\ 0, & \text{otherwise;} \end{cases}$$

where

$$\delta(t) = \prod_{k=1}^{n-1} |t_k|_p^{k(n-k)},$$

$$\log|t|_p = (\log|t_1|_p, \dots, \log|t_{n-1}|_p)$$

and

$$s_\lambda(\alpha) = \frac{\begin{vmatrix} \alpha_1^{n-1+\lambda_1+\lambda_2+\cdots+\lambda_{n-1}} & \alpha_2^{n-1+\lambda_1+\lambda_2+\cdots+\lambda_{n-1}} & \cdots & \alpha_n^{n-1+\lambda_1+\lambda_2+\cdots+\lambda_{n-1}} \\ \alpha_1^{n-2+\lambda_1+\lambda_2+\cdots+\lambda_{n-2}} & \alpha_2^{n-2+\lambda_1+\lambda_2+\cdots+\lambda_{n-2}} & \cdots & \alpha_n^{n-2+\lambda_1+\lambda_2+\cdots+\lambda_{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{1+\lambda_1} & \alpha_2^{1+\lambda_1} & \cdots & \alpha_n^{1+\lambda_1} \\ 1 & 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}}$$

is the Schur polynomial, where $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\lambda_i \in \mathbb{Z}$.

Proof. See [Shintani, 1976]. For the relation between the L-function $L(s, \pi)$ and the Whittaker function W_π see Subsection 3.1.3 [Cogdell, 2008]. □

Remark 3.2.6. *We will also use the notation W_α, for $\alpha \in (\mathbb{C}\backslash\{0\})^n$, to denote the function*

$$W_\alpha(t) = \begin{cases} \delta^{1/2}(t) s_{\text{-}\log|t|_p}(\alpha), & \text{if } t_i \in \mathbb{Z}_p \text{ for } i = 1, \dots, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

This coincides with the definition of W_π when $\alpha = (\alpha_1, \ldots, \alpha_n)$ are the Langlands parameters associated to π.

We are now able to define the p-adic Whittaker transform precisely.

Definition 3.2.7 (p-adic Whittaker Transform). *Let $h : Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p) \to \mathbb{C}$. Define the Whittaker transform $h^\sharp : \mathbb{C}^n \to \mathbb{C}$ by*

$$h^\sharp(\alpha) = \int\limits_{T(\mathbb{Z}_p)\backslash T(\mathbb{Q}_p)} h(t) W_\alpha(t) d^\times t,$$

provided the integral converges, where

$$d^\times t = \prod_{k=1}^{n-1} |t_k|_p^{-k(n-k)} \frac{dt_k}{|t_k|_p}.$$

Remark 3.2.8. *A sufficient condition for the convergence of the above integral is that h satisfies the decay condition*

$$|h(t)|_p \ll \delta^{1/2}(t) |t_1 \cdots t_{n-1}|_p^{\tau+\varepsilon}$$

for any $\varepsilon > 0$ and $\tau = \max\limits_{1 \le i \le n-1} -\log|\alpha_i|_p$.

Definition 3.2.9 (Inverse Whittaker Transform). *Let $H : \mathcal{S}^n \to \mathbb{C}$ holomorphic, where $\mathcal{S}$ is an open annulus containing the unit circle. Further, assume that H is invariant under the transformations $(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$, for all $(\alpha_1, \ldots, \alpha_n) \in \mathcal{S}^n$ and for every permutation $\sigma \in S_n$. Let*

$$W_\alpha(t) = \begin{cases} \delta^{1/2}(t) s_{\text{-}\log|t|_p}(\alpha), & \text{if } t_i \in \mathbb{Z}_p \text{ for } i = 1, \ldots, n-1, \\ 0, & \text{otherwise;} \end{cases}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$. Define the inverse Whittaker transform of H, $H^\flat : Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p) \to \mathbb{C}$ by

$$H^\flat(t) = \frac{1}{n!(2\pi i)^{n-1}} \int\limits_{\mathbb{T}} H(\beta) W_{1/\beta}(t) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (\beta_i - \beta_j) \frac{d\beta_1 \cdots d\beta_{n-1}}{\beta_1 \cdots \beta_{n-1}},$$

where $\mathbb{T} = \{(\beta_1, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1} : |\beta_i|_\mathbb{C} = 1\}$, $1/\beta = (1/\beta_1, \ldots, 1/\beta_n)$ and $\beta_n = \frac{1}{\beta_1 \cdots \beta_{n-1}}$.

The inverse transform is given in the following theorem.

Theorem 3.2.10 (Inversion Formula). *Let $H : \mathcal{S}^n \to \mathbb{C}$ holomorphic, where $\mathcal{S}$ is an open annulus containing the unit circle. Let D be a domain where $(H^\flat)^\sharp$ converges, and assume $\big\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n : |\alpha_i|_\mathbb{C} = 1\big\} \subset \mathcal{D} \subset \mathcal{S}^n$. Then,*

$$(H^\flat)^\sharp(\alpha) = H(\alpha)$$

for $\alpha_1 \cdots \alpha_n = 1$.

Corollary 3.2.11. *Let $h : Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p) \to \mathbb{C}$. For*

$$t = \begin{pmatrix} t_1 \cdots t_{n-1} & & & \\ & \ddots & & \\ & & t_1 & \\ & & & 1 \end{pmatrix}$$

assume $h(t)$ is compactly supported on the region $\{t_i \in \mathbb{Z}_p\backslash\{0\} : i = 1, \ldots, n-1\}$, and that $h(t) = f(-|t_1|_p, \ldots, -|t_{n-1}|_p)$ for some function $f : \mathbb{Z}_{\geq 0}^{n-1} \to \mathbb{C}$. Then

$$(h^\sharp)^\flat = h.$$

We postpone the proofs of the results stated above to the last subsection.

3.2.2 Background

Here we show two of the ingredients needed to prove the inverse transform theorem. Both results can also be found in Section 4 of [Bump, 2005]. They are included here as their proofs are concise and improve the exposition. The first of these is a version of Cauchy's identity.

Lemma 3.2.12 (Cauchy's Identity). *Let s_λ be the Schur polynomial defined in the previous section. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{C}$ such that $|\alpha_i \beta_j| < 1$ for every $1 \leq i, j \leq n$. Then,*

$$\sum_{m_1, \ldots, m_{n-1} \geq 0} s_{\mathbf{m}}(\alpha)\, s_{\mathbf{m}}(\beta) = \frac{1 - \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_n}{\prod_{i,j=1}^{n} (1 - \alpha_i \beta_j)},$$

where $\mathbf{m} = (m_1, \ldots, m_{n-1})$.

Proof. This proof follows [Goldfeld, 2006] and [Macdonald, 1979]. We start by stating the Cauchy determinant identity (lemma 7.4.18 in [Goldfeld, 2006])

$$\begin{vmatrix} \frac{1}{1-\alpha_1\beta_1} & \cdots & \frac{1}{1-\alpha_n\beta_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{1-\alpha_1\beta_n} & \cdots & \frac{1}{1-\alpha_n\beta_n} \end{vmatrix} = \frac{\prod\limits_{1\le i<j\le n}(\alpha_i-\alpha_j)\prod\limits_{1\le i<j\le n}(\beta_i-\beta_j)}{\prod\limits_{i,j=1}^{n}(1-\alpha_i\beta_j)}.$$

Expanding each term $\frac{1}{1-\alpha_i\beta_j}$ as $1+\alpha_i\beta_j+\alpha_i^2\beta_j^2+\cdots$, we obtain

$$\begin{vmatrix} \frac{1}{1-\alpha_1\beta_1} & \cdots & \frac{1}{1-\alpha_n\beta_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{1-\alpha_1\beta_n} & \cdots & \frac{1}{1-\alpha_n\beta_n} \end{vmatrix} = \sum_{l_1,\ldots,l_n\ge 0}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\alpha_{\sigma(1)}^{l_1}\cdots\alpha_{\sigma(n)}^{l_n}\beta_1^{l_1}\cdots\beta_n^{l_n}.$$

Notice that if the l_i are not all distinct then the sum over S_n vanishes. Therefore

$$\begin{aligned}\begin{vmatrix} \frac{1}{1-\alpha_1\beta_1} & \cdots & \frac{1}{1-\alpha_n\beta_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{1-\alpha_1\beta_n} & \cdots & \frac{1}{1-\alpha_n\beta_n} \end{vmatrix} &= \sum_{\substack{l_1>\ldots>l_n\ge 0\\ \sigma,\tau\in S_n}}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\alpha_{\sigma(1)}^{l_1}\cdots\alpha_{\sigma(n)}^{l_n}\beta_{\tau(1)}^{l_1}\cdots\beta_{\tau(n)}^{l_n} \\ &= \sum_{l_1>\ldots>l_n\ge 0}\begin{vmatrix}\alpha_1^{l_1} & \cdots & \alpha_n^{l_1}\\ \vdots & \ddots & \vdots \\ \alpha_1^{l_n} & \cdots & \alpha_n^{l_n}\end{vmatrix}\begin{vmatrix}\beta_1^{l_1} & \cdots & \beta_n^{l_1}\\ \vdots & \ddots & \vdots \\ \beta_1^{l_n} & \cdots & \beta_n^{l_n}\end{vmatrix}.\end{aligned}$$

