Constant Scalar Curvature of Toric Fibrations

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Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2014

ABSTRACT

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We study the conditions under which a fibration of toric varieties, fibered over a flag variety, admits a constant scalar curvature Kähler metric. We first provide an introduction to toric varieties and toric fibrations and derive the scalar curvature equation. Next we derive interior a priori estimates of all orders and a global L^∞-estimate for the scalar curvature equation. Finally we extend the theory of K-Stability to this setting and construct test-configurations for these spaces.

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Acknowledgments

First I would like to thank my advisor Professor Duong H. Phong. Professor Phong knew exactly when I needed to be pushed, but was patient and caring whenever I grew frustrated. I continually learn from his thoughtfulness, wisdom, and strong convictions and will never forget all the help and support he has provided me during my time at Columbia.

Next I would like to thank all my fellow students and colleagues. I would especially like to thank Professor Melissa Liu. I cannot overstate how kind and helpful she has been over the years. I would also like to thank Professors Jacob Sturm, Jian Song, Mu-Tao Wang, Xiangwen Zhang, Valentino Tosatti, and Gábor Székelyhidi for all their help and friendliness. I cannot do justice to my fellow students here, but I would especially like to thank Tristan Collins, Adam Jacob, Daniel Rubin, Ye-Kai Wang, and Connor Mooney for their mathematical help as well as Heather Macbeth for helping revise my thesis manuscript. In addition to those already named, I would like to thank Jillian Carson, Karol Kozial, Zachary Maddock, Andre Carneiro, Alice Rizzardo, João Guerreiro, Vivek Pal and many others for all the good times over the years. Finally I would like to thank Bianca Santoro for all her smiles and for a feijoada that warms the soul.

I would especially like to thank both Terrance Cope and Mary Young who are always working hard behind the scenes to keep the building from falling down.

I would also like to thank all my family and friends. My wonderful parents Lisa and Anders, my brothers Eric and Daniel, and my sisters Linda and Jenelle have all shown me more love and support than I deserve. Tusen tack till Farmor och hela familjen hemma i Sverige and many thanks to Grandma and the rest of the family in California. Kyle, Wes, thanks for keeping it real. Finally I would like to thank my wonderful Chen-Yun for always being there and keeping the important things in life in perspective.

To my family

Chapter 1

Introduction

Complex projective manifolds admit two very rich structures. On one hand, they are Kähler manifolds and can be described in differential geometric terms. On the other hand, by Chow's theorem, they are projective algebraic subvarieties, and can also be described in algebraic-geometric terms. In particular, at the most intrinsic and fundamental level, there should be a complete correspondence between differential-geometric properties, such as whether they admit a constant scalar curvature metric, and algebraic-geometric properties, such as their stability in the sense of geometric invariant theory. This is the essence of the well-known conjecture of Yau [Yau, 1993], formulated first for Kähler-Einstein metrics, and extended since to constant scalar curvature metrics by Donaldson [Donaldson, 2002]. A corresponding notion of stability, called K-stability, has been proposed in different versions by Tian [Tian, 1997] and Donaldson [Donaldson, 2002]. The necessity of K-stability for the existence of a metric of constant scalar curvature has been proved by Donaldson [Donaldson, 2005b] and Stoppa [Stoppa, 2009]. For a survey of other notions of stability, the reader is referred to [Phong and Sturm, 2009].

Fix an integral Kähler class $c_1(L)$, where L is a positive holomorphic line bundle over a compact manifold X, and fix a representative Kähler form $\omega_0 \in c_1(L)$. By the $\partial\bar\partial$ Lemma, a Kähler metric ω is in the class $c_1(L)$ if and only if $\omega = \omega_0 + \frac{i}{2}\partial\bar\partial\varphi$, for some smooth function φ on X. Since the scalar curvature $R(\omega)$ is just

$$R(\omega) = -g^{j\bar k}\partial_j\partial_{\bar k} \log \omega^n, \tag{1.0.1}$$

the equation of constant scalar curvature is the following fourth-order non-linear elliptic equation in the potential φ

$$-g^{j\bar{k}}\partial_j\partial_{\bar{k}} \log\det\left((g_0)_{\bar{p}q} + \partial_q\partial_{\bar{p}}\varphi\right) = A, \tag{1.0.2}$$

where A is a constant. As such, it can be viewed as a composition of a complex Monge-Ampère equation with a linearized complex Monge-Ampère equation. Real analogues of such equations have arisen independently, in different contexts, in works of Trudinger and X.J. Wang [Trudinger and Wang, 2000; 2002], and of Caffarelli and Gutierrez [Caffarelli and Gutiérrez, 1997]. But the complex version required by the constant scalar curvature problem in Kähler geometry is still largely unexplored in its most general form. It is also important to note that the constant scalar curvature problem admits a variational formulation, namely, there exists a functional $\varphi \to K(\varphi)$, called the Mabuchi K-energy, so that the equation (1.0.2) holds if and only if

$$\frac{\delta K}{\delta \varphi} = 0. \tag{1.0.3}$$

This variational formulation also suggests a basic link between the existence of metrics of constant scalar curvature, and the behavior of the functional $K(\varphi)$, as φ tends to the boundary of the space of Kähler potentials in $c_1(L)$.

There is however a particular class of Kähler manifolds in which a lot of progress has been made, namely toric manifolds. A toric manifold X is an n-dimensional compact Kähler manifold admitting a $(\mathbb{C}^*)^n$ action with a dense, fixed point free orbit. As such, it admits a moment map μ which maps it into a convex polytope P in $\mathbb{R}^n$. A key simplifying feature of toric manifolds is that each invariant Kähler metric ω on X can be characterized by its symplectic potential u, which is a smooth strictly convex function on the polytope P, smooth in the interior of P. Guillemin [Guillemin, 1994] has identified the precise conditions under which a smooth strictly convex function u on P arises from a Kähler metric on X which we now explain. Delzant [Delzant, 1988] showed that for each face F of P one can choose a minimal vector l_F, inward-pointing normal to F, such that $\lambda_F \in \mathbb{Z}^n$. Let l_F be the affine linear function whose derivative is λ_F and such that $l_F(F) = \{0\}$. Guillemin showed that u arises from a Kähler metric on if there exists a smooth function f on P such that

$$u = \frac{1}{2} \sum_F l_F \log l_F + f, \tag{1.0.4}$$

such that u is strictly convex on the interior of P as well as when restricted to any of the faces of P. Furthermore, Abreu [Abreu, 1998] has shown how the scalar curvature $R(\omega)$ can be expressed in terms of u,

$$S(u) = -\frac{1}{2}(u^{jk})_{jk}. \tag{1.0.5}$$

Here u^{jk} is the inverse of the Hessian u_{jk} of u and here (as well as throughout this thesis) we use Einstein's summation notation. This means in particular that, for toric manifolds, the constant scalar curvature problem can be transformed into a differential equation for a symplectic potential u on a polytope with the Guillemin boundary conditions. Furthermore, the form (1.0.5) of Abreu's equation also allows a direct use of the works of Trudinger-Wang [Trudinger and Wang, 2000; 2002] and Caffarelli-Gutierrez [Caffarelli and Gutiérrez, 1997] for related real equations.

In a series of papers [Donaldson, 2002; 2005a; 2008a; 2009], Donaldson makes an essential use of the reformulation of the constant scalar curvature problem in terms of symplectic potentials to solve the problem for toric varieties in dimension $\dim|, X = 2$. Namely, he showed that the K-stability of the toric surface X implies the existence of an invariant constant scalar curvature Kähler metric on X. Now, in analogy with the Hilbert-Mumford numerical criterion of geometric invariant theory, the condition of K stability is formulated as positivity of a certain invariant, the Futaki invariant, for all non-trivial test configurations (see [Tian, 1997] and [Donaldson, 2002]). A first task, accomplished in [Donaldson, 2002], is to complete the translation of the problem from Kähler metrics on toric varieties to symplectic potentials on polytopes by showing that the K-energy can also be expressed directly in terms of the symplectic potential u,

$$\mathcal{M}(u) = -\int_P \log\det(u_{jk})d\mu + \mathcal{L}(u),$$

where $\mathcal{L}$ is a certain linear functional. The functional $\mathcal{M}$ is a convex functional whose critical points are solutions to (1.0.5). Similarly, the K-stability conditions also admit a completely equivalent and explicit formulation in terms of u, with test configurations corresponding to piecewise linear functions on the polytope. With the reformulation of both the equation and the solvability conditions now complete, Donaldson can then complete the proof by producing a series of difficult a priori estimates, some building on [Trudinger and Wang, 2000; 2002; Caffarelli and Gutiérrez, 1997], but many others requiring novel arguments making full use of the formulation of K-stability in terms of piecewise linear functions.

Even with this success for two-dimensional toric varieties, the general case seems out of reach at the present time. In [Donaldson, 2008b], Donaldson suggests studying an intermediate class of manifolds, namely those which can be viewed as toric varieties fibered over flag varieties, more precisely fiber products $G \times_T M$, where G is a compact semisimple Lie group and $T \subset G$ is a maximal torus. The related problem of Kähler-Ricci solitons has been considered by Podesta and Spiro [Podestà and Spiro, 2010]. Building off the work of Raza [Raza, 2006], Donaldson noted that the scalar curvature S of a toric metric on such a fibration should be given by

$$S(u) = -\frac{1}{2} W^{-1} (W u^{jk})_{jk} + f_G, \tag{1.0.6}$$

where W is the Duistermaat-Heckman polynomial and f_G is a smooth function given locally by $4 \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (\log W)$. Thus the constant scalar curvature problem for such fibrations reduces to a twisted version of Abreu's equation for toric varieties.

The goal of this work is to extend Donaldson's approach for toric surfaces to the setting of toric fibrations. For this purpose, we found it useful to provide a complete derivation of the expression (1.0.6) for the scalar curvature of such fibrations. The reason is that Raza [Raza, 2006] did not determine the function f_G, while Donaldson [Donaldson, 2008b] just wrote down the formula. Our main results consist of extending to the case of toric fibrations the expression for the K-energy in terms of the symplectic potentials; necessary conditions for the solvability of the equation (1.0.6); how to interpret these conditions as K-stability; and some a priori estimates resulting from K-stability. They are contained in Theorems 1-5 below.

Theorem 1. *Let $(G \times_T M, \Omega)$ be a toric Kähler fibration and let (M, ω) be the restriction to the identity fiber. There exists a unique moment map $\mu : M \to P \subset \mathbb{R}^n$ for ω such that* *the push-forward of the scalar curvature $S(\Omega)$ under μ is given by*

$$\mu_*(S) = -\frac{1}{2}W^{-1}(Wu^{jk})_{jk} + f_G, \tag{1.0.7}$$

where u is the symplectic potential on P corresponding to ω and $f_G = 2\sum_{i=j}^{n} \frac{\partial}{\partial x_j}(\log W)$.

Theorem 2. *Define the convex functional $\mathcal{F}$ on symplectic potentials u by the equation*

$$\mathcal{F}(u) = -\int_P \log\det(u_{jk})W d\mu + \int_{\partial P} uW d\sigma - \int_P (a - f_G)uW d\mu. \tag{1.0.8}$$

Then $\mathcal{F}$ is the Mabuchi functional for symplectic potentials u. Hence a symplectic potential u is a critical point of $\mathcal{F}$ if and only if u corresponds to a G-invariant Kähler metric on $G \times_T M$ of constant scalar curvature.

Theorem 3. *Assume $n = 2$. Let u be a normalized solution to $S(u) = A$ and assume that u satisfies Condition 1. Let $K \subset\subset P$. Then there exist uniform constants C and C_j, for $j = 0, 1, \ldots$ such that $C^{-1} < (u_{jk}) < C$ and $||u||_{C^k} \leq C_k$ on K.*

Theorem 4. *Let $p \in P$ be fixed and assume that u is normalized at p. Furthermore, assume that u solves $S(u) = A$ and that (P, σ, A) satisfies Condition 1. Then there exists a universal constant C, such that $||u||_{L^\infty} \leq C$, where C depends only on on the geometry of $(P, \sigma), ||A||_{L^\infty(P)}, ||W||_{L^\infty(P)}$, and the point $p \in P$.*

Theorem 5. *Let $(G \times_T M, L)$ be a G-invariant polarized pair and assume that $(G \times_T M, \Omega)$ is a toric fibration with $\Omega \in c_1(L)$. Let P be the corresponding polytope. Given any rational piecewise linear function f on P, there exists a test-configuration $\mathcal{X}$ for $G \times_T M$ with Futaki-Invariant F given by*

$$F = -\frac{1}{2\int_P W d\mu}\left(\int_{\partial P} fW d\sigma - a\int_P fW d\mu + \int_P f f_G W d\mu\right) = -\frac{1}{2\int_P W d\mu}\mathcal{L}(f).$$

Theorem 6. *Let $P \subset \mathbb{R}^n$ be an integer polytope and let h be a convex function in $C^2(P)$. Then we have*

$$\sum_{p \in \overline{P} \cap \frac{1}{k}\mathbb{Z}^n} h(p) = \left(\int_P h d\mu\right) k^n + \left(\frac{1}{2}\int_{\partial P} h d\sigma\right) k^{n-1} + o(k^{n-2}). \tag{1.0.9}$$

This thesis is organized as follows. In Chapter 2, we provide an introduction to toric varieties. This introduction is mostly from a metric differentiable viewpoint, and is self-contained. The material there is very classical, but we hope that our presentation will still be useful. In particular, we provided a program written in Python for how to compute the relations in the algebraic construction of a toric variety from a polytope. Chapter 3 explains how to construct toric fibrations from toric varieties. It explains how to extend toric metrics to the fibrations and then explicitly derives (1.0.6). Chapter 4 finds the Mabuchi functionals corresponding to (1.0.6) and gives certain necessary conditions for the solvability of the equation. Chapter 5 uses the restrictions imposed by K-stability to derive a-priori estimates on solutions to (1.0.6). Finally, Chapter 6 explains how to realize these necessary conditions as an extension of Donaldson's K-stability.

Chapter 2

Toric Varieties

Toric varieties lie in the intersection of algebraic, symplectic, and complex geometry and can therefore be studied from many different perspectives. This thesis concerns complex geometry and PDEs and therefore takes the perspective of a smooth compact Kähler manifold as the starting point. For other perspectives, the reader is recommended any number of other great expositions on the subject (c.f. [Fulton, 1993; da Silva, 2001; Audin, 2004]). We take the opportunity to stress that the material in this chapter is not original. If there is any merit, it lies rather in the presentation and style.

The most important result from toric geometry is that to any n-dimensional toric Kähler manifold M with fixed Kähler form, there exists a convex polytope $P \subset \mathbb{R}^n$ and a moment map $\mu : M \to P$. As we will see, much of the geometry of M can be understood in terms of P and the main goal of this chapter is to understand how M and P are related. The major results we present are the two theorems of Delzant, the first being the following:

Theorem (Delzant 1988)**.** *The polytope P corresponding to a smooth, compact Kähler toric variety satisfies the following properties:*

(i) There are n edges meeting at each vertex of P,

(ii) The edges meeting at a vertex p are all of the form $p + tu_i, t \geq 0$, with $u_i \in \mathbb{Z}^n$,

(iii) For each vertex p, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Delzant also considered the converse. The second of his results that we present is the following:

Theorem (Delzant 1988)**.** *Let P be a polytope satisfying properties $(i)-(iii)$ of Theorem 2. Then there exists a smooth, compact Kähler toric variety whose moment polytope is P.*

The chapter is outlined as follows. In the first section, we derive as much structure as possible by only considering the compact holomorphic properties of M. In the second section, we consider Kähler properties of M and show how to construct the moment map and the moment polytope P. Furthermore, we show how to identify Kähler metrics with smooth convex functions on P called *symplectic potentials.* In the third section, we give a proof of Delzant's theorem and study the algebraic properties of the polytope. In the final section, we provide the source code for a convenient program that can be used to compute algebraic relations of toric varieties coming from P.

2.1 Complex Differential Aspects of Toric Varieties

Let (M, ω) be a compact n-dimensional Kähler manifold with Kähler form ω. Let $T_{\mathbb{C}}$ denote an n-dimensional commutative complex Lie group and let $T \subset T_{\mathbb{C}}$ denote a maximal compact real torus.

Definition 2.1.1. *(M, ω) is a compact **Kähler toric variety** if there is a holomorphic group action of $T_{\mathbb{C}}$ on M which has a dense, free orbit, and such that the maximal real torus T preserves the form ω.*

Remark 2.1.2. By choosing bases for the Lie algebras $\mathfrak{t} := \mathrm{Lie}(T)$ and $\mathfrak{t}_{\mathbb{C}} := \mathrm{Lie}(T_{\mathbb{C}})$, we can identify $\mathfrak{t}_{\mathbb{C}}$ with $\mathbb{C}^n$, $\mathfrak{t}$ with $\mathbb{R}^n$ (the imaginary part of $\mathbb{C}^n$), $T_{\mathbb{C}}$ with $(\mathbb{C}^*)^n$, and $(S^1)^n$ with T. We will almost always do this, but there are times when we will need to change bases and hence it is more natural to give Definition 2.1.1 in an invariant fashion.

In this section we explore the geometry of a toric manifold M without yet making use of the Kähler metric. The fixed points of the action of $T_{\mathbb{C}}$ on M will turn out to be very important. The main results are the following three propositions:

Proposition 2.1.3. *Let $q \in M$ be a fixed point. Then there is an open set $X \subset M$, containing q, and a biholomorphism $\chi : X \to T_q(M)$, such that χ is $T_{\mathbb{C}}$-equivariant.*

Proposition 2.1.4. *The set of fixed points of the action of $T_{\mathbb{C}}$ on M is finite.*

Proposition 2.1.5. *Assume that q_1 and q_2 are two fixed points of M. Then for each fixed point q_i, there exists a trivialization $\nu_{q_i} : X_{q_i} \to \mathbb{C}^n$, and coordinates identifying $T_{\mathbb{C}}$ with $(\mathbb{C}^*)^n$ such that the action of $T_{\mathbb{C}}$ on X_q is identified with the standard action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$. Moreover, $U \subset X_{q_i}$, where U is the open, dense complex torus in M, and $\nu_{q_i}(U) = (\mathbb{C}^*)^n$. Finally, if one restricts the transition mapping $\tau_{q_1 q_2} := \nu_{q_1} \circ (\nu_{q_2})^{-1}$ to $(\mathbb{C}^*)^n$, then the mapping takes the form*

$$(z_1, \ldots, z_n) \mapsto (z_1^{a_{11}} \cdots z_n^{a_{1n}}, \ldots, z_n^{a_{n1}} \cdots z_n^{a_{nn}}), \tag{2.1.1}$$

where (a_{ij}) is a matrix in $GL(n, \mathbb{Z})$.

2.1.1 Coordinates on the Open Torus

Definition 2.1.1 requires M to have an open, dense, free orbit of $T_{\mathbb{C}}$. In this section we will construct a natural coordinate system to understand its structure. Following Remark 2.1.2, let us work in a basis. If we choose a point $p \in M$ such that $U = (\mathbb{C}^*)^n \cdot p$ is a dense, free orbit, then we have a holomorphic embedding $(\mathbb{C}^*)^n \hookrightarrow M$, $(z^1, \ldots, z^n) \mapsto (z^1, \ldots, z^n) \cdot p$, whose image U is dense in M. If we identify U with $(\mathbb{C}^*)^n$ by this embedding, then the action of $(\mathbb{C}^*)^n$ on M restricts to the standard multiplicative action of $(\mathbb{C}^*)^n$ on itself. Let us use this to understand the geometry of the complement of U in M.

Lemma 2.1.6. *The set $D := M \setminus U$ is an analytic subvariety of M. Consequently, around any point $p \in D$, there is a ball B in M containing p, such that $B \cap U$ is connected.*

Proof. Let V_k be the holomorphic vector field on M generated by the action of the one-dimensional complex subgroup $\mathbb{C}^*$, given by $\mathbb{C}^* \ni z \mapsto (1, \ldots, z, \ldots, 1) \in (\mathbb{C}^*)^n$ (where z is mapped to the k^{th} coordinate). In local coordinates, V_k is given as $z^k \frac{\partial}{\partial z^k}$. Let $s \in \Gamma(M, K_M^{-1})$ be the section given by

$$s := z^1 \frac{\partial}{\partial z^1} \wedge \cdots \wedge z^n \frac{\partial}{\partial z^n}.$$

We see that s is everywhere non-zero on U. If there were some point q in D where s did not vanish, there would necessarily be a ball contained entirely in D containing q where s also did not vanish. But this would contradict the requirement in Definition 2.1.1 that U be dense. Therefore we have concluded that $s^{-1}(0) = D$ and hence that D is an analytic subvariety. □

The lemma tells us that our embedding $(\mathbb{C}^*)^n \hookrightarrow M$ not only captures the group structure, but also covers all but a complex subvariety of M. Unfortunately, our embedding's choice of a point q to realize the free, dense orbit of $(\mathbb{C}^*)^n$ is a little ad-hoc. This construction quickly allows one get an explicit handle on the geometry of M, but there are downsides to such an approach. By its very construction, this open set contains no fixed points of the action of $T_{\mathbb{C}}$ and it is unclear if and how they relate to this set. Though we do not yet know it, the action actually has fixed points. Therefore we would like to have an open trivialization that both respects the action of $(\mathbb{C}^*)^n$ and contains a fixed point, supposing such a point exists.

2.1.2 Local Geometry of M Near a Fixed Point

In this section we prove Propositions 2.1.3 and 2.1.4. We will assume that $q \in M$ is a fixed point of the action of $T_{\mathbb{C}}$ and try to find a nice local trivialization of M which contains q and respects the action of $T_{\mathbb{C}}$. It will be easier to step back and once again work in an invariant fashion, and therefore we will write $T_{\mathbb{C}}$ instead of $(\mathbb{C}^*)^n$. In order to understand the action near a fixed point, we will pick an arbitrary trivialization and "linearize" it. This linearization argument can be found in [Ishida and Karshon, 2012], but we reproduce it for completeness below.

Proof of Proposition 2.1.3. First, let $\eta : U \to \mathbb{C}^n$ be a holomorphic trivialization mapping q to 0. The domain of η has an action of $T_{\mathbb{C}}$, but the range does not. We rectify this by considering the derivative of η at q given by $d\eta_q : T_q(M) \to \mathbb{C}^n$, where we identify $T_0(\mathbb{C}^n)$ with $\mathbb{C}^n$. Since $d\eta_q$ is invertible, we define a new map $\nu : U \to T_q(M)$ by $\nu := (d\eta_q)^{-1} \circ \eta$. The important properties of ν are that both its domain and range have an action of $T_{\mathbb{C}}$, that $d\nu_q : T_q(M) \to T_q(M)$ is the identity mapping, that $\nu(q) = 0$, that q and 0 are fixed points of their respective actions, and finally that ν is holomorphic.

The action of $T_{\mathbb{C}}$ on the domain and range of ν need not be related in any way. We would like to fix this by somehow averaging ν over $T_{\mathbb{C}}$. This, however, would require for U to be closed under $T_{\mathbb{C}}$. Considering that $T_{\mathbb{C}}$ has a dense, open orbit in M, it is unlikely that U is closed under its action. Furthermore, since $T_{\mathbb{C}}$ is not compact, any averaging procedure would be fraught with difficulties. However, the maximal torus $T \subset T_{\mathbb{C}}$ *is* compact and will be enough for our purposes. Employing this idea, we define a new set V by

$$V := \bigcap_{g \in T} g(U).$$

Since q is a fixed point, we certainly know that $q \in V$. Let U^c be the complement of U in M and note that it is closed and hence compact. This means that $T \times U^c$ is a compact subset of $T \times M$. Since the group action is continuous, the image of $T \times U^c$ under the action is compact as well. But this image is exactly V^c and hence V is open.

If we now restrict the map ν to V, we have a holomorphic map whose domain and range are both preserved by the action of $T \subset T_{\mathbb{C}}$. This allows us to integrate ν over this action, to get a T-equivariant holomorphism. More specifically, let T act on maps like ν by

$$(g \cdot \nu)(x) = g^{-1} \cdot \nu(g \cdot x),$$

for $g \in T$. Next let dg be a left-invariant measure on T. Since the range of ν, $T_q(M)$, is a vector space, it has a linear structure which we can use to integrate. Define the map χ by

$$\chi(x) = \int_{g \in T} g^{-1} \cdot \nu(g \cdot x) dg.$$

By construction, χ satisfies the property that $\chi(g \cdot x) = g \cdot \chi(x)$, for all $x \in V$. Furthermore, since $d\nu_q$, is the identity map, $d(g \cdot \nu)_q$ is the identity for all $g \in T$. This implies that $d\chi_q$ is the identity map. As a result, by restricting V to a smaller T-invariant neighborhood if necessary, we have that $\chi : V \to T_q(M)$ is a T-equivariant holomorphic embedding.

