The Effect of Dissipation on Spatially Growing Nonlinear Baroclinic Waves*

L. M. POLVANI

MIT-WHOI Joint Program in Physical Oceanography, Center for Meteorology and Physical Oceanography, MIT, Cambridge, Massachusetts

J. PEDLOSKY

Woods Hole Oceanograhic Institution, Woods Hole, Massachusetts

(Manuscript received 19 August 1987, in final form 5 January 1988)

ABSTRACT

The question of convective (i.e., spatial) instability of baroclinic waves on an f-plane is studied in the context of the two-layer model. The viscous and inviscid marginal curves for linear convective instability are obtained. The finite-amplitude problem shows that when dissipation is O(1) it acts to stabilize the waves that are of Eady type. For very small dissipation the weakly nonlinear analysis reveals that at low frequencies, contrary to what is known to occur in the temporal problem, in addition to the baroclinic component a barotropic correction to the "mean" flow is generated by the nonlinearities, and spatial equilibration occurs provided the ratio of shear to mean flow does not exceed some critical value. In the same limit, the slightly dissipative nonlinear dynamics reveals the presence of large spatial vacillations immediately downstream of the source, even if asymptotically (i.e., very far away from the source) the amplitudes are found to reach steady values. No case of period doubling or aperiodic behavior was found. The results obtained seem to be qualitatively independent of the form chosen to model the dissipation.

1. Introduction

In recent years, much attention has been given to the question of absolute instability of baroclinic flows. Thacker (1976) investigated the familiar two-layer model in the context of Gulf Stream meander growth, and Merkine (1977) revisited the same problem in connection with cyclogenesis. More recently, Farrell (1982, 1983) and Pierrehumbert (1986) have examined the question for the Charney problem (Charney 1947).

The heart of the matter is the response of a stratified rotating fluid to a localized source of perturbations. If the perturbations are found to decay in time, the system is said to be stable. If the perturbations grow, two situations are possible. In the case of convective instability a disturbance will grow as it moves away from the source but will propagate faster than it spreads so that at any given point in space the response will vanish after a long enough time. Alternatively, when the system is absolutely unstable, an initial disturbance will, at any given point, grow faster than it moves away from that point, with the net result being that the response of the system will amplify with time at every point in space.

For perturbations that are independent of the meridional coordinate, the two-layer model on an f-plane was found by Thacker (1976) to possess absolute instability when the ratio of the shear to the mean flow exceeds the value of $1/\sqrt{2}$. Merkine (1977) extended this result to include disturbances with meridional variations and derived an approximate condition for absolute instability. If l is the meridional wavenumber, F the Froude number of the system and

$$\eta = (V_1 - V_2)/(V_1 + V_2),$$

where V_1 and V_2 are the dimensional mean zonal velocities in the upper and lower layer respectively, the two-layer model is absolutely unstable for

$$\eta > (2 - l^2F)^{-1/2}. \tag{1.1}$$

However, for the Charney problem, in the case when the surface velocity is greater or equal to zero, Pierrehumbert (1986) has shown that no absolute instability is present. This would suggest that the presence of absolute instability may only be a property of the two-layer model, and that spatially growing modes, which are the consequence of convective instability, may in fact play a more important role than previously thought.

In the two-layer model, as Merkine (1977) points out, the spatial modes possess several properties that

* Woods Hole Oceanographic Institution Contribution No. 6635.

Corresponding author address: Dr. L. M. Polvani, Center for Meteorology and Physical Oceanography, MIT 54-1421, Cambridge, MA 02139.

distinguish them from the more familiar temporal modes: they are weakly dispersive, their phase speed is not equal to the speed of the mean flow and their spatial growth rate is not linear in the shear, as is the case for the temporal modes. Moreover, one can show from the energetics that, in addition to the familiar conversion of available potential into kinetic energy, spatially growing modes can extract energy from the mean flow through the work of ageostrophic pressure gradients and Coriolis forces.

In order to investigate further the dynamics of spatially amplifying modes, Merkine (1978) examined their finite-amplitude dynamics in the two-layer model and found that weak nonlinearity may lead to both amplitude equilibration and subcritical destabilization depending on the values of F and η. He found no spatial vacillation or aperiodic behavior, however, because he was considering an inviscid system.

In the presence of Ekman dissipation, the temporal evolution of finite-amplitude baroclinic waves in the two-layer model is well known to exhibit a rich variety of behaviors (Pedlosky and Frenzen 1980). For large dissipation, the amplitude evolves to a steady value. As dissipation is decreased periodic vacillations appear and, in some ranges, the solution is found to be aperiodic. As dissipation is decreased even further, the solution becomes again periodic, but the amplitude of the vacillation becomes independent of the value of the dissipation.

For the temporal problem, it is also known that the form chosen to model the dissipation plays a crucial role in the presence or absence of nonsteady solutions (Pedlosky and Polvani 1987). When dissipation is taken to be proportional to potential vorticity (as opposed to the Ekman case) the amplitude vacillation due to wave–mean flow interactions disappears.

In the light of these results, the principal goal of the present work is therefore to investigate the role played by dissipation in the weakly nonlinear dynamics of spatially growing baroclinic waves, in particular with respect to the possible existence of spatial finite-amplitude vacillations. The existence of periodic and possibly aperiodic variations in the eddy activity downstream of a source of perturbations might then be ascribed uniquely to the nonlinear character of the dynamics without the need to invoke other physical mechanisms.

2. The linear convective stability problem

We consider the well-known two-layer model (Pedlosky 1970) in a zonal channel between $y = 0$ and $y = 1$, on an f-plane. The geostrophic perturbation streamfunctions ψ_i on the zonal mean flow U_i are known to evolve according to the quasi-geostrophic potential vorticity equations, which can be written in nondimensional form:

$$[\partial_t + U_i\partial_x + J(\psi_i,\cdot)]\Pi_i + F(U_i - U_j)\partial_x\psi_i = -r\nabla\psi_i - rqF(\psi_j - \psi_i),$$
$$i = 1, 2 \quad \text{and} \quad j = 3 - i \qquad (2.1)$$

where

$$\Pi_i = \nabla^2\psi_i + F(\psi_j - \psi_i)$$

is the potential vorticity. Four nondimensional parameters appear in these equations: r measures the strength of the dissipation and F is the Froude number. Because we have chosen the mean flow $(V_1 + V_2)/2$ as the velocity scale, the nondimensional quantities U_1 and U_2 are related to the third parameter η (defined in the previous section) by the relations: $U_1 = 1 + \eta$ and $U_2 = 1 - \eta$.

The fourth parameter q has been introduced in order to represent two kinds of dissipation: For $q = 1$, dissipation will be proportional to the potential vorticity itself, while, for $q = 0$ only Ekman dissipation will be present. As mentioned earlier, the form of the dissipation is known to play a crucial role in determining the behavior of the finite-amplitude solutions for the temporal problem; it is therefore important to be able to incorporate both types in the same formalism for the spatial problem of interest here.

Upon substitution of exponential solutions of the form:

$$\psi_i = A_i \exp[i(kx + ly - \omega t)]$$

into (2.1) and linearization, one obtains the dispersion relation for the two-layer model, with can be written as follows:

$$k^4[U_1U_2] - 2k^3\omega + k^2[\omega^2 + F(U_1{}^2 + U_2{}^2) + l^2U_1U_2] - 2k\omega[l^2 + 2F] + \omega^2[l^2 + 2F] - 2ir[k - \omega][k^2 + l^2 + (q + 1)F] - r^2[k^2 + l^2 + 2qF] = 0. \qquad (2.2)$$

(Notice that this is a fourth-order polynomial in k but is only second order in ω. On a β-plane the two-layer model dispersion relation would have been a sixth-order polynomial in k. As will become apparent in what follows, it is for reasons of simplicity that we have chosen to limit ourselves to the case $\beta = 0$ in this study).

