A MINKOWSKI-TYPE INEQUALITY FOR HYPERSURFACES IN THE REISSNER-NORDSTRÖM-ANTI-DESITTER MANIFOLD

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ABSTRACT

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We prove a sharp Minkowski-type inequality for hypersurfaces in the n-dimensional Reissner-Nordström-Anti-deSitter(AdS) manifold for $n \geq 3$. This inequality generalizes the one for hypersurfaces in the uncharged AdS-Schwarzschild manifold proved in [5]. With the Minkowski inequality, we prove a charged Gibbons-Penrose inequality for a large class of $(n-1)$-dimensional spacelike surfaces in the Reissner-Nordström spacetime.

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I dedicate this thesis to Professor Mu-Tao Wang.

Chapter 1

INTRODUCTION

The Minkowski-type inequality proved in [5] states that for a compact mean convex, star-shaped hypersuface Σ in the n-dimensional AdS-Schwarzschild space, we have

$$\int_\Sigma fH d\mu - n(n-1)\int_\Omega f d\text{vol}$$
$$\geq (n-1)|S^{n-1}|^{\frac{1}{n-1}}(|\Sigma|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}})$$

where f is the static potential, H is the mean curvature of Σ, and Ω is the region bounded by Σ and the horizon ∂M. Throughout this paper, we denote the area of the surface Σ by $|\Sigma|$, the area of the horizon ∂M by $|\partial M|$, and the area of the $(n-1)$ dimensional unit sphere S^{n-1} by $|S^{n-1}|$. For other related work, see [13], [17], [22].

In this paper, we extend the above inequality to the case of hypersurfaces in the Reissner-Nordström-AdS manifold.

Let us recall the definition of the Reissner-Nordström-AdS manifold. We fix three positive numbers m, q and κ, where $q < m, \kappa \ll \infty$ such that the equation $1+\kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n} = 0$ has positive solutions, and let s_0 be the larger one. We consider the manifold $M = S^{n-1} \times [s_0, \infty)$ equipped with the Riemannian metric

$$\bar{g} = \frac{1}{1+\kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n}} ds \otimes ds + s^2 g_{S^{n-1}},$$

where $g_{S^{n-1}}$ is the standard round metric on the unit sphere S^{n-1}. The boundary $\partial M = S^{n-1} \times \{s_0\}$ will be referred to as the horizon.

We define

$$f(s) = \sqrt{1 + \kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n}}. \tag{1.1}$$

Then the function f satisfies

$$(\bar{\Delta} f)\bar{g} - \bar{D}^2 f + f \operatorname{Ric} = (n-2)(n-1)q^2 f s^{4-2n} g_{S^{n-1}}, \tag{1.2}$$

where $\bar{\Delta}$ and $\bar{D}^2$ are the Laplacian and Hessian operators of $\bar{g}$.

Recall that the scalar curvature $R = -n(n-1)\kappa^2 + (n-1)(n-2)q^2 s^{2-2n}$. Taking the trace in (1.2) we have $\bar{\Delta} f = n\kappa^2 f + (n-2)^2 q^2 f s^{2-2n}$.

Now we state the main result of this paper:

Theorem 1. *Let Σ be a compact mean convex, star-shaped hypersurface in the Reissner-Nordström-AdS manifold, and let Ω denote the region bounded by Σ and the horizon ∂M. Then*

$$\begin{aligned}\int_\Sigma f H d\mu - n(n-1)\kappa^2 \int_\Omega f d\mathrm{vol} \geq {} & (n-1)|S^{n-1}|^{\frac{1}{n-1}} \left(|\Sigma|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}}\right) \\ & + (n-1)q^2 |S^{n-1}|^{\frac{2n-3}{n-1}} \left(|\Sigma|^{-\frac{n-2}{n-1}} - |\partial M|^{-\frac{n-2}{n-1}}\right).\end{aligned}$$

Equality holds if and only if $\Sigma = S^{n-1} \times \{s\}$ for some $s \in [s_0, \infty)$.

If we let $q = 0$, Theorem 1 reduces to the Minkowski inequality proved in [5]. We remark that other special cases ($\kappa = 0$, $m = 0$) are discussed in [5].

As a matter of fact, in Chapter 2, we will prove a Minkowski inequality in the following more general setting, where the Reissner-Nordström-AdS manifold is a special case.

Consider a continuous function f defined on $[s_0, \infty)$ with $s_0 > 0$ which satisfies

(H1) f is differentiable and positive on (s_0, ∞) and $f(s_0) = 0$.

(H2) $1 + \kappa^2 s^2 - f^2 = 2ms^{2-n} + O(s^{4-2n})$.

(H3) $f' > 0$.

(H4) $f(f'^2 + f f'') + (n-3)\frac{f^2 f'}{s} + (n-2)\frac{(1-f^2)f}{s^2} \geq 0$.

(H5) $P(x) = R(x^{\frac{1}{n-1}})x$, where $R(s) = 2f(s)f'(s) - 2\kappa^2 s$, is nondecreasing and concave.

We consider the asymptotically hyperbolic n-dimensional space $(M, \bar{g})$, where $M = S^{n-1} \times [s_0, \infty)$, and $\bar{g} = \frac{1}{f^2} ds^2 + s^2 g_{S^{n-1}}$.

Remark 2. *(H4) corresponds to* $(\bar{\Delta} f)\bar{g} - \bar{D}^2 f + f\,\mathrm{Ric} \geq 0$.

Remark 3. *The Reissner-Nordström-AdS manifold is one principal example of such asymptotically hyperbolic manifolds with* $P(x) = 2(n-2)m - 2(n-2)q^2 x^{-\frac{n-2}{n-1}}$.

The generalized version of Theorem 1 will be

Theorem 4. *Let* Σ *be a compact mean convex, star-shaped hypersurface in* $(M, \bar{g})$ *defined as above with* f *satisfying (H1)-(H5). Then*

$$\int_\Sigma f H d\mu - n(n-1)\kappa^2 \int_\Omega f d\mathrm{vol} \geq (n-1) f(\bar{s})^2 \bar{s}^{n-2} |S^{n-1}| - (n-1)\kappa^2 \bar{s}^n |S^{n-1}| + (n-1)\kappa^2 s_0^n |S^{n-1}|,$$

where $\bar{s} = \left(\frac{|\Sigma|}{|S^{n-1}|}\right)^{\frac{1}{n-1}}$ *is the areal radius of* Σ.
Equality holds if and only if $\Sigma = S^{n-1} \times \{s\}$ *for some* $s \in [s_0, \infty)$.

Remark 5. *To compare with Theorem 1,* $s_0 = \left(\frac{|\partial M|}{|S^{n-1}|}\right)^{\frac{1}{n-1}}$ *is the areal radius of the horizon* $|\partial M|$. *If* f *takes the form as in* (1.1)*, Theorem 4 reduces to Theorem 1, which is discussed in the beginning of Chapter 3.*

Remark 6. *As suggested by Professor Mu-Tao Wang, we remark that the above theorem might be related to solutions of static spherical symmetric Einstein/Yang-Mills Equation considered in [19].*

To prove Theorem 4, we follow the basic strategy in [5]. We start with the given mean convex, star-shaped hypersurface Σ, denoted by Σ_0 from now on, and evolve it by the inverse mean curvature flow, which was also used in the proof of the Riemannian Penrose inequality due to Huisken and Ilmanen [14]. We prove the long time existence of the flow by estimating the mean curvature and the second fundamental form of the evolving surfaces Σ_t. More precisely, we prove that $|h_i^j - \delta_i^j| \leq O(t^2 e^{-\frac{2}{n-1}t})$. Some of the computation can be found in [8], [9], [10].

Next, to accommodate the new situation, we consider the quantity

$$Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \Big(\int_{\Sigma_t} f H d\mu - n(n-1)\kappa^2 \int_{\Omega_t} f d\,\mathrm{vol} - (n-1)\left(f(\bar{s}_t)\right)^2 |\Sigma_t|^{\frac{n-2}{n-1}} |S^{n-1}|^{\frac{1}{n-1}} + (n-1)\kappa^2 \bar{s}_t^n |S^{n-1}| - (n-1)\kappa^2 s_0^n |S^{n-1}| \Big),$$

where f is the potential function defined above and $\bar{s}_t = \left(\frac{|\Sigma_t|}{|S^{n-1}|}\right)^{\frac{1}{n-1}}$ is the areal radius of $|\Sigma_t|$. The involvement of the numerical value of f at the areal radius of $|\Sigma_t|$ is a new phenomenon, maybe related to a new type of mass-area-charge inequality. See [16].

We can give a lower bound for the liminf of $Q(t)$ as $t \to \infty$, which is

$$\liminf_{t\to\infty} Q(t) \geq 0$$

Finally, we show that $Q(t)$ is monotone decreasing along the inverse mean curvature flow. The proof of the monotonicity formula uses Remark 2 and the Heintze-Karcher type inequality

$$(n-1)\int_{\Sigma_t} \frac{f}{H} d\mu \geq n \int_{\Omega_t} f \, d\,\mathrm{vol} + s_0^n |S^{n-1}|$$

(cf.[3]), as in [5]. This inequality was used in [3] to prove a generalization of Alexanderov's theorem (see also[4]). In addition, the important hypothesis (H5) helps us extend the monotonicity of Q to this more general setting. Thus, we conclude that $Q(0) \geq 0$. From this, Theorem 4 follows easily.

Our study leads to Conjecture 24 on charged Gibbons-Penrose inequality. For backgrounds and motivations, see [2], [6], [7], [11], [12], [18], [20], [21]. Here we state the conjecture for $n = 3$, which is of particular physics interest.

Conjecture. *Let Σ be a closed embedded orientable spacelike 2-surface in the Reissner-Nordström spacetime equipped with the metric*

$$-(1 - 2ms^{-1} + q^2 s^{-2})dt^2 + \frac{1}{1 - 2ms^{-1} + q^2 s^{-2}} ds^2 + s^2 dg_{S^2}. \tag{1.3}$$

Suppose that $|\Sigma| \geq (m - \sqrt{m^2 - q^2})^2 |S^2|$ and the past null hypersurface generated by Σ is smooth. Then

$$-\frac{1}{16\pi}\int_\Sigma \left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle d\mu + m \geq \sqrt{\frac{|\Sigma|}{16\pi}} + \frac{1}{4} q^2 \sqrt{\frac{16\pi}{|\Sigma|}},$$

where $\vec{J}$ is the dual mean curvature vector of Σ.

