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Decentralized College Admissions

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Abstract

We study decentralized colleges admissions in the face of uncertain student preferences. Enrollment uncertainty causes colleges to strategically target their admissions, forgoing students sought after by others and seeking students overlooked by others. When students' types are multidimensional, colleges avoid head-on competition by placing excessive weights on less correlated dimensions. Restricting the number of applications or allowing for wait-listing alleviates enrollment uncertainty, but the resulting assignments of decentralized matching are inefficient and unfair. A centralized matching via Gale and Shapley's deferred acceptance algorithm attains efficiency and fairness, but some colleges can be worse off relative to decentralized matching.

1 Introduction

The standard market design research on matching focuses on how best to design a centralized matching mechanism, taking the societal consensus on centralization as a given. While such a consensus exists in a number of markets (e.g., medical residency matching and public school matching), many markets remain decentralized with college admissions and graduate school admissions being notable cases in point. Decentralized markets often exhibit congestion and do not operate efficiently (Roth and Xing, 1997). Although it is widely believed that these markets will benefit from improved coordination or centralization, it is not well understood why they remain decentralized and what welfare benefits would be gained by improving coordination possibly via a centralized clearinghouse.

At least part of the problem is the lack of an analytical grasp of decentralized matching markets. Often treated as a black box, the equilibrium and welfare implications of decentralized matching markets have not been understood well in the literature. Indeed, we have yet to develop a workhorse model of decentralized matching that could serve as a useful benchmark for comparison with a

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centralized system.[1]

The current paper develops an analytical framework for understanding decentralized matching markets in the context of college admissions. In many countries, such as Japan, Korea, and the US, college admissions are organized similarly to decentralized labor markets, with exploding and binding admissions made by schools during a short window of time, among other things.

With limited offers and acceptances to clear the markets, decentralized matching provides only a limited chance for colleges to learn students' preferences and to condition their admission decisions on them. This presents a challenge for colleges in managing its yield. Inability to forecast yield accurately could result in too many or too few students enrolling in a college relative to its capacity. Either mistake is costly. For instance, 1,415 freshmen accepted Yale's invitation to join its incoming class in 1995-96, although the university had aimed for a class of 1,335. At the same year, Princeton also reported 1,100 entering students, the largest in its history. The college set up mobile homes in fields and built new dorms to accommodate the students (Avery, Fairbanks and Zeckhauser, 2003).[2]

The yield management problem becomes increasingly important in many countries. In Korea, for example, students apply for departments and not for colleges. Since each department has a small quota and there are many potential choices for students, it is critical for departments to predict yield rates accurately to ensure that they fill their capacities. The US colleges have recently experienced a dramatic increase in applications they receive, in part due to the introduction of common applications (which reduced the application cost).[3] As a consequence, the average yield rate of four-year colleges in the US has declined significantly over the past decade, from 49 percent in 2001 to 38 percent in 2011 (Clinedinst, Hurley and Hawkins, 2012). Declining rates signal greatly increased uncertainty for colleges:

> *Trying to hit those numbers is like trying to hit hot tub when you are skydiving 30,000 feet. I'm going to go to church every day in April.* – Jennifer Delahunty (Dean of admissions and financial aid at Kenyon College in Ohio)[4]

Importantly, the uncertainty facing a college with respect to an admitted student's enrollment depends not just on the student's preference but also on what other set of admissions she receives.

[1]The main exceptions are two excellent works by Chade and Smith (2006) and Chade, Lewis and Smith (2011). As we discuss more fully later, they focus on the portfolio decisions students face in application and colleges' inference of students' abilities based on imperfect signals. By contrast, the current paper focuses on the matching implications of college admissions, paying special attention to the yield management problem arising from (aggregately) uncertain students' preferences.

[2]The cost may also take the form of an explicit sanction imposed on the admitting unit (e.g., department) by the government (as in Korea) or by the college (as in Australia).

[3]The average number of applications per institution increased 60 percent between 2002 and 2011. Seventy-nine percent of Fall 2011 freshmen applied to three or more colleges and twenty-nine percent of them submitted seven or more applications. (Clinedinst, Hurley and Hawkins, 2012)

[4]"In Shifting Era of Admissions, Colleges Sweat," *NY Times*, March 8, 2009

This makes a college's admission policy a strategic tool for managing its yield. We provide a simple model of colleges' strategic admissions game and characterize its equilibrium. The explicit analysis of equilibrium allows us to evaluate the resulting assignment in terms of welfare and fairness and to compare this with outcomes that arise from other coordinated admissions and centralized matching.

In our baseline model, there are two colleges, each with limited capacity, and a unit mass of students with "scores" that are common for both colleges (e.g., high school GPA or SAT scores). Students apply to colleges at no cost. Colleges prefer students according to their scores, but they do not know students' preferences toward them. This uncertainty takes an aggregate form: The mass of students preferring one college over the other varies across states that are unknown to the colleges. Over-enrollment costs a college in proportion to the enrollment in excess of its capacity. Our baseline model involves a simple time line: Initially, students simultaneously apply to colleges. Each college observes only the scores of those students who apply to it. Next, the two colleges simultaneously offer admissions to sets of students. Finally, the students who are admitted by either or both colleges decide on which admission they accept.

Given that application is costless, students have a (weak) dominant strategy of applying to both colleges. Hence, the main focus of the analysis is the college's admission decisions. Our main finding is that the colleges engage in "strategic targeting": In equilibrium, each college may forgo good students who are sought after by the other college and may admit less attractive students who appear overlooked by the other college. The reason for this is that the students who attract competing admissions from the other college present greater enrollment uncertainty and add to capacity cost. Randomization in admissions for students may also emerge. We then provide existence of these equilibria. Next, we show that the assignment is typically unfair; that is, it entails justified envy among students and fails to achieve efficiency among students, among colleges and among all parties including colleges and students.

These results can be illustrated via a simple example. Suppose there are only two students, 1 and 2, applying to colleges A and B. Each college has one seat to fill and faces a prohibitively high cost of having two students. Student i has score v_i, $i = 1, 2$, where $0 < v_2 < v_1 < 2v_2$. Each student has an equal probability of preferring either school, which is private information (unknown to the other student and to the colleges). Each college values having student i at v_i. The applications are free of cost, and the timing is the same as that explained above.

Given the large cost of over-enrollment, each college admits only one student. Their payoffs are described as follow.

A's strategy\ B's strategy	Admit 1	Admit 2
Admit 1	$\frac{1}{2}v_1,\ \frac{1}{2}v_1$	v_1, v_2
Admit 2	v_2, v_1	$\frac{1}{2}v_2,\ \frac{1}{2}v_2$

This game has a battle of the sexes' structure (with asymmetric payoffs), so there are two different types of equilibria. First, there are two asymmetric pure-strategy equilibria in which one college admits student 1 and the other admits student 2. There is also a mixed-strategy equilibrium in which each college admits 1 with probability $\gamma := \frac{2v_1 - v_2}{v_1 + v_2} > 1/2$ and admits 2 with probability $1 - \gamma$, where γ is chosen such that the other college is indifferent. Both types of equilibria show the pattern of strategic targeting. In the pure-strategy equilibria, colleges manage to avoid competition and thus randomness in enrollment by targeting different students. The mixed-strategy equilibrium also arises from the targeting motive, i.e., colleges' attempt to avoid students sought after by the other, although it does not result in perfect coordination.

This example, while extremely simple, suggests problems with decentralized matching in terms of welfare and fairness. First, the student with high score (student 1) may go to a less preferred school (in both types of equilibria) even though both colleges prefer that student; that is, justified envy arises. Second, it could be the case that student 1 prefers A and student 2 prefers B, but the former is assigned B and the latter is assigned A, showing that the equilibrium outcome is inefficient among students. Lastly, the mixed-strategy equilibrium is Pareto inefficient because both colleges may admit the same student, and it would be Pareto improving for the unmatched college to match with the other student.

We next study the admissions problem when students have multidimensional types. Some measures, such as students' academic performances or system-wide test like SAT, are highly correlated among colleges, but others measures, such as students' college-specific essays and tests or their extracurricular activities, are less correlated among them. Clinedinst, Hurley and Hawkins (2012) report that private colleges place emphasis on many factors other than standard test scores, including essay/writing samples and extracurricular activities. We show that colleges' desire to avoid head-on competition, and thus to lessen enrollment uncertainty, leads them to bias their evaluation toward less correlated measures by placing excessive weights on theses dimensions.

We also study two common ways for colleges to alleviate enrollment uncertainty. One is to restrict the number of applications each student can submit. This form of coordination is observed in many countries; for instance, students in the UK cannot apply to both Cambridge and Oxford, students in Japan can apply to at most one public university, and students in Korea face a similar restriction. Restricted application reduces enrollment uncertainty for the colleges, and thus alleviates their yield management burden. Yet, we show that this method may not completely eliminate the yield management problem and justified envy, and it may also fail to achieve efficiency.

Colleges also seek to cope with enrollment uncertainty by admitting students in sequence, or "wait-listing": Some students are admitted outright and others are placed in a wait list in each of multiple rounds, and some students in the wait list are later admitted when some offers are declined and seats open up. This method is also observed in many countries, including France, Korea and the US. Wait-listing alleviates colleges' yield management problem, since colleges may adjust their admission offers based on the students' acceptance behavior and the information they may learn over the course of the process. We show, however, that they still engage in strategic targeting by admitting "non-contiguous" set of students in scores, and the welfare and fairness problems still remain. The predicted pattern of equilibrium is observed in the admissions data from Hanyang university in Korea. See Section 6.

Finally, we consider centralized matching via Gale and Shapley's Deferred Acceptance algorithm (DA in short). This eliminates colleges' yield management problem and justified envy completely and attains efficiency. At the same time, it is possible for a college to be worse off relative to the decentralized matching. For instance, in the above example, suppose a pure-strategy equilibrium in which college i always gets student 1 is played. Then, that college will clearly be worse off from a switch to a centralization via DA because it will not always attract student 1. This may explain a possible lack of consensus toward centralization and may underscore why college admissions remain decentralized in many countries.

The paper is organized as follows. Section 1.1 discusses the related literature. The model is introduced in Section 2. Equilibrium characterization and its existence are established in Section 3. Section 3.1 discusses welfare and fairness implications of equilibria. Section 4 studies admissions problem when students' types are multidimensional. In Section 5, restriction on the number of applications is studied, and in Section 6, wait-listing is studied. Centralized matching via DA is considered in Section 7. Section 8 concludes the paper. Proofs are provided in the Appendix unless stated otherwise. The Appendix also extends the baseline model to allow for more than two colleges and shows that our analysis in the two-college model carries over.

1.1 Related Literature

Several papers in the matching literature have considered decentralized matching markets. Roth and Xing (1997) study the entry-level market for clinical psychologists in which firms make offers to workers sequentially within a day and workers can accept, reject or hold an offer. They find that, mainly based on simulations, such a decentralized (but coordinated) market exhibits *congestion*, i.e., not enough offers and acceptances could be made to clear the market, and the resulting outcome is unstable. Neiderle and Yariv (2009) study a decentralized one-to-one matching market in which firms make offers sequentially through multiple periods. They provide sufficient conditions under which such decentralized markets generate stable outcomes in equilibrium in the presence of market friction (namely, time discounting) and preference uncertainty. Coles, Kushnir and Neiderle (2013) also consider one-to-one matching in decentralized markets and show that introducing a signaling device alleviates congestion and increases welfare in terms of the number of matches and the workers' payoffs.

Like these models, our model concerns the consequence of congestion arising from decentralized matching, but unlike Roth and Xing (1997), we study participants' strategic responses, by analyzing colleges' admissions decisions and their welfare and fairness implications. This framework identifies strategic targeting as an important new implication of decentralized matching. Moreover, the explicit analysis of equilibria permits a clear comparison with the outcome that would arise from a centralized matching.

The college admissions problem has recently received attention in the economics literature. Chade and Smith (2006) study students' application decision as a portfolio choice problem. Chade, Lewis and Smith (2011) analyze colleges' admission decisions together with the students' application decisions. In their model, students with heterogeneous abilities make application decisions subject to application costs, and colleges set admission standards based on noisy signals on students' abilities. Avery and Levin (2010) and Lee (2009) study early admissions. Unlike our model, these models have no aggregate uncertainty with respect to students' preferences, so colleges face no enrollment uncertainty in these models. Hence, colleges do not employ strategic targeting; they instead use cutoff strategies.

Some aspects of our equilibrium are related to political lobbying behavior studied by Lizzeri and Persico (2001, 2005). Just as colleges target students in our model, politicians in these models target voters in distributing their favors. In their models, voters are homogeneous, and a voter votes for the candidate that offers her the largest favor. In our model, however, students have heterogeneous abilities and preferences. Thus, colleges' admission decisions are more complicated—admission probabilities vary according to students' scores. Aggregate uncertainty plays a unique role in shaping competition in our model, whereas how the spoils of office are split among candidate (either *winner-take-all* or *proportional* rule) is crucial in their model.

Our model also shares some similarities with directed search models, such as Montgomery (1991) and Burdett, Shi and Wright (2001). In these studies, each firm (seller) posts a wage (price), and each worker (buyer) decides which job to apply for. Firms have a fixed number of job openings and cannot hire more than the capacity, and workers can only apply to one firm. Workers' inability to precisely coordinate their search decisions causes a "search friction," so they randomize on application decisions. Just like the workers in these models, colleges in our model can be seen to engage in "directed searches" on students. The difference is that the colleges in our model offer admissions to many students subject to aggregate uncertainty. This leads to strategic targeting, a novel feature of our model.

2 Model

There is a unit mass of students with score v distributed from $\mathcal{V} \equiv [0, 1]$ according to an absolutely continuous distribution $G(\cdot)$. There are two colleges, A and B, each with capacity $\kappa < \frac{1}{2}$.[5] Each

[5] Appendix B will extend the model to include more than two colleges, showing that our main results carry over to that extension.

college values a student with score v at v and faces a cost $\lambda \geq 1$ for each incremental enrollment exceeding the quota.[6] Each student has a preference over the two colleges, which is private information. A state of nature s, drawn from $[0, 1]$ according to the uniform distribution, determines the fraction of students who prefer A over B. In state s, a fraction $\mu(s) \in [0, 1]$ of students prefers A to B, where $\mu(\cdot)$ is strictly increasing and continuous in s.[7] While we shall consider a general form of $\mu(\cdot)$, for some result, we will consider a **symmetric environment** in which $\mu(s) = 1 - \mu(1 - s)$ for all $s \in [0, 1]$. In a symmetric environment, the measure of students who prefer A over B is symmetric around $s = \frac{1}{2}$.

The timing of the game is as follows. First, Nature draws the (aggregate uncertainty) state s. Next, all students simultaneously apply to colleges. Each college observes the scores of only those students who apply to it, and based on the scores, colleges simultaneously decide which applicants to admit. Last, students who have received at least one admission offer decide which offer to accept.

We assume that there is no application cost for the students, so it is a weak dominant strategy for each student to apply to both colleges. Throughout this paper, we focus on a perfect Bayesian equilibrium in which students play the weak dominant strategy.[8]

Colleges distribute admissions based on students' scores. Let $\sigma_i : \mathcal{V} \to [0, 1]$, $i = A, B$, be college i's admission strategy in terms of the fraction of students with score v that the college admits. For given $\sigma_i(\cdot)$, let $\mathcal{V}_i := \{v \in [0, 1] \mid \sigma_i(v) > 0\}$ be the types of students college i admits. Let $\mathcal{V}_{AB} := \mathcal{V}_A \cap \mathcal{V}_B$. If $\mathcal{V}_{AB}$ has a positive measure in an equilibrium, this means that a positive measure of students has admissions from both colleges. We call such an equilibrium **competitive**. An equilibrium in which $\mathcal{V}_{AB}$ has zero measure is called **non-competitive**.

Consider the students with score v. A fraction $\sigma_i(v)\,(1 - \sigma_j(v))$ of them, where $i, j = A, B$ and $i \neq j$, is admitted only by college i, and a fraction $\sigma_i(v)\sigma_j(v)$ of them is admitted by both colleges. The former group accepts i's admissions, but only a fraction $\mu_i(s)$ of the latter group accepts i's offer, where $\mu_i(s) = \mu(s)$ if $i = A$ and $\mu_i(s) = 1 - \mu(s)$ if $i = B$. Thus, the mass of students who attend college i in state s, given strategies $\sigma_i(\cdot)$ and $\sigma_j(\cdot)$, is

$$m_i(s) := \int_0^1 \sigma_i(v)\,(1 - \sigma_j(v) + \mu_i(s)\sigma_j(v))\; dG(v). \tag{2.1}$$

Each college enjoys the scores of enrolled students as its gross payoff and incurs cost λ for each

[6] The common preference assumption is later relaxed when we allow colleges to consider students' attributes that are imperfectly correlated. See Section 4.

[7] There is no loss of generality to assume the uniform distribution, because for a distribution $F(\cdot)$ of s, we can simply relabel s, and the popularity of a college over the other is captured by $\mu(\cdot)$.

[8] The strategy of applying to both colleges can be made a strictly dominant strategy if students have some uncertainty about their scores, which is realistic in case the scores are either not publicly observable or depend on multiple dimensions of attributes, the weighting of which may be unknown to the students.

incremental enrollment beyond its capacity. Thus, college i's ex ante payoff is

$$\pi_i := \mathbb{E}\Big[\int_0^1 v\sigma_i(v)\,(1-\sigma_j(v)+\mu_i(s)\sigma_j(v))\;dG(v) - \lambda\max\{m_i(s)-\kappa,0\}\Big].$$

One immediate observation is that each college's payoff is concave in its own admission strategy; that is, $\pi_i(\eta\sigma_i+(1-\eta)\sigma_i') \geq \eta\,\pi_i(\sigma_i)+(1-\eta)\,\pi_i(\sigma_i')$ for any feasible strategies σ_i and σ_i' and for any $\eta \in [0,1]$. Therefore, mixing over σ_i's is unprofitable for college i. For this reason, any equilibrium is characterized by a pair (σ_A, σ_B). Of course, this does not mean that the equilibrium is in pure-strategies; the values of σ_A and σ_B may be strictly interior, in which case the admission strategies would involve randomization.

In the following section, we characterize different types of equilibria and establish their existence. We then provide welfare and fairness properties of equilibria.

3 Characterization of Equilibria

We analyze colleges' admission decisions in this section. To this end, we fix any equilibrium (σ_A, σ_B) and explore the properties it must satisfy. Later, we shall establish existence of the equilibria. We begin with the following observations, whose proofs are in Appendix A.1.

Lemma 1. *In any equilibrium (σ_A, σ_B), the following results hold.*

(i) $m_A(0) \leq \kappa \leq m_A(1)$ and $m_B(1) \leq \kappa \leq m_B(0)$.

(ii) $\mathcal{V}_A \cup \mathcal{V}_B$ is a connected interval with $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$ and $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$.

(iii) If the equilibrium is competitive (i.e., $\mathcal{V}_{AB}$ has a positive measure), then there exists a unique $(\hat{s}_A, \hat{s}_B) \in (0,1)^2$ such that $m_A(\hat{s}_A) = \kappa$ and $m_B(\hat{s}_B) = \kappa$.

(iv) If the equilibrium is non-competitive (i.e., $\mathcal{V}_{AB}$ has zero measure), then $m_A(s) = m_B(s) = \kappa$ for all $s \in [0,1]$. Further, almost every student with $v \geq G^{-1}(1-2\kappa)$ receives an admission offer from exactly one college.

Part (i) of the lemma states that in equilibrium, colleges cannot have strict over-enrollment or strict under-enrollment in all states. This is obvious since if there were over-enrollment in all states for a college, then since $\lambda \geq 1$, it will profitably deviate by rejecting some students with $v < 1$, and if there were under-enrollment in all states, a college will likewise profitably deviate by accepting more students. Part (ii) suggests that if a student with score v is admitted by either college, then all students with scores higher than such v must be admitted by some college at least with positive probability, and there is a positive mass of students in the low tail who are never admitted by either college. Part (iii) suggests that in a competitive equilibrium, the colleges will suffer from under-enrollment in some states and over-enrollment in other states. This is intuitive since given (aggregately) uncertain preferences on the part of students, the presence of students who receive admissions from both colleges presents non-trivial enrollment uncertainty. Each college will deal with uncertainty by optimally trading off the cost of over-enrollment with the loss from under-enrollment, thus entailing both types of mistakes depending on the states. Part (*iv*) states that in a non-competitive equilibrium, colleges avoid the over- and under-enrollment problems, and almost every top 2κ students receive admissions from only one college. This is, again, intuitive since the colleges in this case face no enrollment uncertainty, so they will fill their capacities exactly in all states with students whose scores are within the top 2κ.

In what follows, we shall focus on competitive equilibria. There are several reasons for this. It will be seen that competitive equilibria always exist (see Theorem 2). By contrast, non-competitive equilibria can be ruled out if either λ is not too large or κ is not too small (see Appendix A.2). Finally, even if a noncompetitive equilibrium exists, the characterization provided in Lemma 1-(*iv*) is sufficient for our welfare and fairness statements, as will be seen later.

Fix any competitive equilibrium (σ_A, σ_B). For ease of notation, let $S_A := \{s \,|\, s \geq \hat{s}_A\}$ and $S_B := \{s \,|\, s \leq \hat{s}_B\}$ be the sets of states that the colleges are over-enrolled. It is convenient to rewrite college i's payoff at the equilibrium as follows:[9]

$$\begin{aligned}\pi_i &= \int_0^1 v\sigma_i(v)\big(1 - \sigma_j(v) + \mathbb{E}[\mu_i(s)]\,\sigma_j(v)\big)\,dG(v) - \lambda\,\mathbb{E}\big[m_i(s) - \kappa \,|\, s \in S_i\big]\mathrm{Prob}(s \in S_i)\\ &= \int_0^1 \sigma_i(v)\,H_i(v, \beta(v))\,dG(v) + \lambda\,\kappa\,\mathrm{Prob}(s \in S_i)\end{aligned}$$

where

$$H_i(v, \sigma_j(v)) := v\,\big(1 - \sigma_j(v) + \mathbb{E}[\mu_i(s)]\,\sigma_j(v)\big) - \lambda\,\mathrm{Prob}(s \in S_i)\,\big(1 - \sigma_j(v) + \mathbb{E}[\mu_i(s)|s \in S_i]\,\sigma_j(v)\big)$$

is college i's marginal payoff from admitting a student with score v for given $\hat{s}_i$ and $\sigma_j(\cdot)$[10] in equilibrium. H_i captures college i's local incentive; that is, what the college gains by admitting v, holding fixed its opponent's decision and *its own decisions for the rest of the students* at $\sigma_i(\cdot)$.

Lemma 2. *A strategy profile (σ_A, σ_B) is a competitive equilibrium if and only if (i) $H_i(v, \sigma_j(v)) > 0$ implies $\sigma_i(v) = 1$, (ii) $H_i(v, \sigma_j(v)) < 0$ implies $\sigma_i(v) = 0$, (iii) $H_i(v, \sigma_j(v)) = 0$ implies $\sigma_i(v) \in [0, 1]$, where $i, j = A, B$ and $j \neq i$.*

Proof. See Appendix A.3. ■

Lemma 2 states that in equilibrium, each college admits students if their marginal payoffs are positive and rejects them if their marginal payoffs are negative. This lemma also suggests that a strategy profile satisfying the stated conditions forms an equilibrium. Note that these conditions ensure marginal incentive compatibility: that is, each college has no incentive to deviate its decision

[9] $\mathrm{Prob}(s \in S_A) = 1 - \hat{s}_A$ and $\mathrm{Prob}(s \in S_B) = \hat{s}_B$ (by the assumption of the uniform distribution).

[10] We shall suppress its dependence on $\hat{s}_i$ unless it is important.

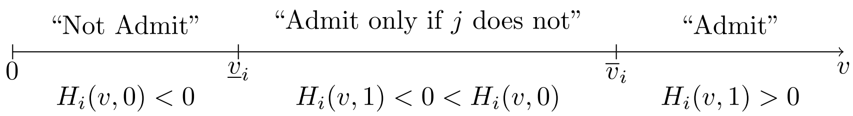


Figure 3.1: College i's Admission Decision

on a given student, *holding fixed it own admissions decisions on all other students.* Even with these conditions satisfied, a college may still profitably deviate on a mass of students. In Appendix A.3, we show that no such global deviation is profitable. That is, a strategy profile satisfying local incentives is indeed an equilibrium.

Inspection of colleges' marginal payoffs together with Lemma 2 reveals their admission decisions in more detail. Note that H_i can be rewritten as

$$H_i(v, \sigma_j(v)) = (1 - \sigma_j(v))\,(v - \underline{v}_i) + \sigma_j(v)\,\mathbb{E}[\mu_i(s)]\,(v - \overline{v}_i),$$

where

$$\underline{v}_i := \lambda\,\text{Prob}(s \in S_i) \quad \text{and} \quad \overline{v}_i := \lambda\,\text{Prob}(s \in S_i)\,\frac{\mathbb{E}[\mu_i(s)|s \in S_i]}{\mathbb{E}[\mu_i(s)]}$$

are college i's capacity costs of admitting a student when she does not receive an admission offer from college j and when she does, respectively. Recall that the college incurs capacity cost only when there is over-enrollment. If the student does not receive a competing offer from college j, then she accepts i's admission for sure. Hence, over-enrollment occurs with probability $\text{Prob}(s \in S_i)$, entailing the marginal cost $\underline{v}_i$. If the student receives a competing offer from college j, then she accepts i's offer only when she prefers i to j. Hence, conditional on acceptance, the over-enrollment arises with probability $\text{Prob}(s \in S_i)\,\frac{\mathbb{E}[\mu_i(s)|s\in S_i]}{\mathbb{E}[\mu_i(s)]}$, entailing the marginal cost $\overline{v}_i$.