Consider first the temporal inviscid problem, for which $r = 0$. One chooses k to be real, solves the quadratic for ω as a function of k and obtains the result:

$$\omega(k) = k \pm \eta\left[\frac{k^2 + l^2 - 2F}{k^2 + l^2 + 2F}\right]^{1/2}. \qquad (2.3)$$

When $\text{Im}(\omega) > 0$ (<0), the system is unstable (stable).

The critical value of F is then found by setting $\mathrm{Im}(\omega) = 0$. This yields the marginal inviscid temporal stability curve:

$$F_{it}(k) = \frac{k^2 + l^2}{2}. \quad (2.4)$$

One would like to proceed in an analogous way to determine the spatial (i.e., convective) instability. The procedure in this case is not as simple, however. First, since the dispersion relation is a fourth order polynomial in k, one cannot solve analytically for $k(\omega)$ in a useful way. Second, one cannot deduce from the sign of $\mathrm{Im}(k)$ whether a given mode is amplifying or evanescent (and thereby deduce the stability of the system).

Briggs (1964) has developed a straightforward procedure that allows one to distinguish between spatial amplification and evanescence directly from the dispersion relation. For a given fixed $\mathrm{Re}(\omega)$, one follows the trajectory, in the complex k plane, of the roots $k(\omega)$ of the dispersion relation as $\mathrm{Im}(\omega)$ is varied from positive infinity to zero. The modes whose trajectory crosses the real k axis are amplifying; the other are evanescent.

By carrying out this procedure for many values of F and ω, one can construct the inviscid spatial marginal curve $F_{is}(\omega)$. In practice one finds that, for a given real ω, two of the four roots are complex and evanescent for all values of F; the other two roots are real for $F < F_{is}$. When $F > F_{is}$ they acquire a nonzero imaginary part and are complex conjugates. Contrary to the claim of Merkine (1977) only one of these two roots is amplifying, namely the one which grows in the direction of the mean flow; the other one is evanescent.

In Fig. 1a we show both F_{it} and F_{is} as functions of ω for the values $\eta = 0.8$ and $l = \pi$; $F_{it}(\omega)$ is obtained from $F_{it}(k)$ by noting that on the temporal marginal curve $\omega \equiv k$. Note that the two curves become identical in the limit of small ω and converge to $F = l^2/2$ for $\omega \to 0$. In Fig. 1b we have plotted the values of k on the marginal curves.

Surprisingly enough, it turns out that when $r \neq 0$ the problem of finding the marginal curves is actually simpler than in the inviscid case. Indeed, by inspection of the dispersion relation (2.2), one can see that the only way it can be satisfied with ω and k both real is when the imaginary term proportional to $2ir$ vanishes identically; for this to happen one must require that $\omega = k$ on the marginal curve. From the other terms one then derives an analytic expression for the viscous marginal curve $F_v(\omega)$, which is the same for both the temporal and spatial problem:

$$F_v(\omega) = \frac{\dfrac{\omega^2 + l^2}{2} + r^2 \dfrac{\omega^2 + l^2}{2\omega^2\eta^2}}{1 - q\dfrac{r^2}{\omega^2\eta^2}}. \quad (2.5)$$

In Fig. 2a and 2b, the viscous marginal curves $F_v(\omega)$ are plotted versus ω at several values for r for $\eta = 0.8$, $l = \pi$ and $q = 0$ and $q = 1$, respectively, together with the inviscid spatial marginal curve F_{is}. The region above the marginal curves is unstable. When the dissipation is proportional to potential vorticity (i.e., q

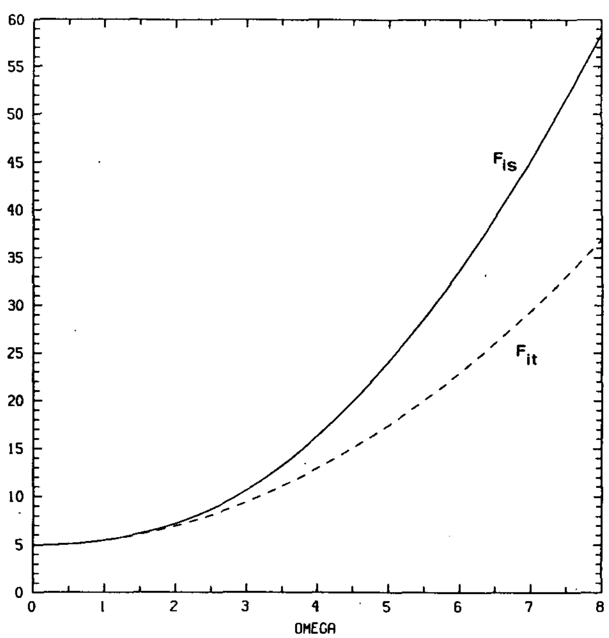

FIG. 1a. The inviscid spatial (F_{is}) and temporal (F_{it}) marginal stability curves, for $\eta = 0.8$ and $l = \pi$.

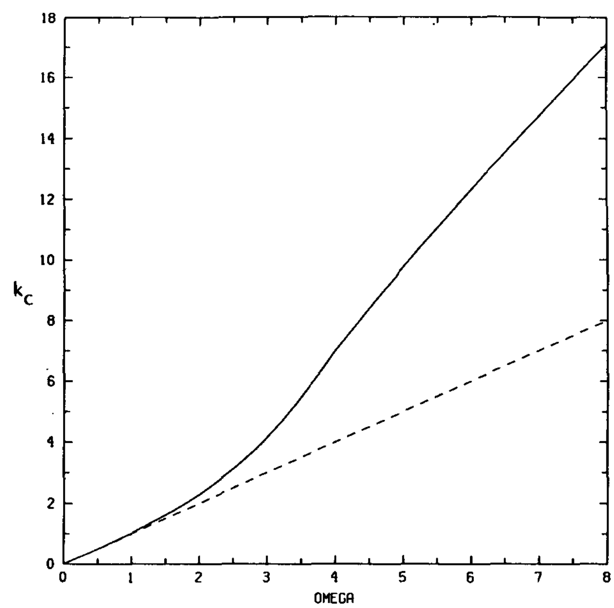

FIG. 1b. The marginal inviscid spatial wavenumber k_c vs ω for $\eta = 0.8$ and $l = \pi$. The dashed line is the temporal inviscid marginal $k = \omega$.

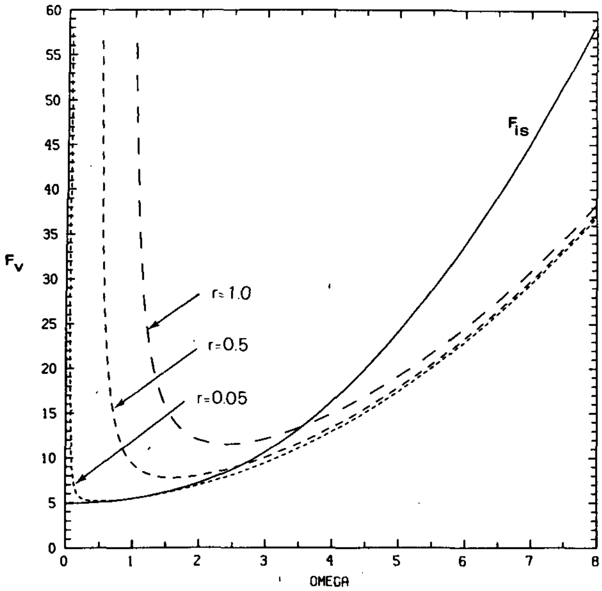

FIG. 2a. The viscous spatial marginal stability curve F_v for $\hat{r} = 0.05$, 0.5 and 1.0, for $q = 0$, $\eta = 0.8$ and $l = \pi$. The solid line is the inviscid spatial marginal stability curve F_{is}.