Now that we have that χ is T-equivariant, we'd like to take this one step further. Since χ is holomorphic, we have that $d\chi \circ J = J \circ d\chi$, where J is the complex structure on $T_{\mathbb{C}}$ and $T_q(M)$ respectively. Let $V \in J(\mathfrak{t})$ and let $g_t = e^{tV} \cdot x$ be the infinitessimal action of V at x. Let $W = -J(V)$ and let $h_t = e^{tW}$ be its corresponding infinitessimal action. Then the T-equivariance and holomorphicity of χ tell us that

$$\frac{d}{dt}(\chi(g_t)) = J\frac{d}{dt}(\chi(h_t)) = J\frac{d}{dt}(e^{tW}\cdot(\chi(x))) = \frac{d}{dt}(e^{tV}\cdot\chi(x)).$$

Hence χ infinitesimally respects the action of all of $T_{\mathbb{C}}$. Since $T_{\mathbb{C}}$ is simply-connected, this tells us that $\chi(g\cdot x) = g\cdot\chi(x)$ for all x such that $g\cdot x \in V$.

The final step is to extend the action of χ beyond V. Let $X = T_{\mathbb{C}}\cdot V$—i.e. all points in M that are reachable by the action of $T_{\mathbb{C}}$ on elements of V. Extend the map χ to all of X by $T_{\mathbb{C}}$-equivariance. The commutativity of $T_{\mathbb{C}}$ guarantees that this is a well-defined map. If the mapping were not injective, the original mapping on V would not be either. Finally the $T_{\mathbb{C}}$-equivariance gives us that this mapping is still a holomorphism. Since $T_{\mathbb{C}}$ acts linearly, and the image contains an open set around the origin, the mapping must be surjective and is hence a biholomorphism. In conclusion, we have proved Proposition 2.1.3. □

2.1.3 An Example

A corollary of the above argument is the following: Any holomorphic map from $\mathbb{C}^n \to \mathbb{C}^n$ that fixes the identity and is invariant under the action of $(S^1)^n$ must be a linear map. This may be a bit surprising so it is good to consider an example. Let $f(z) = z + z^2$. Then we have that

$$\int_0^{2\pi} e^{-i\theta}\cdot f(e^{i\theta}\cdot z)dt = \int_0^{2\pi} e^{-i\theta}(e^{i\theta}z + e^{2i\theta}z^2)dt = \int_0^{2\pi}(z + e^{i\theta}z^2) = 2\pi z.$$

The case of multiple variables is similar. In fact, one can see that this result even applies to meromorphic functions as well.

The construction of the previous proposition allows to deduce more properties of the fixed points.

Lemma 2.1.7. *The set X of the previous proposition contains no fixed point other than q.*

Proof. Consider the possibility that $T_q(M)$ has a fixed point other than 0. In that case, by linearity, the action would fix an entire line $L \subset T_q(M)$. Since $T \subset T_{\mathbb{C}}$ is compact, we can pick a Hermitian metric on $T_q(M)$ which is invariant under T. Let $L^{\perp}$ be the perpendicular set to L with respect to this metric. Since the metric is T-invariant, we have that T fixes

$L^\perp$. Since $L^\perp$ is a complex subspace of $T_q(M)$ and is preserved by T, it must be preserved by all $T_\mathbb{C}$ for the reasons as in the proof of the previous proposition.

Hence we have that $L \oplus L^\perp = T_q(M)$ and that the action of $T_\mathbb{C}$ respects this decomposition. Next let $x \in T_q(M)$ and let $x = x_1 + x_2$ with $x_1 \in L$ and $x_2 \in L^\perp$. Then we have that for any $\alpha \in T_\mathbb{C}$, $\alpha \cdot x = \alpha \cdot (x_1 + x_2) = x_1 + \alpha \cdot x_2$. This means that $\alpha(x_1 + L^\perp) \subset x_1 + \alpha(L^\perp) = x_1 + L^\perp$ which implies that the orbit of any point in $T_q(M)$ is at least of codimension 1. This, however, contradicts the fact that M has a free, dense orbit of $T_\mathbb{C}$. □

Proof of Proposition 2.1.4. Since the set X is open, the previous lemma implies immediately that the set of fixed points of M is discrete. Since M is compact, we have that the set of fixed points must be finite. □

Proof of Proposition 2.1.5. Much of this proposition was contained in Proposition 2.1.3. The coordinate description follows directly from general Lie theory. The fact that the sets X_q cover all M is a result of the polytope description in Lemma 2.2.8. The fact that $U \subset X_q$ for all q is due to the fact that X_q is open and that it is closed under $T_\mathbb{C}$. The image of U in $\mathbb{C}^n$ must be $(\mathbb{C}^*)^n$ given our local description of the action. Finally the mapping $\tau_{qq'} : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is a biholomorphic automorphism of $(\mathbb{C}^*)^n$ and hence must be of the form (2.1.1) for some matrix $(a_{ij}) \in GL(n, \mathbb{Z})$ by general theory. □

This entire section is predicated on the assumption that there exists a fixed point in M. All of our conclusions are the result of purely topological and holomorphic properties of $T_\mathbb{C}$ and M. In the next section, we will make use of Kähler properties to prove that fixed points do exist and we demonstrate their importance to the overall structure of M.

2.2 Metric Properties of Toric Varieties

In the previous section, we studied toric varieties without considering any extra conditions that we get from the metric. We now study the properties of the Kähler form of our toric variety. We will employ a tool from symplectic geometry called a *moment map* (see

Definition 2.2.4) to better understand the symmetries of the metric. Our first important result is the following:

Proposition 2.2.1. *Let (M, ω) be a toric variety. Then there is a moment map μ of ω. The image of this moment map is a convex polytope P in $\mathbb{R}^n$. Furthermore, except for an additive constant, the polytope is uniquely defined by the Kähler class $[\omega]$.*

By using the moment map, we are finish up the work in the previous section involving fixed points:

Proposition 2.2.2. *The action of $T_{\mathbb{C}}$ on M has a non-empty finite set of fixed points. The moment map sends the fixed points bijectively to the extreme points of the moment polytope.*

The moment map allows us to understand the geometry of V in terms of data on P. The most important idea will be use the *Legendre transform* to associate to ω a smooth convex map on P called the *symplectic potential* (see Definitions 2.2.9 and 2.2.11). The most important result is the following which tells us the form of such potentials:

Proposition 2.2.3. *Let (M, ω) be a toric, Kähler manifold and let P be its polytope. Then the space of symplectic potentials corresponding to metrics in the class $[\omega]$ is given by the set of functions $u_P + f$, where u_P is a fixed function defined by (2.2.10) and f is a function smooth function on P such that the derivatives of f are continuous up to the boundary.*

2.2.1 Constructing the Moment Map

Consider the open, dense torus U in M and choose coordinates identifying U with $(\mathbb{C}^*)^n$ and $T_{\mathbb{C}}$ with $(\mathbb{C}^*)^n$ so that the action is standard. (See the construction in the beginning of the previous section.) As a reminder, since we do not yet know that M has any fixed points, we cannot extend this trivialization U to a set isomorphic to $\mathbb{C}^n$ as done in the latter part of the previous section.

Let $z = (z_1, \dots, z_n)$ be the coordinates on $(\mathbb{C}^*)^n$. Working in these local coordinates, we have that ω is a Kähler form on $(\mathbb{C}^*)^n$ which is invariant under the compact action of $(S^1)^n \subset (\mathbb{C}^*)^n$. Then ω takes the form $\omega = \omega_{\overline{k}j} i dz^j \wedge d\overline{z}^k$. To simplify the action, we instead use log-coordinates on M. Define the holomorphic covering map $\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ by

$$\exp(y^1, \dots, y^n) = (e^{y^1}, \dots, e^{y^n}) = (z^1, \dots, z^n). \tag{2.2.2}$$

Let $\eta = \exp^*\omega$ be the pull-back of ω under this map. In local coordinates, $\eta = \eta_{\overline{k}j} i dy^j \wedge d\overline{y}^k$. Let $y^j = w^j + i\theta^j$ be the standard real coordinates. The fact that ω is invariant under the multiplicative $(S^1)^n$-action implies that η is invariant under the additive action of $(\theta^1, \dots, \theta^n)$. Since η is a Kähler form on $\mathbb{C}^n$, it has a Kähler potential. This means that there exists a smooth function $\phi : \mathbb{C}^n \to \mathbb{R}$ such that

$$\eta_{\overline{k}j} i dy^j \wedge d\overline{y}^k = 2\frac{\partial^2 \phi}{\partial y^j \partial \overline{y}^k} i dy^j \wedge d\overline{y}^k. \tag{2.2.3}$$

Since η is invariant under the additive θ-action, we can assume that ϕ is independent of θ. Now since we have a θ-invariant Kähler potential ϕ of η, the push-forward $\varphi := \exp_*(\phi)$ is well-defined and is an $(S^1)^n$-invariant Kähler potential for ω on $(\mathbb{C}^*)^n$.

By changing to real-coordinates with $dy^j = dw^j + i d\theta^j$ and $\frac{\partial}{\partial y^j} = \frac{1}{2}(\frac{\partial}{\partial w^j} - i\frac{\partial}{\partial \theta^j})$, one can use the θ-invariance of ϕ and anti-symmetrization to compute that

$$\eta = 2\frac{\partial^2 \phi}{\partial y^j \partial \overline{y}^k} i dy^j \wedge d\overline{y}^k = \frac{\partial^2 \phi}{\partial w^j \partial w^k} dw^j \wedge d\theta^k. \tag{2.2.4}$$

Next let $\tilde{\mu} : \mathbb{C}^n \to \mathbb{R}^n$ be the gradient map of ϕ in the w-coordinates—i.e. $\tilde{\mu}_j = \frac{\partial \phi}{\partial w^j}$. Since $\tilde{\mu}$ is θ-invariant, its push-forward is a well-defined $(S^1)^n$-invariant map $\exp_*(\tilde{\mu}) : (\mathbb{C}^*)^n \to \mathbb{R}^n$. Note that (2.2.4) says that $-\iota_{\frac{\partial}{\partial \theta^k}}(\eta) = \frac{\partial \tilde{\mu}_k}{\partial w^j} dw^j = d(\tilde{\mu}_k)$. Since ϕ is only determined by its Hessian, $\tilde{\mu}_k$ is only determined up to a constant.

Now fix k and consider the smooth function $\mu_k := (\exp_*(\tilde{\mu}))_k : (\mathbb{C}^*)^n \to \mathbb{R}$. Let V_k be the vector field on M defined in the proof of Lemma 2.1.6. We have that V_k is a globally-defined vector field on M and hence $-\iota_{V_k}(\omega)$ is a globally-defined 1-form. Thus μ_k is a smooth function defined on U such that $d(\mu_k) = -\iota_{V_k}(\omega)|_U$. Next let $p \in D$. By Lemma 2.1.6 there is a ball $B \subset M$ containing p, such that $B \cap U$ is connected. Therefore $-\iota_{V_k}(\omega)$ has a local potential f on B and $df = d\mu_k$ on $B \cap U$. But the connectedness of $B \cap U$ means then that μ_k and f differ by a constant on $B \cap U$ and hence μ_k can be extended locally to a smooth function on all B. Since this holds for any point in D, we have that μ_k can be extended to a smooth function on all of M.

In this section, we have followed Remark 2.1.2 and used a specific basis in our local computations. Let us now consider the invariant nature of this problem. The moment map $\mu = \nabla\phi = (\frac{\partial \phi}{\partial w^1}, \dots, \frac{\partial \phi}{\partial w^n})$ is the same as the exterior derivative $d(\phi)$ in the basis given by

$dw^1, \ldots, dw^n$. The complex structure J on $\mathbb{C}^n$ satisfies $J(dw^j) = d\theta^j$ for all j. This means that μ in our local coordinates is the same as the mapping $J(d\phi)$ in the basis given by $d\theta^1, \ldots, d\theta^n$—which is certainly an invariant global object whose range is $\mathfrak{t}^*$.

Hence let us think of μ as a mapping $M \to \mathfrak{t}^*$. Given any $V \in \mathfrak{t}$, let $V^\#$ be the real vector field on M generated by the infinitessimal action of V. Let $\langle \cdot, \cdot \rangle$ denote the natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. Then $\langle \mu, V \rangle$ is a smooth function on M. In this setting, we still have the fundamental equation $d\langle \mu, V \rangle = -\iota_{V^\#}(\omega)$. This leads to the following definition.

Definition 2.2.4. *Let (M, ω) be a symplectic manifold with a left-action by a compact commutative Lie group T which preserves the symplectic form ω. A* ***moment map*** *of (M, ω), is a smooth function $\mu : M \to \mathfrak{t}^*$ satisfying the property that for any $X \in \mathfrak{t}$, $d\langle \mu, X \rangle = -\iota_{X^\#}(\omega)$—where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$.*

Thus, we have shown:

Lemma 2.2.5. *Any smooth, compact Kähler toric variety (M, ω) has a moment map μ which is unique up to the addition of an additive constant.*

2.2.2 Example: $\mathbb{P}^n$

The standard example for all these computations is n-dimensional projective space. Choose coordinates $(z^1, \ldots z^n) \mapsto [1 : z^1 : \cdots : z^n] \in \mathbb{P}^n$. Next take ω to be the Fubini-Study metric. I.e. $\omega = i\partial\overline{\partial} \log(1 + |z^1|^2 + \cdots + |z^n|^2)$. This means that $\phi(w^1, \ldots, w^n) = \frac{1}{2} \log(1 + e^{2w^1} + \cdots + e^{2w^n})$. The gradient map then gives the relation to the x-coordinates:

$$x_j = \frac{\partial \phi}{\partial w^j} = \frac{e^{2w^j}}{1 + e^{2w^1} + \cdots + e^{2w^n}}. \tag{2.2.5}$$

Hence the moment map in the z-coordinates is given by

$$\mu(z) = \frac{1}{1 + |z_1|^2 + \cdots + |z_n|^n} (|z_1|^2, \cdots, |z_n|^2), \tag{2.2.6}$$

which shows that the image of μ is the standard n-simplex of $\mathbb{R}^n$.

Remark 2.2.6. In the previous example , we saw that the image of the moment map of $\mathbb{P}^n$ with the Fubini-Study metric is the standard n-simplex in $\mathbb{R}^n$. This is no accident and a similar geometric result will, in fact, hold in the general case as well. We next study the geometry of $\mu(M)$ for general (M, ω).

2.2.3 Geometry of the Image of the Moment Map

Let us return to the mapping $\tilde{\mu} : (\mathbb{C}^*)^n \to \mathbb{R}^n$. Since $\tilde{\mu}$ is the gradient map of a strictly convex function, the image $\tilde{\mu}(\mathbb{R}^n) = \mu(U) \subset \mathbb{R}^n$ is a convex set. The fact that μ is smooth and M is compact means that $\mu(M)$ is compact as well. Next we would like to better understand the boundary of $\mu(M)$ in $\mathbb{R}^n$. The goal is to prove the following:

Lemma 2.2.7. *Let $p \in D$ and let O_p be the orbit of p under the action of $(\mathbb{C}^*)^n$. Then O_p is biholomorphic to $(\mathbb{C}^*)^k$ for some integer k and $\mu(O_p)$ is a convex set in $\mathbb{R}^n$, contained in an affine subspace of dimension k.*

Proof. Let $W \subset \mathbb{C}^n$ be the subspace of holomorphic tangent vectors whose infinitessimal actions fix p and let k be the complex dimension of W. Hence $\exp(W) =: S \subset (\mathbb{C}^*)^n$ is a holomorphic subgroup. S can be identified as the image of a mapping $(\mathbb{C}^*)^k \to (\mathbb{C}^*)^n$ given by

$$(s^1, \ldots, s^k) \mapsto ((s^1)^{a_{11}} \cdots (s^k)^{a_{1k}}, \ldots, (s^1)^{a_{n1}} \cdots (s^k)^{a_{nk}}),$$

where $a := (a_{pq})$ is a matrix of integers. Furthermore, when a is viewed as a linear mapping from $\mathbb{C}^k \to \mathbb{C}^n$ it takes the standard basis vectors in $\mathbb{C}^k$ to a minimal integral basis of $W \subset \mathbb{C}^n$. This minimal integral basis can be completed to a complete integral basis of $\mathbb{C}^n$. I.e. there is a matrix $A \in GL(n, \mathbb{Z})$ such that the first k columns of A coincide with a.

After using A to change coordinates, we can assume that the stabilizer of the $(\mathbb{C}^*)^n$-action on M at p is given by $\{(\alpha_1, \cdots, \alpha_n) \in (\mathbb{C}^*)^n \mid \alpha_j = 1, \text{ for all } j > k\}$. This means that the mapping $(\mathbb{C}^*)^{n-k} \hookrightarrow (\mathbb{C}^*)^k \times (\mathbb{C}^*)^{n-k} \to U$ is a holomorphic embedding with image O_p. Under this embedding, the standard $(\mathbb{C}^*)^{n-k}$ action agrees with the action on O_p. This means that $(O_p, \omega|_{O_p})$ is an open $(n-k)$-dimensional Kähler manifold such that O_p is biholomorphic to $(\mathbb{C}^*)^{n-k}$ and such that $(\mathbb{C}^*)^{n-k}$ acts by the standard fashion and $(S^1)^k \subset (\mathbb{C}^*)^{n-k}$ preserves the form $\omega|_{O_p}$. I.e. $(O_p, \omega|_{O_p})$ satisfies all the conditions of a Kähler toric manifold as before except that it is not compact. Regardless, all of the analysis at the beginning of the section can be redone exactly to produce a moment map μ_p for $\omega|_{O_p}$ which maps O_p to a convex set in $\mathbb{R}^{n-k}$ (compactness was never used up until that point). Next note that μ_p agrees with the final $n-k$ coordinate functions of μ. The first k coordinate functions of μ map O_p to a constant in $\mathbb{R}^n$. Taken together, this implies that $\mu(O_p)$ is a convex set contained in an affine subspace of codimension k in $\mathbb{R}^n$ which is what we needed to prove. □

We can apply this lemma to break down and understand the geometry of $\mu(M)$. The compact convex set $\mu(M)$ is the convex hull of its extreme points. The previous lemma implies that if $p \in M$ is *not* a fixed point, then $\mu(p)$ is *not* an extreme point. As a result, we have that $\mu(M)$ is the convex hull of the image of the fixed points of the action on M. This proves Proposition 2.2.2. Furthermore, we have concluded to the following:

Lemma 2.2.8. *The image of the moment map μ of a compact Kähler manifold is a compact convex polytope P in $\mathfrak{t}^*$. The polytope P is the convex hull of the image under μ of the fixed points of the action of $T_{\mathbb{C}}$.*

The next natural question is how much of our construction depends upon the specific metric ω? Let $\omega_0 \in [\omega]$ be an $(S^1)^n$-invariant metric cohomologous to ω. By the $\partial\overline{\partial}$-Lemma, there is a real, smooth $(S^1)^n$-invariant function f on M such that $\omega_0 = \omega + i2\partial\overline{\partial}f$. Let μ_0 be the moment map constructed similarly as above except now with respect to ω_0. This means that $(\tilde{\mu}_0)_k = \tilde{\mu}_k + \frac{\partial \tilde{f}}{\partial w^k} + c_k$, where $\tilde{f} = \exp^*(f)$ as with previous notation and $(c_1, \ldots, c_n)$ is a constant vector coming from the fact that our construction of $\tilde{\mu}$ from ω is only well-defined up to a constant. For the time being, let us assume that $c = 0$. Hence on M we have that $(\mu_0)_k = \mu_k + V_k(f)$, where $V_k = \exp_*(\frac{\partial}{\partial w^k})$. The vector fields $V_1, \ldots, V_k$ all vanish at the fixed points of the $(\mathbb{C}^*)^n$-action on M. Since f is smooth and globally defined on M, this means that $V_k(f) = 0$ for all k at those fixed points. But this means that μ and μ_0 agree at the fixed points. Since the images of both maps is the convex hull of the images of the fixed points, we have that $\mu(M) = \mu_0(M)$. As a result, we have proved Proposition 2.2.1.

2.2.4 Space of Symplectic Potentials

Let us return our attention to equation (2.2.4). If we restrict ϕ to the w-coordinates, then ϕ is a smooth strictly convex function on $\mathbb{R}^n$. Similarly, if we restrict $\tilde{\mu}$ to the w-coordinates, then $\tilde{\mu} : \mathbb{R}^n \to \mathbb{R}^n$ is the gradient map of ϕ. Since ϕ is strictly convex, $\tilde{\mu} : \mathbb{R}^n \to \tilde{\mu}(\mathbb{R}^n)$ is a diffeomorphism onto its image. Next we consider a concept from convex geometry:

Definition 2.2.9. *Let $\phi(w)$ be a smooth, strictly convex function on $\mathbb{R}^n$. The* ***Legendre Transform*** *u of f is a function defined on $\nabla f(\mathbb{R}^n)$—the image of the gradient mapping of f—defined by the requirement that $u(x) = \sup_{w\in\mathbb{R}^n}(w \cdot x - \phi(w))$, for any $x \in \nabla f(\mathbb{R}^n)$.*

Remark 2.2.10. Since ϕ is smooth, we have that the supremum of $(w \cdot x - \phi(w))$ occurs at the point w where the gradient vanishes. This occurs at the point w where $x_j = \frac{\partial \phi}{\partial w^j} = \tilde{\mu}_j$, for all j. This means that $u(x) = w \cdot x - \phi(w(x))$, where we implicitly define w as a function of x by way of the diffeomorphism given by $\tilde{\mu}$.

Definition 2.2.11. *Let (M, ω) be a Kähler toric variety and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be as in* (2.2.3)*. Then the Legendre transform u of ϕ is called the* ***symplectic potential*** *of ω.*

Remark 2.2.12. The domain of the symplectic potential u is the range of ϕ, however the range of $\nabla\phi$ is not uniquely fixed by ω. Furthermore, if one replaces ϕ with $\phi + k$, the range of $\nabla\phi$ remains the same, but the symplectic potential changes from u to $u - k$. When working with the symplectic potential, one must consider these subtleties.

2.2.5 Example: $\mathbb{P}^n$ Revisited

Let us now pick up the previous example at equation 2.2.6. Equation (2.2.5) can be solved for w^j:

$$w^j = \frac{1}{2}\log\left(\frac{x_j}{1 - x_1 - \cdots - x_n}\right).$$

Next we can write out the Legendre transform of ϕ explicitly:

$$\begin{aligned} u(x) &= w \cdot x - \phi(w(x)) \\ &= \frac{1}{2}\left(\sum_j x_j \log(x_j) + (1 - \sum_j x_j)\log(1 - \sum_j x_j)\right), \end{aligned}$$

which is a strictly convex function defined on the standard n-simplex in $\mathbb{R}^n$.

In the previous section we showed that cohomologous Kähler metrics give rise to the same moment polytope. In Definition 2.2.11, associated symplectic potentials to Kähler metrics which are functions defined on the polytope. The main goal of this section is to understand how these symplectic potentials are related.

