Delta I Equals Three Halfs Kaon To Two Pion Decays Using Lattice
Quantum Chromodynamics with Domain Wall Fermions

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ABSTRACT

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We calculate matrix elements for kaon to two pion decays in the $\Delta I = 3/2$ channel using lattice gauge theory simulations. From these we can extract the decay amplitude A_2 , for which the real part is related to the decay rate and can be compared to the experimental result $\text{Re}(A_2) = 1.484 \times 10^{-8}$ GeV, and for which the imaginary part is related to direct charge-parity violation in the neutral kaon system. We report the results of one simulation with nearly physical particle masses and kinematics, specifically $m_K = 509.0(9.1)$ MeV, $m_\pi = 142.8(2.5)$ MeV, and $E_{\pi\pi} = 485.7(8.0)$ MeV. This simulation was performed on RBC/UKQCD $32^3 \times 64$, $L_s = 32$ lattices, using 2+1 dynamical flavors of domain wall fermions and a Dislocation Suppressing Determinant Ratio plus Iwasaki gauge action, and with an inverse lattice spacing $a^{-1} = 1.373(24)$ GeV so that the spatial extent of the lattice is 4.60 fm and $m_\pi L = 3.3$. We find that $\text{Re}(A_2) = 1.461(87)_{\text{stat}}(200)_{\text{sys}} \times 10^{-8}$ GeV, in good agreement with the experimental value. We also find $\text{Im}(A_2) = -8.67(45)_{\text{stat}}(1.95)_{\text{sys}} \times 10^{-13}$ GeV, and $\text{Im}(A_2)/\text{Re}(A_2) = -5.93(27)_{\text{stat}}(1.42)_{\text{sys}} \times 10^{-5}$, however the value of $\text{Im}(A_2)$ depends on a rough hypothesis for some of the renormalization constants which have not yet been calculated, and

thus we quote a large systematic error.

We also report the results of a simulation involving a variety of kaon and pion masses and momenta, which was conducted in order to study the dependence of the decay amplitude on particle masses and kinematics, and to study the effect of not having exactly physical masses and kinematics in the first simulation. The use of the quenched approximation and smaller spatial volume in this second simulation allowed for multiple masses to be simulated in a reasonable amount of time, but introduced an uncontrolled approximation and forced us to use pion masses a bit larger than the physical mass. The study was conducted on $24^3 \times 64$, $L_s = 16$ lattices, with the quenched Doubly Blocked Wilson 2 gauge action, and an inverse lattice spacing of $a^{-1} = 1.31(2)$ GeV. We find that an extrapolation to physical masses and kinematics yields values $\text{Re}(A_2) = 2.25(18)_{\text{stat}} \times 10^{-8}$ GeV and $\text{Im}(A_2) = -13.44(84)_{\text{stat}} \times 10^{-13}$ GeV. These results are significantly larger than those of the full dynamical simulation and of experiment. We attribute this mainly to the an inaccurate determination of the lattice spacing a using the rho mass, since it comes in as a^{-3} in the calculation of A_2 .

Finally, a third simulation is performed with 2+1 dynamical flavors of domain wall fermions on a finer $32^3 \times 64$, $L_s = 16$ lattice, but only with pions that have nearly zero momentum. It, and the quenched simulation, are used mainly to estimate the systematic error in the first simulation, which is taken as the final result.

Contents

Acknowledgments	xxxiii
1 Introduction	1
2 The Phenomenology of $K \rightarrow \pi\pi$ Decays	7
2.1 The Standard Model of Particle Physics	8
2.2 Kaons, Pions, and Isospin symmetry	13
2.3 Direct and Indirect CP Violation	15
3 Lattice QCD	23
3.1 QCD Lagrangian	24
3.2 Gauge Actions	28
3.3 Fermion Actions	30
3.4 Numerical Methods	35
3.5 Quenched and Partially Quenched QCD	38
4 Weak Decays on the Lattice	41
4.1 Operator Product Expansion	43
4.2 Non-Perturbative Renormalization	51

4.3	Converting Between Regularization Schemes	56
5	Finite Volume Effects	57
5.1	Two Particle States in Finite Volume - One Dimension	59
5.2	Two Particle States in Finite Volume - Three Dimensions . . .	63
5.3	The Lellouch-Lüscher Factor	77
6	Computational Methods	89
6.1	Coarse 32^3 Lattices	90
6.2	Fine 32^3 Lattices	117
6.3	Quenched 24^3 Lattices	121
7	Results	126
7.1	Coarse 32^3 Lattices	126
7.2	Fine 32^3 Lattices	136
7.3	Quenched 24^3 Lattices	139
7.4	Extrapolations and Interpolations to Physical Kinematics . . .	142
7.5	Discussion of Errors	155
8	Conclusion	163
	Tables	166
	Figures	260
A	Normalization Conventions for A_2	400

A.1	Relation to Decay Rates	400
A.2	Calculating A_2 from the Lattice Matrix Elements	403
B	Estimate of Finite Volume Systematic Error	413
B.1	Chiral Perturbation Theory Formulae	413
B.2	Numerical Results	417
B.3	Numerical Values of the Functions ℓ , β , $\Delta_L\ell$, and $\Delta_L\beta$	419

List of Tables

1	Quark contents and masses of the kaons and pions.	167
2	Fits to the pion and kaon correlators on the coarse 32^3 lattice.	167
3	Fitting ranges for the pion and kaon correlators on the coarse 32^3 lattice.	168
4	Check of the pion and kaon dispersion relation on the coarse 32^3 lattice.	168
5	Results of the fits to the two pion correlators on the coarse 32^3 lattice.	168
6	Fitting ranges for the two pion correlators on the coarse 32^3 lattice.	168
7	Results of the fits to the ratio $C_{\pi\pi}/C_\pi^2$ on the coarse 32^3 lattice.	169
8	Fitting ranges for the ratio $C_{\pi\pi}/C_\pi^2$ on the coarse 32^3 lattice. .	169
9	Two pion energy on the coarse 32^3 lattice as obtained by fitting the ratio $C_{\pi\pi}/C_\pi^2$	169
10	Gamma matrix structures for the different weak operators. . .	169
11	Results of fitting the quotient of correlators $C_{K\rightarrow\pi\pi}/(C_K C_{\pi\pi})$ to a constant on the coarse 32^3 lattices, $p = 0$	170

12	Results of fitting the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant on the coarse 32^3 lattices, $p = \sqrt{2}\pi/L$	171
13	List of configurations analyzed for the fine 32^3 ensembles. . . .	172
14	Results of the fits to the pion correlators on the fine 32^3 lattices.	173
15	Results of the fits to the kaon correlators on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices.	174
16	Results of the fits to the kaon correlators on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices.	175
17	Results of the fits to the kaon correlators on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices.	176
18	Results of the fits to the two pion correlators on the fine 32^3 lattices.	177
19	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (27,1) weak operator on the fine 32^3 lattices.	178
20	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) weak operator on the fine 32^3 lattices.	179
21	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) mixed weak operator on the fine 32^3 lattices.	180
22	List of configurations analyzed for the quenched 24^3 ensemble.	181

23	Results of the fits to the pion correlators on the quenched 24^3 lattices.	181
24	Results of the fits to the zero momentum kaon correlators on the quenched 24^3 lattices.	182
25	Results of the fits to the kaon correlators with $m_l=0.0023$ and $m_s=0.062$ for different momenta, on the quenched 24^3 lattices.	182
26	Fit ranges for the pion correlators on the quenched 24^3 lattices.	183
27	Fit ranges for the kaon correlators with $m_l=0.0023$ and $m_s=0.062$ on the quenched 24^3 lattices.	183
28	Check of the dispersion relation for the pion correlators on the quenched 24^3 lattices.	184
29	Check of the dispersion relation for the kaon correlators with $m_l=0.0023$ and $m_s=0.062$ on the quenched 24^3 lattices. . . .	184
30	Results of the fits to the two pion correlators on the quenched 24^3 lattices.	185
31	Fit ranges for the two pion correlators on the quenched 24^3 lattices.	185
32	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (27,1) weak operator with zero momentum on the quenched 24^3 lattices.	186
33	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (27,1) weak operator with momentum $p = \pi/L$ (one twist) on the quenched 24^3 lattices.	187

34	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (27,1) weak operator with momentum $p = \sqrt{2}\pi/L$ (two twists) on the quenched 24^3 lattices.	188
35	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (27,1) weak operator with momentum $p = \sqrt{3}\pi/L$ (three twists) on the quenched 24^3 lattices.	189
36	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) weak operator with zero momentum on the quenched 24^3 lattices.	190
37	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) weak operator with momentum $p = \pi/L$ (one twist) on the quenched 24^3 lattices.	191
38	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) weak operator with momentum $p = \sqrt{2}\pi/L$ (two twists) on the quenched 24^3 lattices.	192
39	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) weak operator with momentum $p = \sqrt{3}\pi/L$ (three twists) on the quenched 24^3 lattices.	193
40	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) mixed weak operator with zero mo- mentum on the quenched 24^3 lattices.	194

41	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) mixed weak operator with momentum $p = \pi/L$ (one twist) on the quenched 24^3 lattices.	195
42	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) mixed weak operator with momentum $p = \sqrt{2}\pi/L$ (two twists) on the quenched 24^3 lattices. . .	196
43	Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant for the (8,8) mixed weak operator with momentum $p = \sqrt{3}\pi/L$ (three twists) on the quenched 24^3 lattices. .	197
44	Two pion energies in physical units and s wave phase shifts from the 32^3 coarse lattices.	198
45	Wilson coefficients needed for the 32^3 coarse and fine lattice calculations.	199
46	Wilson coefficients for the (27,1), (8,8), and (8,8) mixed operators.	199
47	Lellouch-Lüscher factor for the 32^3 coarse lattices.	199
48	Physical constants needed to calculate A_2	200
49	Lattice matrix elements with zero twists ($p = 0$) for the 32^3 coarse lattices.	200
50	Lattice matrix elements with two twists ($p = \sqrt{2}\pi/L$) for the 32^3 coarse lattices.	201
51	Results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ for different separations between the kaon and two pions, on the 32^3 coarse lattices. . . .	202

52	Contributions to $\text{Re}(A_2)$ and $\text{Im}(A_2)$ from the different weak operators for the 32^3 coarse lattices.	203
53	Pion mass in MeV on the fine 32^3 lattices.	204
54	Kaon mass in MeV on the fine 32^3 lattices.	205
55	Two pion energies in physical units from the fine 32^3 lattices. .	206
56	The s wave phase shifts from the fine 32^3 lattices.	207
57	Lellouch-Lüscher factor over free field normalization factor for the 32^3 fine lattices.	208
58	Lellouch-Lüscher factor for the 32^3 fine lattices.	209
59	Lattice matrix element for the (27,1) weak operator on the 32^3 fine lattices.	210
60	Lattice matrix element for the (8,8) weak operator on the 32^3 fine lattices.	211
61	Lattice matrix element for the (8,8) mixed weak operator on the 32^3 fine lattices.	212
62	Results for $\text{Re}(A_2)$ on the 32^3 fine lattices.	213
63	Results for $\text{Im}(A_2)$ on the 32^3 fine lattices.	214
64	Contributions to $\text{Re}(A_2)$ and $\text{Im}(A_2)$ from the different weak operators for $m_l^{\text{sea}} = 0.004$, $m_l^{\text{val}} = 0.002$, and $m_s^{\text{val}} = 0.03$ on the 32^3 fine lattices.	215
65	Pion mass in MeV on the quenched 24^3 lattices for the three different light quark masses.	216

66	Kaon mass in MeV on the quenched 24^3 lattices for the different light and strange quark masses.	216
67	Two pion energies in physical units and s wave phase shifts for $m_l = 0.0023$ on the 24^3 quenched lattices.	217
68	Two pion energies in physical units and s wave phase shifts for $m_l = 0.0047$ on the 24^3 quenched lattices.	218
69	Two pion energies in physical units and s wave phase shifts for $m_l = 0.0071$ on the 24^3 quenched lattices.	219
70	Wilson coefficients for the (27,1), (8,8), and (8,8) mixed operators for the 24^3 quenched lattices.	220
71	Renormalization factors for the (27,1), (8,8), and (8,8) mixed operators on the 24^3 quenched lattices.	220
72	Results on the quenched 24^3 quenched lattices for fit of δ_0 vs. q	221
73	Lellouch-Lüscher factor over free field normalization factor for the 24^3 quenched lattices.	221
74	Lellouch-Lüscher factor for the 24^3 quenched lattices.	222
75	Lattice matrix elements for the (27,1) operator with zero twists ($p = 0$) on the 24^3 quenched lattices.	223
76	Lattice matrix elements for the (27,1) operator with one twist ($p = \pi/L$) on the 24^3 quenched lattices.	224
77	Lattice matrix elements for the (27,1) operator with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices.	225

78	Lattice matrix elements for the (27,1) operator with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices.	226
79	Lattice matrix elements for the (8,8) operator with zero twists ($p = 0$) on the 24^3 quenched lattices.	227
80	Lattice matrix elements for the (8,8) operator with one twist ($p = \pi/L$) on the 24^3 quenched lattices.	228
81	Lattice matrix elements for the (8,8) operator with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices.	229
82	Lattice matrix elements for the (8,8) operator with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices.	230
83	Lattice matrix elements for the (8,8) mixed operator with zero twists ($p = 0$) on the 24^3 quenched lattices.	231
84	Lattice matrix elements for the (8,8) mixed operator with one twist ($p = \pi/L$) on the 24^3 quenched lattices.	232
85	Lattice matrix elements for the (8,8) mixed operator with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices.	233
86	Lattice matrix elements for the (8,8) mixed operator with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices.	234
87	Results for $\text{Re}(A_2)$ with zero twists ($p = 0$) on the 24^3 quenched lattices.	235
88	Results for $\text{Re}(A_2)$ error weighted average with zero twists ($p = 0$) on the 24^3 quenched lattices.	236

89	Results for $\text{Re}(A_2)$ with one twist ($p = \pi/L$) on the 24^3 quenched lattices.	237
90	Results for $\text{Re}(A_2)$ error weighted average with one twist ($p = \pi/L$) on the 24^3 quenched lattices.	238
91	Results for $\text{Re}(A_2)$ with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices.	239
92	Results for $\text{Re}(A_2)$ error weighted average with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices.	240
93	Results for $\text{Re}(A_2)$ with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices.	241
94	Results for $\text{Re}(A_2)$ error weighted average with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices.	242
95	Results for $\text{Im}(A_2)$ with zero twists ($p = 0$) on the 24^3 quenched lattices.	243
96	Results for $\text{Im}(A_2)$ error weighted average with zero twists ($p = 0$) on the 24^3 quenched lattices.	244
97	Results for $\text{Im}(A_2)$ with one twist ($p = \pi/L$) on the 24^3 quenched lattices.	245
98	Results for $\text{Im}(A_2)$ error weighted average with one twist ($p = \pi/L$) on the 24^3 quenched lattices.	246
99	Results for $\text{Im}(A_2)$ with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices.	247

100	Results for $\text{Im}(A_2)$ error weighted average with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices.	248
101	Results for $\text{Im}(A_2)$ with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices.	249
102	Results for $\text{Im}(A_2)$ error weighted average with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices.	250
103	Results for the fit of pseudoscalar meson mass as a function of valence quark masses on the 32^3 fine lattices.	251
104	Results for the fit of $E_{\pi\pi}^2$ as a function of valence light quark masses on the 32^3 fine lattices.	252
105	Results for the fit of $\text{Re}(A_2)$ as a function of valence quark masses on the 32^3 fine lattices.	252
106	Same as Table 105 but for $\text{Im}(A_2)$	253
107	Error weighted averages over different sea light quark masses of the results of extrapolations/interpolations on the 32^3 fine lattices.	253
108	Results for the fit of pseudoscalar meson mass as a function of valence quark masses on the 24^3 quenched lattices.	254
109	Results for the fit of $E_{\pi\pi}^2$ as a function of light quark mass for different numbers of twists on the 24^3 quenched lattices. . . .	255
110	Results for the fit of $\text{Re}(A_2)$ as a function of quark masses for different numbers of twists on the 24^3 quenched lattices. . . .	255
111	Same as Table 110 but for $\text{Im}(A_2)$	256

112	Final results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ extrapolated to the masses and kinematics of the coarse lattice and of the physical decay on the 24^3 quenched lattices.	256
113	$\text{Re}(A_2)$ and $\text{Im}(A_2)$ calculated using m_Ω , f_K , and r_0 to determine the lattice spacing on the 32^3 coarse lattices.	257
114	Systematic error budget for the 32^3 coarse calculation.	258
115	Values used in the numerical estimate of finite volume corrections from chiral perturbation theory.	258
116	Terms contributing to the sum of Eq. B.33. In the header row of the table $\delta_\beta(ABC)$ stands for $\delta_\beta(m_AL, m_BL, m_CL)$	259

List of Figures

1	Illustration of gauge links on a two-dimensional slice of the lattice.	261
2	Schematic diagram of the OPE.	262
3	Diagrams in the full theory corresponding to the four quark operators.	263
4	Form of diagrams that contribute at lowest order to a change in $\pi\pi$ phase shift when the weak interaction is turned on. . . .	264
5	Effective mass plot of the wall-point pion correlator with 0 momentum on the coarse 32^3 lattices.	265
6	Effective mass plot of the wall-wall pion correlator with 0 momentum on the coarse 32^3 lattices.	266
7	Effective mass plot of the wall-point pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. . . .	267
8	Effective mass plot of the wall-wall pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. . . .	268
9	Effective mass plot of the wall-point kaon correlator with 0 momentum on the coarse 32^3 lattices.	269

10	Effective mass plot of the wall-wall kaon correlator with 0 momentum on the coarse 32^3 lattices.	270
11	Effective mass plot of the wall-point kaon correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices.	271
12	Effective mass plot of the wall-wall kaon correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices.	272
13	Effective mass plot of the two pion correlator with 0 momentum on the coarse 32^3 lattices.	273
14	Effective mass plot of the two pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices.	274
15	Fit range dependence of the fitted $E_{\pi\pi}$ for the two pion correlator with $p = \sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices.	275
16	Effective mass plot of the ratio $C_{\pi\pi}/C_\pi^2$ with zero momentum on the coarse 32^3 lattices.	276
17	Effective mass plot of the ratio $C_{\pi\pi}/C_\pi^2$ with momentum $p = \sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices.	277
18	Fit range dependence of the fitted $\Delta E(p)$ to the ratio $C_{\pi\pi}/C_\pi^2$ for momentum $p = \sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices.	278
19	Quark flow diagram of the $\Delta I = 3/2$ $K \rightarrow \pi\pi$ correlator.	279
20	The kaon source and pion sink can't be too close together or too far apart if we want to get a good signal for the $K \rightarrow \pi\pi$ correlator.	280

21	Folding of $K \rightarrow \pi\pi$ correlators.	281
22	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with 0 momentum, the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	282
23	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with 0 momentum, the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	283
24	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with 0 momentum, the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	284
25	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	285
26	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	286
27	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	287
28	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 20$, on the coarse 32^3 lattices.	288

29	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	289
30	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 28$, on the coarse 32^3 lattices.	290
31	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 32$, on the coarse 32^3 lattices.	291
32	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 36$, on the coarse 32^3 lattices.	292
33	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 40$, on the coarse 32^3 lattices.	293
34	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 44$, on the coarse 32^3 lattices.	294
35	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with 0 momentum, the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	295

36	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with 0 momentum, the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	296
37	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with 0 momentum, the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	297
38	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	298
39	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	299
40	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices.	300
41	Effective mass plot of the pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices.	301
42	Effective mass plot of the pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices.	302
43	Effective mass plot of the pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices.	303

44	Effective mass plot of the kaon correlator with valence strange quark mass $m_s^{\text{val}} = 0.03$ and valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices.	304
45	Effective mass plot of the kaon correlator with valence strange quark mass $m_s^{\text{val}} = 0.03$ and valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices.	305
46	Effective mass plot of the kaon correlator with valence strange quark mass $m_s^{\text{val}} = 0.03$ and valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices.	306
47	Effective mass plot of the two pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices.	307
48	Effective mass plot of the two pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices.	308
49	Effective mass plot of the two pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices.	309
50	The two $\pi\pi$ correlators that are folded together on the fine 32^3 lattices.	310
51	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices.	311

52	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices.	312
53	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices.	313
54	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices.	314
55	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices.	315
56	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices.	316
57	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices.	317
58	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices.	318

59	Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices.	319
60	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices.	320
61	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices.	321
62	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices.	322
63	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices.	323
64	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices.	324
65	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices.	325

66	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices.	326
67	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices.	327
68	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices.	328
69	Effective mass plot of the pion correlator with 0 momentum and $m_l = 0.0023$ on the quenched 24^3 lattices.	329
70	Effective mass plot of the pion correlator with momentum π/L (one twist) and $m_l = 0.0023$ on the quenched 24^3 lattices. . . .	330
71	Effective mass plot of the pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) and $m_l = 0.0023$ on the quenched 24^3 lattices.	331
72	Effective mass plot of the pion correlator with momentum $\sqrt{3}\pi/L$ (three twists) and $m_l = 0.0023$ on the quenched 24^3 lattices.	332
73	Effective mass plot of the kaon correlator with 0 momentum and $m_l = 0.0023$, $m_s = 0.062$, on the quenched 24^3 lattices. . . .	333

74	Effective mass plot of the kaon correlator with momentum π/L (one twist) and $m_l=0.0023$, $m_s=0.062$, on the quenched 24^3 lattices.	334
75	Effective mass plot of the kaon correlator with momentum $\sqrt{2}\pi/L$ (two twists) and $m_l=0.0023$, $m_s=0.062$, on the quenched 24^3 lattices.	335
76	Effective mass plot of the kaon correlator with momentum $\sqrt{3}\pi/L$ (three twists) and $m_l=0.0023$, $m_s=0.062$, on the quenched 24^3 lattices.	336
77	Effective mass plot of the two pion correlator with zero momentum and $m_l=0.0023$ on the quenched 24^3 lattices.	337
78	Effective mass plot of the two pion correlator with momentum $p = \pi/L$ (one twist) and $m_l=0.0023$ on the quenched 24^3 lattices.	338
79	Effective mass plot of the two pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists) and $m_l=0.0023$ on the quenched 24^3 lattices.	339
80	Effective mass plot of the two pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists) and $m_l=0.0023$ on the quenched 24^3 lattices.	340

81	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	341
82	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	342
83	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. . . .	343
84	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	344
85	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	345

86	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	346
87	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	347
88	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	348
89	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	349
90	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. . . .	350

91	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	351
92	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	352
93	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 20$ on the quenched 24^3 lattices.	353
94	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	354
95	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 28$ on the quenched 24^3 lattices.	355

96	Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 32$ on the quenched 24^3 lattices.	356
97	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	357
98	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	358
99	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	359
100	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	360

101	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	361
102	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	362
103	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	363
104	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	364
105	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	365

106	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	366
107	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	367
108	Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices.	368
109	Plot of $I = 2$ two pion s wave phase shift as a function of k , comparing 32^3 coarse results to experiment.	369
110	Zoom in on the top left corner of Figure 109 so that the result from zero twists can be seen more clearly.	370
111	Plot of $I = 2$ two pion s wave phase shift as a function of q for $m_l = 0.0023$ on the quenched 24^3 lattices.	371
112	Plot of $I = 2$ two pion s wave phase shift as a function of q for $m_l = 0.0047$ on the quenched 24^3 lattices.	372
113	Plot of $I = 2$ two pion s wave phase shift as a function of q for $m_l = 0.0071$ on the quenched 24^3 lattices.	373

114	Plot of $E_{\pi\pi}^2$ vs. number of twists for the 32^3 coarse lattices, and extrapolation to energy conserving kinematics.	374
115	Plot of $\text{Re}(A_2)$ vs. number of twists for the 32^3 coarse lattices, and extrapolation to energy conserving kinematics.	375
116	Plot of $\text{Im}(A_2)$ vs. number of twists for the 32^3 coarse lattices, and extrapolation to energy conserving kinematics.	376
117	Linear fit to pseudoscalar meson mass as a function of the sum of valence quark masses for $m_l^{\text{sea}} = 0.004$ on the 32^3 fine lattices.	377
118	Linear fit to $E_{\pi\pi}^2$ as a function of the light valence quark mass m_l^{val} , for $m_l^{\text{sea}} = 0.004$ on the 32^3 fine lattices.	378
119	Linear fit to $\text{Re}(A_2)$ as a function of the valence quark masses for $m_l^{\text{sea}} = 0.004$ on the 32^3 fine lattices.	379
120	Same as Figure 119, but for $\text{Im}(A_2)$	380
121	Illustration showing that $E_{\pi\pi}$ will be lower in a larger volume.	381
122	Linear fit to pseudoscalar meson mass as a function of the sum of valence quark masses on the 24^3 quenched lattices.	382
123	Linear fit to $E_{\pi\pi}^2$ as a function of the light quark mass on the 24^3 quenched lattices.	383
124	Linear fit to $\text{Re}(A_2)$ as a function of the quark masses for zero twists ($n_{\text{tw}} = 0$) on the 24^3 quenched lattices.	384
125	Same as Figure 124 but for one twist ($n_{\text{tw}} = 1$).	385
126	Same as Figure 124 but for two twists ($n_{\text{tw}} = 2$).	386
127	Same as Figure 124 but for three twists ($n_{\text{tw}} = 3$).	387

128	Same as Figure 124 but for $\text{Im}(A_2)$	388
129	Same as Figure 128 but for one twist ($n_{\text{tw}} = 1$).	389
130	Same as Figure 128 but for two twists ($n_{\text{tw}} = 2$).	390
131	Same as Figure 128 but for three twists ($n_{\text{tw}} = 3$).	391
132	Plot of $E_{\pi\pi}^2$, extrapolated to the pion mass of the coarse lattice, as a function of n_{tw} for the 24^3 quenched lattices.	392
133	Plot of $\text{Re}(A_2)$, extrapolated to the pion and kaon masses of the coarse lattice, as a function of n_{tw} for the 24^3 quenched lattices.	393
134	Same as Figure 133 but for $\text{Im}(A_2)$	394

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1 Introduction

All current knowledge of particle physics is incorporated into a framework called the *standard model* which has a theoretical and mathematical basis in *quantum field theory*. In this framework, various “matter” particles interact with each other by exchanging other “force” particles. For example, the electron is a fundamental matter particle, as are the quarks that make up the protons and neutrons in the nuclei of atoms. These building blocks constitute all known matter in nature,¹ and the different force particles give rise to all of the forces felt by matter (electromagnetic, strong, and weak), except for gravity. Gravity has so far evaded being incorporated consistently into the standard model, but it is extremely weak compared to the other forces and has not yet had any observable effects at the energy scales in particle accelerators. In the standard model there are also *anti-particles* corresponding to each particle, that have the same mass and opposite charge. However, anti-matter is found to be much less abundant than matter in our universe, otherwise there would be only photons in the universe and we would

¹This does *not* include so-called dark matter and dark energy which are known to make up the vast majority of the mass and energy of the universe, but whose compositions are still unknown.

not exist! This “missing anti-matter” remains a puzzle in particle physics and cosmology.

In this connection, we describe a series of experiments and theoretical advances in understanding the weak interaction. In 1956, Lee and Yang first suggested that the weak interaction might not respect parity symmetry (P) [1], which is the invariance of physics under the transformation of a system into its mirror image. This was confirmed shortly after in experimental measurements of beta decay [2, 3]. It still appeared that CP was conserved by the weak interaction, where C refers to charge conjugation (the replacement of all particles with their anti-particles). However, in 1964 CP symmetry violation in the weak interaction was observed to occur in neutral kaon decay [4]. It has also been observed more recently in B decays [5]. Note that CPT invariance is always expected to hold in a quantum field theory [6], where T refers to time reversal. If CP is violated, then T must also be violated, and indeed this is also found to be the case in neutral kaon decays. All experimental tests for CPT violation have come up negative [5]. CP violation is a necessary condition for the dominance of matter in the universe [7], hence it is of great interest to understand the mechanism and amount of CP violation in the standard model.

Another active area of investigation in the standard model is strong dynamics. Quarks are always bound together as *hadrons*, either in groups of three called *baryons* such as the proton and neutron, or in quark anti-quark pairs called *mesons* such as the kaon and pion. They are held together by

the strong force, which also mediates many of the scattering and decay processes of these particles. The quantum field theory of the strong interaction is called Quantum Chromodynamics (QCD), but the strength of the interaction means that it cannot be described by perturbation theory at low energies. The only analytical methods available are phenomenological models and effective field theories such as chiral perturbation theory, which cannot give a complete, first principles description of strong dynamics. However, the numerical technique of *lattice gauge theory* is a widely used method for doing non-perturbative calculations of strongly coupled field theories, especially QCD. Lattice gauge theory was first proposed by Wilson [8] and involves discretizing space and time into a lattice of points and numerically evaluating physical quantities using the path integral formulation of quantum field theory. Chapter 3 outlines various aspects of the method. Calculations in lattice QCD usually require large scale parallel computing machines, such as QCDOC [9] and the Blue Gene P which were used in the calculation presented in this report. Successful lattice calculations have been performed of a variety of quantities relating to mesons and baryons such as masses, decay constants, and form factors.

The subject of this report is the decay of a single kaon into two pions ($K \rightarrow \pi\pi$ decay). A calculation of this decay must be done using the technique of lattice gauge theory since it involves kaons and pions. However, the decay is caused by the weak interaction and for neutral kaons it is one of the decays which can be used to probe CP violation. Thus $K \rightarrow \pi\pi$ calculations

on the lattice are extremely interesting for several reasons. In Chapter 2 we discuss the phenomenology of these decays, their connection to CP violation, and we specify what needs to be computed on the lattice.

A number of theoretical and technical advances over the last thirty years have made $K \rightarrow \pi\pi$ calculations on the lattice possible. The problem of incorporating the vastly different energy scales of the weak interaction and the strong interaction between kaons and pions into a single lattice calculation is the subject of Chapter 4. There the use of the *operator product expansion* to represent the weak interaction is discussed, as well as the technique of *non-perturbative renormalization* to convert the weak operators into a continuum renormalization scheme. In Chapter 5 we discuss the method for dealing with two pion states in finite volume, and then go on to discuss the related problem of extracting $K \rightarrow \pi\pi$ matrix elements from correlation functions in finite volume and Euclidean time. We also make use of different boundary conditions in spatial directions [10, 11, 12], as discussed in Chapter 5 and 6, in order to simulate pions with the correct, non-zero momentum dictated by conservation of energy, and without introducing the large statistical noise inherent in dealing with excited states. Finally, state of the art computing resources allow us to perform a calculation in which the pion mass is close to physical in a reasonable amount of time.