Note that we have $\overline{v}_i > \underline{v}_i$. This is because while acceptance by students without offer from college j is independent of the state, the acceptance by students with offer from college j is more concentrated in the high demand state for college i, making the latter more costly. This explains why a college finds it optimal to favor the students without a competing offer than those with a competing offer. [11]

Lemma 3. *In any competitive equilibrium, $H_i(v, x)$, $i = A, B$, is strictly increasing in v for each x. Moreover, for each v, $H_i(v, x)$ satisfies the single crossing property: If $H_i(v, x) \leq 0$, then $H_i(v, x') < 0$ for any $x' > x$.*

Proof. See Appendix A.4. ∎

[11] Legacy admissions can be seen as a way for colleges to avoid uncertainty. Legacies (those who are children or close relatives of alumni) for a college are likely to have loyalty for that college; that is, their preferences (and hence their acceptance decisions) are not much various across states comparing to the other students. Thus, colleges will have incentive to favor those students. See Section 8 for more discussion.

Lemma 3 implies that $H_i(v, \beta(v))$ partitions the students' type space into three intervals, as depicted in Figure 3.1. Notice first that $H_i(\overline{v}_i, 1) = 0$ and $H_i(\underline{v}_i, 0) = 0$. Since $H_i(v, 1) > 0$ for $v > \overline{v}_i$ and $H_i(v, 0) < 0$ for $v < \underline{v}_i$ (recall H_i is strictly increasing in v), college i admits all students with $v > \overline{v}_i$ *even if college j admits all of them* and rejects all students with $v < \underline{v}_i$ *even if college j rejects all of those students.*

For the students with $v \in (\underline{v}_i, \overline{v}_i)$, we have $H_i(v, 1) < 0 < H_i(v, 0)$. This means that college i's incentive for admitting these students depends on college j's admission decisions toward them. The single crossing property established in Lemma 3 implies that for each v, there exists $\hat{\sigma}_j(v) \in (0, 1)$ such that $H_i(v, x) > 0$ if $x < \hat{\sigma}_j(v)$, $H_i(v, x) < 0$ if $x > \hat{\sigma}_j(v)$, and $H_i(v, \hat{\sigma}_j(v)) = 0$ if $x = \hat{\sigma}_j(v)$. Hence, college i admits (rejects) all students with v if college j admits less (greater) than fraction $\hat{\sigma}_j(v)$ of them and admits any fraction of those students if college j admits exactly fraction $\hat{\sigma}_j(v) \in (0, 1)$ of them. In particular, college i admits all of them if college j does not admit any of them, but does not admit them if college j admits all of them.

Combining the two colleges' admission decisions leads to the following characterization of equilibria.

Theorem 1. *In any competitive equilibrium, there exist $\underline{v}_i < \overline{v}_i$, $i = A, B$, such that college i admits students with $v > \overline{v}_i$ and $v \in [\underline{v}_i, \underline{v}_j]$ and rejects students with $v < \underline{v}_i$ and $v \in [\overline{v}_j, \overline{v}_i]$, where $j \neq i$. At least one college admits a positive fraction of students with $v \in [\max\{\underline{v}_A, \underline{v}_B\}, \min\{\overline{v}_A, \overline{v}_B\}]$.*

Theorem 1 describes the structure of competitive equilibrium. Figure 3.2 depicts a typical pure-strategy equilibrium. Here, the students at the top with $v > \overline{v}_A = \max\{\overline{v}_A, \overline{v}_B\}$ receive admissions from both colleges, because their scores are above the high cutoffs of both colleges. And the students at the bottom below $\underline{v}_B = \min\{\underline{v}_A, \underline{v}_B\}$ do not receive any admissions. Strategic targeting occurs with students in the middle with $v \in [\underline{v}_B, \overline{v}_A]$. The students with $v \in [\overline{v}_B, \overline{v}_A]$ are admitted only by B, since A finds them admission-worthy only if B does not admit them, but B admits them no matter what A does. Each of the students in the intermediate range of scores, i.e., $[\underline{v}_A, \overline{v}_B]$, receives an admission from only one college. The students with scores $v \in [\underline{v}_B, \underline{v}_A]$ receive admissions only from B, since that college alone finds them admission-worthy given that they are not admitted by A. This pattern of strategic targeting — i.e., forgoing good students sought after by the other college but admitting less attractive ones neglected by others — stands in stark contrast with the cutoff strategy equilibrium found by the existing literature (see Chade, Lewis and Smith, 2011).

The particular pattern of strategic targeting, namely how the two colleges coordinate exactly on the students in $[\underline{v}_A, \overline{v}_B]$, is indeterminate, and the figure depicts one possible coordination.[12]

[12] As noted, there may be many ways for colleges to coordinate their admissions for students with $v \in [\check{v}, \hat{v}]$, where $\check{v} := \max\{\underline{v}_A, \underline{v}_B\}$ and $\hat{v} := \min\{\overline{v}_A, \overline{v}_B\}$. The range of different pure-strategy equilibria can be summarized by two extreme types of equilibria. We call a competitive equilibrium an **A-priority equilibrium** if $\sigma_A(v) = 1$ for all $v \in [\check{v}, \hat{v}]$, and a **$B$-priority equilibrium** if $\sigma_B(v) = 1$ for all $v \in [\check{v}, \hat{v}]$. In words, in an i-priority equilibrium, the

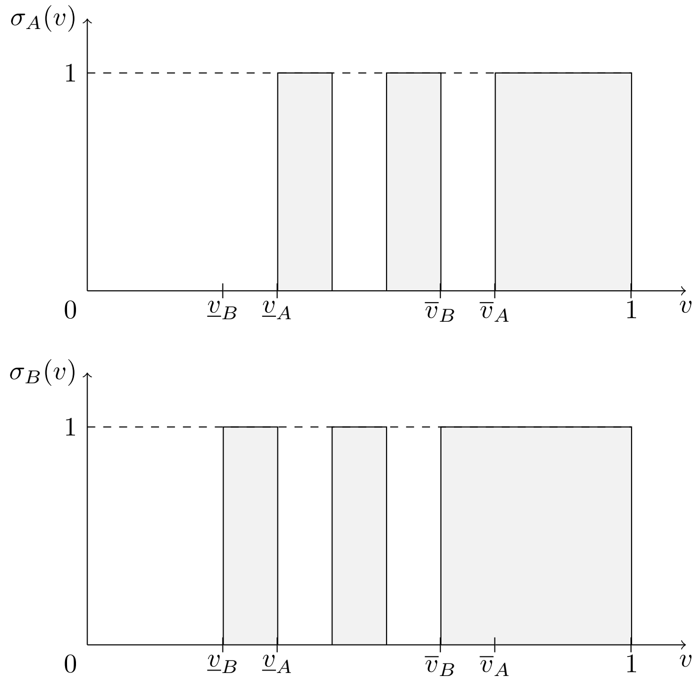


Figure 3.2: Pure-Strategy Equilibrium

In practice, it is implausible for colleges to achieve the kind of precise coordination described in the pure-strategy equilibria. It seems much more plausible for colleges to randomize its admission over students with the intermediate range of scores $v \in [\check{v}, \hat{v}]$, where $\check{v} := \max\{\underline{v}_A, \underline{v}_B\}$ and $\hat{v} := \min\{\overline{v}_A, \overline{v}_B\}$.[13] A typical mixed-strategy equilibrium is depicted in Figure 3.3.

Notice that the admission strategies outside the intermediate range is similar to that in the above pure-strategy equilibrium, as completely pinned down by Theorem 1. For the intermediate range of scores, college i admits a fraction $\sigma_i^o(v)$ of students with score v, where $\sigma_i^o(v)$ satisfies $H_j(v, \sigma_i^o(v)) = 0$ for $j \neq i$. Given this behavior, college j is indifferent and therefore finds it a best response to admit a fraction $\sigma_j^o(v)$ of students with v, where $s_j^o(v)$ satisfies $H_i(v, \sigma_j^o(v)) = 0$.

The fractions of admitted students are thus pinned down in equilibrium, but the identities of the chosen students are not; each college randomizes on the students it admits. In practice, colleges could use extraneous or nonessential students' attributes as their randomization device. Examples may include extracurricular activities or non-academic performances. Although in many cases colleges's interests in these aspects are genuine, our theory is consistent with colleges placing excessive weights on them in their admission decisions, as will be discussed at length in the next section.

Observe that each college admits a higher fraction of students with higher scores, since $\sigma_i^o(\cdot)$

coordination is tilted in favor of college i. Clearly, between these two equilibria, one can construct (infinitely) many equilibria.

[13] It is important to note that the thresholds are not necessarily the same as in the pure-strategies, since different equilibria involve different cutoff states, $(\hat{s}_A, \hat{s}_B)$, which affect the marginal payoff functions H_A and H_B.

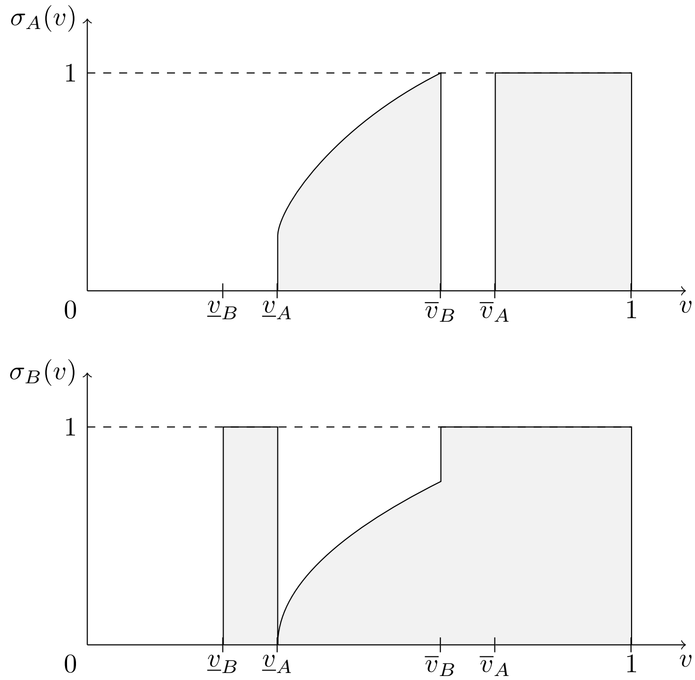


Figure 3.3: Mixed-Strategy Equilibrium

is increasing in v. This is intuitive: Higher score students are more valuable all else equal, so admitting a higher fraction of those students is necessary to keep the opponent college indifferent. It is also interesting to observe discrete jumps in this figure — $\sigma_A^o(\underline{v}_A) > 0$ and $\sigma_B^o(\overline{v}_B) < 1$. The former follows from the fact that $\underline{v}_A > \underline{v}_B$ which implies $H_B(\underline{v}_A, 0) > 0$, and the latter follows from $\overline{v}_A > \overline{v}_B$ which implies $H_A(\overline{v}_B, 1) < 0$.

There could be many ways for colleges to play mixed-strategies: For instance, colleges could coordinate to use a pure-strategy for some students, say $[\tilde{v}, \hat{v}]$ for some $\tilde{v} \in (\check{v}, \hat{v})$, and use mixed-strategies for $v \in [\check{v}, \tilde{v}]$. Consistent with our selection, we focus on the **maximally mixed equilibrium** (MME, in short) in which both colleges play mixed-strategies (σ_A^o, σ_B^o) for students with $v \in [\check{v}, \hat{v}]$ and according to Theorem 1 for outside that range.

Theorem 2. *There exists a competitive equilibrium with maximal mixing.*

Proof. See Appendix A.6. ∎

We sketch the proof here.[14] The proof constructs equilibrium strategies (σ_A, σ_B) with maximal mixing in terms of threshold states $(\hat{s}_A, \hat{s}_B)$. Since the latter space is Euclidean (whereas the former is functional), we can simply appeal to the Brouwer's fixed point theorem to establish the existence. To begin, fix any candidate threshold states $\hat{s} = (\hat{s}_A, \hat{s}_B)$ for the two colleges. Next, we construct

[14]The same argument proves the A- or B-priority equilibrium. Note that existence of an (arbitrary) equilibrium follows from the Glicksberg-Fan theorem, since each college's strategy space is compact and convex, and each college's payoff function is concave in its own strategy. A proof is required here only because the special structure of behavior we impose on MME (or A- or B-priority) we insist upon.

the colleges' mutual best-responses (σ_A, σ_B) corresponding to the chosen $\hat{s}$ following the algorithm described earlier. Formally, we set for $i, j = A, B$ and $i \neq j$,

$$\sigma_i(v;\hat{s}) = \begin{cases} 1 & \text{if } H_i(v,1;\hat{s}) > 0 \\ 0 & \text{if } H_i(v,1;\hat{s}) < 0, \;\; H_j(v,1;\hat{s}) > 0 \\ \sigma_i^o(v;\hat{s}) & \text{if } H_i(v,1;\hat{s}) < 0 < H_i(v,0;\hat{s}), \;\; H_j(v,1;\hat{s}) < 0 < H_j(v,0;\hat{s}) \\ 1 & \text{if } H_i(v,0;\hat{s}) > 0, \;\; H_j(v,0;\hat{s}) < 0 \\ 0 & \text{if } H_i(v,0;\hat{s}) < 0 \end{cases} \tag{3.1}$$

where $\sigma_i^o(\cdot)$ satisfies $H_j(v, \sigma_i^o(v); \hat{s}) = 0$ for $v \in [\check{v}, \hat{v}]$.

Since the threshold states $(\hat{s}_A, \hat{s}_B)$ are arbitrary, there is no guarantee that the constructed strategies reproduce them as the correct thresholds. In fact, they will reproduce another possible threshold states $\tilde{s} = (\tilde{s}_A, \tilde{s}_B)$:[15]

$$\tilde{s}_A = \inf\{s \in [0,1] \,|\, m_A(s;\hat{s}) - \kappa > 0\} \quad \text{and} \quad \tilde{s}_B = \inf\{s \in [0,1] \,|\, m_B(s;\hat{s}) - \kappa > 0\}, \tag{3.2}$$

where m_A and m_B are derived from the formula (2.1).

But this process defines a mapping $T : [0,1]^2 \to [0,1]^2$ such that $T(\hat{s}) = \tilde{s}$. In Appendix A.6, we apply the Brouwer's fixed point theorem to show that T admits a fixed point $\hat{s}^* = (\hat{s}_A^*, \hat{s}_B^*)$ such that $T(\hat{s}^*) = \hat{s}^*$. By Lemma 2, the strategies (σ_A, σ_B) constructed as above based on this fixed point $\hat{s}^* = (\hat{s}_A^*, \hat{s}_B^*)$ does form mutual best responses for the colleges, given the accurate thresholds.

It is important to recognize that a randomization by each college arises from its attempt to avoid competition for students in the intermediate range of scores. In this sense, as long as a competitive equilibrium admits the intermediate region, i.e., if $\check{v} = \max\{\underline{v}_A, \underline{v}_B\} < \hat{v} = \min\{\overline{v}_A, \overline{v}_B\}$, one can say that equilibrium involves strategic targeting, regardless of whether the colleges play a mixed- or a pure-strategy. We say *competitive equilibrium exhibits* ***strategic targeting*** if $\check{v} < \hat{v}$.

When do competitive equilibria exhibit strategic targeting? Note that MME does not preclude a competitive equilibrium in which $\hat{v} < \check{v}$. Figure 3.4 depicts such a possibility with $\underline{v}_B < \overline{v}_B < \underline{v}_A < \overline{v}_A$. As before, college i admits students with $v > \overline{v}_i$ and rejects those with $v < \underline{v}_i$. Observe that college A does not admit any student with $v \in [\underline{v}_A, \overline{v}_A]$, since college B admits them for sure (because $\overline{v}_B < \underline{v}_A$). Even though colleges have targeting incentives in this example, the resulting equilibrium is indistinguishable from the cutoff equilibria featured in the existing research.

A natural question is when such an equilibrium can be ruled out. The exact condition for its existence appears difficult to find, but we show next that the symmetric environment is sufficient to guarantee strategic targeting behavior.

[15] As usual, these formulae are valid only if the associated sets in (3.2) are nonempty. If they are empty, then threshold values are set equal to one for $\tilde{s}_A$ and zero for $\tilde{s}_B$.

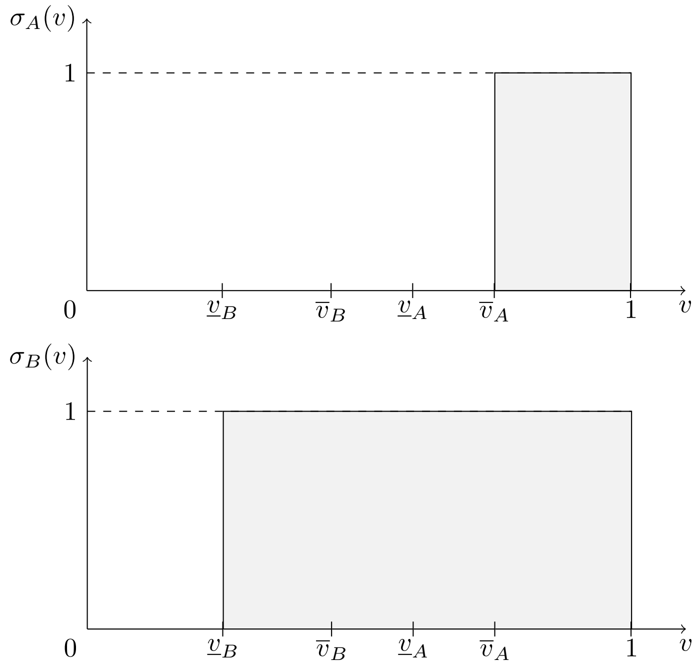


Figure 3.4: Cutoff Equilibrium

Theorem 3. *If the environment is symmetric (i.e., $\mu(s) = 1 - \mu(1 - s)$ for all s), then every competitive equilibrium exhibits strategic targeting.*

Proof. See Appendix A.5. ■

Note that the result of Theorem 3 does not just apply to symmetric equilibria. Strategic targeting occurs in any MME so long as the environment is symmetric. The intuition can be seen more clearly, though, in a symmetric equilibrium. Suppose to the contrary that a symmetric equilibrium were to involve a (common) cutoff $\hat{v} = \check{v}$. Then, a college faces competition with types $[\hat{v}, \hat{v} + \varepsilon)$ but would avoid the competition by admitting instead types $[\hat{v} - \varepsilon', \hat{v})$ for small enough ε and ε', which are chosen to keep the expected yield to remain unchanged. The resulting drop in the quality of admission pool is negligible but the benefit in reducing the uncertainty is of first order importance. Hence, the (symmetric) cutoff equilibrium cannot be sustained.

3.1 Properties of Equilibria

We have seen that the equilibrium outcome involves strategic targeting. We now consider the properties of the equilibria in welfare and fairness.

A few definitions are necessary. For each state s, an **assignment** is a mapping from $\mathcal{V} \times \{A, B\}$ into $[0, 1]$ that specifies the fraction of students of given type that is assigned to each college. An **outcome** is a mapping from a state to an assignment, i.e., the realized allocation in state s.

We say that a student has a **justified envy** at state s if at that state she prefers a college to the one that enrolls her, even though the former enrolls a student with a lower score. An outcome is said to be **fair** if for almost every state, the assignment it selects has no justified envy for almost all students. Next, an outcome is **Pareto efficient** if for almost every state, the assignment it selects is not Pareto dominated, i.e., there is no other assignment in which both colleges and all students are weakly better off and either at least one college or a positive measure of students is strictly better off relative to the initial assignment.

It is also useful to study the welfare of one side, taking the other side simply as resources. We say that an outcome is **student efficient** if for almost every state, there is no other assignment in which all students are weakly better off and a positive measure of students is strictly better off relative to the assignment that the outcome selects. An outcome is said to be **college efficient** if for almost every state, no other assignment can make both colleges weakly better off and at lease one college strictly better off relative to the assignment that the outcome selects. Notice that even if an outcome is Pareto efficient, this need not imply student efficiency or college efficiency.

The next theorem states properties of equilibria that arise in decentralized matching.

Theorem 4. *(i) Every competitive equilibrium is student, college and Pareto inefficient.*

(ii) Every competitive equilibrium is unfair if and only if it exhibits strategic targeting.

(iii) Every non-competitive equilibrium is unfair, student inefficient, but college efficient.

(iv) Every non-competitive equilibrium is Pareto inefficient unless almost every student admitted by one college has higher score than those admitted by the other college.

Proof. See Appendix A.7 ∎

4 Multidimensional Performance Measures and Evaluation Distortion

In the baseline model, we have assumed that colleges assess students based on a common performance measure. In practice, colleges consider multiple dimensions of students' attributes and performances, academic as well as non-academic. Some dimensions are common among colleges; for instance, SAT scores or grade points average of students are commonly observed and interpreted virtually the same by colleges. Others are not so common; for instance, many colleges require college-specific essays and testing.[16] Non-academic measures are likely to be less correlated among colleges since they are likely to focus on different aspects and interpret them differently. For instance, students' community service or leadership activities weigh heavily for some colleges,

[16]In the US, colleges require essays on topics that are often very differentiated. In Japan, there is a nation-wide exam, called National Center Test (NCT), and each university has its own exam. Public universities usually use both NCT and their own exams, and private ones use their own exams only. Similarly, students in Korea take a nationwide exam and each college often has its own essay tests and/or oral interviews.

whereas extracurricular activities such as musical or athletic talents may be important for others. We show that strategic targeting takes a particular form in this environment: *Colleges bias their admissions criteria toward non-common performances.*

To illustrate this point, we extend our model as follows. A student's type is described as a triple $(v, e_A, e_B) \in \mathcal{V} \times E_A \times E_B \equiv [0,1]^3$, where v is the common measure or score for both colleges, and e_A and e_B are college specific measures considered respectively by colleges A and B. One interpretation is that v is a student's test score of the nationwide exam, and e_A and e_B are her performances on college-specific essays or tests. Alternatively, v is an academic performance measure observed commonly to both colleges, and e_A and e_B correspond to different dimensions of extracurricular activities that the two colleges focus on.

As before, v is distributed according to $G(\cdot)$, and e_i, $i = A, B$, is conditionally independent on v and is distributed according to $X_i(\cdot|v)$ which admits a density $x_i(\cdot|v)$. We also assume that $\frac{\partial}{\partial v} X_i(e_i|v) < 0$ for any $e_i \in [0,1]$. That is, a student with high v has a higher probability of scoring high e_i. We also assume full support of G and X_i for all i. College i only values (v, e_i). Specifically, it derives payoff $U_i(v, e_i)$ from matriculating student with type (v, e_A, e_B), where U_i is strictly increasing and differentiable in both arguments.

College i's strategy is now described as a mapping $\sigma_i : \mathcal{V} \times E_i \to [0,1]$ with interpretation that it admits a fraction σ_i of students with type (v, e_i). Enrollment uncertainty facing college i with regard to students with type (v, e_i) depends on whether those students receive an admission offer from college j, $j \neq i$. Since e_j is conditionally uncorrelated with e_i, the probability of such event is $\overline{\sigma}_j(v) := \mathbb{E}[\sigma_j(v, e_j)|v]$.

For given $\overline{\sigma}_A(\cdot)$ and $\overline{\sigma}_B(\cdot)$, the mass of students enrolling in college i in state s is

$$m_i(s) = \int_0^1 \int_0^1 \sigma_i(v, e_i) \left(1 - \overline{\sigma}_j(v) + \mu_i(s)\, \overline{\sigma}_j(v)\right) \, dX_i(e_i|v)\, dG(v).$$

Let $\hat{s}_i$ be such that $m_i(\hat{s}_i) = \kappa$ as before. Then, college i's payoff is descried as follows:

$$\begin{aligned} \pi_i = & \int_0^1 \int_0^1 U_i(v, e_i)\sigma_i(v, e_i)\big(1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\, \overline{\sigma}_j(v)\big)\, dX_i(e_i|v)\, dG(v) \\ & - \lambda\, \mathbb{E}_s[m_A(s) - \kappa \,|\, s \in S_i]\, \mathrm{Prob}(s \in S_i) \\ = & \int_0^1 \int_0^1 \sigma_i(v, e_i)\, H_i(v, e_i, \overline{\sigma}_j(v))\, dX_i(e_i|v)\, dG(v) + \lambda\, \kappa\, \mathrm{Prob}(s \in S_i), \end{aligned}$$

where $S_A := \{s \,|\, s \geq \hat{s}_A\}$ and $S_B := \{s \,|\, s \leq \hat{s}_B\}$, and

$$\begin{aligned} H_i(v, e_i, \overline{\sigma}_j(v)) := & U_i(v, e_i)\big(1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\, \overline{\sigma}_j(v)\big) \\ & - \lambda\, \mathrm{Prob}(s \in S_i)\, \big(1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)|s \in S_i]\, \overline{\sigma}_j(v)\big). \end{aligned} \tag{4.1}$$

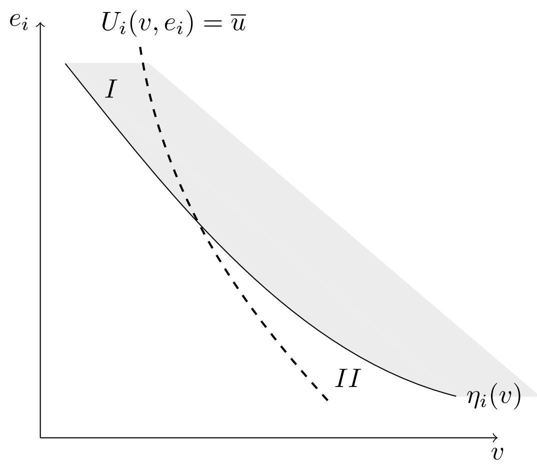


Figure 4.1: College's Cutoff Locus

As before, this marginal payoff consists of a student's value $U_i(v, e_i)$ to college i multiplied by the probability of the student accepting i's admission minus the capacity cost the student adds to i. It is worth noting that the capacity cost depends on the common measure v but not on the non-common measure e_i. This is because a student with high v is more likely to have a competing offer, so her acceptance decision is likely to be correlated with the (aggregate uncertainty) state, but conditional on v, e_i is independent with e_j. Intuitively, a student scoring high in e_i is less likely to be subject to enrollment uncertainty than a student scoring high in v.