Let ω_0 and ω_1 be two toric Kähler metrics on M and let P be their corresponding moment polytope. Let u_0 and u_1 be the corresponding symplectic potentials defined on the interior of P. By the $\partial\overline{\partial}$-Lemma, we have that $\omega_1 = \omega_0 + 2i\partial\overline{\partial}f$, for some $(S^1)^n$-invariant function f defined on all M. If we consider the functions ϕ_0 and ϕ_1 as defined in (2.2.3), we have that $\phi_1 = \phi_0 + h$, where $h = f \circ \exp$. Let $\phi_t = \phi_0 + th$ be the linear path connecting ϕ_0 to ϕ_1. Corresponding to this we have a path of symplectic potentials u_t. In [Donaldson, 2002], Donaldson proves the following:

Lemma 2.2.13. *Let u_t, ϕ_t and h be defined as above. Then we have*

$$\left.\frac{d}{dt}\right|_{t=0} u_t = -(\nabla\phi_0)_* h,$$

Proof. The main difficulty of this proof is the fact that the coordinates x and w are related implicitly by the gradient of ϕ_t which makes all of this extremely confusing. We have that $u_t(x) = w(t)\cdot x - \phi_t(w(t))$, where $w(t)$ is defined by the requirement that $(\phi_t)_i(w(t)) = x_i$. We need to sort out the following limit

$$\begin{aligned}
\lim_{t\to 0}\frac{u_t(x)-u_0(x)}{t} &= \lim_{t\to 0}\frac{(w(t)-w(0))\cdot x - (\phi_t(w(t)) - \phi_0(w(0)))}{t}\\
&= w'(0)\cdot x - \lim_{t\to 0}\frac{\phi_t(w(t)) - \phi_0(w(0))}{t}\\
&= w'(0)\cdot x - \lim_{t\to 0}\frac{\phi_t(w(t)) - \phi_0(w(t)) + \phi_0(w(t)) - \phi_0(w(0))}{t}\\
&= w'(0)\cdot x - \lim_{t\to 0}\frac{\phi_t(w(t)) - \phi_0(w(t))}{t} + (\phi_0)_i(w(0))\cdot (w'(0))^i\\
&= -\lim_{t\to 0}\frac{\phi_t(w(t)) - \phi_0(w(t))}{t},
\end{aligned}$$

where in the last step we used that $(\phi_0)_i(w(0)) = x_i$. Next note that $\phi_t = \phi_0 + th$ and thus we conclude that

$$\lim_{t\to 0}\frac{u_t(x)-u_0(x)}{t} = -\lim_{t\to 0} h(w(t)) = -h(w(0)).$$

Finally, note that $(\nabla\phi_0)_* h(x) = h(w(0))$, by the definition of $w(0)$, which proves the lemma. □

We can apply this lemma to any point along the path between ϕ_0 and ϕ_1 to conclude that $\frac{d}{dt}|_{t=t_0} u_t(x) = (\mu_t)_* f(x)$, for function f that is smooth and $(S^1)^n$-invariant on all M, and where μ_t is the moment map corresponding to ϕ_t. Next we would like to understand how such functions look when pushed forward to the moment polytope.

Lemma 2.2.14. *Given any $(S^1)^n$-invariant smooth function f on (M,ω), the push-forward $\mu_*(f)$ of f to P under the moment map μ is a smooth function in the interior of P and all of its derivatives are continuous on $\overline{P}$.*

Proof. The proof of this result makes strong use of Proposition 2.1.5 in order to see exactly how the moment map looks like at the complement of the open torus. Proposition 2.1.5 allows us to restrict our attention to open sets of the form $\mathbb{C}^n$ with the standard action of $(\mathbb{C}^*)^n$ and metrics ω on $\mathbb{C}^n$ which are preserved by $(S^1)^n \subset (\mathbb{C}^*)^n$.

First we consider simplified case of one dimension. Let $M = \mathbb{C}$ and let $\omega = idz \wedge d\overline{z} + 2i\partial\overline{\partial}\phi$ be an S^1-invariant metric. (On $\mathbb{C}$, all metrics are of this form up to scaling.) Written out, we have

$$\omega = idz \wedge d\overline{z} + 2\frac{\partial^2\phi}{\partial z\partial\overline{z}} idz \wedge d\overline{z} = \left(1 + 2\frac{\partial^2\phi}{\partial z\partial\overline{z}}\right) idz \wedge d\overline{z}. \tag{2.2.7}$$

Since ϕ is S^1-invariant, $2\frac{\partial^2\phi}{\partial z\partial\overline{z}} = \frac{1}{2}r^{-1}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r})$, where $r = |z|$. We have that $\frac{\partial^k\phi}{\partial r^k}$ exists for all k at the origin, and furthermore, $\frac{\partial^k\phi}{\partial r^k} = 0$ at the origin for all odd k, due to the S^1 symmetry.

To further simplify, let us now assume that $\phi(r) = \frac{c}{2}r^2$ for some c. In this case, $\frac{1}{2}r^{-1}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) = c$. The restriction that ω be a metric tells us that $1 + c > 0$ and hence that $c > -1$. In the case where ϕ is an arbitrary smooth function, the same argument tells us that the Taylor expansion of $\phi(r) = \frac{c}{2}r^2 + O(r^4)$, where $c > -1$ as before (we can safely assume that ϕ's constant term is 0).

The moment map μ corresponding to this metric is given by $r \mapsto x = r^2 + r\frac{\partial\phi}{\partial r}$. In the case where $\phi = \frac{c}{2}r^2$, this simply becomes the map $r \mapsto x = (1+c)r^2$. The restriction that $c > -1$ tells us that this mapping is a bijective map from $[0,\infty)$ to itself. Furthermore, it tells us that the pullback of polynomials is given by

$$\mu^*\left(\sum_{j=0}^{n} a_j x^j\right) = \sum_{j=0}^{n} a_j (1+c)^j r^{2j}. \tag{2.2.8}$$

In the case where $\phi(r) = \frac{c}{2}r^2 + O(r^4)$, the pullback of polynomials is instead given by

$$\mu^*\left(\sum_{j=0}^{n} a_j x^j\right) = \sum_{j=0}^{n} a_j (1+c+O(r^2))^j r^{2j}. \tag{2.2.9}$$

This means that, $(1+c)^{-n}x^n$ pulls back to $(1+O(r^2))^n r^{2n} = r^{2n} + O(r^{2n+2})$. This is enough to conclude inductively that any polynomial of degree $2n$ in r which is made up of only even monomials can be approximated by the pullback of a polynomial of degree n in x up to order $2n+2$. This means that for any smooth S^1-invariant function f on $\mathbb{C}$, $\frac{\partial^k}{\partial x^k}(\mu_* f)$ is continuous up to $x = 0$ for all k, which proves the result for the one-dimensional case.

This argument extends straight-forwardly to higher dimensions. Since the result is local, we are done. □

Next let us make use of this lemma. Let u_0 and u_1 be the symplectic potentials from the beginning of this section and let u_t be the path that connects them. Lemma 2.2.13 tells us that $\frac{d}{dt}|_{t=t_0}(u_t(x))$ is the pushforward of a smooth $(S^1)^n$-invariant function on M. Next, Lemma 2.2.14 tells us that this function is smooth in the interior or P and that all of its derivatives are continuous up to the boundary of P. By integrating this result, we conclude that u_0 and u_1 differ by a smooth function which is smooth up to the boundary of P.

To finish this story, we would like to find a sort of canonical symplectic potential on P. Let F be a face of the polytope P. Let l_F be the smallest integral, inward-pointing, normal vector to F (i.e. l_F is affine linear: $l_F(F) = 0$). The polytope $P = \{x \mid l_F(x) \geq 0, \forall F\}$. Finally define

$$u_P(x) = \frac{1}{2}\sum_F l_F(x)\log\left(l_F(x)\right). \tag{2.2.10}$$

The range of the gradient of u_P is all of $\mathbb{R}^n$. Furthermore, the local arguments of the last proposition show that u_P actually is the symplectic potential of some toric Kähler metric on M. To conclude, we have proved Proposition 2.2.3.

2.3 Algebraic Structure of the Polytope

In the previous sections, we showed that to any Kähler toric variety M there is associated a polytope P in $\mathfrak{t}^*$. Proposition 2.2.2, in addition to some basic convex geometry, implies that the collection $\{X_q\}$, where q are the fixed points of M, is an open cover of M. Hence Proposition 2.1.5 immediately implies the following proposition:

Proposition 2.3.1. *P satisfies the following properties:*

(1) There are n faces meeting at each vertex q of P,

(2) Given each face F_i, there is a corresponding minimal inward-pointing vector λ_i normal to F_i with $\lambda_i \in \mathbb{Z}^n$,

(3) For each vertex q, the n vectors $\lambda_1, \ldots, \lambda_n$ corresponding to the n faces meeting at q generate $\mathbb{Z}^n$ over $\mathbb{Z}$.

The previous proposition is a linear-algebraic dual version of Delzant's theorem and hence we have proved the first of Delzant's theorems in the introduction. The structure of P described in Delzant's theorem is quite rigid and has been named after him.

Definition 2.3.2. *Let P be a convex polytope in $\mathbb{R}^n$. Then P is said to be* ***Delzant*** *if it satisfies conditions $(i) - (iii)$ in Theorem 2 or, equivalently, it satisfies conditions $(1) - (3)$ in Proposition 2.3.1.*

Hence in this language, Delzant proved that the polytope corresponding to any toric Kähler manifold is in fact a Delzant polytope. However, as mentioned in the introduction, he proved more than just this. He showed that given any Delzant polytope P, one can construct a toric Kähler manifold whose moment polytope is exactly P. We will prove this by starting with a Delzant polytope and explicitly constructing a corresponding toric variety. This construction has the advantage of also equipping the toric variety with a polarization which we will strongly use later in this thesis. The goal of the rest of this section is to prove the following:

Proposition 2.3.3. *Let $P \subset \mathbb{R}^n$ be a Delzant polytope and assume that the vertices of P lie on the lattice. Then there exists a polarized smooth n-dimensional variety (V, L), admitting a compatible $(\mathbb{C}^*)^n$-action, and an $(S^1)^n$-invariant Kähler metric $\omega \in c_1(L)$, such that (V, ω) is a toric Kähler variety and P is the moment polytope corresponding to (V, ω). Furthermore, there is a basis for $H^0(V, L)$ given by $\{s_\lambda \mid \lambda \in P \cap \mathbb{Z}^n\}$ so that $(\mathbb{C}^*)^n$ acts on $\Gamma^0(V, L)$ by*

$$(\alpha_1, \ldots, \alpha_n) \cdot s_\lambda = \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n} s_\lambda.$$

As a result, $h^0(V, L) = \#(P \cap \mathbb{Z}^n)$.

2.3.1 Constructing Polarized Varieties from P

First note that properties (i) - (iii) of Proposition 2.3.1 do not uniquely specify P. Not only is P only fixed up to an arbitrary position in $\mathbb{R}^n$ (which is an ambiguity one must always accept in this theory), but the "scale" of P is also not specified. The polytope satisfies $P = \{x \mid \lambda_F \geq c_F, \forall F\}$, for some constants c_F, but the constants are not uniquely specified.

Assume for the moment that each $c_F \in \mathbb{Z}$. This restriction implies that the convex hull of $P \cap \mathbb{Z}^n$ is equal to P—i.e. the extreme points of P lie in the lattice $\mathbb{Z}^n$. Let $\Lambda = P \cap \mathbb{Z}^n$ be the integer lattice points in P. Denote the lattice points by $\nu_0, \dots, \nu_N$, where $N = \#(P \cap \mathbb{Z}^n) - 1$. We define the following relation R by declaring that $x_j x_k = x_l x_m$ if $\nu_j + \nu_k = \nu_l + \nu_m$ as elements in $\mathbb{Z}^n$. Define V to be the subvariety of $\mathbb{P}^N$ cut out by the relations given by R. The Delzant conditions allow us to conclude the following:

Lemma 2.3.4. *The subvariety $V \subseteq X$ is smooth.*

Proof. Fix a vertex q of P. Assume that ν_0 is the lattice point corresponding to q and that $\nu_1, \dots, \nu_n$ are the n lattice points lying closest to q. Next let $X_q = (x_q \neq 0) \subset X$. To simplify the computations, we make the harmless assumption that q is the origin of $\mathbb{Z}^n$.

Next we parametrize $V \cap X_q$ explicitly. For each $\nu_k \in P \cap \mathbb{Z}^n$, we have that $\nu = M_k^j \nu_j$, where (M_k^j) is a matrix of non-negative integers. Consider the mapping

$$(z_1, \dots, z_n) \mapsto \left[1 : \prod_{j=1}^{n} z_j^{M_1^j} : \cdots : \prod_{j=1}^{n} z_j^{M_N^j} \right] = [x_0 : \cdots : x_N]. \tag{2.3.11}$$

Since $M_j^k = \delta_{jk}$ for $j, k = 1, \dots, n$, this is a smooth embedding of $\mathbb{C}^n$ into V. The set of such mappings for each extreme point q show that V is a smooth subvariety. □

2.3.2 Example: The First Hirzebruch Surface

To make this all a little more specific, consider the polytope P given by the convex hull of the points $(0,0), (2,0), (0,1)$, and $(1,1)$. There are 5 lattice points in $P \cap \mathbb{Z}^2$. In addition to the extremal points already listed, we also have the point $(1,0)$. Label the lattice points as follows:

$$\nu_0 \longleftrightarrow (0,0), \qquad \nu_1 \longleftrightarrow (1,0), \qquad \nu_2 \longleftrightarrow (2,0),$$

$$\nu_3 \longleftrightarrow (0,1), \qquad \nu_4 \longleftrightarrow (1,1).$$

Following the general construction from earlier in the section, note that the two relations we get come from $(0,0)+(2,0)=(1,0)+(1,0)$ and $(0,0)+(1,1)=(1,0)+(0,1)$. Written in terms of our variables, this says that our relations are $x_0x_2 = x_1^2$ and $x_1x_3 = x_0x_4$. This means that the subvariety V we get is

$$(x_0x_2 - x_1^2, x_1x_3 - x_0x_4) \subset \mathbb{P}^4.$$

which is none other than the first Hirzebruch surface.

2.3.3 Kähler Toric Structure of V

Next let us put a toric structure on the variety V given in Lemma 2.3.4. Notice that the standard action of $(\mathbb{C}^*)^n$ on $(z_1, \ldots, z_n)$ in (2.3.11) preserves V. Next let $L = \mathcal{O}(1)|_V$ and note that we have a section x_j for each lattice point ν_j. Furthermore, $(\mathbb{C}^*)^n$ acts on x_j by

$$(\alpha_1, \ldots, \alpha_n) \cdot x_j = \left(\prod_{k=1}^{n} \alpha_k^{M_j^k} \right) x_j,$$

then this action lifts the action of $(\mathbb{C}^*)^n$ on V. Next let ω be the restriction of the Fubini-Study metric on $\mathcal{O}(1)$ to V. We have that $(\mathbb{C}^*)^n$ acts on the polarized pair (V, L) and that (V, ω) is a toric Kähler manifold with $\omega \in c_1(L)$. One can compute locally that the moment polytope of ω is exactly P and hence we have proved Proposition 2.3.3.

2.3.4 Example: The First Hirzebruch Surface Revisited

Consider again the setting of the previous example. Next let U be the set $U = \{x_0 \neq 0\} \subset \mathbb{P}^4$ and consider $V_0 := V \cap U$. In coordinates $(x_1, \ldots, x_4) \mapsto [1 : x_1 : \cdots : x_4]$, we have that V_0 is given by the equations $(x_2 - x_1^2, x_1x_3 - x_4)$ in $\mathbb{C}^4$. We have that $\mathbb{C}^2$ embeds by

$$(z_1, z_2) \mapsto [1 : z_1 : z_1^2 : z_2 : z_1z_2].$$

If we restrict the Fubini-Study metric we see that ω takes the form

$$\omega = -i\partial\overline{\partial} \log\left(1 + |z_1|^2 + |z_1|^4 + |z_2|^2 + |z_1|^2|z_2|^2\right).$$

2.4 Appendix: Automatically Computing Toric Relations

In the previous section, we considered an algebraic construction of a toric variety from a moment polytope. The following is a program written in Python that one can use to compute the relations used in that construction. There nothing especially complicated about this procedure, but having a convenient means to quickly compute examples makes the study of toric varieties a little less painful and less error prone.

```
#!/usr/bin/env python
from optparse import OptionParser
from collections import defaultdict
from sys import stdin, stdout


def main():
    r"""
    DESCRIPTION
    -----------
    Returns the relations of the projective embedding of a toric variety coming
    from it toric polytope.

    NOTES
    -----
    The lattice points of the polytope should be given in an input text file or
    they should be piped in (but care must be taken to add newline
    characters--i.e. '\n'--in between the lattice points. The names of the
    variables are chosen in the order the lattice points are given.

    EXAMPLES
    ---------
    Return the relations for P^1 polarized by O(2)--which corresponds to the
    polytope [0,2]. (The echo function simulates piping in a file.)
    $ echo -e '0\n1\n2' | python relations.py
    """
```

```
usage = "usage: %prog [options] dataset"
usage += '\n'+main.__doc__
parser = OptionParser(usage=usage)
parser.add_option(
    "-n", "--variable_start",
    help="Start variable count at this number. [default: %default]",
    action="store", dest="variableStart", default=0)
parser.add_option(
    "-i", "--infile",
    help="Filename for input file.",
    action="store", dest="infile", default=None)
parser.add_option(
    "-o", "--outfile",
    help="Filename for output file.",
    action="store", dest="outfile", default=None)

(options, args) = parser.parse_args()
assert len(args) <= 1

if options.infile:
    infile = open(options.infile)
else:
    infile = stdin

if options.outfile:
    outfile = open(options.outfile, 'w')
else:
    outfile = stdout

relations(infile, outfile, variableStart=options.variableStart)

infile.close()
outfile.close()
```

```
def relations(infile, outfile, variableStart):
    latticePoints = loadPoints(infile, delimiter=',')
    variableDict = getVariableDict(latticePoints, variableStart)
    variables = variableDict.keys()
    variables.sort()
    relations = getRelations(variableDict)
    numRelations = reduce(lambda x, y: x + len(y) - 1, relations, 0)
    prettyPrint(outfile, variables, variableDict, relations, numRelations)


def makeStringRep(point):
    """
    Convert list of integers to comma-separated string for hashing.
    """
    point = [str(num) for num in point]
    rep = ",".join(point)
    return rep


def loadPoints(infile, delimiter=','):
    """
    Take a file object whose lines are comprised of delimiter-separated list of
    integers and return a list of lists of those lines split at the dilimiters
    and convert the strings to integer objects. Check to make sure values are
    valid integers and that each line has the same number of items in the
    comma-separated list.
    """
    latticePoints = []
    for vector in infile:
        point = vector.strip().split(delimiter)
        try:
            point = [int(num) for num in point]
        except ValueError:
```

```
            raise Exception("Must input integer vector")
        latticePoints.append(point)
    dim = len(latticePoints[0])
    for point in latticePoints:
        if len(point) != dim:
            raise Exception("All points must be same dimension")
    return latticePoints


def getRelations(variableDict):
    """
    Return a list of pairs of variables whose corresponding pair of points sum
    to the same point. (This probably makes no sense.)
    """
    relationsDict = defaultdict(list)
    variables = variableDict.keys()
    for var1 in variables:
        for var2 in variables:
            if var1 <= var2:
                key = addPoints(variableDict[var1], variableDict[var2])
                key = makeStringRep(key)
                relationsDict[key].append([var1, var2])
    relations = []
    for key in relationsDict:
        relation = relationsDict[key]
        if len(relation) > 1:
            relations.append(relation)
    return relations


def addPoints(point1, point2):
    """
    Return the vector sum of two lists of integers.
    """
```

```
    result = []
    for num1, num2 in zip(point1, point2):
        result.append(point1 + point2)
    return result

def prettyPrint(outfile, variables, variableDict, relations, numRelations):
    """
    Print out result statistics to outfile.
    """
    outfile.write("Variables:\n")
    for variable in variables:
        outfile.write('\t' + variable + ' <----> ')
        outfile.write(makeStringRep(variableDict[variable]) + '\n')
    outfile.write(str(numRelations) + " Relations:\n")
    if not relations:
        outfile.write('\tNone\n')
    else:
        for relation in relations:
            relation.sort()
            tempString = '\t'
            for var1, var2 in relation:
                tempString = tempString + var1 + var2 + ' = '
            outfile.write(tempString[:-3] + '\n')


def getVariableDict(latticePoints, variableStart):
    """
    Take a list of lattice points and return a dictionary mapping a formal
    variable to each lattice point.
    """
    variableStart = int(variableStart)
    variableDict = {}
    for num, point in enumerate(latticePoints, variableStart):
        variableDict['x' + str(num)] = point
```

```
    return variableDict



if __name__=='__main__':
    main()
```

Chapter 3

Toric Fibrations

In the previous chapter, we reviewed the theory of toric varieties. In this chapter, we extend the theory to certain spaces called toric fibrations which we now describe. Let M be a compact toric variety whose open complex torus $U \subset M$ is of complex dimension n. Let G be a real compact semi-simple Lie group and let $T \subset G$ be a maximal torus of real dimension n. Consider simultaneously the action of T on M and the right-action of T on G. The manifold $G \times_T M$ is defined as the space $G \times M$ modulo the relation $(gh, x) = (g, hx)$ for all $g, h \in G$ and $x \in M$. As will be explained below, $G \times_T M$ has a holomorphic Kähler structure such that the G-action, given by $g \cdot [h, x] = [gh, x]$, acts by biholomorphisms. It is therefore natural to consider Kähler metrics Ω on $G \times_T M$ which are invariant under the G-action. This leads to our main definition.

Definition 3.0.1. *A* ***Kähler toric fibration*** *is a pair $(G \times_T M, \Omega)$, where the Kähler metric Ω is invariant under the G action.*

Consider the projection mapping $\pi : G \times_T M \to G/T$, given by $[g, x] \mapsto gT$. Equipped with π, $G \times_T M$ is a fiber bundle over G/T, each of whose fibers is isomorphic to M. Consider the embedding $\iota : M \to G \times_T M$ given by $x \mapsto [e, x]$. We have that $\iota(M) = \pi^{-1}(eT)$ and that $t \cdot \iota(x) = \iota(t \cdot x)$—i.e. the action of T on $\iota(M)$ agrees with the original action of T on M as a toric variety. To simplify notation, we denote $\iota(M)$ by M and the restriction of Ω to M by ω. This result of this discussion is that (M, ω) is a toric Kähler variety.

We will see that we can understand much of the geometry of $(G \times_T M, \Omega)$ by the studying (M, ω). We make use of this fact to derive the scalar curvature equation for Ω in terms of the toric polytope structure coming from (M, ω). Let $\mu : M \to P$ be a moment map corresponding to ω. We define the extension of μ to $G \times_T M$ by requiring that $\mu([g, x]) = \mu(x)$ (to simplify notation, we call the extension μ as well). Note that, by construction, $\mu : G \times_T M \to P$ is G-invariant. Since the scalar curvature $S(\Omega)$ is G-invariant, we may consider its push-forward to P under μ. Let N be the complex dimension of $G \times_T M$. Consider the push-forward of Ω^N (the volume form on $G \times_T M$) under μ. We have that $\mu_*(\Omega^N) = cW d\mu$, where $d\mu$ is the Lebesgue volume form, W is the Duistermaat-Heckman polynomial, and c is a positive multiplicative constant. Given this setup, the main result of this chapter is Theorem 1.

Theorem 1 is an extension of Abreu's well-known scalar curvature equation for toric varieties (see [Abreu, 1998]). In his thesis, Raza derives a similar equation, but his variational approach precludes him from determining the function f_G (see [Raza, 2006]). In [Donaldson, 2008b], Donaldson states, but does not prove Equation (1.0.7). For our work it is very important to know the exact form of the scalar curvature and therefore we explicitly work out Equation (1.0.7).

The chapter is outlined as follows. In the first section, we will explain some necessary background from Lie theory and then use it to give a good local trivialization of $G \times_T M$. In the second section, we will show how to translate the geometry of $G \times_T M$ to $G \times P$, giving explicit formulas for both the Duistermaat-Heckman polynomial and the Laplacian. In the final section, we will give a proof of Theorem 1.

3.1 A Local Description of $(G \times_T M, \Omega)$

In the previous chapter, we understood most of the toric geometry of M by restricting to the dense open complex torus $U \subset M$. We will develop an analogue of this to the setting of toric fibrations. Before continuing, however, we need to explain the Lie theory we will be using.

3.1.1 Theory of Semi-simple Lie Groups

The exposition here is not a complete treatment of the subject, but it does set down terminology and provides a reasonable review. For those interested, see [Humphreys, 1972; Sepanski, 2007] for a more detailed treatment of the theory.

Let G be a semisimple Lie group and let $T \subset G$ be a maximal torus of dimension n. Let $\mathfrak{t} \subset \mathfrak{g}$ denote the corresponding Lie algebras. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$. Let κ be the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Since G is semisimple, κ is a non-degenerate bilinear form, which means that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^{\perp},$$

where $\mathfrak{t}^{\perp}$ is the perpendicular space to $\mathfrak{t}$ with respect to κ. By $\mathbb{C}$-linearity, we also have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}^{\perp}.$$

Let $\Delta \subset \mathfrak{t}_{\mathbb{C}}^{*}$ be the finite set of *roots* of $\mathfrak{g}_{\mathbb{C}}$ and let

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} \right),$$

be the weight space decomposition of $\mathfrak{g}_{\mathbb{C}}$. By choosing a system of positive roots Δ^{+} and negative roots Δ^{-}, we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in \Delta^{+}} \mathfrak{g}_{\alpha} \right) \oplus \left(\bigoplus_{\alpha \in \Delta^{-}} \mathfrak{g}_{\alpha} \right). \tag{3.1.1}$$

For each $\alpha \in \Delta$ there are *real* elements $V_{\alpha}, W_{\alpha}, H_{\alpha} \in (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]) \cap \mathfrak{g}$, such that

$$[W_{\alpha}, V_{\alpha}] = 2H_{\alpha}, \qquad [V_{\alpha}, H_{\alpha}] = 2W_{\alpha}, \qquad [H_{\alpha}, W_{\alpha}] = 2V_{\alpha}. \tag{3.1.2}$$

Furthermore, $\alpha(-H_{\alpha}) = 2$, for each $\alpha \in \Delta^{+}$. Let $\{\alpha_1, \ldots, \alpha_n\} \subset \Delta^{+}$ be the set of *simple roots* and not that given any $\alpha \in \Delta^{+}$, we have that $\alpha = M_{\alpha}^{j} \alpha_j$ for some non-negative integers M_{α}^{j}. Let $H_j = H_{\alpha_j}$ and note that

$$H_{\alpha} = M_{\alpha}^{j} H_j. \tag{3.1.3}$$

These are invariants which uniquely define the Lie algebra G and we will strongly make use of them in our computations.

