There Exists a Problem Whose Computational Complexity is Any Given Function of the Information Complexity

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Abstract: We present an information-based complexity problem for which the computational complexity can be any given increasing function of the information complexity, and the information complexity can be any non-decreasing function of ε^{-1}, where ε is the error parameter.

Introduction

Information-based complexity, IBC, studies the computational complexity of continuous problems. The ε-complexity of a problem is the infimum cost among all information and algorithms that compute an ε-approximation to the solution, and we denote it by $\text{comp}(\varepsilon)$, see TWW[88].

As defined in WW[92], the *information complexity*, $\text{comp}^{\text{Info}}(\varepsilon)$, is the minimal cost of computing an ε-approximation when there is no charge for combining the information. On the other hand, the *combinatory complexity*, $\text{comp}^{\text{Comb}}(\varepsilon)$, is the minimal cost of computing an ε-approximation when there is no charge for computing the information.

Assume we are charged a constant c for the computation of each information evaluation and unity for each arithmetic and comparison operation. Usually $c \gg 1$. For a given problem, let $m(\varepsilon)$ be the minimal information, i.e., the minimal number of information evaluations, needed to compute an ε-approximation. The *information complexity* is thus equal to $c\,m(\varepsilon)$. Obviously, $\text{comp}(\varepsilon) \geq c\,m(\varepsilon)$.

We know tight bounds on the complexity for many problems. To date, for problems for which the complexity has been obtained, the complexity is a linear function of $m(\varepsilon)$; that is,

$$\text{comp}(\varepsilon) = c_1\, m(\varepsilon) \quad \text{for} \quad c_1 = c + O(1).$$

This means that the complexity is dominated by the information complexity. We stress, however, that $m(\varepsilon)$ can be any increasing function of ε^{-1}, see TWW[88] for examples.

In this note, we will construct a problem for which the complexity is dominated by the combinatory complexity. More precisely, we prove that for a given increasing function g, we define a problem for which

$$\mathrm{comp}(\varepsilon) = c\,m(\varepsilon) + c_1\, g(m(\varepsilon)),$$

where $1 \le c_1 < 3$. Hence, if $g(x)/x$ goes to $+\infty$ as x goes to $+\infty$, then the combinatory complexity $c_1\, g(m(\varepsilon))$ dominates the information complexity $c\,m(\varepsilon)$. Furthermore we can define the problem in such a way that $m(\varepsilon) = q(\varepsilon^{-1})$ for any given increasing function q from $\mathbf{N}$ to $\mathbf{N}$ if $\varepsilon^{-1} \in \mathbf{N}$.

Basic Definitions

We consider the problem of approximating $S(x)$ where

$$S : F \subseteq F_1 \mapsto G$$

is a given operator, F is a subset of a linear space F_1 over $\mathbf{R}$, and G is a metric linear space with metric ρ over $\mathbf{R}$.

Let Λ be a given class of linear functionals $L : F_1 \mapsto \mathbf{R}$. By an approximation $U(x)$ to $S(x)$ using the information

$$N(x) = [L_1(x), L_2(x), \ldots, L_n(x)]$$

we mean the mapping

$$U(x) = \phi(N(x)), \text{ where } \phi : N(F) \mapsto G,$$

and the L_i are from Λ. The choice of L_i as well as the number $n = n(x)$ of functionals may depend adaptively on the values of already computed functionals. The *cardinality* of N, denoted by $\mathrm{card}\, N$, is defined as $\sup_{x \in F} n(x)$.

We wish to compute U with error at most ε. The worst case error of U is defined by

$$e(U) = \sup_{x \in F} \rho(S(x), U(x)).$$

If $e(U) \le \varepsilon$ then $U(x)$ is called an ε-approximation of $S(x)$.

Our model of computation is based on the real number model for which we can exactly perform the four arithmetic operations and the comparison operation on real numbers. In addition, we can add two elements of G and multiply a real number by an element of G. We are charged unity for each such operation.

We assume that *precomputation* is allowed. That is, we can precompute a finite number $n = n(\varepsilon)$ of elements from G and use them for free. We are also charged a constant c for computing each $L(x)$ for $L \in \Lambda$. Usually $c \gg 1$.

The worst case cost of U, denoted by $\text{cost}(U)$, is defined by

$$\text{cost}(U) = \sup_{x \in F}(\text{cost}(N, x) + \text{cost}(\phi, N(x)),$$

where $\text{cost}(N, x)$ denotes the cost of computing $N(x)$, and $\text{cost}(\phi, N(x))$ denotes the cost of computing $\phi(y)$ for given $y = N(x)$.

