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Phoebus J. Dhrymes

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Department of Economics
Columbia University
New York, NY 10027

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PHOEBUS J. DHRYMES
Columbia University

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Abstract

This paper deals with the problem of testing for the presence of autocorrelation in a system of general linear models (Seemingly Unrelated Regressions, SUR) when the model is formulated as a vector autoregression (VAR) with exogenous variables. The solution presented in this paper is a generalization of the h-statistic for the single equation single parameter case given in Durbin (1970).

KEY WORDS: Autocorrelation, h-test, H^2-test, VAR

1 Introduction

Consider the model

$$y_{t\cdot} = y_{t-1\cdot}A + x_{t\cdot}B + u_{t\cdot}, \quad t = 1, 2, 3, \ldots, T, \tag{1}$$

where $y_{t\cdot}$ is an m-element row vector of dependent, and $x_{t\cdot}$ is a k-element vector of independent variables, respectively; $u_{t\cdot}$, $t = 1, 2, \ldots, T$ is the structural error vector. We assume

i. $\{u_{t\cdot} : t = 1, 2, 3, \ldots, T\}$ is a sequence of independent identically distributed (i.i.d.) random vectors with

$$E\, u_{t\cdot} = 0, \qquad \mathrm{Cov}(u_{t\cdot}) = \Sigma > 0, \tag{2}$$

defined on some probability space $(\Omega, \mathcal{A}, \mathcal{P})$.

ii. It is further assumed that

$$\operatorname*{plim}_{T\to\infty} \frac{X'X}{T} = M_{xx} > 0, \tag{3}$$

and that the elements in X and U are mutually independent.

iii. The system of Eq. (1) is **stable**, i.e. the characteristic roots of A are less than one in absolute value.

Regarding the errors, the alternative hypothesis we entertain is

$$u_{t\cdot} = u_{t-1\cdot}R + \epsilon_{t\cdot}. \tag{4}$$

We require, for stationarity, the following assumptions:

1. The matrix R is non-singular and stable, i.e. its characteristic roots are less than one in absolute value;

2. With little loss of generality, and certainly no loss of relevance, we further assume that the matrix R is diagonalizable, i.e. it has the representation $R = P\Lambda P^{-1}$, where Λ is the (diagonal) matrix of its characteristic roots.

This problem, for the case $m = 1$, (and R a scalar) was dealt with by Durbin (1970). A search of widely used econometrics textbooks such as Greene (1999) and Davidson and MacKinnon (1993) discloses no mention of its generalization to VARs.

Remark 1. If one were to write down a VAR one would normally not be concerned about the behavior of the "error", since **by definition** the errors in such a system are assumed to be i.i.d. If not, one simply specifies a VAR of a higher order, in empirical applications. Notwithstanding this observation, in many applied contexts the logic of the economic model requires the presence of a specific number of lagged endogenous variables. In such a case, the problem we are examining here may arise.

Remark 2. When the structural error, $u_{t\cdot}$, is a first order autoregression, the OLS estimators for the parameters of the model in Eq. (1) are inconsistent **because of the presence of lagged endogenous variables**, which are therefore correlated with the structural error.

Thus, if we suspect that the form given in Eq. (4) may be appropriate, we may wish to test the hypothesis

$H_0: \quad R = 0\,,$
as against the alternative
$H_1: \quad R \neq 0\,,$

when least squares (OLS) is used to estimate the unknown parameters of Eq. (1).

2 Derivation of the Test Statistic

Writing the sample as

$$Y = Y_{-1}A + XB + U = ZC + U, \quad Z = (Y_{-1}, X), \quad C = (A', B')', \tag{5}$$

the OLS estimator of C is given by

$$\tilde{C} = (Z'Z)^{-1}Z'Y = C + (Z'Z)^{-1}Z'U. \tag{6}$$

Assuming that a central limit theorem (CLT), such as the Lindeberg CLT, see Dhrymes (1989), pp. 271 ff, we may write the limiting distribution of the OLS estimator as

$$\sqrt{T}(\tilde{c} - c) \xrightarrow{\mathrm{d}} N(0, \Sigma \otimes M_{zz}^{-1}), \tag{7}$$

where $c = \mathrm{vec}(C)$ and $M_{zz} = \mathrm{plim}_{T\to\infty}[Z'Z/T]$.

From this it is easily verified that

$$\sqrt{T}(\tilde{a} - a) \xrightarrow{\mathrm{d}} N(0, \Sigma \otimes S_{11}), \quad \text{where} \quad M_{zz}^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{8}$$

and S_{11} is the (principal) submatrix of M_{zz}^{-1}, consisting of its first m rows and columns.