3.1.2 $SU(2)$-example

Let us elucidate these objects by considering the case of $G = SU(2)$. One computes that $su(2)$ is given by complex 2×2 matrices M such that $\mathrm{tr}(M) = 0$ and $M^* = -M$ (where $*$ is the conjugate transpose). We have that $su(2)$ is given by matrices N of the form

$$N = \begin{bmatrix} i\theta & \alpha \\ -\overline{\alpha} & -i\theta \end{bmatrix}, \tag{3.1.4}$$

where $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{C}$. The space $\mathfrak{t}$ is given by matrices N given in (3.1.4) with $\alpha = 0$. The space $su(2) \otimes_{\mathbb{R}} \mathbb{C} = sl(2, \mathbb{C})$ is the space of 2×2 trace-free complex matrices. The space $\mathfrak{t}_{\mathbb{C}}$ is given by diagonal trace-free 2×2 complex matrices.

The Killing form κ on $su(2)$ is given by $\kappa(M, N) = 4\mathrm{tr}(MN)$ and the Killing form on $sl(2, \mathbb{C})$ is given by the same formula. One can directly compute that the space $\mathfrak{t}^{\perp}$ is given by matrices of the form N in (3.1.4) with $\theta = 0$. This implies that $\mathfrak{t}_{\mathbb{C}}^{\perp}$ is given by 2×2 complex matrices with zero diagonal.

To compute the roots of the semi-simple Lie algebra $sl(2, \mathbb{C})$, we need to find the weights of the adjoint action of $\mathfrak{t}_{\mathbb{C}}$ on $sl(2, \mathbb{C})$. Define a $\mathbb{C}$-basis $\{h, e, f\}$ of $sl(2, \mathbb{C})$ by the following

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

One can directly compute that $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. This shows that if $\alpha_+ \in \mathfrak{t}_{\mathbb{C}}^*$ is defined by $\alpha_+(h) = 2$ and $\alpha_- \in \mathfrak{t}_{\mathbb{C}}^*$ is defined by $\alpha_-(h) = -2$, then the root space decomposition of $sl(2, \mathbb{C})$ is given by

$$sl(2, \mathbb{C}) = \mathbb{C} \cdot h \oplus \mathbb{C} \cdot e \oplus \mathbb{C} \cdot f.$$

Define H, V, and W in $su(2)$ by

$$H = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad W = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{3.1.5}$$

One can check that the same relations as in (3.1.2) hold for these vectors.

Remark 3.1.1. The vectors H, V, and W in (3.1.2) are called the *standard $SU(2)$ triple*. Hence (3.1.2) is just a reflection of the important fact that such a triple exists for every weight in semi-simple Lie algebras. Using (3.1.5), we see that this allows us to identify subalgebras in $\mathfrak{g}$ which have the form of $SU(2)$. The $SU(2)$-triples are commonly used to extend arguments from $SU(2)$ to arbitrary semi-simple Lie algebras and are of paramount importance to the theory.

3.1.3 The Complex Structure on $G \times_T M$

As stated in the introduction, $G \times_T M$ and G/T are in fact complex manifolds. To see this let $G_{\mathbb{C}}$ be a complexification of G and let $T_{\mathbb{C}}$ be the corresponding complexification of T. Next let $B := T_{\mathbb{C}}A$ be the Borel subgroup of $G_{\mathbb{C}}$, where A defined by

$$A = \exp\left(\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha\right),$$

where $\mathfrak{g}_\alpha$ is as in (3.1.1). As differentiable manifolds we have that $G/T \cong G_{\mathbb{C}}/B$ and that the left action of G acts by biholomorphisms. The action of $T_{\mathbb{C}}$ on M extends to an action of B on M by requiring $g \cdot x = x$ for all $g \in A$ and $x \in M$. With this action we have that $G \times_T M \cong G_{\mathbb{C}} \times_B M$ as differentiable manifolds and have once again that the left G-action acts by biholomorphisms.

Though one can simultaneously employ both descriptions of $G \times_T M$, it is easier to stay in the real setting. To do that, however, we will need to understand how the complex structure J on $G \times_T M$ acts in terms of the real geometry. This is more naturally worked out in a nice local description which we will first explain in the next section.

3.1.4 The Local Description of $G \times_T M$

The geometry of toric varieties is simplified greatly by the fact that they contain an open dense complex torus. We use this fact to find a analogous local description of toric fibrations.

Let $\{H_j, V_\alpha, W_\alpha \mid 1 \le j \le n, \alpha \in \Delta^+\}$ be the real basis of $\mathfrak{g}$ as described in (3.1.2). The elements $e^{tH_j} \in T \subset G$, for $t \in \mathbb{R}$, provide coordinates for the T-action on M. Let $U \subset M$ be the dense open complex torus, and choose coordinates $(z_1, \ldots, z_n)$ on U identifying U with $(\mathbb{C}^*)^n$ with the extra restriction that

$$e^{tH_j} \cdot (z_1, \ldots, z_n) = (z_1, \ldots, e^{it} z_j, \ldots, z_n).$$

Using these coordinates, we can identify $G \times_T U$ with the space $G \times (\mathbb{R}^+)^n$ by the mapping

$$[g, x] = \Big[g, (r^1 e^{it^1}, \dots, r^n e^{it^n})\Big] \mapsto \Big(g \cdot e^{t^1 H_1} \cdots e^{t^n H_n}, (r^1, \dots, r^n)\Big).$$

Note that the G-action on $G \times_T U$ becomes the standard left G-action on the first coordinate of $G \times (\mathbb{R}^+)^n$. This means that we can extend $\{H_j, V_\alpha, W_\alpha \mid 1 \le j \le n, \alpha \in \Delta^+\}$ to left-invariant vector fields on $G \times (\mathbb{R}^+)^n$. Denote by $\{\nu^j, dV^\alpha, dW^\alpha\}$ the left-invariant 1-forms on G dual to $\{H_j, V_\alpha, W_\alpha\}$. Note that $\nu^1, \dots, \nu^n$ are called the *fundamental weights*. Let $\frac{1}{r^j}\frac{\partial}{\partial r^j}$ and $r^j dr^j$ be the left-invariant vector fields and one-forms on $G \times (\mathbb{R}^+)^n$ coming from the second coordinate.

We are now in the position to describe the complex structure J on $G \times_T M$. Since G acts by biholomorphisms, we only need to understand J's action on the vector fields H_j, V_α, W_α, and $\frac{1}{r^j}\frac{\partial}{\partial r^j}$. Local computations result in the following:

Lemma 3.1.2. *The complex structure J on $G \times_T M$ satisfies $J(V_\alpha) = W_\alpha$ and $J(\frac{1}{r^j}\frac{\partial}{\partial r^j}) = H_j$.*

Next we would like to better understand the geometry of $G \times_T M$ coming from the left-invariant 1-forms.

Lemma 3.1.3. *We have that $d\nu^j = 2M^j_\alpha dV^\alpha \wedge dW^\alpha$, where M^j_α is given by* (3.1.3).

Proof. Let Y_1, Y_2 be two left-invariant vector fields on $G \times_T U$. One can compute that $d\nu^j(V, W) = V(\nu^j(W)) - \nu^j([V, W]) = -\nu^j([V, W])$, where we used the fact that $V(\nu^j(W)) = 0$, since $\nu^j(W)$ is G-invariant. Next note that out of the vector fields H_j, V_α, W_α, and $\frac{1}{r^j}\frac{\partial}{\partial r^j}$, the only non-zero commutation relations are given by (3.1.2). This means that for a fixed α,

$$d\nu^j(V_\alpha, W_\alpha) = -\nu^j([V_\alpha, W_\alpha]) = -\nu^j(H_\alpha) = -\nu^j(M^k_\alpha H_k) = 2M^j_\alpha,$$

which proves the lemma. □

We are finally in the position to give a local description of Ω.

Proposition 3.1.4. *There exists a moment map μ for ω such that on $G \times_T U$ we have*

$$\Omega = d\mu_j \wedge \nu^j + 2M^j_\alpha \mu_j dV^\alpha \wedge dW^\alpha. \tag{3.1.6}$$

Proof. First let μ' be any moment map for ω and extend it to $G \times_T U$ by G-symmetry as above. Let $\nu = d(\mu'_j \nu^j) = d\mu'_j \wedge \nu^j + \mu'_j d\nu^j$. By combining Lemmas 3.1.2 and 3.1.3 we see that ν is in fact a G-invariant (1, 1)-form. This implies that $\chi = \Omega - \nu$ is also a G-invariant (1, 1)-form. Furthermore, χ restricts to 0 on U. This means that $\chi = \pi^*(\eta)$ for some G-invariant (1, 1)-form on G/T. Combining the facts that G/T is a homogeneous space and our knowledge of the complex structure given in Lemma 3.1.2, we have that $\chi = c_j d\nu^j = c_j M^j_\alpha dV^\alpha \wedge dW^\alpha$ for some constants c_j. If we define $\mu = \mu' + c$, we have that $\Omega = d(\mu_j \nu^j)$ which combined with Lemma 3.1.3 proves the proposition. □

3.2 The Geometry of $G \times P$

Our next goal is to extend much of the theory of Section 2.2.1 to the setting of toric fibrations. In the previous section, we restricted $G \times_T M$ to $G \times_T U$ which allowed us to conclude many useful properties about Ω. However, we can also use μ to identify $G \times_T M$ with $G \times P$. The goal of this section is to entirely transplant the geometry of $G \times_T M$ to this setting.

Define the mapping $\exp(w^1, \ldots, w^n) = (e^{w^1}, \ldots, e^{w^n}) = (r^1, \ldots, r^n)$ as in Section 2.2.1. In these coordinates, Ω is given by

$$\begin{aligned}\Omega &= \frac{\partial \mu_k}{\partial w^j} dw^j \wedge \nu^j + 2M^j_\alpha \mu_j dV^\alpha \wedge dW^\alpha \\ &= \frac{\partial^2 \phi}{\partial w^j \partial w^k} dw^j \wedge \nu^j + 2M^j_\alpha \mu_j dV^\alpha \wedge dW^\alpha,\end{aligned}$$

where $\phi(w)$ is the convex function satisfying $\phi_{jk} = \frac{\partial \mu_k}{\partial w^j}$. When pushed-forward to $G \times P$ under the map $\mu_k = x_k$, Ω takes the form

$$\Omega = dx_j \wedge \nu^j + 2M^j_\alpha x_j dV^\alpha \wedge dW^\alpha.$$

Since Ω is a Kähler form, the corresponding Riemannian metric g is related to Ω be $g(\cdot, \cdot) = \Omega(\cdot, J(\cdot))$. Hence we need to understand the complex structure J on $G \times P^\circ$. Translating Lemma 3.1.2 into the w-coordinates, we have that $J(\frac{\partial}{\partial w^j}) = H_j$. The push-forward of $\frac{\partial}{\partial w^j}$ to P° is given by

$$\mu_* \left(\frac{\partial}{\partial w^j} \right) = \frac{\partial x_l}{\partial w^j} \frac{\partial}{\partial x_l} = \sum_l \frac{\partial^2 \phi}{\partial w^l \partial w^j} \frac{\partial}{\partial x_l}. \tag{3.2.7}$$

The fact that the final equation involves the function ϕ is not very satisfactory, since we would prefer to have a result given entirely in terms of the polytope and functions defined on it. To that end, we have the following simple lemma relating ϕ to its symplectic potential u.

Lemma 3.2.1. *The matrices* $(\frac{\partial^2\phi}{\partial w^j \partial w^k})_p$ *and* $(\frac{\partial^2 u}{\partial x_j \partial_k})_q$ *are inverse to each other at points* p *and* q *with* $\mu(p) = q$.

Proof. Recall by Remark 2.2.10 that u is defined by $u(x) = x \cdot w - \phi(w(x))$, where w and x are identified by μ. Hence if we differentiate we see that

$$\frac{\partial u}{\partial x_j} = w^j + x_l \frac{\partial w^l}{\partial x_j} - \frac{\partial \phi}{\partial w^l}\frac{\partial w^l}{\partial x_j} = w^j,$$

where we used that $x_l = \frac{\partial \phi}{\partial w^l}$. Thus if we differentiate again we see that $\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial w^j}{\partial x_k}$. However, we know that $\frac{\partial x_k}{\partial w^j} = \frac{\partial^2 \phi}{\partial w^k \partial w^j}$. Finally, the chain rule tells us that $(\frac{\partial x_k}{\partial w^j})$ and $(\frac{\partial w^j}{\partial x_k})$ are inverse to each other as matrices and the proof is complete. □

Combining the previous lemma with Equation 3.2.7, we see that on $G \times P^\circ$ the complex structure is given by $J\left(u^{jk}\frac{\partial}{\partial x_k}\right) = H_j$ and $J(V_\alpha) = W_\alpha$. This allows us to recognize the Riemannian metric g on $G \times P$ which corresponds to Ω by the requirement that $g(\cdot,\cdot) = \Omega(\cdot, J(\cdot))$. This implies that $g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) = \Omega(\frac{\partial}{\partial x_j}, u_{kl}H_l) = u_{jk}$. Similarly, $g(H_j, H_k) = u^{jk}$. Furthermore, we have that $g(V_\alpha, V_\beta) = g(W_\alpha, W_\beta) = 0$, for all α, β, and that $g(V_\alpha, W_\beta) = \delta_{\alpha\beta} 2M^j_\alpha x_j$. From this we conclude that

$$g = \sum_{j,k} u_{jk} dx_j dx_k + \sum_{j,k} u^{jk} dH^j dH^k + 2x_j M^j_\alpha \left(dV^\alpha dV^\alpha + dW^\alpha dW^\alpha\right). \tag{3.2.8}$$

The Duistermaat-Heckman polynomial W satisfies the property that $\Omega^{n+N} = cW d\mu$, where $d\mu$ is the standard Lebesgue volume form and c is a positive constant. In our case we can easily compute that

$$\Omega^N = cW \left(\bigwedge_j dx_j \wedge dH^j\right) \wedge \left(\bigwedge_\alpha dV^\alpha \wedge dW^\alpha\right), \tag{3.2.9}$$

where

$$W = \prod_{\alpha \in D^+} M^j_\alpha x_j, \tag{3.2.10}$$

giving us the local form of the Duistermaat-Heckman polynomial.

Finally, we would like to find the Laplacian Δ_g for G-invariant functions on $G \times P$. In the $(\frac{\partial}{\partial x}, H, V, W)$-frame we see that $\sqrt{|\det(g)|} = cW$, where W is the Duistermaat-Heckman polynomial and c is a positive constant. Hence the Laplacian Δ_g on G-invariant functions h is given by

$$\Delta_g(h) = -W^{-1}\frac{\partial}{\partial x_k}\left(Wu^{jk}\frac{\partial h}{\partial x_j}\right). \tag{3.2.11}$$

3.3 Scalar Curvature Equation

We are finally in the position to employ our work in the previous sections to give a proof of Theorem 1.

Let g be the Riemannian metric corresponding to Ω. In local holomorphic coordinates, the scalar curvature S of g is

$$S = -g^{j\bar{k}}\frac{\partial^2}{\partial z^j \partial \bar{z}^k}\big(\log\det(g_{\bar{b}a})\big).$$

Working in holomorphic coordinates on $G \times_T M$ brings many difficulties and hence we would like to translate this equation into the real setting on $G \times P$. First recall that the operator $-g^{j\bar{k}}\frac{\partial^2}{\partial z^j \bar{z}^k} = \frac{1}{2}\Delta_g$—the Riemannian Laplacian. Next we would like to write the function $\log\det(g_{\bar{b}a})$ in terms more compatible with the vector fields $(H_j, V_\alpha, W_\alpha, \frac{\partial}{\partial x_j})$. If we let χ be a local, non-vanishing, holomorphic $(N,0)$-form on $G \times P$—i.e. a holomorphic section of the anti-canonical bundle—then $\left|\frac{\Omega^N}{\chi \wedge \bar{\chi}}\right|$ is a smooth function and

$$S = \frac{1}{2}\Delta_g\big(\log\det(g_{\bar{b}a})\big) = \frac{1}{2}\Delta_g\left(\log\left|\frac{\Omega^N}{\chi \wedge \bar{\chi}}\right|\right). \tag{3.3.12}$$

We computed Ω^N in (3.2.9) and Δ_g in (3.2.11) and thus the final piece we need is to find a candidate for χ.

3.3.1 An Explicit Section of the Anti-canonical Bundle of $G \times_T M$

In order to find a candidate for χ, we will first find a holomorphic $(N,0)$ vector field. Note that on $G \times_T U$, the holomorphic vector fields have nothing to do with any specific metric g. However, we can use the fact that the metric g is Kähler to find χ. To simplify the computations we may assume that our original Kähler form equals $\omega_E = \sum_{j=1}^n i dz^j \wedge d\bar{z}^j$ corresponding to the standard Euclidean metric. In that case, $\phi_E(w^1, \ldots, w^n) = \frac{1}{2}(e^{2w^1}, \ldots, e^{2w^n})$ and the moment map $D\phi_E : G \times (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n$ is given by $D\phi_E = (e^{2w^1}, \ldots, e^{2w^n}) = (y_1, \ldots, y_n)$. (The coordinates are denoted by $(y_1, \ldots, y_n)$ to stress that we are no longer working on the original polytope P.) The Legendre transform of ϕ_E is given by $u_E(y_1, \ldots, y_n) = \frac{1}{2}\sum_{j=1}^n (y_j \log(y_j) - y_j)$. The Hessian $H(u_E)$ of u_E is given by the diagonal matrix

$$H(u_E) = \mathrm{Diag}\left(\frac{1}{2y_1}, \ldots, \frac{1}{2y_n}\right).$$

The moment map sends the vector field $\frac{\partial}{\partial w^j}$ to $2y_j\frac{\partial}{\partial y_j} =: Y_j$. The vector fields V_α, W_α, H_j all get sent to themselves. Recall that $J(Y_j) = H_j$ and $J(V_\alpha) = W_\alpha$. In the $(Y_j, H_j, V_\alpha, W_\alpha)$-frame, we have

$$g_E = 2y_j(dY^j dY^j + dH^j dH^j) + 2y_j M_\alpha^j (dV^\alpha dV^\alpha + dW^\alpha dW^\alpha),$$

where M_α^j is given by 3.1.3.

By computing the Christoffel symbols of the Levi-Civita connection D, one sees that

$$D_{Y_j}Y_k = \delta_{jk}Y_j, \qquad D_{Y_j}H_k = D_{H_k}Y_j = \delta_{jk}H_j, \qquad D_{H_j}H_k = -\delta_{jk}Y_j. \tag{3.3.13}$$

Further computations show (here k and α are fixed!)

$$D_{Y_k}(V_\alpha) = \frac{M_\alpha^k y_k}{\sum_j M_\alpha^j y_j}V_\alpha, \qquad D_{Y_k}(W_\alpha) = \frac{M_\alpha^k y_k}{\sum_j M_\alpha^j y_j}W_\alpha. \tag{3.3.14}$$

This shows that

$$D_{(\sum_j Y_j)}(V_\alpha) = V_\alpha, \qquad D_{(\sum_j Y_j)}(W_\alpha) = W_\alpha, \qquad D_{(\sum_j Y_j)}H_\alpha = H_\alpha. \tag{3.3.15}$$

The vector fields H_j, V_α, W_β do not commute, and hence the Christoffel symbols involving them depend on the Lie algebra structure. One can compute that (j and α are fixed!)

$$D_{V_\alpha}H_j = \frac{M_\alpha^j y_j}{\sum_k M_\alpha^k y_k}W_\alpha, \qquad D_{W_\alpha}H_j = -\frac{M_\alpha^j y_j}{\sum_k M_\alpha^k y_k}V_\alpha. \tag{3.3.16}$$

Next we use that $J(Y_j) = H_j$ and $J(V_\alpha) = W_\alpha$ to define smooth sections s_j and t_α of the holomorphic tangent bundle of $G \times (\mathbb{R}^+)^n$ by $s_j = Y_j - iH_j$ and $t_\alpha = V_\alpha - iW_\alpha$. A smooth $(N, 0)$-vector field is given by $\rho = (\bigwedge_j s_j) \wedge (\bigwedge_\alpha t_\alpha)$. We would like to find a smooth function f on $G \times (\mathbb{R}^+)^n$ such that $f\rho$ is holomorphic. Since g_E is Kähler we have that the Chern and Levi-Civita connections coincide. Hence we need to find a function f such that $D_{\bar{s}_j}(f\rho) = 0$ for all j and $D_{\bar{t}_\alpha}(f\rho) = 0$ for all α.

Equations (3.3.13)-(3.3.16) show that $D_{\bar{s}_j} s_k = D_{\bar{t}_\alpha} s_k = 0$ for all j and all α—i.e. that the sections s_j are holomorphic. Further computations show that $D_{\bar{s}_j}(t_\alpha) = J[H_j, V_\alpha] + i[H_j, V_\alpha]$. Inspection of the Christoffel symbols shows that for all $\alpha \neq \beta$, there exist smooth functions h^γ, with $h^\beta = 0$, such that $D_{\bar{t}_\alpha}(t_\beta) = h^\gamma t_\gamma$. Furthermore, $D_{\bar{t}_\alpha}(t_\alpha) = -2s_\alpha =: -2\sum_k s_k$. Taken together, these facts imply that $D_{\bar{t}_\alpha}(\rho) = 0$ for all α and that $D_{\bar{s}_j}\rho = c_j\rho$ for some constant c_j. This means that the function f must satisfy the requirement that $\bar{s}_j(f) = -c_j f$ for all j and that $\bar{t}_\alpha(f) = 0$ for all α. If we prescribe that f be H-invariant, what we need is for $2y_j \frac{\partial f}{\partial y_j} = -c_j f$. The function $f = e^{-\frac{1}{2}\sum_l c_l \log(y_l)}$ satisfies these requirements.