We focus on a cutoff strategy equilibrium in which college i admits student type (v, e_i) if and only if $e_i \geq \eta_i(v)$ for some η_i nonincreasing in v. Figure 4.1 depicts a typical cutoff strategy. In the figure, the solid line represents the locus of η_i, so the shaded area depicts the types of students college i admits under a cutoff strategy, where as the dotted line is i's true indifference curve. Appendix A.8 provides a condition under which cutoff equilibrium exists. Such an equilibrium is quite plausible since the use of non-common performance measure by the colleges lessens their head-on competition and associated enrollment uncertainty.

We shall now show that the colleges further reduce head-on competition and enrollment uncertainty by placing more weight on the non-common measures relative to their common preferences. The reasoning of Lemma 2 implies that a college i must accept student type (v, e_i) if and only if $H_i(v, e_i, \overline{\sigma}_j(v)) \geq 0$. In particular, the cutoff locus $e_i = \eta_i(v)$ must satisfy $H_i(v, \eta_i(v), \overline{\sigma}_j(v)) = 0$ whenever $\eta_i(v) \in (0, 1)$. Its slope $-\eta_i'(v)$ shows the "relative worth" of the student's common performance v in college i's evaluation, as measured by the units of the student's non-common performance that it is willing to give up to obtain a unit increase in her common performance. The higher this value is, the larger weight the college places on the common performance. In particular, we shall say that the college *under-weights a student's common performance v and over-weights* *her non-common performance* e_i if for all v,

$$-\eta'(v) \leq \frac{\partial U_i(v, \eta_i(v))/\partial v}{\partial U_i(v, \eta_i(v))/\partial e_i}$$

and the inequality is strict for a positive measure of v.[17] Since students with high v contributes more to enrollment uncertainty than students with high e_i, as seen from (4.1), evaluation distortion arises in a cutoff equilibrium:

Theorem 5. *In a cutoff equilibrium, each college under-weights a student's common performance and over-weights her non-common performance.*

Proof. See Appendix A.8. ∎

This particular form of strategic targeting again entails justified envy in a positive measure of states. Among students who prefer A to B, those who are in region II (at the bottom between dotted and solid lines) in Figure 4.1 have justified envy toward those students in region I (at the top between solid and dotted lines). Pareto inefficiency also arises, since a college also underfills its seats in a positive measure of states, which could have been filled (Pareto improvingly) with unmatched students.

Theorem 6. *A cutoff equilibrium is unfair and students, college and Pareto inefficient.*

5 Restriction on Applications: Self-Targeting

So far, we have studied decentralized college admissions in the most stylized format. In the current and the following sections, we study two common ways for colleges to manage their enrollment uncertainty. We assume, as with the baseline model, that students' type is single dimensional.

One common method used in many countries is to limit the number of applications that students can submit. For instance, students cannot apply to both Cambridge and Oxford in the UK, and applicants in Japan can only apply to one public university.[18] In Korea, all schools (more precisely, college-department pairs) are partitioned into three groups, and students are allowed to apply to only one in each group.[19]

[17] Suppose for instance $U_i(v, e_i) = (1-\rho)\, v + \rho\, e_i$. Then, the condition means $-\eta'(v) \leq \frac{1-\rho}{\rho}$, so the college places a weight less than $1-\rho$ to common performance v and the weight more than ρ to non-common performance e_i.

[18] Public colleges in Japan may hold three exams. The first one is called "zenki(former period)-exam" and the last one is called "koki(later period)-exam". There are very small number of schools that have exam between these two exams. Students can apply to at most one public school at each exam date but the deadline for registering to the school that a student is admitted at zenki-exam is earlier than the date for applying the koki-exam.

[19] Although there is no such restriction in the US, high application fees may serve this role. See Chade and Smith (2006) and Chade, Lewis and Smith (2011) for students application decisions subject to application costs, without aggregate uncertainty.

Limiting the number of applications a student can make forces her to "self-target" colleges. Since students are likely to apply to schools they are most likely to accept when admitted, this method improves the odds of enrollment for colleges and reduces their yield management burden. In our model with two colleges, if the number of applications is restricted to one, colleges face no enrollment uncertainty because a student admitted by a college will never turn down its offer, so their admission behavior is non-strategic; namely, they admit students in the order of v until their capacities are filled. As will be seen, however, students' application behavior will be strategic. Thus, the overall welfare effects are not clear a priori.

We now provide a simple model showing students' application behavior when students can apply to only one college. To this end, we introduce students' cardinal preferences for colleges.[20] Each student has a taste $y \in \mathcal{Y} \equiv [0, 1]$, which is independent of score $v \in [0, 1]$. A student with taste y obtains payoff y from attending college A and $1 - y$ from attending college B. Thus, students with $y \in [0, \frac{1}{2}]$ prefer B to A, and those with $y \in [\frac{1}{2}, 1]$ prefer A to B. To facilitate the analysis, we assume that colleges observe an applicant's score v but not her preference y, while each student knows her preference y but not her score v.[21] In reality, even though students submit their records to colleges, they do not know precisely how they are ranked by colleges. See Avery and Levin (2010) for the same treatment.

A student's taste y is drawn according to a distribution that depends on the underlying state. For a given s, let $K(y|s)$ be the distribution of y with a density function $k(y|s)$, which is continuous and obeys (strict) monotone likelihood ration property: For any $y' > y$ and $s' > s$,

$$\frac{k(y'|s')}{k(y|s')} > \frac{k(y'|s)}{k(y|s)}, \tag{5.1}$$

meaning that a student's taste is more likely to be high in a high state. We further assume that there is $\delta > 0$ such that $\left|\frac{k_y(y|s)}{k(y|s)}\right| < \delta$ for any $y \in [0, 1]$ and $s \in [0, 1]$, which means that students' tastes change moderately according to the state. Each student with taste y forms a posterior belief about the state s, given by the following conditional density:

$$l(s|y) := \frac{k(y|s)}{\int_0^1 k(y|s)ds}.$$

Before proceeding, we make the following observations: First, for each student, applying to a school dominates not applying at all. Second, since a student does not know her score and her preference is independent of the score, her application depends only on the preference. Third, since each student's preference depends on the state, the mass of students applying to each college varies across states. Let $n_i(s)$ be the mass of students who apply to college $i = A, B$ in state s.

[20] Note that this does not alter the previous analyses, because even if students have cardinal preferences, it is still a weak dominant strategy for students to apply to both colleges in the previous model.

[21] Note that our analysis in Section 3 remains valid with this assumption.

We next consider a college's admission strategy. Since a college faces no enrollment uncertainty, it is optimal to admit all students up to a cutoff

$$c_i(s) := \inf \{c \in [0,1] \,|\, n_i(s)[1 - G(c)] \leq \kappa\}.$$

If $n_i(s) \geq \kappa$ in state s, then college i will set its cutoff so as to admit students up to its capacity. Otherwise, it will admit all applicants.

Consider now students' application decisions. Fix any $\sigma : \mathcal{Y} \to [0,1]$ which maps from taste to a probability of applying to A. This induces the mass of students applying to A in each state s,

$$n_A(s) := \int_0^1 \sigma(y) k(y|s)\, dy.$$

Clearly, $n_B(s) = 1 - n_A(s)$. A student with taste y expects to be admitted by college i with probability

$$P_i(y|\sigma) \equiv \mathbb{E}[1 - G(c_i(s)) \,|\, y, \sigma] = \int_0^1 q_i(s|\sigma) l(s|y) ds,$$

where $q_i(s|\sigma) := \min \{\kappa / n_i(s|\sigma), 1\}$ for $i = A, B$. This probability depends on the student's preference intensity y since it is informative about the underlying states. Note that a student with taste y will apply to A if and only if

$$yP_A(y|\sigma) \geq (1-y) P_B(y|\sigma).$$

We show that students follow a cutoff strategy in any equilibrium, given a moderate value of δ.

Lemma 4. *Suppose $\delta \leq \frac{1}{2}$. In any equilibrium, there exists a cutoff $\hat{y}$ such that students with $y \geq \hat{y}$ apply to A and those with $y < \hat{y}$ apply to B. And such an equilibrium exists.*

Proof. See Appendix A.9. ■

We now show that an equilibrium involves strategic application by students if one school is more popular than the other.

Theorem 7. *Suppose $\mu(s) > \frac{1}{2}$ $\left(\mu(s) = \frac{1}{2}\right)$ for almost all s. Then, $\hat{y} \in (\frac{1}{2}, 1)$ $\left(\hat{y} = \frac{1}{2}\right)$, where $\hat{y}$ is the equilibrium cutoff.*

Proof. See Appendix A.9. ■

The intuition behind Theorem 7 is clear. Suppose college A is popular so that a student expects to be admitted more easily by college B than college A, when all others employ a cutoff at $\frac{1}{2}$. Then, a student who prefers B ($y \leq \frac{1}{2}$) must apply to B. But, for a student who mildly prefers A (i.e., y is greater than but close to $\frac{1}{2}$), there is a trade-off since her payoff is higher if she goes to A, but

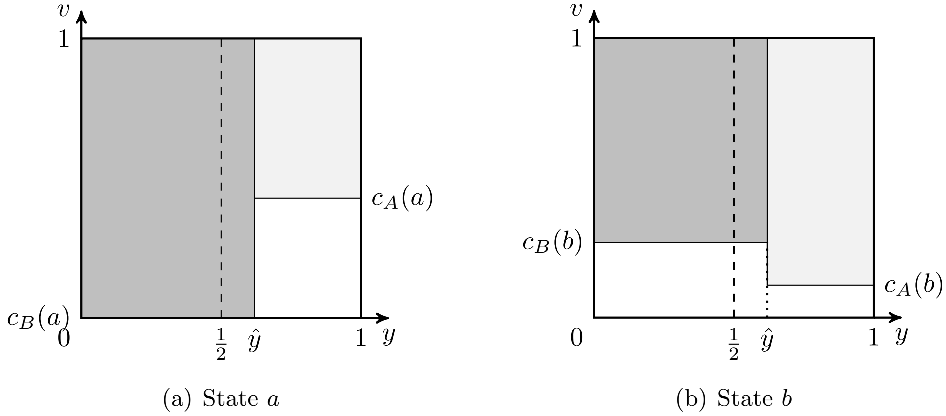


Figure 5.1: Equilibrium Assignment when $\kappa = 0.4$

there is a higher chance of admission at B. Thus, she may apply to B instead of A, leading to a cutoff $\hat{y} > \frac{1}{2}$ as depicted in Figure 5.1.

Example 1. Suppose there are two states a and b, each arising with probability $\frac{1}{2}$. Let $K(y|a) = y^2$, $K(y|b) = y$ and $\kappa = 0.4$. Then, we have

$\hat{y}$	$n_A(a)$	$n_B(a)$	$c_A(a)$	$c_B(a)$	$n_A(b)$	$n_B(b)$	$c_A(b)$	$c_B(b)$
0.547	0.701	0.299	0.429	0	0.453	0.547	0.116	0.269

Observe that if $n_i(s) \geq \kappa$ for all s and all $i = A, B$, then the self-targeting eliminates colleges' yield management problem, since each college fills its capacity with the best students among those who applied to it. But, a college may be undersubscribed; for instance, the mass of applicants to college B in state a is smaller than its capacity ($n_B(a) = 0.299 < \kappa = 0.4$).

Let us now consider welfare and fairness properties of the equilibrium outcome. First, the equilibrium is unfair. That is, justified envy arises in that (i) students who happen to have applied to a more popular college for a given state may be unassigned even though their scores would have been good enough for the other college to accept (the area on the bottom right below the light-shaded area of Figure 5.1(a)); and (ii) students who prefer ex ante more popular college may apply to and get into an ex ante less popular college, although they could have gotten into the former when it is ex post less popular (the dark-shaded area between $\frac{1}{2}$ and $\hat{y}$ of Figure 5.1(b)).

Second, a college may be undersubscribed in equilibrium so that its capacity is not filled even though there are unassigned, acceptable students. By assigning those students to unfilled seats of that college, students and college will be all better off. Thus, the equilibrium outcome is still student, college and Pareto inefficient.

Theorem 8. *The outcome of the restricted applications is unfair. Suppose $K(\hat{y}|s) < \kappa$ for a positive measure of states. Then, college B suffers from under-subscription, and the outcome is student, college and Pareto inefficient.*

Proof. See Appendix A.9. ■

6 Sequential Admissions: Wait-listing

Colleges also manage enrollment uncertainty by offering admissions sequentially. According to this method, a college would admit initially some applicants and wait-list others and later admit students from the latter group when some of the former group decline admissions, and this process may repeat. Wait-listing is adopted by most colleges in France, Korea, and the US.[22] In a typical application, the acceptance decisions are not deferred and/or the number of iterations is limited. Hence, even though wait-listing allows for more admission offers and acceptances than the baseline model or restricted applications, it does not fully eliminate congestion. For this reason, strategic targeting remains an issue as well.

To see this, we consider a simple extension of our baseline model. There are three colleges, A, B and C, each with a mass $\kappa < \frac{1}{3}$ capacity. There is a unit mass of students with score v, where v is distributed from $[0, 1]$ according to $G(\cdot)$ as before. All students prefer A and B to C, but C is sufficiently better than not attending any school. Colleges' preferences are given by students' scores, but for each student, there is a probability ε that each of colleges A and B finds the student unacceptable. College C admits students simply based on their scores.

There are two states, a and b, each arising with probability $\frac{1}{2}$. In state $i = a, b$, a fraction s_i of students gets utility u from A and $u'(< u)$ from B, and the remaining $1 - s_i$ students have the opposite preference, where $s_a = 1 - s_b > \frac{1}{2}$. In either state, a student gets utility u'' from C, where $(1 - \varepsilon)u < u'' < u$ so that entering C with certainty is better than entering A with probability $1 - \varepsilon$. In state a, the mass of students who prefer A to B is larger than that of those who prefer B to A ($s_a > \frac{1}{2} > 1 - s_a$), and in state b, the opposite is true ($s_b < \frac{1}{2} < 1 - s_b$).

Suppose also the capacity cost is prohibitively high so that whenever a college makes an admission decision, it must make sure that the capacity constraint is not violated. Wait-listing has the following feature. In each round, each college admits a set of students and wait-lists the remaining. A student who has received an offer must accept or reject the offer immediately; that is, the acceptance decision cannot be deferred. After the first round, colleges A and B learn the state, so the game effectively ends in two rounds.

We show that there is no symmetric equilibrium in which both colleges A and B use a cutoff strategy (i.e., admit the top κ acceptable students) in the first round.

Theorem 9. *There is no symmetric equilibrium in which both A and B offer admissions to the top κ students (excluding those whom they find unacceptable) in the first round.*

Proof. See Appendix A.10. ■

[22]In the US, nearly 45 percent of four-year colleges utilize wait lists in 2011 (Clinedinst, Hurley and Hawkins, 2012).

The intuition behind this result is as follows. Suppose colleges A and B admit the most preferred candidates up to their capacities with a plan to approach the next best students in case some of those first group students turn their offers down. The problem with this strategy is that when some of those admitted turn down their offers, the second-best students may not be available for the colleges. The reason is that those latter students are uncertain about whether A or B find them acceptable, hence if they receive an admission offer from college C, they would simply accept it. This means that the students who remain after the first round are likely to be far worse than the second-best group. Hence, a college would deviate profitably by leapfrogging some of the top κ students and preemptively admitting some of the second-best students.

Theorem 9 implies that strategic targeting must occur in any symmetric equilibrium. The strategic targeting here can be traced to the uncertainty facing the colleges about what students will remain after each round. This uncertainty in turn arises from the uncertainty students face about the offers they will receive in case they turn down current offers. Without a deferral of decisions, either by colleges in admitting students or by students in accepting offers, the uncertainties result in strategic targeting.[23]

Again, strategic targeting—i.e., a non-cutoff nature of equilibrium—means that the equilibrium outcome involves justified envy and is thus unfair. It is also student inefficient because there are two groups of students, among those in the second-best group, one preferring A but are admitted only by B and the other preferring B but are admitted only by A. In sum, the undesirable properties of decentralized matching are not eliminated by wait-listing.

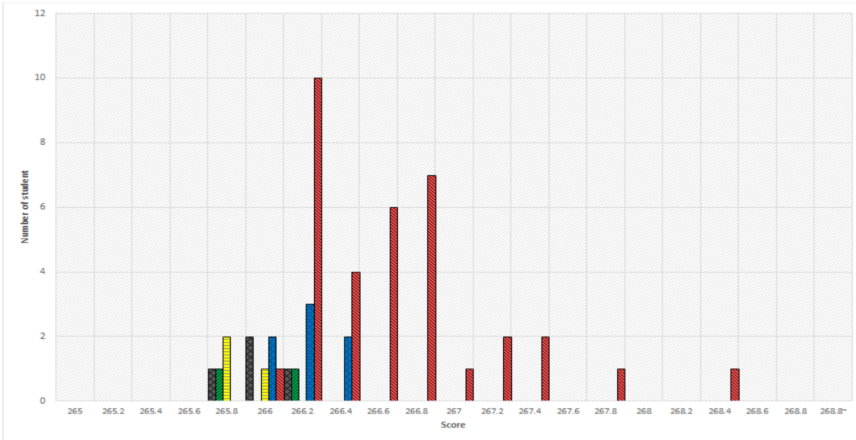
Such a pattern of strategic targeting is observed in practice. We present one evidence with the admissions decision employed by Hanyang University in Korea.[24] Figure 6.1 depicts the distribution of the nation-wide College Scholastic Ability Test (CSAT) scores earned by the students who are admitted by the Department of Economics and Finance (DEF) at different sequential rounds from 2011 to 2013.[25] The horizontal axis represents students' CSAT scores,[26] and the vertical axis is the number of students admitted in each round of wait lists. The figure reveals a pattern of strategic targeting for each year. In 2011, 133 students applied, and DEF admitted 35 students in the first round, 7 students in the second round and additional 9 students in the subsequent rounds. The average score of the top four students admitted in the second round (266.155) is higher than that

[23] As will be seen in the next section, the deferral of decisions allowed in the Gale-Shapley's algorithm solves this problem.

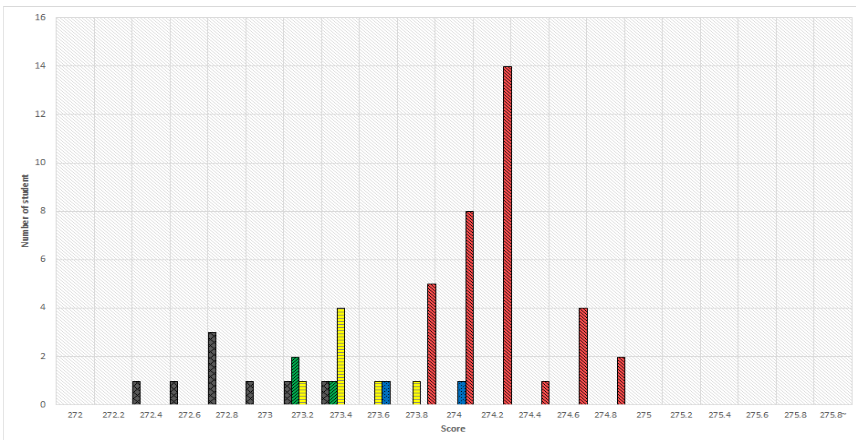
[24] We gratefully acknowledge the Hanyang University for providing their admissions data.

[25] As noted in Section 5, each college-department pair (unit) in Korea is divided into three groups, called groups *Ga*, *Na* and *Da*, and students can apply to one in each group. Thus, an unit in each group competes with other units in the same group but does not face competition with units in other groups. DEF divided its quota into two groups, *Ga* and *Na*, and admitted students separately for the two groups. We focus on admissions decision on the group *Ga*, which is the primary target group of DEF. The quota assigned for group *Ga* is twice as many as the quota for group *Na*.

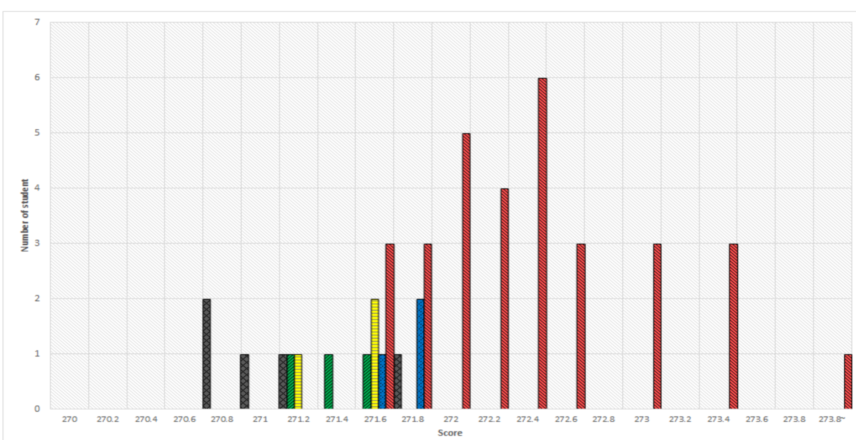
[26] The total score of CSAT is normalized as 280 by the admission office, while the actual score may depend on the subjects taken by students.



(a) 2011



(b) 2012



(c) 2013

1st 2nd 3rd 4th >5th

Note: "ith," $i = 1, .., 4$, means the ith round of wait lists and ">5th" includes all rounds after 5th round.

Figure 6.1: Admissions on Wait Lists

of the bottom four students in the first round (266.027). In 2012, DEF admitted 54 students in 11 rounds among 103 applicants. It admitted 34 students in the first round and 2 students in the second rounds. The highest score in the second round (273.844) is the same as the 27th highest one in the first round. In 2013, 45 students are admitted in 8 rounds among 124 applicants. DEF admitted 31 and 3 students in the first and the second rounds, respectively. The scores of the top two students in the second round are higher than those of the bottom three students in the first round. The reason for the observed non-monotonicity is that the DEF awards a significant number of its admissions based on a measure that "garbles" a student's CSAT score by another measure that is regarded as less informative of the student's ability.[27]

7 Centralized Matching via Deferred Acceptance

The most systemic response to enrollment uncertainty would be to have a central clearinghouse coordinate the preferences on both sides of the market. Such a centralized procedure is adopted in some countries, such as Australia, China, Germany, Taiwan, Turkey and the UK.[28] In this section, we consider a centralized matching with a Gale and Shapley's Deferred Acceptance algorithm (henceforth DA). Not only is the DA employed in many centralized markets, such as public school admissions and medical residency assignments, but it has a number of desirable properties compared with the outcomes of decentralized matching, as we shall highlight below.

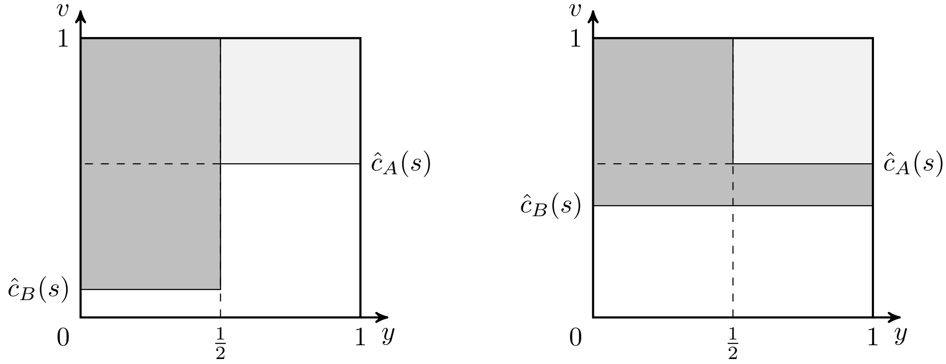
In the DA algorithm, students and colleges report their ordinal preferences to the clearinghouse, which then uses the information to simulate the following multi-round procedure. In each round, students propose to the best schools that have not yet rejected them. The colleges then accept tentatively the applicants in the order of their scores up to their capacities and rejects the rest. This process is repeated until no further proposals are made, in which case each student is assigned to a college that has tentatively accepted her proposal.[29]

Figure 7.1 illustrates the process for the case $\mu(s) \geq \frac{1}{2}$. In the first round, a fraction $\mu(s)$ of

[27]A college in Korea typically considers an applicant's CSAT score and his/her high school grade point average (GPA). But for the high school GPA, a college is prohibited by law from adjusting it for the quality of the high school the student is attending. Since the quality of high schools differs significantly across regions and between "special-purpose" schools and regular schools, a student's CSAT score is widely regarded as a more reliable indicator of his/her ability than his/her high school GPA. In keeping with this, DEF, as well as many other departments, at Hanyang University awards a small number of the so-called "priority" admissions in its first round admission based solely on applicants' CSAT scores. But for the remaining admissions in the first and the subsequent rounds, DEF makes selection based on the sum of an applicant's CSAT score and his/her GPA (unadjusted for high school quality). Accordingly, the students receiving priority admissions have higher CSAT scores than all other admitted students. But, because of the "garbling" of the CSAT scores by the high school GPA for the remaining admissions, the students admitted in the earlier rounds need not have higher CSAT scores than those admitted in later rounds. Figure 6.1 includes the students receiving priority admissions.