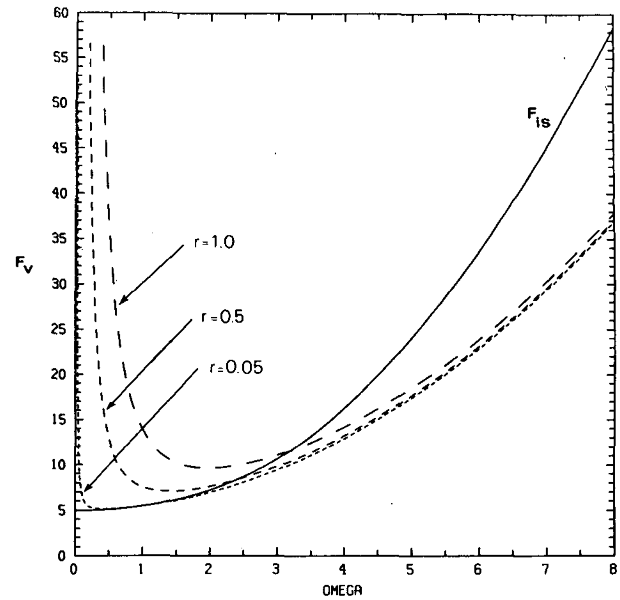

FIG. 2b. As in Fig. 2a, but with $q = 1$.

$= 1$) F_v has a singularity at $\omega_s = r/\eta$, and all frequencies smaller than ω_s are stable.

It is important to note that, while for the temporal problem, as $r \rightarrow 0$ the viscous marginal curve tends toward the inviscid curve, this is not the case for the spatial problem. Indeed, as $r \rightarrow 0$, F_v tends toward $(\omega^2 + l^2)/2$ which does not correspond to F_{is}. In other words, as dissipation in decreased to zero the *spatial viscous* marginal curve approaches the *temporal inviscid* curve and, even for $r \ll 1$, is different by an O(1) amount from the spatial inviscid marginal curve. This fact has crucial consequences on the degree of the spatial amplitude equation that we wish to obtain.

More specifically, consider first the region where ω is large, so that the viscous marginal curve differs by an O(1) amount from the inviscid curve (even for $r \ll 1$). A similar situation occurs for the temporal problem with $\beta \neq 0$ (Romea 1977). One can expect that, since the marginal curve is not the point where two roots coalesce, the amplitude equation will be of first order.

On the other hand, when $\omega \ll 1$ and $r \ll 1$, where the spatial viscous and spatial inviscid curves are identical at O(1), i.e., their difference is of O(r^2), we expect a second-order amplitude equation, since two roots of the dispersion relation do coalesce on the marginal curve.

Before proceeding to the finite-amplitude problem, we point out that the concept of convective instability is only meaningful when the system is free from absolute instabilities. By establishing the convective marginal stability curve, we have determined that, when F exceeds the some critical value, the system possesses a spatially amplifying mode. However, where the system is absolutely unstable for that value of F, the existence of a spatially growing mode would be irrelevant since the response of the system would be growing with time at every point in space.

In practice there exists an upper bound on F beyond which the two-layer model becomes absolutely unstable. For the inviscid case, the upper bound on F is approximately given by the condition (1.1). The finite-amplitude theory that we present next will only be applicable for those values of F for which the system is free from absolute instabilities.

3. The convective finite-amplitude problem for $r = O(1)$

We now turn our attention to the spatial finite-amplitude evolution of a slightly supercritical baroclinic wave when the dissipation is O(1), in which case F_v and F_{is} differ by O(1). We take the Frounde number to be

$$F = F_v + \Delta \qquad (3.1)$$

where $\Delta \ll 1$ is the value of the supercriticality. It is easy to show from the dispersion relation (2.2) that for such an unstable wave the spatial growth rate will be O(Δ). The geostrophic fields can then be considered to depend on a long space variable $X \equiv \Delta x$, in addition to the short space variables x and y and the time variable t. After the appropriate change of variables and substitution of (3.1) into (2.1) the potential vorticity equation takes the form:

$$(\partial_t + U_i\partial_x)[\nabla^2\psi_i + F_v(\psi_j - \psi_i)] + F_v(U_i - U_j)\partial_x\psi_i + r[\nabla^2\psi_i + qF_v(\psi_j - \psi_i)]$$
$$= -\Delta\{U_i\partial_X[\nabla^2\psi_i + F_v(\psi_j - \psi_i)] + (\partial_t + U_i\partial_x)[2\partial_x\partial_X\psi_i + (\psi_j - \psi_i)] + F_v(U_i - U_j)\partial_X\psi_i$$
$$+ (U_i - U_j)\partial_x\psi_i + r[2\partial_x\partial_X\psi_i + q(\psi_j - \psi_i)]\} - J\{\psi_i, [\nabla^2\psi_i + F_v(\psi_j - \psi_i)]\} + \text{hot} \quad (3.2)$$

for $i = 1, 2$ and $j = 3 - i$; hot ≡ higher order terms.

Now the fields themselves must be expanded as power series in $\Delta^{1/2}$. We therefore write:

$$\psi_i = \Delta^{1/2}[\psi_i^{(0)} + \Delta^{1/2}\psi_i^{(1)} + \Delta\psi_i^{(2)} + \text{hot}]. \quad (3.3)$$

Substituting (3.3) into (3.2), one obtains a series of problems in powers of $\Delta^{1/2}$. At $O(\Delta^{1/2})$, the linear problem itself is recovered. Since we are interested in the spatial evolution of a single baroclinic wave we choose

$$\psi_1^{(0)} = A(X)\exp[i(kx - \omega t)]\sin(ly) + cc$$
$$\psi_2^{(0)} = \gamma A(X)\exp[i(kx - \omega t)]\sin(ly) + cc \quad (3.4)$$

where ν is given by:

$$\gamma = \frac{\dfrac{k^2 + l^2}{F_v} - 1 - ir\dfrac{k^2 + l^2 + qF_v}{\omega\eta F_v}}{1 - \dfrac{irq}{\omega\eta}} \quad (3.5)$$

and F_v is from (2.5). We note that, although γ is a complex number, which implies a phase shift between the upper and lower layer, one can show that $|\gamma| = 1$, as in the temporal problem. Note that the solution (3.4) automatically satisfies the boundary condition of no meridional flow at $y = 0$ and $y = 1$, provided l is a multiple of π.

At $O(\Delta)$ the following problem emerges:

$$(\partial_t + U_i\partial_x)[\nabla^2\psi_i^{(1)} + F_v(\psi_j^{(1)} - \psi_i^{(1)})]$$
$$+ F_v(U_i - U_j)\partial_x\psi_i^{(1)} + r[\nabla^2\psi_i^{(1)} + qF_v(\psi_j^{(1)}$$
$$- \psi_i^{(1)})] = -J[\psi_i^{(0)}, F_v\psi_j^{(0)}]. \quad (3.6)$$

The solution of (3.6) is easily obtained and is given by

$$\psi_i^{(1)} = (-1)^{i+1}p|A|^2\sin(2ly) \quad (3.7)$$

where

$$p = \frac{\omega F_v\gamma_i}{2rl(1 + qF_v/2l^2)} \quad (3.8)$$

and γ_i is the imaginary part of γ. It is important to note that at this order, as is the case for the temporal problem, the fields have only a baroclinic component since $\psi_2^{(1)} = -\psi_1^{(1)}$, and that they represent $O(\Delta)$ corrections to the x and t independent flow (by analogy with the temporal problem we are going to refer to this component of the streamfunction with the expression "mean" flow). The difference here is that we have applied the boundary condition $\partial_X\psi_i^{(1)} = 0$ at $y = 0, 1$ instead of the condition $\partial_y\partial_t\bar{\psi}_i = 0$ at $y = 0, 1$ that one uses in the temporal problem.

Indeed, provided the Rossby number is small, one can easily show that the meridional velocity induced by the variation of the geostrophic streamfunction on the long spatial scale X is dominant (in an asymptotic sense) when compared to the ageostrophic component. It is therefore the former that must be set to zero at this order to enforce the condition of no meridional flow at $y = 0, 1$.