By making the substitution $(l_1,\ldots,l_n)=(m_0+\cdots+m_{n-1}+(n-1),m_0+\cdots+m_{n-2}+(n-2),\ldots,m_0)$ and dividing by $\prod\limits_{1\le i<j\le n}(\alpha_i-\alpha_j)\prod\limits_{1\le i<j\le n}(\beta_i-\beta_j)$ we obtain

$$\frac{1}{\prod\limits_{i,j=1}^{n}(1-\alpha_i\beta_j)} = \sum_{m_0,m_1,\ldots,m_{n-1}\ge 0} s_{\mathbf{m}}(\alpha)\, s_{\mathbf{m}}(\beta)\,(\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n)^{m_0},$$

where right hand side simplifies to

$$\frac{1}{1-\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n}\sum_{m_1,\ldots,m_{n-1}\ge 0} s_{\mathbf{m}}(\alpha)\, s_{\mathbf{m}}(\beta).$$

Multiplying out by $1-\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n$ finishes the proof. □

The second necessary ingredient is an identity for the integral of a product of two Whittaker function, as in [Stade, 2001].

Proposition 3.2.13. *Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{C}$ and $\epsilon > 0$. Let W_α, W_β be Whittaker functions as defined in Definition 3.2.9. Then*

$$\int\limits_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} W_\alpha(t) W_\beta(t) \prod_{k=1}^{n-1} |t_k|_p^{\epsilon(n-k)} d^\times t = \frac{1 - \frac{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_n}{p^{\epsilon n}}}{\prod\limits_{i,j=1}^{n} \left(1 - \frac{\alpha_i \beta_j}{p^\epsilon}\right)}.$$

Proof. We start by applying Theorem 3.2.5 to write down the Whittaker functions explicitly:

$$\begin{aligned}
&\int\limits_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} W_\alpha(t) W_\beta(t) \prod_{k=1}^{n-1} |t_k|_p^{\epsilon(n-k)} d^\times t \\
&= \int\limits_{\mathbb{Z}_p^{n-1}} \delta(t) s_{\text{-log}\,|t|_p}(\alpha) s_{\text{-log}\,|t|_p}(\beta) \prod_{k=1}^{n-1} |t_k|_p^{-(k-\epsilon)(n-k)} \frac{dt_k}{|t_k|_p} \\
&= \int\limits_{\mathbb{Z}_p^{n-1}} s_{\text{-log}\,|t|_p}(\alpha) s_{\text{-log}\,|t|_p}(\beta) \prod_{k=1}^{n-1} |t_k|_p^{\epsilon(n-k)} \frac{dt_k}{|t_k|_p}.
\end{aligned}$$

Breaking up the region of integration by absolute value we get

$$\begin{aligned}
&\int\limits_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} W_\alpha(t) W_\beta(t) \prod_{k=1}^{n-1} |t_k|_p^{\epsilon(n-k)} d^\times t \\
&= \sum_{m_1,\dots,m_{n-1} \geq 0} \overbrace{\int \cdots \int}^{n-1 \text{ times}}_{p^{m_k}\mathbb{Z}_p^\times} s_{\text{-log}\,|t|_p}(\alpha) s_{\text{-log}\,|t|_p}(\beta) \prod_{k=1}^{n-1} |t_k|_p^{\epsilon(n-k)} \frac{dt_k}{|t_k|_p} \\
&= \sum_{m_1,\dots,m_{n-1} \geq 0} s_{\mathbf{m}}(\alpha) s_{\mathbf{m}}(\beta)\, p^{-\epsilon(n-1)m_1 - \epsilon(n-2)m_2 - \cdots - \epsilon m_{n-1}} \\
&= \sum_{m_1,\dots,m_{n-1} \geq 0} s_{\mathbf{m}}\left(\frac{\alpha}{p^\epsilon}\right) s_{\mathbf{m}}(\beta) \\
&= \frac{1 - \frac{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_n}{p^{\epsilon n}}}{\prod\limits_{i,j=1}^{n} \left(1 - \frac{\alpha_i \beta_j}{p^\epsilon}\right)},
\end{aligned}$$

where the last equality follows from Cauchy's identity 3.2.12. □

3.2.3 Proof of Inversion Formula

We are now ready to proceed to the proof of the main theorem in this section. Restate the theorem as

$$H(\alpha) = \int\limits_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} H^\flat(t) W_\alpha(t) d^\times t.$$

Recall that $\alpha \in \mathcal{D}$ and $\alpha_1 \cdots \alpha_n = 1$. Further assume that $|\alpha_1|_{\mathbb{C}} = \cdots = |\alpha_n|_{\mathbb{C}} = 1$. We can later remove this assumption as both sides of the equality are holomorphic. Note that H is invariant under permutations of $(\alpha_1, \ldots, \alpha_n)$ as the Whittaker function also has those symmetries. Define

$$H_\epsilon(\alpha) = \int\limits_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} H^\flat(t) W_\alpha(t) \prod_{k=1}^{n-1} |t_k|_p^{\epsilon(n-k)} d^\times t.$$

Then

$$\lim_{\epsilon \to 0} H_\epsilon(\alpha) = \int\limits_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} H^\flat(t) W_\alpha(t) d^\times t.$$

It suffices to show that $\lim\limits_{\epsilon \to 0} H_\epsilon(\alpha) = H(\alpha)$ as well.
Replacing $H^\flat(t)$ by its explicit formula and swapping the order of integration we get

$$\begin{aligned} H_\epsilon(\alpha) = & \frac{1}{n!(2\pi i)^{n-1}} \int\limits_{\mathbb{T}} \left(\int\limits_{Z(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} W_\alpha(t) W_{1/\beta}(t) \prod_{k=1}^{n-1} |t_k|_p^{\epsilon(n-k)} d^\times t \right) \\ & \times H(\beta) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (\beta_i - \beta_j) \frac{d\beta_1 \cdots d\beta_{n-1}}{\beta_1 \cdots \beta_{n-1}}. \end{aligned}$$

We use Proposition 3.2.13 to compute the innermost integrals and get

$$H_\epsilon(\alpha) = \frac{1}{n!(2\pi i)^{n-1}} \int\limits_{\mathbb{T}} \frac{H(\beta)\left(1 - \frac{1}{p^{\epsilon n}}\right)}{\prod\limits_{i,j=1}^{n} \left(\beta_j - \frac{\alpha_i}{p^\epsilon}\right)} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (\beta_i - \beta_j) \frac{d\beta_1 \cdots d\beta_{n-1}}{\beta_1 \cdots \beta_{n-1}}.$$

Now shift each variable β_i $(i = 1, \ldots, n-1)$ to the contour given by the equation $|z|_{\mathbb{C}} = \frac{1}{p^{\epsilon + \epsilon'}}$, in order. For the sake of clearness, we shall assume that all the α_i are distinct. When the contour of integration for β_1 is shifted one picks up several residues at

$$\beta_1 = \frac{\alpha_1}{p^\epsilon}, \ldots, \beta_1 = \frac{\alpha_n}{p^\epsilon}.$$