[28]See Chen and Kesten (2011) for Shanghai mechanism and Westkamp (2013) for Germany medical school matchings.

[29]The outcome of college-proposing DA is the same as that of student-proposing DA in our model, since colleges have a uniform rank on students. See also Abdulkadiroğlu, Che and Yasuda (2012) and Azevedo and Leshno (2012) for a model of DA in which a continuum mass of students is matched to a finite number of schools.



(a) Round 1

(b) Round 2

Figure 7.1: Deferred Acceptance Algorithm

students proposes to college A, and the remaining students propose to college B. Each college tentatively admits the top κ students among the applicants. Thus, colleges' cutoffs in this round, denoted by $\hat{c}_i(s)$, $i = A, B$, satisfy $\mu(s)[1 - G(\hat{c}_A(s))] = \kappa$ and $(1 - \mu(s))[1 - G(\hat{c}_B(s))] = \kappa$ (see Figure 7.1(a)). Unassigned students then propose to another college at the second round, and again, colleges reselect the top κ students from those tentatively admitted and from the new applicants. Thus, colleges' cutoffs in this round satisfy $\mu(s)[1 - G(\hat{c}_A(s))] = \kappa$ and $1 - G(\hat{c}_B(s)) = 2\kappa$ (see Figure 7.1(b)). Since there are no more colleges to which unassigned students can apply, the assignment is finalized in the second round in our model.

Consider now the equilibrium properties of the DA outcome. Under DA, the matching is strategy proof for the students, so the students have a dominant strategy of reporting their preferences truthfully (Dubins and Freedman, 1981; Roth, 1982). In addition, colleges in our model also report their rankings and capacities truthfully in an ex post equilibrium, namely to form a Nash equilibrium for any profile of preferences students may report.

Lemma 5. *Given the common college preferences, it is an ex post equilibrium for colleges to report their rankings and capacities truthfully.*

Proof. See Appendix A.11. ∎

The matching in the equilibrium involves no justified envy (Gale and Shapley, 1962; Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003) and is student efficient (because colleges' preferences are acyclic in the sense of Ergin (2002)) and Pareto efficient (an implication of stability). It also eliminates colleges' yield management problem completely. Colleges never exceed their capacities (because it is never allowed by the algorithm) and have no seats left unfilled in the presence of acceptable unmatched students (a consequence of stability).

In fact, given the homogeneous preferences of the colleges, there exists a single cutoff such that a student is assigned a college under DA if and only if her score exceeds that cutoff. In order words, only those with the top 2κ scores are assigned to a college. This outcome is jointly optimal for the two colleges, in the sense that it would be selected if the two colleges were to merge. In particular, the outcome is college efficient.

By contrast, a competitive equilibrium in decentralized matching entails unfilled seats for colleges in low-demand states and overfilled seats in high-demand states, so the assignment is far from jointly optimal. This observation suggests that at least one college must be strictly better off from a shift from decentralized matching to a centralized matching via the deferred acceptance algorithm. Despite the overall benefit from switching centralization via DA, it is possible for one college to be worse off. To see this, consider the following example.

Example 2. Let $v \sim U[0,1]$, $\lambda = 5$, $\kappa = 0.45$ and $\mu(s) = \frac{2}{5}s + \frac{3}{5}$. Then, in a decentralized admission, there is a MME such that $\check{v} = \underline{v}_A < \overline{v}_B = \hat{v}$ and colleges' payoffs in the equilibrium are $\pi_A = 0.283$ and $\pi_B = 0.180$. Suppose now that the DA is in use. Then, their payoffs are $\pi_A^{DA} = 0.321$ and $\pi_B^{DA} = 0.174$. Notice that $\pi_A^{DA} + \pi_B^{DA} = 0.495 > \pi_A + \pi_B = 0.463$ (overall benefit for the two colleges), $\pi_A^{DA} > \pi_A$ (college A is strictly better off), but $\pi_B^{DA} < \pi_B$ (college B is worse off).

In this example, college A is more popular than B for all states. Yet, in a decentralized matching, strategic targeting enables college B to attract good students whom it would otherwise not be able to attract under DA. This may explain why centralized matching is not as common in college admissions as in other contexts such as public high school admissions. In the latter, the schools are largely under the control of the school system which serves the interest of the students. In contrast, colleges are independent strategic players with their own interest to pursue.

Equilibrium properties of the outcome under DA are summarized in the follow.

Theorem 10. *Under DA, the equilibrium outcome is fair and student, college and Pareto efficient. However, some college may be worse off relative to decentralized matching.*

8 Conclusion

The current paper has introduced and analyzed a new model of decentralized college admissions. In the model, colleges make admission decisions subject to aggregate uncertainty about students' preferences and linear costs for any enrollment exceeding the capacity. We find that colleges' admission decisions become a tool for strategic yield management and in equilibrium, colleges seek to manage their enrollment uncertainty by strategically targeting their admissions to students who are likely overlooked by their competitors. When colleges also consider students' performance in college-specific essays or tests, or their non-academic performance or extracurricular activities, strategic targeting takes the form of colleges' overweighting those non-common performance measures and underweighting common academic measures such as GPA or SAT scores.

We also obtain the welfare and fairness implications of the equilibrium outcomes. We show that the equilibrium outcome under decentralized matching entails justified envy and is Pareto inefficient. Our analytical model also permits a comparison of the outcomes that would arise when students are forced to self-target by the restricted applications, when admissions are made sequentially, and when the market is centralized via DA. Both self-targeting and wait-listing alleviate colleges' yield management burden, but strategic targeting and enrollment uncertainty remain. And so do inefficiencies and justified envy. Centralized matching via DA completely eliminates the yield management problem and justified envy, and it also achieves efficiency. At the same time, not all colleges may benefit from such a centralized matching. This last observation may explain why college admissions remain decentralized in many countries.

Our analyses have several other implications.

Early Admissions. Early admissions are widely used in the US and Korea. In these countries, students can apply early often to selected schools, and the schools early-admit them (with binding or non-binding requirements for students to accept them). The remaining students and seats are then allocated through regular admissions, operating much as in our baseline model. The early admissions process thus involves both the elements of sequential admissions, as studied in Section 6, and of restricted applications, as studied in Section 5. While the process is too complicated to analyze in our framework (especially with aggregate uncertainty), our analyses suggest an important purpose the early admissions program may serve. By restricting the number of applications, the early admissions programs induce students to reveal their preferences for colleges. This, together with sequential admissions, allows colleges to manage enrollment uncertainty more effectively than they could without the program. We believe this is an important function of early admissions, in addition to those recognized by other recent papers (Avery and Levin, 2010; Lee, 2009). Regardless of the motives, the programs restrict choices for students and force colleges to make decisions based on less than full information that becomes eventually available to them. As seen in Section 5 and in Avery and Levin (2010), students are likely to respond strategically, which will likely entail justified envy and inefficiencies.

Colleges' Preferences for Loyalty and Enthusiasm. It is well documented that colleges favor students who are eager to attend them. Students who convey seriousness of their interests through campus visits, essays, and webcam interviews are known to be marginally favored, especially by small liberal arts colleges. Early admissions, as Avery and Levin (2010) argue, also serve as a tool for colleges to identify enthusiastic applicants and favor them in the admission. It is entirely plausible that these preferences by colleges are intrinsic, as postulated by Avery and Levin (2010). But, our theory suggests that such a preference by colleges could also arise endogenously from their desire to manage enrollment uncertainty. Like the students without an competing offer in our baseline model, those who credibly demonstrate their seriousness of preferences for a college are less likely to be subject to aggregate uncertainty, and are thus less likely to contribute to enrollment uncertainty for a college. This suggests that even a college with no intrinsic preference for the former students has a reason to favor them. For instance, legacy students (who have a family history with the school) are likely to have loyalty for schools, making their preferences less subject to fads and whims that may sway preferences of other students, and hence they contribute less to enrollment uncertainty. Espenshade, Chung and Walling (2004) show that legacy applicants have nearly three times the likelihood of being accepted as nonlegacies. Our theory provides a rationale for why colleges favor those students. Similarly, campus visits, essays and webcam interviews all serve as a device for screening students' seriousness of preferences.

Specialized Requirements and College Specific Investments. Colleges often have special requirements for their applicants to fulfill. These requirements range from specialized essay questions, college-specific entrance exams to specialized admissions tracks requiring specific qualifications. For example, colleges in Korea admit a number of students through specialized tracks that require specific qualifications, such as foreign language skills, awards in contests in science, music, invention or information technologies. Such requirements help colleges to identify students with serious interests. More demanding requirements encourage students to make college-specific investments well in advance of application. Our theory suggests that these investments serve as a means by which colleges can target and secure enrollment of students even in early stages.

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A Appendix A: Proofs

A.1 Proof of Lemma 1

Claim 1. *Suppose $\mathcal{V}_{AB}$ has zero measure. Then, the following results hold.*

(i) $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$.

(ii) Almost every student with $v \geq G^{-1}(1 - 2\kappa)$ receives a admission.

Proof. (i) Since $\mathcal{V}_{AB}$ is a measure zero set, $m_i(s)$ is constant across states for all $i = A, B$. If $m_i(s) < \kappa$, then college i can benefit by admitting some students with measure less than $\kappa - m_i(s)$. Similarly, if $m_i(s) > \kappa$, then it can benefit by rejecting some students with measure less than $m_i(s) - \kappa$.

(ii) Observe that $\mathcal{V}_A \cup \mathcal{V}_B$ cannot have a gap, otherwise at least one college can benefit by replacing a positive measure of low score students with the same measure of students in the gap. So, it must be a connected interval with $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$. Since $m_A(s) = m_B(s) = \kappa$ for all s by Part (i), this means that almost every top 2κ students are admitted. ■

Note that the proofs for Parts (i), (ii) and (iv) of the lemma for noncompetitive equilibrium follow from Claim 1. We thus consider competitive equilibrium in what follows. We prove in the sequence of Parts (i), (iii) and (ii).

Proof of Part (i). Consider a competitive equilibrium. Suppose $m_A(1) < \kappa$. Let college A admit a mass $\kappa - m_A(1)$ of students. Then, the mass of students attending A in this case, denoted by $\widetilde{m}_A(s)$, satisfies that for any $s < 1$,

$$m_A(s) < m_A(s) + \mu(s)[\kappa - m_A(1)] \leq \widetilde{m}_A(s) \leq m_A(s) + [k - m_A(1)] < \kappa,$$

where the first and the last inequality follow from the fact that $m_A(s) < m_A(1)$ for $s < 1$ (since $\mu(\cdot)$ is strictly increasing in s). Observe that A benefits from such deviation since it admits more students without having over-enrollment. Hence, we must have $\kappa \leq m_A(1)$ in equilibrium. Similarly, if $m_A(0) > \kappa$, then A can benefit by rejecting a mass $m_A(0) - \kappa$ of students. Therefore, we must have $m_A(0) \leq \kappa \leq m_A(1)$ in any competitive equilibrium. The proof for college B is analogous. ■

Proof of Part (iii). We consider college A here. The proof for college B will be analogous. Since $\mu(\cdot)$ is strictly increasing and continuous in s, so is $m_A(\cdot)$. Thus, there exists $\hat{s}_A \in [0, 1]$ such that $m_A(\hat{s}_A) = \kappa$ by Part (i). We show that $\hat{s}_A \neq 0, 1$ in what follows.

Suppose $\hat{s}_A = 0$. Then, $m_A(s) > m_A(0) = \kappa$ for all $s > 0$. Thus, college A's payoff is

$$\pi_A = \int_0^1 v\sigma_A(v)(1 - \sigma_B(v) + \overline{\mu}\sigma_B(v))dG(v) - \lambda \int_0^1 [m_A(s) - \kappa]ds,$$

where $\overline{\mu} := \mathbb{E}[\mu(s)]$ and

$$m_A(s) = \int_0^1 \sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v).$$

Let A reject a positive measure of students, say $(c, c+\delta) \in \mathcal{V}_A$. Then, its payoff is

$$\widetilde{\pi}_A = \int_{[0,1]\setminus(c,c+\delta)} v\sigma_A(v)(1 - \sigma_B(v) + \overline{\mu})\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 [\widetilde{m}_A(s) - \kappa]ds,$$

where

$$\widetilde{m}_A(s) = m_A(s) - \int_c^{c+\delta} \sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \tag{A.1.1}$$

and $\tilde{s}_A$ is such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A > \hat{s}_A = 0$ since $\widetilde{m}_A(s) < m_A(s)$. Now, we can choose δ such that $\tilde{s}_A < \varepsilon$ for sufficiently small $\varepsilon > 0$. Then, A's net payoff from the deviation is

$$\begin{aligned}
&- \int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \overline{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 [\widetilde{m}_A(s) - \kappa]ds + \lambda \int_0^1 [m_A(s) - \kappa]ds \\
=& - \int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \overline{\mu}\sigma_B(v))dG(v) + \lambda \int_{\tilde{s}_A}^1 [m_A(s) - \widetilde{m}_A(s)]ds + \lambda \int_0^{\tilde{s}_A} [m_A(s) - \kappa]ds \\
=& - \int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \Big) ds \\
&- \int_0^{\tilde{s}_A} \Big(\int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \Big) ds \\
&+ \lambda \int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta} \sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \Big) ds + \lambda \int_0^1 [m_A(s) - \kappa]ds \\
=& \int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta} (\lambda - v)\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \Big) ds \\
&- \int_0^{\tilde{s}_A} \Big(\int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \Big) ds + \lambda \int_0^{\tilde{s}_A} [m_A(s) - \kappa]ds \\
>& 0,
\end{aligned}$$

where the second equality follows from (A.1.1) and the last inequality holds for sufficiently small ε.

Next, suppose $\hat{s}_A = 1$. Then, $m_A(s) < m_A(1) = \kappa$ for all $s < 1$. Let A admit all students in $(c, c+\delta) \notin \mathcal{V}_A$ for some $c < 1$. Then, the mass of students attending A becomes

$$\widetilde{m}_A(s) = m_A(s) + \int_c^{c+\delta} (1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v). \tag{A.1.2}$$

Let $\tilde{s}_A$ be such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A < \hat{s}_A = 1$ since $\widetilde{m}_A(s) > m_A(s)$. We can choose δ such that $1 - \tilde{s}_A < \varepsilon$ for sufficiently small ε. Then, A's net payoff from the deviation is

$$\begin{aligned}
&\int_c^{c+\delta} v(1-\sigma_B(v)+\overline{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 (\widetilde{m}_A(s)-\kappa)ds \\
=&\int_c^{c+\delta} v(1-\sigma_B(v)+\overline{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 \Big(m_A(s) + \int_c^{c+\delta}(1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v) - \kappa\Big)ds \\
=&\int_c^{c+\delta} v(1-\sigma_B(v)+\overline{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta}(1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v)\Big)ds \\
&+\lambda \int_{\tilde{s}_A}^1 [\kappa - m_A(s)]ds \\
=&\int_0^{\tilde{s}_A} \Big(\int_c^{c+\delta} v(1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v)\Big)ds \\
&-\int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta} (\lambda - v)(1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v)\Big)ds + \lambda \int_{\tilde{s}_A}^1 [\kappa - m_A(s)]ds \\
>&0
\end{aligned}$$

where the first equality follows from (A.1.2) and the last inequality holds for sufficiently small ε. ■

Proof of Part (ii). We first show $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$ and then show that $\mathcal{V}_A \cup \mathcal{V}_B$ is a connected interval and $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$.

Step 1. $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$.

Proof . Suppose to the contrary that $\overline{c} := \sup\{\mathcal{V}_A \cup \mathcal{V}_B\} < 1$. We show that at least one college can benefit by rejecting some students in favor of those with $[\overline{c}, 1]$.

Suppose $\mathcal{V}_i \setminus \mathcal{V}_{AB}$ contains an open interval with positive measure for some $i = A, B$. Then, it is clear that college i can benefit by rejecting a positive measure of students from the bottom of $\mathcal{V}_i \setminus \mathcal{V}_{AB}$ and admits the same measure of students from 1.

Suppose now it is not the case. Let college A reject students in $(c, c+\delta) \in \mathcal{V}_{AB}$ and admit those in $(1-\varepsilon, 1]$ instead, where δ and ε satisfy

$$\int_{1-\varepsilon}^1 v\, dG(v) = \int_c^{c+\delta} v\, dG(v) \tag{A.1.3}$$

and

$$\int_{1-\varepsilon}^1 1\, dG(v) = \int_c^{c+\delta} \sigma_A(v)[1-\sigma_B(v)+\mu(\hat{s}_A)\sigma_B(v)]dG(v), \tag{A.1.4}$$

for given $\hat{s}_A$ such that $m_A(\hat{s}_A) = \kappa$. The mass of students attending A from this deviation is

$$\widetilde{m}_A(s) = m_A(s) + \int_{1-\varepsilon}^1 1\, dG(v) - \int_c^{c+\delta} \sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v).$$

Note that $\widetilde{m}_A(\hat{s}_A) = m_A(\hat{s}_A)$. Denote college A's payoff from the deviation by $\widetilde{\pi}_A$. Then, its net payoff from the deviation, $\widetilde{\pi}_A - \pi_A$, is

$$
\begin{aligned}
&\int_{1-\varepsilon}^{1} v\,dG(v) - \int_{c}^{c+\delta} v\sigma_A(v)[1-\sigma_B(v)+\overline{\mu}\sigma_B(v))dG(v) - \lambda\,\mathbb{E}[\widetilde{m}_A(s) - m_A(s)|s \in S_A]\hat{P}_A \\
\geq &\int_{1-\varepsilon}^{1} v\,dG(v) - \int_{c}^{c+\delta} v\,dG(v) - \lambda\int_{\hat{s}_A}^{1}\Big(\int_{1-\varepsilon}^{1} 1\,dG(v) - \int_{c}^{c+\delta}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v)\Big)ds \\
= &-\lambda\int_{\hat{s}_A}^{1}\Big(\int_{1-\varepsilon}^{1} 1\,dG(v) - \int_{c}^{c+\delta}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v)]dG(v)\Big)ds \\
> &-\lambda\int_{\hat{s}_A}^{1}\Big(\int_{1-\varepsilon}^{1} 1\,dG(v) - \int_{c}^{c+\delta}\sigma_A(v)[1-\sigma_B(v)+\mu(\hat{s}_A)\sigma_B(v)]dG(v)\Big)ds \\
= &0,
\end{aligned}
$$

where the first inequality holds since $\sigma_A(v)$, $\sigma_B(v)$, $\overline{\mu} = \mathbb{E}[\mu(s)] \leq 1$ for any v, the first equality follows from (A.1.3), and the last inequality follows from the fact that $\mu(\cdot)$ is strictly increasing in s, and the last equality follows from (A.1.4). □

Step 2. $\mathcal{V}_A \cup \mathcal{V}_B$ *is a connected interval.*

Proof . Suppose that there is gap in $\mathcal{V}_A \cup \mathcal{V}_B$. The proof is analogous to Step 1, where $(1-\varepsilon, 1]$ is now replaced by the gap in $\mathcal{V}_A \cup \mathcal{V}_B$. We omit the details. □

Step 3. $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$

Proof . Suppose to the contrary that $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} = 0$. Suppose $\inf\{\mathcal{V}_A\} = 0$. Let A reject a small fraction of students at the bottom, $[0, \varepsilon)$, where $2\varepsilon < 1 - \hat{s}_A$ and $\hat{s}_A$ is such that $m_A(\hat{s}_A) = \kappa$. Then, the mass of students attending A from the deviation is

$$\widetilde{m}_A(s) = \int_{\varepsilon}^{1} \sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v).$$

Denote $\tilde{s}_A$ be the state such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A > \hat{s}_A$ since $\widetilde{m}_A(s) < m_A(s)$. Hence, we can choose ε such that $\tilde{s}_A - \hat{s}_A < \varepsilon$. Then, A's net payoff from the deviation is

$$\widetilde{\pi}_A - \pi_A = \underbrace{-\int_{0}^{\varepsilon} v\sigma_A(v)[1-\sigma_B(v)+\overline{\mu}\sigma_B(v)]dG(v)}_{(*)} + \underbrace{\lambda\left[\int_{\hat{s}_A}^{1}(m_A(s)-\kappa)ds - \int_{\tilde{s}_A}^{1}(\widetilde{m}_A(s)-\kappa)ds\right]}_{(**)}.$$

Note that

$$
\begin{aligned}
(*) &= \int_{0}^{1}\Big(\int_{0}^{\varepsilon} v\sigma_A(v)(1-\sigma_B(v)+\mu(s)\sigma_B(v))dG(v)\Big)ds \\
&< \varepsilon\int_{\tilde{s}_A}^{1}\Big(\int_{0}^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v)]dG(v)\Big)ds
\end{aligned}
$$

$$+\varepsilon\int_0^{\tilde{s}_A}\Big(\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v)]dG(v)\Big)ds$$

and

$$\begin{aligned}
(**) &= \int_{\tilde{s}_A}^1 (m_A(s)-\kappa)ds + \int_{\hat{s}_A}^{\tilde{s}_A}(m_A(s)-\kappa)ds - \int_{\tilde{s}_A}^1(\widetilde{m}_A(s)-\kappa)ds \\
&= \int_{\tilde{s}_A}^1\Big(\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v)]dG(v)\Big)ds + \int_{\hat{s}_A}^{\tilde{s}_A}(m_A(s)-\kappa)ds \\
&> \int_{\tilde{s}_A}^1\Big(\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v)]dG(v)\Big)ds
\end{aligned}$$

where the last inequality holds since $m_A(s) > \kappa$ for any $s \in (\hat{s}_A, \tilde{s})$. Thus, we have

$$\begin{aligned}
\widetilde{\pi}_A - \pi_A >&(\lambda-\varepsilon)\int_{\tilde{s}_A}^1\Big(\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v)]dG(v)\Big)ds \\
&-\varepsilon\int_0^{\tilde{s}_A}\Big(\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(s)\sigma_B(v)]dG(v)\Big)ds \\
&> (\lambda-\varepsilon)(1-\tilde{s}_A)\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(\tilde{s}_A)\sigma_B(v)]dG(v) \\
&-\varepsilon\,\tilde{s}_A\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(\tilde{s}_A)\sigma_B(v)]dG(v) \\
&= \big(\lambda(1-\tilde{s}_A)-\varepsilon\big)\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(\tilde{s}_A)\sigma_B(v)]dG(v) \\
&> \varepsilon(\lambda-1)\int_0^{\varepsilon}\sigma_A(v)[1-\sigma_B(v)+\mu(\tilde{s}_A)\sigma_B(v)]dG(v) \\
&\geq 0
\end{aligned}$$

where the penultimate inequality holds since $\lambda(1-\tilde{s}_A)-\varepsilon = \lambda((1-\hat{s}_A)-(\tilde{s}_A-\hat{s}_A))-\varepsilon > \lambda\varepsilon-\varepsilon = \varepsilon(\lambda-1)$ because $\tilde{s}_A-\hat{s}_A < \varepsilon$ and $2\varepsilon < 1-\hat{s}_A$. □ ■

A.2 Non-Competitive Equilibrium

In this section, we show that when $\kappa < \frac{1}{2}$ is not too small or $\lambda > 1$ is not too large, there does not exist a non-competitive equilibrium.

Lemma A1. *Suppose that $\mathcal{V}_{AB}$ has zero measure. Then, we have the followings:*

(i) There is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, one college has an incentive to deviate.

(ii) There is $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, one college has an incentive to deviate.

Proof. Since $\mathcal{V}_{AB}$ has zero measure, $m_i(s) = \kappa$ for all s and

$$\pi_i = \int_{\mathcal{V}_i} v\,dG(v), \quad i = A, B.$$

Now, let $\underline{c}_i := \inf\{\mathcal{V}_i\}$ and $\overline{c}_i := \sup\{\mathcal{V}_i\}$.

Proof of (i) . Let $\underline{c}_A = \inf\{\mathcal{V}_A \cup \mathcal{V}_B\}$, without loss of generality. Then, $\underline{c}_A = G^{-1}(1-2\kappa)$ by Lemma 1. We show that college A has an incentive to deviate. Suppose A rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but accepts those in $[\overline{c}_B - \varepsilon, \overline{c}_B]$, where ε and δ are such that

$$G(\overline{c}_B) - G(\overline{c}_B - \varepsilon) = G(\underline{c}_A + \delta) - G(\underline{c}_A). \tag{A.2.1}$$

Note that the mass of students attending A under this deviation is

$$\begin{aligned}
\widetilde{m}_A(s) &= \int_{\overline{c}_B - \varepsilon}^{\overline{c}_B} \mu(s)\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} 1\, dG(v) \\
&= \mu(s)[G(\overline{c}_B) - G(\overline{c}_B - \varepsilon)] + \kappa - [G(\underline{c}_A + \delta) - G(\underline{c}_A)] \\
&\leq \kappa,
\end{aligned}$$

where the second equality holds since $m_A(s) = \kappa$ for all s, and the last inequality follows from (A.2.1) and the fact that $\mu(s) \leq 1$ for all s.