The worst case ε-complexity, $\text{comp}(\varepsilon)$, is defined by

$$\text{comp}(\varepsilon) = \inf\{ \ \text{cost}(U) \mid U \text{ such that } e(U) \le \varepsilon\}.$$

Finally, the minimal information, $m(\varepsilon)$, is defined by

$$m(\varepsilon) = \inf\{ \ n \mid \exists \ U = (N, \phi) \text{ such that } \text{card}\, N \le n \text{ and } e(N, \phi) \le \varepsilon\}.$$

Obviously, $\text{comp}^{\text{Info}}(\varepsilon) = c\, m(\varepsilon)$.

Main Result

We now state the main result of this paper. For simplicity we vary ε such that $\varepsilon^{-1} \in \mathbb{N}$.

Theorem: *Let $g,\ q : \mathbf{N} \mapsto \mathbf{N}$ be any given increasing functions with $g(1) = q(1) = 1$. Then there is a problem such that*

1. $m(\varepsilon) = q(\varepsilon^{-1})$, $\forall \varepsilon^{-1} \in \mathbf{N}$;

2. *comp*$(\varepsilon) = c\, m(\varepsilon) + c_1\, g(m(\varepsilon))$, $\forall\, \varepsilon^{-1} \in \mathbf{N}$, *where* $1 \leq c_1 < 3$.

Proof:

For the given functions $g(\cdot)$ and $q(\cdot)$ we define the problem as follow. Let

$$X_1 = \{x \in \mathbf{R}^\infty \mid \text{ the number of nonzero components of } x \text{ is finite } \}.$$

Now we define the metric ρ on X_1.

Denote $k_i = g(i)$ and $q_i = q(i)$ for $i \geq 1$. Then $K = \{k_i\}$ and $Q = \{q_i\}$ are subsequences of $\mathbf{N}$ with $q_1 = k_1 = 1$.

We construct a sequence $\{\lambda_i\}$ such that $2 > \lambda_1 > \lambda_2 > \ldots > 0$ in the following way. First we define a subsequence of $\{\lambda_i\}$ by

$$\lambda_{k_{q_j}} = 1/j, \ \text{ for } j = 1, 2, \ldots .$$

Note that $1 = \lambda_{k_{q_1}} > \lambda_{k_{q_2}} > \ldots > 0$. Then for each $i \in \mathbf{N} - \{k_{q_j}\}$, define λ_i so that $1 \geq \lambda_1 > \lambda_2 > \ldots > 0$.

For $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in X_1$ we define the metric on X_1 by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 2 & \text{if } x_1 \neq y_1; \\ \lambda_i & \text{if } x_j = y_j \text{ for } j = 1, \ldots, i \text{ and } x_{i+1} \neq y_{i+1}. \end{cases}$$

Then (X_1, ρ) forms a metric linear space over $\mathbf{R}$.

We define $F_1 = X_1$, and $G = X_1$ with metric $\rho(\cdot, \cdot)$. Thus, F_1 is a linear space over $\mathbf{R}$ and G is a metric linear space over $\mathbf{R}$.

Define $S : F = F_1 \mapsto G$ by

$$S(x) = (x_1^1, x_1^2, \ldots, x_1^{k_2-k_1}, x_2^1, x_2^2, \ldots, x_2^{k_3-k_2}, \ldots, x_i^1, x_i^2, \ldots, x_i^{k_{i+1}-k_i}, \ldots).$$

This means, S has the form

$$S(x) = \sum_{i=1}^{\infty} \sum_{j=k_i}^{k_{i+1}-1} x_i^{j-k_i+1} e_j,$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots)$ with the 1 on the jth place. Note that S is not linear if $K \neq \mathbf{N}$.

We assume that the classs Λ consists of all linear functionals ¿From F_1 to $\mathbf{R}$. That is, $L \in \Lambda$ is of the form $L(x) = \sum_{i=1}^{\infty} v_i x_i$, for some $\mathrm{v} \in \mathbf{R}^{\infty}$. This concludes the definition of our problem.

We now prove that for our problem we have $m(\varepsilon) = q(\varepsilon^{-1})$ for $\varepsilon^{-1} \in \mathbf{N}$.

Indeed, denote $j = \varepsilon^{-1}$. Then $\lambda_{k_{q_j}} = \varepsilon$ by the definition of $\lambda_{k_{q_j}}$. Suppose $U(x) = \phi(N(x))$ is an ε-approximation to $S(x)$. We want to show that $\operatorname{card} N \geq q_j$.

We have $e(U) \leq \varepsilon$. By the discreteness of the metric ρ on G and the definition of $e(U)$, there is $n \in \mathbf{N}$ such that $\lambda_n = e(U)$. Hence the first n components of $U(x)$ are exactly equal to the corresponding components of $S(x)$ for any $x \in F$.

Since $\lambda_n \leq \varepsilon$ and $\{\lambda_i\}$ is a monotonically decreasing sequence, $\lambda_{k_{q_j}} \geq \lambda_n$, which implies that $k_{q_j} \leq n$.