Let

$$\tilde{U} = Y - Z\tilde{C} = U - Y_{-1}(\tilde{A} - A) - X(\tilde{B} - B), \tag{9}$$

be the matrix of OLS residuals and consider the estimator of R

$$\tilde{R} = (\tilde{U}_{-1}'\tilde{U}_{-1})^{-1}\tilde{U}_{-1}'\tilde{U} \tag{10}$$

Using Eq. (10), and omitting terms that converge to zero in probability, we may write, see Dhrymes (1989), pp 161 ff.[1]

$$\sqrt{T}(\tilde{R} - R) \sim (\tilde{U}_{-1}'\tilde{U}_{-1})^{-1}\frac{1}{\sqrt{T}}\left(U_{-1}'U - U_{-1}'Y_{-1}(\tilde{A} - A)\right), \tag{11}$$

either because $Z'Z/T$ converges, or because $\tilde{C}$ is consistent and has a well defined limiting distribution, or both. Moreover, using the result again, and bearing in mind that

$$\frac{1}{T}U_{-1}'Y_{-1} \xrightarrow{\mathrm{P}} \Sigma$$

we finally obtain

$$\sqrt{T}(\tilde{R} - R) \sim \Sigma^{-1}\frac{1}{\sqrt{T}}U_{-1}'U - \sqrt{T}(\tilde{A} - A). \tag{12}$$

Let

$$M_{zz}^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

[1] The notation $X \sim W$ below means X has the same limiting distribution as W.

and note that

$$\sqrt{T}(\tilde{A}-A) \sim S_1 \frac{1}{\sqrt{T}} Z'U.$$

Vectorizing, we have the expression

$$\sqrt{T}\text{vec}(\tilde{R}-R) \sim \frac{1}{\sqrt{T}} \left((I_m \otimes [\Sigma^{-1}U_{-1}' - S_1 Z'] \right) u, \quad u = \text{vec}(U), \tag{13}$$

which obeys the conditions of the Lindeberg CLT, see Dhrymes (1989), pp. 271 ff. Let

$$\mathcal{A}_t = \sigma(u_{s\cdot}, s \leq t), \tag{14}$$

i.e. it is the σ-algebra generated by the u's up to t. To evaluate the covariance matrix of the limiting distribution of the left member of Eq. (13), we need to find the expectation of terms like $(I_m \otimes K)uu'(I_m \otimes K)$. We shall do so by first conditioning with respect to $\mathcal{A}_{t-1}$. Thus we need to evaluate

$$E_{\mathcal{A}_{t-1}} E \left([I_m \otimes (\Sigma^{-1}U_{-1}' - S_1 Z')]uu'[I_m \otimes (U_{-1}\Sigma^{-1} - ZS_1')] | \mathcal{A}_{t-1} \right)$$

$$= E_{\mathcal{A}_{t-1}} \left(\Sigma \otimes [\Sigma^{-1}U_{-1}'U_{-1}\Sigma^{-1} - \Sigma^{-1}U_{-1}'ZS_1' - S_1 Z'U_{-1}\Sigma^{-1} + S_1 Z'ZS_1'] \right)$$

$$= \Sigma \otimes [\Sigma^{-1}\Sigma\Sigma^{-1} - \Sigma^{-1}(\Sigma, 0)S_1' - S_1(\Sigma, 0)'\Sigma^{-1} + S_{11}]$$

$$= \Sigma \otimes (\Sigma^{-1} - S_{11}). \tag{15}$$

Hence,

$$H^2 = T\text{vec}(\tilde{R})'[\tilde{\Sigma}^{-1} \otimes (\tilde{\Sigma}^{-1} - \tilde{S}_{11})^{-1}]\text{vec}(\tilde{R}) \xrightarrow{\text{d}} \chi^2_{m^2}. \tag{16}$$

Remark 3. Evidently if, in a given application, the estimated matrix $\tilde{\Sigma}^{-1} - \tilde{S}_{11}$ is **not at least positive semi-definite**, the test fails. If the matrix itself (not only the estimated one) is positive semi-definite **but not** positive definite, the distribution is still asymptotically χ^2, but with degrees of freedom equal to the rank $\Sigma^{-1} - S_{11}$.