Two pioneering $K \rightarrow \pi\pi$ calculations were performed in [13, 14] in which $K \rightarrow \pi\pi$ matrix elements were extracted indirectly from other single hadron matrix elements using chiral perturbation theory. These calculations also

employed the *quenched approximation*, (an uncontrolled approximation in which the fermion determinant is neglected in the path integral), and pions which were unphysically heavy. Much of the formalism from [13] is used in the current report. The method of indirectly calculating $K \rightarrow \pi\pi$ matrix elements was used again in [15], this time with fully dynamical ensembles (i.e. not quenched) and with next-to-leading order expressions in partially quenched $SU(3)$ chiral perturbation theory. However, it was found that the next-to-leading order terms were often equal in size to the leading order terms, calling into question the convergence of the series. Direct $K \rightarrow \pi\pi$ calculations have been performed in [16, 17], the former using methods similar to those of the current report, and the latter using a reference frame in which the initial kaon is not at rest. Both of these calculations used the quenched approximation and large extrapolations of results down to the physical pion mass. The result for the decay rate in [16] was several times larger than the experimental value, although agreement was found in [17].

The main calculation of this report significantly improves on all of the shortcomings of the previous calculations. It is a full dynamical, direct $K \rightarrow \pi\pi$ calculation with essentially physical kaon and pion masses. Two companion calculations, one quenched, and one on a finer lattice with only zero momentum pions, are also presented and are helpful for estimating the systematic error in the first calculation. The setup and results of the three calculations are presented in Chapters 6 and 7 respectively. The final result for the decay amplitude represents a significant milestone in solving the long-

standing problem of calculating two body hadronic decays on the lattice, and is an important step in understanding CP violation in the standard model.

2 The Phenomenology of $K \rightarrow \pi\pi$ Decays

$K \rightarrow \pi\pi$ decays can proceed either through the $\Delta I = 1/2$ or $\Delta I = 3/2$ channel (to be defined in this chapter). The amplitudes for these processes in continuum quantum field theory are denoted A_0 and A_2 respectively, and can be related directly to the observed rates for these processes. The results of lattice calculations can in principle be used to find A_0 and A_2 in terms of CKM matrix elements through a procedure described in Chapters 4 and 5. (Calculating A_0 on the lattice is considerably more difficult since disconnected diagrams must be computed; this report only presents a calculation of A_2). Thus lattice calculations can be used to calculate kaon decay rates into two pions in terms of CKM matrix elements. It is interesting in and of itself to see if lattice QCD can accurately predict such decay rates, particularly to see if QCD effects alone can account for the experimentally observed suppression of the $\Delta I = 3/2$ channel relative to the $\Delta I = 1/2$ channel, which does not have a simple phenomenological explanation [18]. This suppression is called the “ $\Delta I = 1/2$ rule”, and is regarded as an important figure of merit for how realistic a $K \rightarrow \pi\pi$ calculation is.

These decays are also of great interest because they can be related to

CP violation in the standard model. In the present chapter we will see that direct CP violation in $K \rightarrow \pi\pi$ decays is parametrized by a quantity called ϵ' , whose magnitude can be found from experimental measurements of different $K \rightarrow \pi\pi$ decay rates. The magnitude of ϵ' can also be obtained from the complex amplitudes A_0 and A_2 . In this way ϵ' can be related to the results of lattice calculations and CKM matrix elements, in particular the CP violating phase of the CKM matrix. Lattice calculations of $K \rightarrow \pi\pi$ decays, as well as of neutral kaon oscillation, thus represent an extremely valuable way of putting constraints on the CP violating phase when compared to experimental results.

2.1 The Standard Model of Particle Physics

We give a brief description of the standard model of particle physics, focusing on the sectors that are important for $K \rightarrow \pi\pi$ decays. For more detailed introductions to quantum field theory and the standard model of particle physics the reader is referred to any of several introductory texts [6, 18, 19, 20, 21]. We focus on the strong interaction, which describes the dynamics of quarks and gluons and thus the dynamics of the quarks bound together inside the kaons and pions. We are equally interested in the weak interaction which is responsible for the kaon decay into two pions. In the calculation of this report, electromagnetic effects (such as the Coulomb forces between charged pions) are neglected as they are arguably a small perturbation to the $K \rightarrow \pi\pi$ process and require more sophistication and computational

resources to incorporate into lattice calculations in general. Many lattice calculations of simpler phenomena have begun to incorporate the effects of electromagnetism [22, 23, 24, 25, 26].

The fundamental particles in the standard model are the fermions, which consist of leptons (electron, muon, tau, and associated neutrinos) and quarks (up, down, charm, strange, top, and bottom, denoted u , d , c , s , t , and b), as well as the gauge bosons which mediate the forces between particles. Each particle has an associated anti-particle as well, although in some cases the particle and anti-particle are identical. Among the fermions we will only be interested in the quarks here. The gauge bosons consist of gluons which mediate the strong force, the $W^\pm$ and Z which mediate the weak force, and the photon which mediates electromagnetism. Finally there is the Higgs boson which gives other particles mass, and has not yet been directly detected. Quarks can in general interact via all three forces, whereas leptons can only interact weakly and electromagnetically. Free quarks do not exist in nature as they are always bound together either in quark-antiquark pairs called mesons (for example the kaons and pions that are the main subject of this report), or in states consisting of three quarks called baryons (proton, neutron, etc.). This phenomenon, called *confinement*, is a feature of the strong interaction, and is in contrast to the leptons which can be found by themselves in nature since they do not interact strongly.

Mathematically, the structure of the standard model is determined by the $SU(3) \otimes SU(2) \otimes U(1)$ gauge symmetry that it possesses. The fermions

transform under the fundamental representations of these symmetries, and gauge bosons must exist to allow these symmetries to be local rather than global. In general there will be one gauge boson for each generator of the symmetry. The $SU(3)$ symmetry is associated with the strong force, meaning that the gauge bosons of this symmetry are the gluons. Thus there are 8 gluons, and 3 different *colors* of each each quark. Their specific interactions are specified by the standard model Lagrangian, and are what we simulate directly in lattice calculations. The name for this mathematical theory of the strong interaction is Quantum Chromodynamics (QCD). More details on the strong interaction and lattice QCD are found in Chapter 3.

The $SU(2) \otimes U(1)$ symmetry is called the electroweak symmetry, and it is spontaneously broken resulting in the $W^\pm$ and Z bosons of the weak interaction and the photon of the electromagnetic interaction. The spontaneous symmetry breaking involves the Higgs field. It acquires a vacuum expectation value which allows for mass terms for the quarks, leptons, and the $W^\pm$ and Z bosons. These mass terms would otherwise be forbidden by the electroweak symmetry. Spontaneous symmetry breaking also affects the form of the interaction terms in the weak Lagrangian, which can be written

$$\frac{g_W}{\sqrt{2}} J_\mu^+ W^{+\mu} + \text{Hermitian conjugate} \quad (2.1)$$

where g_W is the weak coupling constant, $W^{+\mu}$ is the field that represents the

W^+ particle, and J_μ^+ is the following “current” term:

$$J_\mu^+ = \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} \end{pmatrix} \gamma_\mu V^{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \quad (2.2)$$

In Eq. 2.2, u , c , t are the fields representing the up, charm, and top quark respectively, while d , s , b are the fields representing the down, strange and bottom quarks. V^{CKM} is a 3×3 matrix called the Cabibbo-Kobayashi-Maskawa (CKM) matrix which mixes the d , s , and b quarks:

$$V^{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \quad (2.3)$$

That is, the mass eigenstates d , s , b are not eigenstates of the weak interaction.

The CKM matrix must be unitary to maintain the form of the kinetic terms in the standard model Lagrangian. Due to the possibility of quark field redefinitions there are only four real degrees of freedom in the CKM matrix that actually have physical significance. In most parametrizations these are organized into three “mixing angles” and one “CP violating phase”, such that if the CP violating phase were 0 then all of the CKM matrix elements would

be real and there would be no CP violation in the quark sector.¹ These four parameters are input parameters to the standard model, (i.e. they cannot be calculated in the framework of the standard model, only measured), just like the particle masses. The CKM matrix can be thought to parametrize low energy weak physics in the quark sector in the sense that most measurements in low energy weak physics can be interpreted as putting bounds on CKM matrix elements and thus the four parameters. New physics beyond the standard model could potentially show up as contradictory bounds from different experiments, and in particular as apparent non-unitarity of the CKM matrix [27].

The standard parametrization [5] of the CKM matrix is

$$V^{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (2.4)$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$, so that θ_{12} , θ_{13} , and θ_{23} are the three mixing angles and δ is the CP violating phase. This is the parametrization we will assume in this report.

¹It should be mentioned that CP violation in the quark sector could theoretically occur via the strong interaction through the so-called θ term, but experiments have constrained this term to be extremely small.

2.2 Kaons, Pions, and Isospin symmetry

The pions are the lightest mesons in nature, being made up of up and down quarks and anti-quarks, while the kaons are the next lightest, being made up of a strange quark or anti-quark and an up or down quark or anti-quark.² The quark content and masses of the kaons and pions are shown in Table 1. The up and down quark are nearly degenerate, which manifests itself in the nearly equal masses of the charged and neutral kaons, and similarly for the pions. This leads to an approximate *isospin symmetry*, an invariance under rotating between the up and down quark, which becomes exact when their masses are equal³. This is exactly analogous to physical rotations in a spin 1/2 system, (the symmetry group is SU(2) in both cases), and the formalism of angular momentum algebra can be used in the same way. For isospin we label the quantum numbers as I and m_I rather than l and m_l , so that an up quark state (and an anti-down quark state) has $I = 1/2$, $m_I = 1/2$ and a down quark state (and an anti-up quark state) has $I = 1/2$, $m_I = -1/2$. The strange quark and all other quarks have $I = 0$, $m_I = 0$ (i.e. they are singlets under isospin). The pions are the $I = 1$ states that result from combining two $I = 1/2$ states, with $m_I = -1, 0, 1$ for the π^- ,

²Both the kaon and pion decay in fractions of a second however, and hence they do not make up the matter of our every day experience.

³Isospin symmetry and the equality of the up and down quark masses, (and thus the equality of neutral and charged kaon masses as well as for the pions), is assumed in the calculation of this report. Isospin breaking effects are generally included in those lattice calculations which attempt to include electromagnetism.

π^0 , π^+ respectively. Kaons are $I = 1/2$ states since they have one strange or anti-strange quark, and therefore only one up, down, anti-up, or anti-down quark.

Two pion states can have $I = 0, 1$, or 2 , with all of the possible m_I values for each. Since isospin is not even an approximate symmetry of the weak interaction, kaons can decay weakly into these two pion states, however they cannot decay to the $I = 1$ state as a consequence of Bose symmetry. Since kaons have $I = 1/2$, decays to the $I = 0$ two pion state are called $\Delta I = 1/2$ channel decays, and decays to the $I = 2$ two pion state are called $\Delta I = 3/2$ channel decays. The amplitudes for these decays are denoted by A_0 and A_2 respectively. Specifically,

$$\langle \pi\pi(I, m_I = 0) | \mathcal{L}_W(0) | K^0 \rangle = \sqrt{2} A_I e^{i\delta_I} \quad (2.5)$$

where $I = 0, 2$, and δ_I is the s wave scattering phase shift for the two pion state. The reason for the form of this definition is that if time reversal symmetry holds then Watson's theorem implies that A_I will be purely real [28]. The factor of $\sqrt{2}$ is convention when the two pion state consists of pions with definite momentum and has a relativistic normalization (see Eq. 5.66 and Eq. A.2).

Experimentally it is found that $|A_2|/|A_0| \approx 1/22$. This suppression of the decay to the $I = 2$ state is called the “ $\Delta I = 1/2$ rule”, and has no simple phenomenological explanation. One motivation for lattice calculations of A_0

and A_2 is to reproduce this result, and thus explain the origin of the $\Delta I = 1/2$ rule as originating solely from strong dynamics.

2.3 Direct and Indirect CP Violation

We begin with a discussion of CP violation in the neutral kaon system, and use this as a foundation to discuss CP violation in $K \rightarrow \pi\pi$ decays. From Table 1 it can be seen that there are two neutral kaons, K^0 and $\bar{K}^0$. We choose conventions in which the K^0 and $\bar{K}^0$ states are related through the CP operator by

$$CP|K^0\rangle = -|\bar{K}^0\rangle \quad (2.6)$$

$$CP|\bar{K}^0\rangle = -|K^0\rangle. \quad (2.7)$$

In general, the K^0 and $\bar{K}^0$ states will mix with each other over time and also decay into other states (such as two pions) due to the weak interaction. Second order time-dependent perturbation theory can be used to describe this situation [29, 30]. If we work with a two-component column matrix

$$|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = a(t)|K^0\rangle + b(t)|\bar{K}^0\rangle \quad (2.8)$$

then the time-development can be described by

$$i\frac{d}{dt}|\psi(t)\rangle = \left(M - \frac{i}{2}\Gamma\right)|\psi(t)\rangle \quad (2.9)$$

where the *mass matrix* is given by

$$M - \frac{i}{2}\Gamma = \begin{pmatrix} A & p^2 \\ q^2 & A \end{pmatrix}. \quad (2.10)$$

Second order perturbation theory allows one to write the elements of the mass matrix in terms of matrix elements of the weak Hamiltonian. The diagonal elements of the mass matrix can then be shown to be equal to each other due to CPT symmetry.

The eigenstates of the mass matrix (which do not mix with time) are

$$|K_{L/S}\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}} [p|K^0\rangle \pm q|\overline{K^0}\rangle]. \quad (2.11)$$

On the other hand, the eigenstates of the CP operator are

$$|K_{\pm}^0\rangle = \frac{1}{\sqrt{2}} [|K^0\rangle \mp |\overline{K^0}\rangle] \quad (2.12)$$

which, from Eq. 2.6 and 2.7, have eigenvalues under CP of ± 1 . If CP were a good symmetry of the system it can be shown from the second order perturbation theory expression that $p = q$ and the states of Eq. 2.11 and Eq. 2.12 would be the same. That is,

$$|K_S\rangle \xrightarrow{p=q} |K_+^0\rangle \quad |K_L\rangle \xrightarrow{p=q} |K_-^0\rangle. \quad (2.13)$$

Since Eq. 2.11 actually only depends on p/q (neglecting an irrelevant

overall phase factor), we re-express it in terms of the parameter $\bar{\epsilon}$,

$$|K_{L/S}\rangle = \frac{1}{\sqrt{1+|\bar{\epsilon}|^2}}[|K_{\mp}^0\rangle + \bar{\epsilon}|K_{\pm}^0\rangle] \quad (2.14)$$

where

$$\frac{p}{q} = \frac{1+\bar{\epsilon}}{1-\bar{\epsilon}} \quad (2.15)$$

and where we have used Eq. 2.12 to work in terms of CP eigenstates. If CP violation is small in the neutral kaon system, (which it is observed to be), then $|\bar{\epsilon}| \ll 1$. If there is no CP violation at all then we would have $\bar{\epsilon} = 0$, and Eq. 2.14 would reduce to Eq. 2.13.

The eigenvalues corresponding to the eigenstates of Eq. 2.11 are $A \pm pq$, and thus the time development of the eigenstates would be

$$|K_{L/S}(t)\rangle = e^{-i(A \pm pq)t} |K_{L/S}\rangle. \quad (2.16)$$

Since $A \pm pq$ are independent quantities, and both are independent from p/q , we write them in terms of their real and imaginary parts:

$$A \pm pq = m_{L/S} - \frac{i}{2}\Gamma_{L/S} \quad (2.17)$$

so that the time development is

$$|K_{L/S}(t)\rangle = e^{-\Gamma_{L/S}t/2} e^{-im_{L/S}t} |K_{L/S}\rangle \quad (2.18)$$

and thus m_L and m_S can be interpreted as masses, and Γ_L and Γ_S as decay rates of the K_L and K_S particles respectively. In summary, the neutral kaon system is parametrized by m_L , m_S , Γ_L , and Γ_S describing the K_L and K_S particles respectively, and also by $\bar{\epsilon}$ which describes the amount of CP violation in the system.

Since the two pion states and three pion states into which neutral kaons can decay are CP even and CP odd respectively, if CP were conserved then only the K_+^0 could decay to two pions and only the K_-^0 could decay to three pions. When this does not hold, we have *direct* CP violation in the decay. Since the phase space for the two pion decay is much larger, this decay happens much more quickly. To the extent which CP violation is small, we see from Eq. 2.14 that the K_S state is mostly made up of the K_+^0 state, and the K_L state is mostly made up of the K_-^0 state. Thus the K_S decays much more quickly than the K_L ($\Gamma_S \gg \Gamma_L$), hence the subscripts S and L which stand for “short” and “long” respectively.

When CP is violated, the K_S does have a small component of the K_-^0 state and the K_L has a small component of the K_+^0 state, these components being proportional to $\bar{\epsilon}$. This allows the K_S and K_L to decay to three and two pions respectively even if there is no direct CP violation. This phenomenon is called *indirect* CP violation in the decay since it is due to the initial kaon state not being an eigenstate of CP rather than an actual transition between states of different CP. We see that the amount of indirect CP violation is related to $\bar{\epsilon}$, whereas the amount of direct CP violation has nothing to do

with this parameter.

In an experiment one would typically start out with a pure K^0 beam or a pure $\overline{K}^0$ beam, since these are eigenstates of the strong interaction which likely produced the particles in the first place. The time development of the particle states in such beams would be

$$|K^0(t)\rangle = g_+(t)|K^0\rangle + \frac{q}{p}g_-(t)|\overline{K}^0\rangle \quad (2.19)$$

$$|\overline{K}^0(t)\rangle = \frac{p}{q}g_-(t)|K^0\rangle + g_+(t)|\overline{K}^0\rangle \quad (2.20)$$

$$g_{\pm} = \frac{1}{2}e^{-\Gamma_L t/2}e^{-im_L t} [1 \pm e^{-\Delta\Gamma t/2}e^{i\Delta m t}] \quad (2.21)$$

where $\Delta\Gamma = \Gamma_S - \Gamma_L$ and $\Delta m = m_L - m_S$. Such experiments provide measurements of Δm and $\bar{\epsilon}$. In particular it has been found that $\Delta m = 3.483(6) \times 10^{-12}$ MeV [5], much less than the neutral kaon mass of 497.648(22) MeV itself.

It is possible to measure both direct and indirect CP violation by looking at the decay products of the neutral kaons. We describe how this works for the decay into two pions. It was stated below Eq. 2.5 that A_0 and A_2 will be real if time-reversal invariance (T) holds. Since CPT is always conserved in quantum field theory, CP violation implies a violation of time-reversal invariance. Thus a non-zero imaginary part of A_0 or A_2 would be due to direct CP violation in the $K \rightarrow \pi\pi$ decay.

Experimentally one has the following measures of CP violation in $K \rightarrow \pi\pi$

decay:

$$\eta_{+-} = \epsilon + \epsilon' = \frac{\langle \pi^+ \pi^- | \mathcal{L}_W(0) | K_L \rangle}{\langle \pi^+ \pi^- | \mathcal{L}_W(0) | K_S \rangle} \quad (2.22)$$

$$\eta_{00} = \epsilon - 2\epsilon' = \frac{\langle \pi^0 \pi^0 | \mathcal{L}_W(0) | K_L \rangle}{\langle \pi^0 \pi^0 | \mathcal{L}_W(0) | K_S \rangle}. \quad (2.23)$$

We now wish to substitute Eq. 2.14 into Eq. 2.22 and 2.23. We also use the isospin decompositions

$$|\pi^0 \pi^0\rangle = \sqrt{\frac{2}{3}} |\pi\pi(I=2, m_I=0)\rangle - \frac{1}{\sqrt{3}} |\pi\pi(I=0, m_I=0)\rangle \quad (2.24)$$

$$\frac{1}{\sqrt{2}} (|\pi^+ \pi^- \rangle + |\pi^- \pi^+ \rangle) = \frac{1}{\sqrt{3}} |\pi\pi(I=2, m_I=0)\rangle + \sqrt{\frac{2}{3}} |\pi\pi(I=0, m_I=0)\rangle \quad (2.25)$$

and the definition of Eq. 2.5 to write things in terms of A_0 and A_2 . We need the following relation for matrix elements containing a $\overline{K^0}$ as well

$$\langle \pi\pi(I, m_I=0) | \mathcal{L}_W(0) | \overline{K^0} \rangle = -\sqrt{2} A_I^* e^{i\delta_I} \quad (2.26)$$

which is a result of CPT symmetry.

Upon substituting everything into Eq. 2.22 and 2.23, and solving for the parameters ϵ and ϵ' , we find

$$\epsilon = \bar{\epsilon} + i \frac{\text{Im}(A_0)}{\text{Re}(A_0)} \quad (2.27)$$

$$\epsilon' = \frac{ie^{i(\delta_2 - \delta_0)}}{\sqrt{2}} \frac{\text{Re}(A_2)}{\text{Re}(A_0)} \left[\frac{\text{Im}(A_2)}{\text{Re}(A_2)} - \frac{\text{Im}(A_0)}{\text{Re}(A_0)} \right]. \quad (2.28)$$

To obtain these results we used the fact that CP violation is small so that we can work to first order in $\bar{\epsilon}$, $\text{Im}(A_0)$, and $\text{Im}(A_2)$. We also used the observation of the $\Delta I = 1/2$ rule that $|A_2|/|A_0| \approx 1/22 \ll 1$. Since CP violation is small it must be that $|A_I|$ is dominated by $\text{Re}(A_I)$, and thus $\text{Re}(A_2)/\text{Re}(A_0) \ll 1$. We used this to neglect $\text{Re}(A_2)$ in comparison to $\text{Re}(A_0)$ to arrive at Eq. 2.27 and 2.28.

Since $\text{Im}(A_0)$ and $\text{Im}(A_2)$ are non-zero only when there is direct CP violation, we see in Eq. 2.27 and 2.28 that if there were no direct CP violation we would have $\epsilon = \bar{\epsilon}$ and $\epsilon' = 0$. Thus ϵ' is the experimental measure of direct CP violation and ϵ is the experimental measure of indirect CP violation in $K \rightarrow \pi\pi$ decays (although the imaginary part of ϵ does have a contribution due to direct CP violation). The measured values for these parameters are [5]

$$|\epsilon| = 2.228(11) \times 10^{-3} \quad (2.29)$$

$$\text{Re}(\epsilon'/\epsilon) \approx \epsilon'/\epsilon = 1.65(26) \times 10^{-3} \quad (2.30)$$

$$\phi_\epsilon = \text{phase}(\epsilon) = 43.5(7)^\circ \quad (2.31)$$

$$\phi_{\epsilon'} = \text{phase}(\epsilon') = \delta_2 - \delta_0 + \frac{\pi}{2} = 42.3(1.5)^\circ. \quad (2.32)$$

We see that direct CP violation is a much smaller effect than indirect CP violation (which is itself small).

The role of the lattice is to enable us to perform first principles calculations of A_0 and A_2 (real and imaginary parts), as well as of $\bar{\epsilon}$. We will see in Chapter 4 that A_0 and A_2 can be written in terms of products of CKM

matrix elements and matrix elements involving kaon and two pion initial and final states. The latter can be calculated on the lattice, (and in fact the lattice is the only known way to calculate them since they involve non-perturbative, strong dynamics). The CKM matrix elements can be taken from a world average of experiments, as found in [5], and combined with the lattice results to arrive at A_0 and A_2 . Thus we can calculate ϵ' and compare with the experimental results.⁴ Any discrepancy could be a sign of new physics. Alternatively, one can use the experimental result for ϵ' and the lattice results to put constraints on CKM matrix elements. In particular, this calculation puts a valuable constraint on the CP violating phase of the CKM matrix. (We only calculate A_2 in the present report, and thus a full calculation of this sort cannot be done, but we have nevertheless made significant progress towards this goal). The situation is similar for ϵ for which lattice calculations of neutral kaon mixing can be used. (From Eq. 2.27 we see that the results of $K \rightarrow \pi\pi$ calculations are also needed to obtain the imaginary part of ϵ). In particular, the quantity B_K , which is proportional to a matrix element with K^0 and $\overline{K}^0$ initial and final states, can be calculated on the lattice [31, 32, 33, 34, 35, 36, 37, 38].

⁴The two pion phase shifts δ_0 and δ_2 can be taken from experiment or from the lattice calculation, since it is necessary to calculate the phase shifts in order to perform the full $K \rightarrow \pi\pi$ calculation.

3 Lattice QCD

We describe here the method by which quantities in QCD are evaluated numerically. Space time is discretized into a *lattice* of points. Fermion fields are associated with each space-time point, while gauge fields are associated with the *links* that connect one point to another. This technique is termed *lattice QCD*, and we provide here only a brief introduction. More details can be found in many introductory texts on the subject, such as [39, 40].

A non-perturbative treatment of QCD, such as lattice QCD, is essential in describing the dynamics of the quarks that are bound together in the kaons and pions, and also in describing the interaction of the two final state pions with each other in $K \rightarrow \pi\pi$ decay. Perturbation theory is insufficient for this purpose since it can only describe weakly interacting quarks, rather than quarks confined in mesons. In Chapter 4 we will discuss how to incorporate the weak interaction that causes the kaon to decay to two pions. The decay will be separated into a high energy (short distance) and low energy (long distance) part. Computing the latter requires the evaluation of a QCD matrix element which can only be achieved through a non-perturbative calculation.

3.1 QCD Lagrangian

Our starting point is the continuum QCD Lagrangian, before anything has been discretized

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i \not{D} - m_f) \psi_f - \frac{1}{4} \text{Tr} F^{\mu\nu} F_{\mu\nu}. \quad (3.1)$$

The sum is over the different quark flavors being described (up, down, charm, strange, top, bottom in the real world). This Lagrangian describes only quarks and gluons as they interact via the strong interaction, which is what we simulate on the lattice. The quarks are represented by the field ψ_f which is a four component Dirac spinor and also carries a color index (not shown) which can take on three different values.

The field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \sum_a \lambda^a F_{\mu\nu}^a \quad (3.2)$$

where λ^a are the generators of the color $SU(3)$ group (e.g. they can be chosen to be the Gell-Mann matrices), and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_0 f^{abc} A_\mu^b A_\nu^c. \quad (3.3)$$

Here the A_μ^a are the fields that represent the gluons, f^{abc} are the structure constants of $SU(3)$, and g_0 is the bare coupling constant. We also use the

notation A_μ which denotes the following 3×3 matrix:

$$A_\mu = \sum_a \lambda^a A_\mu^a. \quad (3.4)$$

The covariant derivative $\mathcal{D}$ in Eq. 3.1 is given by

$$\mathcal{D} = \gamma^\mu (\partial_\mu - ig_0 A_\mu). \quad (3.5)$$

An important property of the QCD Lagrangian is *gauge invariance*. It is invariant under the following local transformation

$$\psi(x) \rightarrow \psi'(x) = G(x)\psi(x) \quad (3.6)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = G(x)A_\mu(x)G^{-1}(x) - \frac{i}{g_0}\partial_\mu G(x)G^{-1}(x) \quad (3.7)$$

where $G(x)$ is an $SU(3)$ matrix. It is important that the discretized version of QCD have this symmetry as well. Gauge invariance, and the fact that $SU(3)$ is non-Abelian, are what necessitate the last term in Eq. 3.3, which would not appear in a theory like QED that has an Abelian gauge group. This last term is what makes QCD interesting since it leads to gluon self-interaction (for example, three gluon and four gluon vertices). These self-interactions are the source of non-trivial features like confinement and asymptotic freedom.

In the path integral formulation, all of the physics contained in a quantum field theory is extracted by evaluating expectation values of operators, which can be written in the form of path integrals. For the case of QCD, we need

to evaluate the expectation value of operators $O(A_\mu, \psi, \bar{\psi})$:

$$\langle O(A_\mu, \psi, \bar{\psi}) \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi O(A_\mu, \psi, \bar{\psi}) e^{iS_{\text{QCD}}} \quad (3.8)$$

where

$$Z = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS_{\text{QCD}}} \quad (3.9)$$

and

$$S_{\text{QCD}} = \int d^4x \mathcal{L}_{\text{QCD}}(x) \quad (3.10)$$

with $\mathcal{L}_{\text{QCD}}(x)$ given by Eq. 3.1. The integrations in Eq. 3.8 and 3.9 are *functional integrals* over all possible field configurations, and with a measure that is invariant under the $SU(3)$ gauge group. There is no known way to evaluate these integrals analytically in general. Note that the A_μ variables are just ordinary numbers rather than operators, and that ψ and $\bar{\psi}$ are Grassmann variables that anti-commute with each other.

Before discretizing the functional integral of Eq. 3.8 it is necessary Wick rotate to imaginary time. This is accomplished by putting

$$x^0 \rightarrow -ix_4 \quad (3.11)$$

$$A^0 \rightarrow +iA_4. \quad (3.12)$$

Eq. 3.8 becomes

$$\langle O(A_\mu, \psi, \bar{\psi}) \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi O(A_\mu, \psi, \bar{\psi}) e^{-S_{\text{QCD}}^E} \quad (3.13)$$

(with a similar modification to Eq. 3.9), where the Euclidean Lagrangian is

$$\mathcal{L}_{\text{QCD}}^E = \sum_f \bar{\psi}_f (\not{D}^E + m_f) \psi_f + \frac{1}{4} \text{Tr} F_{\mu\nu} F_{\mu\nu} \quad (3.14)$$

and the Euclidean covariant derivative is given by

$$\not{D}^E = \gamma_\mu^E (\partial_\mu - ig_0 A_\mu) \quad (3.15)$$

with the Euclidean gamma matrices defined by $\gamma_4^E = \gamma^0$, $\gamma_i^E = -i\gamma^i$. Note that the Euclidean gamma matrices are Hermitian and satisfy

$$\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu}. \quad (3.16)$$

We will henceforth omit the E superscript and assume all quantities are Euclidean.