Remark 7. *If we let $q = 0$, the assumption $|\Sigma| \geq (m - \sqrt{m^2 - q^2})^2 |S^2|$ is vacuum, and inequality* (1) *recovers the Gibbons-Penrose inequality for surfaces in the Schwarzschild spacetime, which is discussed in [6].*

In particular, if $\Sigma = \partial M$, inequality (1) *takes form of the Penrose inequality.*

Remark 8. *When $q \neq 0$ and $\Sigma = \partial M$, inequality* (1) *recovers inequality (1.2) in [16] and inequality (1.3) in [15]. For other discussion on the Penrose inequality with charge, see [23].*

As an application of the main theorem, we prove Theorem 27, i.e. a charged Gibbons-Penrose inequality holds for a large class of hypersurfaces in the Reissner-Nordström spacetime of any dimension. The proof is motivated by that of Theorem 4 in [6]. For completeness, we include them in the last section.

Chapter 2

PROOF OF THE MINKOWSKI-TYPE INEQUALITY

2.1 Asymptotic geometry of the manifold $(M, \bar{g})$.

Lemma 9. *By a change of variable, the metric can be rewritten as*

$$\bar{g} = dr \otimes dr + \lambda(r)^2 g_{S^{n-1}},$$

where $\lambda(r)$ satisfies the ODE

$$\lambda'(r) = f(\lambda), \tag{2.1}$$

and the asymptotic expansion

$$\lambda(r) = \kappa^{-1}\sinh(\kappa r) + \frac{m}{n}\kappa^{n-3}\sinh^{-n+1}(\kappa r) + \kappa^{n-3}O(\sinh^{-n}(\kappa r)).$$

Proof. We define

$$r(s) = \int_{s_0}^{s} \frac{1}{f(t)}dt - b,$$

where

$$b = \int_{s_0}^{\infty} \left(\frac{1}{f(t)} - \frac{1}{\sqrt{1+\kappa^2 t^2}} \right) dt.$$

Then the metric $\bar{g}$ can be written as $\bar{g} = dr \otimes dr + \lambda(r)^2 g_{S^{n-1}}$, where $\lambda(r(s)) = s$.

Using (H2), the function $r(s)$ can be rewritten as

$$\begin{aligned} r(s) &= \int_0^s \frac{1}{\sqrt{1+\kappa^2 t^2}} dt - \int_s^\infty \left(\frac{1}{f(t)} - \frac{1}{\sqrt{1+\kappa^2 t^2}} \right) dt \\ &= \frac{1}{\kappa} \operatorname{arcsinh}(\kappa s) - \frac{m}{n} \kappa^{-3} s^{-n} + \kappa^{-4} O(s^{-n-1}). \end{aligned}$$

By Taylor expansion, we have

$$\begin{aligned} \sinh(\kappa r(s)) &= \kappa s - \frac{m}{n} \kappa^{-1} s^{1-n} + \kappa^{-2} O(s^{-n}) \\ &= \kappa s - \frac{m}{n} \kappa^{n-2} \sinh^{-n+1}(\kappa r) + \kappa^{n-2} O(\sinh^{-n}(\kappa r)), \end{aligned}$$

from which the assertion follows. □

In the next lemma, we calculate the asymptotic expansion of Riemannian curvature tensors.

Lemma 10. *Let $e_\alpha, \alpha = 1, 2, \cdots, n$ be an orthonormal frame and $R_{\alpha\beta\gamma\mu}$ is the Riemannian curvature tensor. Then*

$$R_{\alpha\beta\gamma\mu} = -\kappa^2 \delta_{\beta\mu} \delta_{\alpha\gamma} + \kappa^2 \delta_{\beta\gamma} \delta_{\alpha\mu} + O(e^{-n\kappa r}), \tag{2.2}$$

and

$$\bar{D}_\rho R_{\alpha\beta\gamma\mu} = O(e^{-n\kappa r}). \tag{2.3}$$

The Ricci tensor satisfies

$$\operatorname{Ric}(\partial_r, \partial_r) = -(n-1)\kappa^2 + O(\kappa^n \sinh^{-n}(\kappa r)),$$

and

$$\lambda^{-2} \operatorname{Ric}(\partial_{\theta^i}, \partial_{\theta^j}) = -(n-1)\kappa^2 \sigma_{ij} + O(\kappa^n \sinh^{-n}(\kappa r)),$$

where $\sigma_{ij} = g_{S^{n-1}}(\partial_{\theta^i}, \partial_{\theta^j})$.

Proof. Each level set of r is a round sphere with induced metric $\lambda(r)^2 g_{S^{n-1}}$ and second fundamental form $\lambda(r)\lambda'(r) g_{S^{n-1}}$. Using the Gauss equation,we have

$$R(\partial_{\theta^i}, \partial_{\theta^j}, \partial_{\theta^k}, \partial_{\theta^l}) = \lambda(r)^2 (1 - \lambda'(r)^2)(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}).$$

Since the level set of r is umbilic, from the Codazzi equation, we have

$$R(\partial_{\sigma^i}, \partial_{\sigma^j}, \partial_{\sigma^k}, \partial_r) = 0.$$

The remaining components of the curvature tensors are

$$\begin{aligned} R(\partial_{\theta^i}, \partial_r, \partial_{\theta^j}, \partial_r) &= \left\langle (\bar{\nabla}_i \bar{\nabla}_r - \bar{\nabla}_r \bar{\nabla}_i) \partial_r, \partial_{\theta^j} \right\rangle \\ &= - \left\langle \bar{\nabla}_r \bar{\nabla}_i \partial_r, \partial_{\theta^j} \right\rangle \\ &= - \left\langle \bar{\nabla}_r \left(\frac{\lambda'}{\lambda} \partial_{\theta^i} \right), \partial_{\theta^j} \right\rangle \\ &= -\lambda(r)\lambda''(r)\sigma_{ij}. \end{aligned}$$

From this, (2.2) and (2.3) follow easily.

Moreover, we have

$$\begin{aligned} \operatorname{Ric}(\partial_r, \partial_r) &= -(n-1)\frac{\lambda''(r)}{\lambda(r)} \\ &= -(n-1)\kappa^2 + O(\kappa^n \sinh^{-n}(\kappa r)), \end{aligned}$$

and

$$\begin{aligned} \lambda^{-2} \operatorname{Ric}(\partial_{\theta^i}, \partial_{\theta^j}) &= (n-2)\frac{1-\lambda'^2}{\lambda^2}\sigma_{ij} - \frac{\lambda''}{\lambda}\sigma_{ij} \\ &= \left(-(n-1)\kappa^2 + O(\lambda^{-n})\right)\sigma_{ij} \\ &= -(n-1)\kappa^2\sigma_{ij} + O(\kappa^n \sinh^{-n}(\kappa r)). \end{aligned}$$

□

2.2 Star-shaped hypersurfaces in $(M, \bar{g})$.

Let $\theta = \{\theta^j\}_{j=1,2,\ldots,n-1}$ be a coordinate system on S^{n-1} and ∂_θ^j be the corresponding coordinate vector field in M. A star-shaped hypersurface $\Sigma \subset M$ can be parametrized by

$$\Sigma = \{(r(\theta), \theta) : \theta \in S^{n-1}\}$$

for a smooth function r on S^{n-1}. We define $\varphi : S^{n-1} \to \mathbb{R}$ by

$$\varphi(\theta) = \Phi(r(\theta)),$$

where $\Phi(r)$ is a positive function that satisfies $\Phi'(r) = \frac{1}{\lambda(r)}$.

Let $\varphi_i = \nabla_i \varphi$ and $\varphi_{ij} = \nabla_i \nabla_j \varphi$ denote the covariant derivatives of φ with respect to the round metric $g_{S^{n-1}}$. Let

$$v = \sqrt{1 + |\nabla \varphi|^2_{S^{n-1}}}.$$

In the next proposition, we write the metric and second fundamental form of Σ in terms of covariant derivatives of φ:

Proposition 11. *Let g_{ij} be the induced metric on Σ and h_{ij} be the second fundamental form in terms of the coordinates. Then*

$$g_{ij} = \lambda^2(\sigma_{ij} + \varphi_i\varphi_j),$$

and

$$h_{ij} = \frac{\lambda}{v}\left(\lambda'(\sigma_{ij} + \varphi_i\varphi_j) - \varphi_{ij}\right).$$

Proof. We choose a basis of tangent vector fields of Σ to be $r_j\partial_r + \partial_{\theta^j}$. We compute

$$\begin{aligned} g_{ij} &= \langle r_i\partial_r + \partial_{\theta^i}, r_j\partial_r + \partial_{\theta^j}\rangle \\ &= \lambda^2(r)\sigma_{ij} + r_ir_j \\ &= \lambda^2(r)(\sigma_{ij} + \varphi_i\varphi_j). \end{aligned}$$

The unit normal vector ν is given by

$$\nu = \frac{1}{v}\left(\partial_r - \frac{r^j}{\lambda^2}\partial_{\theta^j}\right).$$

Thus, the second fundamental form is given by

$$\begin{aligned} h_{ij} &= -\left\langle \bar{\nabla}_{r_i\partial_r + \partial_{\theta^i}}(r_j\partial_r + \partial_{\theta^j}), \nu\right\rangle \\ &= -\left\langle (r_{ij} - \lambda\lambda')\partial_r + \frac{\lambda'}{\lambda}r_j\partial_{\theta^i} + \frac{\lambda'}{\lambda}r_i\partial_{\theta^j}, \nu\right\rangle \\ &= \frac{1}{v}\left(\lambda\lambda'\sigma_{ij} + \frac{2\lambda'}{\lambda}r_ir_j - r_{ij}\right) \\ &= \frac{\lambda}{v}\left(\lambda'(\sigma_{ij} + \varphi_i\varphi_j) - \varphi_{ij}\right). \end{aligned}$$

□

2.3 The inverse mean curvature flow.

Let Σ_0 be a mean convex star-shaped hypersurface in M which is given by the embedding

$$F_0 : S^{n-1} \to M.$$

Let $F_t : S^{n-1} \to M, t \in [0,T)$ be the solution of inverse mean curvature flow with initial data F_0, in other words,

$$\frac{\partial F}{\partial t} = \frac{1}{H}\nu, \tag{2.4}$$

where ν is the unit outer normal vector and H is the mean curvature. We call (2.4) the parametric form of the flow.

Also we can write Σ_0 as the graph of a function $\tilde{r}_0$ defined on the unit sphere:

$$\Sigma_0 = \{(\tilde{r}_0(\theta), \theta) : \theta \in S^{n-1}\}.$$

If Σ_t is star-shaped, it can be parametrized as the graph

$$\Sigma_t = \{(\tilde{r}(\theta, t), \theta) : \theta \in S^{n-1}\},$$

in which case, the inverse mean curvature flow can be written as a parabolic PDE for $\tilde{r}$. As long as the solution of (2.4) exists and remains star-shaped, it is equivalent to

$$\frac{\partial \tilde{r}}{\partial t} = \frac{v}{H}. \tag{2.5}$$

We will refer to the equation (2.5) as the non-parametric form of the inverse mean curvature flow.