Remark 4. Note that in the case $m = 1$, and consequently $\tilde{R} = \tilde{\rho}$, the test statistic of Eq. (16) reduces to

$$T\text{vec}(\tilde{R})'[\tilde{\Sigma}^{-1} \otimes (\tilde{\Sigma}^{-1}\tilde{S}_{11})^{-1}]\text{vec}(\tilde{R}) = \frac{T\tilde{\rho}^2}{1 - \text{AVar}(\tilde{a}_{11})}, \tag{17}$$

where $\mathrm{Avar}(\tilde{a}_{11})$ is the variance of the limiting distribution of the OLS estimated coefficient of the lagged dependent variable. Thus, the H^2 statistic reduces to the square of the h-statistic, as given by Durbin (1970), because basically $\Sigma \otimes \Sigma^{-1}$ reduces to unity in the case $m = 1$. Thus, the case where $\Sigma^{-1} - S_{11}$ **is not at least positive semi-definite** corresponds to the case where the asymptotic variance in question is equal to or greater than 1. When this is so one should employ an alternative procedure to be derived below.

3 An Alternative Test when the H^2 Test Fails

When the H^2 statistic yields inadmissible results we may employ the following procedure.

Write the model in Eq. (1) as

$$Y = U_{-1}R + ZC + E = WD + E, \quad E = (\epsilon_{t\cdot}), W = (U_{-1}, Y_{-1}, X), \quad D = (R', C')', \tag{18}$$

where we have merely made use of the alternative specification in Eq. (4). If we could observe U_{-1} we would simply estimate R by OLS and then carry out a test on R as we would with any other OLS-estimated parameter. Since we do not, we shall use the OLS residuals from the regression of Y on Z. The estimator thus obtained is

$$\tilde{D} = (\tilde{W}'\tilde{W})^{-1}\tilde{W}'Y = (\tilde{W}'\tilde{W})^{-1}\tilde{W}'WD + (\tilde{W}'\tilde{W})^{-1}\tilde{W}'E, \quad \tilde{W} = (\tilde{U}_{-1}, Z). \tag{19}$$

Noting that, under the null, $W - \tilde{W} = [Z(\tilde{C} - C), 0]$, we find

$$(\tilde{W}'\tilde{W})^{-1}\tilde{W}'W = I + (\tilde{W}'\tilde{W})^{-1}\tilde{W}'[Z(\tilde{C} - C), 0] \xrightarrow{\mathrm{P}} I.$$

Specifically, note that $W - \tilde{W} = (Z(\tilde{C} - C), 0)$, so that

$$(\tilde{W}'\tilde{W})^{-1}\tilde{W}'[W - \tilde{W}]D = 0,$$

because under the null $R = 0$. Consequently, under the null,

$$\tilde{D} \xrightarrow{\mathrm{P}} D, \text{and, moreover}$$

$$\sqrt{T}(\tilde{D} - D) \sim \left(\frac{\tilde{W}'\tilde{W}}{T}\right)^{-1} \frac{\tilde{W}'E}{\sqrt{T}}. \tag{20}$$

so that D is estimated consistently and has a well defined limiting distribution.

Concentrate now on the estimator of R, viz.

$$\sqrt{T}\tilde{R} = \tilde{S}_1^* \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{w}_{t\cdot}' \epsilon_{t\cdot}, \quad \left(\frac{\tilde{W}'\tilde{W}}{T}\right)^{-1} = \tilde{S}^* = \begin{bmatrix} \tilde{S}_{11}^* & \tilde{S}_{12}^* \\ \tilde{S}_{21}^* & \tilde{S}_{22}^* \end{bmatrix} = \begin{bmatrix} \tilde{S}_1^* \\ \tilde{S}_2^* \end{bmatrix}. \tag{21}$$

Letting S_{ij}^* represent the corresponding blocks in the probability limit of $\tilde{S}^*$, and vectorizing the expression in Eq. (21) we find

$$\sqrt{T}\,\mathrm{vec}(\tilde{R}) = \sqrt{T}\,\tilde{r} \sim (I_m \otimes S_1^*) \sum_{t=2}^{T} (I_m \otimes \tilde{w}_{t\cdot}') \epsilon_{t\cdot}'. \tag{22}$$

Since this model too obeys the condition of the Lindeberg theorem, we therefore conclude

$$\sqrt{T}\tilde{r} \xrightarrow{\mathrm{d}} N(0, \Sigma \otimes S_{11}^*). \tag{23}$$

Consequently, we may test

$H_0 : \quad R = 0$

as against the alternative

$H_1 : \quad R \neq 0$

by means of the statistic

$$H^{*2} = T\tilde{r}'(\tilde{\Sigma} \otimes \tilde{S}_{11}^*)^{-1}\tilde{r} \xrightarrow{\mathrm{d}} \chi^2_{m^2}. \tag{24}$$

4 Diagonal R

When the autoregression matrix R is diagonal, the situation is more complex than that of the simple Durbin context, **unless**

$$\mathrm{Cov}(\epsilon_{t\cdot}) = \mathrm{diag}(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}), \tag{25}$$

in which case we are reduced to doing m h-tests *seriatim*.