The Lagrangian of Eq. 3.14 is quadratic in ψ_f , and the integration over fermion fields can be done explicitly. Eq. 3.13 becomes

$$\langle O(A_\mu, \psi, \bar{\psi}) \rangle = \frac{1}{Z} \int \mathcal{D}A O'(A_\mu) e^{-S_G} \prod_f \det(\not{D} + m_f) \quad (3.17)$$

where $O'(A_\mu)$ is obtained from $O(A_\mu, \psi, \bar{\psi})$ by Wick contracting all of the ψ and $\bar{\psi}$ fields so that $\psi_f \bar{\psi}_f$ gets replaced by $(\not{D} + m_f)^{-1}$. S_G is the pure gauge part of the action, i.e. the action resulting from the last term of Eq. 3.14. Eq. 3.17 shows that QCD is equivalent to a statistical mechanical system

with Boltzmann weight $e^{-S_G} \prod_f \det(\not{D} + m_f)$, and hence numerical and Monte Carlo methods similar to those used in statistical mechanics can be used to evaluate QCD once it is discretized. We now discuss this discretization.

3.2 Gauge Actions

On the lattice we prefer to work with “gauge links” $U(x, x + a\hat{\mu}) = U_\mu(x)$ instead of the gauge fields $A_\mu(x)$. Here $\hat{\mu}$ is a unit vector in the μ direction, a is the lattice spacing, and the links are $SU(3)$ matrices that are pictured as pointing from the space-time point x to the point $x + a\hat{\mu}$ which is one lattice unit forward in the μ direction. The situation is depicted in Figure 1. The links are related to the gauge fields by

$$U(x, x + a\hat{\mu}) = U_\mu(x) = e^{iag_0 A_\mu(x)}. \quad (3.18)$$

The gauge links transform via

$$U_\mu(x) \rightarrow G(x)U_\mu(x)G^{-1}(x + a\hat{\mu}) \quad (3.19)$$

under a gauge transformation. We define a backwards pointing link by

$$U(x + a\hat{\mu}, x) = U_\mu^\dagger(x) \quad (3.20)$$

which is consistent the transformation law of Eq. 3.19.

It can be seen from Eq. 3.19 that products of links that go around a

closed path will be gauge invariant. One such product is the “plaquette” (also pictured in Figure 1)

$$P_{\mu\nu}(x) = \text{ReTr}[U(x, x + a\hat{\mu})U(x + a\hat{\mu}, x + a\hat{\mu} + a\hat{\nu}) \\ \times U(x + a\hat{\mu} + a\hat{\nu}, x + a\hat{\nu})U(x + a\hat{\nu}, x)] \quad (3.21)$$

$$= \text{ReTr}[U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x)]. \quad (3.22)$$

The plaquette traces out a 1×1 square in the μ and ν directions. The simplest gauge invariant gauge action we could have is basically a sum over all possible plaquettes:

$$S_G = \beta \sum_{x, \mu < \nu} \left[1 - \frac{1}{3} P_{\mu\nu}(x) \right] \quad (3.23)$$

where $\beta = 6/g_0^2$. This is called the *Wilson gauge action*, and indeed reproduces the continuum gauge action when the limit $a \rightarrow 0$ is taken.

However, it is possible to write other gauge actions that are gauge invariant and go to the continuum action in the limit $a \rightarrow 0$. A larger class of gauge actions that have these properties is

$$S_G = -\frac{\beta}{3} \left[(1 - 8c_1) \sum_{x, \mu < \nu} P_{\mu\nu}(x) + c_1 \sum_{x, \mu \neq \nu} R_{\mu\nu}(x) \right] + \text{const.} \quad (3.24)$$

where $R_{\mu\nu}(x)$ is a product of gauge links that traces out a 1×2 rectangle in the μ and ν directions. The choice $c_1 = 0$ gives back the Wilson gauge

action, while the choices $c_1 = -1.4069$ and $c_1 = -0.331$ correspond to the *DBW2* [41, 42] and *Iwasaki* [43, 44] gauge actions respectively. In this report, the DBW2 gauge action is used with the 24^3 quenched lattices and the Iwasaki gauge action is used with the 32^3 coarse and fine lattices (these different lattices are described in detail in Chapter 6). In principle none of the choices of c_1 are better than any other in so far as the action remains gauge invariant and has the correct continuum limit, but in practice different choices of c_1 can more faithfully reproduce different kinds of continuum behavior for the non-zero values of a that are necessary in a real calculation.

3.3 Fermion Actions

A naive discretization of the fermionic action (the action corresponding to just the first term in Eq. 3.14) can be obtained replacing derivatives with a central symmetric difference. We get

$$S_F^{\text{naive}} = a^4 \sum_x \left\{ \sum_\mu \frac{1}{2a} \bar{\psi}(x) \gamma_\mu [U_\mu(x) \psi(x + a\hat{\mu}) - U_\mu^\dagger(x - a\hat{\mu}) \psi(x - a\hat{\mu})] + m_f \bar{\psi}(x) \psi(x) \right\} \quad (3.25)$$

for a single flavor. The insertion of the gauge links ensures that the action is gauge invariant.

This action leads to the notorious “fermion doubling” problem, where the action corresponds to sixteen different fermions with the same mass instead

of one. This can be seen by looking at the fermion propagator in momentum space of Eq. 3.25 in the absence of gauge fields ($U_\mu(x) = 1$):

$$G_F(p) = \left(m_f + \frac{i}{a} \sum_{\mu} \gamma_{\mu} \sin p_{\mu} a \right)^{-1}. \quad (3.26)$$

On the lattice each component of the momentum can take on a value in the range $-\frac{\pi}{a} < p_{\mu} \leq \frac{\pi}{a}$. For small m_f Eq. 3.26 has poles near $p_{\mu} = 0$ and $p_{\mu} = \frac{\pi}{a}$ because of the sine function. Each component can be near 0 or $\frac{\pi}{a}$ meaning there are sixteen different poles in four dimensions, corresponding to sixteen different particles. Clearly we do not want to simulate sixteen different quarks in most cases.

The doubling problem is actually a result of the symmetric difference used to discretize the derivative [39]. One way of fixing the problem is to add a term that breaks the symmetry and vanishes in the continuum limit. This leads to the *Wilson fermion action*:

$$S_F^{\text{Wilson}} = S_F^{\text{naive}} - \frac{r}{2a} a^4 \sum_{x,\mu} \bar{\psi}(x) [U_{\mu}(x) \psi(x + a\hat{\mu}) + U_{\mu}^{\dagger}(x - a\hat{\mu}) \psi(x - a\hat{\mu}) - 2\psi(x)] \quad (3.27)$$

$$\begin{aligned} &= \frac{am_f + 4r}{a} a^4 \sum_x \bar{\psi}(x) \psi(x) \\ &\quad - \frac{1}{2a} a^4 \sum_{x,\mu} \bar{\psi}(x) [(r - \gamma_{\mu}) U_{\mu}(x) \psi(x + a\hat{\mu}) \\ &\quad \quad + (r + \gamma_{\mu}) U_{\mu}^{\dagger}(x - a\hat{\mu}) \psi(x - a\hat{\mu})] \end{aligned} \quad (3.28)$$

where $0 < r \leq 1$ ($r = 0$ corresponds to the naive action). The propagator in momentum space is now

$$G_F(p) = \left[\frac{am_f + 4r}{a} + \frac{i}{a} \sum_{\mu} \gamma_{\mu} \sin p_{\mu} a - \frac{r}{a} \sum_{\mu} \cos p_{\mu} a \right]^{-1}. \quad (3.29)$$

For small m_f there is still a pole near $p_{\mu} = 0$, but not near $p_{\mu} = \frac{\pi}{a}$ where there is now a term proportional $\frac{r}{a}$ in the denominator which actually grows large in the limit $a \rightarrow 0$. Thus all of the doublers are suppressed at the expense of introducing a term that breaks chiral symmetry in the same way that the mass term does. This makes the Wilson fermion action less desirable for calculations where chiral symmetry is important, so we prefer to use an action that breaks chiral symmetry less severely.

Another fermion action that is in common use is the *staggered fermion* action [45, 46]. A redefinition of the fermion fields in the naive action causes it to be diagonal with respect to the four spin components of ψ . Three of the four components are thrown away so that the action now represents four fermions rather than sixteen when doublers are taken into account. These four fermions can be further reduced to a single flavor by taking a fourth root of the determinant in Eq. 3.17.

In this report we will be using the *domain wall fermion* action [47, 48, 49], which we now discuss. A closely related fermion action is the *overlap fermion* action [50, 51]. The starting point for domain wall fermions is Kaplan's observation [47] that a theory of massive fermions in $2n + 1$ dimensions with

a step function-like mass term can give rise to massless chiral modes on the $2n$ dimensional domain wall. The form of domain wall fermions used here was proposed by Shamir [48]. The discretized fermion action is given by

$$S_F^{\text{DWF}} = \sum_{x,x',s,s'} \bar{\Psi}(x,s) D_{x,s;x',s'} \Psi(x',s') \quad (3.30)$$

where Ψ is a fermion field that lives in five dimensions, x denotes the coordinates in the four physical dimensions, and s denotes the coordinate in a fictitious fifth dimension which runs from 0 to $L_s - 1$, where L_s is the size of the fifth dimension.

The five-dimensional Dirac operator is given by

$$D_{x,s;x',s'} = \delta_{s,s'} D_{x,x'}^{\parallel} + \delta_{x,x'} D_{s,s'}^{\perp} \quad (3.31)$$

$$\begin{aligned} D_{x,x'}^{\parallel} = & -\frac{1}{2} \sum_{\mu=1}^4 [(1 - \gamma_{\mu}) U_{\mu}(x) \delta_{x+\hat{\mu},x'} + (1 + \gamma_{\mu}) U_{\mu}^{\dagger}(x') \delta_{x-\hat{\mu},x'}] \\ & + (4 - M_5) \delta_{x,x'} \end{aligned} \quad (3.32)$$

$$\begin{aligned} D_{s,s'}^{\perp} = & -\frac{1}{2} [(1 - \gamma_5) \delta_{s+1,s'} + (1 + \gamma_5) \delta_{s-1,s'} - 2\delta_{s,s'}] \\ & + \frac{m_f}{2} [(1 - \gamma_5) \delta_{s,L_s-1} \delta_{0,s'} + (1 + \gamma_5) \delta_{s,0} \delta_{L_s-1,s'}]. \end{aligned} \quad (3.33)$$

Here we have used the conventions of [52]. The quantity m_f is a parameter

that becomes the fermion mass in four dimensions and M_5 is a new parameter called the *domain wall height*. The gauge field lives in four dimensions which means that it is independent of s .

This action gives rise to a left-handed chiral fermion mode localized near $s = 0$ and a right-handed mode localized near $s = L_s - 1$. These modes decay exponentially with s and their overlap vanishes in the limit of large L_s , meaning that chiral symmetry is restored in this limit. Four dimensional fermion fields can be constructed from the five dimensional ones, essentially by taking the left handed component from $s = 0$ and the right-handed component from $s = L_s - 1$, and these four dimensional fields can be used to compute the correlation functions necessary for the desired physics.

We would like to take $L_s \rightarrow \infty$, but in practice this is not possible. With finite L_s we will see some chiral symmetry breaking. At low energy and near the continuum limit, the effect of finite L_s and the extent of the chiral symmetry breaking can be described by a universal parameter m_{res} , known as the *residual mass*, which can be extracted from correlation functions in five dimensions. It also shows up as an effective shift of the fermion mass, so that the chiral limit point (where the pseudoscalar meson masses vanish) becomes $m_f = -m_{\text{res}}$ rather than $m_f = 0$. It has been found that m_{res} vanishes roughly exponentially with increasing L_s [48]. Thus chiral symmetry breaking can be controlled by adjusting L_s , independently of all other parameters like the lattice spacing. This comes at the expense of an increase in computational cost due to introducing a fifth dimension. The domain wall

fermion action also does not suffer from doublers.

3.4 Numerical Methods

The general goal of the numerical part of lattice QCD is to calculate expectation values like that in Eq. 3.17 with the discretized actions discussed in Sections 3.2 and 3.3. The strategy is to generate a set of gauge configurations $U_\mu^i(x)$, where i labels the different gauge configurations in the set, in such a way that for each configuration generated the probability $P[U]$ of obtaining a particular configuration U is given by

$$P[U] \propto e^{-S_G[U]} \prod_f \det(\mathcal{D}[U] + m_f) \quad (3.34)$$

where we have explicitly shown which quantities depend on the gauge configuration U for emphasis. The gauge configurations in the set must also be statistically uncorrelated to each other. Such a set is called an ensemble. The integral of Eq. 3.17 can then be approximated by an average of $O'[U]$ over the gauge configurations in the ensemble. (The approximation becomes exact in the limit of an infinite number of configurations in the ensemble). The two most numerically demanding parts of the process are generating the gauge configurations in the ensemble, and inverting $\mathcal{D}[U] + m_f$ as required in evaluating $O'[U]$ (keeping in mind that since $\mathcal{D}[U]$ depends on the gauge field this inversion must be done for each gauge configuration in the ensemble).

An ensemble of configurations is created using a Markov chain method [39].

We start with some initial configuration and then make a change that we accept or reject according to some rule. Both the change and the acceptance test generally involve random variables making this a Monte Carlo method. The change and acceptance test are repeated continually to evolve the configuration. Provided that the change and acceptance test meet certain criteria such as *detailed balance*, it can be shown that after a large number of steps the probability of obtaining a given configuration will be given by Eq. 3.34 as desired. At this point the gauge configuration is said to have *equilibrated*. Gauge configurations from one step to the next are naturally correlated to each other since they are only different by one small change. This is known as *autocorrelation*. Thus, even after the process has equilibrated, it is important to take configurations that are separated by a sufficient number of steps to include in the ensemble in order to avoid autocorrelations.

Specific examples of Markov chain algorithms that can be used in lattice QCD to generate gauge configurations are the *Metropolis*, *heat bath*, and *molecular dynamics* (MD) algorithms [39, 40]. More sophisticated algorithms such as the *Hybrid Monte Carlo* (HMC) and *Rational Hybrid Monte Carlo* (RHMC) [53] algorithms are used in many modern lattice calculations. In this report the heat bath algorithm is used for the 24^3 quenched lattices and versions of the RHMC algorithm are used for the 32^3 coarse and fine lattices.

Inverting $\not{D}[U] + m_f$ with a discretized Dirac operator amounts to inverting a very large matrix. This is achieved with a standard conjugate gradient algorithm. Due to the sizes of the matrices, and the fact that inversions

must be done with each gauge configuration, this becomes a computationally intensive task comparable to the configuration generation itself. The time needed to perform an inversion increases with decreasing m_f , which is one reason that simulating lighter particles is more difficult.

The inverse of $\mathcal{D} + m_f$ is called a *propagator* and is denoted by $S(x, x')$ in this report, where x and x' are coordinates of points on the lattice in four dimensions. (Note that $\mathcal{D}(x, x') + m_f$, being the discrete version of a differential operator, is generally non-zero only when x and x' are neighboring points. However, this is generally not the case for its inverse $S(x, x')$). The point x' is referred to as the “source” point and x the “sink” point. An inversion with the conjugate gradient algorithm allows one to compute $S(x, x')$ for all x , but with a fixed x' . More generally, with a single inversion one can compute

$$\sum_{x'} S(x, x') s(x') \quad (3.35)$$

for a fixed function $s(x')$. We speak of computing a propagator with a “source” $s(x')$. An example of a source function used in this report is the “wall source”

$$s_{\text{wall}}(x) = \delta_{t,0}. \quad (3.36)$$

That is, a wall source function is equal to 1 at all spatial points on the time slice $t = 0$, and is equal to 0 elsewhere. Note that this definition of a wall source is not gauge invariant and that more sophisticated definitions which are gauge invariant are also possible.

3.5 Quenched and Partially Quenched QCD

The evaluation of the fermion determinant $\det(\not{D}[U] + m_f)$ in Eq. 3.34 is quite computationally intensive compared to the evaluation of the gauge action. As a result, much early work in lattice QCD employed the *quenched* approximation in which the fermion determinant is set to 1, thus greatly speeding up the generation of gauge ensembles. This approximation is used for the 24^3 quenched ensemble in this report. The quenched approximation is an uncontrolled approximation that violates unitarity, but seems to keep other essential non-perturbative features of QCD like confinement. In [54] the authors investigated possible theoretical issues in performing a quenched direct $K \rightarrow \pi\pi$ calculation (as opposed to the indirect method of [13, 14, 15]). They found that the non-unitarity of the quenched approximation poses serious problems for $\Delta I = 1/2$ decays, such as non-standard behavior of correlation functions with volume and time, but that these problems were not present for $\Delta I = 3/2$ decays like the one simulated in this report. Quenched direct calculations of $\Delta I = 3/2$ $K \rightarrow \pi\pi$ decay have also been done in the past [16, 17]. However, we do find in this report, as was found in [16], that the quenched approximation gives a value of $\text{Re}(A_2)$ that is significantly larger than the value from experiment *and* the values from the other calculations in this report, (which are only partially quenched and not fully quenched; see below). This is discussed further in Section 7.4.3.

It is also possible to have *partially quenched* QCD in which the fermion

determinant is not eliminated, but the mass m_f in the fermion determinant is different from the mass used when calculating propagators for the same quark. The mass used for the propagators is called the *valence quark mass* (m^{val}), and the mass in the fermion determinant the *sea quark mass* (m^{sea}). These names come from the picture that propagators correspond to quarks that are external legs in diagrams (so-called *valence quarks*), whereas the fermion determinant corresponds to quarks inside loops (*sea quarks*). In this picture the quenched approximation amounts to eliminating quarks in loops altogether. Partially quenched QCD is non-unitary like quenched QCD, however, since the quenched approximation can also be thought of as taking the sea quark mass to infinity, partial quenching is thought to be a less severe approximation than complete quenching. When discussing a set of calculations each with given valence and sea quark masses, we refer to an unquenched calculation where the sea and valence quark masses are equal as being *unitary* or being at the *unitary point*. We also refer to any ensemble that does include the fermion determinant, and to the quarks simulated by that ensemble, as being *dynamical*, whether we go on to do a unitary or partially quenched calculation with these gauge configurations.

Since lighter masses in the fermion determinant increase the computational cost of generating gauge configurations, partial quenching has the practical appeal of reducing this cost by using a sea quark mass that is not as light as the valence quark mass, thus shifting the cost to calculation of propagators alone. This approach is used with the 32^3 coarse lattices to

simulate a pion mass that is close to physical (whereas a unitary calculation with the same sea quark mass would give rise to a somewhat heavier pion). Many physical quantities are expected to be determined largely by the valence quark mass and to depend only mildly on the sea quark mass [55], justifying this procedure. Furthermore, partial quenching allows us to examine more easily the mass dependence of physical quantities by only varying valence quark masses and keeping the sea quark masses fixed, which means that it is only necessary to calculate more propagators and not to generate an entirely new gauge ensemble. The approach of repeating the calculation with many different valence quark masses while keeping the sea quark mass fixed is used with the 32^3 fine lattices, which are actually three different ensembles with three different sea quark masses. The three different sea quark masses allow us to examine the effects of partial quenching on $K \rightarrow \pi\pi$ decays, and indeed in Section 7.4.2 we find no visible effect.

4 Weak Decays on the Lattice

The weak interaction is responsible for the kaon decay into two pions, as explained in Chapter 2. In principle it is possible to incorporate the weak interaction directly into the lattice calculation by representing the W and Z particles as dynamical fields on the lattice, as we have done for the quarks and gluons in Chapter 3. However, the masses of the W and Z particles are of order 100 GeV. In contrast, the largest energy scale that can be accurately represented on the lattice is of order a^{-1} , the inverse lattice spacing, which is of order 1-2 GeV for the lattices used in this report. Furthermore, the *smallest* energy scale that can be represented on the lattice is L^{-1} , the inverse of the side length of the lattice. (This is discussed further in Chapter 5). Modern computing capabilities limit us to lattices that have of order 30 points in each spatial direction, so that $L^{-1} \sim a^{-1}/30$. Thus it is impossible to simultaneously represent the widely disparate energy scales of the weak particles (~ 100 GeV) and the kaons and pions ($\sim 100 - 500$ MeV), on the lattice.

The solution is to use the technique of the *operator product expansion* (OPE) [56, 57]. We represent the effects of the weak interaction as an ef-

effective Hamiltonian consisting of operators with four quark fields multiplied by quantities called *Wilson coefficients*. For $K \rightarrow \pi\pi$ decay the desired amplitude can be calculated as a matrix element of the effective Hamiltonian between kaon and two pion initial and final states, and the OPE reduces this to the task of calculating matrix elements of four quark operators once the Wilson coefficients are known. In essence the OPE is a factorization that separates the low energy physics at the scale of the kaon and pion, which is encoded in the four quark operators and their matrix elements, and the high energy physics at the scale of the W and the Z , which is encoded in the Wilson coefficients. Intuitively, this can be pictured as shrinking the W and Z interactions, which are at high energy and thus take place within a short distance, to a point-like interaction as in Figure 2. The OPE is thus a modern, formalized version of the classic “four-Fermi” interaction. The W and Z have been removed as dynamical degrees of freedom, and we refer to this as *integrating out* these particles. We use the OPE technique to integrate out the top, bottom and charm quarks as well since they are also significantly heavier.

The Wilson coefficients are determined by matching amplitudes in the effective theory, embodied by the effective Hamiltonian, to amplitudes in the full theory. Due to the asymptotic freedom property of QCD, such matching calculations can be done using perturbation theory at energies much higher than Λ_{QCD} (the scale below which non-perturbative effects become important), but still much lower than the weak scale. More details are given in

Section 4.1, and a comprehensive review can be found in [57]. We must then calculate matrix elements of the four quark operators. This is a low energy QCD calculation which must be done on the lattice, and is the subject of the original work of this report (see Chapter 6). Finally, the matrix elements and Wilson coefficients can be assembled to obtain the physical $K \rightarrow \pi\pi$ amplitude. We carry out this final step in Chapter 7 to obtain A_2 . It is also necessary to renormalize the four quark operators so that there is consistency with the renormalization scheme employed in calculating the Wilson coefficients. This is done using the technique of *non-perturbative renormalization* (NPR) [58] which is discussed in Sections 4.2 and 4.3.

4.1 Operator Product Expansion

For $K \rightarrow \pi\pi$ decays we need only operators in the OPE that change strangeness by 1 ($\Delta S = 1$ operators) because the kaon contains one strange quark and the two pions contain none. Thus the four quark operators of interest will contain one strange quark field (s). For a general OPE we would start with an effective Hamiltonian involving all quarks (u , d , s , c , and b) other than the top quark. The top quark is integrated out along with the W and Z particles since they are all of comparable mass. Wilson coefficients for the different operators are determined by matching to the full theory at first order in $\alpha_s = g_s^2/(4\pi)$, (where g_s is the strong coupling constant), where diagrams are evaluated at a scale μ_W well below the W mass. The effective

Hamiltonian at this scale can be written generically as

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{\text{CKM}} \vec{Q}^T(\mu_W) \vec{C}(\mu_W) \quad (4.1)$$

where $\vec{Q}(\mu)$ and $\vec{C}(\mu)$ are vectors containing the four quark operators and Wilson coefficients respectively, evaluated at the scale μ , V_{CKM} stands for some combination of CKM matrix elements, and G_F is the Fermi constant.

The Wilson coefficients are then run down to a scale μ_b well below the bottom quark mass using the renormalization group equation. The renormalization group equation is

$$\frac{d}{d \ln \mu} \vec{C}(\mu) = \gamma^T(\alpha_s) \vec{C}(\mu) \quad (4.2)$$

where $\gamma^T(\alpha_s)$ is the transpose of the anomalous dimension matrix (which is obtained from an operator renormalization matrix Z), and $\alpha_s = \alpha_s(\mu)$ also depends on the scale. The running of $\alpha_s(\mu)$ can be determined from its own renormalization group equation in terms of the QCD beta function. Eq. 4.2 can be used to run the Wilson coefficients down to the scale μ_b from the scale μ_W . Due to the linearity of Eq. 4.2 this can be written in terms of an evolution matrix $U(\mu_1, \mu_2)$ as

$$\vec{C}(\mu_b) = U(\mu_b, \mu_W) \vec{C}(\mu_W). \quad (4.3)$$

At the scale μ_b we start using a four flavor effective theory involving

u , d , s , and c , thus integrating out the b quark which has become a heavy particle at this scale. We must therefore match onto an effective Hamiltonian containing only u , d , s , and c fields and a new set of Wilson coefficients. We then run the Wilson coefficients down to a scale μ_c well below the charm quark mass and match onto a three flavor effective Hamiltonian containing only u , d , and s . Finally, the Wilson coefficients are run down to a scale μ where the matrix elements of the four quark operators can actually be evaluated. Since the matrix elements are evaluated on the lattice, this scale will be on the order of the inverse lattice spacing.¹ The entire process can be summarized by

$$\vec{C}(\mu) = U_3(\mu, \mu_c)M(\mu_c)U_4(\mu_c, \mu_b)M(\mu_b)U_5(\mu_b, \mu_W)\vec{C}(\mu_W) \quad (4.4)$$

where $U_f(\mu_1, \mu_2)$ is the evolution matrix for f active flavors and $M(\mu)$ is a matching matrix at the threshold μ . It should be noted that the number of Wilson coefficients can in general be different as you switch from one effective theory to another, so the matching matrix $M(\mu)$ is not necessarily a square matrix. Note also that the evolution matrices do depend on f , the number of active flavors of the effective theory with which we are working, because the anomalous dimension matrix and beta function depend on f .

¹We have some freedom to choose this scale as long as it is on the order of the inverse lattice spacing. Results will be independent of scale as long as the renormalization constants of Section 4.2 are evaluated at the same scale as the Wilson coefficients.

The $\Delta S = 1$ effective Hamiltonian relevant to $K \rightarrow \pi\pi$ decay is given by

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} [z_i(\mu) + \tau y_i(\mu)] Q_i(\mu) \quad (4.5)$$

for three active flavors. Here V_{us} and V_{ud} are CKM matrix elements (c.f. Eq. 2.3), $z_i(\mu)$ and $y_i(\mu)$ are the Wilson coefficients, and τ is given by

$$\tau = \frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}}. \quad (4.6)$$

Note that in the standard parametrization of the CKM matrix, given in Eq. 2.4, V_{us} and V_{ud} are purely real and the CP violating phase only enters into the effective Hamiltonian through the quantity τ .

The four quark operators $Q_i(\mu)$ in Eq. 4.5 are given by

$$Q_1 = (\bar{s}_\alpha d_\alpha)_{V-A} (\bar{u}_\beta u_\beta)_{V-A} \quad (4.7)$$

$$Q_2 = (\bar{s}_\alpha d_\beta)_{V-A} (\bar{u}_\beta u_\alpha)_{V-A} \quad (4.8)$$

$$Q_3 = (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V-A} \quad (4.9)$$

$$Q_4 = (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V-A} \quad (4.10)$$

$$Q_5 = (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V+A} \quad (4.11)$$

$$Q_6 = (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V+A} \quad (4.12)$$

$$Q_7 = \frac{3}{2} (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\beta)_{V+A} \quad (4.13)$$

$$Q_8 = \frac{3}{2} (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\alpha)_{V+A} \quad (4.14)$$

$$Q_9 = \frac{3}{2} (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\beta)_{V-A} \quad (4.15)$$

$$Q_{10} = \frac{3}{2} (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\alpha)_{V-A}. \quad (4.16)$$

Here α and β are color indices, and spinor indices contract in the normal way. We have also used the notation e_q for the electric charge of quark q , and

$$(\bar{q}_1 q_2)_{V-A} = \bar{q}_1 \gamma_\mu (1 - \gamma_5) q_2 \quad (4.17)$$

$$(\bar{q}_1 q_2)_{V+A} = \bar{q}_1 \gamma_\mu (1 + \gamma_5) q_2. \quad (4.18)$$

Operators Q_1 - Q_2 are called *current-current operators*, operators Q_3 - Q_6 are called *QCD penguin operators*, and operators Q_7 - Q_{10} are called *electroweak penguin operators*. These groups of operators derive their names from the fact that they correspond to the diagrams in the full theory shown in Figure 3.

Eq. 4.5 is the effective Hamiltonian for three active flavors u , d , and s , and for scales $\mu < \mu_c$. It is thus the final result we wish to use. For scales $\mu_c < \mu < \mu_b$ and four active flavors the effective Hamiltonian is

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left[(1 - \tau) \sum_{i=1}^2 z_i(\mu) (Q_i(\mu) - Q_i(\mu)^c) + \tau \sum_{i=1}^{10} v_i(\mu) Q_i(\mu) \right] \quad (4.19)$$

where we have introduced two new operators that involve the charm quark:

$$Q_1^c = (\bar{s}_\alpha c_\alpha)_{V-A} (\bar{c}_\beta d_\beta)_{V-A} \quad (4.20)$$

$$Q_2^c = (\bar{s}_\alpha c_\beta)_{V-A} (\bar{c}_\beta d_\alpha)_{V-A} \quad (4.21)$$

and where we have written the expression in terms of the Wilson coefficients $v_i(\mu)$ rather than $y_i(\mu)$ which are related by

$$y_i(\mu) = v_i(\mu) - z_i(\mu). \quad (4.22)$$

Furthermore, in Eq. 4.19 the operators Q_3 - Q_{10} are given as in Eq. 4.9-4.16 but with the sums over q running over all *four* active flavors u , d , s , and c .

The Wilson coefficients $z_i(\mu)$ and $v_i(\mu)$ are computed at the final scale

$\mu < \mu_c$ of Eq. 4.5 by using Eq. 4.4, with initial values at the scale μ_W given in [57]. Expressions for the evolution and matching matrices are also found in [57]. Note that when running down through the threshold μ_c the $z_i(\mu)$ go from 2 to 10 in number, (as we can see by comparing Eq. 4.19 and 4.5), and thus the matrix $M(\mu_c)$ has dimensions 10×2 for the z coefficients.