Since A is never over-demanded, its payoff from the deviation is

$$\widetilde{\pi}_A = \overline{\mu} \int_{\overline{c}_B - \varepsilon}^{\overline{c}_B} v\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} v\, dG(v) = \overline{\mu} \int_{\overline{c}_B - \varepsilon}^{\overline{c}_B} v\, dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v\, dG(v),$$

where $\overline{\mu} = \mathbb{E}[\mu(s)]$. Therefore,

$$\begin{aligned}
\tilde{\pi}_A - \pi_A &= \overline{\mu} \int_{\overline{c}_B - \varepsilon}^{\overline{c}_B} v\, dG(v) - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v\, dG(v) \\
&= \overline{\mu} \left[\overline{c}_B G(\overline{c}_B) - (\overline{c}_B - \varepsilon) G(\overline{c}_B - \varepsilon) - \int_{\overline{c}_B - \varepsilon}^{\overline{c}_B} G(v) dv \right] \\
&\quad - \left[(\underline{c}_A + \delta) G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \int_{\underline{c}_A}^{\underline{c}_A + \delta} G(v) dv \right] \\
&> \overline{\mu} \left[\overline{c}_B G(\overline{c}_B) - (\overline{c}_B - \varepsilon) G(\overline{c}_B - \varepsilon) - \varepsilon G(\overline{c}_B) \right] - \left[(\underline{c}_A + \delta) G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \delta G(\underline{c}_A) \right] \\
&= \left[G(\overline{c}_B) - G(\overline{c}_B - \varepsilon) \right] \left[\overline{\mu}\, \overline{c}_B - \underline{c}_A - \overline{\mu}\, \varepsilon - \delta \right],
\end{aligned} \tag{A.2.2}$$

where the first equality follows from the integration by parts, and the last equality follows from (A.2.1). Observe that if $\overline{\mu} > \frac{\underline{c}_A}{\overline{c}_B}$, then (A.2.2) is strictly positive for sufficiently small ε and δ, hence $\tilde{\pi}_A > \pi_A$. Note that since $\underline{c}_A = G^{-1}(1-2\kappa)$ and $m_i(s) = \kappa$ for all s and $i = A, B$, we have that $G(\overline{c}_B) \geq 1 - \kappa$; that is, $\overline{c}_B \geq G^{-1}(1-\kappa)$. (Otherwise, college A must be admitting more than measure κ of students.) Therefore,

$$\frac{\underline{c}_A}{\overline{c}_B} \leq \frac{G^{-1}(1-2\kappa)}{G^{-1}(1-\kappa)}. \tag{A.2.3}$$

Since the RHS of (A.2.3) is continuous in κ and converges to zero as κ approaches to $\frac{1}{2}$, there is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, $\overline{\mu} > \frac{\underline{c}_A}{\overline{c}_B}$ for any given $\overline{\mu}$. $\square$

Proof of (ii) . Let $\overline{c}_B = \sup\{\mathcal{V}_A \cup \mathcal{V}_B\}$, without loss of generality. Then, $\overline{c}_B = 1$ by Lemma 1. We show that college A has an incentive to deviate. Suppose A rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but admits students in $[1-\varepsilon, 1]$, where ε and δ satisfy

$$\mu(1-\underline{c}_A)[1-G(1-\varepsilon)] = G(\underline{c}_A + \delta) - G(\underline{c}_A). \tag{A.2.4}$$

The mass of students attending A in state s under the deviation is

$$\widetilde{m}_A(s) = \int_{1-\varepsilon}^{1} \mu(s)\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A+\delta]} 1\, dG(v) = \mu(s)[1-G(1-\varepsilon)] + \kappa - [G(\underline{c}_A+\delta) - G(\underline{c}_A)].$$

Let $\hat{s}_A$ be such that $\widetilde{m}_A(\hat{s}_A) = \kappa$, i.e., $\mu(\hat{s}_A)[1-G(1-\varepsilon)] = [G(\underline{c}_A+\delta) - G(\underline{c}_A)]$. Since $\mu(\cdot)$ is strictly increasing in s, $\hat{s}_A = 1 - \underline{c}_A$ by (A.2.4).

Thus, A's payoff from the deviation is

$$\begin{aligned}
\widetilde{\pi}_A &= \overline{\mu} \int_{1-\varepsilon}^{1} v\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A+\delta]} v\, dG(v) - \lambda \int_{\hat{s}_A}^{1} [m(s) - \kappa]\, ds \\
&= \overline{\mu} \int_{1-\varepsilon}^{1} v\, dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A+\delta} v\, dG(v) \\
&\quad - \lambda \Big[(1-G(1-\varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\underline{c}_A+\delta) - G(\underline{c}_A)](1-\hat{s}_A) \Big].
\end{aligned}$$

and the net payoff from the deviation is

$$\begin{aligned}
\tilde{\pi}_A - \pi_A &= \overline{\mu} \int_{1-\varepsilon}^{1} v\, dG(v) - \int_{\underline{c}_A}^{\underline{c}_A+\delta} v\, dG(v) \\
&\quad - \lambda \Big[(1-G(1-\varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\underline{c}_A+\delta) - G(\underline{c}_A)](1-\hat{s}_A) \Big] \\
&> \overline{\mu}(1-\varepsilon)[1-G(1-\varepsilon)] - (\underline{c}_A+\delta)[G(\underline{c}_A+\delta) - G(\underline{c}_A)] \\
&\quad - \lambda \Big[(1-G(1-\varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\underline{c}_A+\delta) - G(\underline{c}_A)](1-\hat{s}_A) \Big] \\
&= [1-G(1-\varepsilon)] \Big(\overline{\mu} - \eta\, \underline{c}_A - \overline{\mu}\, \varepsilon - \eta\, \delta - \lambda \Big[\int_{\hat{s}_A}^{1} \mu(s) ds - \eta(1-\hat{s}_A) \Big] \Big),
\end{aligned} \tag{A.2.5}$$

where $\eta = \mu(1-\underline{c}_A)$ and the last equality follows from (A.2.4).

Observe that if $\overline{\mu} - \eta\,\underline{c}_A - \lambda\Big[\int_{\hat{s}_A}^1 \mu(s)ds - \eta(1-\hat{s}_A)\Big] > 0$, then (A.2.5) is strictly positive for sufficiently small ε and δ. Note that

$$\overline{\mu} - \eta\,\underline{c}_A - \lambda\Big[\int_{\hat{s}_A}^1 \mu(s)ds - \eta(1-\hat{s}_A)\Big] = \overline{\mu} - \lambda\int_{\hat{s}_A}^1 \mu(s)\,ds + (\lambda-1)\,\eta\,\underline{c}_A,$$

Since $\overline{\mu} = \int_0^1 \mu(s)\,ds > \int_{\hat{s}_A}^1 \mu(s)\,ds$ (which follows from the fact that $\hat{s}_A < 1$), there exists $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, $\tilde{\pi}_A > \pi_A$. $\square$ ■

A.3 Proof of Lemma 2

"If" part. We show that the strategy profile satisfying the stated conditions forms a best response. First, define $\sigma_i(v;t) := t\tilde{\sigma}_i(v) + (1-t)\sigma_i(v)$ for $t \in [0,1]$ and for $i = A, B$. Let $\hat{s}_i(t)$ be the cutoff state in equilibrium for given $\sigma_i(v;t)$, and $S_i(t)$ be such that $S_A(t) := \{s \,|\, s \geq \hat{s}_A(t)\}$ and $S_B(t) := \{s \,|\, s \leq \hat{s}_B(t)\}$. Next, let

$$\begin{aligned} W(t, \hat{s}_i(t)) := &\int_0^1 v\sigma_i(v;t)[1-\sigma_i(v) + \overline{\mu}_i\,\sigma_j(v)]dG(v) \\ &- \lambda\int_{S_i(t)}\Big[\int_0^1 \sigma_i(v;t)[1-\sigma_j(v) + \mu_i(s)\sigma_j(v)]dG(v) - \kappa\Big]ds, \end{aligned}$$

where $\overline{\mu}_i = \mathbb{E}[\mu_i(s)]$, and denote it by $V(t) := W(t, \hat{s}_i(t))$. Observe that $\pi_i(\tilde{\sigma}_i) = V(1)$ and $\pi_i(\sigma_i) = V(0)$. Therefore, the proof is completed by showing $V(1) \leq V(0)$. Because $\tilde{\sigma}_i(\cdot)$ is arbitrary, this proves that $\sigma_i(\cdot)$ is a best response for a given $\sigma_j(\cdot)$, where $j \neq i$. To do this, we establish the following lemmas.

Lemma A2. *$V(\cdot)$ is concave in t for any $t \in [0,1]$.*

Proof. Observe that $\sigma_i(\cdot;t)$ is linear in t, clearly. Hence, it suffices to show that π_i is concave in σ_i because if so, we have for any $\eta \in [0,1]$ and $t, t' \in [0,1]$,

$$\begin{aligned} V(\eta\,t + (1-\eta)\,t') &= \pi_i(\sigma_i(v;\eta\,t + (1-\eta)\,t')) = \pi_i(\eta\,\sigma_i(v;t) + (1-\eta)\,\sigma_i(v;t')) \\ &\geq \eta\,\pi_i(\sigma_i(v;t)) + (1-\eta)\,\pi_i(\sigma_i(v;t')) = \eta V(t) + (1-\eta)\,V(t'), \end{aligned}$$

where the second equality follows from the linearity of $\sigma_i(\cdot;t)$. For the concavity of π_i, recall that

$$\begin{aligned} \pi_i = &\int_0^1 \Big[\int_0^1 v\sigma_i(v)[1-\sigma_j(v) + \mu_i(s)\sigma_j(v)]dG(v)\Big]ds \\ &- \lambda\int_0^1 \max\Big\{\int_0^1 \sigma_i(v)[1-\sigma_j(v) + \mu_i(s)\sigma_j(v)]dG(v) - \kappa, 0\Big\}\,ds. \end{aligned} \tag{A.3.1}$$

Consider any feasible σ_i and σ_i'. Note that the first part of (A.3.1) is linear in σ_i clearly, and the second part is convex in σ_i, since for $\eta \in [0,1]$,

$$\begin{aligned}
&\max\left\{\int_0^1 [\eta\sigma_i(v) + (1-\eta)\sigma_i'(v)][1-\sigma_j(v)+\mu_i(s)\sigma_j(v)]dG(v) - \kappa, 0\right\} \\
&= \max\left\{\eta\Big[\int_0^1 \sigma_i(v)[1-\sigma_j(v)+\mu_i(s)\sigma_j(v)]dG(v) - \kappa\Big]\right. \\
&\quad \left. + (1-\eta)\Big[\int_0^1 \sigma_i'(v)[1-\sigma_j(v)+\mu_i(s)\sigma_j(v)]dG(v) - \kappa\Big], 0\right\} \\
&\leq \eta \max\left\{\int_0^1 \sigma_i(v)[1-\sigma_j(v)+\mu_i(s)\sigma_j(v)]dG(v) - \kappa, 0\right\} \\
&\quad + (1-\eta)\max\left\{\int_0^1 \sigma_i'(v)[1-\sigma_j(v)+\mu_i(s)\sigma_j(v)]dG(v) - \kappa, 0\right\}.
\end{aligned}$$

Therefore, we have $\pi_i(\eta\sigma_i + (1-\eta)\sigma_i') \geq \eta\, \pi_i(\sigma_i) + (1-\eta)\pi_A(\sigma_i')$. ∎

Lemma A3. $V'(0) \leq 0$.

Proof. Observe that

$$V'(t) = W_1(t, \hat{s}_i(t)) + W_2(t, \hat{s}_i(t))\hat{s}_i'(t),$$

where

$$W_1(t, \hat{s}_i(t)) = \int_0^1 (\tilde{\sigma}_i(v) - \sigma_i(v))\Big[v[1-\sigma_j(v) + \overline{\mu}_i\sigma_j(v)] - \lambda\int_{S_i(t)}[1-\sigma_j(v)+\mu_i(s)\sigma_j(v)]ds\Big]\, dG(v)$$

and

$$W_2(t, \hat{s}_i(t)) = \lambda\Big[\int_0^1 \sigma_i(v;t)[1-\sigma_j(v)+\mu_i(\hat{s}_i(t))\sigma_j(v)]\, dG(v) - \kappa\Big].$$

Notice that $W_2(0, \hat{s}_i(0)) = 0$ by definition of $\hat{s}_i$. Therefore, we have

$$V'(0) = W_1(0, \hat{s}_i(0)) = \int_0^1 (\tilde{\sigma}_i(v) - \sigma_i(v))H_i(v, \sigma_j(v))\, dG(v) \leq 0,$$

where the inequality holds since if $H_i(v, \sigma_j(v)) > 0$ for some v, then $\sigma_i(v) = 1$ and $\tilde{\sigma}_i(v) \leq 1$; if $H_i(v, \sigma_j(v)) < 0$ for some v, then $\sigma_i(v) = 0$ and $\tilde{\sigma}_i(v) \geq 0$; and $H_i(v, \sigma_j(v)) = 0$ otherwise. ∎

Now, note that

$$\pi_i(\tilde{\sigma}_i) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_i(\sigma_i),$$

where the first inequality follows from the concavity of $V(\cdot)$ and the second inequality follows from Lemma A3. This completes the proof.

"Only if" part. We now show that in any competitive equilibrium, the strategy profile must satisfy the stated conditions. Let $\mathcal{V}_+ := \{v \,|\, H_i(v, \sigma_j(v)) > 0\}$ and $\mathcal{V}_- := \{v \,|\, H_i(v, \sigma_j(v)) < 0\}$. Suppose to the contrary that in equilibrium, $\sigma_i(\cdot)$ doest not satisfy either *(i)* or *(ii)* (or both); that is, either $\sigma_i(v) < 1$ for $v \in \mathcal{V}_+$, or $\sigma_i(v) > 0$ for $v \in \mathcal{V}_-$ (or both). Consider a deviating strategy $\tilde{\sigma}_i(\cdot)$ such that $\tilde{\sigma}_i(v) = 1$ for every $v \in \mathcal{V}_+$, $\tilde{\sigma}_i(v) = 0$ for every $v \in \mathcal{V}_-$, and $\tilde{\sigma}_i(v) = \sigma_i(v)$ for all other v's. Now, define $\sigma_i(v;t) := t\tilde{\sigma}_i(v) + (1-t)\sigma_i(v)$ for $t \in [0,1]$ and $V(t)$ similar as above. In what follows, we show that $V'(0) > 0$ so there exists $\alpha(\cdot;t)$ for small t that will be profitable. To see this, observe that

$$\begin{aligned}
V'(0) &= W_1(0, \hat{s}_A(0)) \\
&= \int_{\mathcal{V}_+} (\tilde{\sigma}_i(v) - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) + \int_{\mathcal{V}_-} (\tilde{\sigma}_i(v) - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) \\
&\quad + \int_{\mathcal{V}\setminus\{\mathcal{V}_+\cup\mathcal{V}_-\}} (\tilde{\alpha}(v) - \alpha(v)) H_i(v, \sigma_j(v)) dG(v) \\
&= \int_{\mathcal{V}_+} (1 - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) - \int_{\mathcal{V}_-} \sigma_i(v) H_i(v, \sigma_j(v)) dG(v) \\
&> 0,
\end{aligned}$$

where the last equality follows from the construction of $\tilde{\sigma}_i(\cdot)$, and the inequality holds since $\tilde{\sigma}_i(v) = 1 > \sigma_i(v)$, $H_i(v, \sigma_j(v)) > 0$ for $v \in \mathcal{V}_+$, and $\tilde{\sigma}_i(v) = 0 < \sigma_i(v)$, $H_i(v, \sigma_j(v)) < 0$ for $v \in \mathcal{V}_-$.

A.4 Proof of Lemma 3

Observe that $H_i(\cdot, x)$ is strictly increasing in v, since for $v' > v$,

$$H_i(v', x) - H_i(v, x) = (1 - x + \mathbb{E}[\mu_i(s)]\, x)(v' - v) > 0,$$

where the inequality holds since $\mathbb{E}[\mu_i(s)] > 0$ since $\mu(\cdot)$ is strictly increasing in s.

Next, $H_i(v, x)$ satisfies the strict single crossing property with respect to x; that is, if $H_i(v, x) \leq 0$ for some $x \in (0,1)$, then $H_i(v, x') < 0$ for any $x' > x$. Suppose for any $x \in (0,1)$,

$$H_i(v,x) = (1-x)\big(v - \lambda \operatorname{Prob}(s \in S_i)\big) + x\,\mathbb{E}[\mu_i(s)]\Big(v - \lambda \operatorname{Prob}(s \in S_i)\frac{\mathbb{E}[\mu_i(s)|s \in S_i]}{\mathbb{E}[\mu_i(s)]}\Big) \leq 0. \quad \text{(A.4.1)}$$

Consider any $x' > x$. If $v < \lambda \operatorname{Prob}(s \in S_i)$, then

$$H_i(v,x') = (1-x')\big(v - \lambda \operatorname{Prob}(s \in S_i)\big) + x'\,\mathbb{E}[\mu_i(s)]\Big(v - \lambda \operatorname{Prob}(s \in S_i)\frac{\mathbb{E}[\mu_i(s)|s \in S_i]}{\mathbb{E}[\mu_i(s)]}\Big) < 0,$$

where the inequality follows from (A.4.1) and the facts that $x' > x$ and $\mathbb{E}[\mu_i(s)|s \in S_i] > \mathbb{E}[\mu_i(s)]$. If $v \geq \lambda \operatorname{Prob}(s \in S_i)$, then

$$\begin{aligned}
H_i(v,x) - H_i(v,x') &= (x' - x)\Big[v\big(1 - \mathbb{E}[\mu_i(s)]\big) - \lambda \operatorname{Prob}(s \in S_i)\big(1 - \mathbb{E}[\mu_i(s)|s \in S_i]\big)\Big] \\
&> (x' - x)\big(v - \lambda \operatorname{Prob}(s \in S_i)\big)\big(1 - \mathbb{E}[\mu_i(s)|s \in S_i]\big)
\end{aligned}$$

$$\geq 0,$$

where the first inequality holds since $x' > x$ and $\mathbb{E}[\mu_i(s)|s \in S_i] > \mathbb{E}[\mu_i(s)]$, and the second inequality holds since $v \geq \lambda\,\text{Prob}(s \in S_i)$. Since $H_i(v, x) \leq 0$, we thus have $H_i(v, x') < 0$.

A.5 Proofs of Theorem 3

Suppose to the contrary that $\hat{v} \leq \check{v}$ in a competitive equilibrium. Suppose further that

$$\underline{v}_B < \overline{v}_B \leq \underline{v}_A < \overline{v}_A, \tag{A.5.1}$$

without loss of generality, where the first and the last strict inequalities hold since $(\hat{s}_A, \hat{s}_B) \in (0,1)^2$ by Lemma 1-(*iii*). Note that we must have $\overline{v}_A \in (0,1)$ in equilibrium, since if $\overline{v}_A = 1$, then $m_A(s) = 0$ for all s, and if $\overline{v}_A = 0$, then $\underline{v}_B = \overline{v}_B = \underline{v}_A = \overline{v}_A = 0$, so $m_B(s) = 0$ for all s. In equilibrium, we have

$$m_A(\hat{s}_A) = \mu(\hat{s}_A)[1 - G(\overline{v}_A)] = \kappa, \tag{A.5.2}$$

and

$$m_B(\hat{s}_B) = (1 - \mu(\hat{s}_B))[1 - G(\overline{v}_A)] + G(\overline{v}_A) - G(\underline{v}_B) = \kappa. \tag{A.5.3}$$

From (A.5.2), $1 - G(\overline{v}_A) = \frac{\kappa}{\mu(\hat{s}_A)}$. Substituting this into (A.5.3), we have

$$G(\overline{v}_A) - G(\underline{v}_B) = \kappa\left(\frac{\mu(\hat{s}_A) + \mu(\hat{s}_B) - 1}{\mu(\hat{s}_A)}\right).$$

Since $\underline{v}_B < \overline{v}_A$, this implies

$$\mu(\hat{s}_A) + \mu(\hat{s}_B) > 1 \Leftrightarrow \mu(\hat{s}_B) > 1 - \mu(\hat{s}_A) = \mu(1 - \hat{s}_A),$$

where the last equality follows from the symmetry of $\mu(\cdot)$. Since $\mu(\cdot)$ is strictly increasing, we have $\hat{s}_B > 1 - \hat{s}_A$, and so $\underline{v}_B = \lambda\,\hat{s}_B > \lambda\,(1 - \hat{s}_A) = \underline{v}_A$ which contradicts (A.5.1).

A.6 Proof of Theorem 2

Step 1: Existence of a profile of admission strategies (σ_A, σ_B) that forms local best responses.

We first establish existence of a profile of admission strategies $(\sigma_A, \sigma_B) : [0,1]^2 \to [0,1]^2$ such that for each $v \in [0,1]$, $\sigma_i(v)$ is given by (3.1) for $i = A, B$.[30] Now, fix any $\hat{s} = (\hat{s}_A, \hat{s}_B) \in S \equiv [0,1]^2$ and consider the resulting profile $(\sigma_A(\cdot;\hat{s}), \sigma_B(\cdot;\hat{s}))$. This strategy profile in turn induces the mass

[30] One can also structure the strategy profile to satisfy the requirements of an A-priority equilibrium by replacing $\sigma_A^o(\cdot)$ and $\sigma_B^o(\cdot)$ with 1 and 0, respectively, and of a B-priority equilibrium by replacing them with 0 and 1, respectively.

of students enrolling in each college i:

$$m_i(s;\hat{s}) = \int_0^1 \sigma_i(v;\hat{s})[1-\sigma_j(v;\hat{s}) + \mu_i(s)\sigma_j(v;\hat{s})]\,dG(v).$$

Observe that $m_i(\cdot;\hat{s})$ and $m_B(\cdot;\hat{s})$ in turn yield a new profile of cutoff states:

$$\tilde{s}_A = \inf\{s \in [0,1] | m_A(s;\hat{s}) - \kappa > 0\}, \tag{A.6.1}$$

if the set in the RHS is nonempty, or else $\tilde{s}_A \equiv 1$, and

$$\tilde{s}_B = \sup\{s \in [0,1] | m_B(s;\hat{s}) - \kappa > 0\}, \tag{A.6.2}$$

if the set in the RHS is nonempty, or else $\tilde{s}_B \equiv 0$.

Next, define a mapping T such that $T(\hat{s}) = \tilde{s}$, where $\tilde{s} = (\tilde{s}_A, \tilde{s}_B)$ is given by (A.6.1) and (A.6.2). The next lemma shows that T is continuous. Therefore, it has a fixed point by the Brouwer's fixed point theorem. From the construction of T, it is immediate that given the fixed point, say $\hat{s}^* = (\hat{s}^*_A, \hat{s}^*_B)$, the profile $(\sigma_A(\cdot;\hat{s}^*), \sigma_B(\cdot;\hat{s}^*))$ satisfies the local incentives.

Lemma A4. *$T(\cdot)$ is continuous in s for $s \in S$.*

Proof. Note that $\overline{v}_A$ and $\underline{v}_A$ are continuous in $\hat{s}_A$, and $\overline{v}_B$ and $\underline{v}_B$ are continuous in $\hat{s}_B$. Now, let

$$\underline{v} := \min\{\underline{v}_A, \underline{v}_B\}, \quad \check{v} := \max\{\underline{v}_A, \underline{v}_B\}, \quad \hat{v} := \min\{\overline{v}_A, \overline{v}_B\}, \quad \overline{v} := \max\{\overline{v}_A, \overline{v}_B\}.$$

For any given $\hat{s}$, $T(\hat{s}) = \tilde{s}$ is given by (A.6.1) and (A.6.2). Consider now any $\hat{s}' = (\hat{s}'_A, \hat{s}'_B) \in S$, where $\hat{s}' \neq \hat{s}$. Then, $\sigma'_A \equiv \sigma_A(\cdot;\hat{s}')$ and $\sigma'_B \equiv \sigma_B(\cdot;\hat{s}')$ are defined by (3.1). Let

$$\underline{v}' := \min\left\{\underline{v}'_A, \underline{v}'_B\right\}, \quad \check{v}' := \max\left\{\underline{v}'_A, \underline{v}'_B\right\}, \quad \hat{v}' := \min\left\{\overline{v}'_A, \overline{v}'_B\right\}, \quad \overline{v}' := \max\left\{\overline{v}'_A, \overline{v}'_B\right\}.$$

Again, $\tilde{s}' = (\tilde{s}'_A, \tilde{s}'_B) \in S$ is defined by T through (A.6.1) and (A.6.2). Next, let

$$\begin{aligned} &v_1 := \min\left\{\underline{v}, \underline{v}'\right\}, \quad v_2 := \max\left\{\underline{v}, \underline{v}'\right\}, \quad v_3 := \min\left\{\check{v}, \check{v}'\right\}, \quad v_4 := \max\left\{\check{v}, \check{v}'\right\}, \\ &v_5 := \min\left\{\hat{v}, \hat{v}'\right\}, \quad v_6 := \max\left\{\hat{v}, \hat{v}'\right\}, \quad v_7 := \min\left\{\underline{v}, \overline{v}'\right\}, \quad v_8 := \max\left\{\overline{v}, \overline{v}'\right\}, \end{aligned}$$

and consider a partition of $[0,1]$ such that

$$\mathcal{V}_1 = (\cup_{i=2,4,6,8}[v_{i-1}, v_i]) \cap [0,1], \quad \mathcal{V}_2 = [v_4, v_5] \cap [0,1], \quad \mathcal{V}_3 = [0,1] \setminus (\mathcal{V}_1 \cup \mathcal{V}_2).$$

Consider σ_i and σ_i' for $i = A, B$. For any $v \in [0,1]$, we have

$$\int_0^1 \left|\sigma_i'(v) - \sigma_i(v)\right| \, dG(v) = \sum_{k=1}^{3} \int_0^1 \left|\sigma_i'(v) - \sigma_i(v)\right| \mathbb{1}_{\mathcal{V}_k}(v) \, dG(v),$$

where $\mathbb{1}_{\mathcal{V}_k}(v)$ is 1 if $v \in \mathcal{V}_k$ or 0 otherwise.