As expected from the arguments given in section 2, the removal of secular terms at $O(\Delta^{3/2})$ yields a first-order spatial amplitude equation which we write is the following form:

$$\frac{dA}{dX} = a_0A + a_1A|A|^2 \quad (3.9)$$

where $a_0 = -b_0/c$ and $a_1 = -b_1/c$ and

$$b_0 = i\omega\eta(\gamma + 1)^2$$
$$b_1 = [i\omega(\omega^2 + l^2)/2\eta]\{(\omega^2 + l^2 + F_v)(1 + \gamma^2) - 2\gamma F_v - 2(2l^2 + F_v)(1 + \gamma^2)\}$$
$$c = 2\gamma F_v\eta - (\omega^2 + l^2 + F_v)(U_1 - \gamma^2U_2) + 2\eta(F_v - \omega^2)(1 + \gamma^2) + 2ri\omega(1 - \gamma^2). \quad (3.10)$$

Since the system is linearly unstable, we know that $\mathrm{Re}(a_0)$ is positive. The crucial quantity that determines whether spatial equilibration takes place is the sign of $\mathrm{Re}(a_1)$. Figure 3a shows the plotted values of $\mathrm{Re}(a_1)$ as functions of ω for $\eta = 0.8$ and $l = \pi$ for both types of dissipation. The amplitudes of the equilibrated solutions are shown in Fig. 3b. It is interesting to remark that the amplitudes decrease with increasing frequency.

Notice that, qualitatively, both Ekman and potential vorticity dissipation give similar results. The important point is that for all ω to the right of the minimum value of F_v, i.e., on the "Eady" branch of the marginal curve, spatial equilibration will take place. With reference to the inviscid results of Merkine (1978), we then conclude that O(1) viscosity will equilibrate all ω except those on the "viscous" branch of the marginal curve (i.e., those to the left of the minimum F_v).

For very small ω, i.e., those on the viscous branch of F_v, the effect of dissipation is in fact destabilizing; however this may not be important in practice, since in that part of the marginal curve F has quite large values, and therefore the system is already absolutely unstable.

4. The convective finite-amplitude problem for $r \ll 1$ and $\omega \ll 1$

We finally consider the case when dissipation is very small ($r \ll 1$), for which there exists a region where the viscous marginal curve F_v and the inviscid spatial mar-

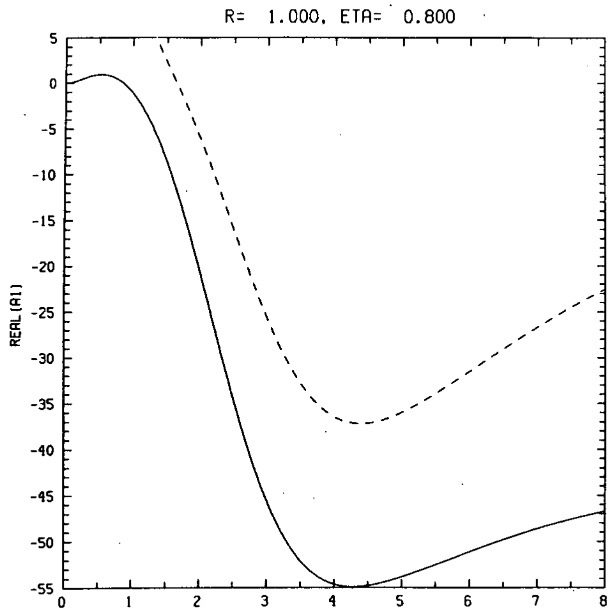

FIG. 3a. Re (a_1) for $\hat{r} = 1$, $\eta = 0.8$ and $l = \pi$. The solid line corresponds to Ekman dissipation ($q = 0$); the dashed line to potential vorticity dissipation ($q = 1$).

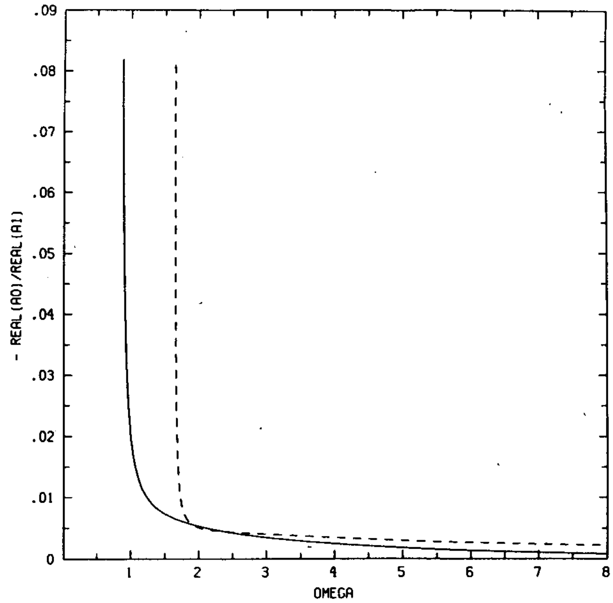

FIG. 3b. The amplitude of the equilibrated solution $-\mathrm{Re}(a_0/a_1)$ for $\hat{r} = 1$, $\eta = 0.8$ and $l = \pi$. The solid line is for $q = 0$; the dashed one for $q = 1$.

ginal curve F_{is} differ only by an amount $O(r^2)$. The curves F_v and F_{is} overlap at O(1) for $\omega \ll 1$ when $r \ll 1$. We note that since, as can be seen from (2.5) and Fig. 2a and 2b, F_v has a boundary layer near $\omega = 0$ whose width is $O(r)$, we must choose $\omega \gg r$ to be sure we are in the overlap region.

In order to determine how the space and time variables are to be scaled one examines the dispersion relation (2.2) for $\omega \ll 1$. If Δ is the order of the supercriticality, it is easy to determine that the interesting region is the one where $\omega = O(\Delta^{1/2})$, and, since $k = O(\omega)$ near the marginal curve, x must also be rescaled by $O(\Delta^{-1/2})$. Moreover we must introduce an even longer scale $X = \Delta x$ over which the amplitude varies. Finally, to make sure that the boundary layer of F_v is well to the left of the region of interest, we choose $r = O(\Delta)$.

After rescaling the variables as indicated above and setting $F = F_0 + \delta\Delta$, where $F_0 = l^2/2$, $\Delta \ll 1$ is the order of the supercriticality and δ is an O(1) number that will be determined later, the potential vorticity Eq. (2.1) can be written:

$$\begin{aligned}(\partial_t + U_i\partial_x)\pi_i + F_0(U_i - U_j)\partial_x\psi_i &= -\Delta^{1/2}\{U_i\partial_X\pi_i + F_0(U_i - U_j)\partial_X\psi_i + \hat{r}\partial_y{}^2\psi_i \\ &\quad + \hat{r}qF_0(\psi_j - \psi_i)\} - \Delta\{(\partial_t + U_i\partial_x)[\delta(\psi_j - \psi_i) \\ &\quad + \partial_x{}^2\psi_i] + (U_i - U_j)\partial_x\psi_i\} - J(\psi_i, \pi_i) \\ &\quad - \Delta^{1/2}J_X(\psi_i, \pi_i) + \text{hot.}\end{aligned} \quad (4.1)$$

where

$$J(A, B) = \partial_x A\partial_y B - \partial_y A\partial_x B$$

$$J_X(A, B) = \partial_X A\partial_y B - \partial_y A\partial_X B$$

and

$$\pi_i = \partial_y{}^2\psi_i + F_0(\psi_j - \psi_i).$$

The new O(1) viscosity parameter $\hat{r}$ is defined by the identity $r = \hat{r}\Delta$. Again we must now expand the fields in powers of $\Delta^{1/2}$, as in (3.3), to obtain a sequence of problems.

At $O(\Delta^{1/2})$ we have the inviscid marginal wave in the low frequency limit, namely:

$$\psi_1{}^{(0)} = A(X)\exp[i\omega(x - t)]\sin(ly) + cc$$
$$\psi_2{}^{(0)} = \psi_1{}^{(0)}. \quad (4.2)$$

At $O(\Delta)$ we must introduce corrections to the x and t independent ("mean") flow, which we denote by $\Psi_i(X, y)$. Up to this order the problem is identical to the temporal case, provided we substitute the long space scale with a long time scale. The $\psi_i{}^{(1)}$ fields are given by

$$\psi_1{}^{(1)} = \Psi_1(X, y)$$

$$\psi_2{}^{(1)} = \Psi_2(X, y) - \frac{2i}{\omega\eta}\left[\frac{dA}{dX} + \hat{r}A\right]\exp[i\omega(x - t)]\sin(ly) + cc$$

Imposition of a solvability condition at $O(\Delta^{3/2})$ yields the spatial amplitude equations. There are two types of resonant terms that must be set to zero at that order.