The operators of Eq. 4.7-4.16 are not all linearly independent, so we wish to work with a smaller set of operators that are linearly independent. For this purpose it is convenient to work in terms of operators that cause definite changes in isospin, $\Delta I = 1/2$ or $\Delta I = 3/2$, and that transform irreducibly under $SU(3)_L \otimes SU(3)_R$, where $SU(3)$ refers to a rotation among the three quark flavors u , d , and s , and L and R refer to the left and right handed spinor components of those quark fields. The method for decomposing Q_1 - Q_{10} in terms of these new operators this is presented in detail in Appendix B of [13]. Since we are only dealing with the $\Delta I = 3/2$ decay in this report, we reproduce the results only for those operators containing $\Delta I = 3/2$ operators

which are Q_1, Q_2, Q_7 - Q_{10} :

$$Q_1 = \frac{1}{10}Q_{(8,1),1/2}^{LL,S} + \frac{1}{2}Q_{(8,1),1/2}^{LL,A} + \frac{1}{15}Q_{(27,1),1/2}^{LL,S} + \frac{1}{3}Q_{(27,1),3/2}^{LL,S} \quad (4.23)$$

$$Q_2 = \frac{1}{10}Q_{(8,1),1/2}^{LL,S} - \frac{1}{2}Q_{(8,1),1/2}^{LL,A} + \frac{1}{15}Q_{(27,1),1/2}^{LL,S} + \frac{1}{3}Q_{(27,1),3/2}^{LL,S} \quad (4.24)$$

$$Q_7 = \frac{1}{2}Q_{(8,8),3/2}^{LR,S} + \frac{1}{2}Q_{(8,8),1/2}^{LR,A} \quad (4.25)$$

$$Q_8 = \frac{1}{2}Q_{(8,8),3/2}^{LR,S,\text{mix}} + \frac{1}{2}Q_{(8,8),1/2}^{LR,A,\text{mix}} \quad (4.26)$$

$$Q_9 = -\frac{1}{10}Q_{(8,1),1/2}^{LL,S} + \frac{1}{2}Q_{(8,1),1/2}^{LL,A} + \frac{1}{10}Q_{(27,1),1/2}^{LL,S} + \frac{1}{2}Q_{(27,1),3/2}^{LL,S} \quad (4.27)$$

$$Q_{10} = -\frac{1}{10}Q_{(8,1),1/2}^{LL,S} - \frac{1}{2}Q_{(8,1),1/2}^{LL,A} + \frac{1}{10}Q_{(27,1),1/2}^{LL,S} + \frac{1}{2}Q_{(27,1),3/2}^{LL,S}. \quad (4.28)$$

The subscripts of the operators give the irreducible representation of $SU(3)_L \otimes SU(3)_R$ as an ordered pair in brackets, followed by the value of ΔI .

The three unique $\Delta I = 3/2$ operators in Eq. 4.23-4.28 are given by

$$\begin{aligned} Q_{(27,1),3/2}^{LL,S} &= (\bar{s}_\alpha d_\alpha)_{V-A} (\bar{u}_\beta u_\beta)_{V-A} + (\bar{s}_\alpha u_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V-A} \\ &\quad - (\bar{s}_\alpha d_\alpha)_{V-A} (\bar{d}_\beta d_\beta)_{V-A} \end{aligned} \quad (4.29)$$

$$\begin{aligned} Q_{(8,8),3/2}^{LR,S} &= (\bar{s}_\alpha d_\alpha)_{V-A} (\bar{u}_\beta u_\beta)_{V+A} + (\bar{s}_\alpha u_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V+A} \\ &\quad - (\bar{s}_\alpha d_\alpha)_{V-A} (\bar{d}_\beta d_\beta)_{V+A} \end{aligned} \quad (4.30)$$

$$\begin{aligned} Q_{(8,8),3/2}^{LR,S,\text{mix}} &= (\bar{s}_\alpha d_\beta)_{V-A} (\bar{u}_\beta u_\alpha)_{V+A} + (\bar{s}_\alpha u_\beta)_{V-A} (\bar{u}_\beta d_\alpha)_{V+A} \\ &\quad - (\bar{s}_\alpha d_\beta)_{V-A} (\bar{d}_\beta d_\alpha)_{V+A}. \end{aligned} \quad (4.31)$$

The label “mix” refers to the color indices α and β which are interchanged in the otherwise identical expressions of Eq. 4.30 and 4.31.

The effective Hamiltonian of Eq. 4.5 can be written in terms of the new operators by substituting the results of Eq. 4.23-4.28. We drop all of the $\Delta I = 1/2$ operators to get the effective Hamiltonian that is relevant for the $\Delta I = 3/2$ decays:

$$\mathcal{H}_{\text{eff}}^{\Delta I=3/2} = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_J C_J(\mu) Q_J(\mu) \quad (4.32)$$

where in the sum the index J runs over the three values “(27,1)”, “(8,8)”, and “(8,8) mixed”, corresponding to the three operators of Eq. 4.29-4.31. The new Wilson coefficients are given by

$$C_{(27,1)}(\mu) = \frac{C_1(\mu) + C_2(\mu)}{3} + \frac{C_9(\mu) + C_{10}(\mu)}{2} \quad (4.33)$$

$$C_{(8,8)}(\mu) = \frac{C_7(\mu)}{2} \quad (4.34)$$

$$C_{(8,8) \text{ mixed}}(\mu) = \frac{C_8(\mu)}{2} \quad (4.35)$$

where we have used the shorthand

$$C_i(\mu) = z_i(\mu) + \tau y_i(\mu). \quad (4.36)$$

4.2 Non-Perturbative Renormalization

In the previous section we saw that the Wilson coefficients can be calculated at some low energy scale μ by calculating them at the scale μ_W and using the renormalization group to run them down to the scale μ . Such a calculation

must be done in a particular regularization scheme. The $\overline{\text{MS}}$ scheme is a standard one for this purpose. However, this means that it is also necessary for the four quark operators to be renormalized in the same scheme at the same scale μ . In contrast, when calculating correlators on the lattice corresponding to matrix elements of a four quark operator one is effectively using a “bare” operator. This means that the operators have not been renormalized and the result will therefore depend on the lattice spacing, even if the lattice spacing is very small.

One approach is to use *lattice perturbation theory*, in which analytic expressions to some order in a^2 can be found to convert bare lattice operators to renormalized ones. However, in the cases where other non-perturbative methods are possible, comparisons with lattice perturbation theory have revealed significant discrepancies. This led to the development of the technique of non-perturbative renormalization by the Rome-Southampton group [58]. The method makes use of Green’s functions computed directly on the lattice rather than perturbative expressions, and we now give more details.

We choose a scheme in which a certain Green’s function of the operator we wish to renormalize is set equal to its value at tree level. This will be a Green’s function in a fixed gauge of the operator between external off-shell quark and gluon states. One must specify the external states and their off-shell momenta in order to fully specify the scheme. Such a scheme is called a *regularization independent* (RI) scheme. It is important that the external momentum scale μ obey $\Lambda_{QCD} \ll \mu \ll a^{-1}$, where Λ_{QCD} is the scale at

which strong dynamics occur ($\Lambda_{QCD} \sim 200$ MeV). We must have $\mu \gg \Lambda_{QCD}$ to be able to apply perturbation theory on the continuum side (see Section 4.3), and we must have $\mu \ll a^{-1}$ to avoid large lattice artifacts in the lattice calculation of the Green's function.² NPR calculations are typically done in the Landau gauge.

We first show how NPR is done for a single bilinear operator that does not mix with any others under renormalization. The renormalized operator $\mathcal{O}(\mu)$ is related to the bare operator on the lattice $\mathcal{O}(a)$ by

$$\mathcal{O}(\mu) = Z_{\mathcal{O}}(\mu a, g(a)) \mathcal{O}(a). \quad (4.37)$$

The conversion factor $Z_{\mathcal{O}}$ is given by imposing the renormalization condition

$$Z_{\mathcal{O}}(\mu a, g(a)) Z_q^{-1}(\mu a, g(a)) \Gamma_{\mathcal{O}}(pa) \Big|_{p^2=\mu^2} = 1. \quad (4.38)$$

Here $Z_q^{1/2}$ is the quark field renormalization factor (which we will describe how to calculate) and $\Gamma_{\mathcal{O}}(pa)$ is the amputated Green's function, given by

$$\Gamma_{\mathcal{O}}(pa) = \frac{1}{12} \text{Tr} \left(\Lambda_{\mathcal{O}}(pa) \hat{P}_{\mathcal{O}} \right) \quad (4.39)$$

where

$$\Lambda_{\mathcal{O}}(pa) = S(pa)^{-1} G_{\mathcal{O}}(pa) S(pa)^{-1}. \quad (4.40)$$

²In practice, in particular when μ represents the magnitude of external momenta, it is found that μ can actually be on the order of a^{-1} , as long as the momenta don't begin to approach the largest momentum allowed on the lattice.

The quantity $S(pa)$ is the quark propagator, $G_{\mathcal{O}}(pa)$ is the non-amputated Green's function, and $\hat{P}_{\mathcal{O}}$ is a suitable projector on the tree level operator. For example $\hat{P}_{\mathcal{O}} = 1, \gamma_5$ for $\mathcal{O} = \bar{q}q, \bar{q}\gamma_5 q$ (the scalar and pseudoscalar density).

The quark field renormalization $Z_q^{1/2}$ can be found for a DWF action by defining a five dimensional, non-local, conserved axial current $\mathcal{A}_{\mu}^a(x)$ (see [52]). This current must approach the renormalized partially conserved local axial current in the continuum limit. The bare partially conserved local axial current $A_{\mu}^a(x) = \bar{q}(x)t^a\gamma_{\mu}\gamma_5 q(x)$ is proportional to $\mathcal{A}_{\mu}^a(x)$ in the continuum limit, i.e.

$$\mathcal{A}_{\mu}^a(x) = Z_A A_{\mu}^a(x). \quad (4.41)$$

Thus Z_A must be the renormalization factor for $A_{\mu}^a(x)$. It can be determined from ratios of correlators of $A_{\mu}^a(x)$ and $\mathcal{A}_{\mu}^a(x)$. Once Z_A has been determined, we can use Eq. 4.38 with $\mathcal{O} = A_{\mu}^a$, but now $Z_{\mathcal{O}} = Z_A$ will be known and we can solve for Z_q .

The procedure for four quark operators is similar, and we will now outline it. We also wish to take into account the possibility of mixing between operators under renormalization, meaning there will be a matrix of renormalization constants Z^{ij} and Eq. 4.37 becomes

$$\mathcal{O}^i(\mu) = Z^{ij}(\mu a, g(a)) \mathcal{O}^j(a) \quad (4.42)$$

where the indices i and j label the different operators that can mix with each other. As before, we must compute amputated Green's functions of the

operators $\mathcal{O}^i$

$$\Lambda_{\alpha\beta\gamma\delta}^i = \langle \mathcal{O}^i q_\alpha(p_1) \bar{q}_\beta(p_2) q_\gamma(p_1) \bar{q}_\delta(p_2) \rangle_{\text{amp}} \quad (4.43)$$

where the momenta are chosen such that

$$p_1^2 = p_2^2 = (p_1 - p_2)^2 = \mu^2 \quad (4.44)$$

where μ is the renormalization scale as always. Again we apply a projector to Λ

$$P^j \{ \Lambda^i \} = \Gamma_{\alpha\beta\gamma\delta}^j \Lambda_{\beta\alpha\delta\gamma}^i \equiv M^{ij} \quad (4.45)$$

but notice from this equation that we must calculate projections of a given operator $\mathcal{O}^i$ onto different operators $\mathcal{O}^j$ as well. With this set-up, the renormalization constants are given in matrix form by

$$Z = Z_q^2 F M^{-1} \quad (4.46)$$

where F is the free field value of M .

For the four quark operators relevant for $\Delta I = 3/2$ $K \rightarrow \pi\pi$ decay, we work in the basis of the three operators of Eq. 4.29-4.31. Because they transform as irreducible representations of $SU(3)_L \otimes SU(3)_R$ and of isospin, we know that the (27,1) operator of Eq. 4.29 will not mix with any others, and the (8,8) operators of Eq. 4.30 and 4.31 will only mix with each other [13]. It has been verified in actual lattice NPR calculations that such mixing between representations is indeed highly suppressed [15, 59].

4.3 Converting Between Regularization Schemes

The Wilson coefficients of Section 4.1 are calculated from continuum perturbation theory, and this is generally done in the $\overline{\text{MS}}$ renormalization scheme using dimensional regularization. This scheme is labeled “NDR”. However, the renormalization constants of Section 4.2 are calculated using NPR and will therefore be in an RI scheme. It is therefore necessary to convert the Wilson coefficients from the NDR scheme to the RI scheme. This can be accomplished via a conversion factor

$$\vec{C}_{RI}(\mu) = \left(1 - \frac{\alpha_s(\mu)}{4\pi} (\Delta r_{\lambda^*=0}^{NDR})^T\right) \vec{C}_{NDR}(\mu) \quad (4.47)$$

to first order in α_s . The details are given in [60], see for example Table VIII.

5 Finite Volume Effects

One aspect of lattice simulations that is significantly different from continuum physics is that they take place in a finite spatial volume. Specifically, particles are generally confined to a cubic box of side length L in which opposite faces are identified by periodic or antiperiodic boundary conditions. In fact, the length L is by no means “very large” as we might hope (e.g. 4.60 fm for the 32^3 coarse lattices of this report, compared to the proton radius of ~ 0.8 fm), so it is necessary to have a clear picture of the effects of finite volume and a systematic way of relating finite volume results to infinite volume physics.

It can be shown that for single particle states, corrections due to finite volume vanish exponentially with increasing $m_\pi L$ [61]. A rough figure of merit that has been adopted for good suppression of this type of finite volume effect is $m_\pi L \gtrsim 4$. The calculation on the coarse 32^3 lattice of this report has $m_\pi L = 3.3$ which is still reasonable, and more is said about possible finite volume effects in Chapter 7.

In this chapter we will focus on how finite volume affects two particle states and particle decays. The allowed energies for two particle states in

which the particles are not bound to each other become quantized in finite volume, and this quantization is non-trivial when the two particles interact with each other. In contrast, any energy (greater than or equal to the sum of the rest masses of the two particles) is possible for two particle unbound states in infinite volume. However, it was shown by Lüscher [62] that the allowed energies for two interacting particles in a given volume are simply related to the phase shifts characterizing the scattering of the two particles at these energies in infinite volume. The derivation of this relation will be discussed in Sections 5.1 and 5.2. Since lattice simulations can yield the energy of the two pion ground state in a given volume, the relation can be used to extract the s wave scattering phase shift of the two pions at this particular energy, which can be compared to experiment.

An apparent difficulty in performing lattice calculations of $K \rightarrow \pi\pi$ decay amplitudes is the so-called *Maiani-Testa no-go theorem* [63], which states that there is no simple relation between $K \rightarrow \pi\pi$ correlation functions in Euclidean time and the desired decay amplitude in Minkowski space. However, this statement applies only to large or infinite volumes where the two pion spectrum is continuous. The discreteness of the two pion spectrum in finite volume actually comes to the rescue! Lellouch and Lüscher showed in [64] that in this case the $K \rightarrow \pi\pi$ amplitude in infinite volume can be related to a $K \rightarrow \pi\pi$ matrix element in finite volume by a simple multiplicative factor. The derivation of this result will be discussed in Section 5.3. The correction factor is just due to the fact that states are normalized differently in infinite

volume and finite volume (which is largely a matter of convention), but the conversion factor between the two normalizations becomes non-trivial for the two pion final state due to the interaction between the two pions.

5.1 Two Particle States in Finite Volume - One Dimension

First we derive the relationship between the phase shift and allowed two pion energies in a one dimensional box as it is very illustrative and because the three dimensional case proceeds in a similar fashion only with some additional complications. We treat the problem in non-relativistic quantum mechanics and represent the interaction between the two pions by a periodic potential that depends only on the distance between the two pions

$$V(x_1, x_2) = \sum_{n=-\infty}^{\infty} V(|x_1 - x_2 + nL|). \quad (5.1)$$

We also suppose that the potential has a finite range R smaller than the length of the box

$$V(x) = 0 \text{ for } |x| > R \quad (5.2)$$

where $R < L/2$. The wave function obeys periodic or antiperiodic boundary conditions

$$\psi(x_1 + L, x_2) = \psi(x_1, x_2 + L) = \pm \psi(x_1, x_2). \quad (5.3)$$

Upon changing to center of mass coordinates, and performing separation

of variables

$$x = x_1 - x_2 \quad (5.4)$$

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad (5.5)$$

$$\psi(X, x) = \Psi(X)\phi(x) \quad (5.6)$$

we find that the CM wave function $\Psi(X)$ obeys the free Schrödinger equation and thus we restrict ourselves to the 0 total momentum sector. Then we find that $\phi(x)$ obeys the Schrödinger equation with the potential V

$$-\frac{1}{2m_{eff}} \frac{\partial^2 \phi}{\partial x^2} + V(x)\phi = \epsilon\phi \quad (5.7)$$

and with the same periodic/antiperiodic boundary conditions as $\psi(x_1, x_2)$

$$\phi(x + L) = \pm\phi(x). \quad (5.8)$$

We now effectively have a one particle problem for $\phi(x)$. Assuming that the potential is symmetric about $x = 0$ (which amounts to assuming rotational symmetry), and since the boundary conditions also have this symmetry, solutions will be either even or odd. If the particles are identical bosons then only the even solutions will be physically acceptable, if they are identical fermions only the odd solutions will be physically acceptable, and if they are distinguishable then either are possible.

First, looking at the problem in infinite volume, we at least know what the form of the solution must be for $|x| > R$:

$$\phi_S(x) = \begin{cases} e^{-ikx} + A_s e^{ikx} & x > R \\ e^{ikx} + A_s e^{-ikx} & x < -R \end{cases} \quad (5.9)$$

$$\phi_A(x) = \begin{cases} e^{-ikx} + A_a e^{ikx} & x > R \\ -e^{ikx} - A_a e^{-ikx} & x < -R \end{cases} \quad (5.10)$$

where ϕ_S is an even (symmetric) solution and ϕ_A is an odd (antisymmetric) solution. Any k value (related to the energy by $E = k^2/2m$) is allowed. The coefficients A_s, A_a must have modulus 1 due to the probability current conservation condition

$$\left[\phi^* \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi^*}{\partial x} \right]_{x=x_0} = \left[\phi^* \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi^*}{\partial x} \right]_{x=-x_0} \quad (5.11)$$

so we set

$$A_s = e^{2i\delta_s} \quad (5.12)$$

$$A_a = -e^{2i\delta_a}. \quad (5.13)$$

The quantities δ_s and δ_a are the phase shifts and depend on the particular potential. They are 0 in the case of 0 potential. The solution in the region $|x| < R$ depends on the particular potential, but is unique and is in one-to-one correspondence with the solution in $|x| > R$.

In finite volume the solutions of Eq. 5.9 and 5.10 must still hold in the unit cell $-L/2 < x < L/2$, and must be repeated periodically or antiperiodically in adjacent cells (Eq. 5.8). In order for such a construction to be continuous, and continuously differentiable we must have

$$\phi(L/2) = \pm\phi(-L/2) \quad (5.14)$$

$$\phi'(L/2) = \pm\phi'(-L/2). \quad (5.15)$$

Imposing this constraint on Eq. 5.9, 5.10 leads to the desired relationship between the energy and the phase shift

$$e^{i(kL+2\delta)} = \pm 1 \quad (5.16)$$

or alternatively

$$n\pi - \delta(k) = \pi q \quad (5.17)$$

for periodic boundary conditions and

$$n\pi - \delta(k) = \pi \left(q + \frac{1}{2} \right) \quad (5.18)$$

for antiperiodic boundary conditions, where

$$q = \frac{kL}{2\pi} \quad (5.19)$$

and where we have explicitly indicated the dependence of the phase shift on

k for emphasis.

Eq. 5.17 and 5.18 can be used in two ways: (i) if the phase shift as a function of k is known for the potential, the equations can be solved for k in order to find the allowed energies in the volume L , and (ii) if the allowed energies of the system are found in volume L then the equations allow you to find the phase shifts at these energies. It is in this second way, in which finite volume results yield phase shifts that can be compared to results of infinite volume scattering experiments, that the three dimensional equivalent of these formulae will be used with lattice results.

5.2 Two Particle States in Finite Volume - Three Dimensions

We follow the derivation of Lüscher [62], and generalize his discussion to the case where there can be antiperiodic boundary conditions in some or all of the spatial directions. This generalization is also discussed in [16].

The setup is similar to the one dimensional case in that we represent the interaction by a periodic potential with a finite range $R < L/2$, and which depends only on the distance between the two particles.¹ There is the additional complication that each of the three coordinate directions can have either periodic or antiperiodic boundary conditions. As in the one dimensional case we can change to center of mass coordinates and restrict ourselves to the 0 total momentum sector, so that we are dealing with a wave

¹Lüscher also argues in [62] that the results of this section, derived in the framework of non-relativistic quantum mechanics, should continue to hold in the framework of quantum field theory.

function $\psi(\mathbf{x})$ that obeys the single particle Schrödinger equation. This will be even in $\mathbf{x}$ for bosons and odd for fermions, but here we treat the general case (i.e. distinguishable particles).

As in the one dimensional case, we first have to look at the infinite volume case, because the solution $\psi(\mathbf{r})$ inside the box must also be an infinite volume solution in that region. For each energy, there is a unique solution for each l, m , where l, m label the angular momentum:

$$\psi_{lm}(\mathbf{r}) = \psi_{lm}(r)Y_{lm}(\theta, \phi). \quad (5.20)$$

Furthermore, for $r > R$, the solution obeys the Schrödinger equation with no potential, i.e. it obeys the Helmholtz equation

$$(\Delta + k^2)\psi(\mathbf{r}) = 0 \quad (5.21)$$

where Δ is the Laplacian. This means that

$$\psi_{lm}(r; k) = \alpha_l(k)j_l(kr) + \beta_l(k)n_l(kr) \quad r > R \quad (5.22)$$

where j_l and n_l are spherical Bessel functions, and $\alpha_l(k)$, $\beta_l(k)$ are uniquely determined by the potential (there is a standard overall normalization in the definition of $\alpha_l(k)$ and $\beta_l(k)$, which is irrelevant here). They are related to the phase shift by

$$e^{2i\delta_l(k)} = \frac{\alpha_l(k) + i\beta_l(k)}{\alpha_l(k) - i\beta_l(k)}. \quad (5.23)$$

Thus the general solution in infinite volume is

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l b_{lm} \psi_{lm}(r) Y_{lm}(\theta, \phi) \quad (5.24)$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^l b_{lm} (\alpha_l(k) j_l(kr) + \beta_l(k) n_l(kr)) Y_{lm}(\theta, \phi) \quad r > R \quad (5.25)$$

where the coefficients b_{lm} are arbitrary, but everything else is completely determined by the potential. This is analogous to the one dimensional problem in which the most general solution is a linear combination of Eq. 5.9 and 5.10.

We introduce an angular momentum cutoff Λ , (Λ can be taken to ∞ at the end or can be kept if we want to make the approximation that scattering of higher angular momentum modes is negligible). In particular we define the operator Q_{Λ} so that it cuts off all terms with $l > \Lambda$ in the expansion of a function in spherical harmonics

$$Q_{\Lambda} f(\mathbf{r}) = \sum_{l=0}^{\Lambda} \sum_{m=-l}^l b_{lm} f_{lm}(r) Y_{lm}(\theta, \phi) \quad (5.26)$$

where the expansion of a function $f(\mathbf{r})$ in spherical harmonics is given by Eq. 5.24 if you replace ψ with f . The problem then involves a modified Hamiltonian

$$H_{\Lambda} = -\frac{1}{2\mu} \Delta + V(r) Q_{\Lambda} \quad (5.27)$$

which basically means that angular momentum modes with $l > \Lambda$ do not see

the potential. Although Q_Λ is a non-local operator with respect to the θ and ϕ variables, it is still local with respect to the r variable (if you operate with Q_Λ on $\delta(r - r_0)f(\theta, \phi)$ the result will still be proportional to $\delta(r - r_0)$). Thus the potential still has a finite range R , and the solution in infinite volume to this problem outside of that range is

$$\begin{aligned} \psi(\mathbf{r}) = & \sum_{l=0}^{\infty} \sum_{m=-l}^l b_{lm} \alpha_l(k) j_l(kr) Y_{lm}(\theta, \phi) \\ & + \sum_{l=0}^{\Lambda} \sum_{m=-l}^l b_{lm} \beta_l(k) n_l(kr) Y_{lm}(\theta, \phi) \quad r > R. \end{aligned} \quad (5.28)$$

Furthermore, because of the locality in r , Eq. 5.28 must hold inside the box in finite volume. Thus we are still in basically the same situation as Eq. 5.25, the only difference being that $\beta_l(k) = 0$ for $l > \Lambda$. The wave function is still a solution of the Helmholtz equation Eq. 5.21 for $r > R$.

The next step is to impose periodic/antiperiodic boundary conditions on Eq. 5.28. Unlike in one dimension, we do *not* expect solutions with fixed l, m to obey the boundary conditions because the cubic box breaks the rotational symmetry, (whereas in one dimension the finite box does not break the parity symmetry). Thus we expect that the result of imposing the boundary conditions on Eq. 5.28 will be to yield a system of equations for the coefficients b_{lm} whose consistency will lead to a condition analogous to Eq. 5.17 and 5.18.

Instead of trying to impose boundary conditions directly, we follow an

indirect route in which we construct all periodic/antiperiodic solutions to the Helmholtz equation (Eq. 5.21) and then require that Eq. 5.28 be equal to some linear combination of these. Since we only need the Helmholtz equation to be satisfied outside of some radius R , we allow for these solutions to be singular at the origin. We restrict ourselves to values of k not in the “singular set” defined by

$$k = \frac{2\pi}{L} |\mathbf{n}| \quad \mathbf{n} \in S \quad (5.29)$$

where S is the set of three component vectors with integer components in directions with periodic boundary conditions and half integer (but not whole integer) components in directions with antiperiodic boundary conditions. In this singular set there are plane wave solutions to the Helmholtz equation that meet the boundary conditions

$$\exp \left[i \frac{2\pi}{L} \mathbf{n} \cdot \mathbf{r} \right] \quad (5.30)$$

and are not singular at the origin. (Solutions that are not singular at the origin are also referred to as *regular*). In fact, these constitute *all* of the non-trivial regular solutions to the Helmholtz equation. Thus if we stay away from the singular set, then our solutions to the Helmholtz equation *must* be singular at the origin. (Lüscher treats the singular set separately and finds that all of the results are continuous with respect to k , so we do not need to worry about these isolated points in k space. In any case, the singular set corresponds to the allowed energies for non-interacting pions, so it is very

likely that the interaction will perturb k away from these values).

The solutions to the Helmholtz equation that satisfy the boundary conditions (and that are singular at the origin) are the following

$$G_{lm}(\mathbf{r}; k^2) = \mathcal{Y}_{lm}(\nabla)G(\mathbf{r}; k^2) \quad (5.31)$$

where

$$G(\mathbf{r}; k^2) = L^{-3} \sum_{\mathbf{n} \in S} \frac{e^{i\mathbf{p} \cdot \mathbf{r}}}{\mathbf{p}^2 - k^2} \quad (5.32)$$

$$\mathbf{p} = \frac{2\pi}{L} \mathbf{n} \quad (5.33)$$

$$\mathcal{Y}_{lm}(\mathbf{r}) = r^l Y_{lm}(\theta, \phi). \quad (5.34)$$

Note that the $\mathcal{Y}_{lm}$ are just polynomials in x, y, z . It can be seen that Eq. 5.32 is a solution to the Helmholtz equation that satisfies the boundary conditions, and since the gradient commutes with the Helmholtz operator (see Eq. 5.21) and derivatives do not change the periodicity/antiperiodicity of a function, Eq. 5.31 is also a solution that satisfies the boundary conditions.

Lüscher proves the linear independence and completeness of the G_{lm} functions by analyzing the form of their singularities. The proof relies on the fact that $G(\mathbf{r}; k^2)$ of Eq. 5.32 is a Green's function of the Helmholtz operator (when you operate on it with the Helmholtz operator you get back the delta function Fourier expanded into the eigenfunctions of the Helmholtz operator of Eq. 5.30), and this still holds for the more general boundary conditions

discussed here. As part of this analysis it is found that the G_{lm} functions can be separated into parts that are singular and parts that are regular at the origin:

$$G_{lm}(\mathbf{r}; k^2) = \frac{(-1)^l}{4\pi} k^{l+1} n_l(kr) Y_{lm}(\theta, \phi) + \hat{G}_{lm}(\mathbf{r}; k^2) \quad (5.35)$$

where $\hat{G}_{lm}$ is the regular part. To compare this to the solutions of Eq. 5.28, we want to expand the regular part in spherical harmonics, and since it is a regular solution of the Helmholtz equation the expansion coefficients must be proportional to $j_l(kr)$, i.e.

$$\begin{aligned} G_{lm}(\mathbf{r}; k^2) = & \frac{(-1)^l}{4\pi} k^{l+1} n_l(kr) Y_{lm}(\theta, \phi) \\ & + \frac{(-1)^l}{4\pi} k^{l+1} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \mathcal{M}_{lm,l'm'}(k) j_{l'}(kr) Y_{l'm'}(\theta, \phi). \end{aligned} \quad (5.36)$$

An explicit expression for the coefficients $\mathcal{M}_{lm,l'm'}(k)$ is given in [62]. We will only need the expression for $\mathcal{M}_{00,00}(k)$, which is

$$\mathcal{M}_{00,00}(k) = \frac{\mathcal{Z}_{00}(1; q^2)}{\pi^{3/2} q}. \quad (5.37)$$

Here $q = kL/2\pi$ as in the previous section, and the zeta function $\mathcal{Z}_{lm}(s; q^2)$ is defined as

$$\mathcal{Z}_{lm}(s; q^2) = \sum_{\mathbf{n} \in S} \mathcal{Y}_{lm}(\mathbf{n}) (\mathbf{n}^2 - q^2)^{-s} \quad (5.38)$$

and is defined by analytic continuation at $s = 1$. Note that the zeta function

depends on the boundary conditions since the sum is over $\mathbf{n} \in S$. Methods for evaluating the zeta function numerically can be found in [62, 65].

The last step is to require that the solution of Eq. 5.28 be equal to a linear combination of the G_{lm} functions, thus imposing the boundary conditions on Eq. 5.28.