For which residue integral obtained in this manner one can shift the contour of integration for β_2 picking up $n-1$ residues in the process. Repeating this process for which β_i, $i = 3, \ldots, n-1$ one obtains

$$H_\epsilon(\alpha) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathcal{R}_\epsilon^\sigma + \mathcal{I}_{\epsilon,\epsilon'},$$

where

$$\mathcal{R}_\epsilon^\sigma = \frac{H\left(\frac{\alpha_{\sigma(1)}}{p^\epsilon}, \dots, \frac{\alpha_{\sigma(n)}}{p^\epsilon}\right)\left(1 - \frac{1}{p^{\epsilon n}}\right)\prod_{i=1}^{n-1}\left(\frac{\alpha_{\sigma(i)}}{p^\epsilon} - p^{(n-1)\epsilon}\alpha_{\sigma(n)}\right)}{\left(p^{(n-1)\epsilon}\alpha_{\sigma(n)} - \frac{\alpha_{\sigma(n)}}{p^\epsilon}\right)\prod_{i=1}^{n-1}\left(\frac{\alpha_{\sigma(i)}}{p^\epsilon} - \frac{\alpha_{\sigma(n)}}{p^\epsilon}\right)\prod_{i=1}^{n-1}\frac{\alpha_{\sigma(i)}}{p^\epsilon}}$$

and $\mathcal{I}_{\epsilon,\epsilon'}$ is a sum of residue integrals with each integrand bounded (in absolute value) by

$$C_H \frac{\left(1 - \frac{1}{p^{n\epsilon}}\right)p^{\epsilon(n^3+n)}}{1 - \frac{1}{p^{\epsilon'}}}$$

for some constant $C_H > 0$ depending only on the function H. Therefore,

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathcal{R}_\epsilon^\sigma &= \lim_{\epsilon\to 0}\frac{H\left(\frac{\alpha_{\sigma(1)}}{p^\epsilon}, \dots, \frac{\alpha_{\sigma(n)}}{p^\epsilon}\right)}{\prod_{i=1}^{n}\frac{\alpha_{\sigma(i)}}{p^\epsilon}}\frac{\left(1 - \frac{1}{p^{\epsilon n}}\right)}{\left(p^{(n-1)\epsilon} - \frac{1}{p^\epsilon}\right)}\frac{\prod_{i=1}^{n-1}\left(\frac{\alpha_{\sigma(i)}}{p^\epsilon} - p^{(n-1)\epsilon}\alpha_{\sigma(n)}\right)}{\prod_{i=1}^{n-1}\left(\frac{\alpha_{\sigma(i)}}{p^\epsilon} - \frac{\alpha_{\sigma(n)}}{p^\epsilon}\right)} \\
&= \frac{H\left(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}\right)}{\prod_{i=1}^{n}\alpha_{\sigma(i)}}\frac{\prod_{i=1}^{n-1}\left(\alpha_{\sigma(i)} - \alpha_{\sigma(n)}\right)}{\prod_{i=1}^{n-1}\left(\alpha_{\sigma(i)} - \alpha_{\sigma(n)}\right)} \\
&= H\left(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}\right) \\
&= H(\alpha)
\end{aligned}$$

and

$$\lim_{\epsilon\to 0}|\mathcal{I}_{\epsilon,\epsilon'}| \ll_H \lim_{\epsilon\to 0}\frac{\left(1 - \frac{1}{p^{n\epsilon}}\right)p^{\epsilon(n^3+n)}}{1 - \frac{1}{p^{\epsilon'}}} = 0.$$

In conclusion, we obtain

$$\lim_{\epsilon\to 0}H_\epsilon(\alpha) = \frac{1}{n!}\sum_{\sigma\in S_n}H(\alpha) = H(\alpha),$$

as desired. If the α_i are not all distinct then the original integrand would have higher order poles. This means that the residue sum would have a small number of residue but some residues would contribute to the sum with a multiple of $H(\alpha)$.

To prove the corollary one simply needs to compute the image of the inverse transform $H \mapsto H^\flat$ as any function h in the image of this mapping will satisfy $(h^\sharp)^\flat = h$. Simply choose H such that $h = H^\flat$, and apply the inversion formula 3.2.10 to H, to obtain $(h^\sharp)^\flat = ((H^\flat)^\sharp)^\flat = H^\flat = h$. We start by noting that any function h satisfying the conditions in the corollary is a finite sum of scalar multiples of function of the form:

$$f_{\lambda_1,\ldots,\lambda_{n-1}}(t) = \begin{cases} 1, & \text{if } \log|t_i|_p = -\lambda_i \text{ for } i = 1,\ldots,n-1, \\ 0, & \text{otherwise;} \end{cases}$$

where $\lambda_1, \ldots, \lambda_{n-1}$ are nonnegative integers. Therefore, it suffices to show that all function $f_{\lambda_1,\ldots,\lambda_{n-1}}$ for $\lambda_1, \ldots, \lambda_{n-1}$ nonnegative integers are in the image of the map of the inverse transform $H \mapsto H^\flat$. Let $H(\beta) = \frac{W_\beta(p^\lambda)}{\delta(p^\lambda)}$, where

$$p^\lambda = \begin{pmatrix} p^{\lambda_1} \cdots p^{\lambda_{n-1}} & & & \\ & \ddots & & \\ & & p^{\lambda_1} & \\ & & & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
&H^\flat(t) \\
&= \frac{1}{\delta(p^\lambda) n! (2\pi i)^{n-1}} \int\limits_{\mathbb{T}} W_\beta(p^\lambda) W_{1/\beta}(t) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (\beta_i - \beta_j) \frac{d\beta_1 \cdots d\beta_{n-1}}{\beta_1 \cdots \beta_{n-1}} \\
&= \frac{\delta^{1/2}(t)}{\delta^{1/2}(p^\lambda) n! (2\pi i)^{n-1}} \int\limits_{\mathbb{T}} s_\lambda(\beta) s_{\text{-}\log|t|_p}(1/\beta) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (\beta_i - \beta_j) \frac{d\beta_1 \cdots d\beta_{n-1}}{\beta_1 \cdots \beta_{n-1}} \\
&= \frac{\delta^{1/2}(t)}{\delta^{1/2}(p^\lambda) n! (2\pi i)^{n-1}} \int\limits_{\mathbb{T}} \sum_{\sigma,\tau \in S_n} \left(\beta_{\sigma(1)}^{(n-1)+\lambda_1+\cdots+\lambda_{n-1}} \cdots \beta_{\sigma(n-1)}^{1+\lambda_1} \right) \\
&\qquad \times \left(\beta_{\tau(1)}^{-(n-1)+\log|t_1|_p+\cdots+\log|t_{n-1}|_p} \cdots \beta_{\tau(n-1)}^{-1+\log|t_1|_p} \right) \frac{d\beta_1 \cdots d\beta_{n-1}}{\beta_1 \cdots \beta_{n-1}} \\
&= \frac{\delta^{1/2}(t)}{\delta^{1/2}(p^\lambda) n! (2\pi i)^{n-1}} \\
&\qquad \times \int\limits_{\mathbb{T}} \sum_{\sigma \in S_n} \left(\beta_{\sigma(1)}^{\lambda_1+\cdots+\lambda_{n-1}+\log|t_1|_p+\cdots+\log|t_{n-1}|_p} \cdots \beta_{\sigma(n-1)}^{\lambda_1+\log|t_1|_p} \right) \frac{d\beta_1 \cdots d\beta_{n-1}}{\beta_1 \cdots \beta_{n-1}} \\
&= \frac{\delta^{1/2}(t)}{\delta^{1/2}(p^\lambda)} f_{\lambda_1,\ldots,\lambda_{n-1}}(t) \\
&= f_{\lambda_1,\ldots,\lambda_{n-1}}(t),
\end{aligned}$$

as desired.

3.2.4 Local L-function of a Symmetric Square Lift

Let $L(s,\pi)$ be the L-function associated to an irreducible automorphic representation of $\mathrm{GL}(3,\mathbb{A}_{\mathbb{Q}})$ with local factors (at the unramified places p) given by

$$L_p(s,\pi) = \left(1-\alpha_1^2 p^{-s}\right)^{-1}\left(1-\alpha_1\alpha_2 p^{-s}\right)^{-1}\left(1-\alpha_2^2 p^{-s}\right)^{-1} = h_{s,p}(\alpha),$$

with $\alpha_1\alpha_2 = 1$ and $\Re(s)$ large enough. These are the L-factors that occur when π is the symmetric square lift of an irreducible automorphic representation of $\mathrm{GL}(3,\mathbb{A}_{\mathbb{Q}})$. We shall obtain an integral representation for the L-factors $L_p(s,\pi)$ using Theorem 3.2.10.

By Theorem 3.2.10 we can write

$$L_p(s,\pi) = \int\limits_{T(\mathbb{Z}_p)\backslash T(\mathbb{Q}_p)} (h_{s,p})^\flat(t) W_\alpha(t) d^\times t,$$

with $\alpha = (\alpha_1,\alpha_2)$, assuming $\Re(s)$ is large enough. We will now compute $(h_{s,p})^\flat$ explicitly.

$$\begin{aligned}
(h_{s,p})^\flat(t) &= \frac{1}{4\pi i}\int\limits_{|\beta_1|=1} h_{s,p}(\beta) W_{1/\beta}(t)(\beta_1-\beta_1^{-1})(\beta_1^{-1}-\beta_1)\frac{d\beta_1}{\beta_1}\\
&= \left(\frac{|t_1|_p^{1/2}}{1-p^{-s}}\right)\frac{1}{4\pi i}\\
&\times \int\limits_{|\beta_1|_{\mathbb{C}}=1}\left(\frac{1}{1-\beta_1^2p^{-s}}\right)\left(\frac{1}{1-\beta_1^{-2}p^{-s}}\right) W_{1/\beta}(t)\left(\beta_1^{\lambda+1}-\beta_1^{-(\lambda+1)}\right)\left(\beta_1^{-1}-\beta_1\right)\frac{d\beta_1}{\beta_1},
\end{aligned}$$

with $t = \begin{pmatrix} t_1 & 0\\ 0 & 1\end{pmatrix}$ and $|t_1|_p = p^{-\lambda}$.