We now examine the case where Σ is unrestricted, i.e. we produce the analogue of the H^2-statistic when R is diagonal but the elements of $u_{t\cdot}$ are cross correlated. Specifically, the alternative dealt with is

$$u_{t\cdot} = u_{t-1\cdot}R + \epsilon_{t\cdot}, \quad R = \mathrm{diag}(r_{11}, r_{22}, \ldots, r_{mm}), \quad \mathrm{Cov}(\epsilon_{t\cdot}) = \Sigma \tag{26}$$

where

$$\Sigma = (\sigma_{ij}), \quad \sigma_{ij} \neq 0, \quad \text{for} \quad i \neq j\,.$$

If the u's could be observed, we would write the model as

$$u = Vr + e, \quad u = \mathrm{vec}(U), \quad V = \mathrm{diag}(v_{\cdot 1}, v_{\cdot 2}, \ldots, v_{\cdot m}),, \tag{27}$$

where $v_{\cdot i}$ is the ith column of U_{-1}, $r = (r_{11}, r_{22}, \ldots, r_{mm})'$, and estimate

$$\hat{r} = [V'(\Sigma^{-1} \otimes I_T)V]^{-1}V'(\Sigma^{-1} \otimes I_T)u; \tag{28}$$

the limiting distribution of the entity above is given by

$$\sqrt{T}\hat{r} \xrightarrow{\mathrm{d}} N(0, \Omega_*^{-1}), \qquad \Omega_* = (\sigma^{ij}\sigma_{ij}). \tag{29}$$

Since they are not known, we may substitute the corresponding OLS residuals, instead of U and U_{-1}. When we do so we have, under the null,

$$\sqrt{T}\tilde{r} = [\tilde{V}'(\tilde{\Sigma}^{-1} \otimes I_T)\tilde{V}]^{-1}\tilde{V}'(\tilde{\Sigma}^{-1} \otimes I_T)\tilde{u} \sim \Omega_*^{-1}\frac{1}{\sqrt{T}}[\sum_{j=1}^{m} \sigma^{ij}\tilde{v}'_{\cdot i}\tilde{u}_{\cdot j}]. \tag{30}$$

But,

$$\frac{1}{\sqrt{T}}\tilde{v}'_{\cdot i}\tilde{u}_{\cdot j} \sim \frac{1}{\sqrt{T}}\left(v'_{\cdot i}u_{\cdot j}\right) - \left(\frac{1}{T}v'_{\cdot i}Y_{-1}\right)\sqrt{T}(\hat{a}_{\cdot j} - a_{\cdot j}) \sim \frac{1}{\sqrt{T}}\left(v_{\cdot i} - ZS_1'\sigma_{\cdot i}\right)' u_{\cdot j}. \tag{31}$$

Defining the matrix

$$V^* = \mathrm{diag}(v_{\cdot 1} - ZS_1'\sigma_{\cdot 1},\ v_{\cdot 1} - ZS_1'\sigma_{\cdot 2}, \ldots, v_{\cdot m} - ZS_1'\sigma_{\cdot m}), \tag{32}$$

we may finally write

$$\sqrt{T}\tilde{r} \sim \Omega_*^{-1}\frac{1}{\sqrt{T}}V^{*\prime}(\Sigma^{-1} \otimes I_T)u \xrightarrow{\mathrm{d}} N(0, \Phi), \quad \Phi = \Omega_*^{-1}[\sigma^{ij}(\sigma_{ij} - \sigma'_{\cdot i}S_{11}\sigma_{\cdot j})]\Omega_*^{-1}. \tag{33}$$

Remark 5. The matrix Ω_* is non-singular as the following demonstration easily shows. Let $e_{\cdot i}$ be an m-element column vector all of hose elements are zero, except the ith which is one. Then note that

$$\Omega_* = H'(\Sigma^{-1} \otimes \Sigma)H, \qquad H = \mathrm{diag}(e_{\cdot 1}, e_{\cdot 2}, \ldots, e_{\cdot m}).$$

The non-singularity of Ω_* follows from the non-singularity of Σ and the fact that H is evidently of rank m. Since the generalized inverse of H and H' are given respectively by

$$H_g = H', \quad H_g' = H, \quad \text{because } HH' = I_m$$

it follows that

$$\Omega_g^* = H_g(\Sigma \otimes \Sigma^{-1})H_g' = H'(\Sigma \otimes \Sigma^{-1})H,$$

which is non-singular and, thus, it is the **inverse** of Ω^*.