Observe, first, that by the continuity of $\underline{v}_i$ and $\overline{v}_i$, there is a $\delta_1 > 0$ such that for any $\varepsilon > 0$, if $\|\hat{s}' - \hat{s}\| < \delta_1$, then

$$\int_0^1 \mathbb{1}_{\mathcal{V}_1}(v) \, dG(v) < \frac{\varepsilon}{6}. \tag{A.6.3}$$

Second, for any $v \in \mathcal{V}_2$, the continuity of $\sigma_i^o(\cdot)$, given by

$$\sigma_i^o(v) := \frac{v - \lambda \operatorname{Prob}(s \in S_j)}{v\,(1 - \mathbb{E}[\mu_i(s)]) - \lambda \operatorname{Prob}(s \in S_j)\Big(1 - \mathbb{E}[\mu_i(s) | s \in S_j]\Big)},$$

implies that there is δ_2 such that $\|\hat{s}' - \hat{s}\| < \delta_2$ implies

$$\left|\sigma_i'(v) - \sigma_i(v)\right| = \left|\sigma_i^{o'}(v) - \sigma_i^o(v)\right| < \frac{\varepsilon}{6}, \tag{A.6.4}$$

Lastly, for any $v \in \mathcal{V}_3$, $\alpha'(v)$ and $\alpha(v)$ are either 0 or 1 at the same time, hence we have that

$$\left|\sigma_i'(v) - \sigma_i(v)\right| = 0. \tag{A.6.5}$$

Now, let $\delta := \min\{\delta_1, \delta_2\}$ and suppose $\|\hat{s}' - \hat{s}\| < \delta$. Then, we have

$$\begin{aligned}\int_0^1 \left|\sigma_i'(v) - \sigma_i(v)\right| \, dG(v) &= \int_0^1 \left|\sigma_i'(v) - \sigma_i(v)\right| \mathbb{1}_{\mathcal{V}_1}(v) \, dG(v) + \int_0^1 \left|\sigma_i^{o'}(v) - \sigma_i^o(v)\right| \mathbb{1}_{\mathcal{V}_2}(v) \, dG(v) \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3},\end{aligned} \tag{A.6.6}$$

where the equality follows from (A.6.5) and the inequality follows from (A.6.3) and (A.6.4).

Observe that

$$\begin{aligned}&\Big|m_i(s; \hat{s}') - m_i(s; \hat{s})\Big| \\ &= \left|\int_0^1 \sigma_i'(v)[1 - \sigma_j'(v) + \mu_i(s)\sigma_j'(v)] \, dG(v) - \int_0^1 \sigma_i(v)[1 - \sigma_j(v) + \mu_i(s)\sigma_j(v)] \, dG(v)\right| \\ &= \left|\int_0^1 \Big[[\sigma_i'(v) - \sigma_i(v)] - (1 - \mu_i(s))[\sigma_i'(v)\sigma_j'(v) - \sigma_i(v)\sigma_j(v)]\Big] \, dG(v)\right| \\ &\le \int_0^1 \left|\sigma_i'(v) - \sigma_i(v)\right| \, dG(v) + (1 - \mu_i(s)) \int_0^1 \left|\sigma_i'(v)\sigma_j'(v) - \sigma_i(v)\sigma_j(v)\right| \, dG(v)\end{aligned} \tag{A.6.7}$$

The first part of (A.6.7) is smaller than $\varepsilon/3$ by (A.6.6). The second part of (A.6.7) is

$$\begin{aligned}\int_0^1 \left|\sigma_i(v)\sigma_j'(v) - \sigma_i(v)\sigma_j(v)\right| dG(v) &= \int_0^1 \left|\sigma_i'(v)\sigma_j'(v) - \sigma_i'(v)\sigma_j(v) + \sigma_i'(v)\sigma_j(v) - \sigma_i(v)\sigma_j(v)\right| dG(v) \\ &\leq \int_0^1 \left|\sigma_j'(v) - \sigma_j(v)\right| dG(v) + \int_0^1 \left|\sigma_i'(v) - \sigma_i(v)\right| dG(v) \\ &< \frac{2}{3}\varepsilon,\end{aligned}$$

where the first inequality holds since $\sigma_i'(v), \sigma_j(v) \leq 1$, and the last inequality follows from (A.6.6). Therefore, if $\|\hat{s}' - \hat{s}\| < \delta$, then

$$\left|\int_0^1 \sigma_i'(v)[1 - \sigma_j'(v) + \mu_i(s)\sigma_j'(v)]dG(v) - \int_0^1 \sigma_i(v)[1 - \sigma_j(v) + \mu_i(s)\sigma_j(v)]dG(v)\right| < \varepsilon. \quad \text{(A.6.8)}$$

Hence, we conclude that there is $\delta > 0$ such that for any $\varepsilon > 0$, if $\|\hat{s}' - \hat{s}\| < \delta$, then $\|\tilde{s}' - \tilde{s}\| < \varepsilon$. Since $\hat{s}$ is chosen arbitrary, T is continuous on S. ■

Step 2: $\mathcal{V}_{AB}$ has a positive measure in the strategy profile identified in Step 1.

Suppose to the contrary that $\mathcal{V}_{AB}$ has measure zero. Then, $\hat{s}_B^* = 0$ and $\hat{s}_A^* = 1$. But in that case, $H_A(v, 1) > 0$ and $H_B(v, 1) > 0$ for all v. Hence, $\overline{v}_A = \overline{v}_B = 0$. Therefore, we cannot have a non-competitive equilibrium.

A.7 Proof of Theorem 4

Proof of Part (i). Recall that there are cutoff states $(\hat{s}_A, \hat{s}_B)$ such that colleges have a mass of unfilled seats in a positive measure of states, $[0, \hat{s}_A)$ for A and $(\hat{s}_B, 1]$ for B, despite the fact that there are unmatched and acceptable students ($\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$ in Lemma 1-(ii)). By assigning those unmatched students to a college with excess capacity, both the students and college are better off. Thus, it is student, college and Pareto inefficient. ■

Proof of Part (ii). Suppose a competitive equilibrium exhibits strategic targeting; i.e., $\check{v} < \hat{v}$. Fix a state s such that $\mu(s) \neq 0, 1$. For those students in $[\check{v}, \hat{v}]$, there is a positive measure of students who are assigned to a college, say B, but prefer A, and their scores are higher than those of a positive measure of students who are assigned to A, even though both colleges prefer the high-score students. Moreover, students in $[\check{v}, \hat{v}]$ get no admission from either college with positive probabilities even if their scores are high. Thus, it entails justified envy for a positive measure of states for almost every state.

Suppose now a competitive equilibrium does not exhibit strategic targeting; i.e., $\hat{v} < \check{v}$. Let $\overline{v}_B < \underline{v}_A$, as depicted in Figure 3.4, without loss of generality, so students in $[\underline{v}_B, \overline{v}_A]$ admitted only by B and those in $(\overline{v}_A, 1]$ are admitted by both colleges. Observe that only the students who are not admitted by either college or admitted only by college B may have envies. However, the students whom they envy have higher scores. So, no justified envy arises in any state s, making the outcome fair. ■

Proof of Part (iii). Consider any non-competitive equilibrium. For each state s except $\mu(s) = 0$ or 1, the equilibrium must admit a positive measure of students who prefer A but are assigned to B and a positive measure of students who are assigned to A but have scores lower than those of the first group of students; that is, justified envy arises. Since justified envy arises for a positive measure of students for almost every state,[31] the outcome is unfair. Also, for almost every state, there must be a positive measure of students assigned to A but prefer B and a positive measure of students assigned to B but prefer A. Thus, the outcome is student inefficient.

Next, the equilibrium is college efficient. To see this, recall that in any non-competitive equilibrium, almost all top 2κ students are assigned to either college. Suppose to the contrary that for a given state, there is another assignment that makes both colleges weakly better off and at least one college strictly better off. Then, it must also admit almost all top 2κ students, or else at least one college is strictly worse off. Therefore, it is a reallocation of the initial assignment, hence if one college is strictly better off, then the other college must be strictly worse off. Thus, we reach a contraction. ■

Proof of Part (iv). Suppose that almost all top κ students are assigned to one college, and the next top κ students are assigned to the other college. Then, any change of assignments by positive measure of students will leave the former college strictly worse off, hence it is Pareto efficient.

Suppose this is not the case in a non-competitive equilibrium. Note that for a fixed s, there are some $\mathcal{V}_i', \mathcal{V}_i'' \subset \mathcal{V}_i$ and $\mathcal{V}_j' \subset \mathcal{V}_j$, $i \neq j$, all with positive measures, such that $v' < \hat{v} < v''$ whenever $v' \in \mathcal{V}_i'$, $v'' \in \mathcal{V}_i''$ and $\hat{v} \in \mathcal{V}_j'$. Let $i = A$ and $j = B$ without loss of generality. We can choose $\mathcal{V}_A'$, $\mathcal{V}_A''$ and $\mathcal{V}_B'$ that satisfy

$$\frac{\int_{\mathcal{V}_A' \cup \mathcal{V}_A''} v \, dG(v)}{\int_{\mathcal{V}_A' \cup \mathcal{V}_A''} 1 \, dG(v)} = \frac{\int_{\mathcal{V}_B'} v \, dG(v)}{\int_{\mathcal{V}_B'} 1 \, dG(v)} \tag{A.7.1}$$

and

$$(1 - \mu(s)) \int_{\mathcal{V}_A' \cup \mathcal{V}_A''} 1 \, dG(v) = \mu(s) \int_{\mathcal{V}_B'} 1 \, dG(v). \tag{A.7.2}$$

(If either (A.7.1) or (A.7.2) is violated, we can adjust $\mathcal{V}_A'$, $\mathcal{V}_A''$ and/or $\mathcal{V}_B'$ by adding or subtracting a positive mass of students.) Note that the LHS (resp. RHS) of (A.7.2) is the measure of students who prefer B (resp. A) in $\mathcal{V}_A' \cup \mathcal{V}_A''$ (resp. $\mathcal{V}_B'$). From (A.7.1), we have

$$\frac{\int_{\mathcal{V}_A' \cup \mathcal{V}_A''} v \, dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}_A' \cup \mathcal{V}_A''} 1 \, dG(v)} = \frac{\int_{\mathcal{V}_B'} v \, dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}_B'} 1 \, dG(v)}$$

[31] since $\mu(\cdot)$ is strictly increasing and continuous in s, $\mu(s) \in (0, 1)$ for almost every state.

$$\Leftrightarrow \frac{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v\, dG(v)}{\mu(s) \int_{\mathcal{V}'_B} 1\, dG(v)} = \frac{\int_{\mathcal{V}'_B} v\, dG(v)}{(1-\mu(s)) \int_{\mathcal{V}'_B} 1\, dG(v)}$$

$$\Leftrightarrow (1-\mu(s)) \int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v\, dG(v) = \mu(s) \int_{\mathcal{V}'_B} v\, dG(v),$$

where the first equivalence follows from (A.7.2). The last equivalence shows that the average value of students who prefer college B in $\mathcal{V}'_A \cup \mathcal{V}''_A$ is the same as that of students who prefer college A in $\mathcal{V}'_B$. Thus, in state s, a fraction $1-\mu(s)$ of students in $\mathcal{V}'_A \cup \mathcal{V}''_A$ who prefer B to A can be swapped with a fraction of $\mu(s)$ of students in $\mathcal{V}'_B$ who prefer A to B. This reassignment leaves both colleges the same in welfare and makes all students weakly better off and some positive measure of students strictly better off. Since this argument holds for all s except $\mu(s) = 0$ or 1, the outcome is Pareto inefficient. ■

A.8 Proofs of Theorem 5 and Existence of a Cutoff Equilibrium

A.8.1 Proof of Theorem 5

Suppose there is a cutoff equilibrium with strategy profiles (σ_A, σ_B) where $\sigma_i(v, e_i) = \mathbb{1}_{\{e_i \geq \eta(v)\}}$, $i = A, B$, for some $\eta_i(\cdot)$ which is nonincreasing. Since $\partial U_i / \partial e_i > 0$, by the Implicit Function Theorem, $H_i(v, e_i, \overline{\sigma}_j(v)) = 0$, $j \neq i$, implicitly defines $\eta_i(v)$. Since $\mathbb{E}[\mu_i(s) | s \in S_i] > \mathbb{E}[\mu_i(s)]$, we must have

$$1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)|s \in S_i]\, \overline{\sigma}_j(v) > 1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\, \overline{\sigma}_j(v).$$

Then, $H_i(v, \eta_i(v), \overline{\sigma}_j(v)) = 0$ implies that by (4.1),

$$U_i(v, \eta_i(v)) > \lambda \,\text{Prob}(s \in S_i). \tag{A.8.1}$$

Next, totally differentiate H_i to obtain:

$$\begin{aligned} &\frac{\partial U_i(v, \eta_i(v))}{\partial v} + \frac{\partial U_i(v, \eta_i(v))}{\partial e_i} \eta_i'(v) \\ =& \frac{\left[U_i(v, \eta_i(v))(1 - \mathbb{E}[\mu_i(s)]) - \lambda \,\text{Prob}(s \in S_i)(1 - \mathbb{E}[\mu_i(s)|s \in S_i])\right] \overline{\sigma}_j'(v)}{1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\, \overline{\sigma}_j(v)}. \end{aligned} \tag{A.8.2}$$

Since college j adopts a cutoff strategy, $\overline{\sigma}_j(v) = 1 - X_j(\eta_j(v)|v)$, we have that

$$\overline{\sigma}_j'(v) = -x_j(\eta_j(v)|v)\eta_j'(v) - \frac{\partial X_j(\eta_j(v)|v)}{\partial v} > 0, \tag{A.8.3}$$

where the inequality holds since $\eta_j'(v) \leq 0$ and $\frac{\partial}{\partial v}X_j(e|v) < 0$.[32] Further, $\mathbb{E}[\mu_i(s)] < \mathbb{E}[\mu_i(s)|s \in S_i] \leq 1$, so it follows from (A.8.1) that the RHS of (A.8.2) is strictly positive for any v such that $\eta_i(v) \in (0,1)$. Hence, for all v,

$$-\eta_i'(v) \leq \frac{\partial U_i(v,\eta_i(v))/\partial v}{\partial U_i(v,\eta_i(v))/\partial e_i},$$

and the inequality is strict for a positive measure of v.

A.8.2 Existence of a Cutoff Equilibrium

Step 1: Existence of a profile of cutoff strategies for A and B.

Define

$$\delta := \max_{v,e_A,e_B}\left\{x_A(e_A|v)\left(\frac{\frac{\partial U_A(v,e_A)}{\partial v}}{\frac{\partial U_A(v,e_A)}{\partial e_A}}\right) - \frac{\partial X_A(e_A|v)}{\partial v} \bigvee x_B(e_B|v)\left(\frac{\frac{\partial U_B(v,e_B)}{\partial v}}{\frac{\partial U_B(v,e_B)}{\partial e_B}}\right) - \frac{\partial X_B(e_B|v)}{\partial v}\right\}.$$

Let $\mathcal{M}$ be the set of Lipschitz-continuous function from [0,1] to [0,1] with Lipschitz bound given by δ. We define an operator $T : [0,1]^2 \times \mathcal{M}^2 \to [0,1]^2 \times \mathcal{M}^2$ as follows.

For any $(\hat{s}_A, \hat{s}_B, \overline{\sigma}_A, \overline{\sigma}_B) \in [0,1]^2 \times \mathcal{M}^2$, the third component of $T(\hat{s}_A, \hat{s}_B, \overline{\sigma}_A, \overline{\sigma}_B)$ is a function $\overline{\alpha}$ defined as follows. First, $\eta_A(v)$ is implicitly defined via $H_A(v, \eta_A(v), \overline{\sigma}_B(v)) = 0$ according to the Implicit Function Theorem (since $\partial U_A/\partial e_A > 0$). For v such that $\eta_A(v) \in (0,1)$, the same argument as in the proof of Theorem 5 implies that

$$-\eta_A'(v) \leq \frac{\partial U_A(v,\eta_A(v))/\partial v}{\partial U_A(v,\eta_A(v))/\partial e_A}. \tag{A.8.4}$$

We now define $\sigma_A(v,e) := \mathbb{1}_{\{e_A \geq \eta_A(v)\}}$. Let $\alpha(v) = \mathbb{E}[\sigma_A(v,e_A)|v]$. Then,

$$\overline{\alpha}(v) = 1 - X_A(\eta_A(v)|v)$$

and

$$\overline{\alpha}'(v) = -x_A(\eta_A(v)|v)\eta_A'(v) - \frac{\partial X_A(\eta_A(v)|v)}{\partial v} \leq x_A(\eta_A(v)|v)\frac{\frac{\partial U_A(v,\eta_A(v))}{\partial v}}{\frac{\partial U_A(v,\eta_A(v))}{\partial e_A}} - \frac{\partial X_A(\eta_A(v)|v)}{\partial v} \leq \delta,$$

where the first inequality follows from (A.8.4). It thus follows that $\overline{\alpha} \in \mathcal{M}$.

The fourth component of $T(\hat{s}_A, \hat{s}_B, \overline{\sigma}_A, \overline{\sigma}_B)$, labeled $\overline{\beta}$, is analogously constructed via $e_B =$

[32] When v and e_i are independent, $\overline{\sigma}_j'(v) = -x_j(\eta_j(v))\eta_j'(v) \geq 0$. This implies that each college under-weights a students' common performance and over-weights her non-common performance at least weakly and one college does so strictly. Further, together with college j's condition (total differentiation of H_j), one can show that $\overline{\sigma}_j'(v) > 0$ for a positive measure of v, generically.

$\eta_B(v)$ determined implicitly by $H_B(v, \eta_B(v), \overline{\sigma}_A) = 0$, analogously, and belongs to $\mathcal{M}$.

The first two components $(\hat{s}'_A, \hat{s}'_B)$ are determined by the $m_A(\hat{s}'_A) = m_B(\hat{s}'_B) = \kappa$, much as in the earlier proofs, using σ_A and σ_B, along with $(\hat{s}_A, \hat{s}_B)$ as input.

In sum, the operator T maps from $(\hat{s}_A, \hat{s}_B, \overline{\sigma}_A, \overline{\sigma}_B) \in [0,1]^2 \times \mathcal{M}^2$ to $(\hat{s}'_A, \hat{s}'_B, \overline{\alpha}, \overline{\beta}) \in [0,1]^2 \times \mathcal{M}^2$. By Arzela-Ascoli theorem, the set $\mathcal{M}$ endowed with sup norm topology is compact, bounded and convex. Hence, the same holds for the Cartesian product $[0,1]^2 \times \mathcal{M}^2$. Following the techniques used in Appendix B, the mapping T is continuous (with respect to sup norm). Hence, by the Schauder's theorem, T has a fixed point. The fixed point then identifies a profile of cutoff strategies (σ_A, σ_B) via $\sigma_A(v, e_A) = \mathbb{1}_{\{e_A \geq \eta_A(v)\}}$ and $\sigma_B(v, e_B) = \mathbb{1}_{\{e_B \geq \eta_B(v)\}}$. See Appendix B for technical details.

Step 2: The cutoff strategies identified in Step 1 form an equilibrium under a condition.

Consider the following conditions:

$$\left(\frac{\partial U_i(v, e_i)/\partial v}{U_i(v, e_i)} + \frac{\partial X_j(\eta_j(v)|v)}{\partial v} \Psi_i(s) \right) \frac{\partial U_j(v, \tilde{e})/\partial e_j}{\partial U_B(v, \tilde{e})/\partial v} \geq x_j(\tilde{e}|v) \Psi_i(s)$$

for all v, e_i, $\tilde{e}$, s, where

$$\Psi_i(s) := \frac{\mathbb{E}[\mu_i(s)|s \in S_i] - \mathbb{E}[\mu_i(s)]}{\mathbb{E}[\mu_i(s)|s \in S_i] \, \mathbb{E}[\mu_i(s)]}. \tag{A.8.5}$$

Since the RHS of each inequality is bounded by some constant, the conditions can be interpreted as requiring that each college values the non-common performance sufficiently highly. For instance, if $U_i(v, e_i) = (1-\rho)v + \rho e_i$ for all $i = A, B$, then the LHS of each inequality will be no less than $\rho - \gamma$, where $\gamma := \max_{v, e_A, e_B} \left\{ \left| \frac{\partial X_A(e_A|v)}{\partial v} \right|, \left| \frac{\partial X_B(e_B|v)}{\partial v} \right| \right\}$, whenever $\mathbb{E}[\mu_i(s)] \geq \rho$. So the condition will hold if the RHS of each inequality is less than $\rho - \gamma$.

We now show the cutoff strategies identified by Step 1 form an equilibrium, given this condition. For the proof, it suffices to show that

$$\frac{\partial H_i(v, e_i, \overline{\sigma}_j(v))}{\partial v} \geq 0 \text{ whenever } H_i(v, e_i, \overline{\sigma}_j(v)) = 0.$$

This result holds since

$$\begin{aligned}
&\operatorname{sgn}\left(\frac{\partial H_i(v, e_i, \overline{\sigma}_j(v))}{\partial v} \right) \\
&= \frac{\partial U_i(v, e_i)}{\partial v} - \frac{[U_i(v, e_i)(1 - \mathbb{E}[\mu_i(s)]) - \lambda \operatorname{Prob}(s \in S_i)(1 - \mathbb{E}[\mu_i(s)|s \in S_i])]}{1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\overline{\sigma}_j(v)} \overline{\sigma}'_j(v) \\
&= \frac{\partial U_i(v, e_i)}{\partial v} - \frac{U_i(v, e_i)}{1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\overline{\sigma}_j(v)} \times \\
&\qquad \left((1 - \mathbb{E}[\mu_i(s)]) - \frac{1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\overline{\sigma}_j(v)}{1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)|s \in S_i]\overline{\sigma}_j(v)} (1 - \mathbb{E}[\mu_i(s)|s \in S_i]) \right) \overline{\sigma}'_j(v) \\
&= \frac{\partial U_i(v, e_i)}{\partial v} - U_i(v, e_i) \frac{\mathbb{E}[\mu_i(s)|s \in S_i] - \mathbb{E}[\mu_i(s)]}{(1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\overline{\sigma}_j(v))(1 - \overline{\sigma}_j(v) + \mathbb{E}[\mu_i(s)|s \in S_i]\overline{\sigma}_j(v))} \overline{\sigma}'_j(v)
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\partial U_i(v,e_i)}{\partial v} - U_i(v,e_i)\Psi_i(s)\overline{\sigma}_j'(v) \\
&= \frac{\partial U_i(v,e_i)}{\partial v} + U_i(v,e_i)\Psi_i(s)\Big(x_j(\eta_j(v)|v)\eta_j'(v) + \frac{\partial X_j(\eta_j(v)|v)}{\partial v}\Big) \\
&\geq \frac{\partial U_i(v,e_i)}{\partial v} - U_i(v,e_i)\Psi_i(s)\Big(x_j(\eta_j(v)|v)\frac{\partial U_j(v,\eta_j(v))/\partial v}{\partial U_j(v,\eta_j(v))/\partial e_j} - \frac{\partial X_j(\eta_j(v)|v)}{\partial v}\Big) \\
&\geq 0,
\end{aligned}
$$

where $\Psi_i(s)$ is given by (A.8.5), the second equality is obtained by substituting $H_i(v, e_i, \overline{\sigma}_j(v)) = 0$, the first inequality follows since $\mathbb{E}[\mu_i(s)], \mathbb{E}[\mu_i(s)|s \in S_i] \leq 1$, the penultimate equality follows from the fact that $\overline{\sigma}_j(v) = 1 - X_j(\eta_j(v)|v)$, the second inequality follows from (A.8.4), and the last inequality follows from the above conditions.

We last show that the identified strategies are nonincreasing in v. Note that

$$
\frac{d\eta_i(v)}{dv} = -\frac{\partial H_i(v,e_i,\overline{\sigma}_j(v))/\partial v}{\partial H_i(v,e_i,\overline{\sigma}_j(v))/\partial e_i} \leq 0,
$$

where the equality follows from the Implicit Function Theorem, and the inequality holds since $\partial H_i(v, e_i, \overline{\sigma}_j(v))/\partial v \geq 0$ and

$$
\frac{\partial H_i(v,e_i,\overline{\sigma}_j(v))}{\partial e_i} = \frac{\partial U_i(v,e_i)}{\partial e_i}\big(1 - \overline{\sigma}_j + \mathbb{E}[\mu_i(s)]\overline{\sigma}_j\big) > 0.
$$

A.9 Proofs of Lemma 4 and Theorems 7 and 8

It is convenient to define $T(y|\sigma) := y\, P_A(y|\sigma) - (1-y)\, P_B(y|s)$ for the proofs.