Associated with $\tilde{r}$, we define the function

$$\varphi(\theta, t) := \Phi(\tilde{r}(\theta, t)),$$

where $\Phi(r)$ is a positive function satisfying $\Phi'(r) = \frac{1}{\lambda(r)}$. Then φ satisfies

$$\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda H}. \tag{2.6}$$

In the sequel, we will use the non-parametric form (2.5) to derive C^0 and C^1 estimates of $\tilde{r}$.

Lemma 12. *Let* $\bar{r}(t) = \sup_{S^{n-1}} \tilde{r}(\cdot, t)$ *and* $\underline{r}(t) = \inf_{S^{n-1}} \tilde{r}(\cdot, t)$*. Then*

$$\lambda(\bar{r}(t)) \leq e^{\frac{1}{n-1}t} \lambda(\bar{r}(0)),$$

and

$$\lambda(\underline{r}(t)) \geq e^{\frac{1}{n-1}t} \lambda(\underline{r}(0)).$$

Proof. We have that

$$\frac{\partial \tilde{r}}{\partial t} = \frac{v}{H},$$

and

$$H = \frac{(n-1)\lambda'}{\lambda v} - \frac{\tilde{\sigma}^{ij}}{\lambda v}\varphi_{ij},$$

where $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}$. At the point where the function $\tilde{r}(\cdot, t)$ attains its maximum, we have $H \geq \frac{(n-1)\lambda'}{\lambda}$. This implies

$$\frac{d}{dt}\bar{r}(t) \leq \frac{\lambda(\bar{r}(t))}{(n-1)\lambda'(\bar{r}(t))},$$

hence

$$\frac{d}{dt}\lambda(\bar{r}(t)) \leq \frac{\lambda(\bar{r}(t))}{n-1}.$$

The first statement follows easily. We may prove the second one similarly. □

Proposition 13. *We have* $H \leq (n-1)\kappa + O(e^{-\frac{2}{n-1}t})$.

Proof. We work in the parametric setting. The evolution equation of the mean curvature is

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - 2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\mathrm{Ric}(\nu,\nu)}{H}.$$

By Lemma 10, we have$|\operatorname{Ric} + (n-1)\kappa^2 g| \leq O(\kappa^n e^{-\frac{n}{n-1}t})$ on Σ_t. This gives

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - 2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} + \frac{(n-1)\kappa^2}{H} + \frac{1}{H}O(\kappa^n e^{-\frac{n}{n-1}t}). \tag{2.7}$$

Using (2.7) and the inequality $|A|^2 \geq \frac{1}{n-1}H^2$, we obtain

$$\frac{d}{dt}H^2_{\max} \leq -\frac{2}{n-1}H^2_{\max} + 2(n-1)\kappa^2 + O(\kappa^n e^{-\frac{n}{n-1}t}).$$

This implies

$$H_{\max}(t)^2 \leq (n-1)^2\kappa^2 + O(e^{-\frac{2}{n-1}t}).$$

□

We next establish a gradient bound for the function φ. We define

$$F = \frac{\lambda H}{v} = \frac{(n-1)\lambda' - \tilde{\sigma}^{ij}\varphi_{ij}}{v^2},$$

and

$$G_k = F\varphi_k - \frac{1}{v^2}\varphi^i\varphi_{ik} + \frac{1}{v^4}\varphi_k\varphi^i\varphi^j\varphi_{ij},$$

where $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$. Note that the variation of F with respect to φ_k is $-\frac{2}{v^2}G_k$.

Proposition 14. *We have* $\sup_{S^{n-1}} |\nabla\varphi|_{g_{S^{n-1}}} = O(e^{-\frac{1}{n-1}t})$.

Proof. The non-parametric form of the equation can be written as

$$\frac{\partial\varphi}{\partial t} = \frac{1}{F}. \tag{2.8}$$

Let $\omega = \frac{1}{2}|\nabla\varphi|^2_{g_{S^{n-1}}}$. We differentiate the identity (2.8) with respect to $\varphi^k\nabla_k$ to get

$$\begin{aligned}\frac{\partial\omega}{\partial t} &= -\frac{1}{F^2}\varphi^k\nabla_k F\\ &= \frac{1}{v^2F^2}\left(\tilde{\sigma}^{ij}\varphi_{ijk}\varphi^k + 2G^k\omega_k - 2(n-1)\lambda\lambda''\omega\right).\end{aligned}$$

Observe that

$$\begin{aligned}\omega_{ij} &= \varphi_{kij}\varphi^k + \varphi_{ki}\varphi^k{}_j\\ &= \varphi_{ijk}\varphi^k + (\sigma_{ij}\sigma_{kp} - \sigma_{ik}\sigma_{jp})\varphi^p\varphi^k + \varphi_{ki}\varphi^k{}_j\\ &= \varphi_{ijk}\varphi^k + \sigma_{ij}|\nabla\varphi|^2_{g_{S^{n-1}}} - \varphi_i\varphi_j + \varphi_{ki}\varphi^k{}_j,\end{aligned}$$

where the covariant derivatives are taken with respect to $g_{S^{n-1}}$.

Notice that

$$\tilde{\sigma}^{ij}(\sigma_{ij}|\nabla\varphi|^2_{g_{S^{n-1}}} - \varphi_i\varphi_j) = 2(n-2)\omega.$$

It follows that

$$\tilde{\sigma}^{ij}\omega_{ij} = \tilde{\sigma}^{ij}\varphi_{ijk}\varphi^k + 2(n-2)\omega + \tilde{\sigma}^{ij}\varphi_{ki}\varphi^k{}_j.$$

Putting the above facts together, we conclude

$$\begin{aligned}\frac{\partial\omega}{\partial t} = &\frac{1}{v^2F^2}(\tilde{\sigma}^{ij}\omega_{ij} + 2G^k\omega_k - 2(n-2)\omega - 2(n-1)\lambda\lambda''\omega)\\ &- \frac{1}{v^2F^2}\tilde{\sigma}^{ij}\sigma^{kl}\varphi_{ik}\varphi_{jl}.\end{aligned}$$

From (H3), we know that $\lambda'' > 0$. Using Proposition 13, we obtain

$$\frac{(n-1)\lambda\lambda''}{v^2F^2} = \frac{(n-1)\lambda\lambda''}{\lambda^2H^2} \geq \frac{1}{n-1} - Ce^{-\frac{2}{n-1}t}.$$

Therefore,

$$\frac{d}{dt}\omega_{\max} \leq -2\left(\frac{1}{n-1} - Ce^{-\frac{2}{n-1}t}\right)\omega_{\max},$$

where $\omega_{\max} = \frac{1}{2}\sup_{S^{n-1}} |\nabla\varphi|^2_{g_{S^{n-1}}}$.

Thus $\omega_{\max} = O(e^{-\frac{2}{n-1}t})$. □

Proposition 15. *The function $\dot\varphi = \frac{v}{\lambda H}$ satisfies $\sup_{S^{n-1}} \dot\varphi \le Ce^{-\frac{1}{n-1}t}$.*

Proof. If we differentiate (2.8) with respect to t, we obtain

$$\begin{aligned}\frac{\partial\dot\varphi}{\partial t} &= -\frac{1}{F^2}\frac{\partial F}{\partial t}\\ &= \frac{1}{v^2F^2}\left(\tilde\sigma^{ij}\dot\varphi_{ij} + 2G^k\dot\varphi_k - (n-1)\lambda\lambda''\dot\varphi\right).\end{aligned} \tag{2.9}$$

As above, we have

$$\frac{(n-1)\lambda\lambda''}{v^2F^2} = \frac{(n-1)\lambda\lambda''}{\lambda^2H^2} \ge \frac{1}{n-1} - Ce^{-\frac{2}{n-1}t}.$$

Applying the maximum principle, we obtain

$$\sup_{S^{n-1}} \dot\varphi \le Ce^{-\frac{1}{n-1}t},$$

as claimed. □

Corollary 16. *The mean curvature H is bounded from below by a positive constant.*

Proof. By Proposition 15, we have $\frac{v}{\lambda H} \le Ce^{-\frac{1}{n-1}t}$ for some uniform constant C. Since $v \ge 1$ and $\lambda \le Ce^{\frac{1}{n-1}t}$, the assertion follows. □

Proposition 17. *The norm of the second fundamental form is uniformly bounded globally in time.*

Proof. We work with the parametric formulation. We compute

$$\begin{aligned}\frac{\partial h_i^j}{\partial t} &= \frac{1}{H^2}\nabla^j\nabla_i H - 2\frac{\nabla_i H\nabla^j H}{H^3} - \frac{h_i^k h_k^j}{H} - \frac{1}{H}g^{mj}R_{\nu i\nu m}\\ &= \frac{\Delta h_i^j}{H^2} - 2\frac{\nabla_i H\nabla^j H}{H^3} + \frac{|A|^2}{H^2}h_i^j - 2\frac{h_i^k h_k^j}{H}\\ &+ \frac{2}{H^2}g^{kl}g^{sj}R_{miks}h_l^m - \frac{1}{H^2}g^{kl}g^{sj}R_{mksl}h_i^m - \frac{1}{H^2}g^{kl}R_{mkil}h^{mj}\\ &+ \frac{1}{H^2}\operatorname{Ric}(\nu,\nu)h_i^j - \frac{2}{H}g^{mj}R_{\nu i\nu m}\\ &- \frac{1}{H^2}g^{kl}g^{mj}\bar\nabla_m R_{\nu kil} - \frac{1}{H^2}g^{kl}g^{mj}\bar\nabla_k R_{\nu iml}.\end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned}\frac{\partial h_i^j}{\partial t} &= \frac{\Delta h_i^j}{H^2} - 2\frac{\nabla_i H \nabla^j H}{H^3} + \frac{|A|^2}{H^2}h_i^j - 2\frac{h_i^k h_k^j}{H} \\ &+ (n-1)\kappa^2 \frac{h_i^j}{H^2} + \frac{|A|+1}{H^2}O(e^{-\frac{n}{n-1}t}).\end{aligned} \tag{2.10}$$

Combining (2.7) and (2.10), we have the following evolution equation for $M_i^j = H h_i^j$:

$$\begin{aligned}\frac{\partial M_i^j}{\partial t} &= \frac{\Delta M_i^j}{H^2} - 2\frac{\nabla^k H \nabla_k M_i^j}{H^3} - 2\frac{\nabla_i H \nabla^j H}{H^2} \\ &- 2\frac{M_i^k M_k^j}{H^2} + 2(n-1)\kappa^2 \frac{M_i^j}{H^2} + \frac{|M|+H}{H^2}O(e^{-\frac{n}{n-1}t}).\end{aligned} \tag{2.11}$$

Let μ denote the largest eigenvalue of the tensor M_i^j, and let $\mu_{\max}(t)$ denote the maximum of μ at a given time t. Since the trace of M is positive, we have $|M| \leq C\mu$ for some constant C. Since H is uniformly bounded from above and below, we obtain

$$\frac{d}{dt}\mu_{\max} \leq -\frac{1}{C}\mu_{\max}^2 + C\mu_{\max} + C,$$

for some uniform constant C. Therefore, $\mu_{\max} \leq C$ for some uniform constant C, which implies $|M| \leq C$. Since H is uniformly bounded from below, we conclude that $|A|$ is uniformly bounded. □

Corollary 18. *The solution of the inverse mean curvature flow is defined on* $[0, \infty)$.