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{m=-l}^l b_{lm} \alpha_l(k) j_l(kr) Y_{lm}(\theta, \phi) + \sum_{l=0}^{\Lambda} \sum_{m=-l}^l b_{lm} \beta_l(k) n_l(kr) Y_{lm}(\theta, \phi) \\ = \sum_{l=0}^{\infty} \sum_{m=-l}^l v_{lm} G_{lm}(\mathbf{r}; k^2). \end{aligned} \quad (5.39)$$

Substituting Eq. 5.36 into the right hand side,

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{m=-l}^l b_{lm} \alpha_l(k) j_l(kr) Y_{lm}(\theta, \phi) + \sum_{l=0}^{\Lambda} \sum_{m=-l}^l b_{lm} \beta_l(k) n_l(kr) Y_{lm}(\theta, \phi) \\ = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left[\sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} v_{l'm'} \frac{(-1)^{l'}}{4\pi} k^{l'+1} \mathcal{M}_{l'm',lm}(k) \right] j_l(kr) Y_{lm}(\theta, \phi) \\ + \sum_{l=0}^{\infty} \sum_{m=-l}^l v_{lm} \frac{(-1)^l}{4\pi} k^{l+1} n_l(kr) Y_{lm}(\theta, \phi) \end{aligned} \quad (5.40)$$

where the unprimed dummy indices l and m have been interchanged with the primed dummy indices l' and m' in the first term on the right hand side.

The Y_{lm} 's are linearly independent, and $j_l(kr)$ is linearly independent from $n_l(kr)$ (the latter is singular at the origin while the former is not). Thus coefficients of $j_l(kr) Y_{lm}(\theta, \phi)$ can be equated on both sides of Eq. 5.40, and so can coefficients of $n_l(kr) Y_{lm}(\theta, \phi)$. Equating coefficients of $n_l(kr) Y_{lm}(\theta, \phi)$,

we get

$$v_{lm} \frac{(-1)^l}{4\pi} k^{l+1} = \begin{cases} b_{lm} \beta_l(k) & l \leq \Lambda \\ 0 & l > \Lambda \end{cases}. \quad (5.41)$$

Substituting this expression for the v_{lm} coefficients back into Eq. 5.40, we get

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{m=-l}^l b_{lm} \alpha_l(k) j_l(kr) Y_{lm}(\theta, \phi) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \left[\sum_{l'=0}^{\Lambda} \sum_{m'=-l'}^{l'} b_{l'm'} \beta_{l'}(k) \mathcal{M}_{l'm',lm}(k) \right] j_l(kr) Y_{lm}(\theta, \phi). \end{aligned} \quad (5.42)$$

Now we can equate coefficients of $j_l(kr) Y_{lm}(\theta, \phi)$ and we see that

$$b_{lm} \alpha_l(k) = \sum_{l'=0}^{\Lambda} \sum_{m'=-l'}^{l'} b_{l'm'} \beta_{l'}(k) \mathcal{M}_{l'm',lm}(k). \quad (5.43)$$

This equation holds for all l and m , but if we consider the set of equations with $l \leq \Lambda$ then we have a finite number of linear homogeneous equations in the b_{lm} variables, with as many equations as there are variables. This set of equations will have a non-trivial solution for the b_{lm} 's if the determinant of the coefficient matrix is 0. The equation that results from setting the determinant to 0 will be an equation in k , and is exactly the quantization condition that determines the allowed values of k that we've been seeking (like Eq. 5.17, 5.18 in one dimension). Once this condition is met, non-trivial solutions for the b_{lm} 's with $l \leq \Lambda$ can be found using the same set of

equations. If we then want to know the b_{lm} 's for $l > \Lambda$, we can set l and m to the desired value in Eq. 5.43, and by looking at the right hand side of this equation we can see that b_{lm} will be given in terms of $b_{l'm'}$'s with $l' \leq \Lambda$ only, which have already been found. However, the quantization condition on k is what is of real interest to us here.

The result of Eq. 5.43 can be manipulated so that it involves only the phase shift $\delta_l(k)$ rather than $\alpha_l(k)$ and $\beta_l(k)$. To do so, picture this equation for $l \leq \Lambda$ as a matrix equation on the set of components $\{b_{lm}, l \leq \Lambda, -l \leq m \leq l\}$. A matrix P acts on these components in the following way:

$$[Pb]_{lm} = \sum_{l'=0}^{\Lambda} \sum_{m'=-l'}^{l'} P_{lm,l'm'} b_{l'm'}. \quad (5.44)$$

Thus Eq. 5.43 can be written in matrix form

$$b^T(A - BM) = 0 \quad (5.45)$$

where the matrix elements of A , B , and M are

$$A_{lm,l'm'} = \alpha_l(k) \delta_{ll'} \delta_{mm'} \quad (5.46)$$

$$B_{lm,l'm'} = \beta_l(k) \delta_{ll'} \delta_{mm'} \quad (5.47)$$

$$M_{lm,l'm'} = \mathcal{M}_{lm,l'm'}(k). \quad (5.48)$$

If we take the transpose of Eq. 5.45, we get

$$(A - BM)^T b = 0 \quad (5.49)$$

and we see that the condition for a non-trivial solution for b is

$$\det((A - BM)^T) = \det(A - BM) = 0. \quad (5.50)$$

To put Eq. 5.50 in terms of the phase shift we divide both sides by

$$\det(A - iB)\det(M - i) \quad (5.51)$$

and arrive at

$$\det(e^{2i\delta} - U) = 0 \quad (5.52)$$

where

$$U = (M + i)(M - i)^{-1} \quad (5.53)$$

and

$$e^{2i\delta} = (A + iB)(A - iB)^{-1}. \quad (5.54)$$

From Eq. 5.23 we see that

$$[e^{2i\delta}]_{lm,l'm'} = e^{2i\delta_l(k)} \delta_{ll'} \delta_{mm'} \quad (5.55)$$

that is, this matrix just consists of exponentials of the phase shifts $\delta_l(k)$ in

its diagonal elements. Therefore in Eq. 5.52 we have succeeded in finding an equation involving only the phase shifts and the $\mathcal{M}_{lm,l'm'}(k)$ which are known functions of k .

Eq. 5.52 is in principal the final result, but it is still not very useful because it involves all of the $\delta_l(k)$ for $l \leq \Lambda$, and it involves an arbitrary angular momentum cutoff Λ . One way to simplify the situation is to assume that the energy of the two pion system is low enough (as is the case for the lattice calculation presented in this thesis) so that only s wave scattering occurs, i.e. all other phase shifts besides $\delta_0(k)$ are very close to 0. This is possible because, to leading order in the low energy/low k expansion for the phase shifts we have

$$\delta_l(k) = a_l k^{l+1} + \dots \quad (5.56)$$

Thus we can set the angular momentum cutoff $\Lambda = 0$ (i.e. allow for only scattering of the $l = 0$ mode), and Eq. 5.52 becomes

$$e^{2i\delta_0(k)} = \frac{\mathcal{M}_{00,00} + i}{\mathcal{M}_{00,00} - i}. \quad (5.57)$$

Some algebra, and using the fact that $\mathcal{M}_{00,00}$ is real, allows this to be rewritten as

$$\tan \delta_0(k) = \frac{1}{\mathcal{M}_{00,00}} \quad (5.58)$$

or

$$n\pi - \delta_0(k) = \phi(q) \quad (5.59)$$

where

$$\tan \phi(q) = -\frac{1}{\mathcal{M}_{00,00}(k)} = -\frac{\pi^{3/2}q}{\mathcal{Z}_{00}(1;q^2)} \quad (5.60)$$

in which Eq. 5.37 was used. Eq. 5.59 is exactly analogous to Eq. 5.17 and 5.18 in one dimension, and will be used in Chapter 7 to find the two pion phase shift from the two pion energies measured on the lattice.

It turns out that we do not have to set the cutoff as low as $\Lambda = 0$ to arrive at Eq. 5.59. In the case that the boundary conditions are periodic in all three directions or antiperiodic in all three directions then the symmetries of the two pion system are the same as those of the cube (the octahedral group, which consists of various rotations and reflections that leave the cube invariant). In the case that one or two directions have antiperiodic boundary conditions, two opposite faces of the cube become distinct from the four others, let's say the ones perpendicular to the z axis. These two faces must be mapped back onto themselves or each other by a symmetry transformation, and the symmetry group becomes smaller, consisting only of some rotations about the z axis, some reflections in planes which contain the z axis, and a reflection in the xy plane.

In both cases the two pion states will transform irreducibly under representations of the symmetry group, and we can examine separately sectors corresponding to different irreducible representations. In our lattice calculation the pion sources, (i.e. the cosine sources of Eq. 6.18 and the wall sources which are special cases of the cosine sources with $\mathbf{p} = 0$), are all invariant un-

der the symmetry transformations. This causes the results of the calculation to be restricted to the sector of two pion states which transform under the trivial representation of the symmetry group (call this the “trivial sector”).

In Eq. 5.52 we can change to a basis in which the matrices are block diagonal and the different blocks correspond to different irreducible representations of the symmetry group, and indeed this is done by Lüscher. To see which phase shifts δ_l will be involved for a given sector/irreducible representation, we need to know which spherical harmonics, or equivalently, which of the harmonic polynomials of Eq. 5.34, will be part of that sector and in particular we need to know their l values. For the trivial sector this amounts to finding if there are harmonic polynomials of degree l that are invariant under the symmetry transformations. Because inversion ($\mathbf{r} \rightarrow -\mathbf{r}$) is always one of the symmetry transformations, a polynomial will have to be of even degree to be invariant. Obviously a polynomial of degree 0, i.e. a constant, is invariant. The next possibility is a polynomial of degree 2. When the boundary conditions in one direction, let’s say the z direction, are distinct, then there is one such harmonic polynomial:

$$x^2 + y^2 - 2z^2. \tag{5.61}$$

However, if all three directions have the same boundary conditions which means that they symmetry group is larger, then after the degree 0 polynomial the next harmonic polynomial that is invariant under the symmetries has

degree 4:

$$x^4 + y^4 + z^4 - \frac{3}{5}r^4. \quad (5.62)$$

Thus, since we are dealing with the trivial sector, when boundary conditions are different in one direction we can actually set the cutoff to $\Lambda = 1$, ignoring the phase shifts δ_l with $l \geq 2$, and the result of Eq. 5.59 will still be valid. When the boundary conditions are the same in all directions then we can set the cutoff as high as $\Lambda = 3$, ignoring the phase shifts δ_l with $l \geq 4$, and Eq. 5.59 will still be valid. Note that if we were dealing with a different sector then instead of Eq. 5.59 we would have a relationship involving phase shifts $\delta_l(k)$ with $l > 0$, rather than $\delta_0(k)$. Different sectors have been examined recently in [66], and were used to calculate $\delta_2(k)$ on the lattice for two pions with isospin $I = 2$.

5.3 The Lellouch-Lüscher Factor

We now wish to find a relationship between the $K \rightarrow \pi\pi$ matrix element in finite volume and the infinite volume amplitude. We present the derivation of this relation by Lellouch and Lüscher as found in [64]. We define the matrix element in finite volume as

$$M = \langle \pi\pi | H_W | K \rangle \quad (5.63)$$

where H_W is an effective Hamiltonian for the weak interactions which give rise to $K \rightarrow \pi\pi$ decay (see Section 4.1), and the kaon and two pion states

are normalized to unity. We define the infinite volume amplitude through

$$\langle \pi(p_1)\pi(p_2) | \mathcal{L}_W(0) | K(p) \rangle = A e^{i\delta_0} \quad (5.64)$$

(c.f. Eq. 2.5), where $\mathcal{L}_W$ is the Lagrange density corresponding to H_W , and δ_0 is the s wave phase shift of the two pions in the final state. (Here the subscript on δ refers to the value of l , whereas in the notation of Eq. 2.5 the subscript is used to refer to the isospin I).

The states in Eq. 5.64 are normalized according to

$$\langle p | p' \rangle = 2E(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \quad (5.65)$$

for one particle states, and

$$\begin{aligned} \langle p_1, p_2 | p'_1, p'_2 \rangle = 4E_1 E_2 (2\pi)^6 & \left[\delta^3(\mathbf{p}_1 - \mathbf{p}'_1) \delta^3(\mathbf{p}_2 - \mathbf{p}'_2) \right. \\ & \left. + \delta^3(\mathbf{p}_1 - \mathbf{p}'_2) \delta^3(\mathbf{p}_2 - \mathbf{p}'_1) \right] \end{aligned} \quad (5.66)$$

for two particle states. Note that we consider the two pions in Eq. 5.64 to be identical particles, hence the second term in the square brackets in Eq. 5.66.

We desire a relationship between A in Eq. 5.64 and M in Eq. 5.63. Lellouch and Lüscher find this relationship in an indirect manner which we now describe. The weak interaction, described by H_W , is considered to be a small perturbation to the strong Hamiltonian. Two pions will interact with each other and scatter due to the strong Hamiltonian alone, but a kaon will not

decay to two pions in the absence of the weak interaction. The $K \rightarrow \pi\pi$ decay is caused entirely by the weak Hamiltonian. Two pion scattering will also be slightly perturbed by the weak Hamiltonian H_W . In infinite volume this perturbation will result in a small change in the phase shifts at a given energy.

In finite volume, the situation is more complicated. In the absence of H_W we have two pion states with certain allowed energies. As in Section 5.2 we restrict ourselves to states that are invariant under the symmetries of the box so that Eq. 5.59 applies. There is also a kaon state which must have the same energy as the two pion state so that we are describing an energy conserving decay.² Turning on the weak interaction will mix the kaon and two pion states in a way that can be described by degenerate perturbation theory, and the energies of the new states will be slightly shifted from the common energy of the original kaon and two pion state. Degenerate perturbation theory allows one to calculate this energy shift. Alternately, we could calculate in infinite volume the change in the phase shift due to turning on H_W and use Eq. 5.59 to determine the very same energy shift. Comparison of the results obtained from the two different methods yields the desired relationship between A and

²Thus, strictly speaking, the results of this section do not apply to an energy non-conserving $K \rightarrow \pi\pi$ decay. Of course there is no such thing as an energy non-conserving decay in nature. Lattice calculations of energy non-conserving decays can be done, and are done in this report, but these are only meaningful because the decay amplitudes and matrix elements are assumed to be continuous in the particle masses and energies. Therefore results for energy non-conserving decays are either extrapolated or interpolated to energy-conserving kinematics, or we use results for decays that are close to energy-conserving and attempt to estimate the systematic error in doing so. In this way we will be justified in using the results of the present section.

M .

We first obtain the energy shift from a direct degenerate perturbation theory calculation involving the finite volume states that appear in Eq. 5.63. The part of the weak Hamiltonian with which we are concerned, because it gives rise to $K \rightarrow \pi\pi$ decay, is the strangeness changing part. Thus

$$\langle \pi\pi | H_W | \pi\pi \rangle = \langle K | H_W | K \rangle = 0. \quad (5.67)$$

The shifted energies of the two new states that are mixtures of the original kaon and two pion states can now be written in terms of M :

$$W = m_K \pm |M| + O(M^2) \quad (5.68)$$

where $m_K = E_{\pi\pi}$ is the common energy of the original kaon and two pion states. The energy of the state is related to the quantity k in Eq. 5.59 via

$$W = 2\sqrt{m_\pi^2 + k^2}. \quad (5.69)$$

Thus the shift in k corresponding to Eq. 5.68 is

$$k = k_\pi + \Delta k + O(M^2), \quad \Delta k = \pm \frac{m_K}{4k_\pi} |M| \quad (5.70)$$

where k_π is the value of k before the weak interaction is turned on, i.e. $m_K = 2\sqrt{m_\pi^2 + k_\pi^2}$.

Now we calculate the change in the phase shift in infinite volume due to H_W . Diagrams that change the $\pi\pi$ scattering amplitude to first order in H_W are all of the form shown in Figure 4. The kaon propagator evaluates to

$$\frac{iZ_K}{p^2 - m_K^2} = \pm \frac{iZ_K}{2m_K|M|} + O(M^2) \quad (5.71)$$

at the shifted energies of Eq. 5.68. The vertices are the diagrams of the $K \rightarrow \pi\pi$ decay itself and thus contribute a factor proportional to A^2 . Thus the diagram is overall first order in H_W (since A and M are first order in H_W). The resultant change in the phase shift as a function of k due to turning on H_W is then found to be

$$\Delta\delta_0 = \mp \frac{k_\pi |A|^2}{32\pi m_K^2 |M|} + O(H_W^2). \quad (5.72)$$

Finally, we require that Eq. 5.59 hold at the unshifted energy m_K , corresponding to k_π ,

$$n\pi - \delta_0(k_\pi) = \phi(q_\pi) \quad (5.73)$$

and that it hold at the shifted energy W , corresponding to k in Eq. 5.70, and with the modified phase shift function of Eq. 5.72,

$$n\pi - \delta_0(k_\pi) - \frac{\partial\delta_0(k_\pi)}{\partial k} \Delta k - \Delta\delta_0 = \phi(q_\pi) + \frac{\partial\phi(q_\pi)}{\partial q} \Delta q. \quad (5.74)$$

Combining Eq. 5.73 and Eq. 5.74 gives

$$-\frac{\partial\delta_0(k_\pi)}{\partial k}\Delta k - \Delta\delta_0 = \frac{\partial\phi(q_\pi)}{\partial q}\Delta q. \quad (5.75)$$

Substituting Eq. 5.70 and Eq. 5.72, (and also recalling Eq. 5.19, the relationship between q and k), we arrive at the desired relationship between A and M

$$|A|^2 = 8\pi \left\{ q \frac{\partial\phi}{\partial q} + k \frac{\partial\delta_0}{\partial k} \right\}_{k=k_\pi} \left(\frac{m_K}{k_\pi} \right)^3 |M|^2 \quad (5.76)$$

where we have solved for $|A|^2$ in terms of $|M|^2$.

We call the proportionality factor between $|A|^2$ and $|M|^2$ in Eq. 5.76 the “Lellouch-Lüscher” factor:

$$\text{LL factor} = 8\pi \left[q \frac{\partial\phi}{\partial q} + k \frac{\partial\delta_0}{\partial k} \right] \left(\frac{m_K}{k} \right)^3 \quad (5.77)$$

where k is evaluated at the energy of the two pion state in finite volume in the absence of weak interactions. (This value of k is called k_π in the discussion above). This is indeed the energy we obtain in lattice simulations of two pion states which do not incorporate the weak interactions (which are a small effect if we are only concerned with two pion scattering). The weak interactions are incorporated only in the $K \rightarrow \pi\pi$ decay calculation, and then only by calculating the matrix element of Eq. 5.63 using an effective Hamiltonian as described in Section 4.1.

The Lellouch-Lüscher factor essentially just corrects for the different nor-

malization of states in finite and infinite volume. Thus it can be worked out explicitly in the “free field” case where the two pions in the final state do not interact with each other, because then all states are pure plane waves in both finite and infinite volume. It is instructive to do this because the actual Lellouch-Lüscher factor turns out to be numerically relatively close to the factor obtained from this free-field calculation.

A single particle plane wave state in finite volume will have the wavefunction

$$\psi_{\mathbf{p}}^{\text{fin}}(\mathbf{x}) = \frac{1}{L^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}} \quad (5.78)$$

where $\mathbf{p}$ is such that the wave function obeys the appropriate boundary conditions. This state is normalized to unity in the volume L^3 . In infinite volume the state will have the wavefunction

$$\psi_{\mathbf{p}}^{\text{inf}}(\mathbf{x}) = \sqrt{2E} e^{i\mathbf{p}\cdot\mathbf{x}} \quad (5.79)$$

corresponding to the normalization of Eq. 5.65. Thus for one particle states

$$|p\rangle^{\text{inf}} = \sqrt{2EL^3} |p\rangle^{\text{fin}}. \quad (5.80)$$

Two particle states must be properly symmetrized since the two pions

are identical. In finite volume

$$\psi_{\mathbf{p}_1 \mathbf{p}_2}^{\text{fin}}(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} \frac{1}{\sqrt{2}L^3} (e^{i\mathbf{p}_1 \cdot \mathbf{x}_1} e^{i\mathbf{p}_2 \cdot \mathbf{x}_2} + e^{i\mathbf{p}_1 \cdot \mathbf{x}_2} e^{i\mathbf{p}_2 \cdot \mathbf{x}_1}) & \mathbf{p}_1 \neq \mathbf{p}_2 \\ \frac{1}{L^3} e^{i\mathbf{p}_1 \cdot (\mathbf{x}_1 + \mathbf{x}_2)} & \mathbf{p}_1 = \mathbf{p}_2 \end{cases} \quad (5.81)$$

so that the states have unit norm. (For the case $\mathbf{p}_1 \neq \mathbf{p}_2$ we have used the fact that exponentials with different momenta are orthogonal to each other for the allowed momenta in finite volume). In infinite volume,

$$\psi_{\mathbf{p}_1 \mathbf{p}_2}^{\text{inf}}(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2}} \sqrt{2E_1} \sqrt{2E_2} (e^{i\mathbf{p}_1 \cdot \mathbf{x}_1} e^{i\mathbf{p}_2 \cdot \mathbf{x}_2} + e^{i\mathbf{p}_1 \cdot \mathbf{x}_2} e^{i\mathbf{p}_2 \cdot \mathbf{x}_1}) \quad (5.82)$$

corresponds to the normalization of Eq. 5.66. Thus

$$|p_1, p_2\rangle^{\text{inf}} = \begin{cases} \sqrt{2E_1} \sqrt{2E_2} L^3 |p_1, p_2\rangle^{\text{fin}} & \mathbf{p}_1 \neq \mathbf{p}_2 \\ \sqrt{2} \sqrt{2E_1} \sqrt{2E_2} L^3 |p_1, p_2\rangle^{\text{fin}} & \mathbf{p}_1 = \mathbf{p}_2 \end{cases}. \quad (5.83)$$

In the CM frame, $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$ so that it is natural to look at two cases separately: $\mathbf{p} = 0$ (meaning $\mathbf{p}_1 = \mathbf{p}_2$), and $\mathbf{p} \neq 0$ (meaning $\mathbf{p}_1 \neq \mathbf{p}_2$). The first case occurs for the lowest energy state when we have periodic boundary in all directions (two pions with zero momentum is the lowest energy state when there is no interaction between them), otherwise we have the second case. For the first case we substitute Eq. 5.80 and Eq. 5.83 for the case

$\mathbf{p}_1 = \mathbf{p}_2$ into Eq. 5.64,

$$A = \langle \pi(0)\pi(0) |^{\text{inf}} \mathcal{L}_W(0) | K(0) \rangle^{\text{inf}} \quad (5.84)$$

$$= \sqrt{2}(2E_\pi)L^3 \sqrt{2m_K L^3} \langle \pi(0)\pi(0) |^{\text{fin}} \mathcal{L}_W(0) | K(0) \rangle^{\text{fin}} \quad (5.85)$$

$$= 2(m_K L)^{3/2} \langle \pi(0)\pi(0) |^{\text{fin}} H_W | K(0) \rangle^{\text{fin}} \quad (5.86)$$

$$= 2(m_K L)^{3/2} M \quad (5.87)$$

where in the second last line we used the fact that $2E_\pi = m_K$ for energy conserving kinematics, and where we used translation invariance to write

$$\mathcal{L}_W(0) \rightarrow \frac{1}{L^3} \int d^3x \mathcal{L}_W(\mathbf{x}) = \frac{1}{L^3} H_W. \quad (5.88)$$

In the second case we also have to consider the fact that the two pion state with the desired transformation properties under the symmetries of the box may be a linear combination of plane wave states with different $\mathbf{p}$. (These $\mathbf{p}$ must have the same magnitude, but can have different directions). For example, if the x and y directions have antiperiodic boundary conditions then the lowest energy state invariant under the symmetries of the box is a sum of plane waves with $\mathbf{p} = \frac{\pi}{L}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$ and $\mathbf{p} = \frac{\pi}{L}(\hat{\mathbf{x}} - \hat{\mathbf{y}})$. (Note that since the pions are identical the plane wave states are already symmetrized with respect to particle exchange and thus $\mathbf{p} \rightarrow -\mathbf{p}$ does not generate a new state that must be included in the sum). In contrast, the infinite volume two pion state in Eq. 5.64 consists of a single plane wave state. We are interested in

finite volume states that are invariant under the symmetries of the box:

$$|\pi\pi(\text{sum})\rangle^{\text{fin}} = \frac{1}{\sqrt{n_s}} \sum_{\mathbf{p}} |\pi(\mathbf{p})\pi(-\mathbf{p})\rangle^{\text{fin}} \quad (5.89)$$

where the sum is over all distinct plane wave states that are mapped into each by the symmetries of the box, and where n_s is the number of such states. This state has unit norm and is the two pion state that appears in Eq. 5.63.

Thus we have

$$A = \langle \pi(\mathbf{p})\pi(-\mathbf{p}) |^{\text{inf}} \mathcal{L}_W(0) | K(0) \rangle^{\text{inf}} \quad (5.90)$$

$$= (2E_\pi)L^3 \sqrt{2m_K L^3} \langle \pi(\mathbf{p})\pi(-\mathbf{p}) |^{\text{fin}} \mathcal{L}_W(0) | K(0) \rangle^{\text{fin}} \quad (5.91)$$

$$= \sqrt{2}(m_K L)^{3/2} \langle \pi(\mathbf{p})\pi(-\mathbf{p}) |^{\text{fin}} H_W | K(0) \rangle^{\text{fin}} \quad (5.92)$$

$$= \sqrt{2}(m_K L)^{3/2} \frac{1}{n_s} \sum_{\mathbf{p}} \langle \pi(\mathbf{p})\pi(-\mathbf{p}) |^{\text{fin}} H_W | K(0) \rangle^{\text{fin}} \quad (5.93)$$

$$= \sqrt{\frac{2}{n_s}} (m_K L)^{3/2} \langle \pi\pi(\text{sum}) |^{\text{fin}} H_W | K(0) \rangle^{\text{fin}} \quad (5.94)$$

$$= \sqrt{\frac{2}{n_s}} (m_K L)^{3/2} M \quad (5.95)$$

where we used Eq. 5.80 and Eq. 5.83 for the case $\mathbf{p}_1 \neq \mathbf{p}_2$ in the second line, we used the fact that due to rotation invariance the sum in the fourth line is over n_s identical matrix elements equal to the one in the third line, and we used Eq. 5.89 in the fifth line.

We summarize the results of Eq. 5.87 and 5.95 by giving the proportion-

ality factor between $|A|^2$ and $|M|^2$ in the case of non-interacting pions

$$\text{LL free field} = \begin{cases} 4(m_K L)^3 & \mathbf{p} = 0 \\ \frac{2}{n_s}(m_K L)^3 & \mathbf{p} \neq 0 \end{cases}. \quad (5.96)$$

The lattice calculation in this report will always pick out the lowest energy two pion state which is invariant under the symmetries of the box. When all spatial directions have periodic boundary conditions the $\mathbf{p} = 0$ case applies, but when some directions have antiperiodic boundary conditions then the $\mathbf{p} \neq 0$ case applies with $n_s = 2^{n_{\text{tw}}-1}$, where n_{tw} is the number of directions with antiperiodic boundary conditions. Thus all of the cases relevant in this report can be summarized by³

$$\text{LL free field} = \frac{4}{2^{n_{\text{tw}}}}(m_K L)^3. \quad (5.97)$$

We will also like to look at the ratio of the actual normalization factor to the free field normalization factor:

$$\sqrt{\frac{\text{LL factor}}{\text{LL free field}}} = \frac{\sqrt{2^{n_{\text{tw}}}}}{2\pi q^{3/2}} \sqrt{q \frac{\partial \phi}{\partial q} + k \frac{\partial \delta_0}{\partial k}}. \quad (5.98)$$

In summary, the result Eq. 5.76 can be used compute the decay amplitude from the matrix element in finite volume. The ratio of the exact result to the “free-field” approximation in which interactions between the two pions in the

³It can be verified explicitly that Eq. 5.77 becomes Eq. 5.97 in the case of no interaction.

final state are neglected is given in Eq. 5.98. More detail on the conversion factors between lattice matrix elements and infinite volume amplitudes is given in Appendix A, where the results of this section, Sections 4.1 and 4.2, and the conventions used in the lattice simulation are discussed, and explicit directions are given for converting the lattice results to an infinite volume amplitude.

6 Computational Methods

Computations of $K \rightarrow \pi\pi$ decay amplitudes on three different lattices were performed, the details of which will be discussed in this chapter. One has dynamical quarks, dimensions $L^3 \times T = 32^3 \times 64$ (where L is the spatial extent of the lattice and T is the time extent of the lattice), $L_s = 32$, and a coarse lattice spacing of $a^{-1} = 1.373(24)$ GeV. The second also has dynamical quarks, dimensions $32^3 \times 64$, $L_s = 16$, and a fine lattice spacing of $a^{-1} = 2.285(29)$ GeV. The third is a quenched ensemble with dimensions $24^3 \times 64$, $L_s = 16$, and a lattice spacing of $a^{-1} = 1.31(2)$ GeV. These lattices will be referred to as the “coarse 32^3 lattices”, “fine 32^3 lattices”, and “quenched 24^3 lattices”, respectively. The large lattice spacing of the coarse 32^3 lattices means that the volume is large, which allows a $K \rightarrow \pi\pi$ calculation with nearly physical kaon and pion masses, ($m_\pi L = 3.3$ with our nearly physical pion mass), and nearly physical kinematics, while on the fine 32^3 lattices only an unphysical computation in which the two pions are too heavy and nearly at rest is performed in order to investigate the effect of varying the lattice spacing. The quenched 24^3 lattices are used to perform simulations with a variety of kaon and pion masses, and pion momenta, in order to investigate

the dependence of the decay amplitude on these quantities and assess the effect of not having exactly physical masses and momenta in the calculation on the coarse 32^3 lattices.

6.1 Coarse 32^3 Lattices

6.1.1 Gauge Ensembles

The ensembles were generated by the RBC/UKQCD collaboration using the Dislocation Suppressing Determinant Ratio (DSDR) [67] plus Iwasaki gauge action, with 2+1 dynamical flavors¹ of Domain Wall Fermions (DWF). The lattice size is $32^3 \times 64$, with a fifth dimension of size $L_s = 32$ and domain wall height $M_5 = 1.8$. The inverse lattice spacing is $a^{-1} = 1.373(24)$ GeV (corresponding to a coupling $\beta = 1.75$), which means the total box size is $(4.60 \text{ fm})^3$ in physical units. The sea strange quark mass is set to $m_s^{\text{sea}} = 0.045$ and ensembles were generated with sea light quark masses of $m_l^{\text{sea}} = 0.0042$ and 0.001 , corresponding to unitary pion masses of approximately $m_\pi \approx 250 \text{ MeV}, 180 \text{ MeV}$ respectively, but only the second sea light quark mass is used in the present report. Measurements were performed on 47 configurations beginning at 500 MD time units at an interval of 8 MD time units. The residual mass is $m_{\text{res}} = 0.0018347(87)$ [68].