First, we must eliminate the terms proportional to $\exp[i\omega(x - t)] \sin(ly)$ that have the same structure as the marginal wave (4.2). This leads, after much algebra, to the spatial evolution equation for the wave amplitude A:

$$\frac{d^2A}{dX^2} + \left(\frac{3+q}{2}\right)\hat{r}\frac{dA}{dX} - \frac{\omega^2\eta^2}{l^2}\left[\delta - \frac{\omega^2}{2} - \frac{\hat{r}^2 l^2}{\omega^2\eta^2}\left(\frac{1+q}{2}\right)\right]A - \frac{\omega^2\eta}{2l}A\int_0^1 dy \sin(2ly)\partial_y{}^2(\Psi_1 - \Psi_2) = 0. \quad (4.3)$$

It is perhaps surprising to realize at this point that (4.3) is identical to the amplitude evolution equation of the temporal problem provided one subtitutes d/dX by d/dT (cf. Eq. 4.10 of Pedlosky 1971 in the limit of $k \ll 1$). Note in particular that the wave A only interacts with the baroclinic component $\Psi_1 - \Psi_2$ of the "mean" flow.

The square bracket in front of the linear term reflects the fact that, in order to be truly supercritical, we must incorporate in the choice of F the $O(\Delta)$ corrections to the marginal curve (i.e., the last two terms inside the square bracket); these can be easily obtained from (2.5). In what follows we have, without loss of generality, chosen δ so that the coefficient of the linear term is identically 1.

The second kind of secularities that must be set to zero consists of those terms that are independent of x and t. Their removal yields the amplitude equations for the "mean" flow corrections $\Psi_i(X, y)$, which are found to take the form:

$$\partial_X[U_1(\partial_y{}^2\Psi_1 + F_0\Psi_2) - F_0U_2\Psi_1] + \hat{r}[\partial_y{}^2\Psi_1 + qF_0(\Psi_2 - \Psi_1)] = \frac{l^3}{\eta}\left[\frac{d|A|^2}{dX} + 2\hat{r}|A|^2\right]\sin(2ly)$$

$$\partial_X[U_2(\partial_y{}^2\Psi_2 + F_0\Psi_1) - F_0U_1\Psi_2] + \hat{r}[\partial_y{}^2\Psi_2 + qF_0(\Psi_1 - \Psi_2)] = -\frac{l^3}{\eta}\left[\frac{d|A|^2}{dX} + 2\hat{r}|A|^2\right]\sin(2ly). \quad (4.4)$$

Equations (4.4) can be integrated immediately with respect to y because the boundary condition for the "mean" flow corrections at $y = 0$ and 1 is $\partial_X\Psi_i = 0$. It is convenient to write:

$$\Psi_i(X, y) = P_i(X)\sin(2ly). \quad (4.5)$$

Upon substitution of (4.5) into (4.4) and (4.3) and after the rescaling

$$P_i = -\frac{1}{2\omega^2 l\eta}B_i$$

$$A = \frac{1}{\omega l}W$$

we obtain a fourth-order system of nonlinear ordinary differential equations which describe the spatial evolution of the wave amplitude W and the barotropic and baroclinic components of the "mean" flow B_T and B_C, respectively:

$$\frac{d^2W}{dX^2} + \left(\frac{3+q}{2}\right)\hat{r}\frac{dW}{dX} - W + WB_C = 0$$

$$\mathcal{D}\frac{dB_T}{dX} = \hat{r}\{[4\eta B_C - 5B_T] + q\eta B_C\} - 4\eta\left[W\frac{dW}{dX} + \hat{r}W^2\right]$$

$$\mathcal{D}\frac{dB_C}{dX} = \hat{r}\{[3\eta B_T - 4B_C] - qB_C\} + 4\left[W\frac{dW}{dX} + \hat{r}W^2\right] \quad (4.6)$$

where B_C and B_T are defined by

$$B_C = \frac{B_1 - B_2}{2}, \quad B_T = \frac{B_1 + B_2}{2}$$

and the determinant $\mathcal{D} = 5 - 3\eta^2$ arises when the two equations of (4.4) are diagonalized to obtain the form (4.6). When $\eta = \eta_c = \pm\sqrt{5/3}$, for which $\mathcal{D} = 0$, the "mean" flow corrections, whose meridional structure is $\sin(2ly)$, are resonant with the marginal wave (we note that, because of our choice of the velocity scale, when $\eta = \pm\eta_c$, U_1 and U_2 have opposite sign).

This is not a dissipative effect, and, although he failed to recognize it, is also present in the inviscid analysis of Merkine (1978) (consider, for instance, the denominator in Merkine's Eq. 2.16). If one wanted to analyze this resonance, one could do a local analysis near $\eta = \eta_c$, and include the "mean" flow corrections at lowest order with the marginal wave as part of a triad.

It is also important to note that in (4.6) the nonlinear interaction of the wave with itself forces both a baroclinic and a *barotropic* component of the "mean" flow. In the temporal problem only the baroclinic component is generated, and the set of equations analogous to (4.6) is one degree lower.

The presence of a nonzero barotropic "mean" flow correction in the spatial problem is a direct consequence of the existence of pressure gradients with respect to the long space scale X. The absence of such gradients in the temporal problem forces the "mean" flow correction to be purely baroclinic in order for mass conservation to be satisfied.

System (4.6) possesses steady (i.e., space independent solutions). They are:

$$W = \pm(1 + q/4)^{1/2}, \quad B_C = 1, \quad B_T = 0. \quad (4.7)$$

In this case B_T is identically zero. Only when there are spatial variations can the system have a nonzero barotropic "mean" flow correction. Note also that when disspation is taken to be proportional to potential vorticity ($q = 1$) the steady solutions have a somewhat larger amplitude than when only Ekman dissipation is present.

The linear stability of solutions (4.7) can be investigated analytically. It is easy to show that, irrespective of the values of $\hat{r}$ and q, the steady states are stable for $|\eta| < |\eta_c|$ and unstable otherwise. This is due to the fact that one of the eigenvalues of the system obtained by linearizing (4.6) about the steady solution (4.7) can be shown to be always proportional to $1/\mathcal{D}$.

In the inviscid limit, $\hat{r} = 0$, the set of equations (4.6) can be integrated twice to yield the following system:

$$\frac{d^2W}{dX^2} - W + \frac{4}{\mathcal{D}} W[W^2 - W_0^2] = 0$$

$$B_C = \frac{4}{\mathcal{D}}[W^2 - W_0^2], \; B_T = -\frac{4\eta}{\mathcal{D}}[W^2 - W_0^2]. \quad (4.8)$$

This shows that spatial equilibration is only possible when $|\eta| < |\eta_c|$ for which $\mathcal{D} > 0$ (compare 4.8 with Eqs. (7.16.47) of Pedlosky 1987). When this condition is satisfied, the inviscid system (4.8) will have limit cycle solutions in the form of elliptic functions.

Therefore, as in the temporal problem, a low frequency inviscid slightly supercritical baroclinic wave can undergo amplitude vacillation, provided the ratio of shear to mean flow is not too large. One example is show in Fig. 4. Notice that, as can be seen from (4.8), the barotropic and baroclinic "mean" flow corrections are 180° out of phase, and that the baroclinic component is in phase with the wave itself.