A.9.1 Proof of Lemma 4

Fix any σ. To prove the optimality of the cutoff strategy, we show that $T'(y|\sigma) > 0$ for any y. Note that

$$
\begin{aligned}
T'(y|\sigma) &= P_A(y|\sigma) + P_B(y|\sigma) + yP_A'(y|\sigma) - (1-y)P_B'(y|\sigma) \\
&\geq y\big[P_A(y|\sigma) + P_A'(y|\sigma)\big] + (1-y)\big[P_B(y|\sigma) - P_B'(y|\sigma)\big] \\
&= y\int_0^1 q_A(s)\big[l(s|y) + l_y(s|y)\big]ds + (1-y)\int_0^1 q_B(s)\big[l(s|y) - l_y(s|y)\big]ds.
\end{aligned}
$$

Observe that

$$
l(s|y) + l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds}\left[1 + \frac{k_y(y|s)}{k(y|s)} - \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds}\right] > \frac{k(y|s)}{\int_0^1 k(y|s)ds}(1-2\delta),
$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} > -\delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)} k(y|s)ds}{\int_0^1 k(y|s)ds} < \delta$$

because $\left|\frac{k_y(y|s)}{k(y|s)}\right| < \delta$. Similarly,

$$l(s|y) - l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds}\left[1 - \frac{k_y(y|s)}{k(y|s)} + \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds}\right] > \frac{k(y|s)}{\int_0^1 k(y|s)ds}(1-2\delta),$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} < \delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)} k(y|s)ds}{\int_0^1 k(y|s)ds} > -\delta.$$

Therefore, we have that $T'(y|\sigma) > 0$ since $\delta \leq \frac{1}{2}$.

It remains to show that there exists an equilibrium in cutoff strategy. Let $\hat{y}$ be a cutoff. Then, we have $n_A(s|\hat{y}) = \int_{\hat{y}}^1 k(y|s) = 1 - K(\hat{y}|s)$. Hence,

$$P_A(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{1-K(\hat{y}|s)}, 1\right\} l(s|y)ds \quad \text{and} \quad P_B(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} l(s|y)ds,$$

Now, let

$$T(y|\hat{y}) := yP_A(y|\hat{y}) - (1-y)P_B(y|\hat{y}).$$

Note that

$$T(0|\hat{y}) = -P_B(0|\hat{y}) = -\int_0^1 \min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} l(s|0)ds < 0,$$

where the inequality holds since $\min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} > 0$ and $l(s|0) \geq 0$ for all s, and $l(s|0) > 0$ for a positive measure of states. Similarly, $T(1|\hat{y}) > 0$. By the continuity of $T(\cdot|\hat{y})$, there is a $\tilde{y}$ such that $T(\tilde{y}|\hat{y}) = 0$. Moreover, such a $\tilde{y}$ is unique since $T'(y|\hat{y})\big|_{y=\tilde{y}} > 0$.

Next, let $\tau : [0,1] \to [0,1]$ be the map from $\hat{y}$ to $\tilde{y}$, which is implicitly defined by $T(\tau(\hat{y})|\hat{y}) = 0$ according to the Implicit Function Theorem (since $T'(y|\hat{y})\big|_{y=\tilde{y}} > 0$). Since $P_A(y|\cdot)$ is nondecreasing and $P_B(y|\cdot)$ is nonincreasing $\hat{y}$, $\tau(\cdot)$ is decreasing. Therefore, there is a fixed point such that $\tau(\hat{y}) = \hat{y}$, and hence there is $\hat{y}$ such that $T(\hat{y}|\hat{y}) = 0$.

A.9.2 Proof of Theorem 7

We first show $\hat{y} < 1$. Suppose $\hat{y} = 1$. Then, $n_A(s|1) = 1 - K(1|s) = 0$, so $P_A(y|\hat{y}) = 1$ for any y. Hence, $T(1|1) = P_A(1|1) = 1$, which contradicts the fact that $T(\hat{y}|\hat{y}) = 0$.

We now show that $\hat{y} > \frac{1}{2}$ whenever $\mu(s) > \frac{1}{2}$. Suppose to the contrary $\hat{y} \leq \frac{1}{2}$. We then have

$\frac{1}{2} < \mu(s) = 1 - K(\frac{1}{2}|s) \le 1 - K(\hat{y}|s)$, so $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. Therefore,

$$P_A(y|\hat{y}) - P_B(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{1-K(\hat{y}|s)}, 1\right\} l(s|y)ds - \int_0^1 \min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} l(s|y)ds \le 0. \quad \text{(A.9.1)}$$

Hence, if $\hat{y} < \frac{1}{2}$, then

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1-\hat{y})P_B(\hat{y}|\hat{y}) < \frac{1}{2}\big[P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})\big] \le 0, \quad \text{(A.9.2)}$$

where the first inequality holds since $\hat{y} < \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) < 0$, a contradiction. Suppose now $\hat{y} = \frac{1}{2}$. Notice that since $K(\hat{y}|s) < 1 - K(\hat{y}|s)$, we have $K(\frac{1}{2}|s) < \frac{1}{2} < 1 - \kappa$, where the the second inequality holds since $\kappa < \frac{1}{2}$. So, $\kappa/(1 - K(\frac{1}{2}|s)) < 1$. Therefore, the last inequality of (A.9.1) becomes strict, and hence

$$T(\hat{y}|\hat{y}) = \frac{1}{2}\big[P_A(\tfrac{1}{2}|\tfrac{1}{2}) - P_B(\tfrac{1}{2}|\tfrac{1}{2})\big] < 0,$$

a contradiction again.

Lastly, let $\mu(s) = \frac{1}{2}$. If $\hat{y} < \frac{1}{2}$, then $\frac{1}{2} = \mu(s) = 1 - K(\frac{1}{2}|s) < 1 - K(\frac{1}{2}|s)$, so we have $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. By (A.9.1) and (A.9.2), we reach a contradiction. If $\hat{y} > \frac{1}{2}$, then $\frac{1}{2} = \mu(s) = 1 - K(\frac{1}{2}|s) > 1 - K(\hat{y}|s)$ and so $K(\hat{y}|s) > 1 - K(\hat{y}|s)$. We then have $P_A(y|\hat{y}) - P_B(y|\hat{y}) \ge 0$ and

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1-\hat{y})P_B(\hat{y}|\hat{y}) > \frac{1}{2}\big[P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})\big] \ge 0,$$

where the first inequality holds since $\hat{y} > \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) > 0$, a contradiction again.

A.9.3 Proof of Theorem 8

For the first part of the theorem, observe that for a given s, justified envy arises whenever $c_A(s) \neq c_B(s)$ as depicted in Figure 5.1. We thus show that there is a positive measure of states in which $c_A(s) \neq c_B(s)$. Suppose to the contrary $c_A(s) = c_B(s)$ for almost all s. Recall that equilibrium admission cutoff of each college satisfies

$$G(c_A(s)) = \max\left\{1 - \frac{\kappa}{1-K(\hat{y}|s)}, 0\right\} \quad \text{and} \quad G(c_B(s)) = \max\left\{1 - \frac{\kappa}{K(\hat{y}|s)}, 0\right\}.$$

Since $G(\cdot)$ is strictly increasing, if $c_A(s) = c_B(s)$, then we must have either $n_i(s) < \kappa$ for all $i = A, B$ (so that $c_A(s) = c_B(s) = 0$) or $n_A(s) = n_B(s) \ge \kappa$.

First, we cannot have $n_i(s) < \kappa$ for all i in equilibrium, since this means that all applicants are admitted by either college, and this contradicts to $2\kappa < 1$. Second, suppose $n_A(s) = n_B(s) \ge \kappa$. This implies that $K(\hat{y}|s) = \frac{1}{2}$ for all s (recall that $n_A(s) = 1 - K(\hat{y}|s)$ and $n_B(s) = K(\hat{y}|s)$). However, by (5.1), we have $K(\hat{y}|s') < K(\hat{y}|s)$ for all $s' > s$. Therefore, we reach a contradiction again.

To see the second part of the theorem, recall that for given $\hat{y}$ in equilibrium, the mass of students applying to B is $K(\hat{y}|s)$. Thus, if there is a positive measure of states in which $K(\hat{y}|s) < \kappa$, college B faces under-subscription in such states. Therefore, the equilibrium outcome is inefficient.

A.10 Proof of Theorem 9

Suppose there is a symmetric equilibrium as described in the theorem. Then, colleges A and B will admit all acceptable students with $v > \tilde{v}$, where $\tilde{v}$ is such that each of A and B fills its capacity in the popular state, i.e., $s_a(1-\varepsilon)[1-G(\tilde{v})] = \kappa$ and $(1-s_b)(1-\varepsilon)[1-G(\tilde{v})] = \kappa$, and wait-lists the remaining students. College C will offer admissions to all of these students (i.e., those whose scores are above $\tilde{v}$), knowing that exactly measure ε^2 of them will accept its offer. It will also offer $\kappa - \varepsilon^2$ admissions to all students with $v \in [\hat{v}, \tilde{v}]$, where $\hat{v}$ is such that $G(\tilde{v}) - G(\hat{v}) = \kappa - \varepsilon^2$.

The students in $[\hat{v}, \tilde{v}]$ now have a choice to make. If a student accepts C, then she will get u'' for sure, but if she turns down C's offer, then with probability $1-\varepsilon$ the less popular one between A and B will offer an admission to her (assuming all other students admitted by C have accepted that offer), and the student will earn the payoff u if she happens to like the college, or u' otherwise. Since $u'' > (1-\varepsilon)u$, she will accept C.

Given this, consider now the incentive for deviation of college A. If it does not deviate, there will be seats left in the less popular state, equal to $\kappa - s_b(1-\varepsilon)[1-G(\hat{v})]$. Thus, A will fill those vacant seats with students whose scores are below $\hat{v}$. Thus, its payoff is

$$\begin{aligned}
\pi_A &= \frac{1}{2}s_a(1-\varepsilon)\int_{\tilde{v}}^{1} v\,dG(v) + \frac{1}{2}\Big[s_b(1-\varepsilon)\int_{\tilde{v}}^{1} v\,dG(v) + (1-\varepsilon)\int_{\check{v}}^{\hat{v}} v\,dG(v)\Big] \\
&= \frac{1}{2}(1-\varepsilon)\Big[\int_{\tilde{v}}^{1} v\,dG(v) + \int_{\check{v}}^{\hat{v}} v\,dG(v)\Big],
\end{aligned}$$

where $\check{v}$ is such that

$$(1-\varepsilon)[G(\hat{v}) - G(\check{v})] = \kappa - s_b(1-\varepsilon)[1-G(\tilde{v})]. \tag{A.10.1}$$

and the second equality follows from $s_a = 1 - s_b$.

Suppose now A admits a small fraction, say δ', of (acceptable) students just below $\tilde{v}$ instead of admitting those who are acceptable and slightly above $\tilde{v}$, say $[\tilde{v}, \tilde{v}+\delta]$, where δ and δ' are such that

$$G(\tilde{v}+\delta) - G(\tilde{v}) = G(\tilde{v}) - G(\tilde{v}-\delta'). \tag{A.10.2}$$

Notice that students in $[\tilde{v}-\delta', \tilde{v}]$ accept A's admission offer, since they prefer it over C. Hence, A's payoff under the deviation is

$$\pi_A^d = (1-\varepsilon)\int_{\tilde{v}-\delta'}^{\tilde{v}} v\,dG(v) + \frac{1}{2}s_a(1-\varepsilon)\int_{\tilde{v}+\delta}^{1} v\,dG(v)$$

$$
\begin{aligned}
&+\frac{1}{2}\Big[s_b(1-\varepsilon)\int_{\tilde v+\delta}^{1} v\,dG(v)+(1-\varepsilon)\int_{\overline{v}}^{\hat v} v\,dG(v)\Big]\\
&=(1-\varepsilon)\int_{\tilde v-\delta'}^{\hat v} v\,dG(v)+\frac{1}{2}(1-\varepsilon)\Big[\int_{\tilde v+\delta}^{1} v\,dG(v)+\int_{\tilde v}^{\hat v} v\,dG(v)\Big],
\end{aligned}
$$

where $\overline{v}$ satisfies

$$
(1-\varepsilon)[G(\hat v)-G(\overline{v})]=\kappa-(1-\varepsilon)[G(\tilde v)-G(\tilde v-\delta')]-s_b(1-\varepsilon)[1-G(\tilde v+\delta)],
$$

that is, $\overline{v}$ is set to meet the capacity in the less popular state. Observe that $\overline{v}>\check v$, since

$$
\begin{aligned}
(1-\varepsilon)[G(\hat v)-G(\overline{v})]&=\kappa-s_b(1-\varepsilon)[1-G(\hat v)]-s_a(1-\varepsilon)[G(\tilde v)-G(\tilde v-\delta')]\\
&=(1-\varepsilon)[G(\hat v)-G(\check v)]-s_a(1-\varepsilon)[G(\tilde v)-G(\tilde v-\delta')], \quad \text{(A.10.3)}
\end{aligned}
$$

where the first equality follows from (A.10.2) and the fact that $s_a=1-s_b$, and the last equality follows from (A.10.1). Thus, we have

$$
\begin{aligned}
\frac{2(\pi_A^d-\pi_A)}{1-\varepsilon}&=2\int_{\tilde v-\delta'}^{\tilde v} v\,dG(v)-\Big[\int_{\tilde v}^{\tilde v+\delta} v\,dG(v)+\int_{\check v}^{\overline{v}} v\,dG(v)\Big]\\
&=2\Big[\tilde v\,G(\tilde v)-(\tilde v-\delta)G(\tilde v-\delta')-\int_{\tilde v-\delta'}^{\tilde v}G(v)dv\Big]\\
&\quad-\Big[(\tilde v+\delta)G(\tilde v+\delta)-\tilde v\,G(\tilde v)-\int_{\tilde v}^{\tilde v+\delta}G(v)dv\Big]-\Big[\overline{v}\,G(\overline{v})-\check v\,G(\check v)-\int_{\check v}^{\overline{v}}G(v)dv\Big]\\
&=\tilde v\Big[G(\tilde v)-G(\tilde v-\delta')\Big]+2\Big[\delta'G(\tilde v-\delta')-\int_{\tilde v-\delta'}^{\tilde v}G(v)dv\Big]\\
&\quad-\Big[\delta G(\tilde v+\delta)-\int_{\tilde v}^{\tilde v+\delta}G(v)dv\Big]-\Big[\overline{v}\,G(\overline{v})-\check v\,G(\check v)-\int_{\check v}^{\overline{v}}G(v)dv\Big]\\
&\geq(\tilde v-2\delta')\Big[G(\tilde v)-G(\tilde v-\delta')\Big]-\delta\Big[G(\tilde v+\delta)-G(\tilde v)\Big]-\overline{v}\Big[G(\overline{v})-G(\check v)\Big]
\end{aligned}
$$

where the second equality follows from the integration by parts, and the third equality follows from (A.10.2). The inequality holds since $\int_{\tilde v-\delta'}^{\tilde v}G(v)\leq\delta'G(\tilde v)$, $\int_{\tilde v}^{\tilde v+\delta}G(v)dv\geq\delta G(\tilde v)$ and $\int_{\check v}^{\overline{v}}G(v)\geq(\overline{v}-\check v)G(\check v)$. Observe that by rearranging (A.10.3), we have $G(\overline{v})-G(\check v)=s_a\Big[G(\tilde v)-G(\tilde v-\delta')\Big]$. Hence, using (A.10.2) again, we get

$$
\frac{2(\pi_A^d-\pi_A)}{1-\varepsilon}\geq[G(\tilde v)-G(\tilde v-\delta')\Big](\tilde v-2\delta'-\delta-\overline{v}\,s_a)=[G(\tilde v)-G(\tilde v-\delta')\Big]\Big(s_a(\tilde v-\overline{v})+s_b\,\tilde v-(2\delta'+\delta)\Big).
$$

Therefore, for sufficiently small δ, we have $\pi_A^d>\pi_A$.

A.11 Proof of Lemma 5

Recall that when both colleges A and B report truthfully up to the capacity, they achieve jointly optimal matching for the two colleges. Now suppose college A unilaterally deviates by either reporting untruthfully about its preferences or its capacity and is strictly better off for some state s. Then, college B must be strictly worse off. Thus, there must exist a positive measure set of students whom A must obtain from the deviation which it prefers to some students it had before the deviation. At the same time, it must be the case that either college B gets a positive measure set of students who are worse than the former set of students or it has some unfilled seats left after A's deviation. Note that students in the former set (who are assigned to A in the new matching) must prefer B, or else the original matching would be not be stable. But then since B prefer each of those students to some students it has in the new matching, this means that the new matching is not stable (given the stated preferences).

B Appendix B: More than Two Colleges

Our main model in Section 2 considers the case with two colleges. In this section, we show that our analysis extends to the case with more than two colleges. While the extension works for any arbitrary number of colleges, we provide the result for the three-college case for expositional simplicity. It will become clear that the method also extends to larger numbers.

Let $\sigma_i : \mathcal{V} \to [0,1]$ be college i's admission strategy, where $i = 1,2,3$. In each state $s \in [0,1]$, let $\mu_{ijk}(s)$, where $i,j,k = 1,2,3$, denote the mass of students whose preference ordering is $i \succ j \succ k$. Define the following notations.

- $\mu_{i\succ j}(s) := \mu_{ijk}(s) + \mu_{ikj}(s) + \mu_{kij}(s)$ (the mass of students who prefer i over j in state s),
- $\mu_{i\succ j,k}(s) := \mu_{ijk}(s) + \mu_{ikj}(s)$ (the mass of students who prefer i over j and k in state s),

and

$$\overline{\mu}_{i\succ j} := \int_0^1 \mu_{i\succ j}(s)\,ds, \quad \overline{\mu}_{i\succ j,k} := \int_0^1 \mu_{i\succ j,k}(s)\,ds.$$

For given $\sigma_i(\cdot)$, $i = 1,2,3$, let $n_i(v)$ be the probability that a student with score v attends college i in state s when she is admitted by i. That is,

$$n_i(v|s) := \prod_{t=j,k}(1-\sigma_t(v)) + \mu_{i\succ j}(s)\sigma_j(v)(1-\sigma_k(v)) + \mu_{i\succ k}(s)\sigma_k(v)(1-\sigma_j(v)) + \mu_{i\succ j,k}(s)\sigma_j(v)\sigma_k(v). \tag{B.0.1}$$

The student will attend college i if she is admitted only by i, which happens with probability $(1-\sigma_j(v))(1-\sigma_k(v))$; or is admitted by college i and one of the less preferred colleges, which happens with probability $\mu_{i\succ j}(s)\sigma_j(v)(1-\sigma_k(v)) + \mu_{i\succ k}(s)\sigma_k(v)(1-\sigma_j(v))$ in state s; or is admitted by both of the other colleges but prefers i the most, which happens with probability $\mu_{i \succ j,k}(s)\sigma_j(v)\sigma_k(v)$ in state s.

Thus, for a given profile of admission strategies, $\sigma = (\sigma_i)_{i=1,2,3}$, in equilibrium, the mass of students who attend college i in state s is

$$m_i(s) := \int_0^1 \sigma_i(v)\, n_i(v|s)\, dG(v),$$

and college i's payoff is

$$\pi_i = \int_0^1 v\, \sigma_i(v)\, \overline{n}_i(v)\, dG(v) - \lambda \int_0^1 \max\{m_i(s) - \kappa, 0\}\, ds, \tag{B.0.2}$$

where

$$\overline{n}_i(v) := \int_0^1 n_i(v|s)\, ds. \tag{B.0.3}$$

Recall that in the two-school case, the monotonicity of $\mu(\cdot)$ yields cutoff states $(\hat{s}_A, \hat{s}_B)$ that trigger over-enrollment for each college, and the set of over-demanded states for each of them is a connected interval, $(\hat{s}_A, 1]$ and $[0, \hat{s}_B)$. Using this, we project the admission strategies to state space in order to establish the existence of MME. This allows us to use the Brouwer's fixed point theorem. When there are more than two colleges, however, we do not know the structure of the set of over-demanded states in general, so we cannot directly define a map from cutoff states to cutoff states. Nonetheless, the main idea of the proof can be carried over, although we use a fixed point theorem (Schauder) in a functional space.

Define a **subdistribution** $F_i : [0,1] \to [0,1]$, $i = 1,2,3$, such that $F_i(0) = 0$ and

$$F_i(s) := \text{Prob}\big(m_i(t) > \kappa \text{ for } t < s\big). \tag{B.0.4}$$

The subdistribution of college i places a positive mass only on the states in which college i is over-demanded. Observe that $F_i(\cdot)$ is nondecreasing and

$$0 \leq F_i(s') - F_i(s) \leq s' - s, \quad \forall\, s' \geq s.$$ [33]

Let $\mathcal{F}_i$ be the set of all such subdistributions and $\mathcal{F} := \times_{i=1}^3 \mathcal{F}_i$. (It will become clear that these

[33] The second inequality holds because

$$\begin{aligned} F_i(s') - F_i(s) &= \text{Prob}\big(m_i(t) > \kappa \text{ for } t < s'\big) - \text{Prob}\big(m_i(t) > \kappa \text{ for } t < s\big) \\ &= \text{Prob}\big(m_i(t) > \kappa \text{ for } s < t < s'\big) \\ &\leq \text{Prob}(s < t < s') \\ &= s' - s. \end{aligned}$$

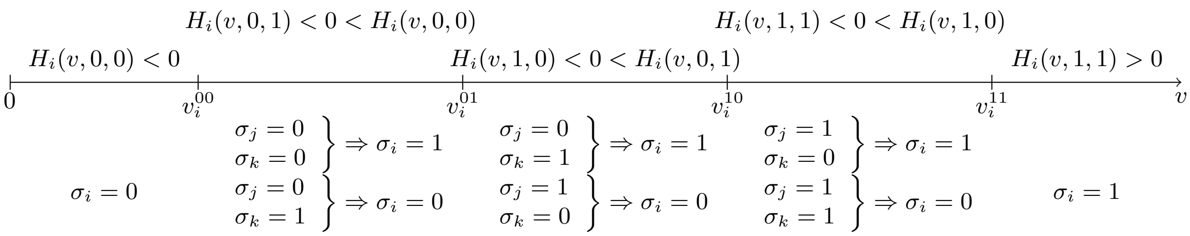


Figure B.1: College i's Admission Decision

subdistributions will play a similar role to the cutoff states in the two-school case.)

Using the subdistributions, each college's payoff is now given by[34]

$$\begin{aligned}\pi_i &= \int_0^1 v\,\sigma_i(v)\,\overline{n}_i(v)\,dG(v) - \lambda \int_0^1 \big(m_i(s) - \kappa\big)\,dF_i(s) \\ &= \int_0^1 \sigma_i(v)\,H_i(v,\sigma_j(v),\sigma_k(v))\,dG(v) + \lambda \int_0^1 \kappa\,dF_i(s),\end{aligned} \tag{B.0.5}$$

where

$$H_i(v,\sigma_j(v),\sigma_k(v)) := v\,\overline{n}_i(v) - \lambda \int_0^1 n_i(v|s)\,dF_i(s) \tag{B.0.6}$$

is college i's marginal payoff from admitting a student with score v. Note that this marginal payoff depends on the subdistribution F_i, as $\overline{n}_i(v)$ is a constant for given admission strategies $(\sigma_i)_{i=1,2,3}$ (by (B.0.3)) and $n_i(v)$ is evaluated by the subdistribution.

Note that (B.0.6) can be decomposed as follow:

$$\begin{aligned}H_i(v,\sigma_j(v),\sigma_j(v)) = {}&(1-\sigma_j(v))(1-\sigma_k(v))H_i(v,0,0) + \sigma_j(v)(1-\sigma_k(v))H_i(v,1,0) \\ &+ (1-\sigma_j(v))\sigma_k(v)H_i(v,0,1) + \sigma_j(v)\sigma_k(v)H_i(v,1,1),\end{aligned}$$

where $H_i(v,0,0)$ is college i's marginal payoff from admitting a student with score v if she is refused by both of the other colleges, $H_i(v,1,0)$ and $H_i(v,0,1)$ are the marginal payoffs if the student is admitted by college j (k) but rejected by k (j, respectively), and $H_i(v,1,1)$ is the marginal payoff if the student is admitted by both of the other colleges.

Let us now define v_i^{11}, v_i^{10}, v_i^{01} and v_i^{00} such that

$$H_i(v_i^{11},1,1) = 0, \quad H_i(v_i^{10},1,0) = 0, \quad H_i(v_i^{01},0,1) = 0, \quad H_i(v_i^{00},0,0) = 0.$$

Similar to the two-school case, $H_i(v,\sigma_j,\sigma_k)$ partitions the students' type space. College i admits type v students for sure if $H_i(v,1,1) > 0$ and rejects them if $H_i(v,0,0) < 0$. In the case $H_i(v,1,1) < 0 < H_i(v,0,0)$, college i admits type v students only when $H_i(v,1,0) > 0$ or

[34]Note that since F_i is Lipschitz continuous, so it is absolute continuous. Thus, the integration is well defined. Observe also that (B.0.5) does not involve $\max\{\cdot,\cdot\}$ in the cost (see (B.0.2) for comparison), as the subdistribution is defined for states where $m_i(s) > \kappa$ by (B.0.4), and the college's cost is evaluated by the subdistribution.

$H_i(v,0,1) > 0$; that is, those students are worthy only in the case that they is admitted by one of the other colleges. This shows that colleges engage in *strategic targeting* for those intermediate range of scores.