To compute c_j we need to better understand $J[H_j, V_\alpha]$. We have that $J[H_j, V_\alpha] = -J\alpha(H_j)W_\alpha = \alpha(H_j)V_\alpha$. Hence we have that $D_{\bar{s}_j} t_\alpha = \alpha(H_j)t_\alpha$. This means that $c_j = \sum_{\alpha\in\Delta^+} \alpha(H_j)$. But we have that $\sum_{\alpha\in\Delta^+} \alpha = 2\rho$, where ρ is the Weyl vector. Since $\rho(H_j) = 1$, for all j, we have $c_j = 2$, for all j. Hence we have

$$f = e^{-\sum_l \log(y^l)}. \tag{3.3.17}$$

The dual of this form gives us a candidate for χ. This means that

$$|\chi \wedge \overline{\chi}| = f^{-2}\left(\bigwedge_{j=1}^{n} dY_j \wedge dH_j\right) \wedge \cdots \wedge \left(\bigwedge_{\alpha\in\Delta^+} dV_\alpha \wedge dW_\alpha\right).$$

If we pull this form back to $G \times P$ and write it in $(\frac{\partial}{\partial x}, H, V, W)$-frame, we see that

$$|\chi \wedge \overline{\chi}| = \det(u_{jk}) f^{-2}\left(\bigwedge_{j=1}^{n} dx^j \wedge dH_j\right) \wedge \cdots \wedge \left(\bigwedge_{\alpha\in\Delta^+} dV_\alpha \wedge dW_\alpha\right). \tag{3.3.18}$$

3.3.2 Finishing the Computation

By inserting (3.2.9), (3.2.11), and (3.3.18) into (3.3.12), we see that the scalar curvature S is given by

$$S = -\frac{1}{2}W^{-1}\frac{\partial}{\partial x_k}\left(Wu^{jk}\frac{\partial}{\partial x_j}\left(\log W - \log\det(u_{ab}) + 2\log(f)\right)\right).$$

First note that

$$-W^{-1}\frac{\partial}{\partial x_k}\left(Wu^{jk}\frac{\partial}{\partial x_j}(\log W)\right) = -W^{-1}(u^{jk})_k W_j - W^{-1}u^{jk}W_{jk}. \tag{3.3.19}$$

Next note that

$$-W^{-1}\frac{\partial}{\partial x_k}\left(Wu^{jk}\frac{\partial}{\partial x_j}(\log\det(u_{ab}))\right) = W^{-1}W_k u^{jk}u^{ab}u_{abj} + (u^{jk}u^{ab}u_{abj})_k. \tag{3.3.20}$$

Finally note that in (3.3.17), f is written in the $\frac{\partial}{\partial y}$-frame. In the $\frac{\partial}{\partial w}$-frame,

$$2\log(f) = -2\sum_l \log(y^l) = -4\sum_l w^l.$$

Furthermore, the operator $u^{jk}\frac{\partial}{\partial x_j}$ transforms to $\frac{\partial}{\partial w^k}$ and hence

$$u^{jk}\frac{\partial}{\partial x_j}(2\log(f)) = -4.$$

This means that

$$-\frac{1}{2}W^{-1}\frac{\partial}{\partial x_k}\left(Wu^{jk}\frac{\partial}{\partial x_j}(2\log(f))\right) = 2\sum_k \frac{\partial}{\partial x_k}\log W. \tag{3.3.21}$$

If we sum (3.3.19) and (3.3.20), we get $-W^{-1}(Wu^{jk})_{jk}$. Hence the scalar curvature is given on P by the equation

$$S = -\frac{1}{2}W^{-1}(Wu^{jk})_{jk} + f_G \tag{3.3.22}$$

where $f_G = 2\sum_k \frac{\partial}{\partial x^k}\log W$, which proves Theorem 1.

3.4 Line Bundles Over $G \times_T M$

Let $P \subset \mathfrak{t}^*$ be a Delzant polytope. In Section 3.1.4, we used the fundamental weights $\nu^1, \ldots, \nu^n$ as a basis to identify $\mathfrak{t}^*$ with $\mathbb{R}^n$. Next assume that the extreme points of P lie on the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. This means that we can apply Proposition 2.3.3 to construct a smooth toric manifold (M, ω) and a positive line bundle L over M with a compatible linearized action such that $\omega \in c_1(L)$. Furthermore, we can find a compatible moment map $\mu : M \to \mathfrak{t}^*$ whose image is exactly P.

Next we would like to fit G into this picture. We have an action of T on M and a compatible action of T on L. This means that $G \times_T L$ is a line bundle over $G \times_T M$

such that the pair has a compatible holomorphic G-action. Note that $\omega = -i\partial\overline{\partial}\log h$ for some T-invariant metric h on L. We can extend this to a metric H on $G \times_T L$ by G-invariance. Let $\Omega = -i\partial\overline{\partial}\log H$ on $G \times_T M$ and note that $\Omega|_M = \omega$. We claim next that $\Omega = d\mu_j \wedge d\nu^j + 2M^j_\alpha \mu_j dV^\alpha \wedge dW^\alpha$. We have immediately the following lemma:

Lemma 3.4.1. *The line bundle $G \times_T L$ is positive if and only if $P \subset \mathbb{R}^n$ lies in the first quadrant.*

Proof. The line bundle $G \times_T L$ is positive if and only if Ω is positive. Since $M^j_\alpha = \delta_{jk}$ if $\alpha = \alpha^k$ is a simple root, we see that Ω is positive if and only if μ_j is strictly positive on all of $G \times_T M$. □

Next we would like to understand the space $H^0(G \times_T M, G \times_T L)$ and more specifically its dimension. Note that $H^0(G \times_T M, G \times_T L)$ is a G-representation. As G-representations, we have that

$$H^0(G \times_T M, G \times_T L) = \bigoplus_{\lambda \in P} H^0(G \times_T M, G \times_T L_\lambda), \tag{3.4.23}$$

where $L_\lambda \subset L$ spanned by x_λ on which T acts by $\alpha \cdot x_\lambda = \alpha^\lambda x_\lambda$. We now cite the Weyl Dimension Formula (see [Sepanski, 2007] Theorem 7.32):

Theorem (Weyl Dimension Formula)**.** *Let G be a compact connected Lie group with a maximal torus G. If $V(\lambda)$ is the irreducible representation of G with highest weight λ, then*

$$\dim V(\lambda) = \prod_{\alpha \in \Delta^+} \frac{\kappa(\rho + \lambda, \alpha)}{\kappa(\rho, \alpha)}, \tag{3.4.24}$$

where κ is the Killing form and the Weyl vector $\rho = \sum_{j=1}^n \nu^j$ is the sum of the fundamental roots.

Note that $\kappa(\nu^j, \alpha^k) = \delta_{jk}$. Also note that $H^0(G \times_T M, G \times_T L_\lambda)$ is the irreducible representation of highest weight λ. Hence combining (3.4.23) and (3.4.24) we conclude the following:

Proposition 3.4.2. *Let $P \subset \mathfrak{t}^*$ be a Delzant polytope. Identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ by taking the fundamental weights $\nu^1, \ldots, \nu^n$ as a basis and assume that the extreme vertices of P lie on* *the lattice* $\mathbb{Z}^n$. *Let* $(G \times_T M, G \times_T L)$ *be the toric fibration and line bundle constructed from* P. *Then we have*

$$h^0(G \times_T M, G \times_T L) = \sum_{\lambda \in P} \left(\prod_{\alpha \in \Delta^+} \frac{(1 + \lambda_j) M_\alpha^j}{\prod_{j=1}^n M_\alpha^j} \right).$$

Chapter 4

Mabuchi Functional

The culmination of the previous chapter's study of toric fibrations is the scalar curvature equation given in Theorem 1. The purpose of the rest of the chapters is to make partial progress in solving it. A common first strategy when studying a PDE is to find a functional whose critical points necessarily solve the PDE in question. In the context of the complex constant scalar curvature equation, Mabuchi found such a functional in [Mabuchi, 1986] for general compact Kähler manifolds. The theory in his work applies to our problem, but since we are formulating the scalar curvature equation in terms of symplectic potentials, we need to write the Mabuchi functional purely in terms of the polytope. P and the symplectic potentials u. In [Donaldson, 2002], Donaldson does exactly this for the case of toric varieties. In this chapter, we extend Donaldson's work to the case of toric fibrations.

Let $S = -W^{-1}(Wu^{jk})_{jk} + f_G$ be the scalar curvature of u as in Theorem 1. As we will show, the vector field $(u^{jk})_j$ is smooth up to the boundary of P. Given a face F of P, let ν_F be the smallest integral, inward-pointing normal vector to F. We will show that the restriction of $(\nu_F)_k(u^{jk})_j$ to F is a constant σ_F. Finally define $d\sigma$ to be the positive measure on ∂P given by the requirement that $d\nu_F \wedge d\sigma = \pm d\mu$ when restricted to the face F, where $d\mu$ is the standard Lebesgue measure on $\mathbb{R}^n$.

We will show that $\int_P SW d\mu = \int_{\partial P} W d\sigma + \int_P f_G W d\mu$ which is a constant independent of u. This implies the average scalar curvature $a = \int_P SW d\mu / \int_P W d\mu$ is independent of u also. Hence the functional $\mathcal{F}$ in 1.0.8 is well-defined. The main result of this chapter is the proof of 2.

The theorem points towards one possible method of solving this equation, which is to find the conditions under which $\mathcal{F}$ is guaranteed to have critical points. A necessary condition for a convex function to have a critical point is that it be bounded below. The boundedness properties of $\mathcal{F}$ are critically linked to the linear functional $\mathcal{L}$ defined for any smooth function f by

$$\mathcal{L}(f) = \int_{\partial P} fW d\sigma - \int_P (a - f_G) fW d\mu. \tag{4.0.1}$$

We will show that the boundedness of $\mathcal{F}$ breaks down to certain vanishing conditions positivity properties of $\mathcal{L}$.

The chapter is outlined as follows. In the first section, we give a slightly more general analytic description of the constant scalar curvature problem. In the second section, we collect some basic computations necessary to study $\mathcal{F}$ and $\mathcal{L}$. In the third section, we give a proof of Theorem 2. In the fourth section, we present some necessary and some sufficient conditions for the boundedness of $\mathcal{F}$.

4.1 Analytic Setup

We know by Theorem 2 (Delzant's Theorem) that the polytope corresponding to $(G \times_T M, \Omega)$ satisfies certain algebraic conditions. However, for our current purposes, these conditions are unnecessarily strong. It is for this reason that, in this chapter, we put no algebraic restrictions on P. The only requirements on P are that it is a convex polytope in $\mathbb{R}^n$ such that at every vertex there are n edges of P meeting there.

Proposition 2.2.3 that if Ω corresponds to the symplectic potential u on P, then u is of the form $u_P + f$, where u_P is a fixed convex function with certain degeneracy at the boundary and where f is a smooth function on all of P. However, after dropping the algebraic assumptions on P, we no longer have such a natural choice of u_P. Therefore we will next specify a new reference potential to use in our work.

Let $F_1, \ldots, F_N$, be the faces of P and let $\sigma = (\sigma_1, \ldots, \sigma_N)$ be a vector of positive real numbers. For each k, let $d\sigma_k$ be the measure on F_k given as $\sigma_k d\mu_k$, where $d\mu_k$ is the standard (restriction of the) Euclidean measure on F_k. Let $d\sigma$ be the measure on ∂P which is given by $d\sigma_k$ when restricted to each face F_k.

For each face F_k, we let l_k be the affine linear function defined by the requirements that $F_k \subset l_k^{-1}(0)$, $l_k > 0$ on P, and the derivative of $D_{\nu_k} l_k = \sigma_k^{-1}$, where ν_k is the inward-facing norm 1 vector orthogonal to F_k. Hence, we have that $P = \{x \in \mathbb{R}^n \mid l_k(x) \geq 0, \forall k\}$. Using these functions l_k, we define the function u_σ by

$$u_\sigma(x) = \sum_k l_k(x) \log l_k(x). \tag{4.1.2}$$

The function u_σ is a smooth, strictly convex function on the interior of P whose derivatives degenerate at the boundary of P. We next define $\mathcal{S}_\sigma$ as the set of all functions $u = u_\sigma + f$, where $f \in C^\infty(P)$ is smooth up to the boundary, with the additional requirement that the restriction of u to P and to any facet of P have strictly positive Hessian. With this setting in place, our goal is to find analytic conditions under which there exists a function $u \in S_\sigma$ solving

$$\begin{cases} -W^{-1}(Wu^{ij})_{ij} = A, \\ W, A \in C^\infty(P), \\ W > 0 \text{ on } P. \end{cases} \tag{4.1.3}$$

Remark 4.1.1. If P is a Delzant polytope corresponding to $(G \times_T M, \Omega)$, u is the symplectic potential corresponding to Ω (which fixes σ), W is the Duistermaat-Heckman polynomial, and $A = 2(a - f_G)$, then Equation (4.1.3) is the regular scalar curvature equation.

4.1.1 Example Given by $\mathbb{P}^2$

Let $P \subset \mathbb{R}^2$ be the interior of the convex hull of the points $(0,0), (1,0)$, and $(0,1)$. As we computed in the examples in Chapter 2, this is the moment polytope of $\mathbb{P}^2$ corresponding to the Fubini-Study metric. The corresponding symplectic potential on P is given by

$$u = \frac{1}{2}\Big(x \log x + y \log y + (1 - x - y) \log (1 - x - y)\Big).$$

Denote the faces by: $F_1 \subset (x = 0)$, $F_2 \subset (y = 0)$, and $F_3 \subset (x + y = 1)$. In this case, one can compute that the corresponding weights are given by $\sigma_1 = \sigma_2 = 2$ and $\sigma_3 = \sqrt{2}$. Note that while the weights are different, $\int_{F_i} d\sigma_i = 2$, for all $i = 1, 2, 3$.

4.2 Some Basic Computations

Before studying general properties of Equation 4.1.3, we first use this section to gather some necessary technical tools.

Lemma 4.2.1. *Let $u \in \mathcal{S}_\sigma$. Then $u^{ij} \in C^\infty(P)$ up to the boundary for all i, j.*

Proof. Near a vertex p of P, we choose local coordinates given by the inward unit normal vectors orthogonal to the faces meeting at p. Assume for convenience that the faces $F_1, \ldots, F_n$ meet at p. In these coordinates, u is given by

$$u = \sum_{i=1}^{n} \sigma_i^{-1} x_i \log(x_i) + f(x),$$

where $f(x)$ is smooth up to the boundary of $(\mathbb{R}^+)^n$. The Hessian of u is given by

$$\mathrm{Hess}(u) = \begin{bmatrix} \sigma_1^{-1}x_1^{-1} + f_{11} & f_{12} & \cdots & f_{1n} \\ f_{12} & \sigma_2^{-1}x_2^{-1} + f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & \sigma_n^{-1}x_n^{-1} + f_{nn} \end{bmatrix}.$$

The cofactor matrix $\mathcal{U}$ has the property that

$$\mathcal{U}^{ij} = \left(\prod_{i \neq l \neq j} \sigma_l^{-1} x_l^{-1} \right) h^{ij},$$

if $i \neq j$ and

$$\mathcal{U}^{ii} = \left(\prod_{l \neq i} \sigma_l^{-1} x_l^{-1} \right) h^{ii},$$

where the h^{ij} are functions that are smooth up to the boundary of the polytope. The determinant D of $\mathrm{Hess}(u)$ is given by

$$D = \left(\prod_{l=1}^{n} \sigma_l^{-1} x_l^{-1} \right) \Delta,$$

where Δ is a positive function that is smooth up to the boundary. We have that the inverse of the Hessian, u^{ij}, is given by $u^{ij} = D^{-1}\mathcal{U}^{ij}$. Hence we have

$$u^{ij} = \sigma_i \sigma_j x_i x_j h^{ij}, \tag{4.2.4}$$

if $i \neq j$ and we have

$$u^{ii} = \sigma_i x_i h^{ii}. \tag{4.2.5}$$

Hence, given any vertex p of P, there is an open neighborhood U of p such that u^{ij} is smooth up to the boundary of $U \cap \overline{P}$. To finish the proof one needs to consider the behavior of u near the faces away from the vertices. This proof can be adapted straight-forwardly to that situation as well. □

Next we give a lemma which explains the choice of normalization of l_k.

Lemma 4.2.2. *Define the vector field $v^k = -W^{-1}(Wu^{jk})_j$. Then v^k is smooth up to the boundary of P. Furthermore, if we restrict v to ∂P, then the dot product $v \cdot \nu_k = d\sigma_k$ in the sense of measures, where ν_k is the unit outward normal vector to F_k.*

Proof. The fact that v^k is a smooth vector field on all of P is a result of the previous lemma. Next we consider local coordinates near a vertex p as in the previous lemma. On the face F_1 we have that $x_1 \equiv 0$. Equation (4.2.4) shows that $\frac{\partial}{\partial x^1}(u^{1j}) \equiv 0$ on the face F_1 if $1 \neq j$. Furthermore by equation (4.2.4), we see that $\frac{\partial}{\partial x^1}(u^{11})|_{F_1} = \sigma_1 h^{11}|_{F_1}$. The same equations show that $u^{1j}|_{F_j} \equiv 0$ for all j and hence we have that $v^1|_{F_1} = -u^{1j}_j|_{F_1}$. Close inspection of the defining equations of h^{ij} shows that $h^{11}|_{F_1} \equiv 1$. The minus sign in the definition of v is canceled by the minus sign coming from ν_1 being the *outward* normal vector and hence we've shown this to be true on the face F_1 near the vertex p. Since this argument works for all faces, we have proved the lemma near any vertex. Similar arguments can be used to show the lemma holds on the other portions of the face F_1. □

Lemma 4.2.2 lets us integrate by parts to find invariants of (4.1.3).

Lemma 4.2.3. *Let $u \in S_\sigma, f \in C^\infty(P)$ be smooth up to the boundary, and let v be defined as in Lemma 4.2.2. We have that*

$$-\int_P W^{-1}(Wu^{ij})_{ij} f W d\mu = \int_{\partial P} f W d\sigma - \int_P v^j f_j W d\mu$$

and

$$\int_P v^j f_j W d\mu = \int_P u^{ij} f_{ij} W d\mu.$$

Proof. Use the previous Lemma and integration by parts to conclude that

$$\begin{aligned}
-\int_P W^{-1}(Wu^{ij})_{ij} f W d\mu &= \int_P -(Wu^{ij})_{ij} f d\mu \\
&= \int_{\partial P} -W u_i^{ij} f \nu_j + \int_P (Wu^{ij})_i f_j d\mu \\
&= \int_{\partial P} f W d\sigma - \int_P v^j f_j W d\mu.
\end{aligned}$$

Next for the second equality we see that

$$\begin{aligned}
\int_P v^j f_j W d\mu &= -\int_P W^{-1}(Wu^{ij})_i f_j W d\mu \\
&= -\int_P (Wu^{ij})_i f_j d\mu \\
&= \int_P W u^{ij} f_{ij} d\mu - \int_{\partial P} W u^{ij} f_j \nu_i,
\end{aligned}$$

and we note that $u^{ij} f_j \nu_i \equiv 0$ on ∂P which can be seen by similar computations as in the previous lemma. □

Remark 4.2.4. If we choose $f \equiv 1$ in the previous lemma, we see that $\int_P AW d\mu = \int_{\partial P} W d\sigma$ which puts a restriction on the total mass of A. Hence when studying (4.1.3) one knows that the total mass of A is independent of u and instead determined by the boundary conditions coming from σ. This is simply a manifestation of the fact that the total scalar curvature of a Kähler manifold is a constant.

4.3 The Mabuchi Functional

After the setup and basic results derived in the last couple sections, we are in the position to provide a proof of Theorem 2. Define the three functionals $\mathcal{L}_A$, $\mathcal{N}$, and $\mathcal{F}_A$ on the space $\mathcal{S}_\sigma$ by

$$\mathcal{L}_A(u) = \int_{\partial P} uW d\sigma - \int_P AuW d\mu, \tag{4.3.6}$$

$$\mathcal{N}(u) = -\int_P \log\det(u_{jk}) W d\mu, \tag{4.3.7}$$

$$\mathcal{F}_A(u) = \mathcal{N}(u) + \mathcal{L}_A(u). \tag{4.3.8}$$

The local computations in the previous section show that these are all well-defined for any σ and A.

First we study the variational properties of these functionals. Since all functions in $\mathcal{S}_\sigma$ differ by a smooth function, we need only consider linear paths $u_t = u + t\delta u$ where $u \in S_\sigma$ and $\delta u \in C^\infty(P)$. $\mathcal{L}_A$ is a linear functional and hence $\delta\mathcal{L}_A = \mathcal{L}_A$. Using the linear path, it is straight-forward to compute the first and second variations of $\mathcal{N}$:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mathcal{N}(u_t) = -\int_P u^{ij}\delta u_{ij} W d\mu, \tag{4.3.9}$$

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathcal{N}(u_t) = \int_P u^{ia}\delta u_{ab} u^{bj}\delta u_{ij} W d\mu. \tag{4.3.10}$$

This shows that $\mathcal{N}$ (and therefore also $\mathcal{F}_A$) is a convex functional on S_σ. Next we assume that u is a critical point of $\mathcal{F}_A$. If we vary $\mathcal{F}$ by u_t as earlier, we can apply Lemma 4.2.3 to conclude

$$\begin{aligned} 0 &= \left.\frac{\partial}{\partial t}\right|_{t=0} \mathcal{F}_A(u_t) \\ &= -\int_P u^{ij}\delta u_{ij} W d\mu + \mathcal{L}_A(\delta u) \\ &= -\int_P W^{-1}(Wu^{ij})_{ij}\delta u W d\mu - \int_P \delta u A W d\mu. \end{aligned}$$

Since δu is allowed to be any smooth function on P, this shows that if u is a critical point of $\mathcal{F}_A$, then u must satisfy (4.1.3). Since the function $\mathcal{F}$ in Equation 1.0.8, is just a special case of $\mathcal{F}_A$, we have proved Theorem 2.

4.4 Conditions Implying Boundedness of the Mabuchi Functional

In the previous section, we demonstrated that a critical point of the convex functional $\mathcal{F}_A$ necessarily solves (4.1.3), but we still do not know what conditions might guarantee that $\mathcal{F}_A$ actually has a critical point. Due to convexity, a basic necessity would be for $\mathcal{F}_A$ to be bounded below. Note then that if δu is an affine-linear function and $u \in \mathcal{S}_\sigma$, then

$$\mathcal{F}_A(u + t\delta u) = \mathcal{F}_A(u) + t\mathcal{L}_A(\delta u).$$

However, the fact that $u + t\delta u \in \mathcal{S}_\sigma$, for all t, implies $\mathcal{L}_A(\delta u)$ must vanish for all affine functions, if $\mathcal{F}_A$ is to be bounded below. This proves the following:

Proposition 4.4.1. *Let $u \in \mathcal{S}_\sigma$ solve Equation* (4.1.3)*. Then $\mathcal{L}_A(f) = 0$ for all affine-linear functions f.*

Note, however, that the definition of $\mathcal{L}_A$ makes no reference to the function u. This means that the previous proposition provides an obstruction to the existence of constant scalar curvature solutions on toric fibrations. This leads us to the following definition:

Definition 4.4.2. *Let (P, A, σ) be a polytope, a smooth function on P and a vector of positive weights (one for each face of P). Let $\mathcal{L}_A$ be defined by Equation* (4.3.6)*. Then we say that the **Futaki Invariant of** (P, A, σ) **vanishes**, if $\mathcal{L}_A(f) = 0$ for all affine linear functions f.*

Hence in this language, Proposition 4.4.1 simply says that a necessary condition for there to exist a solution to Equation (4.1.3) is for the Futaki invariant of (P, A, σ) to vanish.

Now let us continue trying to understand when $\mathcal{F}_A$ is bounded below. Assume that $\delta u \in C^\infty(P)$ is strictly convex. This implies that $u_t = u + t\delta u \in S_\sigma$ for all $t \geq 0$. We have that

$$\mathcal{L}_A(u_t) = \mathcal{L}_A(u) + t\mathcal{L}_A(\delta u).$$

I.e. $\mathcal{L}_A(u_t)$ is just a line with slope $\mathcal{L}_A(\delta u)$. Next we compute

$$\begin{aligned} \mathcal{N}(u_t) &= -\int_P \log\det((u + t\delta u)_{ij}) W d\mu \\ &= \left(-n \int_P W d\mu\right) \log t - \int_P \log\det(t^{-1}u_{ij} + \delta u_{ij}) W d\mu. \end{aligned}$$

This implies that as $t \to \infty$,

$$\mathcal{F}(u_t) \sim B \log t + Ct + D,$$

where

- $B = -n \int_P W d\mu$,
- $C = \mathcal{L}_A(\delta u)$,
- $D = \mathcal{L}_A(u) - \int_P \log\det(\delta u_{ij}) W d\mu$.

Since B is negative, and the linear term dominates the logarithmic term, we see that the only way for $\mathcal{F}_A$ to be bounded below is for $B = \mathcal{L}_A(\delta u)$ to be positive.

Hence another necessary condition for u to solve Equation (4.1.3) is for $\mathcal{L}_A(f) > 0$ for all non-affine strictly convex functions f. For many arguments one would like to have even more control over this lower bound than this provides. First note that if the Futaki invariant of (P, A, σ) vanishes, we see that $\mathcal{L}_A(f + g) = \mathcal{L}_A(f)$ for any affine linear function g. We can use to property to normalize our functions:

Definition 4.4.3. *A smooth convex function f is said to be* ***normalized*** *at a point $p \in P$ if $f(p) = 0$ and $(\nabla f)(p) = 0$.*

With this normalization on hand, we can specify the desired positivity condition on $\mathcal{L}_A$.

Definition 4.4.4. *Let $p \in P$ be fixed. P is said to satisfy* ***Condition 1****, if there exists a constant $\lambda > 0$ (depending on p), such that for any convex function f which is normalized at p, we have*

$$\mathcal{L}_A(f) \geq \lambda \int_{\partial P} f W d\sigma.$$

The first reason one might expect Condition 1 to be important is its relationship to the Mabuchi Functional:

Proposition 4.4.5. *Let A be a smooth function such that the Futaki Invariant of (P, σ, A) vanishes and assume $\mathcal{L}_A$ satisfies Condition 1. Then $\mathcal{F}_A$ is bounded below on $\mathcal{S}_\sigma$.*

Remark 4.4.6. In the following proof, we will use the fact that $\mathcal{F}_A$ and $\mathcal{L}_A$ are well-defined for all σ to make certain scaling arguments. Hence the fact that the scaling cu of a function $u \in \mathcal{S}_\sigma$ does not stay inside the space $\mathcal{S}_\sigma$ does not cause any problems.