¹Meaning two lighter sea quarks of equal mass representing the up and down quarks, and one heavier sea quark representing the strange quark.

6.1.2 Propagators

Using these lattices we generated propagators with $m^{\text{val}} = 0.0001$ and 0.049 for the light quark and strange quark respectively. Inversions were done with a stopping condition of 10^{-10} . In order to improve the independence of measurements, propagators were generated with sources on time slice $t' = 0, 16, 32, 48, 0, \dots$, etc., on each consecutive measured configuration. Both the source and the sink of the propagators are Coulomb gauge fixed wall sources with stopping condition 10^{-8} .

We now describe in more detail the various sources and boundary conditions of the propagators that were obtained. For each quark mass we calculate propagators with wall sources at t' (meaning the source is equal to 1 on the entire time slice t' and is equal to 0 on all other time slices).² These are calculated with both periodic and antiperiodic boundary conditions in

²All sources and sinks are subsequently gauge fixed. This will be implicit in the discussion to follow.

the time direction and will be denoted

$$S^P(\mathbf{x}, t; \text{wall}, t') = \sum_{\mathbf{y}} S^P(\mathbf{x}, t; \mathbf{y}, t') = \text{Quark propagator, wall source at } t',$$

periodic BC in time direction (6.1)

$$S^A(\mathbf{x}, t; \text{wall}, t') = \sum_{\mathbf{y}} S^A(\mathbf{x}, t; \mathbf{y}, t') = \text{Quark propagator, wall source at } t',$$

antiperiodic BC in time direction.

(6.2)

Note that all boundary conditions in the spatial directions are periodic so far.

These are added and subtracted from each other to make propagators with double the effective time length. Specifically, we will use the notation

$$S(\mathbf{x}, t; \text{wall}, t') = \frac{1}{2} [S^P(\mathbf{x}, t; \text{wall}, t') + S^A(\mathbf{x}, t; \text{wall}, t')]. \quad (6.3)$$

The new propagator on the left hand side has period $2N_t$ rather than N_t , where N_t is the period of the original propagator of Eq. 6.1, (i.e. N_t is the time extent of the lattice which is 64 for the configurations discussed in this section). This can be seen from the periodicity properties of the propagators

in Eq. 6.1 and 6.2:

$$S(t + 2N_t; t') = \frac{1}{2} [S^P(t + 2N_t; t') + S^A(t + 2N_t; t')] \quad (6.4)$$

$$= \frac{1}{2} [S^P(t; t') + S^A(t; t')] \quad (6.5)$$

$$= S(t; t') \quad (6.6)$$

shows that the new propagator has period $2N_t$ (the spatial variables have been suppressed for clarity), whereas

$$S(t + N_t; t') = \frac{1}{2} [S^P(t + N_t; t') + S^A(t + N_t; t')] \quad (6.7)$$

$$= \frac{1}{2} [S^P(t; t') - S^A(t; t')] \quad (6.8)$$

$$\neq S(t; t') \quad (6.9)$$

shows that it does not have a period of N_t . Eq. 6.3 and 6.8 also make explicit that the new propagator can be evaluated for $0 \leq t < N_t$ by adding the original periodic and antiperiodic propagators in this range and dividing by 2, while for $N_t \leq t < 2N_t$ it can be evaluated by subtracting the original antiperiodic propagator from the periodic propagator and dividing by 2, where the original propagators are still evaluated in $0 \leq t < N_t$. To evaluate the new propagator at any other t one can make use of its $2N_t$ periodicity.

The use of the propagator of Eq. 6.3 is justified because it is still a solution of the Dirac equation (i.e. it is the inverse of the Dirac operator $\not{D} + m$, c.f. Section 3.4) with a wall source, just like the original propagators of

Eq. 6.1 and 6.2. This can be seen as follows: The propagators of Eq. 6.1 and 6.2 are solutions of the Dirac equation with a wall source at t' and periodic/antiperiodic boundary conditions, which can be written as

$$D[U]S^P(t; t') = \sum_{n=-\infty}^{\infty} \delta_{t, t' + N_t n} \quad (6.10)$$

$$D[U]S^A(t; t') = \sum_{n=-\infty}^{\infty} (-1)^n \delta_{t, t' + N_t n} \quad (6.11)$$

where D is short for the Dirac operator $\mathcal{D} + m$, which depends on the gauge configuration U , and the spatial dependence of the propagators have been suppressed. These equations plus the requirement of periodicity/antiperiodicity, with period N_t , uniquely define the propagators. Consequently, if one applies the Dirac operator to the propagator of Eq. 6.3,

$$D[U]S(t; t') = \frac{1}{2} [D[U]S^P(t; t') + D[U]S^A(t; t')] \quad (6.12)$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} [1 + (-1)^n] \delta_{t, t' + N_t n} \quad (6.13)$$

$$= \sum_{n=-\infty}^{\infty} \delta_{t, t' + 2N_t n}. \quad (6.14)$$

This result, along with the result of Eq. 6.6, shows that S is a solution of the Dirac equation with a wall source at t' and period $2N_t$, and thus it is unique. The only difference between our propagator and a propagator from a lattice that actually has time extent $2N_t$, is the fact that the Dirac operator depends on the gauge configuration U which for us still has period N_t in the

time direction.

We also generate propagators with *spatial* boundary conditions other than periodic. In addition to the wall source propagators with periodic boundary conditions in all spatial directions that have been discussed so far, which correspond to quarks carrying zero momentum, we also calculate propagators with antiperiodic boundary conditions in various spatial directions and “cosine sources” (which will be described below) in order to give the quarks various momenta. For example, the two smallest momenta generated by antiperiodic boundary conditions in the x direction are

$$\mathbf{p} = \left(\pm \frac{\pi}{L}, 0, 0 \right) \quad (6.15)$$

and the four smallest momenta generated by antiperiodic boundary conditions in the y and z direction are

$$\mathbf{p} = \left(0, \pm \frac{\pi}{L}, \pm \frac{\pi}{L} \right). \quad (6.16)$$

Generally, larger momenta correspond to excited states, so we consider only the smallest momenta which correspond to the ground state. Thus we will label these propagators by the smallest momenta $\mathbf{p}$ that they generate, with the appropriate antiperiodic boundary conditions in the spatial directions understood. We also refer to antiperiodic boundary conditions in a given spatial direction as a “twist” in that direction. Other than the wall source propagators which have 0 twists, we calculate propagators with 2 twists for

each of the three possible choices of directions. That is, we obtain propagators for the momenta

$$\mathbf{p} = \left(0, \pm \frac{\pi}{L}, \pm \frac{\pi}{L}\right), \left(\pm \frac{\pi}{L}, 0, \pm \frac{\pi}{L}\right), \left(\pm \frac{\pi}{L}, \pm \frac{\pi}{L}, 0\right). \quad (6.17)$$

The source for a propagator with momentum $\mathbf{p}$ is a “cosine source”, specifically

$$s_{\text{cosine}}(\mathbf{x}; \mathbf{p}) = \cos(p_x x) \cos(p_y y) \cos(p_z z). \quad (6.18)$$

Thus we will label propagators with momentum as

$$\begin{aligned} S^P(\mathbf{x}, t; \mathbf{p}, t') &= \sum_{\mathbf{y}} s_{\text{cosine}}(\mathbf{y}; \mathbf{p}) S^P(\mathbf{x}, t; \mathbf{y}, t') \\ &= \text{Quark propagator with momentum, cosine source at} \\ &\quad t', \text{ periodic BC in time direction} \end{aligned} \quad (6.19)$$

$$\begin{aligned} S^A(\mathbf{x}, t; \mathbf{p}, t') &= \sum_{\mathbf{y}} s_{\text{cosine}}(\mathbf{y}; \mathbf{p}) S^A(\mathbf{x}, t; \mathbf{y}, t') \\ &= \text{Quark propagator with momentum, cosine source at} \\ &\quad t', \text{ antiperiodic BC in time direction.} \end{aligned} \quad (6.20)$$

These are understood to have antiperiodic boundary conditions in the spatial directions appropriate for $\mathbf{p}$ and to have cosine source as in Eq. 6.18. Note that $\mathbf{p}$ and $-\mathbf{p}$ correspond to the same propagator, and that for $\mathbf{p} = 0$ the propagators become just the wall source propagators of Eq. 6.1, 6.2. Using cosine sources, rather than pure exponential sources, saves us from having

to generate two separate propagators with momentum $\mathbf{p}$ and $-\mathbf{p}$ in order to have two pions with equal and opposite momentum (see Section 6.1.3, 6.1.4). We also add or subtract the periodic and antiperiodic non-zero momentum propagators to obtain a propagator with double the effective time length, as in Eq. 6.3. This will be denoted similarly:

$$S(\mathbf{x}, t; \mathbf{p}, t') = \frac{1}{2} [S^P(\mathbf{x}, t; \mathbf{p}, t') + S^A(\mathbf{x}, t; \mathbf{p}, t')] . \quad (6.21)$$

For the light quark mass we generate all of the propagators that have been described. Light quark propagators will be denoted with a subscript l . For the strange quark mass we only generate zero momentum propagators, but in addition to propagators with source time t' we generate propagators with source times $t_K = t' + \Delta t_K$, where $\Delta t_K = 20, 24, 28, 32, 36, 40, 44$. The strange quark propagators will be denoted with a subscript s .

6.1.3 Two Point Functions

We define pseudoscalar meson correlators as

$$C_M^{wp}(\mathbf{p}, t, t') = \sum_{\mathbf{x}, \mathbf{x}', \mathbf{y}'} e^{-i\mathbf{p} \cdot \mathbf{x}} s_{\text{cosine}}(\mathbf{y}'; \mathbf{p}) \langle 0 | T \left(O_{M^+}(\mathbf{x}, \mathbf{x}, t) O_{M^+}^\dagger(\mathbf{x}', \mathbf{y}', t') \right) | 0 \rangle \quad (6.22)$$

$$C_M^{ww}(\mathbf{p}, t, t') = \sum_{\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}'} e^{-i\mathbf{p} \cdot \mathbf{y}} s_{\text{cosine}}(\mathbf{y}'; \mathbf{p}) \langle 0 | T \left(O_{M^+}(\mathbf{x}, \mathbf{y}, t) O_{M^+}^\dagger(\mathbf{x}', \mathbf{y}', t') \right) | 0 \rangle \quad (6.23)$$

where M is either π or K for the pion and kaon correlator respectively. The label “wp” stands for “wall-point” and “ww” stands for “wall-wall”. Both of these types of correlators are calculated because the wall-wall correlator will be needed later to properly normalize the $K \rightarrow \pi\pi$ matrix element, but the wall-point correlator is less noisy and is better to use for a calculation of m_π and m_K on their own. Notice also that the “wall” at the source time t' is actually a cosine source, and the “wall” or “point” at the sink time t is a pure exponential. This is done so that it is only necessary to use propagators with cosine sources (see Eq. 6.27, 6.28, 6.30 and 6.31), rather than having to calculate propagators with pure exponential sources and both the momenta $\mathbf{p}$ and $-\mathbf{p}$. Furthermore, since a propagator calculation yields a result for every $(\mathbf{x}, t)$ on the sink side, we can give the sink any spatial dependence we want. The choice of a pure exponential for this spatial dependence ensures that the correlator only couples to a meson of momentum $\mathbf{p}$ and not $-\mathbf{p}$. Of course, when $\mathbf{p} = 0$ the sources and sinks become ordinary wall sources, wall sinks and point sinks.

The interpolating operators are given by

$$O_{\pi^+}(\mathbf{x}, \mathbf{y}, t) = \bar{d}(\mathbf{x}, t) \gamma^5 u(\mathbf{y}, t) \quad (6.24)$$

$$O_{K^+}(\mathbf{x}, \mathbf{y}, t) = \bar{s}(\mathbf{x}, t) \gamma^5 u(\mathbf{y}, t). \quad (6.25)$$

In light of Eq. 6.22 and 6.23, this definition of the interpolating operators implies that the u quark should be the one obeying the twisted boundary con-

ditions when there is non-zero momentum. This is because the propagators with twisted boundary conditions must be the ones with cosine sources.

We also define the two pion correlator as

$$\begin{aligned}
C_{\pi\pi}(\mathbf{p}, t, t') = & \sum_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}} \sum_{\mathbf{w}', \mathbf{x}', \mathbf{y}', \mathbf{z}'} e^{i\mathbf{p} \cdot \mathbf{w}} e^{-i\mathbf{p} \cdot \mathbf{y}} s_{\text{cosine}}(\mathbf{w}'; \mathbf{p}) s_{\text{cosine}}(\mathbf{y}'; \mathbf{p}) \\
& \times \langle 0 | T \left(O_{\pi^+}(\mathbf{w}, \mathbf{x}, t) O_{\pi^+}(\mathbf{y}, \mathbf{z}, t) O_{\pi^+}^\dagger(\mathbf{w}', \mathbf{x}', t') O_{\pi^+}^\dagger(\mathbf{y}', \mathbf{z}', t') \right) | 0 \rangle.
\end{aligned} \tag{6.26}$$

Here the d quarks are the ones with the twisted boundary conditions.³ This causes no conflict because the two pion correlator will not be involved in any of the same calculations as the non-zero momentum meson correlators. Again, we use cosine sources and pure exponential sinks so that it is only necessary to use propagators with cosine sources. One of the exponential sinks has momentum $\mathbf{p}$ and the other has momentum $-\mathbf{p}$, enforcing zero total momentum of the two pion system.

When Eq. 6.22, 6.23 are Wick contracted, we find that the meson corre-

³One may worry that these boundary conditions break isospin symmetry. However, I_z is still a good symmetry, and since we are simulating a $\pi^+\pi^+$ state which has $m_I = 2$ there is no worry of it mixing with other two pion states. This is essentially the idea of “H-parity” discussed in [11].

lators can be written in terms of propagators as follows:

$$C_K^{wp}(\mathbf{p}, t, t') = \sum_{\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} \text{Tr} [S_l(\mathbf{x}, t; \mathbf{p}, t') S_s^\dagger(\mathbf{x}, t; \text{wall}, t')] \quad (6.27)$$

$$C_K^{ww}(\mathbf{p}, t, t') = \sum_{\mathbf{x}, \mathbf{y}} e^{-i\mathbf{p}\cdot\mathbf{x}} \text{Tr} [S_l(\mathbf{x}, t; \mathbf{p}, t') S_s^\dagger(\mathbf{y}, t; \text{wall}, t')] \quad (6.28)$$

(recall that the subscript l indicates a light quark propagator and the subscript s indicates a strange quark propagator). The pion correlators are the same except all s subscripts are changed to l subscripts because the pion contains two light quarks rather than one light and one strange quark.

When Eq. 6.26 is Wick contracted, we find

$$C_{\pi\pi}(\mathbf{p}, t, t_\pi) = 2C_{\pi\pi}^{\text{TRTR}}(\mathbf{p}, t, t_\pi) - 2C_{\pi\pi}^{\text{TR}}(\mathbf{p}, t, t_\pi) \quad (6.29)$$

where

$$C_{\pi\pi}^{\text{TR}}(\mathbf{p}, t, t') = \sum_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}} e^{i\mathbf{p}\cdot\mathbf{w}} e^{-i\mathbf{p}\cdot\mathbf{y}} \text{Tr} \left[S_l^\dagger(\mathbf{w}, t; \mathbf{p}, t') S_l(\mathbf{x}, t; \text{wall}, t') \right. \\ \left. \times S_l^\dagger(\mathbf{y}, t; \mathbf{p}, t') S_l(\mathbf{z}, t; \text{wall}, t') \right] \quad (6.30)$$

$$C_{\pi\pi}^{\text{TRTR}}(\mathbf{p}, t, t') = \sum_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}} e^{i\mathbf{p}\cdot\mathbf{w}} e^{-i\mathbf{p}\cdot\mathbf{y}} \text{Tr} \left[S_l^\dagger(\mathbf{w}, t; \mathbf{p}, t') S_l(\mathbf{x}, t; \text{wall}, t') \right] \\ \times \text{Tr} \left[S_l^\dagger(\mathbf{y}, t; \mathbf{p}, t') S_l(\mathbf{z}, t; \text{wall}, t') \right]. \quad (6.31)$$

The two-point functions were evaluated with $\mathbf{p} = 0$ and with the non-zero momenta of Eq. 6.17. Since the propagators used to calculate the correlators have period $2N_t$ in the time direction (Eq. 6.6), all of the unique information of the correlators is contained in a time interval of size $2N_t$, which we will take to be $t' \leq t < t' + 2N_t$ so that it begins at the source at t' and ends before the image of the source at $t' + 2N_t$. Furthermore, due to reflection symmetry in the time direction, we expect the two-point functions to have the same value at $t = t' + \Delta t$ and $t = t' + 2N_t - \Delta t$ once we have averaged over gauge configurations. Thus we average together results from these two time slices:

$$C^{\text{ave}}(t' + \Delta t, t') = \frac{1}{2} [C(t' + \Delta t, t') + C(t' + 2N_t - \Delta t, t')] \quad (6.32)$$

for the correlators of Eq. 6.22, 6.23, 6.26. We refer to this as *folding* the correlator. The resulting correlator C^{ave} has unique information in the range $0 \leq \Delta t < N_t$, but, due to the averaging of the two different time slices, the statistics can be thought of as effectively double those of a correlator made from plain periodic propagators with period N_t . (Some correlation may exist between the two time slices since the data is derived from the same gauge configurations. Thus the effect may not be quite as good as doubling the statistics, but the statistical error bars are certainly reduced). For correlators with non-zero momentum we also average over all of the momenta in Eq. 6.17 as we expect all of these to be the same by cubic

symmetry, further improving the statistics. Henceforth, we will work only with these averaged correlators and the correlator will be examined only in the range $t' \leq t < t' + N_t$. Furthermore, it will be more convenient to translate the source time to $t = 0$. That is, t will now stand for $\Delta t = t - t'$, which is the distance from the source, and the source time will be $t' = 0$.

For the kaon and pion correlators of Eq. 6.22 - 6.23 we expect the following asymptotic form

$$C(\mathbf{p}, t, 0) = A(e^{-E(p)t} + e^{-E(p)(2N_t-t)}), \quad 0 \ll t \ll 2N_t \quad (6.33)$$

where the second term arises from the meson propagating “around-the-world”, i.e. from the mirror source at $t = 2N_t$.

In Eq. 6.33, $E(p)$ is the energy of the pion or kaon with the given momentum. We expect

$$E(p) = \sqrt{m^2 + p^2} \quad (6.34)$$

where m is the mass of the pion or kaon. For $\mathbf{p} = 0$, $E(p)$ is just m , so we can actually check the dispersion relation Eq. 6.34 by comparing $\sqrt{m^2 + p^2}$ using m extracted from the 0 momentum correlator with $E(p)$ extracted directly from the non-zero momentum correlator.

In light of Eq. 6.33, we plot the effective mass for the pion and kaon correlators, defined as

$$m_{\text{eff}} = \ln \left[f_{N_t-t}^{-1} \left(\frac{C(t)}{C(t-1)} \right) \right] \quad (6.35)$$

where the function f is given by

$$f_n(x) = \frac{x^{2n+1} + x}{x^{2n+2} + 1} \quad (6.36)$$

and is such that if the correlator $C(t)$ obeys Eq. 6.33 exactly then $m_{\text{eff}} = E(p)$. Effective mass plots for the pion and kaon correlators with 0 and non-zero momentum are shown in Figures 5 - 12.

Fitted values⁴ for A and E from Eq. 6.33 are tabulated in Table 2 and the fitted E values are also indicated on Figures 5 - 12. The fitting ranges that were used are shown in Table 3. The results are used to check the dispersion relation Eq. 6.34, and this check is shown in Table 4. We see no evidence of a violation of the dispersion relation.

For the two pion correlator of Eq. 6.26 we expect the following asymptotic form

$$C(\mathbf{p}, t, t') = A(e^{-E(p)t} + e^{-E(p)(2N_t - t)}) + C, \quad 0 \ll t \ll 2N_t. \quad (6.37)$$

The extra constant C , as compared to Eq. 6.33, must be present to take into account the possibility of one pion propagating around-the-world (i.e. from $t = 2N_t$) and the other pion propagating from $t = 0$. The quantity $E(p)$ is the energy of the two pion state which will *not* be equal to $2\sqrt{m_\pi^2 + p^2}$

⁴Note that all fits performed in this report use an uncorrelated χ^2 . This is because in most cases the number of data points to be fit will be comparable to the number of configurations, leading to a nearly singular covariance matrix.

due to the interaction between the two pions. In fact, the deviation of $E(p)$ from $2\sqrt{m_\pi^2 + p^2}$ is what allows us to measure the two pion phase shift as discussed in Chapter 5.

Effective mass plots for the two pion correlator with zero and non-zero momentum are shown in Figures 13 - 14. Fitted values for A , E , and C from Eq. 6.37 are given in Table 5 and the fitted values of E are also indicated in Figures 13 - 14. The fitting ranges that were used are shown in Table 6.

The two pion correlator with momentum $p = \sqrt{2}\pi/L$ requires a slightly more careful analysis. First, if it is fit to Eq. 6.37 in the fitting range specified in Table 6, then the constant C comes out to be $8.5(58.4) \times 10^{14}$ where the error is much larger than the central value. Furthermore, the central value of C is larger than the correlator at times $t > 27$, which is not possible if the correlator is of the functional form Eq. 6.37. We conclude that we are not truly seeing the constant term in Eq. 6.37 because the fitting range does not extend far enough to the right (where the noise rapidly becomes very large), and thus we fit to Eq. 6.33 instead.

Second, the point in the effective mass plot (Figure 14) at $t = 17$ is significantly lower than the points surrounding it, (although it is still within the error bar). This causes the fit coefficients of Eq. 6.33 to depend significantly on the fitting range chosen, as shown in Figure 15 which is a plot of the fitted two pion energy as a function of $t_{\max}$ of the fitting range with $t_{\min} = 5$. A significant difference can be seen if we use $t_{\max} = 16$ versus $t_{\max} = 30$ for example.

Motivated in part by the concerns of the previous paragraph, we examine the following ratio of correlators

$$r(t) = \frac{C_{\pi\pi}(\mathbf{p}, t, 0)}{[C_{\pi}^{ww}(\mathbf{p}, t, 0)]^2} \quad (6.38)$$

in the hope that some fluctuations will cancel between the numerator and denominator because they are correlated to each other. The asymptotic form of this ratio is

$$r(t) = Ae^{-\Delta E(p)t} \quad 0 \ll t \ll N_t \quad (6.39)$$

where for simplicity around-the-world terms have been neglected, and we take $t \ll N_t$ for consistency. In Eq. 6.39 the quantity $\Delta E(p)$ is related to the two pion energy and the energy of a single pion with momentum p by

$$\Delta E(p) = E_{\pi\pi}(p) - 2E_{\pi}(p). \quad (6.40)$$

This follows from inserting Eq. 6.33, 6.37 (keeping only the first term in each of these because the rest are around-the-world terms) into Eq. 6.38. The strategy is to fit the ratio to an exponential to obtain $\Delta E(p)$ and then add $2E_{\pi}(p)$, or $2\sqrt{m_{\pi}^2 + p^2}$, to obtain a value for $E_{\pi\pi}(p)$ that will hopefully have a smaller error than the value obtained directly from the two pion correlator.

Effective mass plots of the ratio of Eq. 6.38 for zero and non-zero momentum are shown in Figures 16-17. A simpler definition of the effective mass is

used which does not take into account around-the-world effects:

$$m_{\text{eff}} = -\ln \left[\frac{C(t)}{C(t-1)} \right]. \quad (6.41)$$

For comparison with Figure 15, the fit range dependence is shown in the same way for $\Delta E(p)$ with momentum $p = \sqrt{2}\pi/L$ in Figure 18. It can be seen that there is no significant fit range dependence for $\Delta E(p)$. Fitted values for A , ΔE from Eq. 6.39 are given in Table 7 and the fitted values of ΔE are indicated also indicated in Figures 16-17. The fitting ranges that were used are shown in Table 8.

Finally, we must add $2E_\pi(p)$ or $2\sqrt{m_\pi^2 + p^2}$ to $\Delta E(p)$ to obtain $E_{\pi\pi}(p)$. The choice of $2E_\pi(p)$ or $2\sqrt{m_\pi^2 + p^2}$ should not matter since these quantities, as computed from the wall-wall correlators, agree within error according to Table 4. The results of using both to find $E_{\pi\pi}(p)$ are shown in Table 9. The two methods agree with each other within error. The results of Table 9 can also be compared to the results for $E_{\pi\pi}(p)$ found by directly fitting the two pion correlators which are shown in Table 5. We see agreement for $p = 0$, but not for $p = \sqrt{2}\pi/L$, which is not surprising given the fit range dependence of $E_{\pi\pi}$ that was observed with the direct fit of the two pion correlator. We also see that the results obtained from the ratio have smaller errors as expected. Thus we choose to use the results of Table 9 for $E_{\pi\pi}$ for the rest of the analysis, rather than the results of the direct fit to the two pion correlator shown in Table 5. Furthermore, for $p = \sqrt{2}\pi/L$ we will

use $E_{\pi\pi} = \Delta E(p) + 2E_\pi(p)$ rather than $E_{\pi\pi} = \Delta E(p) + 2\sqrt{m_\pi^2 + p^2}$. This choice is made because the non-zero momentum pion correlator is used to find $\Delta E(p)$ in the first place, so it is more consistent to add back $2E_\pi(p)$ which came directly from the fit to the non-zero momentum pion correlator without making use of the dispersion relation.

6.1.4 Three Point Functions

We define the $K \rightarrow \pi\pi$ correlator⁵ (three point function) as

$$C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A, \Gamma_B) = \sum_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}} \sum_{\mathbf{x}', \mathbf{y}'} s_{\text{cosine}}(\mathbf{w}; \mathbf{p}) s_{\text{cosine}}(\mathbf{y}; \mathbf{p}) \\ \times \langle 0 | T \left(O_{\pi^+}(\mathbf{w}, \mathbf{x}, t_\pi) O_{\pi^+}(\mathbf{y}, \mathbf{z}, t_\pi) O_W(t; \Gamma_A, \Gamma_B) O_{K^+}^\dagger(\mathbf{x}', \mathbf{y}', t_K) \right) | 0 \rangle \quad (6.42)$$

where the interpolating operators are given in Eq. 6.24, 6.25, and the weak operator is given generically by

$$O_W^{\text{unmixed}}(t; \Gamma_A, \Gamma_B) = \sum_{\mathbf{x}} \bar{s}_\alpha(\mathbf{x}, t) \Gamma_{A\mu} d_\alpha(\mathbf{x}, t) \bar{u}_\beta(\mathbf{x}, t) \Gamma_B^\mu d_\beta(\mathbf{x}, t) \quad (6.43)$$

⁵The relationship between the unphysical decay $K^+ \rightarrow \pi^+ \pi^+$ being represented in Eq. 6.42 and the physical decay amplitude A_2 is discussed in Appendix A. The unphysical four quark operators of Eq. 6.43 and 6.44 that allow this decay to proceed are also related to the physical four quark operators of Eq. 4.29 - 4.31, and this is also discussed in Appendix A.

or

$$O_W^{\text{mixed}}(t; \Gamma_A, \Gamma_B) = \sum_{\mathbf{x}} \bar{s}_\alpha(\mathbf{x}, t) \Gamma_{A\mu} d_\beta(\mathbf{x}, t) \bar{u}_\beta(\mathbf{x}, t) \Gamma_B^\mu d_\alpha(\mathbf{x}, t) \quad (6.44)$$

where in Eq. 6.44 α, β are color indices and the spin indices are contracted left to right as always. A specific weak operator will have either the unmixed or mixed color/spin structure (Eq. 6.43 or Eq. 6.44), and will have a specific gamma matrix structure substituted for Γ_A and Γ_B . This is specified in Table 10 for the three operators considered here: $O_{(27,1)}^{\text{unmixed}}$, $O_{(8,8)}^{\text{unmixed}}$, and $O_{(8,8)}^{\text{mixed}}$. Note that $O_{(27,1)}^{\text{mixed}}$ does not appear in this list because it is just equal to $O_{(27,1)}^{\text{unmixed}}$ due to the Fierz identity

$$\bar{q}_1 \gamma^\mu (1 - \gamma^5) q_2 \bar{q}_3 \gamma_\mu (1 - \gamma^5) q_4 = \bar{q}_1 \gamma^\mu (1 - \gamma^5) q_4 \bar{q}_3 \gamma_\mu (1 - \gamma^5) q_2. \quad (6.45)$$

Also notice that in Eq. 6.42 the times t_π and t_K are the time slices of the two pion sink and kaon source respectively. These remain fixed while we examine the dependence of the correlator on t , the time of the weak operator. The situation is illustrated in Figure 19.

The cosine sinks for the two pions become sources of propagators when Eq. 6.42 is Wick contracted (see Eq. 6.47, 6.48, 6.49 and 6.50), so that it is only necessary to use cosine source propagators in the calculation. The wall source for the kaon and conservation of total momentum in the decay enforce zero total momentum in the two pion system. Also, the d quarks, as opposed to the u quarks, have twisted boundary conditions for this correlator as in

the two pion correlator of Eq. 6.26.