2.4 The asymptotic behavior of the flow as $t \to \infty$.

In this section, we improve estimates for the mean curvature and the second fundamental form.

Proposition 19. *We have* $H = (n-1)\kappa + O(te^{-\frac{2}{n-1}t})$.

Proof. With Proposition 13, it suffices to bound H from below. We again work in the non-parametric setting. We consider the function

$$\chi = \lambda\dot{\varphi} = \frac{v}{H}.$$

The results in the previous section imply that the function χ is uniformly bounded from above and below. Using (15) and the identity $\dot{\varphi} = \frac{1}{F} = \frac{\chi}{\lambda}$, we obtain

$$\begin{aligned}
\frac{\partial \chi}{\partial t} &= \lambda \frac{\dot{\varphi}}{\partial t} + \lambda\lambda' \dot{\varphi}^2 \\
&= \frac{\chi^2}{v^2 \lambda} \Big(\tilde{\sigma}^{ij} \nabla_j \nabla_i (\frac{\chi}{\lambda}) + 2G^k \nabla_k (\frac{\chi}{\lambda}) - (n-1)\lambda'' \chi \Big) + \frac{\lambda'}{\lambda}\chi^2 \\
&= \frac{\chi^2}{v^2 \lambda^2} \left(\tilde{\sigma}^{ij} \chi_{ij} - \frac{2}{\lambda} \tilde{\sigma}^{ij} \lambda_i \chi_j + 2G^k \chi_k \right) \\
&+ \frac{\chi^2}{v^2\lambda^2} \left(\frac{2\chi}{\lambda^2} \tilde{\sigma}^{ij} \lambda_i \lambda_j - \frac{\chi}{\lambda} \tilde{\sigma}^{ij} \lambda_{ij} - \frac{2\chi}{\lambda} G^k \lambda_k \right) \\
&+ \frac{\lambda'}{\lambda} \chi^2 - \frac{n-1}{v^2} \frac{\lambda''}{\lambda} \chi^3.
\end{aligned}$$

Using Proposition 14, we obtain

$$\tilde{\sigma}^{ij} \lambda_i \lambda_j \leq C e^{\frac{2}{n-1}t}.$$

Moreover, using the identity

$$-\tilde{\sigma}^{ij} \varphi_{ij} = v^2 F - (n-1)\lambda' = v^2 \frac{\lambda}{\chi} - (n-1)\lambda',$$

we obtain

$$\begin{aligned}
-\tilde{\sigma}^{ij} \lambda_{ij} &= \lambda\lambda' \tilde{\sigma}^{ij} \varphi_{ij} - \lambda(\lambda\lambda'' + \lambda'^2) \tilde{\sigma}^{ij} \varphi_i \varphi_j \\
&\leq \lambda\lambda' \left(v^2 \frac{\lambda}{\chi} - (n-1)\lambda' \right) + C e^{\frac{1}{n-1}t}.
\end{aligned}$$

Finally, the second fundamental form is uniformly bounded by Proposition 17. Using Proposition 11, we obtain $|D^2 \varphi| \leq C e^{\frac{1}{n-1}t}$, where $D^2\varphi$ denotes the Hessian of φ with respect to $g_{S^{n-1}}$. Using Proposition 14, we conclude that

$$-G^k \varphi_k = -F |\nabla \varphi|^2_{g_{S^{n-1}}} + \frac{1}{v^4} \varphi^i \varphi^j \varphi_{ij} \leq C e^{-\frac{1}{n-1}t},$$

hence

$$-G^k \lambda_k \leq C e^{\frac{1}{n-1}t}.$$

Putting these facts together, we obtain that

$$\begin{aligned}
\frac{\partial \chi}{\partial t} &\leq \frac{\chi^2}{v^2 \nabla^2} \left(\tilde{\sigma}^{ij} \chi_{ij} - \frac{2}{\lambda} \tilde{\sigma}^{ij} \lambda_i \chi_j + 2G^k \chi_k \right) \\
&+ \frac{2\lambda'}{\lambda} \chi^2 - \frac{2}{v^2} \frac{\lambda\lambda'' + \lambda'^2}{\lambda^2} \chi^3 + C e^{-\frac{2}{n-1}t}.
\end{aligned}$$

Since χ is uniformly bounded and $v = 1 + O(e^{-\frac{2}{n-1}t})$, the maximum of χ satisfies

$$\frac{d}{dt}\chi_{\max} \leq 2\kappa\chi_{\max}^2 - 2(n-1)\kappa^2\chi_{\max}^3 + Ce^{-\frac{2}{n-1}t}.$$

In particular, we have

$$\frac{d}{dt}\chi_{\max} \leq \frac{2}{(n-1)^2\kappa} - \frac{2}{n-1}\chi_{\max} + Ce^{-\frac{2}{n-1}t},$$

whenever $\chi_{\max} \geq \frac{1}{(n-1)\kappa}$. Therefore, $\chi_{\max} \leq \frac{1}{(n-1)\kappa} + O(te^{-\frac{2}{n-1}t})$. Since $v = 1 + O(e^{-\frac{2}{n-1}t})$, we conclude that $H \geq (n-1)\kappa - O(te^{-\frac{2}{n-1}t})$. □

Proposition 20. *We have* $|h_i^j - \kappa\delta_i^j| \leq O(t^2 e^{-\frac{2}{n-1}t})$.

Proof. As above, we define $M_i^j = Hh_i^j$. We have shown that $|M|$ is uniformly bounded, and $H = (n-1)\kappa + O(te^{-\frac{2}{n-1}t})$. Hence, it follows from (2.11) that

$$\begin{aligned}\frac{\partial M_i^j}{\partial t} &= \frac{\Delta M_i^j}{H^2} - 2\frac{\nabla^k H \nabla_k M_i^j}{H^3} - 2\frac{\nabla_i H \nabla^j H}{H^2} \\ &\quad - \frac{2}{(n-1)^2\kappa^2} M_i^k M_k^j + \frac{2}{n-1} M_i^j + O(te^{-\frac{n}{n-1}t}).\end{aligned}$$

Let μ denote the largest eigenvalue of M_i^j, and let $\mu_{\max}(t)$ be the maximum of μ at a given time t. Then

$$\begin{aligned}\frac{d}{dt}\mu_{\max} &\leq -\frac{2}{(n-1)^2\kappa^2}\mu_{\max}^2 + \frac{2}{n-1}\mu_{\max} + O(te^{-\frac{n}{n-1}t}) \\ &\leq 2\kappa^2 - \frac{2}{n-1}\mu_{\max} + O(te^{-\frac{2}{n-1}t}).\end{aligned}$$

Thus,

$$\mu_{\max} \leq (n-1)\kappa^2 + O(t^2 e^{-\frac{2}{n-1}t}).$$

As $M_i^j = Hh_i^j$ and $H = (n-1)\kappa + O(te^{-\frac{2}{n-1}t})$, we conclude that the largest eigenvalue of the second fundamental form is less than $\kappa + O(t^2 e^{-\frac{2}{n-1}t})$. Since $H = (n-1)\kappa + O(te^{-\frac{2}{n-1}t})$, the smallest eigenvalue of the second fundamental form is greater than $1 - O(t^2 e^{-\frac{2}{n-1}t})$. □

2.5 The liminf of $Q(t)$.

We consider a family of star-shaped surfaces Σ_t evolving by inverse mean curvature flow. We define

$$\begin{aligned}Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}}(&\int_{\Sigma_t} fH d\mu - n(n-1)\kappa^2 \int_{\Omega_t} f d\,\mathrm{vol} \\ &- (n-1)\left(f\left(\bar{s}_t\right)\right)^2 |\Sigma_t|^{\frac{n-2}{n-1}} |S^{n-1}|^{\frac{1}{n-1}} + (n-1)\kappa^2 \bar{s}_t^n |S^{n-1}| - (n-1)\kappa^2 s_0^n |S^{n-1}|),\end{aligned}$$

where $\bar{s}_t = \left(\frac{|\Sigma_t|}{|S^{n-1}|}\right)^{\frac{1}{n-1}}$. We first evaluate the limit of $Q(t)$ as $t \to \infty$. We need the following result which can be obtained using the Sobolev inequalities on the sphere:

Proposition 21. *For every positive function u on S^{n-1}, we have*

$$\frac{1}{2}\int_{S^{n-1}} u^{n-4}|\nabla u|^2_{g_{S^{n-1}}} d\,\mathrm{vol}_{S^{n-1}} + \int_{S^{n-1}} u^{n-2} d\,\mathrm{vol}_{S^{n-1}}$$
$$\geq |S^{n-1}|^{\frac{1}{n-1}} \left(\int_{S^{n-1}} u^{n-1} d\,\mathrm{vol}_{S^{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Moreover, equality holds if and only if u is constant.

Proof. It follows from Theorem 4 in [1] that

$$\frac{2}{(n-2)(n-1)}\int_{S^{n-1}} |\nabla l|^2_{g_{S^{n-1}}} d\,\mathrm{vol}_{S^{n-1}} + \int_{S^{n-1}} l^2 d\,\mathrm{vol}_{S^{n-1}}$$
$$\geq |S^{n-1}|^{\frac{1}{n-1}} \left(\int_{S^{n-1}} l^{\frac{2(n-1)}{n-2}} d\,\mathrm{vol}_{S^{n-1}}\right)^{\frac{n-2}{n-1}}$$

for every positive smooth function l. Now if we put $l = u^{\frac{n-2}{2}}$, we obtain

$$\frac{n-2}{2(n-1)}\int_{S^{n-1}} u^{n-4}|\nabla u|_{g_{S^{n-1}}} d\,\mathrm{vol}_{S^{n-1}} + \int_{S^{n-1}} u^{n-2} d\,\mathrm{vol}_{S^{n-1}}$$
$$\geq |S^{n-1}|^{\frac{1}{n-1}} \left(\int_{S^{n-1}} u^{n-1} d\,\mathrm{vol}_{S^{n-1}}\right)^{\frac{n-2}{n-1}}.$$

From this, the assertion follows. □

Proposition 22. *We have* $\liminf_{t\to\infty} Q(t) \geq 0$.