Proof. Let $u \in \mathcal{S}_\sigma$ be fixed and assume that it is normalized. Choose any $v \in \mathcal{S}_\sigma$ and let $B = -W^{-1}(Wv^{ij})_{ij}$ and consider the functional $\mathcal{F}_B$ on $\mathcal{S}_\sigma$. Hence we have set it up so that

$\mathcal{F}_B(v)$ is the minimum of the functional $\mathcal{F}_B$ on all $\mathcal{S}_\sigma$. Next we compute

$$\begin{aligned}
\mathcal{F}_B(v) &\leq \mathcal{F}_B(u) \\
&= \mathcal{F}_B(u) - \mathcal{F}_A(u) + \mathcal{F}_A(u) \\
&= \mathcal{L}_B(u) - \mathcal{L}_A(u) + \mathcal{F}_A(u) \\
&= \int_P (A - B) u W d\mu + \mathcal{F}_A(u) \\
&\leq ||A - B||_{L^\infty} \int_P u W d\mu + \mathcal{F}_A(u) \\
&\leq C||A - B||_{L^\infty} \int_{\partial P} u W d\sigma + \mathcal{F}_A(u) \\
&\leq \lambda^{-1} C ||A - B||_{L^\infty} \mathcal{L}_A(u) + \mathcal{F}_A(u) \\
&= r\mathcal{L}_A(u) - \int_P \log\det(u_{ij}) W d\mu.
\end{aligned}$$

where $r = 1 + \lambda^{-1}C||A - B||_{L^\infty}$. We continue computing

$$r\mathcal{L}(u) - \int_P \log\det(u_{ij}) W d\mu = \mathcal{F}_A(ru) + n\mathrm{Vol}_W(P)\log(r),$$

and hence $\mathcal{F}_A(ru) \geq \mathcal{F}_B(v) - n\mathrm{Vol}_W(P)\log(r)$, and note that r is independent of u. Next use the fact that u is normalized—and hence that $\mathcal{L}_A(u) \geq 0$—to conclude that

$$\frac{d}{dr}\left(\mathcal{F}_A(ru)\right) = -n\mathrm{Vol}_W(P)\frac{1}{r} + \mathcal{L}_A(u) \geq -n\mathrm{Vol}_W(P)\frac{1}{r},$$

which is independent of u. Furthermore,

$$\frac{d^2}{dr^2}\left(\mathcal{F}_A(ru)\right) = n\mathrm{Vol}_W(P)\frac{1}{r^2} \geq 0,$$

meaning that $\mathcal{F}_A(ru)$ is convex as a function of r. Taken together, these facts as well as the fact that $r \geq 1$ imply that $\mathcal{F}_A(u)$ is bounded below by some constant independent of u and hence we are done. □

4.4.1 Independence of Condition 1 on Choice of Where to Normalize

Here we ask the question of how dependent Condition 1 is on the choice of the point $p \in P$. In this section we let p_c denote the center of mass of $(\partial P, W d\sigma)$.

Lemma 4.4.7. *Assume that the Futaki Invariant of (P, σ, A) vanishes and that it satisfies Condition 1 with a constant $\lambda > 0$ for functions normalized at p_c. Let $q \in P$ be arbitrary. Then (P, σ, A) also satisfies Condition 1 with a constant λ_q which we can estimate in terms of λ. Furthermore, this constant λ_q approaches 0 as q approaches the boundary of P.*

Proof. Let f be normalized at p_c. Let l_q be the affine supporting hyperplane of f at the point q. Note that $l_q(q) \geq 0$ and note further that there is a positive constant C_1, depending only on the geometry of (P, σ) and the location of the point q such that $|(\nabla l_q)(q)| \leq C_1 \int_{\partial P} fW d\sigma$. This means that $l_q(p_c) \geq -C \int_{\partial P} fW d\sigma$, for some positive constant C depending only on (P, σ) and q. Define $\lambda_q = \frac{\lambda}{(1+C)}$. Next we have

$$\begin{aligned}
\lambda_q \int_{\partial P} (f - l_q) W d\sigma &= \lambda_q \left(\int_{\partial P} fW d\sigma - l_q(p_c) \right) \\
&\leq \lambda_q \left(\int_{\partial P} fW d\sigma + C \int_{\partial P} fW d\sigma \right) \\
&= \lambda \int_{\partial P} fW d\sigma \\
&\leq \mathcal{L}_A(f) \\
&= \mathcal{L}_A(f - l),
\end{aligned}$$

where in the last step we used that the Futaki Invariant vanishes.

The size of λ_q is dependent on a bound of $|(\nabla f)(q)|$, but this final bound gets worse even for our model functions u_σ as q approaches ∂P and hence λ_q degenerates to 0 at the boundary. □

The previous proof shows in fact that if Condition 1 is satisfied at a single point q in P, then it must also be satisfied at the point p_c (with an even better constant). This fact allows us to conclude the following:

Proposition 4.4.8. *If the Futaki Invariant vanishes and (P, σ, A) satisfies Condition 1 at one point q in P, then it satisfies Condition 1 at every point in P. Furthermore, the constants λ_q can be estimated in terms of each other for different $q \in P$.*

Chapter 5

A Priori Estimates

In the previous chapter, we showed that the scalar curvature equation given in the form of (4.1.3) is the Euler-Lagrange equation of the convex functional $\mathcal{F}_A$ defined in (4.3.8). We then showed in Propositions 4.4.1 and 4.4.5 that if the Futaki Invariant of (P, σ, A) vanishes and Condition 1 holds, then $\mathcal{F}_A$ is bounded below. We will take these results as a starting point in an attempt to solve (4.1.3). More specifically, in this chapter we derive a priori estimates for u solving (4.3.8) under the assumption that the Futaki invariant of (P, σ, A) vanishes and that it satisfies Condition 1.

The chapter is organized as follows. In the first section we focus deriving interior estimates culminating in the proof of Theorem 3. In the second section prove Theorem 4.

5.1 Interior Estimates

Before restricting ourselves to the interior, we have the following L^1-bound on a normalized solution of (4.1.3).

Proposition 5.1.1. *Let u be a normalized solution to* (4.1.3) *and assume* (P, A, σ) *satisfies Condition 1. Then*

$$\int_{\partial P} uW d\sigma \leq \lambda^{-1} n \mathrm{Vol}_W(P).$$

Proof. Applying Condition 1 and integrating by parts as in Lemma 4.2.3 gives

$$\begin{aligned}
\lambda \int_{\partial P} uW d\sigma &\leq \mathcal{L}_A(u) \\
&= \int_{\partial P} uW d\sigma - \int_P AuW d\mu \\
&= \int_{\partial P} uW d\sigma + \int_P (Wu^{ij})_{ij} u d\mu \\
&= \int_{\partial P} uW d\sigma - \int_{\partial P} uW d\sigma + \int_P u^{ij} u_{ij} W d\mu \\
&= n\mathrm{Vol}_W(P).
\end{aligned}$$

□

Due to convexity, Proposition 5.1.1 can immediately be used to derive interior gradient bounds:

Proposition 5.1.2. *Assume that Condition 1 holds and let u be a normalized solution to* (4.1.3)*. Then there exists a constant C depending only on $P, W, n,$ and λ such that ∇u satisfies*

$$|\nabla u| \leq C d^{-(n+1)},$$

where d is the distance to the boundary of P.

Proof. Let x be some point in the interior of P. Due to convexity, the supporting hyperplane to u at x remains below u. Hence there is a constant κ depending on P such that

$$|\nabla u| \leq \kappa d_x^{-(n+1)} \int_P u d\mu.$$

Next note that again due to convexity, $\int_P u d\mu \leq C \int_{\partial P} u d\sigma$, for some C depending on P, σ, and the choice of where u is normalized. Thus if u also satisfies (4.1.3) we can apply Proposition 5.1.1 to get

$$\begin{aligned}
|\nabla u| &\leq \kappa d_x^{-(n+1)} \int_P u d\mu \\
&\leq \kappa d_x^{-(n+1)} C \int_{\partial P} uW d\sigma \\
&\leq \kappa d_x^{-(n+1)} C \lambda^{-1} n \mathrm{Vol}_W(P),
\end{aligned}$$

which proves the result. □

5.1.1 Interior Estimate For Lower Bound of the Determinant

Next we will derive an interior estimate for the lower bound of the determinant of the Hessian of a syplectic potential u solving (4.1.3).

Proposition 5.1.3. *Suppose that u solves* (4.1.3) *for some bounded function A. Then we have that*

$$\det(u_{ij}) \geq C(\sup_P A)^{-n},$$

throughout P, where C is a constant depending on n, P, and W.

Proof. We will focus on getting a lower bound for $L := \log\det(u_{ij})$. One can compute that

$$A = -W^{-1}(Wu^{ij})_{ij} \tag{5.1.1}$$
$$= u^{ij}(L_{ij} - L_iL_j + (\log W)_iL_j + (\log W)_jL_i - W^{-1}W_{ij}). \tag{5.1.2}$$

Let ψ be any function satisfying the properties:

(1) $M_{ij} = \psi_{ij} - \psi_i\psi_j$ is positive definite throughout P,

(2) $L - \psi \to +\infty$ at ∂P

(3) $\log\det(M_{ij}) \geq \psi$.

For example, let $\psi = \frac{\varepsilon}{2}|x|^2 - c$ and note that ε can be chosen small enough (depending on P) so that (1) and (2) hold, and that c can be chosen large enough to make (3) hold as well.

Let p be a minimum of the function $f = L - \log W - \psi$. Thus at p we have $L_i = \psi_i + (\log W)_i$ and that $L_{ij} \geq \psi_{ij} + (\log W)_{ij}$. We can use these two facts to cancel out all of the terms involving W in (5.1.1) when computing at p:

$$A = u^{ij}(L_{ij} - L_iL_j + (\log W)_iL_j + (\log W)_jL_i - W^{-1}W_{ij}) \tag{5.1.3}$$
$$\geq u^{ij}(\psi_{ij} - \psi_i\psi_j) \tag{5.1.4}$$
$$= u^{ij}M_{ij}. \tag{5.1.5}$$

Thus we conclude that $u^{ij}(p)M_{ij}(p) \leq A(p)$. Since u^{ij} and M_{ij} are both positive definite, this forces $A(p) > 0$. By the arithmetic-geometric inequality we have

$$(\det(u^{ij})\det(M_{ij}))^{1/n}(p) \leq n^{-1}u^{ij}M_{ij}(p) \leq n^{-1}A(p).$$

Using the fact that $\det(u^{ij}) = e^{-L}$ we see that

$$e^{-L(p)}\det(M_{ij}(p)) \leq (A(p)/n)^n,$$

or

$$L(p) \geq \log\det(M_{ij}(p)) - n\log\left(A(p)/n\right).$$

For any other point q we have

$$\begin{aligned}(L-\psi-\log W)(q) &\geq (L-\psi-\log W)(p)\\ &\geq \left(\log\det(M_{ij}) - n\log\left(A(p)/n\right) - \psi - \log W\right)(p),\\ &\geq -n\log\left(A(p)/n\right) - \log W(p),\end{aligned}$$

where we used the fact that $\log\det(M_{ij}) \geq \psi$. If we rearrange and exponentiate the previous inequality we see that

$$\det(u_{ij}(q)) \geq n^n A(p)^{-n}\frac{W(q)}{W(p)}e^{\psi(q)} \geq C(\sup_P A)^{-n},$$

where $C = n^n \inf_P W(\sup_P W)^{-1}\inf_P e^{\psi}$. If we choose ψ to be $\psi = \frac{\varepsilon}{2}|x^2| - c$ as explained above, we are done with the proof. □

5.1.2 Interior Estimate For Upper Bound For The Determinant

Proposition 5.1.4. *Let u solve (4.1.3). Assume that $D = \{x \in P \mid u(x) < 0\} \subset\subset P$ and that the boundary ∂D is smooth. Then we have*

$$(\det(u_{ij}))^{1/n} \leq C(-u)^{-1},$$

in D, where C depends on $\max_D(-u), \max_D(0,-A), |\nabla u|^2$, and $|\nabla \log(W)|^2$.

Proof. We use once again the scalar curvature equation as given in (5.1.1) and initially follow the same path as the proof of Proposition 5.1.3. In that proof we studied functions of the form $f = L - \log W - \psi$ and chose ψ in such a way so as to guarantee that f would have a minimum in P where we could apply the maximum principal. In this case, to get the upper bound we instead choose ψ in a way to force f to have a maximum (and this time inside D). Assuming f takes its maximum at a point $p \in D$, at that point we have that

$L_i = \psi_i + (\log W)_i$ and that $L_{ij} \leq \psi_{ij} + (\log W)_{ij}$. The same computation as in (5.1.3) and (5.1.4) (but with the opposite inequality) allows us to conclude that

$$A(p) \leq u^{ij}(p)(\psi_{ij}(p) - \psi_i(p)\psi_j(p)). \tag{5.1.6}$$

We need to find a class of ψ which forces a minimum. Note that since D is bounded away from the boundary of ∂P, L (and $\log W$) is bounded on D. Furthermore, u_i is bounded by a universal constant on D by Proposition 5.1.2. Thus we can make use of u_i in our computations. Let g_{ij} be any constant Euclidean metric with $\det(g_{ij}) = 1$ and let α be some number that will be specified later. We define

$$\psi = -n \log(-u) - \alpha g^{ij} u_i u_j,$$

and note that with this ψ we have that $f(x) \to -\infty$ as $x \to \partial D$. To make use of (5.1.6), we need to compute the derivatives of ψ:

$$u^{ij}\psi_{ij} = \frac{n}{u^2} u^{ij} u_i u_j - \frac{n^2}{u} - 2\alpha g^{ab} u_{ab} - 2\alpha g^{ab} u^{ij} g_a u_{bij}, \tag{5.1.7}$$

$$u^{ij}\psi_i\psi_j = \frac{n^2}{u^2} u^{ij} u_i u_j + \frac{4\alpha n}{u} g^{ab} u_a u_b + 4\alpha^2 g^{ab} g^{cd} u_a u_c u_{bd}. \tag{5.1.8}$$

Note that at the point p, we have

$$u^{ij} u_{ijq} = L_q = \psi_q + (\log W)_q,$$

and hence at p we have

$$g^{pq} u^{ij} u_{ijp} u_q = -n g^{pq} \frac{u_q u_p}{u} - 2\alpha g^{ab} g^{pq} u_a u_{bq} u_p + g^{pq} (\log W)_q u_p. \tag{5.1.9}$$

If we combine equations (5.1.7) - (5.1.9), many terms cancel out and we see that

$$\begin{aligned} A &\leq u^{ij}(\psi_{ij} - \psi_i\psi_j) \\ &= \frac{n - n^2}{u^2} u^{ij} u_i u_j - \frac{n^2}{u} - 2\alpha g^{ab} u_{ab} - \frac{2\alpha n}{u} g^{ab} u_a u_b - 2\alpha g^{pq} (\log W)_q u_p, \end{aligned}$$

at the point p. First note that $\frac{n-n^2}{u^2} u^{ij} u_i u_j$ is a negative term which can be dropped. Next choose $\alpha = \frac{1}{2}$ and let g be the Euclidean metric. Hence our inequality becomes

$$\Delta(u) \leq -A(p) - \frac{n^2}{u} - \frac{n}{u} |\nabla u|^2 - \sum_j (\log W)_j u_j.$$

Using the fact that $\det(u_{ij}) \leq \Delta(u)^n$, we are done. □

5.1.3 Higher Order Estimates

In this section we explain how to combine Propositions 5.1.2, 5.1.3, and 5.1.4 to arrive at a proof of Theorem 3. Throughout this section we assume that $K = B_1 \subset B_2 \subset\subset P$ and that u is a normalized solution to (4.1.3) such that $u(0) = 0$. Define the operator $Q(f) = \mathcal{U}^{jk} f_{jk}$, where $\mathcal{U}^{jk}$ is the cofactor matrix of u. One can compute that (4.1.3) is equivalent to the equation

$$\mathcal{U}^{jk}(F) = g, \tag{5.1.10}$$

where $F = W\det(u_{ab})^{-1}$ and $g = -WA$. Note that by Proposition 5.1.3, we have that $||F||_{L^\infty} \leq C$ for some universal constant. Equations of the form $Q(F) = 0$, $||F||_{L^\infty} \leq C$, were studied in [Caffarelli and Gutiérrez, 1997], where the authors showed that there is a C^α bound on F at the end of Section 4 of their paper. Trudinger and Wang explained that one has a similar bound in the case where $Q(F) = g$, if one has a bound on $||g||_{L^n}$ [Trudinger and Wang, 2008]:

Corollary (Trudinger-Wang)**.** *Assume $u \in B_1$ solves Equation 5.1.10 and that $C_0 \leq \det(u_{jk}) \leq C_1$. Then there exist constants α and C depending only on n, C_0, C_1 such that*

$$||F||_{C^\alpha(B_{1/2})} \leq C(||F||_{L^\infty(B_1)} + ||g||_{L^n(B_1)}).$$

The upper bound on $\det(u_{jk})$ that this result requires does not follow directly from Proposition 5.1.4. However, in dimension 2, an old result of Heinz ([Heinz, 1959]) provides the lower bound (see also Lemmas 2.2 and 2.3 in [Mooney, 2013]).

Lemma (Heinz)**.** *Let u be smooth and convex in $B_2 \subset \mathbb{R}^2$ such that $\det(u_{jk}) \geq 1$ and assume that u is normalized at the origin. Then $u|_{\partial B_2} > c$ for some c.*

We apply this result by letting $\tilde{u} = u - \frac{c}{2}$ and $D = \{x \,|u(x) - \frac{c}{2} \leq 0\}$. We have that $\tilde{u} < 0$ on D and note that by convexity, $B_1 \subset D$. Proposition 5.1.4 can be applied to $\tilde{u}$ to get an upper bound for $\det(u_{ab})$ at x. This means we can apply Trudinger-Wang to get a C^α bound on F which in turn implies a C^α bound on $\det(u_j k)$.

Next we apply Theorem 2 in [Caffarelli, 1990] to get a $C^{2,\alpha}$ bound on u. This means that we have C^α control of the coefficients of Q which means that we have $C^{2,\alpha}$ control of F. Finally, the techniques of Schauder theory complete the proof of the Theorem 3.

5.2 L^∞-Estimate

In this section we will prove Theorem 4. We provide essentially the same proof as Donaldson does in [Donaldson, 2009]. The modifications are quite straightforward. One really just needs to verify that the addition of the function W does not cause problems in his proof. We present the entire proof for completeness.

Before jumping into the proof, we need to set the stage. First, by replacing W with $W/||W||_{L^\infty(P)}$, we can assume that $||W||_{L^\infty(P)} = 1$. Second, if we replace u by $||A||_{L^\infty(P)}u$ (which also forces us to replace σ with $||A||^{-1}_{L^\infty(P)}\sigma$), we can assume that $||A||_{L^\infty(P)} = 6$ (the reason for this odd choice will become clear later). Note that neither of these changes affects the Futaki invariant or Condition 1 (in fact, the constant λ remains the same).

Next let q be a vertex of P. Choose linear coordinates on P, mapping $(0,0)$ to q, so that $P \subset (\mathbb{R}^+)^2$, and so that $u = x_1 \log(x_1) + x_2 \log(x_2) + f$, where f is smooth in a neighborhood of $(0,0)$. Let $l_1 > 0$ be defined by the requirement that the intersection of P with the x_1-axis is given by the interval $(0, 2l_1)$ and define l_2 similarly. Hence the points $(l_1, 0)$ and $(0, l_2)$ are the centers of the edges adjacent to $(0,0)$.

The fact that (P, σ, A) satisfies Condition 1 gives us an a bound on $\int_{\partial P} uW d\sigma$ by Proposition 5.1.1. If one applies the same argument as in Proposition 5.1.2 to each of the two edges adjacent to q, we get a bound for $\frac{\partial u}{\partial x_1}$ at $(l_1, 0)$ and a bound for $\frac{\partial u}{\partial x_2}$ at $(0, l_2)$. Therefore, by adding an affine linear function to u, we can assume that $\min_P(u) = 0$, $\frac{\partial u}{\partial x_1}_{(l_1,0)} = \frac{\partial u}{\partial x_2}|_{(0,l_2)}$ and, very importantly, we still have a bound on $\int_{\partial P} uW d\sigma$.

Now that we have u and (P, σ) in a convenient form, the goal will be to get a bound on $u(0,0)$. Since u is maximized at one of the vertices, this is enough to get the L^∞ bound we seek. Let h_q be the affine supporting hyperplane of u at the point q. Define the set $X(h)$ by

$$X(h) = \{q \in P \mid h_q(0,0) \leq h\}.$$

Note that $X(h) \subset X(h')$ if $h \leq h'$ and that $X(h) = P$ if $h \geq u(0,0)$. Define $\underline{h}_1 = u(l_1, 0) + l_1$ and $\underline{h}_2 = u(0, l_2) + l_2$. This means that if $h \geq h_i$, for $i = 1, 2$, then $\overline{X(h)}$ contains $(l_1, 0)$ and $(0, l_2)$. Due to this we define $\underline{h} = \max(\underline{h}_1, \underline{h}_2)$ and $\overline{h} = u(0,0)$, and we will restrict our attention to h in the range $(\underline{h}, \overline{h})$.

Next define $\Omega(h) = P \setminus X(h)$ and define ξ_1 by the requirement that $\overline{\Omega(h)}$ intersects the x_1-axis in the line from the origin to $(\xi_1, 0)$ and define ξ_2 similarly. Next let $\tau_{1,h}(t)$ be the function of one variable whose graph is the supporting hyperplane of the restriction of u to the x_1-edge at the point $\xi_1(h)$ and define $\tau_{2,h}(t)$ similarly. I.e. $\tau_{1,h}(t) = u(t,0), \tau'_{1,h}(t) = \frac{\partial u}{\partial x_1}|_{(t,0)}$, and $\tau_{1,h}(0) = h$.

We introduced most of this formalism in order to better understand the following function:

$$G_1(h) = \int_0^{\xi_1(h)} (u(t,0) - \tau_{1,h}(t))\, W dt, \tag{5.2.11}$$

and we define G_2 similarly.

Definition 5.2.1. *Let $X \subset\subset P$ be an open set with piecewise smooth boundary. Define $\underline{u}_X$ to be the smallest convex function which agrees with u on X.*

Using this notation, we can rewrite (5.2.11) as

$$\int_{\partial P} (u - \underline{u}_{X(h)}) W d\sigma = G_1(h) + G_2(h).$$

We want to relate this integral to an integral on the interior of P. To that end we need the following lemma.

Lemma 5.2.2. *We have*

$$\int_{\partial P} (u - \underline{u}_X) W d\sigma \leq 2\text{Area}(P \setminus X) + \int_P A(u - \underline{u}_X) W d\mu.$$

Proof. For any smooth function f on $\overline{P}$ we can integrate by parts to get

$$\int_{\partial P} f W d\sigma = \int_P (u^{ij} f_{ij} + Af) W d\mu. \tag{5.2.12}$$

For $\varepsilon > 0$ small let $\underline{u}_{X,\varepsilon}$ be a convolution of $\underline{u}_X$. Note that $\underline{u}_{X,\varepsilon}$ is now smooth and convex and that it agrees with $\underline{u}_X$ on X_ε, where $X_\varepsilon = \{x \in X \mid \text{dist}(x, \partial P) \geq \varepsilon\}$. Thus taking $f = u - \underline{u}_{X,\varepsilon}$ we have that

$$u^{ij} f_{ij} = u^{ij}(u_{ij} - \underline{u}_{X,\varepsilon,ij}) \leq u^{ij} u_{ij} = 2.$$

Hence plugging f into (5.2.12) we get

$$\int_{\partial P} (u - \underline{u}_{X,\varepsilon}) W d\sigma \leq 2\text{Area}(P \setminus X_\varepsilon) + \int_P A(u - \underline{u}_{X,\varepsilon}) W d\mu.$$

Letting ε go to zero gives the lemma. $\square$

Thus Lemma 5.2.2 tells us that

$$G_1(h) + G_2(h) \leq 2\text{Area}(\Omega(h)) + \int_{\Omega(h)} A(u - \underline{u}_{X(h)})W d\mu. \tag{5.2.13}$$

Next we would like to better understand the right-hand side of Inequality (5.2.13). For $h \geq \underline{h}$, write $f_h = u - \underline{u}_{X(h)}$ which is positive and supported in $\Omega(h)$. If we set $\alpha = ||A||_{L^\infty} = 6$ and define $J(h)$ by

$$J(h) = \int_{\Omega(h)} f_h d\mu,$$

then we have that

$$\int_{\Omega(h)} A f_h W d\mu \leq 6J(h).$$

Hence with this notation our main Inequality (5.2.13) becomes

$$G_1(h) + G_2(h) \leq 2\text{Area}(\Omega(h)) + 6J(h). \tag{5.2.14}$$

First we relate $J(h)$ to Area$(\Omega(h))$.