When Eq. 6.42 is Wick contracted, we find that it can be written in terms of propagators as follows:

$$\begin{aligned}
C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A, \Gamma_B) &= 2C_{K \rightarrow \pi\pi}^{TRTR}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A, \Gamma_B) \\
&\quad - 2C_{K \rightarrow \pi\pi}^{TR}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A, \Gamma_B)
\end{aligned}
\tag{6.46}$$

where, for a weak operator with the “unmixed” color/spin structure

$$\begin{aligned}
C_{K \rightarrow \pi\pi}^{TR}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A^\mu, \Gamma_B^\mu) &= \sum_{\mathbf{x}, \mathbf{y}} \text{Tr} \left[S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{x}, t; \text{wall}, t_\pi) \gamma^5 \Gamma_B^\mu \right. \\
&\quad \left. \times S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{y}, t_K; \text{wall}, t_\pi) \gamma^5 S_s^\dagger(\mathbf{x}, t; \text{wall}, t_K) \gamma^5 \Gamma_{A\mu} \right]
\end{aligned}
\tag{6.47}$$

$$\begin{aligned}
C_{K \rightarrow \pi\pi}^{TRTR}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A^\mu, \Gamma_B^\mu) &= \sum_{\mathbf{x}, \mathbf{y}} \text{Tr} \left[S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{x}, t; \text{wall}, t_\pi) \gamma^5 \Gamma_B^\mu \right] \\
&\quad \times \text{Tr} \left[S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{y}, t_K; \text{wall}, t_\pi) \gamma^5 S_s^\dagger(\mathbf{x}, t; \text{wall}, t_K) \gamma^5 \Gamma_{A\mu} \right]
\end{aligned}
\tag{6.48}$$

and for a weak operator with the “mixed” color/spin structure

$$\begin{aligned}
C_{K \rightarrow \pi\pi}^{TR}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A^\mu, \Gamma_B^\mu) &= \sum_{\mathbf{x}, \mathbf{y}} \left[S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{x}, t; \text{wall}, t_\pi) \gamma^5 \Gamma_B^\mu \right]_{\alpha i, \alpha j} \\
&\times \left[S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{y}, t_K; \text{wall}, t_\pi) \gamma^5 S_s^\dagger(\mathbf{x}, t; \text{wall}, t_K) \gamma^5 \Gamma_{A\mu} \right]_{\beta j, \beta i}
\end{aligned} \tag{6.49}$$

$$\begin{aligned}
C_{K \rightarrow \pi\pi}^{TRTR}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A^\mu, \Gamma_B^\mu) &= \sum_{\mathbf{x}, \mathbf{y}} \left[S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{x}, t; \text{wall}, t_\pi) \gamma^5 \Gamma_B^\mu \right]_{\alpha i, \beta i} \\
&\times \text{Tr} \left[S_l(\mathbf{x}, t; \mathbf{p}, t_\pi) S_l^\dagger(\mathbf{y}, t_K; \text{wall}, t_\pi) \gamma^5 S_s^\dagger(\mathbf{x}, t; \text{wall}, t_K) \gamma^5 \Gamma_{A\mu} \right]_{\beta j, \alpha j}
\end{aligned} \tag{6.50}$$

where α and β are color indices and i and j are spin indices for the matrices in square brackets. The quark flow diagram for these $\Delta I = 3/2$ $K \rightarrow \pi\pi$ contractions is shown in Figure 19.

Since O_W in Eq. 6.42 is always linear in Γ_A and Γ_B (see Eq. 6.43, 6.44), we can expand the $K \rightarrow \pi\pi$ correlator of Eq. 6.42 into four different terms when the particular gamma matrix structures of Table 10 are inserted. Explicitly,

showing only the dependence of the $K \rightarrow \pi\pi$ correlator on Γ_A and Γ_B ,

$$\begin{aligned} C_{K \rightarrow \pi\pi}(\gamma^\mu(1 - \gamma^5), \gamma^\mu(1 - \gamma^5)) &= C_{K \rightarrow \pi\pi}(\gamma^\mu, \gamma^\mu) - C_{K \rightarrow \pi\pi}(\gamma^\mu\gamma^5, \gamma^\mu) \\ &\quad - C_{K \rightarrow \pi\pi}(\gamma^\mu, \gamma^\mu\gamma^5) + C_{K \rightarrow \pi\pi}(\gamma^\mu\gamma^5, \gamma^\mu\gamma^5) \end{aligned} \quad (6.51)$$

$$\begin{aligned} C_{K \rightarrow \pi\pi}(\gamma^\mu(1 - \gamma^5), \gamma^\mu(1 + \gamma^5)) &= C_{K \rightarrow \pi\pi}(\gamma^\mu, \gamma^\mu) - C_{K \rightarrow \pi\pi}(\gamma^\mu\gamma^5, \gamma^\mu) \\ &\quad + C_{K \rightarrow \pi\pi}(\gamma^\mu, \gamma^\mu\gamma^5) - C_{K \rightarrow \pi\pi}(\gamma^\mu\gamma^5, \gamma^\mu\gamma^5). \end{aligned} \quad (6.52)$$

The first and last term on the right hand side of Eq. 6.51 and 6.52 are expected to vanish theoretically due to parity symmetry, thus we only compute the second and third terms:

$$C_{K \rightarrow \pi\pi}(\gamma^\mu(1 - \gamma^5), \gamma^\mu(1 - \gamma^5)) \rightarrow -C_{K \rightarrow \pi\pi}(\gamma^\mu\gamma^5, \gamma^\mu) - C_{K \rightarrow \pi\pi}(\gamma^\mu, \gamma^\mu\gamma^5) \quad (6.53)$$

$$C_{K \rightarrow \pi\pi}(\gamma^\mu(1 - \gamma^5), \gamma^\mu(1 + \gamma^5)) \rightarrow -C_{K \rightarrow \pi\pi}(\gamma^\mu\gamma^5, \gamma^\mu) + C_{K \rightarrow \pi\pi}(\gamma^\mu, \gamma^\mu\gamma^5) \quad (6.54)$$

in the actual calculation.

We compute the $K \rightarrow \pi\pi$ correlator of Eq. 6.42 with the pion sink at the times $t_\pi = t' = 0, 16, 32, 48, 0, \dots$ as described in Section 6.1.2, and with the kaon source at the times $t_K = t_\pi + \Delta t_K$ with $\Delta t_K = 20, 24, 28, 32, 36, 40, 44$, also described in Section 6.1.2. In Eq. 6.47 - 6.50 all light quark propagators have source time t_π and all strange quark propagators have source time t_K ,

so the choices for t_π and t_K we have just made are consistent with the source times of the different propagators that were generated according to Section 6.1.2. The correlator is examined as a function of t , the time location of the weak operator, in the range $t_\pi \leq t \leq t_K$, i.e. with the weak operator in between the kaon source and pion sink. Refer again to Figure 19 for a diagram of the situation.

The quantity $\Delta t_K = t_K - t_\pi$ is the time separation between the kaon source and pion sink, and it takes on several different values in this calculation. This is done so that we can determine the optimal separation that gives us the best signal for the correlator. If Δt_K is too small then t will never be far enough away from the kaon source and pion sink to avoid excited state contamination, but if Δt_K is too large then t can never be close enough to both the kaon source and the pion sink so that the signals coming from both of them have not decayed and become buried in the noise. This phenomenon is illustrated in Figure 20.

We use the *folding* method of Eq. 6.32, but some modification is necessary because there are two different source times t_π and t_K . There are images of these two sources at $t_\pi + 2N_t$ and $t_K + 2N_t$. Therefore the correlator is not expected to be identical at $t_\pi + \Delta t$ and $t_\pi + 2N_t - \Delta t$ because the latter time is not situated between a pion and kaon source with time separation $\Delta t_K = t_K - t_\pi$ between them.

However, it is still possible to construct an independent set of propagators with the desired source separation Δt_K between them. In place of

propagators $S(t; t_\pi)$ with sources at t_π , we use

$$S'(t; t_\pi) = S(t + N_t; t_\pi). \quad (6.55)$$

By applying the Dirac operator to Eq. 6.55, as is done in Eq. 6.14, we see that S' actually has sources at $t_\pi + N_t + 2N_t n$. (S' can be evaluated in terms of already available data as shown in Eq. 6.8). We will be focusing on the source at $t_\pi + N_t$ in particular, rather than any of the other images. In place of the propagators $S(t; t_K)$ with sources at $t_K = t_\pi + \Delta t_K$, we use

$$S'(t; t_K) = S(t; t_\pi + N_t - \Delta t_K) \quad (6.56)$$

i.e. a propagator with a source at $t_\pi + N_t - \Delta t_K$, which is possible because for each propagator generated with source $t_\pi + \Delta t_K$ there is a propagator generated with source $t_\pi + N_t - \Delta t_K$ (e.g. $\Delta t_K = 20$ and $\Delta t_K = 44$ are both in the list of Δt_K values).

Let

$$C'_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, t_\pi; \Gamma_A, \Gamma_B) \quad (6.57)$$

be the $K \rightarrow \pi\pi$ correlator with $S'(t; t_\pi)$ and $S'(t; t_K)$ used in place of $S(t; t_\pi)$ and $S(t; t_K)$ respectively. This correlator has a kaon source at $t_\pi + N_t - \Delta t_K$ and a pion sink at $t_\pi + N_t$. It is clear that the correlator of Eq. 6.42 evaluated at $t = t_\pi + \Delta t$, and the correlator of Eq. 6.57 evaluated at $t = t_\pi + N_t - \Delta t$, should be equal due to reflection symmetry, and thus we can define the folded

correlator as

$$C_{K \rightarrow \pi\pi}^{\text{ave}}(t_K, t_\pi + \Delta t, t_\pi) = \frac{1}{2} [C_{K \rightarrow \pi\pi}(t_K, t_\pi + \Delta t, t_\pi) + C'_{K \rightarrow \pi\pi}(t_K, t_\pi + N_t - \Delta t, t_\pi)] \quad (6.58)$$

(where the dependence on $\mathbf{p}$, Γ_A , and Γ_B are not shown). Figure 21 shows a diagram of the situation. It is also clear that C and C' are to some degree statistically independent in the sense that they will not be exactly equal on a given gauge configuration, and thus with the folded correlator of Eq. 6.58 we may have, in effect, as much as doubled the statistics in the same way as with the folded correlator of Eq. 6.32. Eq. 6.58 should be examined in the range $0 \leq \Delta t \leq \Delta t_K$, where $\Delta t = t - t_\pi$, i.e. it should be examined at time slices between the kaon source and pion sink. Just as with Eq. 6.32, we will from now on refer to Δt as simply t , in effect translating the time of the pion sink back to $t = 0$ and the time of the kaon source back to $t = t_K$. And again for the non-zero momentum correlators we will average over all momenta in Eq. 6.17 to further improve statistics.

For the $K \rightarrow \pi\pi$ correlator of Eq. 6.42 we expect the following asymptotic form

$$C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, t_\pi) = A e^{-E_{\pi\pi}(p)|t-t_\pi|} e^{-m_K|t-t_K|} \quad |t-t_\pi|, |t-t_K| \gg 0 \quad (6.59)$$

where $E_{\pi\pi}(p)$ is the energy of the two pion state and m_K is mass of the kaon.

This does not include around the world terms. They generally will not be significant because t is in the range $t_\pi \leq t \leq t_K$, which is usually significantly smaller than $2N_t$. With $t_\pi = 0$ and $t < t_K$ the expression becomes

$$C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, 0) = A' e^{-\Delta E_{K \rightarrow \pi\pi}(p)t} \quad (6.60)$$

where $A' = A e^{-m_K t_K}$ and $\Delta E_{K \rightarrow \pi\pi}(p) = E_{\pi\pi}(p) - m_K$. Note that for an on-shell (energy conserving) decay, $\Delta E_{K \rightarrow \pi\pi} = 0$.

In Figures 22 - 34 we show some effective mass plots of the $K \rightarrow \pi\pi$ correlators for the various weak operators, separations between the kaon and pion, and momenta. The value of the effective mass should be $\Delta E_{K \rightarrow \pi\pi} = E_{\pi\pi} - m_K$, and since $E_{\pi\pi}$ and m_K have already been calculated from the two point functions, the plateaus of the effective mass plots yield no new information. Thus instead of fitting the plateaus we indicate the value of $\Delta E_{K \rightarrow \pi\pi}$ as already determined by the two point functions to show that it is consistent with the plateaus. This serves as a check that the $K \rightarrow \pi\pi$ correlators are indeed coupling to the correct kaon and two pion states. New information, in particular the $K \rightarrow \pi\pi$ matrix element, is obtained from the coefficient A' in Eq. 6.60, but instead of fitting the correlator directly we fit a quotient to be described shortly.

Figure 28 - 34 show the effective mass plot of the $K \rightarrow \pi\pi$ correlator for the (27,1) operator and two twists for all of the different separations $\Delta t_K = 20, 24, 28, 32, 36, 40, 44$ between the kaon source and pion sink. It

can be seen that the amount of noise increases significantly as the separation increases past 32 indicating that these separations are too large (the case of the bottom diagram in Figure 20).

In order to obtain the $K \rightarrow \pi\pi$ matrix element, (which will be computed with the correct normalization in Chapter 7 based on the results of this section), rather than fit the $K \rightarrow \pi\pi$ correlator directly we follow a similar strategy as with the two pion correlator and examine the following quotient:

$$q(t) = \frac{C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, 0)}{C_K(\mathbf{p} = 0, t_K - t, 0)C_{\pi\pi}(\mathbf{p}, t, 0)}. \quad (6.61)$$

According to Eq. 6.33, 6.37, and 6.59, the different correlators in the quotient should have the following asymptotic forms when around-the-world terms are neglected:

$$C_K(\mathbf{p} = 0, t_K - t, 0) = A_K e^{-m_K(t_K - t)} \quad t \ll t_K \quad (6.62)$$

$$C_{\pi\pi}(\mathbf{p}, t, 0) = A_{\pi\pi} e^{-E_{\pi\pi}(p)t} \quad t \gg 0 \quad (6.63)$$

$$C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, 0) = A_{K \rightarrow \pi\pi} e^{-E_{\pi\pi}(p)t} e^{-m_K(t_K - t)} \quad 0 \ll t \ll t_K. \quad (6.64)$$

Inserting these into Eq. 6.61 we see that the quotient has the asymptotic form

$$q(t) = A_q \quad 0 \ll t \ll t_K \quad (6.65)$$

where $A_q = A_{K \rightarrow \pi\pi} / (A_{\pi\pi} A_K)$ is a constant.

We show plots of the quotient of Eq. 6.61 for $t_K = 24$ in Figures 35 - 40.

Fitted values of A_q for all values of t_K are tabulated in Tables 11 and 12, and are also indicated in Figures 35 - 40. The fit range for all quotients is $t_{\min} = 5$, $t_{\max} = t_K - 5$.

6.2 Fine 32^3 Lattices

6.2.1 Gauge Ensembles

The ensembles were generated by the RBC/UKQCD collaboration using the Iwasaki Gauge action, with 2+1 dynamical flavors of Domain Wall Fermions (DWF). The lattice size is $32^3 \times 64$, with a fifth dimension of size $L_s = 16$ and domain wall height $M_5 = 1.8$. The inverse lattice spacing is $a^{-1} = 2.285(29)$ GeV (corresponding to a coupling $\beta = 2.25$), which means the total box size is $(2.76 \text{ fm})^3$ in physical units. The sea strange quark mass is set to $m_s^{\text{sea}} = 0.03$ and ensembles were generated with sea light quark masses of $m_l^{\text{sea}} = 0.004$, 0.006 , and 0.008 , corresponding to unitary pion masses of $m_\pi = 290 \text{ MeV}$, 345 MeV , and 395 MeV respectively. Measurements were performed on 214, 222, and 253 configurations for $m_l^{\text{sea}} = 0.004$, 0.006 , and 0.008 respectively. Measurements begin at no earlier than 260 MD time units for the three different ensembles, and are generally performed at an interval of 10 MD time units, although there are some gaps. The full list configurations that were measured can be found in Table 13. The residual mass is $m_{\text{res}} = 0.0006664(76)$. These ensembles are described in more detail in [69].

6.2.2 Propagators

From these lattices we generated propagators with valence quark masses $m^{\text{val}} = 0.002, 0.004, 0.006, 0.008, 0.025, 0.03$. The intention was for the first four masses to serve as light quark masses, and for the last two to serve as strange quark masses, however, calculations were performed for all possible combinations for which $m_s^{\text{val}} \geq m_l^{\text{val}}$. Notice that for the ensembles with $m_l^{\text{sea}} = 0.004, 0.006$, and 0.008 respectively only the combinations $(m_s^{\text{val}}, m_l^{\text{val}}) = (0.03, 0.004), (0.03, 0.006), (0.03, 0.008)$ correspond to unitary points, and the rest of the combinations are partially quenched.

Inversions were done with a stopping condition of 10^{-8} . As with the 32^3 coarse lattices, propagators were generated with sources on time slices $t' = 0, 16, 32, 48, 0, \dots$, etc., in order to improve the independence of measurements (see Section 6.1.2). The source and sink of the propagators are again Coulomb gauge fixed wall sources with stopping condition 10^{-8} .

We add and subtract propagators with periodic and antiperiodic boundary conditions in the time direction to double the effective time length as we did with the 32^3 coarse lattices (see Eq. 6.3 and the discussion following it). However, only zero momentum propagators were generated for the 32^3 fine lattices, i.e. all spatial boundary conditions are periodic. Also, we only generate propagators with sources at t' , not at the additional times $t' + \Delta t_K$. Making use of the method described by Eq. 6.55 and the discussion following it, we can obtain propagators with sources at $t' + N_t$ as well. In this way

it is possible to have a time separation of N_t between the kaon source and pion sink in the $K \rightarrow \pi\pi$ correlator, but no other separations will be possible. This turns out to be sufficient to obtain a good signal since only zero momentum calculations are done on these ensembles.

6.2.3 Two Point Functions

We calculate two point functions as in Section 6.1.3, but with zero momentum only. Also, we only look at wall-wall meson correlators since there was no significant reduction in the error on the meson masses when the wall-point correlators were fit in Section 6.1.3. Effective mass plots for the two point functions, with $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, for the three different sea light quark masses, are shown in Figures 41 - 49. The results of fitting correlators for all different mass combinations are given in Tables 14 - 18, and the fitted E values are also indicated on Figures 41 - 49. A fitting range of 7 - 63 was used for all two point functions.

6.2.4 Three Point Functions

We calculate three point functions as in Section 6.1.4, but with zero momentum only. As mentioned in Section 6.2.2, only propagators with sources at t' are generated, and not propagators with sources at $t' + \Delta t_K$ for several different kaon-two pion separations Δt_K . But it is at least possible to produce propagators with sources at $t' + N_t$ using the method of Eq. 6.55. This allows us to calculate the $K \rightarrow \pi\pi$ correlator of Eq. 6.42 with $t_\pi = t'$

and $t_K = t' + N_t$, and vice versa. Indeed, Eq. 6.47 - 6.50 show that such a correlator can be evaluated from light quark and strange quark propagators with sources at t' and $t' + N_t$.

The folding of Eq. 6.58 becomes, for the present situation

$$C_{K \rightarrow \pi\pi}^{\text{ave}}(t' + N_t, t' + \Delta t, t') = \frac{1}{2}[C_{K \rightarrow \pi\pi}(t' + N_t, t' + \Delta t, t') + C_{K \rightarrow \pi\pi}(t', t' + N_t - \Delta t, t' + N_t)] \quad (6.66)$$

where $\Delta t = t - t'$ is the distance between the position of the four quark operator (t) and the two pion sink (t'). The situation is illustrated in Figure 50. Again this is expected to, in effect, as much as double the statistics.

In Figures 51 - 59 we show effective mass plots of the $K \rightarrow \pi\pi$ correlator with $m_s^{\text{val}} = 0.03$ and $m_l^{\text{val}} = 0.002$, and for the three different sea light quark masses and three different weak operators. We follow the format of Section 6.1.4 and show on the plots the quantity $\Delta E_{K \rightarrow \pi\pi} = E_{\pi\pi} - m_K$ as determined independently by the fits to the two point functions. Just like in Section 6.1.4, this serves as a consistency check that the $K \rightarrow \pi\pi$ correlator is coupling to the right kaon and two pion states. Figures 51 - 59 show that the $K \rightarrow \pi\pi$ correlators for $m_s^{\text{val}} = 0.03$ and $m_l^{\text{val}} = 0.002$ pass this consistency test, and the consistency also holds for the other valence masses not shown.

In Figures 60 - 68, we show plots of the quotient of correlators of Eq. 6.61 with $m_s^{\text{val}} = 0.03$ and $m_l^{\text{val}} = 0.002$, and for the three different sea light quark masses and three different weak operators. The quotients are fit to a

constant A_q (c.f. Eq. 6.65), and the results are tabulated in Tables 19 - 21, and indicated in Figures 60 - 68 as well. The fit ranges used were 22 - 52 for the (27,1) operator (for all of the masses) and 18 - 58 for the (8,8) and (8,8) mixed operators.

6.3 Quenched 24^3 Lattices

6.3.1 Gauge Ensembles

The ensemble was generated using the Doubly Blocked Wilson (DBW2) action, and Domain Wall Fermions (DWF). The quenched approximation (see Section 3.5) was used in generating this ensemble. The lattice size is $24^3 \times 64$, with a fifth dimension of size $L_s = 16$ and domain wall height $M_5 = 1.8$. The inverse lattice spacing is $a^{-1} = 1.31(2)$ GeV (corresponding to a coupling $\beta = 0.87$), which means the total box size is $(3.62 \text{ fm})^3$ in physical units. Measurements were performed on 50 configurations, with measurements beginning at 32000 MD time units and at an interval of 500 MD time units, although there are some gaps. The full list of configurations that were measured can be found in Table 22. The residual mass was measured to be $m_{\text{res}} = 5.69(26) \times 10^{-4}$ on $16^3 \times 32$ lattices with all other parameters the same [70].

6.3.2 Propagators

Propagators were generated with light quark valence masses of $m_l^{\text{val}} = 0.0023, 0.0047, 0.0071$ and strange quark valence masses of $m_s^{\text{val}} = 0.046, 0.062, 0.078, 0.094, 0.11$. Inversions were done with a stopping condition of 10^{-8} . As with the 32^3 coarse lattices, propagators were generated with sources on time slices $t' = 0, 16, 32, 48, 0, \dots$, etc., in order to improve the independence of measurements (see Section 6.1.2). The source and sink of the propagators are again Coulomb gauge fixed wall sources with stopping condition 10^{-8} .

We add and subtract propagators with periodic and antiperiodic boundary conditions in the time direction to double the effective time length as we did with the 32^3 coarse lattices (see Section 6.1.2). Propagators with twists, i.e. antiperiodic boundary conditions, in various spatial directions were also generated to give momenta to the quarks, as was done for the 32^3 coarse lattices. However, on the present lattices we generate propagators with 0, 1, 2, and 3 twists, whereas only propagators with 0 and 2 twists were generated for the 32^3 coarse lattices. Finally, the strange quark propagators generated have the same source times t_K as those described in Section 6.1.2, so that for the present lattices we will again have the following separations between the kaon and two pions: $\Delta t_K = 20, 24, 28, 32, 36, 40$, and 44.

6.3.3 Two Point Functions

Effective mass plots for kaon and pion correlators with 0 and non-zero momentum are shown in Figures 69 - 76. We show graphs for $m_l=0.0023$ and $m_s=0.062$ (quark masses here refer to valence quark masses without ambiguity because the present lattices are quenched). We examine only wall-wall correlators since the wall-point correlators were not found to reduce the error significantly. The results of fitting the correlators are shown in Tables 23 - 25 for all of the different quark masses, and are indicated in Figures 69 - 76. For the kaons we only computed non-zero momentum correlators for the particular mass combination $m_l=0.0023$ and $m_s=0.062$. The fit ranges used for the fits in Tables 23 and 25 are tabulated in Tables 26 and 27; the fit range for all of the zero momentum kaon correlators in Table 24 is 5 - 63.

In Tables 28 and 29 the results are used to check the dispersion relation (Eq. 6.34) for the pions and for the kaon with $m_l=0.0023$ and $m_s=0.062$. For the kaon we see no evidence of any violation of the dispersion relation, but for the pions with $m_l=0.0023$ and 0.0047 the pion energy with $p = \sqrt{2}\pi/L$ is larger than the value predicted by the dispersion relation by 2.3σ and 1.7σ respectively.

Effective mass plots of the two pion correlators are shown in Figures 77 - 80 for light quark mass $m_l = 0.0023$ and for all of the different momenta. The results of fitting the two pion correlators are given in Table 30 for all light quark masses, and the fitting ranges used can be found in Table 31. For

the non-zero momenta a single exponential fit form

$$C_{\pi\pi}(\mathbf{p} \neq 0, t, t') = Ae^{-E_{\pi\pi}(p)t} \quad 0 \ll t \ll N_t \quad (6.67)$$

is used because there is too much noise at large times to be able to see any of the around-the-world terms. (See the discussion preceding Eq. 6.38 about omitting the constant term in fitting the two pion correlator with momentum $p = \sqrt{2}\pi/L$). For the zero momentum two pion correlator the usual fit form of Eq. 6.37 is used.

6.3.4 Three Point Functions

Effective mass plots for the $K \rightarrow \pi\pi$ correlators are shown in Figures 81 - 92. In particular we show plots for quark masses $m_l=0.0023$ and $m_s=0.062$, for a separation between the kaon and two pions of $\Delta t_K=24$, and for all three weak operators and all of the different momenta. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as obtained independently from the two point functions, is also shown on the plots as a consistency check, as was done with the other ensembles. Figures 93 - 96 show effective mass plots for the same quark masses, and just for the (27,1) operator and momentum $p = \pi/L$, but for different separations $\Delta t_K=20, 24, 28, 32$. This shows that the correlators become noisier with increasing separation. For this reason we do not examine any larger separations than $\Delta t_K = 32$.

We show plots of the quotient of correlators of Eq. 6.61 in Figures 97

- 108 for quark masses $m_l=0.0023$ and $m_s=0.062$, for a separation between the kaon and two pions of $\Delta t_K=24$, and for all three weak operators and all of the different momenta. The quotient is fit to a constant (A_q) and results of this fit are tabulated in Tables 32 - 43 for all masses, separations, weak operators and momenta. The fitted constant is also indicated in Figures 97 - 108. The fit range for all quotients is $t_{\min} = 5$ and $t_{\max} = t_K - 5$.

7 Results

7.1 Coarse 32^3 Lattices

7.1.1 Pion and Kaon Masses

In Section 6.1.3 we fit the pion and kaon correlators, and the results from the zero momentum correlators give the lattice values of the pion and kaon mass. These results are for the only quark masses used for these lattices: $m_l^{\text{sea}} = 0.001$, $m_s^{\text{sea}} = 0.045$, $m_l^{\text{val}} = 0.0001$, $m_s^{\text{val}} = 0.049$, and we use the values obtained from wall-wall correlators in the analysis from now on since values from the wall-point correlators did not have significantly smaller statistical error. We have $m_\pi = 0.10400(38)$ and $m_K = 0.3706(13)$ in lattice units, where these values have been taken from Table 2.

To convert to physical units we must multiply the meson masses by the inverse lattice spacing for these lattices in physical units. This was determined for the DSDR lattices to be $a^{-1} = 1.373(24)$ GeV from a scaling analysis [68] of the 32^3 coarse lattices, the 32^3 fine lattices for the three different sea light quark masses, and of 24^3 dynamical Iwasaki lattices with two different sea light quark masses not discussed in this report. We thus arrive at the meson

masses in physical units for the 32^3 coarse lattices: $m_\pi = 142.8(2.5)$ MeV and $m_K = 509.0(9.1)$ MeV. Note that statistical fluctuations from all of the different lattices in the scaling analysis are included in the error by using the superjackknife method [71, 72].

7.1.2 Two Pion Phase Shift

In Section 6.1.3 we also analyzed two pion correlators with $p = 0$ and $p = \sqrt{2}\pi/L$, and found the energy of the two pion state in lattice units. We can use these results to find the two pion energy in physical units (and check if the decay being simulated is energy conserving), and we can also use the Lüscher relation Eq. 5.59 to find the s wave phase shift δ_0 for two pion scattering (in the $I = 2$ isospin channel) at the energies simulated. As discussed in Section 6.1.3, we choose to use the results obtained from the ratio of correlators (Table 9) rather than the direct fit of the two pion correlators. We also said that for $p = \sqrt{2}\pi/L$ we will use $E_{\pi\pi} = \Delta E(p) + 2E_\pi(p)$ (the last column in the table), rather than $E_{\pi\pi} = \Delta E(p) + 2\sqrt{m_\pi^2 + p^2}$.

For obtaining the two pion energy in physical units we make use of the same scaling analysis described in Section 7.1.1 for determining the inverse lattice spacing. To obtain the phase shift we first find the value k defined by

$$E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}. \quad (7.1)$$

Note that $k \neq p$ due to the interaction between the two pions, where p is

the momentum that a single meson would have with the given number of twists and is equal to 0 and $\sqrt{2}\pi/L$ for zero and two twists respectively. From k we compute q , as defined in Eq. 5.19. Finally, we can use q in the Lüscher relation Eq. 5.59 to obtain the phase shift δ_0 . Note that q and δ_0 are dimensionless so that these results do not depend on the method used to determine the lattice spacing.

The results for $E_{\pi\pi}$, k , q , and δ_0 for zero and two twists are shown in Table 44. Comparing $E_{\pi\pi}$ with two twists from this table to the value of m_K quoted in Section 7.1.1 we see that m_K is slightly greater than $E_{\pi\pi}$ which means that the $K \rightarrow \pi\pi$ decay will not be exactly energy conserving. The effect this has on the $K \rightarrow \pi\pi$ calculation is discussed in Section 7.5. The results from Table 44 for the phase shift δ_0 can be put on a plot of δ_0 versus k and compared with experimental results [73, 74, 75] for the $I = 2$ s wave two pion phase shift. This is done in Figures 109 and 110, and we see good agreement with experiment.

For small values of k we expect that the phase shift is well approximated by the first term in a series expansion in powers of k :

$$\delta_0(k) \approx a_0 k \tag{7.2}$$

where a_0 is called the scattering length. In particular, we expect this to be true for zero twists. Indeed, in Figure 110 we see that for $k \lesssim 20$ MeV, where the zero twist data lies, the phenomenological fit to the experimental data

is linear and agrees quite well with the zero twist data point. (Note that the phenomenological fit curve at small k makes use of chiral perturbation theory fits to a larger class of experimental data not shown in Figure 110 that determine the scattering length; see [73]). Thus we use the zero twist data for k and δ_0 to find the scattering length via Eq. 7.2, i.e.

$$a_0 \approx \frac{\delta_0(k)}{k}. \quad (7.3)$$

We find $a_0 = -0.000319(26) \text{ MeV}^{-1}$, and for the dimensionless quantity $m_\pi a_0$ we find $m_\pi a_0 = -0.0456(37)$. This compares well with the chiral perturbation theory/phenomenological value $m_\pi a_0 = -0.0444(10)$ [73]. We also checked the validity of using Eq. 7.3 by checking that it was consistent with a formula derived by Lüscher relating the two pion energy for zero twists and the scattering length [76].