Proof. Using the inequalities

$$f = \lambda(\kappa + O(e^{-\frac{2}{n-1}t})),$$
$$H - (n-1)\kappa = O(te^{-\frac{2}{n-1}t}),$$
$$\sqrt{\det g} = \lambda^{n-1}\sqrt{\det g_{S^{n-1}}}(1 + O(e^{-\frac{2}{n-1}t})),$$

we obtain

$$\int_{\Sigma_t} f(H-(n-1)\kappa)d\mu = \int_{S^{n-1}} \lambda^n \kappa(H-(n-1)\kappa) d\,\mathrm{vol}_{S^{n-1}} + O(te^{\frac{n-4}{n-1}t}). \tag{2.12}$$

By Proposition 11, the metric and second fundamental form on Σ_t are given by

$$g_{ij} = \lambda^2(\sigma_{ij} + \varphi_i\varphi_j),$$

and

$$h_{ij} = \frac{\lambda'}{\lambda v} g_{ij} - \frac{\lambda}{v}\varphi_{ij}.$$

Here, σ_{ij} is the round metric on S^{n-1} and φ_{ij} is the Hessian of φ with respect to $g_{S^{n-1}}$. By Proposition 2.10, we have $|h - \kappa g|_g \leq O(t^2 e^{-\frac{2}{n-1}t})$. This implies

$$|h - \frac{\lambda'}{\lambda v} g|_g \leq O(t^2 e^{-\frac{2}{n-1}t}),$$

hence

$$\left| h - \frac{\lambda'}{\lambda v} g \right|_{g_{S^{n-1}}} \leq O(t^2).$$

From this, we deduce that $|D^2\varphi|_{g_{S^{n-1}}} \leq O(t^2 e^{-\frac{1}{n-1}t})$, where $D^2\varphi$ denotes the Hessian of φ. Using Proposition 14, we obtain

$$\tilde{\sigma}^{ij}\varphi_{ij} = \Delta_{S^{n-1}}\varphi + O(t^2 e^{-\frac{3}{n-1}t}).$$

This implies

$$\begin{aligned} H &= \frac{(n-1)\lambda'}{\lambda v} - \frac{1}{\lambda v}\tilde{\sigma}^{ij}\varphi_{ij} \\ &= \frac{(n-1)\lambda'}{\lambda v} - \frac{1}{\lambda v}\Delta_{S^{n-1}}\varphi + O(t^2 e^{-\frac{4}{n-1}t}). \end{aligned}$$

Since $\lambda' = \kappa\lambda + \frac{1}{2}\kappa^{-1}\lambda^{-1} + O(e^{-\frac{2}{n-1}t})$ and $\frac{1}{v} = 1 - \frac{1}{2}|\nabla\varphi|^2_{g_{S^{n-1}}} + O(e^{-\frac{4}{n-1}t})$, we conclude that

$$H = (n-1)\kappa + \frac{n-1}{2\kappa\lambda^2} - \frac{n-1}{2}\kappa|\nabla\varphi|^2_{g_{S^{n-1}}} - \frac{1}{\lambda}\Delta_{S^{n-1}}\varphi + O(e^{-\frac{3}{n-1}t}).$$

Substituting this identity into (2.12), we obtain

$$\begin{aligned} &\int_{\Sigma_t} f(H-(n-1)\kappa)d\mu \\ &= \int_{S^{n-1}} \left(\frac{n-1}{2}\lambda^{n-2} - \frac{n-1}{2}\kappa^2\lambda^n|\nabla\varphi|^2_{g_{S^{n-1}}} - \lambda^{n-1}\kappa\Delta_{S^{n-1}}\varphi \right) d\,\mathrm{vol}_{S^{n-1}} + O(e^{\frac{n-3}{n-1}t}) \\ &= \int_{S^{n-1}} \left(\frac{n-2}{2}\lambda^{n-2} - \frac{n-1}{2}\kappa^2\lambda^n|\nabla\varphi|^2_{g_{S^{n-1}}} + (n-1)\kappa\lambda^{n-2}\left\langle \nabla\lambda, \nabla\varphi \right\rangle_{S^{n-1}} \right) d\,\mathrm{vol}_{S^{n-1}} \\ &+ O(e^{\frac{n-3}{n-1}t}). \end{aligned}$$

By Proposition 14, we have $|\nabla\varphi|_{g_{S^{n-1}}} \leq O(e^{-\frac{1}{n-1}t})$. Since $\nabla\lambda = \lambda\lambda'\nabla\varphi$, it follows that $|\nabla\lambda - \kappa\lambda^2\nabla\varphi|_{g_{S^{n-1}}} \leq O(e^{-\frac{1}{n-1}t})$. This implies

$$\begin{aligned} &\int_{\Sigma_t} f(H-(n-1)\kappa)d\mu \\ &= \int_{\Sigma-t} \left(\frac{n-1}{2}\lambda^{n-2} + \frac{n-1}{2}\lambda^{n-4}|\nabla\lambda|^2_{g^{S^{n-1}}} \right) d\,\mathrm{vol}_{S^{n-1}} + O(e^{\frac{n-3}{n-1}t}). \end{aligned} \tag{2.13}$$

On the other hand, the static potential satisfies

$$\begin{aligned}\kappa f - \langle \bar\nabla f, \nu\rangle &\geq \kappa f - |\bar\nabla f| \\ &= \kappa\sqrt{1+\kappa^2\lambda^2 - m\lambda^{2-n} + q^2\lambda^{4-2n}} - (\kappa^2\lambda + \frac{m(n-2)}{2}\lambda^{1-n} + (2-n)q^2\lambda^{3-2n}) \\ &\geq \frac{1}{2}\lambda^{-1} - O(\lambda^{-2}).\end{aligned}$$

This gives

$$(n-1)\int_{\Sigma_t}(\kappa f - \langle\bar\nabla f,\nu\rangle)d\mu \geq \frac{n-1}{2}\int_{S^{n-1}}\lambda^{n-2}d\,\mathrm{vol}_{S^{n-1}} - O(e^{\frac{n-3}{n-1}t}). \tag{2.14}$$

Moreover, using the identity $\bar\Delta f = n\kappa^2 f + (n-2)^2q^2 f\lambda^{2-2n}$ and the divergence theorem, we obtain

$$(n-1)\int_{\Sigma_t}\langle\bar\nabla f,\nu\rangle - n(n-1)\kappa^2\int_{\Omega_t} f\,d\,\mathrm{vol} = O(1). \tag{2.15}$$

Adding (2.13),(2.14), and (2.15), we obtain

$$\begin{aligned}&\int_{\Sigma_t} fH d\mu - n(n-1)\kappa^2\int_{\Omega_t} f\,d\,\mathrm{vol} \\ &\geq \frac{n-1}{2}\int_{S^{n-1}}\lambda^{n-4}|\nabla\lambda|^2_{g_{S^{n-1}}}d\,\mathrm{vol}_{S^{n-1}} \\ &+ (n-1)\int_{S^{n-1}}\lambda^{n-2}d\,\mathrm{vol}_{S^{n-1}} - O(e^{\frac{n-3}{n-1}t}).\end{aligned}$$

Moreover, we have

$$|\Sigma_t| = \int_{S^{n-1}}\lambda^{n-1}d\,\mathrm{vol}_{S^{n-1}} + O(e^{\frac{n-3}{n-1}t}).$$

Using Proposition 21, we conclude that

$$\liminf_{t\to\infty}|\Sigma_t|^{-\frac{n-2}{n-1}}\left(\int_{\Sigma_t} fH d\mu - n(n-1)\kappa^2\int_{\Omega_t} f\,d\,\mathrm{vol}\right) \geq (n-1)|S^{n-1}|^{\frac{1}{n-1}}.$$

Notice that

$$\lim_{t\to\infty}|\Sigma_t|^{-\frac{n-2}{n-1}}\left(-(n-1)f(\bar s_t)^2\bar s_t^{n-2}|S^{n-1}| + (n-1)\kappa^2\bar s_t^n|S^{n-1}|\right) = -(n-1)|S^{n-1}|^{\frac{1}{n-1}}.$$

This completes the proof. □

2.6 The monotonicity formula.

Finally, we show that $Q(t)$ is monotone along the flow:

Lemma 23. *The quantity $Q(t)$ is monotone decreasing in t.*

Proof. The evolution of the mean curvature is given by

$$\frac{\partial}{\partial t}H = -\Delta\left(\frac{1}{H}\right) - \frac{1}{H}\left(|A|^2 + \mathrm{Ric}(\nu,\nu)\right).$$

This implies

$$\frac{\partial}{\partial t}(fH) = -f\Delta\left(\frac{1}{H}\right) - \frac{f}{H}(|A|^2 + \mathrm{Ric}(\nu,\nu)) + \left\langle \bar{\nabla} f, \nu\right\rangle.$$

Using the identity $\Delta f = \bar{\Delta} f - (D^2 f)(\nu,\nu) - H\left\langle \bar{\nabla} f, \nu\right\rangle$, we obtain

$$\begin{aligned}
\frac{d}{dt}\Big(\int_{\Sigma_t} fH d\mu\Big) &= -\int_{\Sigma_t} f\Delta\left(\frac{1}{H}\right)d\mu - \int_{\Sigma_t}\frac{f}{H}(|A|^2 + \mathrm{Ric}(\nu,\nu))d\mu \\
&+ \int_{\Sigma_t}(\left\langle \bar{\nabla} f \nu\right\rangle + fH)d\mu \\
&= -\int_{\Sigma_t}\frac{1}{H}\Delta f d\mu - \int_{\Sigma_t}\frac{f}{H}(|A|^2 + \mathrm{Ric}\,\nu,\nu)d\mu \\
&+ \int_{\Sigma_t}(\left\langle \bar{\nabla} f, \nu\right\rangle + fH)d\mu \\
&= -\int_{\Sigma_t}\frac{1}{H}(\bar{\Delta} f - D^2 f(\nu,\nu) + f\,\mathrm{Ric}(\nu,\nu))d\mu \\
&- \int_{\Sigma_t}\frac{f}{H}|A|^2 d\mu + \int_{\Sigma_t} 2\left\langle \bar{\nabla} f, \nu\right\rangle + fH d\mu \\
&\leq \int_{\Sigma_t}\left(2\left\langle \bar{\nabla} f, \nu\right\rangle + \frac{n-2}{n-1} fH\right)d\mu.
\end{aligned} \tag{2.16}$$