Note that in the last integral the integrand $i(\beta_1)$ has possible poles at inside the unit circle at $\beta = 0, \pm p^{-s/2}$. By the residue theorem,

$$(h_{s,p})^\flat(t) = \frac{|t_1|_p^{1/2}}{2(1-p^{-s})}\left(\mathrm{Res}_{\beta_1=0}\, i(\beta_1) + \mathrm{Res}_{\beta_1=p^{-s/2}}\, i(\beta_1) + \mathrm{Res}_{\beta_1=-p^{-s/2}}\, i(\beta_1)\right),$$

where the residues are equal to

$$\operatorname{Res}_{\beta_1=0} i(\beta_1) = \begin{cases} \frac{p^{s/2}\left(\left(p^{s/2}\right)^{\lambda+1}-\left(p^{s/2}\right)^{-\lambda-1}\right)}{(1+p^{-s})}, & \text{if } \lambda \text{ even,} \\ 0, & \text{if } \lambda \text{ odd;} \end{cases}$$

$$\operatorname{Res}_{\beta_1=p^{-s/2}} i(\beta_1) = \frac{p^{s/2}\left(\left(p^{s/2}\right)^{\lambda+1}-\left(p^{s/2}\right)^{-\lambda-1}\right)}{2\left(1+p^{-s}\right)};$$

$$\operatorname{Res}_{\beta_1=-p^{-s/2}} i(\beta_1) = -\frac{p^{s/2}\left(\left(-p^{s/2}\right)^{\lambda+1}-\left(-p^{s/2}\right)^{-\lambda-1}\right)}{2\left(1+p^{-s}\right)}.$$

Combining all the above terms we obtain

$$(h_{s,p})^{\flat}(t) = \begin{cases} \frac{p^s|t_1|_p^{-(s-1)/2}}{1-p^{-2s}}, & \text{if } \lambda \text{ even,} \\ 0, & \text{if } \lambda \text{ odd.} \end{cases}$$

Chapter 4

Orthogonality Relation

4.1 Introduction

In this chapter, we prove an orthogonality relation for the Fourier coefficients of a "thin" family of Hecke-Maass forms for $\mathrm{SL}(3,\mathbb{Z})$. In addition, we also obtain a Weyl's law type result (weighted by some residues) for the same family of Maass forms. Both results slightly generalize previous work by the author [Guerreiro, 2015].

Let ϕ_j $(j = 1, 2, \ldots)$ be a set of orthogonal Hecke-Maass forms for $\mathrm{SL}(3,\mathbb{Z})$ of type $\nu^{(j)} = \left(\nu_1^{(j)}, \nu_2^{(j)}\right)$, and define $\nu_3^{(j)} = -\nu_1^{(j)} - \nu_2^{(j)}$. Note that for tempered forms the type is purely imaginary. Recall that for a particular Hecke-Maass form ϕ of type $\nu = (\nu_1, \nu_2)$ define $\nu_3 = -\nu_1 - \nu_2$, the Langlands parameters are

$$\alpha_1 = 2\nu_1 + \nu_2, \quad \alpha_2 = \nu_2 - \nu_1, \quad \alpha_3 = -\nu_1 - 2\nu_2.$$

and the Laplace eigenvalue of ϕ is given by

$$\lambda_\phi = 1 - 3(\nu_1^2 + \nu_2^2 + \nu_3^2) = 1 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2).$$

For $T \gg 1$, $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ (with $\nu_3 = -\nu_1 - \nu_2$), $R > 0$ and κ a positive number congruent to 2 (mod 4), we define

$$h_{T,R,\kappa}(\nu) = \left(T^{(\alpha_1/R)^\kappa} + T^{(\alpha_2/R)^\kappa} + T^{(\alpha_3/R)^\kappa}\right)^2 e^{2(\alpha_1^2+\alpha_2^2+\alpha_3^2)/T^2} \times \frac{\left(\prod_{j=1}^{3} \Gamma\left(\frac{2+R+3\nu_j}{4}\right)\Gamma\left(\frac{2+R-3\nu_j}{4}\right)\right)^2}{\prod_{j=1}^{3} \Gamma\left(\frac{1+3\nu_j}{2}\right)\Gamma\left(\frac{1-3\nu_j}{2}\right)}. \tag{4.1}$$

This function is essentially supported on the region where $|\nu_1|, |\nu_2|, |\nu_3| \ll T$. If ϕ is tempered and $|\nu_1|, |\nu_2|, |\nu_3| < T$ then $h_{T,R,\kappa}(\nu)$ is real valued and positive. We estimate this function in three regions. If one of the α_i is equal to 0, then

$$c_R^{(1)} \left[\prod_{i=1}^{3} (1+|\nu_i|)\right]^R \leq h_{T,R,\kappa}(\nu) \leq c_R^{(2)} \left[\prod_{i=1}^{3} (1+|\nu_i|)\right]^R,$$

for some $c_R^{(2)} \geq c_R^{(1)} > 0$. If one of the $|\alpha_i|$ is smaller than R, then

$$h_{T,R,\kappa}(\nu) \gg_R \frac{[(1+|\nu_1|)(1+|\nu_2|)(1+|\nu_3|)]^R}{T^2}.$$

If $|\alpha_1|, |\alpha_2|, |\alpha_3| > R^{1+\epsilon}$, then

$$h_{T,R,\kappa}(\nu) \ll \frac{1}{T^{R^{\epsilon\kappa}}}.$$

Theorem 4.1.1. *For $j = 1, 2, \ldots,$ let ϕ_j be a set of orthogonal Hecke-Maass forms for $SL(3,\mathbb{Z})$ with spectral parameters $\nu^{(j)} = \left(\nu_1^{(j)}, \nu_2^{(j)}\right)$. Let $h_{T,R,\kappa}$ be given as in (4.1). We have that for fixed $R > 50$, $\epsilon > 0$, $\kappa \equiv 2 \pmod 4$ positive and some $c_{R,\kappa} > 0$,*

$$\sum_{j\geq 1} \frac{\left|h_{T,R,\kappa}(\nu^{(j)})\right|}{\mathcal{L}_j} = c_{R,\kappa} \frac{T^{4+3R}}{(\log T)^{1/\kappa}} + \mathcal{O}_{R,\epsilon}(T^{3+3R+\epsilon}), \qquad (T \to \infty).$$

Moreover, for positive integers m_1, m_2, n_1, n_2, we have

$$\sum_{j\geq 1} A_j(m_1, m_2)\overline{A_j(n_1, n_2)} \frac{\left|h_{T,R,\kappa}(\nu^{(j)})\right|}{\mathcal{L}_j} = \begin{cases} \sum_{j\geq 1} \frac{\left|h_{T,R,\kappa}(\nu^{(j)})\right|}{\mathcal{L}_j} + \mathcal{O}_{R,\epsilon}\left((m_1m_2n_1n_2)^6\, T^{3R+3+\epsilon}\right), & \text{if } {m_1=n_1 \atop m_2=n_2}, \\ \mathcal{O}_{R,\epsilon}\left((m_1m_2n_1n_2)^6\, T^{3R+3+\epsilon}\right), & \text{otherwise.} \end{cases}$$

Here the weights $\mathcal{L}_j$ are the L-values $L_{Ad\,\phi_j}(1)$.