To compute the first k_{q_j} components of $S(x)$ exactly, one must know $x_1, x_2, \ldots, x_{q_j}$. ¿From the definition of $S(x)$, one can only obtain the values of $x_1, x_2, \ldots, x_{q_j}$ by using information operators, and one needs at least q_j of them. Hence $\operatorname{card} N \geq q_j = q(\varepsilon^{-1})$. Thus $m(\varepsilon) \geq q(\varepsilon^{-1})$.

To prove the reverse inequality, it suffice to compute $x_1, x_2, \ldots, x_{q_j}$ by using the information operator $N(x) = [L_{e_1}(x), L_{e_2}(x), \cdots, L_{e_{q_j}}(x)]$. Then we obtain the rest of n components (i.e., the lth component of $S(x)$ for $l \leq n$ and $l \notin \{k_1, k_2, \cdots, k_{q_j}\}$) by computing the corresponding powers of $x_1, x_2, \ldots, x_{q_j}$. Finally we define

$$U^*(x) = \sum_{i=1}^{q_j-1} \sum_{l=k_i}^{k_{i+1}-1} x_i^{l+1-k_i} e_l + \sum_{l=k_{q_j}}^{n} x_{q_j}^{l+1-k_{q_j}} e_l.$$

Then $U^*(x)$ is an ε-approximation to $S(x)$. Hence $m(\varepsilon) \leq q(\varepsilon^{-1})$. This proves the first part of the theorem.

We now show the second part of the theorem. Let $m = m(\varepsilon)$ be the minimal information for an ε-approximation to $S(x)$. Then any ε-approximation $U(x) = \phi(N(x))$ to $S(x)$ has to compute at least the first k_m components of $S(x)$ exactly for any $x \in F$. Therefore U has to agree with U^* for the first k_m components. Thus

$$\text{comp}(\varepsilon) \geq c\, m(\varepsilon) + k_m,$$

since k_m is the minimal cost of combining k_m real numbers to form the element $U(x)$ in G. Recall that $k_m = g(m)$, we have

$$\text{comp}(\varepsilon) \geq c\, m(\varepsilon) + g(m(\varepsilon)).$$

Finally, we want to show that $\text{comp}(\varepsilon) < c\, m(\varepsilon) + 3\, g(m(\varepsilon))$.

Since $\varepsilon^{-1} \in \mathbf{N}$, $\varepsilon = 1/j = \lambda_{k_{q_j}}$ for some $j \in \mathbf{N}$. Since $q_j = q(j) = q(\varepsilon^{-1}) = m(\varepsilon)$, $\varepsilon = \lambda_{k_m}$ with $m = m(\varepsilon)$.

Let $N(x) = [L_{e_1}(x), L_{e_2}(x), \cdots, L_{e_m}(x)]$. Using $N(x)$, we can compute

$$U(x) = \sum_{i=1}^{m-1} \sum_{l=k_i}^{k_{i+1}-1} x_i^{l+1-k_i} e_l + x_m e_{k_m}.$$

Note that the above U consists of k_m terms. To compute $U(x)$, we compute $x_i^{l+1-k_i}$ for $l + 1 - k_i > 1$ using $k_m - m$ multiplications, then we multiply

$x_i^{l+1-k_i}$ by e_l using once more k_m multiplications, and finally we form $U(x)$ by adding $x_i^{l+1-k_i} e_l$ together using $k_m - 1$ additions. The total cost of elementary operations is $k_m - m(\varepsilon) + 2\,k_m - 1$, which is less than $3k_m$.

Therefore, we see that $\text{comp}(\varepsilon) < c\,m(\varepsilon) + 3\,k_m = c\,m(\varepsilon) + 3\,g(m(\varepsilon))$.

Hence we have $\text{comp}(\varepsilon) = c\,m(\varepsilon) + c_1\,g(m(\varepsilon))$ for some $1 \le c_1 < 3$, as desired.

Comments

We remark that the construction of the problem presented in the proof can be generalized for

$$S(x) = (h_1(x_1), h_2(x_2), \ldots, h_i(x_i), \ldots)$$

with the same F_1 and G as in the proof. Here $h_i : \mathbf{R} \mapsto \mathbf{R}$ for $i = 1, 2, \ldots$ are the functions such that the computation of $h_i(x_i)$ requires m_i operations in the worst case. Then it can be shown that

$$\text{comp}(\varepsilon) = c\,m(\varepsilon) + c_1 \sum_{i=1}^{m(\varepsilon)} m_i,$$

with $1 \le c_1 < 3$.

Hence, if we choose h_i with large m_i, the complexity can be once more much larger than the information complexity. For example, let $m_i = 2^i$. Then

$$\text{comp}(\varepsilon) = c\,m(\varepsilon) + 2\,c_1\,\left(2^{m(\varepsilon)} - 1\right)$$

and the complexity is the exponential function of $m(\varepsilon)$.

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References

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