Write $\Sigma_t = \{(\omega, s_t(\omega)) : \omega \in S^{n-1}\}$ as a graph over the unit sphere. Using the identity $\bar{\Delta} f = f(f'^2 + ff'') + (n-1)\frac{f^2}{s}f'$ and the divergence theorem, we obtain

$$\begin{aligned}
\int_{\Sigma_t}\left\langle \bar{\nabla} f, \nu\right\rangle d\mu &= \int_{\Omega_t}\bar{\Delta} f d\,\mathrm{vol} + \int_{S^{n-1}_{s_0}}\left\langle \bar{\nabla} f, \nu\right\rangle d\,\mathrm{vol}_{S^{n-1}_{s_0}} \\
&= \int_{S^{n-1}}\int_{s_0}^{s_t(\omega)}\left(f(f'^2 + ff'') + (n-1)\frac{f^2}{s}f'\right)\frac{s^{n-1}}{f} ds d\,\mathrm{vol}_{S^{n-1}} \\
&+ \int_{S^{n-1}_{s_0}}\left\langle \bar{\nabla} f, \nu\right\rangle d\,\mathrm{vol}_{S^{n-1}_{s_0}} \\
&= \int_{S^{n-1}}\int_{s_0}^{s_t(\omega)}(ff's^{n-1})' ds d\,\mathrm{vol}_{S^{n-1}} + \int_{S^{n-1}_{s_0}} ff' d\,\mathrm{vol}_{S^{n-1}_{s_0}} \\
&= \int_{S^{n-1}} ff's^{n-1}|_{s_t(\omega)} d\,\mathrm{vol}_{S^{n-1}}.
\end{aligned}$$

By the definition of the function P in (H5), we have

$$2ff's^{n-1}|_{s_t(\omega)} = P(s_t^{n-1}(\omega)) + 2\kappa^2 s_t^n(\omega),$$

hence,

$$2\int_{S^{n-1}} ff's^{n-1}|_{s_t(\omega)} d\,\mathrm{vol}_{S^{n-1}} = \int_{S^{n-1}} P(s_t^{n-1}(\omega)) d\,\mathrm{vol}_{S^{n-1}} + 2\kappa^2 \int_{S^{n-1}} s_t^n(\omega) d\,\mathrm{vol}_{S^{n-1}}. \tag{2.17}$$

To deal with $\int_{S^{n-1}} P(s_t^{n-1}(\omega)) d\,\mathrm{vol}_{S^{n-1}}$, let $\Delta = \int_{S^{n-1}} (s_t(\omega))^{n-1} d\,\mathrm{vol}_{S^{n-1}} / |S^{n-1}|$. Using the fact that $|\Sigma_t| \geq \int_{S^{n-1}} (s_t(\omega))^{n-1} d\,\mathrm{vol}_{S^{n-1}}$, we obtain

$$\Delta \leq \frac{|\Sigma_t|}{|S^{n-1}|} = \bar{s}_t^{n-1}.$$

By the assumption (H5) that P is nondecreasing and concave, we have

$$\int_{S^{n-1}} P(s_t^{n-1}(\omega)) d\,\mathrm{vol}_{S^{n-1}} \leq P(\Delta)\,|S^{n-1}| \leq P(\bar{s}_t^{n-1})|S^{n-1}|. \tag{2.18}$$

Combining (2.17) and (2.18), along with the fact that P is nondecreasing and concave, we have

$$\begin{aligned} 2\int_{S^{n-1}} ff's^{n-1}|_{s_t(\omega)} d\,\mathrm{vol}_{S^{n-1}} &\leq P\left(\bar{s}_t^{n-1}\right)|S^{n-1}| + 2\kappa^2 \int_{S^{n-1}} s_t^n(\omega) d\,\mathrm{vol}_{S^{n-1}} \\ &= 2f(\bar{s}_t)f'(\bar{s}_t)s_t^{n-1}|S^{n-1}| - 2\kappa^2 \bar{s}_t^n |S^{n-1}| \\ &\quad + 2\kappa^2 \int_{S^{n-1}} s_t^n(\omega) d\,\mathrm{vol}_{S^{n-1}}. \end{aligned}$$

To deal with $\int_{S^{n-1}} s_t^n(\omega) d\,\mathrm{vol}_{S^{n-1}}$, we have the following equation

$$\begin{aligned} \int_{\Omega_t} f d\,\mathrm{vol} &= \int_{S^{n-1}} \int_{s_0}^{s_t(\omega)} f \cdot \frac{s^{n-1}}{f} ds\, d\,\mathrm{vol}_{S^{n-1}} \\ &= \frac{1}{n} \int_{S^{n-1}} s_t^n(\omega) d\,\mathrm{vol}_{S^{n-1}} - \frac{1}{n}|S^{n-1}| s_0^n. \end{aligned}$$

Putting all the above together, we have

$$\begin{aligned} \frac{d}{dt}\left(\int_{\Sigma_t} fH d\mu\right) \leq \frac{n-2}{n-1}\int_{\Sigma_t} fH d\mu + 2n\kappa^2 \int_{\Omega_t} f d\,\mathrm{vol} + 2f(\bar{s}_t)f'(\bar{s}_t)\bar{s}_t^{n-1}|S^{n-1}| \\ + 2\kappa^2 \bar{s}_t^n |S^{n-1}| + 2\kappa^2 s_0^n |S^{n-1}|. \end{aligned} \tag{2.19}$$

Using the evolution equation $\frac{d}{dt}\bar{s}_t = \frac{1}{n-1}\bar{s}_t$, we compute

$$\begin{aligned} \frac{d}{dt}\left((f(\bar{s}_t))^2 \bar{s}_t^{n-2}\right) &= \left(2f(\bar{s}_t)f'(\bar{s}_t)\bar{s}_t^{n-2} + (n-2)(f(\bar{s}_t))^2 \bar{s}_t^{n-3}\right) \cdot \frac{1}{n-1}\bar{s}_t \\ &= \frac{2}{n-1} f(\bar{s}_t)f'(\bar{s}_t)\bar{s}_t^{n-1} + \frac{n-2}{n-1}(f(\bar{s}_t))^2 \bar{s}_t^{n-2}. \end{aligned} \tag{2.20}$$

Using the Heintze-Karcher inequality proved in [3], we have

$$\frac{d}{dt}\int_{\Omega_t} f d\,\mathrm{vol} = \int_{\Sigma_t} \frac{f}{H} d\mu \geq \frac{n}{n-1}\int_{\Omega_t} f d\,\mathrm{vol} + \frac{1}{n-1} s_0^n |S^{n-1}|. \tag{2.21}$$

Combining (2.19) (2.20) and (2.21), we conclude that

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\Sigma_t} fH d\mu - n(n-1)\kappa^2 \int_{\Omega_t} f d\,\mathrm{vol} - (n-1)(f(\bar{s}_t))^2 \bar{s}_t^{n-2}|S^{n-1}| \\
&\quad + (n-1)\kappa^2 \bar{s}_t^n |S^{n-1}| - (n-1)\kappa^2 s_0^n |S^{n-1}|\Big) \\
\leq& \frac{n-2}{n-1}\Big(\int_{\Sigma_t} fH d\mu - n(n-1)\kappa^2 \int_{\Omega_t} f d\,\mathrm{vol} - (n-1) f(\bar{s}_t)^2\, \bar{s}_t^{n-2}|S^{n-1}| \\
&\quad + (n-1)\kappa^2 \bar{s}_t^n |S^{n-1}| - (n-1)\kappa^2 s_0^n |S^{n-1}|\Big).
\end{aligned}$$

Thus, we conclude that $\frac{d}{dt}Q(t) \leq 0$, and equality holds when the surfaces Σ_t are coordinate spheres. □

Next we prove Theorem 4.

Theorem. *With the same assumptions as in Theorem 4, we have*

$$\begin{aligned}
\int_{\Sigma} fH d\mu - n(n-1)\kappa^2 \int_{\Omega} f d\mathrm{vol} \geq (n-1) f(\bar{s})^2 \bar{s}^{n-2}|S^{n-1}| \\
- (n-1)\kappa^2 \bar{s}^n |S^{n-1}| + (n-1)\kappa^2 s_0^n |S^{n-1}|,
\end{aligned} \tag{2.22}$$

where $\bar{s} = \left(\frac{|\Sigma|}{|S^{n-1}|}\right)^{\frac{1}{n-1}}$.
Equality holds if and only if $\Sigma = S^{n-1} \times \{s\}$ *for some* $s \in [s_0, \infty)$.

Proof. $Q(t)$ is monotone decreasing in t, so we have

$$Q(0) \geq \liminf_{t\to\infty} Q(t) \geq 0.$$

(2.22) follows immediately.

In the case of equality, $Q(t)$ is constant and therefore we must have equality in (2.16). It follows that Σ must be a coordinate sphere. □

Chapter 3

THE REISSNER-NORDSTRÖM-ANTI-DESITTER MANIFOLD

Now, we focus back on the Reissner-Nordström-AdS manifold equipped with the metric

$$\bar{g} = \frac{1}{f^2} ds \otimes ds + s^2 g_{S^{n-1}},$$

where $f^2 = 1 + \kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n}$.

For hypersurfaces in the Reissner-Nordström-AdS manifold, we restate our Theorem 1 and prove the following Minkowski-type inequality as a special case of (2.22).

Theorem. *Let Σ be a compact mean convex, star-shaped hypersurface in the Reissner-Nordström-Anti-deSitter manifold, and let Ω denote the region bounded by Σ and the horizon ∂M. Then*

$$\begin{aligned} \int_\Sigma fH d\mu - n(n-1)\kappa^2 \int_\Omega f d\text{vol} \geq (n-1)|S^{n-1}|^{\frac{1}{n-1}} \left(|\Sigma|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}} \right) \\ + (n-1)q^2 |S^{n-1}|^{\frac{2n-3}{n-1}} \left(|\Sigma|^{-\frac{n-2}{n-1}} - |\partial M|^{-\frac{n-2}{n-1}} \right). \end{aligned} \tag{3.1}$$

Equality holds if and only if $\Sigma = S^{n-1} \times \{s\}$ for some $s \in [s_0, \infty)$.