In order to apply our experience with familar dynamical systems to understand the behavior of system (4.6), we have found it useful to consider the case $\eta \ll 1$. If we let

$$W = w^{(0)} + \eta w^{(1)} + O(\eta^2)$$

$$B_{T,C} = b_{T,C}^{(0)} + \eta b_{T,C}^{(1)} + O(\eta^2) \quad (4.9)$$

and substitute the definitions (4.9) into (4.6), the lowest order equations in η obeyed by $w^{(0)}$, $b_T^{(0)}$ and $b_C^{(0)}$ are easily found to be

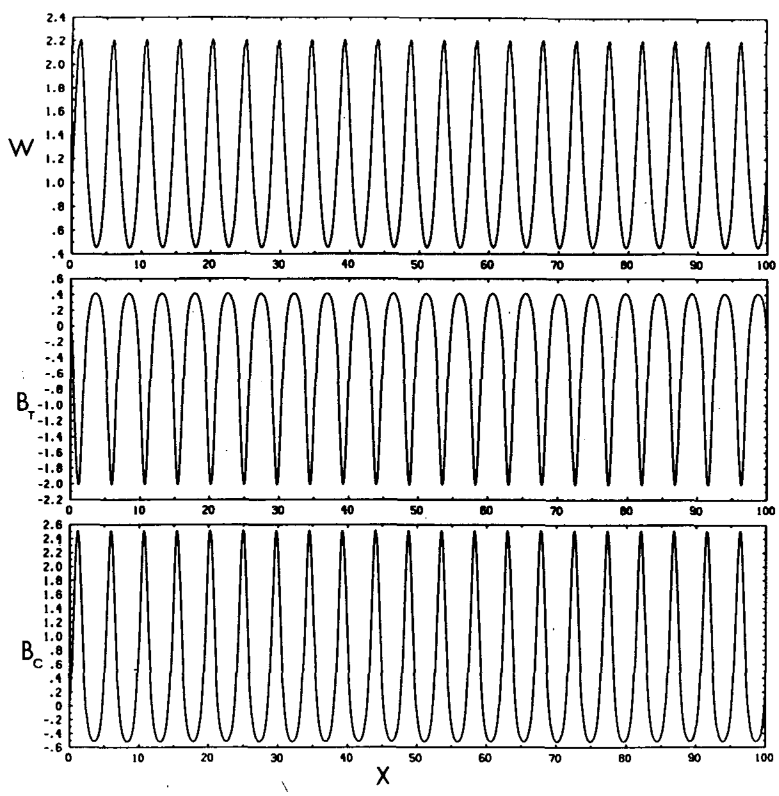

FIG. 4. Inviscid limit cycle solution of (4.6) for $\eta = 0.8$ and $l = \pi$. The initial conditions were: $W = dW/dX = 1$ and $B_T = B_c = 0$.

$$\frac{d^2w^{(0)}}{dX^2} + \left(\frac{3+q}{2}\right)\hat{r}\frac{dw^{(0)}}{dX} - w^{(0)} + w^{(0)}b_C^{(0)} = 0$$

$$\frac{db_C^{(0)}}{dX} = -\frac{4}{5}w^{(0)}\hat{r}b_C^{(0)} + \frac{4}{5}\left[w^{(0)}\frac{dw^{(0)}}{dX} + \hat{r}\{w^{(0)}\}^2\right] \quad (4.10)$$

$$\frac{db_T^{(0)}}{dX} = -\hat{r}b_t^{(0)}. \quad (4.11)$$

In this limit, the barotropic "mean" flow correction decays by itself, and the wave and the baroclinic component obey the third-order system (4.10). Without great effort one can show that system (4.10) is equivalent (i.e., obtainable by a linear transformation) to the familiar Lorenz set (Lorenz 1963):

$$\mathring{x} = \sigma y - \sigma x$$
$$\mathring{y} = Rx - y - xz$$
$$\mathring{z} = xy - bz \quad (4.12)$$

(where (˚) denotes ordinary differentiation with respect to a rescaled independent variable). The linear transformation that maps (4.10) into (4.12), determines the values of the parameters σ (Prandtl number), R (Rayleigh number) and b in terms of the parameters of system (4.10). It is found that

$$\sigma = \frac{2}{1+q}, \quad b = \frac{2}{5}\frac{4+q}{1+q}, \quad R = 1 + \frac{1}{\hat{r}}. \quad (4.13)$$

When dissipation is of Ekman type ($q = 0$) the Prandtl number of the equivalent Lorenz set is equal to 2 and $b = 8/5$, for which values the Lorenz set is known to possess a strange attractor for certain values of R, and to exhibit limit cycle behavior at $R \to \infty$, which, through (4.13) corresponds to very small but nonzero dissipation in (4.10). However, for potential vorticity, dissipation ($q = 1$) $\sigma = 1$ and $b = 1$; at this value of the Prandtl number all initial conditions of the Lorenz set will converge to the stable steady state solutions.

From these observations we may therefore expect that, in the limit of small η, system (4.6) will exhibit vacillations for $\hat{r} \ll 1$ when $q = 0$, but not when $q = 1$. This situation is similar to the one found in the temporal problem (Pedlosky and Polvani 1987). The above analysis, however, gives us no clues as to the behavior we can expect when $\eta = O(1)$, since (4.6) is then veritably a fourth-order system and the analogy with the Lorenz set no longer holds.

For that case we have had to resort to numerical integrations of (4.6). We have implemented the time step with a fourth-order Runge–Kutta algorithm. An alternative fourth-order Taylor scheme was also used, to validate the Runge–Kutta results. In all the runs we have carried out the numerical integrations far enough to determine without possible doubt the behavior of the solution.

As expected from the preceding asymptotic analysis for small η and small dissipation, vacillating solutions were found for the full system (4.6) when $q = 0$. We give one example in Fig. 5a; for the run we chose $\eta = 0.01$ and $\hat{r} = 0.01$; the initial conditions were: $W = dW/dX = 1$ and $B_T = B_C = 0$. The integration was carried out to approximately 500 dissipative *e*-folding distances, and only the last few periods are shown in the figure. Notice how different this limit cycle behavior is from the inviscid case of Fig. 4. In particular, as can be seen from (4.6) itself, the barotropic "mean" flow correction B_T is $O(\eta)$ smaller than either W or B_C.

As $\hat{r}$ was progressively increased, keeping η constant (and very small) and $q = 0$, the limit cycle behavior gradually gave way to steady state solutions. Above $\hat{r} > 0.02$ only steady amplitudes were found. Some windows of amplitude vacillation were detected between regions of steady amplitude, but no period doubling or aperiodic behavior was observed. Figure 5b shows a solution obtained with all the parameters identical to the ones used for Fig. 5a, except for q which had the value 1. As expected, at small η, the choice of potential vorticity dissipation has expunged all vacillations.

By far the most interesting result is the effect of the existence of a nonzero barotropic "mean" flow correction when $\eta = O(1)$. It turns out that, although in that parameter range, even for extremely small dissipations, the solutions of system (4.6) are found to relax to steady amplitudes as $X \to \infty$, independent of the value of q, *large vacillations* persist over *large distances* downstream of the source, with O(1) barotropic "mean" flow corrections.

We give one example in Figs. 6a, b and 7a, b (for $q = 0$ and 1 respectively), for which the initial conditions are the same as for Fig. 5, but with $\eta = 0.8$ and $\hat{r} = 0.001$. In general, we have found that for $\eta \geq 0.1$ steady amplitude behavior occurs in the limit $X \to \infty$ for all values of $\hat{r}$ although, in practice, if the dissipation is extremely small, the system will vacillate very far out before a steady amplitude is reached.

Indeed, for the spatial case as opposed to the temporal problem, one is not as much interested in the asymptotic behavior of the solution as in the response of the system in a large region downstream of the source. Figures 6a and 7a show that, in that region, the nonlinear dynamics of an unstable baroclinic wave can be responsible for the presence of both baroclinic and *barotropic* "mean" flow corrections whose amplitude exhibits rapid spatial vacillation.