Remark 4.1.2. *This result generalizes previous work by the author in [Guerreiro, 2015], where the case $\kappa = 2$ was shown.*

Remark 4.1.3. *Note that an analogous result in [Goldfeld and Kontorovich, 2013] for all Hecke-Maass forms for $SL(3,\mathbb{Z})$ with spectral parameters $|\nu_1|, |\nu_2|, |\nu_3| \ll T$ yields a main term of size T^{5+3R}. This implies that the family of Maass forms picked out by our test function $h_{T,R,\kappa}$ is indeed significantly thinner (by a factor of $T(\log T)^{1/\kappa}$) than the family of all Hecke-Maass forms for $SL(3,\mathbb{Z})$.*

Remark 4.1.4. *The test functions $h_{T,R,\kappa}$ appearing in this theorem are a product of three terms chosen with the following objectives: the first exponential term contributes with polynomial decay when all $|\alpha_1|, |\alpha_2|, |\alpha_3| \gg 0$ and exponential decay when all $|\alpha_1|, |\alpha_2|, |\alpha_3| \gg T^{\epsilon}$; the second exponential term contributes with exponential decay when one of $|\nu_i| > T^{1+\epsilon}$; the product of Gamma factors is already partially present in the Kuznetsov trace formula and its particular form is such that it has polynomial growth in ν. We note that the methods presented here have also been carried out for a different family of test functions in [Goldfeld and Kontorovich, 2013].*

Remark 4.1.5. *Blomer [Blomer, 2013] shows that the weights $\mathcal{L}_j$ are bounded by*

$$\left(\prod_{i=1}^{3}\left(1+\left|\nu_i^{(j)}\right|\right)\right)^{-1} \ll \mathcal{L}_j \ll_\epsilon \left(\prod_{i=1}^{3}\left(1+\left|\nu_i^{(j)}\right|\right)\right)^{\epsilon} \tag{4.2}$$

and, conjecturally, the lower bound is expected to be

$$\left(\prod_{i=1}^{3}\left(1+\left|\nu_i^{(j)}\right|\right)\right)^{-\epsilon} \ll_\epsilon \mathcal{L}_j. \tag{4.3}$$

4.2 Bound for the Inverse Lebedev-Whittaker Transform

Let

$$F^{\#}_{T,R,\kappa}(\nu_1,\nu_2) = \left(T^{(\alpha_1/R)^\kappa} + T^{(\alpha_2/R)^\kappa} + T^{(\alpha_3/R)^\kappa}\right) e^{(\alpha_1^2+\alpha_2^2+\alpha_3^2)/T^2} \times \left(\prod_{j=1}^{3} \Gamma\left(\frac{2+R+3\nu_j}{4}\right)\Gamma\left(\frac{2+R-3\nu_j}{4}\right)\right).$$

As $F^{\#}_{T,R}$ has enough exponential decay on a strip $|\Re(\nu_1)|, |\Re(\nu_2)| < \epsilon$ (see 3.1) then by Lebedev-Whittaker inversion as in Theorem 3.1.3,

$$F_{T,R,\kappa}(y) = \frac{1}{(\pi i)^2} \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} F^{\#}_{T,R,\kappa}(\nu) W^*_\nu(y) \frac{d\nu_1 d\nu_2}{\prod_{j=1}^{3} \Gamma\left(\frac{3\nu_j}{2}\right)\Gamma\left(-\frac{3\nu_j}{2}\right)}.$$

We also have the Parseval-type identity as in the second part of Theorem 3.1.3

$$\langle F_{T,R,\kappa}, F_{T,R,\kappa}\rangle = \frac{1}{(\pi i)^2} \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{\left|F^{\#}_{T,R,\kappa}(\nu)\right|^2 d\nu_1 d\nu_2}{\prod_{j=1}^{3} \Gamma\left(\frac{3\nu_j}{2}\right)\Gamma\left(\frac{-3\nu_j}{2}\right)} = \left\langle F^{\#}_{T,R,\kappa}, F^{\#}_{T,R,\kappa}\right\rangle. \tag{4.4}$$

Proposition 4.2.1. *Fix $C_1, C_2 > 0$, $R > 3\max(C_1, C_2) + 6$ and $\epsilon > 0$. For any $y_1, y_2 > 0$, $T \gg 1$, we have*

$$|F_{T,R,\kappa}(y)| \ll_{C_1,C_2,R,\epsilon} y_1 y_2 T^{3R/2+11/2+C_1/2+C_2/2+\epsilon} \left(\frac{y_1}{T}\right)^{C_1} \left(\frac{y_2}{T}\right)^{C_2}. \tag{4.5}$$

Proof. This proof follows the same steps as the proof of Proposition 3.1 in [Guerreiro, 2015], where it was carried out for $F_{T,R,2}$. We start by writing out the representation of W^*_ν as an inverse Mellin transform as in Proposition 2.3.9,

$$W^*_\nu(y) = \frac{y_1 y_2 \pi^{3/2}}{(2\pi i)^2} \int_{(C_2)}\int_{(C_1)} \frac{\prod_{j=1}^{3} \Gamma\left(\frac{s_1-\alpha_j}{2}\right)\Gamma\left(\frac{s_2+\alpha_j}{2}\right)}{4\pi^{s_1+s_2}\Gamma\left(\frac{s_1+s_2}{2}\right)} y_1^{-s_1} y_2^{-s_2}\, ds_1\, ds_2,$$

for $C_1, C_2 > 0$. Combining this with the Lebedev-Whittaker inverse transform of $F_{T,R,\kappa}$, we observe that

$$F_{T,R,\kappa}(y) = \int\limits_{(0)}\int\limits_{(0)}\int\limits_{(C_2)}\int\limits_{(C_1)} \frac{F^{\#}_{T,R,\kappa}(\nu_1,\nu_2)}{\prod\limits_{j=1}^{3}\Gamma\left(\frac{3\nu_j}{2}\right)\Gamma\left(-\frac{3\nu_j}{2}\right)} \times \frac{\prod\limits_{j=1}^{3}\Gamma\left(\frac{s_1-\alpha_j}{2}\right)\Gamma\left(\frac{s_2+\alpha_j}{2}\right)}{16\pi^{s_1+s_2+5/2}\Gamma\left(\frac{s_1+s_2}{2}\right)} y_1^{1-s_1}y_2^{1-s_2}ds_1ds_2d\nu_1d\nu_2.$$

Assume $2k < C_1 < 2k+2$ for some non-negative interger k and pull s_1 from the contour (C_1) to the contour $(-C_1)$. Then

$$F_{T,R,\kappa}(y) = \mathcal{M} + \sum_{m=0}^{k}\sum_{l=1}^{3}\mathcal{R}_{m,l}, \tag{4.6}$$

where $\mathcal{M}$ is the main term and is given by the same expression as $F_{T,R,\kappa}$ with the contour (C_1) replaced by $(-C_1)$. Here $\mathcal{R}_{m,l}$ are the residues corresponding to the poles of the integrand at $s_1 = \alpha_l - 2m$ for $m = 0,\ldots,k$ and $l = 1,2,3$. Note that there will be no contribution of residues of higher order poles, which could possibly show up when two of the α_i differ by an even integer. In this situation, at least one of $\pm 3\nu_1$, $\pm 3\nu_2$, $\pm 3\nu_3$ is equal to a non-positive even integer, making the integrand identically zero in that region.

Then assume $2k' < C_2 < 2k'+2$ for some non-negative interger k'. For each term in (4.6) shift variable s_2 from the contour (C_2) to $(-C_2)$ to get,

$$F_{T,R,\kappa}(y) = \tilde{\mathcal{M}} + \sum_{m'=0}^{k}\sum_{l'=1}^{3}\mathcal{M}_{m',l'} + \sum_{m,m'=0}^{k,k'}\sum_{l=1}^{3}\sum_{l'\neq l}\mathcal{R}_{m,l,m',l'}, \tag{4.7}$$

where $\tilde{\mathcal{M}}$ is the main term, given by the same expression as $\mathcal{M}$ with the contour (C_2) replaced by the contour $(-C_2)$, and $\mathcal{M}_{m',l'}$ are the residues corresponding to the poles of the integrand of $\mathcal{M}$ at $s_2 = -\alpha_{l'} - 2m'$ for $m' = 0,\ldots,k'$. The terms $\mathcal{R}_{m,l,m',l'}$ are the residues corresponding to the poles of the integrand of $\mathcal{R}_{m,l}$ at $s_2 = -\alpha_{l'} - 2m'$ for $m' = 0,\ldots,k'$ and $l' \neq l$.

Let $\nu_j = it_j$ and $s_j = C_j + iu_j$. Note that, for $\Re(\nu_j) = 0$, the first exponential term in the definition of $F^{\#}_{T,R,\kappa}$ is bounded (independent of κ) and the second one has exponential decay for $|t_j| > T^{1+\delta}$. Using Stirling's formula to estimate the Gamma factors we get

$$|\tilde{\mathcal{M}}| \ll y_1^{1+C_1}y_2^{1+C_2} \iint\limits_{|t_1|,|t_2|<T^{1+\delta}} \iint\limits_{\mathbb{R}^2} \mathcal{P}\cdot\exp\left(\mathcal{E}\right)du_1du_2dt_1dt_2,$$

where the exponential factor is given by

$$\frac{4\mathcal{E}}{\pi} = 3\sum_{j=1}^{3}|t_j| + |u_1+u_2| - \sum_{j=1}^{3}|iu_1-\alpha_j| - \sum_{j=1}^{3}|iu_2+\alpha_j|,$$

and the polynomial term is given by

$$\mathcal{P} = \left(\prod_{j=1}^{3}(1+|t_j|)\right)^{(R+2)/2} (1+|u_1+u_2|)^{(1+C_1+C_2)/2} \times \left(\prod_{j=1}^{3}(1+|iu_1-\alpha_j|)\right)^{(-C_1-1)/2} \left(\prod_{j=1}^{3}(1+|iu_2+\alpha_j|)\right)^{(-C_2-1)/2}.$$