Proof. Applying (2.22), we have

$$\int_\Sigma fH d\mu - n(n-1)\kappa^2 \int_\Omega f d\,\mathrm{vol} \geq (n-1)|\Sigma|^{\frac{n-2}{n-1}}|S^{n-1}|^{\frac{1}{n-1}} - 2m(n-1)|S^{n-1}| \\ + (n-1)q^2|\Sigma|^{-\frac{n-2}{n-1}}|S^{n-1}|^{\frac{2n-3}{n-1}} + (n-1)\kappa^2 s_0^n |S^{n-1}|.$$

Notice that from the definition of s_0, we have

$$f^2(s_0) = 1 + \kappa^2 s_0^2 - 2m s_0^{2-n} + q^2 s_0^{4-2n} = 0,$$

and

$$s_0 = \left(\frac{|\partial M|}{|S^{n-1}|}\right)^{\frac{1}{n-1}}.$$

The inequality (3.1) follows immediately. □

In the sequel, we are going to use this Minkowski inequality to prove a charged Gibbons-Penrose inequality for a large class of spacelike $(n-1)$-surfaces in the Reissner-Nordström spacetime, which is the unique static spherical symmetric solution of the Einstein-Maxwell equation.

Now recall that the Reissner-Nordström spacetime metric is given by

$$-(1 - 2ms^{2-n} + q^2 s^{4-2n})dt^2 + \frac{1}{1 - 2ms^{2-n} + q^2 s^{4-2n}} ds^2 + s^2 g_{S^{n-1}}, \tag{3.2}$$

where m is the mass and q is the charge with $0 < q < m$.

Let Σ be a closed embedded orientable spacelike (n-1)-surface in the Reissner-Nordström spacetime. Let L and $\underline{L}$ be two null normals of Σ with $\langle L, \underline{L}\rangle = 2$. Again we assume L is future-directed and $\underline{L}$ is past-directed.

Since $\frac{\partial}{\partial t}$ is a Killing field, we have

$$\int_\Sigma \left\langle \vec{H}, \frac{\partial}{\partial t} \right\rangle d\mu = 0.$$

This implies

$$-\int_\Sigma \left\langle \vec{H}, L \right\rangle \left\langle \underline{L}, \frac{\partial}{\partial t} \right\rangle d\mu = \int_\Sigma \left\langle \vec{H}, \underline{L} \right\rangle \left\langle L, \frac{\partial}{\partial t} \right\rangle d\mu = -\int_\Sigma \left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle d\mu,$$

where

$$\vec{J} = \frac{1}{2}\left(\left\langle \vec{H}, L \right\rangle \underline{L} - \left\langle \vec{H}, \underline{L} \right\rangle L \right)$$

denotes the dual mean curvature vector.

Conjecture 24. *Let Σ be a spacelike (n-1)-surface in the Reissner-Nordström spacetime such that $|\Sigma| \geq (m - \sqrt{m^2 - q^2})^{n-1}|S^{n-1}|$. Suppose that the past null hypersurface generated by Σ is smooth. Then*

$$\begin{aligned} -\int_\Sigma \left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle d\mu + 2m(n-1)|S^{n-1}| &\geq (n-1)|S^{n-1}|^{\frac{1}{n-1}}|\Sigma|^{\frac{n-2}{n-1}} \\ &\quad + (n-1)q^2|S^{n-1}|^{\frac{2n-3}{n-1}}|\Sigma|^{-\frac{n-2}{n-1}}. \end{aligned} \tag{3.3}$$

Remark 25. *The above conjecture is a natural generalization of Conjecture 2 in [6].*

Definition 26. *Let B be a complete timelike hypersurface in a static spacetime S. We say that B is convex static if $B = \{(t,x) : t \in \mathbb{R}, x \in \hat{\Sigma}\}$ for some (n-1)-surface $\hat{\Sigma} \subset M$, and the second fundamental form $\hat{h}_{ab}$ and the induced metric $\hat{g}_{ab}$ of $\hat{\Sigma}$ satisfies $\hat{h}_{ab} \geq f^{-1}\hat{\nu}(f)\hat{g}_{ab} > 0$. Here, $\hat{\nu}$ denotes the outward-pointing unit normal to $\hat{\Sigma}$.*

Theorem 27. *Let Σ be a closed embedded orientable spacelike (n-1)-surface in the Reissner-Nordström spacetime with mass m and charge q. The charged Gibbons-Penrose inequality* (3.3) *holds in the following cases:*

(1) Σ lies in a totally geodesic spacelike hypersurface and Σ is mean convex and star-shaped.

(2) Σ is mean convex and star-shaped and Σ lies in a totally umbilical (spherically symmetric) spacelike hypersurface with constant mean curvature $n\kappa$ such that $1 - 2ms^{2-n} + q^2s^{4-2n} + \kappa^2 s^2$ has positive roots.

(3) Σ lies in a convex static timelike hypersurface.

3.1 Surfaces in a totally geodesic time slice.

We check the case when Σ lies in a totally geodesic time-slice ($t = 0$) and the induced metric is

$$\frac{1}{1 - 2ms^{2-n} + q^2 s^{4-2n}} ds^2 + s^2 g_{S^{n-1}}.$$

The future timelike unit normal is given by $e_0 = \frac{1}{\sqrt{1-2ms^{2-n}+q^2s^{4-2n}}}\frac{\partial}{\partial t}$. Let $L = e_0 + \nu$ and $\underline{L} = -e_0 + \nu$ be the two null normals where ν is the outward unit normal of Σ in the time-slice. We compute $\vec{H} = -H\nu$ and $\vec{J} = He_0$ and thus have

$$-\int_\Sigma \left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle d\mu = -\int_\Sigma H \left\langle e_0, \frac{\partial}{\partial t} \right\rangle d\mu = \int_\Sigma H\sqrt{1 - 2ms^{2-n} + q^2 s^{4-2n}} d\mu.$$

The inequality 3.3 in this case is equivalent to

$$\begin{aligned}\int_\Sigma H\sqrt{1-2ms^{2-n}+q^2s^{4-2n}}d\mu + 2m(n-1)|S^{n-1}| \geq (n-1)|S^{n-1}|^{\frac{1}{n-1}}|\Sigma|^{\frac{n-2}{n-1}} \\ + (n-1)q^2|S^{n-1}|^{\frac{2n-3}{n-1}}|\Sigma|^{-\frac{n-2}{n-1}}.\end{aligned} \tag{3.4}$$

The inequality (3.4) follows from (3.1) by sending the cosmological constant κ to 0.

3.2 Surfaces in a totally umbilical slice.

We claim that the inequality in Theorem 1 for surfaces in the Reissner-Nordström-AdS manifold corresponds to inequality (3.3) for surfaces in a spherically symmetric umbilical slice of the Reissner-Nordström spacetime.

Consider a function $\rho = \rho(s)$ that satisfies

$$\rho'(s) = \frac{\kappa s}{(1-2ms^{2-n}+q^2s^{4-2n})\sqrt{1-2ms^{2-n}+q^2s^{4-2n}+\kappa^2s^2}}$$

for some constant $\kappa > 0$ such that $(1-2ms^{2-n}+q^2s^{4-2n}+\kappa^2s^2)$ has positive zeros. Take the embedding of $((m+\sqrt{m^2-q^2})^{\frac{1}{n-2}}, \infty) \times S^{n-1}$ into the Reissner-Nordström spacetime by $\Psi(s,\theta_1,\ldots,\theta_{n-1}) = (\rho(s), s, \theta_1, \ldots, \theta_{n-1})$ and denote the image by $\hat{M} = \{(t, r, \theta_1, \ldots, \theta_{n-1}) : t = \rho(s), r = s\}$.

Substituting $t = \rho(s)$ and $r = s$ in (3.2), the induced metric on $\hat{M}$ is given by

$$\frac{1}{1-2ms^{2-n}+q^2s^{4-2n}+\kappa^2s^2}ds^2 + s^2 g_{S^{n-1}},$$

which is isometric to the one on a Reissner-Nordström-AdS manifold M.

Proposition 28. *The hypersurface $\hat{M}$ is umbilical, i.e the second fundamental form is proportional to the induced metric.*

Proof. Let $b(s) = 1-2ms^{2-n}+q^2s^{4-2n}$ and $f(s) = \sqrt{1-2ms^{2-n}+q^2s^{4-2n}+\kappa^2s^2}$. We have the following identity:

$$b^{-1} - (\rho')^2 b = f^{-2}.$$

An orthonormal coframe adapted to the hypersurface $\hat{M}$ is given by

$$
\begin{aligned}
\Theta^0 &= \frac{1}{\sqrt{b^{-1} - b(\rho')^2}}(dt - \rho' dr) = f(s)(dt - \rho' dr),\\
\Theta^1 &= \frac{1}{\sqrt{b^{-1} - b(\rho')^2}}(b\rho' dt - b^{-1} dr) = f(s)(b\rho' dt - b^{-1} dr),\\
\Theta^2 &= s d\theta_1,\\
\Theta^3 &= s \sin\theta_1 d\theta_2,\\
&\cdots\\
\Theta^n &= s \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} d\theta_{n-1},
\end{aligned}
$$

where Θ_0 is the unit conormal that is dual to the unit future timelike normal

$$
e_0 = \frac{f(s)}{b(s)}\frac{\partial}{\partial t} + \kappa s \frac{\partial}{\partial r}. \tag{3.5}
$$

We compute the second fundamental from using the above coframe and get

$$
\begin{aligned}
p_{11} &= \frac{d}{ds}\left(\frac{b\rho'}{\sqrt{b^{-1} - b(\rho')^2}}\right)\\
p_{22} = p_{33} = \cdots = p_{nn} &= \frac{1}{s}\frac{b\rho'}{\sqrt{b^{-1} - b(\rho')^2}}.
\end{aligned}
$$

We have

$$
\frac{b\rho'}{\sqrt{b^{-1} - b(\rho')^2}} = b\rho' f = \kappa s.
$$

Thus, $p_{11} = p_{22} = \cdots = p_{nn} = \kappa$ and $\hat{M}$ is umbilical. □

Proposition 29. *For a spacelike (n-1)-surface Σ in $\hat{M}$ that is mean convex and star-shaped, the inequality* (3.3) *holds.*

Proof. Consider a spacelike (n-1)-surface Σ in the umbilical hypersurface $\hat{M}$. Let ν be the outward unit normal of Σ in $\hat{M}$, and let $L = e_0 + \nu$ and $\underline{L} = -e_0 + \nu$ be the two null normals. The mean curvature $\vec{H}$ is given by $-H\nu + (n-1)\kappa e_0$ where H is the mean curvature of Σ in $\hat{M}$ with respect to ν. Then, the dual mean curvature vector is

$$
\vec{J} = \left\langle \vec{H}, e_0 \right\rangle \nu - \left\langle \vec{H}, \nu \right\rangle e_0 = He_0 - (n-1)\kappa\nu.
$$

This implies

$$
-\int_\Sigma \left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle d\mu = -\int_\Sigma H \left\langle e_0, \frac{\partial}{\partial t} \right\rangle d\mu + (n-1)\kappa \int_\Sigma \left\langle \nu, \frac{\partial}{\partial t} \right\rangle d\mu, \tag{3.6}
$$

where the function $-\langle e_0, \frac{\partial}{\partial t}\rangle = \sqrt{1-2ms^{2-n}+q^2s^{4-2n}} = f(s)$ is the static potential for the Reissner-Nordström space-time.