Lemma 5.2.3. *We have that*

$$\frac{dJ(h)}{dh} = -\frac{1}{3}\text{Area}(\Omega(h)).$$

Next note that $J(\overline{h}) = 0$ and hence that $-\int_h^{\overline{h}} J'(h) = J(h)$. This means that we can rewrite (5.2.14) as

$$G_1(h) + G_2(h) \leq 2\text{Area}(\Omega(h)) + 2\int_h^{\overline{h}} \text{Area}(\Omega(h))dh. \tag{5.2.15}$$

We will estimate Area$(\Omega(h))$ by considering a larger set that is easier to work with. Define $D_i(h)$ to be the point where $\tau_{i,h}$ vanishes. Let Δ_h be the triangle with vertices $(0,0), (D_1(h), 0)$, and $(0, D_2(h))$. We have the following lemma by convexity:

Lemma 5.2.4. *For any $h \in (\underline{h}, \overline{h})$, we have $\Omega(h) \subset \Delta_h$.*

Hence we have that Area$(\Omega(h)) \leq$ Area$(\Delta_h) = \frac{1}{2}D_1 D_2$. Equipped with this, Inequality (5.2.15) becomes

$$G_1(h) + G_2(h) \leq D_1(h)D_2(h) + \int_h^{\overline{h}} D_1(h)D_2(h)dh. \tag{5.2.16}$$

Our goal is now to produce a differential inequality involving each G_i separately. To that end, we will employ the basic fact that $D_1 D_2 \leq \frac{1}{2}\left(D_1^2 + D_2^2\right)$. We define new functions $I_i(h) = \int_h^{\overline{h}} D_i(s)^2 ds$, for $i = 1, 2$. Hence we can rewrite Inequality (5.2.16) as

$$G_1(h) + G_2(h) \leq D_1(h) D_2(h) + \frac{1}{2}(I_1(h) + I_2(h)). \tag{5.2.17}$$

We will next define some quantities that will help us get control over the previous inequality.

Definition 5.2.5. *For $i = 1, 2$, define $\lambda_i(h)$ by the equation*

$$G_i(h) = \frac{\lambda_i(h)}{2} D_i(h)^2 + \frac{1}{2} I_i(h). \tag{5.2.18}$$

Using the definition of λ_i, Inequality (5.2.17) results in

$$\frac{\lambda_1}{2} D_1^2 + \frac{\lambda_2}{2} D_2^2 \leq D_1 D_2,$$

which then gives

$$\frac{1}{2}\left(\lambda_1 \frac{D_1}{D_2} + \lambda_2 \frac{D_2}{D_1}\right) \leq 1,$$

from which we can conclude:

Lemma 5.2.6. *Assuming both $\lambda_1, \lambda_2 > 0$, we have $\lambda_1 \lambda_2 \leq 1$.*

Remark 5.2.7. In the following computations, we drop the subscript i and write, e.g. λ for λ_i, etc.

Note that $\lambda(h)$ is in fact a differentiable function of h. This can be seen by tracing all the definitions backwards to see their dependence on u which is smooth. Hence Equation (5.2.18) can be differentiated to get

$$G' = \lambda D D' + \frac{1}{2}\lambda' D^2 - \frac{1}{2} D^2. \tag{5.2.19}$$

(Note that $I' = -\frac{1}{2} D^2$.) We can also compute G' explicitly by looking back at its definition:

$$G'(h) = \lim_{\varepsilon \to 0} \int_0^{\xi(h+\varepsilon)} \left(\tau_h(t) - \tau_{h+\varepsilon}(t)\right) W dt.$$

By drawing a picture of this limit, it is easy to see that

$$G'(h) \leq -\frac{\min_P(W)}{2} \xi(h). \tag{5.2.20}$$

Combining Equations (5.2.19) and (5.2.20) we get

$$-\frac{\min_P(W)}{2}\xi \geq \lambda D D' + \frac{1}{2}\lambda' D^2 - \frac{1}{2}D^2. \tag{5.2.21}$$

First we would like to relate ξ back to D.

Lemma 5.2.8. *We have that*

$$\xi(h) = \frac{D^2}{D - hD'}. \tag{5.2.22}$$

Proof. We restrict u to the x_1 (or x_2)-axis and consider it a convex function of one variable. Then we compute that

$$h = u(\xi) - \xi u'(\xi). \tag{5.2.23}$$

If one differentiates Equation (5.2.23), then one see that

$$-1 = \xi\xi' u''(\xi). \tag{5.2.24}$$

Next one computes that

$$D = \frac{\xi u'(\xi) - u(\xi)}{u'(\xi)}. \tag{5.2.25}$$

Then one plugs Equation (5.2.25) into the right side of (5.2.22) and uses (5.2.24) to prove the lemma. □

Using the lemma, we can rewrite (5.2.21) as

$$\frac{D^2}{2(hD' - D)} \geq C\left(\lambda D D' + \frac{1}{2}\lambda' D^2 - \frac{1}{2}D^2\right), \tag{5.2.26}$$

where $C = \min_P(W)^{-1}$.

Next if we define $z(h) = D(h)/h$, then $D' = z'h + z$. Note that since D is decreasing in h, z is as well. If we plug this into Equation (5.2.26) we get

$$\frac{z^2}{2z'} \geq C\left(\frac{1}{2}\lambda' z^2 h^2 + \lambda z h(z'h + z) - \frac{1}{2}z^2 h^2\right),$$

which we can rewrite as

$$\frac{z}{2h^2} \leq Cz'\left(\lambda z' + z\left(\frac{\lambda}{h} - \frac{1}{2} + \frac{\lambda'}{2}\right)\right), \tag{5.2.27}$$

where we used the fact that z' is non-positive.

Next we use the fact that $K\sqrt{AB} \le \frac{1}{2}(K^2A+B)$, for any $K, A, B > 0$. Choose $A = -z'$ and $B = -\left(\lambda z' + z(\frac{\lambda}{h} - \frac{1}{2} + \frac{\lambda'}{2})\right)$ and then get

$$\begin{aligned}\frac{\sqrt{z}}{\sqrt{2}h} &\le \sqrt{C}\sqrt{z'\left(\lambda z' + z\left(\frac{\lambda}{h} - \frac{1}{2} + \frac{\lambda'}{2}\right)\right)}\\ &\le \frac{\sqrt{C}}{2K}\left(-K^2 z' - \left(\lambda z' + z\left(\frac{\lambda}{h} - \frac{1}{2} + \frac{\lambda'}{2}\right)\right)\right).\end{aligned}$$

Rearranging, we get

$$K^2 z' + \left(\lambda z' + z\left(\frac{\lambda}{h} - \frac{1}{2} + \frac{\lambda'}{2}\right)\right) \le -\frac{\sqrt{2}K}{\sqrt{C}}\frac{\sqrt{z}}{h}.$$

We take advantage of the uneven scaling in z in this equation, to replace the term z with the term z/C. Since $C = \min_P(W)^{-1}$ is fixed, this will not affect our analysis. Hence, the previous inequality becomes

$$K^2 z' + \left(\lambda z' + z\left(\frac{\lambda}{h} - c + \frac{\lambda}{2}\right)\right) \le -\sqrt{2}K\frac{\sqrt{z}}{h}. \tag{5.2.28}$$

where $c = \frac{1}{2}$ and K may be any positive number. Now we quote Proposition 2 of Donaldson's paper [Donaldson, 2009]:

Proposition 5.2.9 (Donaldson). *Suppose $z(h)$ and $\lambda(h)$ are functions defined on an interval $(h_0, \overline{h})$ with the following properties:*

1. *$z(h) > 0$ and $z'(h) < 0$ for all h.*

2. *z and λ satisfy the differential inequality 5.2.28 for some $c > 0$ and all $K > 0$.*

3. *For some $C > 0$, and all h, we have $z \le Ch^{-2}$.*

4. *$z(h)$ and $\lambda(h)$ tend to 0 as $h \to \overline{h}$.*

Let $b = 2\sqrt{2} - 1$. If we fix any $K > 2c\sqrt{C}$ and if we set $h_1 = \max(h_0, \frac{3K\sqrt{C}}{b})$, then we have $K^2 + \lambda(h) > 0$ for all $h \ge h_1$ and

$$\int_{h_1}^{\overline{h}} \frac{1}{(K^2+\lambda)^{3/4}}\frac{dh}{h} \le \frac{12}{bK}z(h_1)^{1/2}(K^2 + \lambda(h_1))^{1/4}. \tag{5.2.29}$$

We need to show that all the requirements of this proposition are satisfied. We have already noted that the first property is true and the second property holds by design. Next we show that the third property holds.

Lemma 5.2.10. *Assume that $h_0 \geq 2u(l_1, 0)$. Then we have that*

$$z_1(h) \leq 2h^{-2} \int_{\partial P} u d\sigma.$$

The same bound holds similarly for z_2.

Proof. First note that $z_1 h^2/2 = hD_1(h)/2 = \int_0^{D_1(h)} \tau_{1,h}(t)dt$. Since $h_0 \geq 2u(l_1, 0)$, $D_1(h) \leq 2l_1$ which means that $\tau_{1,h}(t) \leq u(t, 0)$ for all t in $[0, D_1(h)]$. Hence $\int_0^{D_1(h)} \tau_{1,h}(t)dt \leq \int_{\partial P} u d\sigma$ and we are done. □

Finally we show that the fourth property holds. It is clear that $z \to 0$ as $h \to \overline{h}$. The following lemma implies that $\lambda \to 0$ as $h \to 0$ as well.

Lemma 5.2.11. *We have*

(i) $\lim_{h \to \overline{h}} \frac{G(h)}{D(h)^2} = 0$

(ii) $\lim_{h \to \overline{h}} \frac{I(h)}{D(h)^2} = 0$

Proof. For (i) note that (here we use that $\max_P(W) = 1$)

$$0 \leq G(h) \leq \frac{1}{2}(\overline{h} - h)\xi(h),$$

which can be seen by drawing a picture of what $G(h)$ represents and noting that the region of the integral is contained in the triangle with vertices $(0, h), (0, \overline{h})$, and $(\xi(h), u(\xi(h)))$. Note also that $D(h) = -\frac{h}{u'(\xi(h))}$. Locally u is given by $u(x) = x \log(x) - x + f(x)$ for some f smooth up to the boundary. Thus $u'(\xi(h)) = \log(\xi(h)) + f'(\xi(h))$ and hence

$$\frac{G(h)}{D(h)^2} \leq \frac{1}{2} \frac{(\overline{h} - h)\xi(h)(\log(\xi(h)) + f'(\xi(h)))^2}{h^2}. \tag{5.2.30}$$

We have that $\xi(h) \log(\xi(h))$ is bounded and hence (5.2.30) converges to 0 as h converges to $\overline{h}$.

For (ii) that $D(h)$ is decreasing as h increases to $\overline{h}$. Hence

$$0 \leq \frac{I(h)}{D(h)^2} = \frac{1}{D(h)^2} \int_h^{\overline{h}} D_i(s)^2 ds \leq \frac{1}{D(h)^2} \int_h^{\overline{h}} D_i(h)^2 ds = \overline{h} - h.$$

□

Hence we are in the position to apply Proposition 5.2.9. In order to apply Lemma 5.2.10, we need to assume that $h_0 \geq 2u(l_1, 0)$. However, as noted at the beginning of this section, $u(l_1, 0) \leq C \int_{\partial P} u d\sigma$, for some constant C depending only on the geometry of (P, σ). We will make this assumption henceforth.

Next note that $\frac{\partial u}{\partial x_1}(t, 0) \leq -1$ for all $t \in (0, l_1)$ and hence we have that $z_1(h) \leq 1$ for $h \in (h_0, \overline{h})$. This holds similarly for z_2 as well. Hence we have that $z_i^2(K^2 + \lambda_i) \leq K^2 + z_i^2 \lambda_i = K^2 + D_i^2 \lambda_i h^{-2}$. However, since I_i is positive, if we look back at (5.2.18), we see that $\lambda_i D_i^2 \leq 2G_i$. Hence

$$z_i^2(K^2 + \lambda_i) \leq K^2 + 2h^{-2} \int_{\partial P} uW d\sigma.$$

Since the right hand side of (5.2.29) only refers to $h = h_1$, we can conclude that

$$\int_{h_1}^{\overline{h}} \frac{1}{(K^2 + \lambda)^{3/4}} \frac{dh}{h} \leq L,$$

where $L = K^2 + 2h_1^{-2} \int_{\partial P} uW d\sigma$ is a constant depending only on (P, σ) and K. If one changes variables by writing $h = e^t$ and $\overline{h} = e^{\overline{t}}$, etc., then one sees that

$$\int_{t_1}^{\overline{t}} \frac{1}{(K^2 + \lambda_i)^{3/4}} dt \leq L.$$

Let $S_i = \{t \in (t_1, \overline{t}) \mid \lambda_i(t) \leq 1\}$. Then the previous inequality tells us that

$$L \geq \int_{t_1}^{\overline{t}} \frac{1}{(K^2 + \lambda_i)^{3/4}} \geq \int_{S_i} \frac{1}{K^2 + 1} = \frac{|S_i|}{K^2 + 1}.$$

Hence the measure of $|S_i| \leq L(K^2 + 1)$. This means that if $\overline{t} - t_1 > 2L(K^2 + 1)$, then there must be a point t where both $\lambda_1(t)$ and $\lambda_2(t)$ are strictly greater than 1. This contradicts the previous lemma and we are done.

Chapter 6

The Donaldson-Futaki Invariant

In the last chapter, we derived a priori estimates for symplectic potentials satisfying the scalar curvature equation (4.1.3) under the assumption that the Futaki invariant of (P, σ, A) vanishes and that it satisfies Condition 1. While the generalized setting introduced in Section 4.1 was convenient for our analytic work, we return now to the standard setting as described in Remark 4.1.1. In this case, the vanishing of the Futaki invariant can readily be seen as a purely algebraic condition on P. The main goal of this chapter is to demonstrate that Condition 1 is an algebraic condition as well.

The idea of relating the solvability of a PDE to an algebraic stability condition has a long history. Yau first suggested this avenue in the setting of Kähler-Einstein metrics. Tian contributed many results and defined a notion called *K-Stability*. In [Donaldson, 2002], Donaldson gave a slightly different definition of K-Stability of polarized varieties. To check whether a polarized variety (X, L) is K-Stable, one needs to construct algebraic degenerations called *test configurations* for (X, L). For each test configuration, one then computes a *Donaldson-Futaki invariant.* The pair (X, L) is defined to be K-Stable, if the Donaldson-Futaki invariants of all non-trivial test configurations of (X, L) are strictly negative. In [Donaldson, 2002], Donaldson conjectured that a smooth polarized projective variety (X, L) admits a Kähler metric of constant scalar curvature in the class $c_1(L)$ if and only if the pair (X, L) is K-stable. In the series of papers [Donaldson, 2002; 2005a; 2008a; 2009], Donaldson verified the conjecture in the case of two-dimensional toric varieties.

In order to extend these results, we need to understand K-Stability in the setting of toric fibrations. In this chapter, we assume that $(G \times_T M, L)$ is a polarized pair with a compatible holomorphic G-action and that Ω is a G-invariant Kähler metric in the class $c_1(L)$. Let P be the corresponding polytope and recall as explained in Section 4.1 that σ is fixed by P. Furthermore, the average scalar curvature is fixed by $a = \int_{\partial P} W d\sigma / \int_P W d\mu$. Note that all the parameters P, a, σ, W, and f_G are algebraic depending on $(G \times_T M, L)$. The main result of this section is Theorem 5.

The proof of Theorem 5 will require understanding the asymptotics of a weighted sum of the lattice lattice points $kP \cap \mathbb{Z}^n$ as k approaches infinity. The key step is to prove Theorem 6 which is a generalization of Pick's Theorem, relating the boundary measure σ to the asymptotics of our sum.

The chapter is outlined as follows. In the first section, we review the general theory of K-Stability. In the second section, we show how to construct test configurations for toric fibrations. In the third section, we assume Theorem 6 to compute the Donaldson-Futaki invariant of our test configurations, proving Theorem 5. Finally in the fourth section, we give a proof of Theorem 6.

6.1 Donaldson-Futaki Invariants and K-Stability

Let X be a smooth compact complex manifold and let L be a positive line bundle over X. A test configuration for (X, L) is a scheme $\mathcal{X}$ with a $\mathbb{C}^*$-action, a $\mathbb{C}^*$-equivariant line bundle $\mathscr{L} \to \mathcal{X}$, and a flat $\mathbb{C}^*$-equivariant map $\pi : \mathcal{X} \to \mathbb{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by standard multiplication. Furthermore, for any fiber $\mathcal{X}_p = \pi^{-1}(p)$, where $p \neq 0$, we require that $(\mathcal{X}_p, \mathscr{L}_p)$ be isomorphic to (X, L).

Given a test configuration $(\mathcal{X}, \mathscr{L})$ for (X, L), note that the $\mathbb{C}^*$-action on $(\mathcal{X}, \mathscr{L})$ restricts to an action on the zero-fiber $(\mathcal{X}_0, \mathscr{L}_0)$. One produces a numerical invariant of $(\mathcal{X}_0, \mathscr{L}_0)$ as follows. For each positive integer k, let $H_k = H^0(\mathcal{X}_0, \mathscr{L}_0^k)$, let $d_k = \dim(H_k)$, and let w_k be the weight of the induced $\mathbb{C}^*$-action on $\Lambda^{d_k} H_k$. Write $F(k) = \frac{w_k}{k d_k}$ and note that by general theory, $F(k)$ is a rational function for large k. We have the expansion

$$F(k) = F_0 + F_1 k^{-1} + \cdots , \tag{6.1.1}$$

for large enough k. The Donaldson-Futaki invariant F of the test configuration $(\mathcal{X}, \mathscr{L})$ is defined to be the rational number F_1 in this expansion.

Given any polarized pair (X, L), one can define $\mathcal{X} = \mathbb{C} \times X$ and $\pi : \mathcal{X} \to \mathbb{C}$ as the projection onto the first coordinate. Next one defines $\mathscr{L} = \pi^*(L)$ and lets $\mathbb{C}$ act trivially on the left by standard multiplication. One can compute that the Donaldson-Futaki invariant of this test configuration is equal to 0. This leads us to the definition of K-Stability.

Definition 6.1.1. *A smooth polarized pair (X, L) is called **K-Stable** if the Donaldson-Futaki invariant of any non-trivial test configuration is strictly negative.*

6.2 Test Configurations for Toric Fibrations

In this section, we show how to construct test configurations for polarized G-invariant pairs $(G \times_T M, L)$. Let P be the polytope corresponding to a G-invariant Kähler metric $\Omega \in c_1(L)$. Let L_P be the line bundle over M corresponding to the polytope P. We have that $L = G \times_T L_P$.

Let f be a convex, continuous, piecewise-linear, rational function defined on $\mathbb{R}^n$ and R a fixed number such that $f(x) \leq R - 1$, for all $x \in P$. Define Q to be the convex polytope in $\mathbb{R}^{n+1}$ given by

$$Q = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid x \in P \text{ and } 0 < t < R - f(x)\}.$$

P can be identified with the "bottom" face of Q.

Corresponding to Q, there is a (possibly singular) polarized toric variety (N, L_Q). Note that M embeds into N and that $(L_Q)|_M = L_P$. Let $G' := G \times S^1$ and let $T' = T \times S^1$. Consider the space $G' \times_{T'} N$ and the line bundle $G' \times_{T'} L_Q$ over it. Note that there is a natural G-equivariant embedding $\iota : (G \times_T M, L) \to (G' \times_{T'} N, G' \times_{T'} L_Q)$.

Proposition 6.2.1. *There is a $\mathbb{C}^*$-equivariant map $p : G' \times_{T'} N \to \mathbb{P}^1$ with $p^{-1}(\infty) = \iota(G \times_T M)$ such that if we define $\mathcal{X} := G' \times_{T'} N \setminus \iota(G \times_T M)$ and define $\mathscr{L}$ as the restriction of $G' \times_{T'} L_Q$ to $\mathcal{X}$, then $(\mathcal{X}, \mathscr{L})$ is a test configuration for $(G \times_T M, L)$.*

Proof. As explained in section 3.4, a basis for the sections of $G' \times_{T'} L_Q \to G' \times_{T'} N$ is given by $s_{\lambda,i,j}$ where λ is a lattice point in $P \cap \mathbb{Z}^n$, $0 \leq i \leq R - f(\lambda)$, and $1 \leq j \leq \dim H^0(G_{\mathbb{C}} \times_B L_\lambda)$. Note that the action of T' on sections $s_{\lambda,i,j}$ and $s_{\lambda,i+1,j'}$ only differs in the last component of $T' = T \times S^1$. Choose a point $p \in G' \times_{T'} N$ where none of these sections vanish. Next rescale the sections to all take the same value in $G' \times_{T'} L_Q$ over the point p. Define the map $p : G' \times_{T'} N \to \mathbb{P}^1$ by

$$x \mapsto [s_{\lambda,i,j}(x) : s_{\lambda,i+1,j'}(x)].$$

This gives a $\mathbb{C}^*$-equivariant map $G' \times_{T'} N \to \mathbb{P}^1$, mapping $i(G \times_T M)$ to $[1, 0]$. Define $\mathcal{X} = G' \times_{T'} N - i(G \times_T M)$ and we have that $G' \times_{T'} L_Q|_{\mathcal{X}} \to \mathcal{X} \to \mathbb{C}, x \mapsto \frac{s_{\lambda,i,j}(x)}{s_{\lambda,i+1,j'}(x)} \in \mathbb{C}$, is a test configuration for $G \times_T M$. □

6.3 Donaldson-Futaki Invariants for Toric Fibrations

Proposition 6.2.1 gives a nice way to construct test configurations for toric fibrations purely in terms of data on the polytope P. In this section we will see how to compute the Donaldson-Futaki invariant of such test configurations giving a proof of Theorem 5. As mentioned in the introduction to this chapter, in this section we will assume that Theorem 6 holds and then prove that result in the final section of the chapter.

Let $(\mathcal{X}, \mathscr{L})$ be a test configuration for $(G \times_T M, L)$ as constructed in Proposition 6.2.1. To compute its Futaki invariant—given by the number F_1 in (6.1.1)—we need to compute d_k and w_k. We require the following two lemmas.

Lemma 6.3.1. *We have that* $d_k = h^0(\mathcal{X}_0, \mathscr{L}|_0^k) = h^0(G \times_T M, L^k)$.

Lemma 6.3.2. *The only sections* $s_{\lambda,i,j}$ *of* $H^0(G' \times_{T'} N, G' \times_{T'} L_Q)$ *which are not identically 0 when restricted to* $\mathcal{X}_0$ *are the sections of the form* $s_{\lambda,R-f(\lambda),j}$. *Consequently, the number* w_k *is given by the sum of the weights on each section* $s_{\lambda,R-f(\lambda),j}$, *for* $1 \le j \le \dim(L_\lambda)$, *and each weight is* $f(\lambda) - R$.