We now show that the exponential factor is non-positive. As the exponential factor is invariant under cyclic permutations of (t_1, t_2, t_3), we may assume, without loss of generality, that t_1 and t_2 have the same sign. Then $|\alpha_1| + |\alpha_3| = 3|t_1| + 3|t_2|$. As $|u_1+u_2| \le |iu_1-\alpha_2| + |iu_2+\alpha_2|$, we get

$$\begin{aligned}\frac{4\mathcal{E}}{\pi} &\le 3\sum_{j=1}^{3}|t_j| - |iu_1-\alpha_1| - |iu_1-\alpha_3| - |iu_2+\alpha_1| - |iu_2+\alpha_3| \\ &\le 6|t_1| + 6|t_2| - 2(|\alpha_1| + |\alpha_3|) \\ &= 0.\end{aligned}$$

For either $|u_1| > 5T^{1+\delta}$ or $|u_2| > 5T^{1+\delta}$, the exponential factor is bounded above by $-T^{1+\delta}$ giving exponential decay to the integral. Integrating first over u_1, u_2 we get

$$|\mathcal{M}| \ll y_1^{1+C_1} y_2^{1+C_2} \iiiint\limits_{\substack{|t_1|,|t_2|<T^{1+\delta} \\ |u_1|,|u_2|<5T^{1+\delta}}} \mathcal{P} du_1 du_2 dt_1 dt_2 \ll y_1^{1+C_1} y_2^{1+C_2} T^{(3R+11-C_1-C_2)/2+\epsilon}$$

by choosing δ appropriately. To bound the residues $\mathcal{M}_{m',l'}$ we start by shifting variables ν_1 and ν_2 to contours (B_1) and (B_2), respectively, where $|B_1|, |B_2| < R/3$ and $B'_j < C_1$ for $j \ne l$, defining $B'_1 = 2B_1 + B_2$, $B'_2 = B_2 - B_1$, $B'_3 = -B_1 - 2B_2$, in a manner similar to the ν_j. Note that the first exponential term is now bounded by $3T^{(\max(B'_1,B'_2,B'_3)/R)^\kappa} \le T$. It follows that

$$|\mathcal{M}_{m',l'}| \ll y_1^{1+C_1} y_2^{1+B'_{l'}+2m'} T \iint\limits_{|t_1|,|t_2|<T^{1+\delta}} \int\limits_{\mathbb{R}} \mathcal{P} \cdot \exp(\mathcal{E})\, du_1 dt_1 dt_2,$$

where the exponential factor is given by

$$\frac{4\mathcal{E}}{\pi} = 3\sum_{j=1}^{3}|t_j| + |u_1 - \Im(\alpha_{l'})| - \sum_{j=1}^{3}|u_1 - \Im(\alpha_j)| - \sum_{j\neq l'}|\Im(\alpha_j - \alpha_{l'})|,$$

and the polynomial term is given by

$$\mathcal{P} = \left(\prod_{j=1}^{3}(1+|t_j|)\right)^{(R+2)/2} (1+|u_1-\Im(\alpha_{l'})|)^{(1+C_1+B'_{l'}+2m')/2}\left(\prod_{j=1}^{3}(1+|u_1-\Im(\alpha_j)|)\right)^{(-C_1-1)/2}$$
$$\times \prod_{j\neq l'}\left((1+|\Im(-\alpha_{l'}+\alpha_j)|)^{(B'_j-B'_{l'}-2m'-1)/2}\right).$$

The exponential factor is again non-positive as

$$\frac{4\mathcal{E}}{\pi} \leq 3\sum_{j=1}^{3}|t_j| - \sum_{j\neq l'}|u_1-\Im(\alpha_j)| - \sum_{j\neq l'}|\Im(\alpha_j-\alpha_{l'})| \leq 0,$$

by using the triangle inequality on the second sum to get a difference of $\Im(\alpha_j)$. We now pick $B'_{l'} = C_2 - 2m'$ and $B'_j < 0$ for $j \neq l'$ to get

$$\begin{aligned} |\mathcal{M}_{m',l'}| &\ll y_1^{1+C_1}y_2^{1+C_2}T \iiint\limits_{\substack{|t_1|,|t_2|<T^{1+\delta}\\ |u_1|<10T^{1+\delta}}} \mathcal{P}du_1dt_1dt_2 \\ &\ll y_1^{1+C_1}y_2^{1+C_2}T^{(3R+11-C_1-C_2)/2+\epsilon}. \end{aligned} \tag{4.8}$$

To bound the residues $\mathcal{R}_{m,l,m',l'}$ we again shift variables ν_1 and ν_2 to contours (B_1) and (B_2), respectively, where $|B_1|, |B_2| < R/3$. To simplify notation, we will assume without loss of generality that $l = 1$ and $l' = 2$. We obtain

$$|\mathcal{R}_{m,1,m',2}| \ll y_1^{1-B'_1+2m}y_2^{1+B'_2+2m'}T \iint\limits_{|t_1|,|t_2|<T^{1+\delta}} \mathcal{P}\cdot\exp(\mathcal{E})\,dt_1dt_2,$$

where the exponential factor $\mathcal{E}$ is given by

$$\frac{4\mathcal{E}}{\pi} = 3\sum_{j=1}^{3}|t_j| - \sum_{j\neq 1}|\Im(\alpha_j-\alpha_1)| - \sum_{j\neq 2}|\Im(\alpha_j-\alpha_2)| + |\Im(\alpha_1-\alpha_2)| = 0,$$

and the polynomial term is given by

$$\mathcal{P} = \left(\prod_{j=1}^{3} (1+|t_j|) \right)^{(R+2)/2} (1+|t_1|)^{(3B_1-1)/2}(1+|t_2|)^{(-3B_2-2m'-1)/2}(1+|t_3|)^{(3B_3-2m-1)/2},$$

where $B_3 = B_1 + B_2$. Then pick $B_1' = -C_1 + 2m$ and $B_2' = C_2 - 2m'$ to get

$$\begin{aligned} |\mathcal{R}_{m,1,m',2}| &\ll y_1^{1+C_1} y_2^{1+C_2}\, T \iint\limits_{|t_1|,|t_2|<T^{1+\delta}} \mathcal{P} dt_1 dt_2 \\ &\ll y_1^{1+C_1} y_2^{1+C_2}\, T^{(3R+9-2C_1-2C_2+2m+2m')/2+\epsilon} \\ &\ll y_1^{1+C_1} y_2^{1+C_2}\, T^{(3R+9-C_1-C_2)/2+\epsilon}. \end{aligned} \tag{4.9}$$

Combining all the bounds we get the desired result.

□

Remark 4.2.2. *It follows from this proposition that $F_{T,R,\kappa}$ satisfies the decay condition 3.1.*

4.3 Kloosterman Terms' Bounds

We start by getting bounds for the Kloosterman integrals $\tilde{\mathcal{J}}_{\varepsilon,F}$ and $\mathcal{J}_{\varepsilon_1,\varepsilon_2,F}$. Break the integral $\tilde{\mathcal{J}}_{\varepsilon,F} = \tilde{\mathcal{J}}_1 + \tilde{\mathcal{J}}_2 + \tilde{\mathcal{J}}_3 + \tilde{\mathcal{J}}_4$ depending on whether y_1 and y_2 are smaller or greater than 1. To estimate $\tilde{\mathcal{J}}_1$, start by taking absolute values inside the integral

$$\begin{aligned} |\tilde{\mathcal{J}}_1| \ll A^{-2} \int_0^1 \int_0^1 \iint\limits_{\mathbb{R}^2} &|F_{T,R,\kappa}(Ay_1, y_2)| \\ &\times \left| F_{T,R,\kappa}\left(y_2 \frac{\sqrt{1+x_1^2+x_2^2}}{1+x_1^2}, \frac{A}{y_1 y_2} \frac{\sqrt{1+x_1^2}}{1+x_1^2+x_2^2} \right) \right| \frac{dx_1 dx_2 dy_1 dy_2}{y_1 y_2^2}. \end{aligned}$$

Use (4.5) with $C_1 = C_2 = 6 - \epsilon/2$ to bound the first instance of $F_{T,R,\kappa}$ and $C_1 = \epsilon$, $C_2 = 6 - \epsilon$ to bound the second instance of $F_{T,R,\kappa}$ in order to get

$$|\tilde{\mathcal{J}}_1| \ll A^{12-3\epsilon/2}\, T^{3R+2+3\epsilon/2} \int_0^1 \int_0^1 \iint\limits_{\mathbb{R}^2} \frac{y_1^{-1+\epsilon/2} y_2^{-1+3\epsilon/2} (1+x_1^2)^{3-3\epsilon/2}}{(1+x_1^2+x_2^2)^{6-3\epsilon/2}} dx_1 dx_2 dy_1 dy_2.$$