5. Conclusion

The investigation of slightly supercritical spatially growing baroclinic waves in the two-layer model has revealed that an O(1) dissipation will act, in combination with the nonlinear self-interaction, to equilibrate

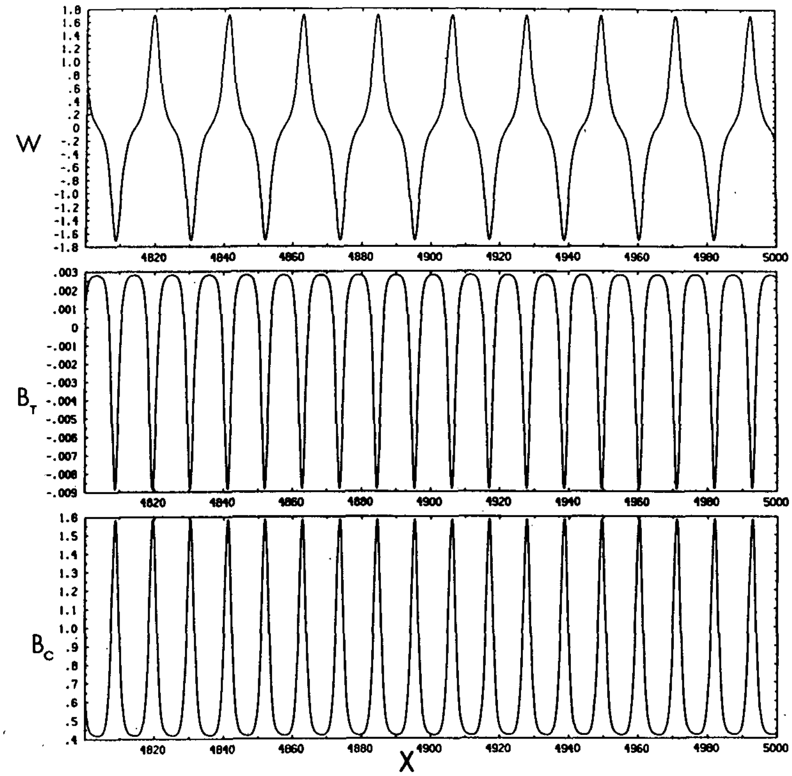

FIG. 5a. Viscous limit cycle solution of (4.6) for $\hat{r} = 0.01$, $\eta = 0.01$, $l = \pi$ and $q = 0$. Initial conditions identical to the ones of Fig. 4.

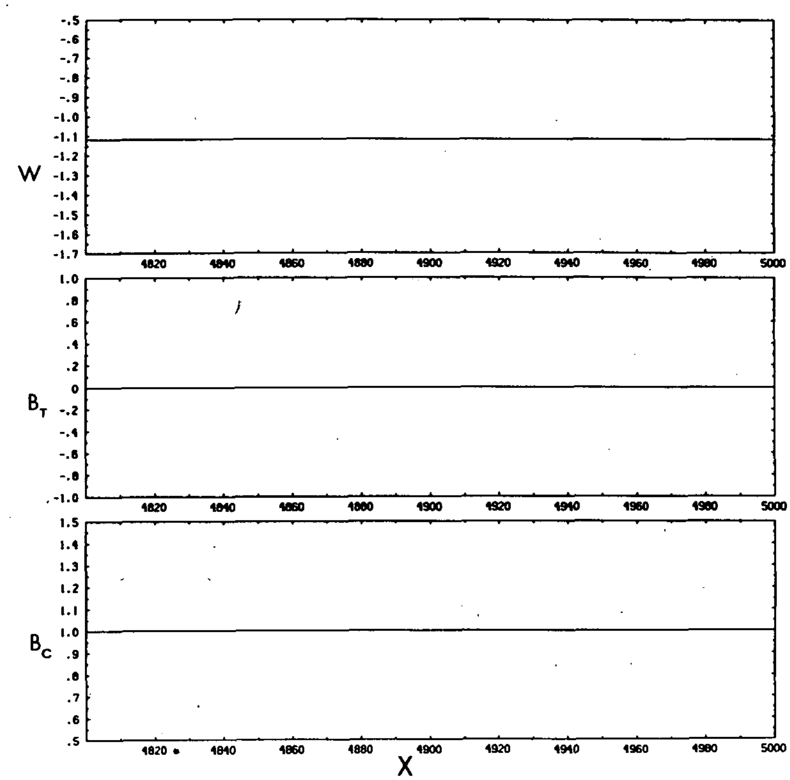

FIG. 5b. All parameters identical to Fig. 5a, except for $\dot{q} = 1$.

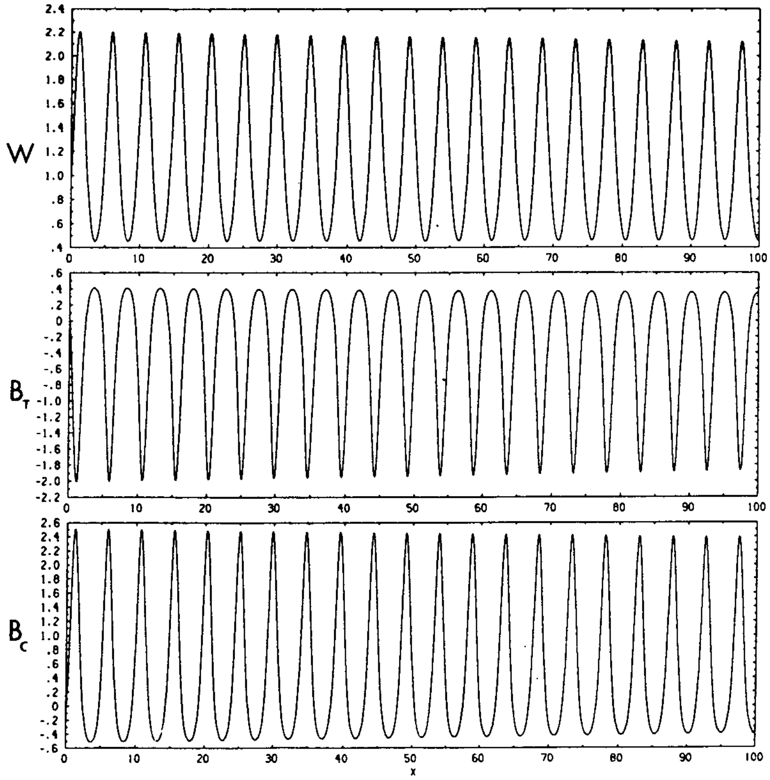

FIG. 6a. Solution of (4.6) between $X = 0$ and $X = 100$ at $\eta = 0.8$, $\hat{r} = 0.001$, $l = \pi$ and $q = 0$. Initial conditions identical to the ones of Fig. 4.

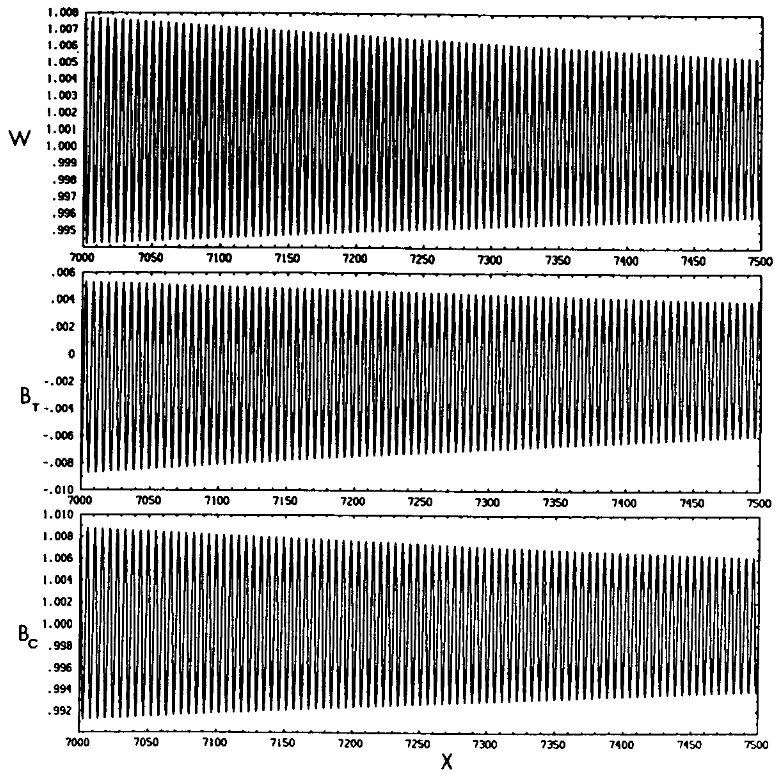

FIG. 6b. Identical solution as for Fig. 6a, but shown between $X = 7000$ and $X = 7500$.