To compute the numbers d_k and w_k we will make use of Proposition 3.4.2. That result tells us that

$$d_k = \sum_{\lambda \in kP \cap \mathbb{Z}^n} h^0(G \times_T L_\lambda) = \sum_{\lambda \in kP} \left(\prod_{\alpha \in \Delta^+} \frac{(1 + \lambda_j) M_\alpha^j}{\prod_{j=1}^n M_\alpha^j} \right).$$

and that the number w_k is given by

$$\begin{aligned} w_k &= \sum_{\lambda \in kP\cap\mathbb{Z}^n} h^0(G\times_T L_\lambda)k(f(\lambda/k)-R) \\ &= \sum_{\lambda\in kQ\cap\mathbb{Z}^{n+1}} h^0(G\times_T L_\lambda) - \sum_{\lambda\in kP\cap\mathbb{Z}^n} h^0(G\times_T L_\lambda) \\ &= \sum_{\lambda\in kQ}\left(\prod_{\alpha\in\Delta^+}\frac{(1+\lambda_j)M_\alpha^j}{\prod_{j=1}^n M_\alpha^j}\right) - \sum_{\lambda\in kP}\left(\prod_{\alpha\in\Delta^+}\frac{(1+\lambda_j)M_\alpha^j}{\prod_{j=1}^n M_\alpha^j}\right). \end{aligned}$$

where $\pi:\mathbb{Z}^{n+1}\to\mathbb{Z}$ is the projection map $(\lambda_1,\cdots,\lambda_{n+1})\mapsto(\lambda_1,\cdots,\lambda_n)$, given in coordinates. We are interested in the ratio $\frac{w_k}{kd_k}$ and hence the common factor of $\prod_{j=1}^n M_\alpha^j$ in the denominator of these formulas can be ignored. This leads us to define the polynomial $q(\lambda)$ by

$$q(\lambda) = \prod_{\alpha\in\Delta^+}(1+\lambda_j)M_\alpha^j = \prod_{\alpha\in\Delta^+}(|M_\alpha|+\lambda_j M_\alpha^j),$$

where $|M_\alpha| = \sum_{j=1}^n M_\alpha^j$. Note that $q(\lambda)$ is an N^{th} degree polynomial in λ—where N is the number of positive roots. The goal is to compute the number F_1 in the following expansion

$$\frac{\sum_{\lambda\in kQ} q(\lambda) - \sum_{\lambda\in kP} q(\lambda)}{k\sum_{\lambda\in kP} q(\lambda)} = F_0 + F_1 k^{-1} + \cdots.$$

This is where we will apply Theorem 6. To apply this this theorem, we need to decompose q into its homogeneous parts. Let q_l be the homogeneous part of q of order l.

$$\begin{aligned} q(\lambda) &= \prod_{\alpha\in\Delta^+}\left(|M_\alpha|+\lambda_j M_\alpha^j\right) \\ &= \left(\prod_{\alpha\in\Delta^+}\lambda_j M_\alpha^j\right) + \left(\sum_{\beta\in\Delta^+}|M_\beta|\prod_{\alpha\neq\beta}\left(\lambda_j M_\alpha^j\right)\right) + r(\lambda) \\ &= q_N(\lambda) + q_{N-1}(\lambda) + r(\lambda), \end{aligned}$$

where r is a polynomial of degree $N-2$. Note that q_N is convex in the positive quadrant. We compute:

$$\begin{aligned} d_k &= \sum_{\lambda\in kP\cap\mathbb{Z}^n}(q_N(\lambda)+q_{N-1}(\lambda)+r(\lambda)) \\ &= k^N\sum_{\lambda\in P\cap\frac{1}{k}\mathbb{Z}^n} q_N(\lambda) + k^{N-1}\sum_{\lambda\in P\cap\frac{1}{k}\mathbb{Z}^n} q_{N-1}(\lambda) + \sum_{\lambda\in P\cap\frac{1}{k}\mathbb{Z}^n} r(k\lambda) \\ &= k^{N+n}\int_P q_N d\mu + k^{N+n-1}\left(\int_P q_{N-1}d\mu + \frac{1}{2}\int_{\partial P} q_N d\sigma\right) + o(N+n-2), \end{aligned}$$

and note that we used Theorem 6 in the final equality. Similarly, we compute w_k:

$$\begin{aligned} w_k &= \sum_{\lambda \in kQ \cap \mathbb{Z}^{n+1}} q(\pi(\lambda)) - \sum_{\lambda \in kP \cap \mathbb{Z}^n} q(\lambda) \\ &= k^{N+n+1}\left(\int_Q q_N d\mu\right) + k^{N+n}\left(\int_Q q_{N-1} d\mu - \int_P q_N d\mu + \frac{1}{2}\int_{\partial Q} q_N d\sigma\right) \\ &\quad + o(K+n-1). \end{aligned}$$

The Fubini Theorem tells us that $\int_Q q_N d\mu = \int_P q_N(R-f)d\mu$ and that $\int_Q q_{N-1} d\mu = \int_P q_{N-1}(R-f)d\mu$. Furthermore,

$$-\int_P q_N d\mu + \frac{1}{2}\int_{\partial Q} q_N d\sigma = \frac{1}{2}\int_{\partial P} q_N(R-f)d\sigma.$$

Hence we have that

$$d_k = Ck^{N+n} + Dk^{N+n-1} + o(N+n-2)$$

and

$$w_k = Ak^{N+n+1} + Ck^{N+n} + o(N+n-1),$$

where the constants A, B, C, and D are given by:

$A = \int_P q_N(R-f)d\mu$

$B = \int_P q_{N-1}(R-f)d\mu + \frac{1}{2}\int_{\partial P} q_N(R-f)d\sigma$

$C = \int_P q_N d\mu$

$D = \int_P q_{N-1} dx + \frac{1}{2}\int_{\partial P} q_N d\sigma$

To compute the Futaki invariant, we need to compute the term $F_1 = C^{-2}(BC - AD)$. Straight-forward computations yield

$$\begin{aligned} F_1 = \frac{-1}{\int_P q_N d\mu}\Bigg\{ &\int_P f q_{N-1} d\mu + \frac{1}{2}\int_{\partial P} f q_N d\sigma \\ &- \frac{\int_P q_{N-1} d\mu + \frac{1}{2}\int_{\partial P} q_N d\sigma}{\int_P q_N d\mu}\int_P f q_N d\mu \Bigg\}. \end{aligned}$$

First note that q_N is the same polynomial as W given by (3.2.10). Next note that $q_{N-1} = \sum_l \frac{\partial}{\partial x^l} p = \frac{1}{4} p f_G$, where f_G is given by (3.3.22).

Lemma 6.3.3. *We have that*

$$\frac{\int_P q_{N-1} d\mu + \frac{1}{2}\int_{\partial P} q_N d\sigma}{\int_P q_N d\mu} = \frac{a}{2},$$

where a is the average scalar curvature of any metric.

Proof. Let u be the symplectic potential of any metric. Then

$$\begin{aligned} a\int_P p(x)dx &= \int_P Sp(x)dx \\ &= \frac{1}{2}\int_P -(p(x)u^{jk})_{jk}dx + \int_P f_G p(x)dx \\ &= \frac{1}{2}\int_{\partial P} p(x)2d\sigma + 2\int_P q_{N-1}dx \\ &= 2\left(\frac{1}{2}\int_{\partial P} p(x)d\sigma + \int_P q_{N-1}dx\right). \end{aligned}$$

Which is what we needed to show. □

Hence we have proved that the Futaki invariant of the test configuration we constructed is equal to

$$-\frac{1}{2\mathrm{Vol}_W(P)}\left(\int_P f f_G W d\mu + \int_{\partial P} fW d\sigma - a\int_P fW d\mu\right),$$

and hence the proof of Theorem 5 is complete.

6.4 Proof of Asymptotic Pick Theorem

The definition of $d\sigma$ given in the beginning of Chapter 4 agrees with that given in [Donaldson, 2002] to avoid possible confusion. However, for our purposes, an equivalent but more useful formulation is given in the following lemma.

Lemma 6.4.1. *The measure $d\sigma(F)$ of a face F of the polytope P is given by*

$$d\sigma(F) = \lim_{k\to\infty} \frac{\#(F \cap \frac{1}{k}\mathbb{Z}^n)}{k^{n-1}}, \qquad (6.4.2)$$

where $\#(S)$ is the number of points in the set S.

Proof. We can assume that P is given as the convex hull of the extreme points $(0,\ldots,0)$, $(p_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, p_n)$, where the p_i are positive integers and p_i and p_j are coprime for $i \neq j$. To verify the lemma, we only need to show that the equation is satisfied for the face F of P that does not include the origin. (The other faces are entirely contained in the standard subsets $(x_i \equiv 0)$ where this lemma is clearly true.) The primitive outward orthogonal vector v to face F is given by

$$v = \sum_{i=1}^{n} \left(\prod_{j \neq i} p_j \right) e_i,$$

where the e_i are the standard basis vectors of $\mathbb{R}^n$. Hence the measure $d\sigma_F$ is given by the following form on $\mathbb{R}^n$ restricted to F:

$$d\sigma_F = \left(\prod_{i=1}^{n-1} p_i \right)^{-1} dx_1 \wedge \cdots \wedge dx_{n-1}.$$

This form can be integrated over the face P given by $(x_n \equiv 0)$ and yields the result

$$d\sigma_F(F) = \frac{1}{(n-1)!}. \tag{6.4.3}$$

Next we would like to verify that we get the same result from equation (6.4.2). The fact that p_i and p_j are coprime for $i \neq j$ tells us that the set $F \cap \mathbb{Z}^n$ has exactly n points and that those are the extreme points of F. This means that the "projection" map π defined by

$$\pi(x_1, \ldots, x_{n-1}, x_n) = \left(\frac{x_1}{p_1}, \ldots, \frac{x_{n-1}}{p_{n-1}} \right),$$

maps the lattice points of $F \cap \mathbb{Z}^n$ to the lattice points of the standard $(n-1)$-simplex S in $\mathbb{R}^{n-1}$. This mapping shows that the number of lattice points in $F \cap \frac{1}{k}\mathbb{Z}^n$ is the same as the number of lattice points in $S \cap \frac{1}{k}\mathbb{Z}^{n-1}$. But the final number is simply $k^n \mathrm{Vol}(S)$ to highest order. One can verify that $\mathrm{Vol}(S)$ agrees with (6.4.3) which proves the lemma. □

Equation (6.4.2) says that the measure of F is given asymptotically by the number of lattice points in kF. Note that this makes it clear that the measure is invariant under transformations in $GL(n, \mathbb{Z})$.

We will prove Theorem 6 by comparing the sum on the left side of (1.0.9) to the integrals on the right side of the equation. This requires some care and leads us to make quite a few definitions. Let $\mathcal{P}_k$ be the set of points $\mathcal{P}_k = P \cap \frac{1}{k}\mathbb{Z}^n$.

For a given point $p \in \mathcal{P}_k$, let $\Box_k(p) = \Box_k(p_1, \cdots, p_n)$ be the *box* defined by

$$\Box_k(p_1, \ldots, p_n) = \left[p_1, p_1 + \frac{1}{k}\right] \times \cdots \times \left[p_n, p_n + \frac{1}{k}\right] \subset \mathbb{R}^n.$$

Given a box $\Box_k(p)$, we will call p the *corner point* of $\Box_k(p)$ and call $p_{k,m} := (p_1 + \frac{1}{2k}, \ldots, p_n + \frac{1}{2k})$ the *midpoint* of $\Box_k(p)$. Furthermore, we will need to partition $\mathcal{P}_k$ into the disjoint sets of *interior*, *face*, and *exterior* points as follows:

$\mathcal{I}_k = \{p \in \mathcal{P}_k \mid \Box_k(p) \cap P = \Box_k(p)\}$

$\mathcal{E}_k = \{p \in \mathcal{P}_k \mid \Box_k(p) \cap P = \{p\}\ \}$

$\mathcal{F}_k = \mathcal{P}_k \setminus (\mathcal{I}_k \cup \mathcal{F}_k)$

Note: $\mathcal{I}_k$ contains points on the boundary of P. Next, define (non-convex) subsets of $\mathbb{R}^n$ as follows

$P_{\mathcal{I},k} = \bigcup_{p \in \mathcal{I}_k} \Box_k(p)$

$P_{\mathcal{F},k} = \bigcup_{p \in \mathcal{F}_k} \Box_k(p)$

$P_{\mathcal{E},k} = \bigcup_{p \in \mathcal{E}_k} \Box_k(p)$

Figure 6.1 illustrates these definitions.

Lemma 6.4.2. *Let h be a C^2 function on $B = \Box_k(0, \ldots, 0)$ and $p_{k,m}$ the midpoint of B. Then*

$$\left| k^n \left(\int_B h d\mu \right) - h(p_{k,m}) \right| \le \frac{1}{k^2} C_n ||h||_{C^2(B)},$$

where C_n only depends on the dimension n.

Proof. First consider the one-dimensional case where $B = [0, \frac{1}{k}]$. Integrating by parts, we

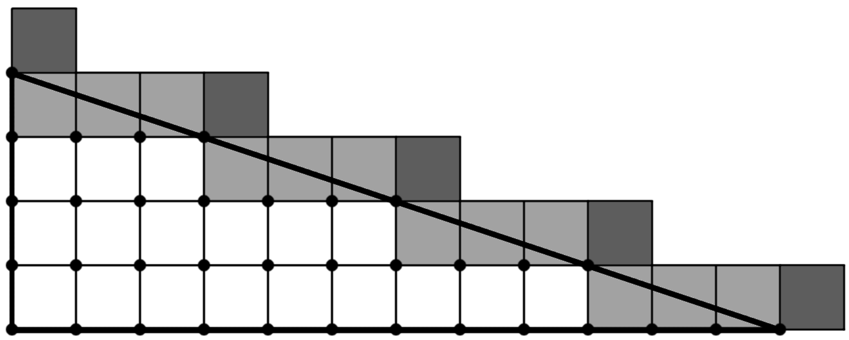

Figure 6.1: The polytope in this example is a triangle with height 1 and base 3. Furthermore, $k = 4$. The dots correspond to lattice points in $\mathcal{P}_3$. The white squares correspond to the set $P_{\mathcal{I},3}$, the light gray squares correspond to the set $P_{\mathcal{F},3}$ and the dark gray squares correspond to the set $P_{\mathcal{E},3}$.

see

$$\begin{aligned}\int_0^{\frac{1}{k}} h(x)dx &= \int_0^{\frac{1}{2k}} h(x)dx + \int_{\frac{1}{2k}}^{\frac{1}{k}} h(x)dx \\ &= \int_0^{\frac{1}{2k}} \left(\frac{x^2}{2} + Ax + B\right) h''(x)dx \\ &\quad + \int_{\frac{1}{2k}}^{\frac{1}{k}} \left(\frac{x^2}{2} + Cx + D\right) h''(x)dx \\ &\quad - \left(\frac{x^2}{2} + Ax + B\right) h'(x)\Big|_0^{\frac{1}{2k}} - \left(\frac{x^2}{2} + Cx + D\right) h'(x)\Big|_{\frac{1}{2k}}^{\frac{1}{k}} \\ &\quad + (x + A)h(x)\Big|_0^{\frac{1}{2k}} + (x + C)h(x)\Big|_{\frac{1}{2k}}^{\frac{1}{k}},\end{aligned}$$

where A, B, C, and D are constants that we can choose freely. By choosing $A = B = 0$, $C = -\frac{1}{k}$, and $D = \frac{1}{2k^2}$, we see that

$$\int_0^{\frac{1}{k}} h(x)dx = \frac{1}{k}h\left(\frac{1}{2k}\right) + \int_0^{\frac{1}{2k}} \frac{x^2}{2}h''(x)dx + \int_{\frac{1}{2k}}^{\frac{1}{k}} \frac{\left(x - \frac{1}{k}\right)^2}{2}h''(x)dx.$$

Hence we have

$$\left|k\int_0^{\frac{1}{k}} h(x)dx - h\left(\frac{1}{2k}\right)\right| \leq \frac{1}{24k^2} \max_{x\in\left[0,\frac{1}{k}\right]} |h''(x)|.$$

The proof is completed by induction. Assume the lemma is true for the n-dimensional case. Let h be a function of $n+1$ variables. Define $\tilde{h}(x) = h(x_1, \ldots, x_n, x)$ and apply the previous argument and the induction hypothesis to get the desired result. □

Lemma 6.4.3. *Let h be a C^2 function on P and $p_{k,m}$ the midpoint of box $\square_k(p)$. Then we have*

$$\left| k^n \int_{P_{\mathcal{I},k}} h(x)dx - \sum_{p \in \mathcal{I}_k} h(p_{k,m}) \right| \leq k^{n-2} C_n K_P ||h||_{C^2(P)|},$$

where C_n is the same constant as the last lemma, and K_P is a constant depending on the geometry of P.

Proof. Sum up the previous lemma over the points in $\mathcal{I}_k$. □

These lemmas yield a sort of asymptotic estimate for

$$k^n \int_{P_{\mathcal{I},k}} h(x)dx,$$

but in order to estimate the right side of (1.0.9), we still need an estimate for

$$k^n \int_{P \setminus P_{\mathcal{I},k}} h(x)dx. \tag{6.4.4}$$

We will compare (6.4.4) to the sum

$$\sum_{p \in \mathcal{F}_k} h(x_{p,k}), \tag{6.4.5}$$

where $x_{p,k}$ is some arbitrary point in $B = \square_k(p)$. Note that if $x_{p,k}, x'_{p,k} \in \square_k(p)$, then

$$|h(x_{p,k}) - h(x'_{p,k})| \leq \frac{\sqrt{n}}{k} ||h||_{C^1(B)}. \tag{6.4.6}$$

Now let $m_k(p) \in \square_k(p)$ be such that $\min_{\square_k(p)} h = h(m_k(p))$ and define $M_k(p)$ similarly to be where h takes its maximum. We have then that

$$\sum_{p \in \mathcal{F}_k} h(m_k(p)) \leq k^n \int_{P_{\mathcal{F},k}} h(x)dx \leq \sum_{p \in \mathcal{F}_k} h(M_k(p)). \tag{6.4.7}$$

Lemma 6.4.4. *There exists a constant C depending only on the dimension n, the geometry of P, and $||h||_{C^1(P)}$ such that*

$$\left| k^n \int_{P \setminus P_{\mathcal{I},k}} h d\mu - \frac{1}{2} \sum_{p \in \mathcal{F}_k} h(x_{p,k}) \right| \leq C k^{n-2}.$$

where, as before, $x_{p,k}$ is any point in $\square_k(k)$.

Proof. Given (6.4.7) and (6.4.6), we need only show

$$\left| k^n \int_{P \setminus P_{\mathcal{I},k}} h d\mu - \frac{1}{2} k^n \int_{P_{\mathcal{F},k}} h d\mu \right| \leq C k^{n-2}. \tag{6.4.8}$$

The idea of this proof is the following observation: Assume we are given a rational plane $H \subset \mathbb{R}^n$ through the origin which cuts $\mathbb{R}^n$ into two pieces S_1 and S_2. Furthermore, assume we are given a hypercube $B = \Box_1(p)$ such that H intersects the interior of B. If B' is the hypercube given by reflecting B about the origin, then the pair (B, B') has the property that $\mathrm{Vol}(B \cap S_1) + \mathrm{Vol}(B' \cap S_1) = 1$. We will use this idea to prove (6.4.8) by considering each of the different lattice points in $\mathcal{F}_k$ as our "origin".

We would ideally like to proceed as follows. Let $p \in \mathcal{F}_k$ be a lattice point and let $q \in \mathcal{E}_k$ be the unique point in $\mathcal{E}_k$ which is closest to $B = \Box_k(p)$. Then let p' be the corner point of the box B' given by reflecting B about the point q.

There are two problems with this approach. The first is that the corresponding point p' may not lie in $\mathcal{F}_k$. This will be true for the boxes B which are close to the boundary of F. We deal with this problem by not considering the points p which have no corresponding point. The second problem is that the point q need not actually be unique. This could be handled multiple ways, but the easiest seems to be to do the following: Let d be the distance from B to the lattice $\mathcal{E}_k$. Let Q_B be the set of $q \in \mathcal{E}_k$ such that $\mathrm{dist}(B, q) = d$. Let N_B the number of elements in Q_B. Finally consider N_B pairs (B, B')—one for each different $q \in Q_B$—and in the end weight each pair by the fraction $\frac{1}{N_B}$. This allows us to compare the two integrals in (6.4.8) with the desired precision. □

Taken together, these lemmas result in the following:

Lemma 6.4.5. *There is a constant C depending only on the geometry of P, $||h||_{C^2}$, and the dimension n so that*

$$\left| k^n \int_P h(x) - \left(\sum_{p \in \mathcal{I}_k} h(p_{k,m}) + \frac{1}{2} \sum_{p \in \mathcal{F}_k} h(x_{p,k}) \right) \right| \leq C k^{n-2}, \tag{6.4.9}$$

where $x_{p,k}$ is an arbitrary point in $\Box_k(p)$ as before.

We are finally in the position to prove Theorem 6.

Proof. To prove this we may assume that P is a "stretched standard simplex". I.e. that there is a vertex v of P, such that if one chooses v as the origin, then P is given as the convex hull of the origin v and the points $p_1e_1, \dots, p_ne_n$, where $p_i > 0$ and e_i is the standard basis vector. Any polytope P can be deconstructed into such stretched standard simplices and then if one applies Theorem 6 to each piece, one gets the result for all of P.

The asymptotic sum S_k we need to approximate is given by

$$S_k = \sum_{p\in\mathcal{P}_k} h(p) = \sum_{p\in\mathcal{I}_k} h(p) + \sum_{p\in\mathcal{F}_k} h(p) + \sum_{p\in\mathcal{E}_k} h(p). \tag{6.4.10}$$

The main idea of the proof is to compare (6.4.10) with the "midpoint rule" for integrals. Given (6.4.9), we only need to understand the asymptotics of the difference $S_k - M_k$, where

$$M_k = \left(\sum_{p\in\mathcal{I}_k} h(p_{k,m}) + \frac{1}{2}\sum_{p\in\mathcal{F}_k} h(p_{k,m}) \right). \tag{6.4.11}$$

Now let p be any lattice point in $\partial P \setminus F$. Let $l_{p,k}(j) = p + \frac{1}{2k}(j, \dots, j)$. I.e. $l_{p,k}(0) = p$, $l_{p,k}(1) = p_{k,m}$, etc. Let $L_{p,k} = l_{p,k}(\mathbb{R})$ be the line through p parallel to the vector $(1, \dots, 1)$.

Now for each p, we will define an alternating sum $A_{p,k}$ as follows. If $p \in \partial P \cap F$, then define $A_p = h(p)$. Otherwise, if $L_{p,k} \cap P$ is a line segment connecting p to an element of $\mathcal{E}_k$, define $A_{p,k}$ as

$$A_{p,k} = h(l_{p,k}(0)) - h(l_{p,k}(1)) + h(l_{p,k}(2)) - h(l_{p,k}(3)) \pm \cdots + h(l_{p,k}(N)),$$

where we have $l_{p,k}(N) \in \mathcal{E}_k$ is that final terminating point. Finally, if p satisfies neither of the preceding requirements, define

$$\begin{aligned} A_{p,k} = h(l_{p,k}(0)) - h(l_{p,k}(1)) + h(l_{p,k}(2)) - h(l_{p,k}(3)) \pm \cdots \\ \cdots + h(l_{p,k}(N-1)) - \frac{1}{2} h(l_{p,k}(N)), \end{aligned}$$

where $l_{p,k}(N)$ is the midpoint of the box $\square_k(q)$ and q is the point in $\mathcal{F}_k$ which lies on the line $L_{p,k}$.

With this setup, we have that

$$S_k - M_k = \sum_{p\in\partial P \setminus F} A_{p,k}.$$

Now if $L_{p,k} \cap P$ is a line terminating in a point in $\mathcal{E}_k$, then we have that

$$A_{p,k} = \frac{1}{2}h(l_{p,k}(0)) + \frac{1}{2}\Big\{[h(l_{p,k}(0)) - h(l_{p,k}(1))] - [h(l_{p,k}(1)) - h(l_{p,k}(2))] + \cdots - [h(l_{p,k}(N-1)) - h(l_{p,k}(N))]\Big\} + \frac{1}{2}h(l_{p,k}(N)).$$

Due to convexity, the middle terms form an alternating series, and hence $A_{p,k} = \frac{1}{2}[h(l_{p,k}(0)) + h(l_{p,k}(N))] + \frac{C}{k}$, for some constant C depending on the derivative of h. Going back to the case where $L_{p,k} \cap P$ does not terminate in a point in $\mathcal{E}_k$, similar arguments show that $A_{p,k} = \frac{1}{2}h(l_{p,k}(0)) + \frac{C}{k}$, with C once again depending upon $||h||_{C^1(P)}$.

Combining these results we have that up to highest order $S_k - M_k = \frac{1}{2}\sum_{p \in \partial P} h(p)$. If we apply Lemma 6.4.1, the proof of Theorem 6 is complete. □

Remark: In the preceding proof we essentially only used the fact that h is convex along lines parallel to the vector $(1, \cdots, 1)$. One may be tempted to conclude that that is all that is necessary for Theorem 6. However, in the previous proof we assumed that P was in the form of a stretched standard simplex. If P is arbitrary, we would need to decompose it into stretched standard simplices and apply this result to each one individually. On those other simplices, we would most likely have to change orientations and consider lines that are going in other directions. Hence in general we do need h to be convex in all directions for Theorem 6 to be true.

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