If the matrix

$$\Omega_1 = [\sigma^{ij}\sigma_{ij} - \sigma^{ij}\sigma_{\cdot i}' S_{11} \sigma_{\cdot j}]$$

is at least positive semi-definite, we may carry out a test of the null by means of the test statistic

$$H_D^2 = T\tilde{r}'\tilde{\Phi}^{-1}\tilde{r} \xrightarrow{\mathrm{d}} \chi_m^2, \quad \text{or, more generally,} \quad H_D^2 \xrightarrow{\mathrm{d}} \chi^2_{\mathrm{rank}(\Omega_1)}. \tag{34}$$

Remark 6. Notice that in the case $m = 1$, H_D^2 reduces to the square of the h-statistic because $\Omega_* = 1$ and $\Omega_1 = 1 - \mathrm{Avar}(\hat{a}_{11})$, as in Durbin (1970).

If the matrix Ω_1 is **indefinite, or negative definite**, the test above is inoperable and an alternative test may be undertaken as follows. Write (the observations on) the equations of the model as

$$y_{\cdot i} = r_{ii} v_{\cdot i} + Z c_{\cdot i} + \epsilon_{\cdot i}, \quad i = 1, 2, 3, \ldots, m, \tag{35}$$

and stack them so that the observations on the entire model may be written as

$$y = Vr + (I_m \otimes Z)c + e, \quad r = (r_{11}, r_{22}, r_{33}, \ldots, r_{mm})', \quad V = \mathrm{diag}(v_{\cdot 1}, v_{\cdot 2}, v_{\cdot 3}, \ldots, v_{\cdot m}). \tag{36}$$

Since V is not observable we use instead the columns of the OLS residuals $\tilde{U}_{-1}$, i.e.

$$\tilde{V} = \mathrm{diag}(\tilde{v}_{\cdot 1}, \tilde{v}_{\cdot 2}, \tilde{v}_{\cdot 3}, \ldots, \tilde{v}_{\cdot m}), \tag{37}$$

to estimate

$$\tilde{d} = [(\tilde{W}'(\tilde{\Sigma}^{-1} \otimes I_T)\tilde{W}]^{-1}\tilde{W}'(\tilde{\Sigma}^{-1} \otimes I_T)y, \;\; \tilde{W} = [\tilde{V}, (I_m \otimes Z)], \;\; d = (r', c')'. \tag{38}$$

As in the discussion above, we can show that, under the null,

$$(W - \tilde{W})d = 0,$$

so the estimator of d, and hence of r, is consistent. Moreover, under the null,

$$\sqrt{T}(\tilde{d} - d) \sim \Phi \frac{1}{\sqrt{T}} W'(\Sigma^{-1} \otimes I_m)e \xrightarrow{\mathrm{d}} N(0, \Phi), \tag{39}$$

where

$$\Phi^{-1} = \operatorname*{plim}_{T \to \infty} \frac{1}{T} W'(\Sigma^{-1} \otimes I_T)W. \tag{40}$$

It follows then, that under the null,

$$\sqrt{T}\tilde{r} \xrightarrow{\mathrm{d}} N(0, \Phi_{11}), \tag{41}$$

where Φ_{11} is the $m \times m$ principal submatrix of Φ. Consequently, to test the null that $r = 0$ we may use the test statistic

$$H_D^{*2} = T\tilde{r}'\tilde{\Phi}_{11}^{-1}\tilde{r} \xrightarrow{\mathrm{d}} \chi_m^2, \tag{42}$$

where

$$\tilde{\Phi}_{11}^{-1} = \frac{1}{T}\left(\tilde{V}'(\tilde{\Sigma}^{-1} \otimes I_T)\tilde{V} - \tilde{V}'(\tilde{\Sigma}^{-1} \otimes Z(Z'Z)^{-1}Z')\tilde{V}\right), \tag{43}$$

and $\tilde{\Sigma} = \tilde{U}'\tilde{U}/T$.

If an estimator for r is obtainable, the matrix of Eq. (41) will be **positive definite** and hence it is always operational in practice.

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