Denote by $\left(\frac{\partial}{\partial t}\right)^\top$ the tangential component of $\frac{\partial}{\partial t}$. From (3.5) and the fact that $\Psi_*(\frac{\partial}{\partial s}) = \rho'(s)\frac{\partial}{\partial t} + \frac{\partial}{\partial r}$, we have

$$\left(\frac{\partial}{\partial t}\right)^\top = -sf(s)\Psi_*\left(\frac{\partial}{\partial s}\right),$$

and

$$\left\langle \nu, \frac{\partial}{\partial t}\right\rangle = -\left\langle \nu, sf(s)\Psi_*\left(\frac{\partial}{\partial s}\right)\right\rangle.$$

Plugging these into (3.6), we obtain that

$$-\int_\Sigma \left\langle \vec{J}, \frac{\partial}{\partial t}\right\rangle d\mu = \int_\Sigma Hf d\mu - (n-1)\kappa^2 \int_\Sigma \left\langle \nu, sf\frac{\partial}{\partial s}\right\rangle d\mu.$$

Through the isometry, this can be viewed as an equation on $(M, \bar{g})$. Applying the divergence theorem on M, we have

$$\begin{aligned}\int_\Sigma \left\langle \nu, sf(s)\frac{\partial}{\partial s}\right\rangle d\mu &= \int_\Omega \operatorname{div}_{\bar{g}}\left(sf(s)\frac{\partial}{\partial s}\right) d\operatorname{vol} + \int_{\partial M}\left\langle \nu, sf(s)\frac{\partial}{\partial s}\right\rangle d\mu \\ &= \int_\Omega nf d\operatorname{vol} + s_0^n|S^{n-1}|,\end{aligned}$$

where ∂M is the horizon and Ω is the region enclosed by ∂M and Σ.

Hence, we have that

$$-\int_\Sigma \left\langle \vec{J}, \frac{\partial}{\partial t}\right\rangle d\mu = \int_\Sigma Hf d\mu - n(n-1)\kappa^2 \int_\Omega f d\operatorname{vol} - (n-1)\kappa^2 s_0^n|S^{n-1}|.$$

Therefore, inequality (3.3) follows easily from Theorem 1. □

3.3 Surfaces in a convex static timelike hypersurface.

Consider a spacelike (n-1)-surface Σ in the Reissner-Nordström spacetime. Let $\hat{\Sigma} = \pi(\Sigma)$, where $\pi : (t, r, \theta_1, \theta_2, \cdots, \theta_{n-1}) \mapsto (r, \theta_1, \theta_2, \cdots, \theta_{n-1})$ is the projection to the $t = 0$ slice along the vector field $\frac{\partial}{\partial t}$. Choose parametrizations F and $\hat{F}$ for Σ and $\hat{\Sigma}$ such that $F(x) = (\tau(x), \hat{F}(x))$. Then we have

$$\frac{\partial F}{\partial x_a} = \frac{\partial \hat{F}}{\partial x_a} + \frac{\partial \tau}{\partial x_a}\frac{\partial}{\partial t},$$

and the induced metric on Σ is related to that on $\hat{\Sigma}$ by

$$\hat{g}_{ab} = g_{ab} + f^2 \partial_a \tau \partial_b \tau,$$

where $f = \sqrt{1 - 2ms^{2-n} + q^2 s^{4-2n}}$.

This gives

$$\hat{g}^{ab} = g^{ab} - \frac{f^2 g^{ac} g^{bd} \partial_c \tau \partial_d \tau}{1 + f^2 |\nabla \tau|^2},$$

where $|\nabla \tau|^2 = g^{ab} \partial_a \tau \partial_b \tau$.

Next we relate the second fundamental form of Σ to that of the projected surface $\hat{\Sigma}$. Consider the timelike hypersurface $B = \{(t, x) : t \in \mathbb{R}, x \in \hat{\Sigma}\}$ and denote by ν the outward-pointing unit normal vector to B. We extend ν to a vector field defined in an open neighborhood of B such that $[\nu, \frac{\partial}{\partial t}] = 0$ and $\left\langle \nu, \frac{\partial}{\partial t} \right\rangle = 0$. Then we have

$$\begin{aligned}
\left\langle \frac{\partial F}{\partial x_a}, D_{\frac{\partial F}{\partial x_b}} \nu \right\rangle &= \left\langle \frac{\partial \hat{F}}{\partial x_a}, D_{\frac{\partial \hat{F}}{\partial x_b}} \nu \right\rangle + \frac{\partial \tau}{\partial x_a} \frac{\partial \tau}{\partial x_b} \left\langle \frac{\partial}{\partial t}, D_{\frac{\partial}{\partial t}} \nu \right\rangle \\
&\quad + \frac{\partial \tau}{\partial x_a} \left\langle \frac{\partial}{\partial t}, D_{\frac{\partial \hat{F}}{\partial x_b}} \nu \right\rangle + \frac{\partial \tau}{\partial x_b} \left\langle \frac{\hat{F}}{\partial x_a}, D_{\frac{\partial}{\partial t}} \nu \right\rangle \\
&= \left\langle \frac{\partial \hat{F}}{\partial x_a}, D_{\frac{\partial \hat{F}}{\partial x_b}} \nu \right\rangle + \frac{\partial \tau}{\partial x_a} \frac{\partial \tau}{\partial x_b} \left\langle \frac{\partial}{\partial t}, D_\nu \frac{\partial}{\partial t} \right\rangle \\
&\quad + \frac{\partial \tau}{\partial x_a} \left\langle \frac{\partial}{\partial t}, D_{\frac{\partial \hat{F}}{\partial x_b}} \nu \right\rangle + \frac{\partial \tau}{\partial x_b} \left\langle \frac{\hat{F}}{\partial x_a}, D_\nu \frac{\partial}{\partial t} \right\rangle \\
&= \hat{h}_{ab} - \frac{\partial \tau}{\partial x_a} \frac{\partial \tau}{\partial x_b} f \nu(f),
\end{aligned}$$

where $\hat{h}_{ab}$ is the second fundamental form of the projected surface $\hat{\Sigma}$. It follows that

$$\begin{aligned}
-\left\langle \vec{H}, \nu \right\rangle &= g^{ab} \left\langle \frac{\partial F}{\partial x_a}, D_{\frac{\partial F}{\partial x_b}} \nu \right\rangle \\
&= g^{ab} \hat{h}_{ab} - |\nabla \tau|^2 f \nu(f) \\
&= \hat{g}^{ab} \hat{h}_{ab} + \frac{f^2 g^{ac} g^{bd} \partial_c \tau \partial_d \tau}{1 + f^2 |\nabla \tau|^2} \hat{h}_{ab} - |\nabla \tau|^2 f \nu(f) \\
&= \hat{H} + \frac{f^2 g^{ac} g^{bd} \partial_c \tau \partial_d \tau}{1 + f^2 |\nabla \tau|^2} (\hat{h}_{ab} - f^{-1} \nu(f) \hat{g}_{ab}),
\end{aligned}$$

where $\hat{H} = \hat{g}^{ab}\hat{h}_{ab}$ is the mean curvature of $\hat{\Sigma}$. Now if B is convex static in the sense of Definition 26, then the tensor $\hat{h}_{ab} - f^{-1}\nu(f)\hat{g}_{ab}$ is positive semidefinite. Hence, we have

$$-\left\langle \vec{H}, \nu \right\rangle \geq \hat{H}.$$

On the other hand, we have

$$-\left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle = -\left\langle \vec{H}, \nu \right\rangle \sqrt{-\left\langle \left(\frac{\partial}{\partial t}\right)^{\perp}, \left(\frac{\partial}{\partial t}\right)^{\perp} \right\rangle} = -\left\langle \vec{H}, \nu \right\rangle f\sqrt{1 + f^2|\nabla\tau|^2},$$

where $|\nabla\tau|^2 = g^{ab}\partial_a\tau\partial_b\tau$. Putting these together, we have the following point-wise inequality

$$-\left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle \geq \hat{H}f\sqrt{1 + f^2|\nabla\tau|^2}.$$

The volume elements of Σ and $\hat{\Sigma}$ are related by $d\mu = \frac{1}{\sqrt{1+f^2|\nabla\tau|^2}}d\hat{\mu}$. It follows that

$$-\int_{\Sigma} \left\langle \vec{J}, \frac{\partial}{\partial t} \right\rangle d\mu \geq \int_{\hat{\Sigma}} \hat{H}f d\hat{\mu}.$$

Since B is convex static, the surface $\hat{\Sigma}$ is star-shaped and convex. Using the above result, we obtain

$$\int_{\Sigma} \hat{H}f d\mu + 2m(n-1)|S^{n-1}| \geq (n-1)|S^{n-1}|^{\frac{1}{n-1}}|\Sigma|^{\frac{n-2}{n-1}} + (n-1)q^2|S^{n-1}|^{\frac{2n-3}{n-1}}|\Sigma|^{-\frac{n-2}{n-1}}.$$

Observing that $|\hat{\Sigma}| \geq |\Sigma|$ and

$$|\hat{\Sigma}||\Sigma| \geq (m + \sqrt{m^2 - q^2})^{n-1}|S^{n-1}|(m - \sqrt{m^2 - q^2})^{n-1}|S^{n-1}|,$$

we have

$$(n-1)|S^{n-1}|^{\frac{1}{n-1}}|\Sigma|^{\frac{n-2}{n-1}} + (n-1)q^2|S^{n-1}|^{\frac{2n-3}{n-1}}|\Sigma|^{-\frac{n-2}{n-1}} \geq (n-1)|S^{n-1}|^{\frac{1}{n-1}}|\hat{\Sigma}|^{\frac{n-2}{n-1}} + (n-1)q^2|S^{n-1}|^{\frac{2n-3}{n-1}}|\hat{\Sigma}|^{-\frac{n-2}{n-1}}.$$

Hence, the desired inequality follows.

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