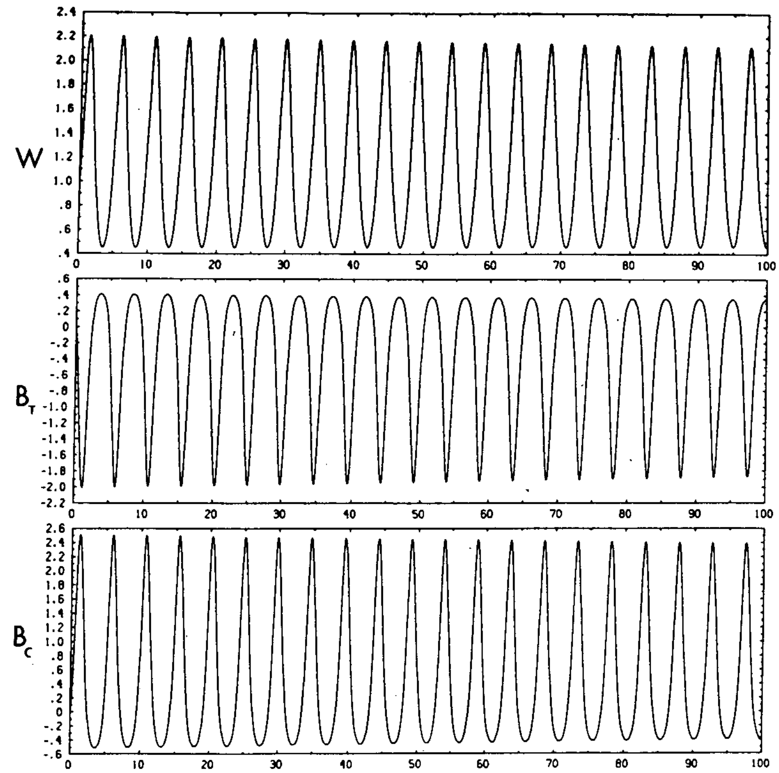

FIG. 7a. All parameters identical to Fig. 6a, except for $q = 1$.

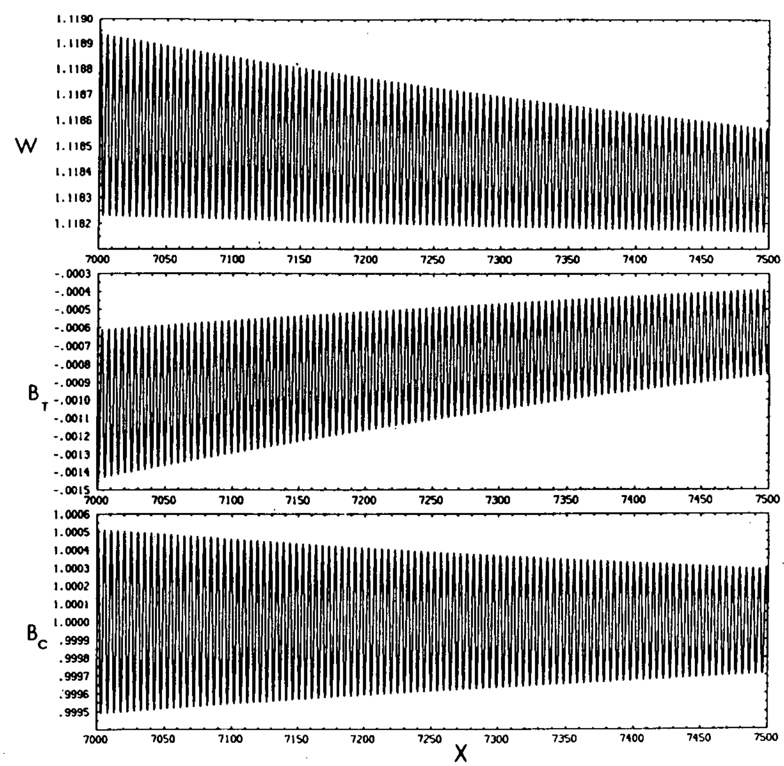

FIG. 7b. All parameters identical to Fig. 6b, except for $q = 1$.

all waves on the "Eady" branch of the spatial marginal curve, for which some cases of inviscid subcritical destabilization were reported earlier (Merkine 1978).

In the limit of small dissipation and low frequency, however, nonlinear instability has been shown to exist when the ratio of shear to mean flow exceeds an O(1) critical value. We have also shown that, in the same limit, baroclinic waves can exhibit spatially vacillating behavior. This means that, downstream of a source of perturbations and due solely to the internal nonlinear baroclinic wave dynamics, one may expect regions of variable eddy activity generated by the self-interaction of the wave. Furthermore, the associated "mean" flow corrections will have both baroclinic and barotropic components.

Perhaps the most unexpected conclusion to emerge from this study is that, in contrast with the temporal problem, the form chosen to model the dissipation is not as crucial in determining the qualitative behavior of these results.

The model used in this study is very idealized and certainly far too simple for a direct comparison with actual geophysical flows such as the Gulf Stream. Among other aspects that should be addressed in future investigations of this problem, we mention the effect of the planetary gradient of potential vorticity and the presence of several forcing frequencies at the source. It is our belief that the main qualitative conclusions of this study, namely the existence of spatial vacillations and of barotropic mean flow corrections, will be confirmed by future more realistic models.

Acknowledgments. This research was supported in part by Grant ATM84-13515 from the National Science Foundation's Division of Atmospheric Science. L.M.P. was also supported, in part, by a Fellowship from the Natural Science and Engineering Research Council of Canada.

REFERENCES

Briggs, R. J., 1964: *Electron-Stream Interaction with Plasmas,* Ch. 2, 8–46, MIT Press.

Charney, J. G., 1946: The dynamics of long waves in a baroclinic westerly current. *J. Meteor.,* **4,** 135–162.

Farrel, B. F., 1982: Pulse asymptotics of the Charney baroclinic instability problem. *J. Atmos. Sci.,* **39,** 507–517.

——, 1983: Pulse asymptotics of three-dimensional baroclinic waves. *J. Atmos. Sci.,* **40,** 2202–2209.

Lorenz, E. N., 1963: Deterministic non-periodic flow. *J. Atmos. Sci.,* **20,** 130–141.

Merkine, L.-O., 1977: Convective and absolute instability of baroclinic eddies. *Geophys. Astrophys. Fluid Dyn.,* **9,** 129–157.

——, 1978: A note on the finite-amplitude dynamics of spatially growing baroclinic waves on an *f*-plane. *Tellus,* **30,** 477–486.

Pedlosky, J., 1970: Finite-amplitude baroclinic waves. *J. Atmos. Sci.,* **27,** 15–30.

——, 1971: Finite-amplitude baroclinic waves with small dissipation. *J. Atmos. Sci.,* **28,** 587–597.

——, 1987: *Geophysical Fluid Dynamics,* 2nd ed., Ch. 7, Springer-Verlag. p. 604.

——, and Frenzen, C., 1980: Chaotic and periodic behavior of finite-amplitude baroclinic waves. *J. Atmos. Sci.,* **37,** 1177–1196.

——, and Polvani, L. M., 1987: Wave-wave interaction of unstable baroclinic waves. *J. Atmos. Sci.,* **44,** 631–647.

Pierrehumbert, R. T., 1986: Spatially amplifying modes of the Charney baroclinic-instability problem. *J. Fluid. Mech.,* **170,** 293–317.

Romea, R. D., 1977: The effects of friction and β on finite-amplitude baroclinic waves. *J. Atmos. Sci.,* **34,** 1689–1695.

Thacker, W. C., 1976: Spatial growth of Gulf Stream meanders. *Geophys. Astrophys. Fluid Dyn.,* **7,** 271–295.