Randomization may emerge for some students. For students with v such that

$$\max_{i=1,2,3} \{H_i(v,1,0), H_i(v,0,1)\} < 0 < \min_{i=1,2,3} \{H_i(v,0,0)\},$$

all three colleges engage in mixed-strategies, where the mixed-strategies satisfy

$$H_i(v, \sigma_j(v), \sigma_k(v)) = 0 \quad \forall i,j,k = 1,2,3.$$

For students with v such that $H_k(v,0,0) < 0$ and

$$\max\{H_i(v,1,0), H_j(v,1,0)\} < 0 < \min\{H_i(v,0,0), H_j(v,0,0)\},$$

college k does not admit such students, but colleges i and j engage in mixed-strategies satisfying

$$H_i(v, \sigma_j, 0) = 0 \quad \text{and} \quad H_j(v, \sigma_i, 0) = 0.$$

A typical mixed-strategy equilibrium is depicted in Figure B.2 when, for instance,

$$v_3^{00} < v_2^{00} < v_1^{00} < v_3^{01} < v_2^{01} < v_1^{01} < v_3^{10} < v_2^{10} < v_1^{10} < v_3^{11} < v_2^{11} < v_1^{11}.$$

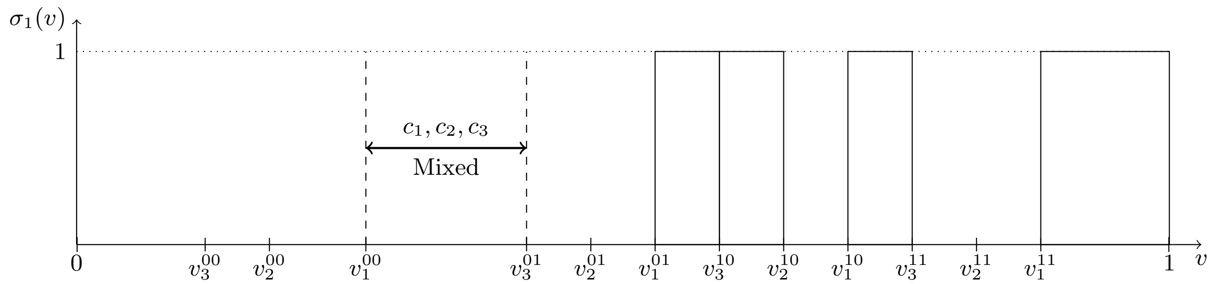
Note that, as in the two-school case, there are many ways that colleges could coordinate (even in a mixed-strategy equilibrium). Hence, we consider the maximally mixed-strategy as before and provide the existence of such equilibrium.

For a given profile of subdistributions $(F_i)_{i=1}^3$, let $\sigma := (\sigma_i)_{i=1}^3$ be the profile of admission strategies that satisfy the local conditions described above. Then, such σ in turn determines a new profile of subdistributions, $(F_i)_{i=1}^3$ via (B.0.4). Next, we define $T : \mathcal{F} \to \mathcal{F}$, a self-map from the set of subdistributions to itself, where $\mathcal{F} = \times_{i=1}^3 \mathcal{F}_i$. The existence of equilibrium is achieved when T has a fixed point (on the functional space of $\mathcal{F}$).

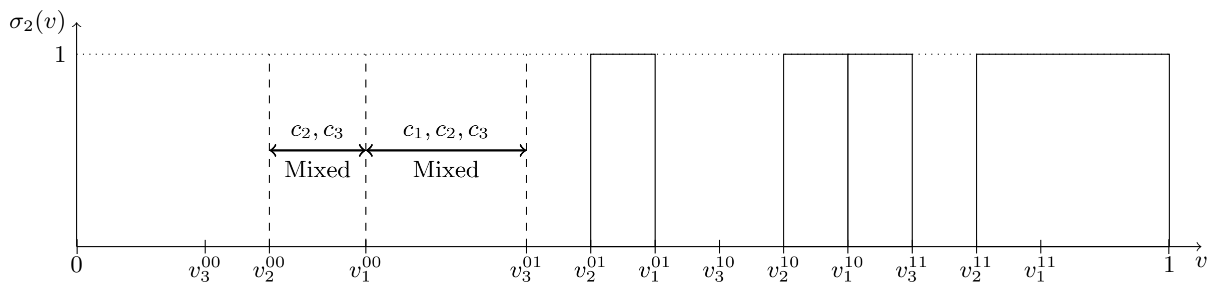
As mentioned earlier, the idea of proving the existence of equilibrium is similar to the idea of Theorem 2, projecting the strategy profile into a simpler space. The difference is that in the two-school case, the strategy profiles are projected into the state space, but in the general case, they are projected into the set of subdistributions $\mathcal{F}$.

Theorem 11. *There exists an equilibrium with maximally mixed-strategies.*

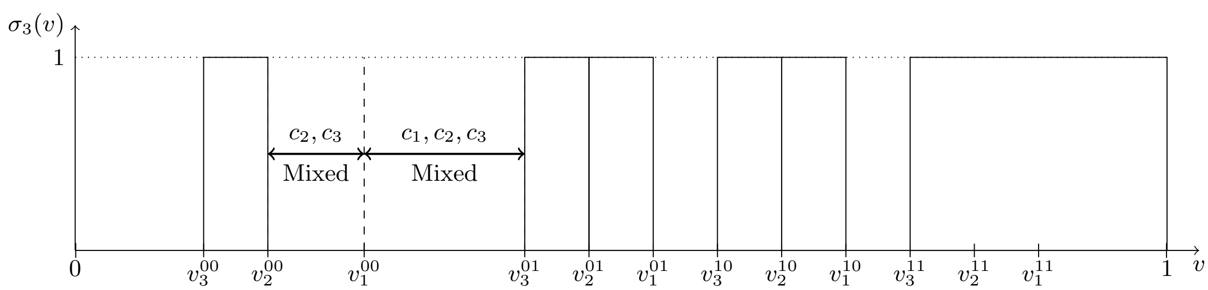
We first show that $\mathcal{F}$ is a compact and convex subset of a normed linear space, and $T : \mathcal{F} \to \mathcal{F}$



(a) College 1 (c_1)



(b) College 2 (c_2)



(c) College 3 (c_3)

Figure B.2: Admission Strategies

is continuous. Then, T has a fixed point by Schauder's fixed point theorem.[35] We then show that the identified strategies indeed constitute mutual (global) best responses. We provide a formal proof in the next subsection.

B.1 Proof of Theorem 11

For given $(F_i)_{i=1,2,3}$, consider colleges' strategy profile $(\sigma_i)_{i=1,2,3}$ which satisfies the following local conditions:

- $\sigma_i(v) = 1$ if $H_1(v,1,1) > 0$, $i = 1,2,3$.

- $\sigma_1(v) = 0$ if $H_1(v,1,1) < 0$, $H_2(v,1,1) > 0$, $H_3(v,1,1) > 0$.
  $\sigma_2(v) = 0$ if $H_1(v,1,1) > 0$, $H_2(v,1,1) < 0$, $H_3(v,1,1) > 0$.
  $\sigma_3(v) = 0$ if $H_1(v,1,1) > 0$, $H_2(v,1,1) > 0$, $H_3(v,1,1) < 0$.

- $\sigma_1(v) = 0$, $\sigma_2(v) = 1$, $\sigma_3(v) = 1$ if $\begin{cases} H_1(v,1,1) < 0 \\ H_2(v,1,1) < 0, H_2(v,0,1) > 0 \\ H_3(v,1,1) < 0, H_3(v,0,1) > 0 \end{cases}$

- $\sigma_1(v) = 1$, $\sigma_2(v) = 0$, $\sigma_3(v) = 1$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,0,1) > 0 \\ H_2(v,1,1) < 0 \\ H_3(v,1,1) < 0, H_3(v,1,0) > 0 \end{cases}$

- $\sigma_1(v) = 1$, $\sigma_2(v) = 1$, $\sigma_3(v) = 0$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,1,0) > 0 \\ H_2(v,1,1) < 0, H_2(v,1,0) > 0 \\ H_3(v,1,1) < 0 \end{cases}$

- $\sigma_1(v) = 1$, $\sigma_2(v) = 0$, $\sigma_3(v) = 0$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,0,0) > 0 \\ H_2(v,1,1) < 0, H_2(v,1,0) < 0 \\ H_3(v,1,1) < 0, H_3(v,1,0) < 0 \end{cases}$

- $\sigma_1(v) = 0$, $\sigma_2(v) = 1$, $\sigma_3(v) = 0$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,1,0) < 0 \\ H_2(v,1,1) < 0, H_2(v,0,0) > 0 \\ H_3(v,1,1) < 0, H_3(v,0,1) < 0 \end{cases}$

[35] Schauder's fixed point theorem is a generalization of Brouwer's theorem on a normed linear space. It guarantees that every continuous self-map on a nonempty, compact, convex subset of a normed linear space has a fixed point (see Ok, 2007).

- $\sigma_1(v) = 0$, $\sigma_2(v) = 0$, $\sigma_3(v) = 1$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,0,1) < 0 \\ H_2(v,1,1) < 0, H_2(v,0,1) < 0 \\ H_3(v,1,1) < 0, H_3(v,0,0) > 0 \end{cases}$

- $\sigma_i(v) = 0$ if $H_1(v,0,0) < 0$, $i = 1,2,3$.

- $\sigma_i(v)$'s satisfy $H_1(v,\sigma_2(v),\sigma_3(v)) = H_2(v,\sigma_1(v),\sigma_3(v)) = H_3(v,\sigma_1(v),\sigma_2(v)) = 0$, if

$$\max_{i=1,2,3} \{H_i(v,1,0), H_i(v,0,1)\} < 0 < \min_{i=1,2,3} \{H_i(v,0,0)\}$$

- $\sigma_i(v)$ and $\sigma_j(v)$ satisfy $H_i(v,\sigma_j,0) = 0$ and $H_j(v,\sigma_i,0) = 0$ if $H_k(v,0,0) < 0$ and

$$\max \{H_i(v,1,0), H_j(v,1,0)\} < 0 < \min \{H_i(v,0,0), H_j(v,0,0)\}$$

Now, let $\mathbf{CB}([0,1])$ be the space of continuous and bounded real maps on $[0,1]$. Then, $\mathbf{CB}([0,1])$ is a normed linear space, with a sup norm $\|\cdot\|$, i.e., for any $F, F' \in \mathbf{CB}([0,1])$,

$$\left\|F - F'\right\| = \sup_{s\in[0,1]} \left|F(s) - F'(s)\right|.$$

Lemma 1. *$\mathcal{F}$ is compact and convex.*

Proof. We first show that $\mathcal{F}_i$, $i = 1,2,3$, is closed. To this end, consider any sequence $\{F_i^n\}$, where $F_i^n \in \mathcal{F}_i$ for each n, such that $\|F_i^n - F_i\| \to 0$ as $n \to \infty$. We prove that $F_i \in \mathcal{F}_i$.

Observe first that F_i is nondecreasing. Suppose to the contrary that $F_i(s') - F_i(s) < 0$ for some $s' > s$. But then,

$$\begin{aligned}
\|F_i^n - F_i\| &\geq \max\left\{\left|F_i^n(s') - F_i(s')\right|, \left|F_i(s) - F_i^n(s)\right|\right\} \\
&\geq \tfrac{1}{2}(\left|F_i^n(s') - F_i(s')\right| + F_i(s) - F_i^n(s)) \\
&\geq \tfrac{1}{2}\left|F_i^n(s') - F_i(s') + F_i(s) - F_i^n(s)\right| \\
&\geq \tfrac{1}{2}\left|F_i(s) - F_i(s')\right| \\
&> 0
\end{aligned}$$

which is a contradiction. Likely, for $s' > s$, we must have that $F_i(s') - F_i(s) \leq s' - s$. If $F_i(s') - F_i(s) > s' - s$, then

$$\begin{aligned}
\|F_i^n - F_i\| &\geq \max\left\{\left|F_i(s') - F_i^n(s')\right|, \left|F_i^n(s) - F_i(s)\right|\right\} \\
&\geq \tfrac{1}{2}(\left|F_i(s') - F_i^n(s')\right| + \left|F_i^n(s) - F_i(s)\right|)
\end{aligned}$$

$$
\begin{aligned}
&\geq \tfrac{1}{2}\left|F_i(s') - F_i(s) + F_i^n(s) - F_i^n(s')\right| \\
&\geq \tfrac{1}{2}\left|F_i(s') - F_i(s) - (s' - s)\right| \\
&> 0,
\end{aligned}
$$

which is a contradiction again. Combining these, we have $F_i \in \mathcal{F}_i$, proving that $\mathcal{F}_i$ is closed.

Next, we show that $\mathcal{F}_i$ is compact. Note that for any $F_i \in \mathcal{F}_i$ and $s, s' \in [0,1]$,

$$\left|F_i(s') - F_i(s)\right| \leq \left|s' - s\right|,$$

Hence, $\mathcal{F}_i$ is Lipschitz continuous and so is equicontinuous and bounded. By the Arzèla-Ascoli theorem,[36] $\mathcal{F}_i$ is compact.

We now show that $\mathcal{F}_i$ is convex. Observe that for any $F_i, F_i' \in \mathcal{F}$ and $s, s' \in [0,1]$, for and $\eta \in (0,1)$,

$$
\begin{aligned}
(\eta F_i + (1-\eta)F_i')(s') - (\eta F_i + (1-\eta)F_i')(s) &= \eta(F_i(s') - F_i(s)) + (1-\eta)(F_i'(s') - F_i'(s)) \\
&\leq \eta(s' - s) + (1-\eta)(s' - s) \\
&= s' - s,
\end{aligned}
$$

which proves that $\mathcal{F}_i$ is convex.

Since $\mathcal{F}_i$ is compact and closed, so is its Cartesian product $\mathcal{F} = \times_{i=1}^{3}\mathcal{F}_i$ (with respect to the product topology). ■

Lemma 2. *T is continuous.*

Proof. The proof involves several steps:

Step 1. *v_i^{jk}'s are continuous on F_1, F_2, F_3.*

Proof . We first show that v_i^{jk}'s are continuous in F_i. Fix any $F_i \in \mathcal{F}_i$ and $\varepsilon > 0$. Take $\delta = \frac{\overline{\mu}_{i\succ j,k}}{2\lambda}\varepsilon$. Then, for any $F_i, F_i' \in \mathcal{F}_i$ such that $\|F_i - F_i'\| < \delta$, we have that

$$
\begin{aligned}
\left|v_i^{jk} - v_i^{jk'}\right| &= \left|\frac{\lambda}{\overline{\mu}_{i\succ j,k}} \int_0^1 \mu(s)_{i\succ j,k}[dF_i(s) - dF_i'(s)]\right| \\
&= \frac{\lambda}{\overline{\mu}_{i\succ j,k}} \left|\mu_{i\succ j,k}(1)[F_i(1) - F_i'(1)] - \int_0^1 \mu_{i\succ j,k}'(s)[F_i(s) - F_i'(s)]ds\right| \\
&\leq 2\left\|F_i(s) - F_i'(s)\right\| \\
&< \varepsilon,
\end{aligned}
$$

[36] Arzèla-Ascoli theorem gives conditions for a set of $\mathbf{C}(T)$ to be compact, where $\mathbf{C}(T)$ is the space of continuous maps on T and T is a compact metric space. A subset of $\mathbf{C}(T)$ is compact if and only if it is closed, bounded, and equicontinuous.

where the third equality follows from the integration by parts and $F_i(0) = F_i'(0) = 0$, and the first inequality holds since $\int_0^1 \mu'_{i \succ j,k}(s)ds = \mu_{i \succ j,k}(1) - \mu_{i \succ j,k}(0) \leq 1$. $\square$

Step 2. *σ_i's in mixed-strategies are continuous.*

Proof . Consider, at first, students with score v such that

$$H_k(v, 0, 0) < 0, \tag{B.1.1}$$

$$H_i(v, 1, 0) < 0 < H_i(v, 0, 0), \tag{B.1.2}$$

$$H_j(v, 1, 0) < 0 < H_j(v, 0, 0). \tag{B.1.3}$$

That is, college k puts zero probability for those students (by (B.1.1)), and colleges i and j use mixed-strategies σ_i and σ_j which satisfy $H_i(v, \sigma_j, 0) = 0$ and $H_j(v, \sigma_i, 0) = 0$.

Now, let $J_i : [0, 1]^2 \times [0, 1]^2 \to [0, 1]$ such that

$$J_i(F_i, F_j, \sigma_i, \sigma_j) \equiv H_i(v, \sigma_j, 0) = v\big[(1-\sigma_j) + \overline{\mu}_{i \succ j}\sigma_j(v)\big] - \lambda \int_0^1 \big[(1-\sigma_j) + \mu_{i \succ j}(s)\sigma_j(v)\big] dF_i(s),$$

$$J_j(F_i, F_j, \sigma_i, \sigma_j) \equiv H_j(v, \sigma_i, 0) = v\big[(1-\sigma_i) + \overline{\mu}_{j \succ i}\sigma_i(v)\big] - \lambda \int_0^1 \big[(1-\sigma_i) + \mu_{j \succ i}(s)\sigma_i(v)\big] dF_j(s).$$

Then, σ_i and σ_j are the solutions to $J_i = 0$ and $J_j = 0$ in terms of F_i and F_j. Observe that

$$J_i = (1 - \sigma_j)H_i(v, 0, 0) + \sigma_j H_i(v, 1, 0).$$

Hence,

$$\frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + H_i(v, 1, 0) < 0,$$

where inequality follows from (B.1.2). Similarly, we also have by (B.1.3)

$$\frac{\partial J_j}{\partial \sigma_i} = -H_j(v, 0, 0) + H_j(v, 1, 0) < 0.$$

Therefore,

$$\Delta_{ij} := \begin{vmatrix} \frac{\partial J_i}{\partial \sigma_i} & \frac{\partial J_i}{\partial \sigma_j} \\ \frac{\partial J_j}{\partial \sigma_i} & \frac{\partial J_j}{\partial \sigma_j} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial J_i}{\partial \sigma_j} \\ \frac{\partial J_j}{\partial \sigma_i} & 0 \end{vmatrix} = -\frac{\partial J_i}{\partial \sigma_j}\frac{\partial J_j}{\partial \sigma_i} < 0.$$

Since $\Delta_{ji} \neq 0$, the Implicit function theorem implies that there are unique σ_i and σ_j such that

$$J_i(F_i, F_j, \sigma_i, \sigma_j) = 0 \quad \text{and} \quad J_j(F_i, F_j, \sigma_i, \sigma_j) = 0.$$

Furthermore, such σ_i and σ_j are continuous.

Consider now the case that $H_1(v, \sigma_2, \sigma_3) = H_2(v, \sigma_1, \sigma_3) = H_3(v, \sigma_1, \sigma_2) = 0$ when

$$\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 < \min_{i=1,2,3} \{H_i(v, 0, 0)\}. \tag{B.1.4}$$

Similar as before, let

$$\begin{aligned}
J_1(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) &\equiv H_1(v, \sigma_2, \sigma_3) = 0,\\
J_2(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) &\equiv H_2(v, \sigma_1, \sigma_3) = 0,\\
J_3(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) &\equiv H_3(v, \sigma_1, \sigma_2) = 0.
\end{aligned}$$

Observe that

$$\begin{aligned}
J_i &= (1-\sigma_j)(1-\sigma_k)H_i(v,0,0) + \sigma_j(1-\sigma_k)H_i(v,1,0) + (1-\sigma_j)\sigma_k H_i(v,0,1) + \sigma_j\sigma_k H_i(v,1,1)\\
&= (1-\sigma_j)H_i(v,0,0) + \sigma_j(1-\sigma_k)H_i(v,1,0) - (1-\sigma_j)\sigma_k H_k(v,1,0) + \sigma_j\sigma_k H_i(v,1,1).
\end{aligned}$$

where the second inequality holds after some rearrangement using the fact that $1 - \mu_{i \succ k}(s) = \mu_{k \succ i}(s)$. Therefore,

$$\frac{\partial J_i}{\partial \sigma_j} = -H_i(v,0,0) + (1-\sigma_k)H_i(v,1,0) + \sigma_k H_k(v,1,0) + \sigma_k H_i(v,1,1) < 0,$$

where the inequality holds since $H_i(v,0,0) > 0$, $H_i(v,1,0) < 0$, $H_k(v,1,0) < 0$ and $H_i(v,1,1) < 0$ by (B.1.4). This implies that

$$\Delta := \begin{vmatrix} \frac{\partial J_1}{\partial \sigma_1} & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\ \frac{\partial J_2}{\partial \sigma_1} & \frac{\partial J_2}{\partial \sigma_2} & \frac{\partial J_2}{\partial \sigma_3} \\ \frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & \frac{\partial J_3}{\partial \sigma_3} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\ \frac{\partial J_2}{\partial \sigma_1} & 0 & \frac{\partial J_2}{\partial \sigma_3} \\ \frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & 0 \end{vmatrix} = \frac{\partial J_1}{\partial \sigma_2}\frac{\partial J_2}{\partial \sigma_3}\frac{\partial J_3}{\partial \sigma_1} + \frac{\partial J_1}{\partial \sigma_3}\frac{\partial J_2}{\partial \sigma_1}\frac{\partial J_3}{\partial \sigma_2} < 0$$

Using the Implicit function theorem again, we conclude that such $\sigma_1, \sigma_2, \sigma_3$ exist and they are continuous. □

Observe that from Step 1 and Step 2, $H_i(v, \sigma_j, \sigma_k)$, $i = 1, 2, 3$, is continuous in $(F_i)_{i=1,2,3}$ for a given s and fixed v.

Step 3. *$m_i(s)$ is continuous.*

Proof . Consider any $F_i, F_i' \in \mathcal{F}_i$ such that $\|F_i - F_i'\| < \delta$ for all $i = 1, 2, 3$. Let σ_i and σ_i' are admission strategies of college i which correspond to F_i and F_i', respectively. Then, for a given s and v, $n_i(v|s)$ is defined by (B.0.1) and $n_i'(v|s)$ is defined similarly using σ_i'.

Let $X := \{v \in [0,1] | \, |\sigma_i(v) - \sigma_i'(v)| \geq \varepsilon/2\}$. Clearly,

$$\left|\sigma_i(v) - \sigma_i'(v)\right| = \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_X(v) + \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_{X^c}(v),$$

where $\mathbb{1}_X(v)$ is the indicator function which is 1 if $v \in X$ or 0 otherwise, and X^c is the complementary set of X. Since v_i^{jk} are continuous by Step 1, we have

$$\int_0^1 \mathbb{1}_X(v)\, dG(v) < \frac{\varepsilon}{2}. \tag{B.1.5}$$

For $v \in X^c$, it must be the case that either $\sigma_i = \sigma_i'$, or σ_i and σ_i' are the mixed-strategies. Thus, we have for $v \in X^c$,

$$\left|\sigma_i(v) - \sigma_i'(v)\right| < \frac{\varepsilon}{2}. \tag{B.1.6}$$

Observe that

$$\begin{aligned}
\int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \, dG(v) &= \int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_X(v)\, dG(v) + \int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_{X^c}(v)\, dG(v) \\
&< \int_0^1 \mathbb{1}_X(v)\, dG(v) + \int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_{X^c}(v)\, dG(v) \\
&< \varepsilon,
\end{aligned}$$

where the first inequality holds since $\sigma_i, \sigma_i' \leq 1$, and the last inequality follows from (B.1.5) and (B.1.6). Thus, there exists δ_1 such that $\|F_i - F_i'\| < \delta_1$, for all $i, i' = 1, 2, 3$, implies

$$\begin{aligned}
&\int_0^1 \left|\sigma_i(1-\sigma_j)(1-\sigma_k) - \sigma_i'(1-\sigma_j')(1-\sigma_k')\right| \, dG(v) \\
&\leq \int_0^1 \left[\left|\sigma_i - \sigma_i'\right| (1-\sigma_j)(1-\sigma_k) + \left|\sigma_j - \sigma_j'\right| \sigma_i'(1-\sigma_k) + \left|\sigma_k - \sigma_k'\right| \sigma_i'(1-\sigma_j') \right] dG(v) \\
&< \frac{\varepsilon}{4}
\end{aligned}$$

Similarly, there are δ_t, $t = 2, 3, 4$, such that $\|F_i - F_i'\| < \delta_t$ respectively imply that

$$\left|\sigma_i\sigma_j(1-\sigma_k) - \sigma_i'\sigma_j'(1-\sigma_k')\right| < \frac{\varepsilon}{4}, \quad \left|\sigma_i\sigma_k(1-\sigma_j) - \sigma_i'\sigma_k'(1-\sigma_j')\right| < \frac{\varepsilon}{4}, \quad \left|\sigma_i\sigma_j\sigma_k - \sigma_i'\sigma_j'\sigma_k'\right| < \frac{\varepsilon}{4}.$$

Now, let $\delta = \min_{t=1,2,3,4} \{\delta_t\}$. We have that $\|F_i - F_i'\| < \delta$ implies

$$\left|m_i(s) - m_i'(s)\right| \equiv \left| \int_0^1 \sigma_i(v)\, n_i(v|s)\, dG(v) - \int_0^1 \sigma_i'(v)\, n_i'(v|s)\, dG(v) \right| < \varepsilon.$$

That is, $m_i(s)$ is continuous on $(F_i)_{i=1,2,3}$. $\square$ ■

Lemma 2 proves the existence admission strategies that satisfy the local conditions. The proof that those strategies are mutual (global) best responses is analogous to that of the two college case.