It is clear that the integral in y converges and the integral in x also converges for small values of ϵ. After a suitable redefinition of ϵ we get

$$|\tilde{\mathcal{J}}_1| \ll_{R,\epsilon} A^{12-\epsilon} T^{3R+2+\epsilon}.$$

For the remaining three integrals the argument is almost identical. The only necessary changes are to pick $C_1 = 6 - 3\epsilon/2$ when $y_1 > 1$ and $C_2 = 6 - 5\epsilon/2$ when $y_2 > 1$, while bounding the first instance of $F_{T,R,\kappa}$. Putting all the bounds together, we obtain

$$|\tilde{\mathcal{J}}_{\varepsilon,F}(A)| \ll_{R,\epsilon} A^{12-\epsilon} T^{3R+2+\epsilon}. \tag{4.10}$$

To bound $\mathcal{J}_{\varepsilon_1,\varepsilon_2,F}$, we also break it into $\mathcal{J}_{\varepsilon_1,\varepsilon_2,F} = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4$ depending on whether y_i is smaller or greater than 1. We will first bound $\mathcal{J}_1$ by taking absolute values and doing the change of variables that sends x_3 to $x_3 + x_1x_2$. We have

$$\begin{aligned} |\mathcal{J}_1| \ll (A_1A_2)^{-2} \int_0^1 \int_0^1 \iiint_{\mathbb{R}^3} &|F_{T,R,\kappa}(A_1y_1, A_2y_2)| \\ &\times \left| F_{T,R,\kappa}\left(\frac{A_2}{y_2} \frac{\sqrt{x_3^2+x_1^2+1}}{1+x_2^2+x_3^2}, \frac{A_1}{y_1} \frac{\sqrt{1+x_2^2+x_3^2}}{x_3^2+x_1^2+1} \right) \right| \frac{dx_1dx_2dx_3dy_1dy_2}{y_1y_2}. \end{aligned}$$

Then apply (4.5) with $C_1 = C_2 = 6 - \epsilon$ to bound the first instance of $F_{T,R,\kappa}$ and $C_1 = C_2 = 6 - 2\epsilon$ to bound the second instance of $F_{T,R,\kappa}$ in order to get

$$|\mathcal{J}_1| \ll (A_1A_2)^{10-3\epsilon} T^{3R-1+2\epsilon} \int_0^1 \int_0^1 \iiint_{\mathbb{R}^3} \frac{(y_1y_2)^{-1+\epsilon} dx_1dx_2dx_3dy_1dy_2}{\left((1+x_2^2+x_3^2)(1+x_1^2+x_3^2)\right)^{3-\epsilon}}.$$

As the integral in y is convergent and the x integral is also convergent for small values of ϵ then, after redefining ϵ

$$|\mathcal{J}_1| \ll_{R,\epsilon} (A_1A_2)^{10-\epsilon} T^{3R-1+\epsilon}.$$

For the remaining three integrals the same argument works by choosing $C_1 = 6 - \epsilon/2$ when $y_1 > 1$ and $C_2 = 6 - \epsilon/2$ when $y_2 > 1$, while bounding the second instance of $F_{T,R,\kappa}$. Combining all four bounds, one obtains

$$|\mathcal{J}_{\varepsilon_1,\varepsilon_2,F}(A_1,A_2)| \ll_{R,\epsilon} (A_1A_2)^{10-\epsilon}\, T^{3R-1+\epsilon}. \tag{4.11}$$

We may now bound the Kloosterman terms Σ_{2a}, Σ_{2b} and Σ_3. The only necessary bounds for the Kloosterman sums will be

$$\tilde{S}(m_1,n_1,n_2,d_1,d_2) \ll_\epsilon (d_1d_2)^{1+\epsilon},$$

$$S(m_1,m_2,n_1,n_2,d_1,d_2) \ll_\epsilon (m_1m_2n_1n_2)^{1/2}(d_1d_2)^{1+\epsilon}.$$

which follow from Proposition 2.6.6 and Proposition 2.6.7, respectively. To bound Σ_{2a} we use (4.10) together with the first bound for Kloosterman sums, to obtain

$$\begin{aligned}
|\Sigma_{2a}| &\ll_{R,\kappa,\epsilon} \sum_{\varepsilon=\pm 1} \sum_{\substack{d_1|d_2 \\ m_2d_1^2=n_1d_2}} \frac{|\tilde{S}(\varepsilon m_1,n_1,n_2,d_1,d_2)|}{d_1d_2} \left|\tilde{\mathcal{J}}_{\varepsilon,F}\left(\sqrt{\frac{m_1n_1n_2}{d_1d_2}}\right)\right| \\
&\ll T^{3R+2+\epsilon} \sum_{d_1,d_2} \frac{(m_1n_1n_2)^6}{(d_1d_2)^{6-3\epsilon/2}} \\
&\ll (m_1n_1n_2)^6\, T^{3R+2+\epsilon}.
\end{aligned}$$

The bound for Σ_{2b} is obtained in the same manner. In order to bound Σ_3 we use (4.11) together with the second bound for Kloosterman sums, to obtain

$$\begin{aligned}
|\Sigma_3| &\ll \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \sum_{d_1,d_2} \frac{|S(\varepsilon_1 m_1,\varepsilon_2 m_2,n_1,n_2,d_1,d_2)|}{d_1d_2} \left|\mathcal{J}_{\varepsilon_1,\varepsilon_2,F}\left(\frac{\sqrt{m_1n_2d_1}}{d_2}, \frac{\sqrt{m_2n_1d_2}}{d_1}\right)\right| \\
&\ll (m_1m_2n_1n_2)^{11/2} T^{3R-1+\epsilon} \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \sum_{d_1,d_2} \frac{1}{(d_1d_2)^{5-3\epsilon/2}} \\
&\ll (m_1m_2n_1n_2)^{11/2}\, T^{3R-1+\epsilon}.
\end{aligned}$$

For future reference, we write down these bounds in the following proposition.

Proposition 4.3.1. *Fix $R > 50$, $T \gg 1$ and $\epsilon > 0$. We have the following bounds for the Kloosterman terms:*

$$|\Sigma_{2a}| \ll_{R,\epsilon} (m_1 n_1 n_2)^6 \, T^{3R+2+\epsilon};$$
$$|\Sigma_{2b}| \ll_{R,\epsilon} (m_2 n_1 n_2)^6 \, T^{3R+2+\epsilon};$$
$$|\Sigma_3| \ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^{11/2} \, T^{3R-1+\epsilon}.$$

4.4 Eisenstein Terms' Bounds

We start by obtaining a bound for the contribution to the Kuznetsov trace formula of the minimal Eisenstein series $\mathcal{E}_{min}$. We will require the de la Vallée Poussin bound for the Riemann zeta function,

$$|\zeta(1+it)| \gg \frac{1}{\log(2+|t|)}.$$

Using the bounds for the Fourier coefficients in Proposition 2.5.5, the de la Vallèe Poussin bound and Stirling's formula for the Gamma factors, we get

$$\begin{aligned}
|\mathcal{E}_{min}| &\ll \int\limits_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} \int\limits_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} (m_1 m_2 n_1 n_2)^{\epsilon} \frac{\prod\limits_{k=1}^{3} \left| \Gamma\left(\frac{2+R+3\nu_k}{4}\right) \Gamma\left(\frac{2+R-3\nu_k}{4}\right) \right|^2}{\prod\limits_{k=1}^{3} \left| \zeta\left(1+3\nu_k\right) \Gamma\left(\frac{1+3\nu_k}{2}\right) \right|^2} |d\nu_1 d\nu_2| \\
&\ll (m_1 m_2 n_1 n_2)^{\epsilon} \int\limits_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} \int\limits_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} \prod_{k=1}^{3} \left((1+|\nu_k|)^R \log(2+|\nu_k|)^2 \right) |d\nu_1 d\nu_2| \\
&\ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^{\epsilon} \, T^{3R+2+\epsilon}.
\end{aligned}$$

To bound the maximal Eisenstein series contribution $\mathcal{E}_{max}$ we require the following lower bounds for L-functions

$$L(1, \mathrm{Ad}\, u_j) \gg_{\epsilon} (1+|r_j|)^{-\epsilon}, \qquad |L(1+3\nu, u_j)| \gg_{\epsilon} (1+|\nu|+|r_j|)^{-\epsilon},$$

where the eigenvalue of u_j is $1/4 + r_j^2$. These lower bounds can be found in [Hoffstein and Lockhart, 1994] and [Hoffstein and Ramakrishnan, 1995]. It follows from the bounds above and Proposition 2.5.5 that

$$|\mathcal{E}_{max}| \ll \sum_{r_j < T^{1+\epsilon}} \int_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} (m_1 m_2 n_1 n_2)^{1/2+\epsilon}$$
$$\times \frac{\left|\Gamma\left(\frac{2+R+3\nu+ir_j}{4}\right)\right|^8 \left|\Gamma\left(\frac{2+R+2ir_j}{4}\right)\right|^4 (1+|r_j|)^{\epsilon}(1+|\nu|+|r_j|)^{2\epsilon}}{\left|\Gamma\left(\frac{1+3\nu-ir_j}{2}\right)\Gamma\left(\frac{1+2ir_j}{2}\right)\Gamma\left(\frac{1+3\nu+ir_j}{2}\right)\right|^2}|d\nu|$$
$$\ll (m_1 m_2 n_1 n_2)^{1/2+\epsilon} \sum_{r_j < T^{1+\epsilon}} \int_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} (1+|r_j|)^{R+\epsilon}(1+|\nu|+|r_j|)^{2R+2\epsilon}|d\nu|$$
$$\ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^{1/2+\epsilon}\, T^{3R+3+\epsilon}.$$

For the last inequality, we use Weyl's law for $GL(2)$ Maass forms which tells us that

$$\left|\{\phi_j : r_j < T\}\right| \sim cT^2$$

for some constant $c > 0$. In summary, we obtain the following proposition.

Proposition 4.4.1. *Fix $R > 50$, $T \gg 1$ and $\epsilon > 0$. We have the following bounds for the Eisenstein terms in the Kuznetsov trace formula:*

$$|\mathcal{E}_{min}| \ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^{\epsilon}\, T^{3R+2+\epsilon}; \qquad |\mathcal{E}_{max}| \ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^{1/2+\epsilon}\, T^{3R+3+\epsilon}.$$

4.5 Main Geometric Term Computation

To estimate the main term on the geometric side, $\Sigma_1 = \langle F_{T,R,\kappa}, F_{T,R,\kappa}\rangle$, we use the Parseval-type identity (4.4) which says that $\Sigma_1 = \left\langle F^{\#}_{T,R,\kappa}, F^{\#}_{T,R,\kappa}\right\rangle$. Hence

$$\left\langle F^{\#}_{T,R,\kappa}, F^{\#}_{T,R,\kappa}\right\rangle = \frac{1}{(\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{\left|F^{\#}_{T,R,\kappa}(\nu)\right|^2 d\nu_1 d\nu_2}{\prod_{j=1}^{3}\Gamma\left(\frac{3\nu_j}{2}\right)\Gamma\left(\frac{-3\nu_j}{2}\right)}$$
$$= \frac{27^{R+1}\pi}{64^R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{3} T^{-(\beta_j/R)^{\kappa}}\right)^2 \exp\left(-2\sum_{j=1}^{3}\beta_j^2/T^2\right)$$
$$\times \prod_{j=1}^{3}\left(|t_j|^{R+1} + O(|t_j|^R + 1)\right) dt_1 dt_2,$$

where $\nu_j = it_j$ and $\alpha_j = i\beta_j$. The second equality follows from Stirling's approximation of the Gamma factors. Making a linear change of variables of integration from t_1, t_2 to $\beta_1 = t_3 - t_1 = 2t_1 + t_2, \beta_2 = t_2 - t_1$ and using the symmetry of the integral in the β_i, one gets

$$\begin{aligned} &\frac{9^{R+1}4\pi}{64^R} \int_0^\infty \int_0^{\beta_2} \left(\sum_{j=1}^3 T^{-(\beta_j/R)^\kappa} \right)^2 \exp\left(-2 \sum_{j=1}^3 \beta_j^2/T^2 \right) \\ &\times \prod_{j \neq k} \left(|\beta_j - \beta_k|^{R+1} + O(|\beta_j - \beta_k|^R + 1) \right) d\beta_1 d\beta_2 \\ &= \frac{9^{R+1}8\pi}{32^R} \int_0^\infty \int_0^{\beta_2} \left(\sum_{j=1}^3 T^{-(\beta_j/R)^\kappa} \right)^2 \exp\left(-2 \sum_{j=1}^3 \beta_j^2/T^2 \right) \\ &\times \left(\beta_2^{3R+3} + O(\beta_2^{3R+2}\beta_1 + \beta_2^{3R+2} + 1) \right) d\beta_1 d\beta_2. \end{aligned}$$

We now multiply out the integrand to get

$$\left\langle F^{\#}_{T,R,\kappa}, F^{\#}_{T,R,\kappa} \right\rangle = M + Error_1 + Error_2,$$

where

$$M = \frac{9^{R+1}8\pi}{32^R} \int_0^\infty \int_0^{\beta_2} \exp\left(-2\log T(\beta_1/R)^\kappa - 2 \sum_{j=1}^3 \beta_j^2/T^2 \right) \beta_2^{3R+3} d\beta_1 d\beta_2,$$

$Error_1$ includes all the terms not involving $T^{-(\beta_1/R)^\kappa}$ and $Error_2$ the remaining terms involving $\beta_2^{3R+2}\beta_1 + \beta_2^{3R+2} + 1$. We can bound the error terms by

$$Error_1 \ll \int_0^\infty \int_0^{\beta_2} T^{-(\beta_2/R)^\kappa} p(\beta_1, \beta_2) d\beta_1 d\beta_2 \ll \int_0^\infty T^{-(\beta_2/R)^\kappa} \tilde{p}(\beta_2) d\beta_2 \ll 1,$$

where $p, \tilde{p}$ are polynomials, and

$$\begin{aligned} Error_2 &\ll \int_0^\infty \int_0^{\beta_2} T^{-(\beta_1/R)^\kappa} \exp\left(-\sum_{j=1}^3 \beta_2^2/T^2 \right) (\beta_2^{3R+2}\beta_1 + \beta_2^{3R+2} + 1) d\beta_1 d\beta_2 \\ &\ll \int_0^\infty \exp\left(-\beta_2^2/T^2\right) (\beta_2^{3R+3} + \beta_2) d\beta_2 \\ &\ll T^{3R+3}. \end{aligned}$$

We then write

$$M = \frac{9^{R+1} 8\pi}{32^R} \int_0^\infty \int_0^{\beta_2} \exp\left(-c_1^2 \beta_1^\kappa - c_2^2(\beta_1^2 + \beta_1\beta_2 + \beta_2^2)\right) \beta_2^{3R+3} d\beta_1 d\beta_2,$$

where $c_1^2 = 2\log T/R^\kappa, c_2^2 = 4/T^2$.

Note that this integral is going to have exponential decay outside the region where $\beta_1 \ll c_1^{-2/\kappa}, \beta_2 \ll c_2^{-1}$. Therefore,

$$\begin{aligned}
M &= \frac{9^{R+1} 8\pi}{32^R} \int_0^\infty \int_0^{\beta_2} \exp\left(-c_1^2 \beta_1^\kappa - c_2^2 \beta_2^2\right) \beta_2^{3R+3} \Bigg(1 + O\left(c_2^2(\beta_1^2 + \beta_1\beta_2)\right)\Bigg) d\beta_1 d\beta_2 \\
&= \frac{9^{R+1} 8\pi}{32^R} \int_0^\infty \int_0^{\beta_2} \exp\left(-c_1^2 \beta_1^\kappa - c_2^2 \beta_2^2\right) \beta_2^{3R+3} d\beta_1 d\beta_2 + O\left(c_2^{-(3R+2)} c_1^{-6/\kappa} + c_2^{-(3R+3)} c_1^{-4/\kappa}\right) \\
&= \frac{9^{R+1} 8\pi}{32^R} \int_0^\infty \int_0^\infty \exp\left(-c_1^2 \beta_1^\kappa - c_2^2 \beta_2^2\right) \beta_2^{3R+3} d\beta_1 + O\left(\exp\left(-c_1^2 \beta_2^\kappa\right) \beta_2^{3R+3}\right) d\beta_2 + O\left(T^{3R+3}\right) \\
&= \frac{9^{R+1} 8\pi}{32^R} c_1^{-2/\kappa} c_2^{-(3R+4)} \int_0^\infty \int_0^\infty \exp\left(-\beta_1^\kappa - \beta_2^2\right) d\beta_1 d\beta_2 + O(T^{3R+3}) \\
&= \frac{9^{R+1} R \pi^{3/2} C_\kappa}{2^{8R+2+1/\kappa}} \frac{T^{3R+4}}{(\log T)^{1/\kappa}} + +O(T^{3R+3}),
\end{aligned}$$

where $C_\kappa = \int_0^\infty \exp\left(-x^\kappa\right) dx$.

Combining the bounds for the Kloosterman terms in Proposition 4.3.1, the bounds for the Eisenstein terms in Proposition 4.4.1 and the computation above we obtain Theorem 4.1.1.

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