7.1.3 Wilson Coefficients

We use Wilson coefficients calculated in [15], which were calculated at the scales $\mu = 1.92 \text{ GeV}$ and $\mu = 2.15 \text{ GeV}$. We perform a linear interpolation to $\mu = 2.0 \text{ GeV}$ to match the scale of the NPR (see Section 7.1.4). The interpolated Wilson coefficients $y_i(\mu)$ and $z_i(\mu)$ needed for the present calculation are given in Table 45 in the RI-MOM scheme [33]. We also show the combination $C_i(\mu)$ defined in Eq. 4.36. The value of τ used is found in Table 48. We tabulate the values of the Wilson coefficients for the (27,1), (8,8),

and (8,8) mixed operators defined in Eq. 4.33-4.35 in Table 46. These last coefficients are the Wilson coefficients that appear in Eq. A.50 which we will use to calculate $\text{Re}(A_2)$ and $\text{Im}(A_2)$.

7.1.4 Non-Perturbative Renormalization

NPR was performed for the (27,1) operator on 32^3 lattices with the same parameters as the coarse lattices except with $L_s = 16$ [77]. The momentum scale was $\mu = 2$ GeV and we take the result that was calculated using the RI-SMOM($\not{q}, \not{q}$) scheme, but convert perturbatively to the RI-MOM scheme to match the Wilson coefficients. We quote the renormalization constant $Z_{(27,1),(27,1)}$, as well as $Z_{B_K} = Z_{(27,1),(27,1)}/Z_A^2$ (the renormalization constant for B_K) and Z_A [68] (the renormalization constant of the local axial current).

$$Z_{(27,1),(27,1)}^{\text{RI-MOM}}(\mu = 2 \text{ GeV}) = 0.44248(90)_{\text{stat}} \quad (7.4)$$

$$Z_{B_K}^{\text{RI-MOM}}(\mu = 2 \text{ GeV}) = 0.934245(69)_{\text{stat}} \quad (7.5)$$

$$Z_A = 0.6882(7)_{\text{stat}}. \quad (7.6)$$

(Statistical errors will later be folded into the systematic error due to the Z factors, see Section 7.5.6). Recall that the (27,1) operator does not mix with the (8,8) or (8,8) mixed operators under renormalization, so values like $Z_{(27,1),(8,8)}$ are 0.

NPR has not been performed for the (8,8) or (8,8) mixed operators on

these lattices. Based on prior experience [13, 15, 16] we adopt a rough hypothesis

$$Z_{JK}^{\text{RI-MOM}}(\mu = 2 \text{ GeV}) \approx 0.9 Z_q^2 \delta_{JK} \quad (7.7)$$

where the indices J and K run over the two values (8,8) and (8,8) mixed. For Z_q we have [77]

$$Z_q = 0.71919(79)_{\text{stat}}. \quad (7.8)$$

7.1.5 Lellouch-Lüscher Factor

In this section we discuss the calculation of the quantity

$$\sqrt{\text{LL factor}} = \frac{1}{\pi q^{3/2}} \sqrt{q \frac{\partial \phi}{\partial q} + k \frac{\partial \delta_0}{\partial k}} \sqrt{m_K E_{\pi\pi} L^{3/2}}. \quad (7.9)$$

This is the square root of Eq. 5.77, thus it is the conversion factor from finite volume matrix element to infinite volume amplitude.

Since we have already calculated q , m_K , and $E_{\pi\pi}$, and since the $\phi(q)$ function can be calculated analytically, the only unknown in Eq. 7.9 is the derivative of the phase shift $\partial \delta_0 / \partial k$. From the lattice calculation we have the phase shift δ_0 itself for the two values of k corresponding to zero and two twists, but not the derivative. As discussed in Section 7.1.2 and shown in Figure 110, $\delta_0(k)$ can be approximated to be linear for small values of k such as that corresponding to zero twists, and $\partial \delta_0 / \partial k$ can therefore be approximated by the scattering length a_0 (see Eq. 7.2, 7.3).

For two twists we use the phenomenological curve shown in Figure 109

to calculate the derivative of the phase shift at the corresponding value of k . This is justified by the agreement shown in Figure 109 between the phenomenological curve, the experimental data, and the phase shifts calculated in the present calculation. Also, Table 47 gives values for the two terms $q\partial\phi/\partial q$ and $k\partial\delta_0/\partial k$ which are added together in Eq. 7.9, and the former is approximately 17 times larger for two twists, so the derivative of the phase shift makes a small contribution in any case. We give an estimate of the systematic error due to using the phenomenological curve to obtain the derivative of the phase shift in Section 7.5.

In Table 47 we give values for the factor of Eq. 7.9 as well as for the ratio of Eq. 5.98. The latter is just the factor of Eq. 7.9 divided by the value it would take if interactions between the two pions were neglected, and thus it shows how much the interactions actually affect the normalization.

7.1.6 $\text{Re}(A_2)$ and $\text{Im}(A_2)$

We first need to compute the *lattice matrix element* $|\mathcal{M}|$, where $\mathcal{M}$ is defined in Eq. A.44. (We write $|\mathcal{M}|$ to remind the reader that the phase of $\mathcal{M}$ is unimportant and is only a matter of convention in Euclidean time, see Appendix A). Comparing Eq. A.31, A.41, and A.44 with Eq. 6.62, 6.63,

6.64, we see that

$$A_K = Z_K Z_K^* \quad (7.10)$$

$$A_{\pi\pi} = Z_{\pi\pi} Z_{\pi\pi}^* \quad (7.11)$$

$$A_{K \rightarrow \pi\pi} = Z_K^* Z_{\pi\pi} \mathcal{M}. \quad (7.12)$$

We also see that when we compute the quotient of Eq. 6.61 and fit to a constant A_q , we obtain

$$A_q = \frac{A_{K \rightarrow \pi\pi}}{A_K A_{\pi\pi}} = \frac{\mathcal{M}}{Z_K Z_{\pi\pi}^*} \quad (7.13)$$

and therefore the lattice matrix element can be obtained from the fit to the quotient by

$$|\mathcal{M}| = A_q |Z_K| |Z_{\pi\pi}| = A_q \sqrt{A_K} \sqrt{A_{\pi\pi}} \quad (7.14)$$

(note that A_q is positive definite because it comes directly from fits to Euclidean correlation functions).

We can thus use the results of Tables 2, 7, 11, and 12 to find $|\mathcal{M}|$. Note that A_K must be taken from the wall-wall kaon correlator at zero momentum in Table 2 so that Z_K in Eq. 7.10 and Eq. 7.12 are the same. Also note that the “ A ” values in Table 7 are for fits to the ratio $C_{\pi\pi}/C_\pi^2$ and are thus equal to $A_{\pi\pi}/A_\pi^2$. They must therefore be multiplied by A_π^2 (for which values can be found for the wall-wall pion correlator at both momenta in Table 2) to obtain $A_{\pi\pi}$ as needed in Eq. 7.14. We give results for $|\mathcal{M}|$ for the various

weak operators, momenta, and separations t_K between the kaon and two pions in Tables 49 and 50.

We can now use Eq. A.50 to compute $\text{Re}(A_2)$ and $\text{Im}(A_2)$ for zero and two twists. Note that for two twists the kinematics are close to being energy conserving with $E_{\pi\pi} = 485.7(8.0)$ MeV, which is very close to $m_K = 509.0(9.1)$ MeV. On the other hand, $E_{\pi\pi} = 288.0(5.0)$ MeV for zero twists, which is quite different from the kaon mass. Thus the results for two twists should be compared to experiment, while the results for zero twists are included for completeness.

To evaluate Eq. A.50 we make use of the factor of Eq. 7.9 for which values are give in Table 47, the physical constants found in Table 48, the Wilson coefficients found in Table 46, the renormalization factors of Eq. 7.4, 7.7, the value $M_5 = 1.8$ for these lattices, and the lattice matrix elements of Tables 49 - 50. Note that the only quantities that are not real positive in Eq. A.50 are the Wilson coefficients, so that they are what give A_2 an imaginary part. Eq. A.50 also requires that we multiply by a^{-3} , which is done using the superjackknife analysis of the lattice spacing described in Section 7.1.1.

Results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ for the two momenta and various separations t_K between the kaon and two pions are given in Table 51. We combine the results from different separations, which should theoretically be the same,

by taking an error weighted average:

$$A_{\text{ave}} = \frac{\sum_{t_K} A(t_K)/(\Delta A(t_K))^2}{\sum_{t_K} 1/(\Delta A(t_K))^2}. \quad (7.15)$$

This reduces the influence of any separations that are not optimal and give noisier results. We show results for the error weighted average of $\text{Re}(A_2)$ and $\text{Im}(A_2)$ in Table 51 and show the contributions from each operator in Table 52. We take the error weighted averages to be our final results for these quantities as calculated from the 32^3 coarse lattice. We also give results for the dimensionless ratio $\text{Im}(A_2)/\text{Re}(A_2)$:

$$\frac{\text{Im}(A_2)(p=0)}{\text{Re}(A_2)(p=0)} = -1.543(20) \times 10^{-4} \quad (7.16)$$

$$\frac{\text{Im}(A_2)(p=\sqrt{2}\pi/L)}{\text{Re}(A_2)(p=\sqrt{2}\pi/L)} = -5.93(27) \times 10^{-5}. \quad (7.17)$$

Again, the results for $p = \sqrt{2}\pi/L$ are close to being energy conserving ($E_{\pi\pi} = 485.7(8.0)$ MeV, compared to $m_K = 509.0(9.1)$ MeV), whereas the results for $p = 0$ are much further from energy conserving and are included for completeness. The pion mass in our calculation is $m_\pi = 142.8(2.5)$ MeV, so comparing to the physical masses $m_K \approx 494$ MeV and $m_\pi \approx 140$ MeV, we see that the calculation for $p = \sqrt{2}\pi/L$ is nearly physical. We can compare the $p = \sqrt{2}\pi/L$ error weighted result from Table 51 of $\text{Re}(A_2) = 1.461(87) \times 10^{-8}$ GeV to the result from experiment $\text{Re}(A_2)_{\text{exp}} = 1.484 \times 10^{-8}$ GeV and already

see good agreement, even before taking systematic error into account. We estimate the systematic error in Section 7.5.

7.2 Fine 32^3 Lattices

7.2.1 Pion and Kaon Masses

The results for the kaon and pion masses in lattice units are given in Tables 14 - 17 for the different valence and sea quark masses. We convert these to physical units and show the results in Tables 53 - 54. We used an inverse lattice spacing of $a^{-1} = 2.285(29)$ GeV, also obtained from a combined analysis of several ensembles [68]. We do not attempt to use a superjackknife analysis to combine the error from the inverse lattice spacing with the error from the meson masses in lattice units. Instead, we just add relative errors in quadrature, neglecting correlations. In any case, for all of the meson masses in physical units the error is dominated by the error in the inverse lattice spacing.

7.2.2 Two Pion Phase Shift

Using the results for $E_{\pi\pi}$ in lattice units in Table 18, we show results for $E_{\pi\pi}$, k , q , and δ_0 in physical units for the various valence and sea light quark masses in Tables 55 and 56. Again, when quoting errors on results in physical units we have just added in quadrature the relative errors of the inverse lattice spacing and of the quantity in lattice units.

7.2.3 Wilson Coefficients and Non-Perturbative Renormalization

We use the same Wilson coefficients as for the 32^3 coarse lattices (see Section 7.1.3 and Tables 45 and 46). For the renormalization constant of the $(27,1)$ operator and Z_A we have [33, 69]

$$Z_{(27,1),(27,1)}^{\text{RI-MOM}}(\mu = 2 \text{ GeV}) = 0.54359(27)_{\text{stat}} \quad (7.18)$$

$$Z_{B_K}^{\text{RI-MOM}}(\mu = 2 \text{ GeV}) = 0.98021(35)_{\text{stat}} \quad (7.19)$$

$$Z_A = 0.74469(13)_{\text{stat}}. \quad (7.20)$$

The NPR was performed on the exact same 32^3 fine lattices as in this report. Note that we again perturbatively convert $Z_{(27,1),(27,1)}$ and Z_{B_K} to the RI-MOM scheme from the RI-SMOM($\not{q}, \not{q}$) scheme. For the renormalization of the $(8,8)$ operators we adopt another rough hypothesis, similar to Section 7.1.3,

$$Z_{JK}^{\text{RI-MOM}}(\mu = 2 \text{ GeV}) \approx 1.03 Z_A^2 \delta_{JK}. \quad (7.21)$$

7.2.4 Lellouch-Lüscher Factor

Since all results are for $p = 0$ for these lattices, we approximate $\partial\delta_0/\partial k$ by the scattering length $a_0 \approx \delta_0(k)/k$ (see Eq. 7.3). This is the same approach as for the $p = 0$ data point on the 32^3 coarse lattices. Other than this, the calculation of the Lellouch-Lüscher factor is straightforward. In Table 57 we give results for the ratio of Eq. 5.98 for the different sea and valence light

quark masses. This ratio depends only on k and thus does not depend on the valence strange quark mass. In Table 58 we show results for the factor of Eq. 7.9 for the different sea light quark masses and the different valence light and strange quark masses. The dependence on the valence strange quark mass is only due to the factor of $\sqrt{m_K}$ in Eq. 7.9.

7.2.5 $\text{Re}(A_2)$ and $\text{Im}(A_2)$

We compute the lattice matrix element $|\mathcal{M}|$ using the results of Tables 15 - 21. We give the value of $|\mathcal{M}|$ for the various quark masses and for the three different operators in Tables 59 - 61.

We give results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ in Tables 62 - 63. These were calculated using Eq. A.50, which requires the factor of Eq. 7.9 for which values are tabulated in Table 58, the physical constants of Table 48, the Wilson coefficients found in Table 46, the renormalization factors of Eq. 7.18 and 7.21, the value $M_5 = 1.8$ for these lattices, and the lattice matrix elements of Tables 59 - 61. Eq. A.50 also requires that we multiply by a^{-3} . We use the value of $a^{-1} = 2.285(29)$ GeV quoted in Section 7.2.1, and combine errors by adding relative errors in quadrature. In Table 64 we show the contributions of the different operators to $\text{Re}(A_2)$ and $\text{Im}(A_2)$ for $m_l^{\text{sea}} = 0.004$, $m_l^{\text{val}} = 0.002$, and $m_s^{\text{val}} = 0.03$.

7.3 Quenched 24^3 Lattices

7.3.1 Pion and Kaon Masses

Results for the pion and kaon masses in lattice units are given in Tables 23 - 24 for the different valence quark masses. We convert these to physical units in Tables 65 - 66. We use an inverse lattice spacing of $a^{-1} = 1.31(2)$ GeV obtained from the ρ mass on 16^3 lattices with the same action and coupling [16]. Relative errors are added in quadrature from the inverse lattice spacing and from quantities in lattice units to find the error in quantities in physical units. This is valid because the measurement of the lattice spacing was performed on a different set of configurations and is statistically independent.

7.3.2 Two Pion Phase Shift

Using the results for $E_{\pi\pi}$ in lattice units in Table 30, we show results for $E_{\pi\pi}$, k , q , and δ_0 in physical units for the various light quark masses and twists in Tables 67 - 69. Again, when quoting errors on results in physical units we have just added in quadrature the relative errors of the inverse lattice spacing and of the quantity in lattice units.

7.3.3 Wilson Coefficients and Non-Perturbative Renormalization

Wilson coefficients and renormalization factors are taken from [16] in which the NPR was performed on 16^3 lattices with the same action and coupling as the 24^3 quenched lattices of the present work. Values at the scale $\mu = 1.44$

GeV are used, and are reproduced in Tables 70 - 71.

7.3.4 Lellouch-Lüscher Factor

In order to evaluate $\partial\delta_0/\partial k$ we fit the phase shift as a function of q to

$$\delta_0 = Aq + Bq^3 \quad (7.22)$$

which is what one would expect for small q (the first term is just the scattering length term). We keep things in terms of the dimensionless variable q because what we need for the Lellouch-Lüscher factor is just the dimensionless quantity $k\partial\delta_0/\partial k = q\partial\delta_0/\partial q$. Plots of the phase shift versus q and the fit to Eq. 7.22 for each light quark mass are shown in Figures 111 - 113, and the fit coefficients A and B are tabulated in Table 72. The data from three twists is excluded from the fit because it is too noisy which causes the phase shifts to be within error of 0.

Because the phase shift data is fit well by Eq. 7.22 we use it to obtain the desired term in the Lellouch-Lüscher factor, i.e.

$$k\frac{\partial\delta_0}{\partial k} = q\frac{\partial\delta_0}{\partial q} = Aq + 3Bq^3. \quad (7.23)$$

We do this for *all* of the different twists. Since the fit is dominated by the linear term in the region of the zero twist data (see Figures 111 - 113), this is indistinguishable from using the scattering length approximation Eq. 7.2 for zero twists. Although there is no a priori reason that the fit form Eq. 7.22

should hold for one and two twists, we believe it should give a satisfactory approximation of the derivative of the phase shift and note that the term $k\partial\delta_0/\partial k$ is generally dominated by $q\partial\phi/\partial q$ in the Lellouch-Lüscher factor. Although the data for three twists is not fit well by Eq. 7.22, it is too noisy to be meaningful in any case and generally will not be included in further analysis. Results for three twists will be quoted for completeness.

In Table 73 we give results for the ratio of Eq. 5.98 for the different light quark masses. Table 74 shows results for the factor of Eq. 7.9 for the different light and strange quark masses.

7.3.5 $\text{Re}(A_2)$ and $\text{Im}(A_2)$

We compute the lattice matrix element $|\mathcal{M}|$ using the results of Tables 24, 30, 32 - 43. In Tables 75 - 86 we give the value of $|\mathcal{M}|$ for the various quark masses, momenta, weak operators and time separations between the kaon and two pions.

We give results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ in Tables 87 - 102 for the various quark masses and momenta. In these tables we show results for each separation t_K between the kaon and two pions, and also the error weighted average over the different separations as in Eq. 7.15. An error weighted average is to be taken as the final result for the given momentum and set of quark masses.

$\text{Re}(A_2)$ and $\text{Im}(A_2)$ were calculated using Eq. A.50, which requires the factor of Eq. 7.9 for which values are tabulated in Table 74, the physical constants of Table 48, the Wilson coefficients found in Table 70, the renor-

malization factors of Table 71, the value $M_5 = 1.8$ for these lattices, and the lattice matrix elements of Tables 75 - 86. Eq. A.50 also requires that we multiply by a^{-3} . We use the value of $a^{-1} = 1.31(2)$ GeV quoted in Section 7.3.1, and combine errors by adding relative errors in quadrature.

7.4 Extrapolations and Interpolations to Physical Kinematics

In this section we will perform extrapolations and interpolations with respect to mass and pion momentum on the data from the 32^3 coarse, 32^3 fine, and 24^3 quenched lattices. These extrapolations and interpolations could be used to obtain results at masses and pion momenta that are closer to physical. However, it will be more appropriate to use these results to estimate the systematic error due to the slightly unphysical masses and momentum in the 32^3 coarse calculation. (Systematic errors will be discussed in Section 7.5).

In general, we will extrapolate/interpolate by performing fits with m_l , m_s , and n_{tw} as independent variables, where n_{tw} is the number of twists which is allowed to take on non-integer values for the purposes of extrapolation or interpolation, but will obviously be an integer for actual data points. (Fits are uncorrelated in general). Meson masses m_{xy} will be fit to the functional form

$$m_{xy}^2 = B_0(m_x + m_y) + B_1 \quad (7.24)$$

where m_x and m_y are quark masses (no distinction between light and strange quark masses are made in this expression). Eq. 7.24 is inspired by leading

order chiral perturbation theory, and the coefficient B_1 allows the effect of the residual mass to be taken into account. This relationship makes it convenient to work in terms of the quark masses when performing fits. The reason for choosing to work with n_{tw} when performing fits is the analogy between Eq. 7.24 and the relationship

$$p^2 = n_{\text{tw}} \frac{\pi^2}{L^2}. \quad (7.25)$$

That is, working with the quark masses and n_{tw} is similar to working with meson masses squared and momentum squared, the latter being the quantities that generally appear in chiral perturbation theory formulae. However, we will not actually fit to chiral perturbation formulae since it was found that this approach failed in a similar setting in [15]. Instead we will attempt to fit to linear functions in m_l , m_s , and n_{tw} , in the hope that this will be sufficient for the relatively small interpolations and extrapolations that are performed. For the 32^3 fine lattices we have data for a variety of masses and for the 24^3 quenched lattices we have data for a variety masses and momenta, thus we can actually see how well these linear functions fit the data.

7.4.1 Coarse 32^3 Lattices

We now summarize the available results for A_2 (real and imaginary part) on the 32^3 coarse lattices which were presented in Section 7.1. We have results for A_2 at one pion and kaon mass, $m_\pi = 142.8(2.5)$ MeV and $m_K = 509.0(9.1)$

MeV, and at two different two pion energies, $E_{\pi\pi} = 288.0(5.0)$ MeV and $E_{\pi\pi} = 485.7(8.0)$ MeV. (Masses quoted for the 32^3 coarse lattices are valence masses, and we ignore the effects of partial quenching for now. Systematic effects of partial quenched will be discussed in Section 7.5). The pion and kaon mass are close to the physical values of $m_\pi = 139.6$ MeV and $m_K = 493.7$ MeV (the charged meson masses), but are both slightly too heavy, and the second two pion energy is close to the kaon mass so that the decay is close to energy conserving, but is still too small. The lattice spacing of the 32^3 coarse lattice is also rather large compared to the 32^3 fine lattice, meaning the results on the former might be more susceptible to discretization errors.

The only thing we can do with this data is extrapolate to energy conserving kinematics with respect to $E_{\pi\pi}$ while m_π and m_K necessarily remain fixed. In particular we first fit

$$E_{\pi\pi}^2 = A_0 n_{\text{tw}} + A_1 \quad (7.26)$$

and then solve for the value of n_{tw} which will make $E_{\pi\pi}$ equal to m_K . This “fit” is only to two data points ($n_{\text{tw}} = 0$ and $n_{\text{tw}} = 2$), so it is really just a linear extrapolation. Eq. 7.26 is inspired by the approximate relation

$$E_{\pi\pi}^2 \approx 4(m_\pi^2 + p^2) \quad (7.27)$$

which ignores two pion interactions. Inserting Eq. 7.25 into Eq. 7.27 gives an explicit expressions for the coefficient A_0 and A_1 in Eq. 7.26. By allowing

these coefficients to be arbitrary we also hope to capture the behavior resulting from two pion interactions that is relevant for the small extrapolation performed here.

The fit to Eq. 7.26 is shown in Figure 114, and it is found that the value

$$n_{\text{tw}} = 2.303(75) \quad (7.28)$$

will make $E_{\pi\pi}$ equal to m_K . We now perform the linear extrapolation

$$A_2 = C_0 n_{\text{tw}} + C_1 \quad (7.29)$$

from the two data points $n_{\text{tw}} = 0, 2$ to the value of n_{tw} in Eq. 7.28 and find

$$\text{Re}(A_2) = 1.58(10) \times 10^{-8} \text{ GeV} \quad (7.30)$$

$$\text{Im}(A_2) = -8.35(46) \times 10^{-13} \text{ GeV} \quad (7.31)$$

$$\frac{\text{Im}(A_2)}{\text{Re}(A_2)} = -5.30(30) \times 10^{-5}. \quad (7.32)$$

The extrapolation is shown in Figures 115 and 116. The percentage difference between the extrapolated values and the values at $n_{\text{tw}} = 2$ are 7.9%, 3.7%, and 10.7% for $\text{Re}(A_2)$, $\text{Im}(A_2)$, and $\text{Im}(A_2)/\text{Re}(A_2)$ respectively. Since we have no way of being sure whether Eq. 7.26 and Eq. 7.29 actually hold, (and in fact they do not for the 24^3 quenched data, see Section 7.4.3), we prefer to look at these percent changes as estimates of systematic error rather than

explicit correction factors.

7.4.2 Fine 32^3 Lattices

We summarize the available results for A_2 (real and imaginary part) on the 32^3 fine lattices which were presented in Section 7.2. We have results for A_2 with valence quark masses of 0.002, 0.004, 0.006, 0.008, 0.025, and 0.03 in all possible combinations where $m_s \geq m_l$. These valence quark masses give several pion masses in the approximate range $220 \text{ MeV} < m_\pi < 740 \text{ MeV}$ and several kaon masses in the approximate range $260 \text{ MeV} < m_K < 710 \text{ MeV}$. The physical kaon mass is certainly contained within the range of the data, but the physical pion mass is somewhat below. Thus we will be performing an interpolation with respect to kaon mass, but an extrapolation down to the physical pion mass.

The results are for three different sea light quark masses: 0.004, 0.006, and 0.008. We will ignore partial quenching at first and perform a separate analysis for each sea quark mass, and then compare end results for the different sea quark masses to see what effect partial quenching has. All results are for zero twists, and thus $E_{\pi\pi}$ is fixed at a single value close to $2m_\pi$ for a given pion mass. This value will be much lower than the kaon mass for the range of masses with which we will be concerned, and thus the data will be for decays that are significantly energy non-conserving. Being restricted to zero twists means that $E_{\pi\pi}$ varies along with m_π in a way that we cannot control, and thus when we extrapolate to the physical pion mass, a value of

$E_{\pi\pi}$ will be forced on us. This value will continue to be close to $2m_\pi$, and will therefore be much less than the physical kaon mass, so that the decay will still be significantly energy non-conserving.

We first fit the meson masses as a function of quark mass to Eq. 7.24. We include all combinations of quark masses (m_x, m_y) in the fit except for $(0.025, 0.025)$, $(0.025, 0.03)$, and $(0.03, 0.03)$, for which $m_x + m_y$ is significantly greater than it is for all other combinations. These three data points that are omitted lie far from the region of interest and also start to deviate from Eq. 7.24. The fit for $m_l^{\text{sea}} = 0.004$ is shown in Figure 117 and fit results are given in Table 103. We use the fit results and Eq. 7.24 to determine the light and strange valence quark masses that would yield the pion and kaon mass of the 32^3 coarse lattices ($m_\pi = 142.8$ MeV and $m_K = 509.0$ MeV) and the light and strange valence quark masses that would yield the physical pion and kaon mass ($m_\pi = 139.6$ MeV and $m_K = 493.7$ MeV). These results are also given in Table 103 and shown in Figure 117 for the case of the meson masses from the 32^3 coarse lattices so that the extent of the extrapolation and interpolation can be seen. Note that the lattice spacing of the 32^3 fine lattice is needed to convert quantities in physical units to the lattice units of the fine lattice, however the error on this lattice spacing is ignored and will be in all similar situations in this section. Also, the errors on the coarse meson masses (both from the coarse lattice spacing and the coarse lattice calculation itself) are ignored. This is justified since our main goal is to estimate the size of systematic errors. Statistical errors within the 32^3 fine

lattice calculation and extrapolation/interpolation *are* propagated using the jackknife method.

We next fit for the two pion energy as a function of light valence quark mass:

$$E_{\pi\pi}^2 = A_0 m_l + A_1. \quad (7.33)$$

This is in analogy to the discussion following Eq. 7.27, recalling that we only have data for zero twists for the 32^3 fine lattices and thus the only parameter on which $E_{\pi\pi}$ can depend is m_l . We do not include $m_l = 0.025$ or $m_l = 0.03$ in the fit because these are far out of the region of interest. We can then use the light quark masses from Table 103 in Eq. 7.33 and thus extrapolate to find the value of $E_{\pi\pi}$ that corresponds to the pion mass of the coarse lattices and to the physical pion mass, for zero twists. The fit and extrapolation to coarse lattice pion mass are shown in Figure 118 for $m_l^{\text{sea}} = 0.004$, and results are presented in Table 104.

Finally, we fit

$$\text{Re}(A_2) = C_0 m_l + C_1 m_s + C_2 \quad (7.34)$$

and similarly for $\text{Im}(A_2)$. Although we have data for all m_l and m_s where $m_s \geq m_l$, we only use the data with $m_s = 0.025, 0.03$ and $m_l = 0.002, 0.004, 0.006, 0.008$, since other quark masses correspond to pions masses that are much heavier or kaon masses that are much lighter than those of interest. With this fit we can then use the light and strange quark masses of Table 103 to find the values of $\text{Re}(A_2)$ and $\text{Im}(A_2)$ that correspond to the pion and kaon

mass of the coarse lattices and to the physical pion and kaon masses. This will be an extrapolation in m_l and an interpolation in m_s . We also know the two pion energy for this extrapolated/interpolated decay from Table 104. The fit and extrapolation/interpolation to coarse lattice meson masses are shown in Figures 119 and 120 for $m_l^{\text{sea}} = 0.004$, and results are presented in Tables 105 and 106. (In this and the following extrapolations/interpolations we will not propagate the error on the lattice spacing into the final results unless otherwise noted). From Figure 119 we see that Eq. 7.34 fits the $\text{Re}(A_2)$ data very well, however, Figure 120 shows that this is *not* the case for $\text{Im}(A_2)$. Thus we will only use the result of the extrapolation for $\text{Re}(A_2)$ in the analysis.

It can be seen from Tables 104 and 105 that results from different sea light quark masses agree within error. Thus there is no sign of partial quenching having any effect on the results, and we take error weighted averages over different sea light quark masses as final results from the 32^3 fine lattices. These are shown in Table 107, which also shows the fractional change of results at the masses of the 32^3 coarse lattices to the results at the physical masses. In particular, there is a change of 7.9% in $\text{Re}(A_2)$. However, bear in mind that although this percent change is from the correct change in pion and kaon mass from coarse lattice values to physical values, the two pion energies are much lower than those in the coarse lattice calculation (two twists) and of the physical decay, thus 7.9% is not necessarily what we expect in going from the kinematics of the coarse lattice to those of the physical decay.

It is perhaps more informative to compare the actual results of the 32^3

coarse calculation to the extrapolation to the coarse masses performed in this section. Looking at Table 107 we see that while the result in the first row is for the same pion and kaon mass as the 32^3 coarse lattice, it is for a two pion energy of $E_{\pi\pi} = 302.7(1.1)$ MeV, compared to the actual coarse lattice calculation which has $E_{\pi\pi} = 288.0(5.0)$ MeV and $E_{\pi\pi} = 485.7(8.0)$ MeV (zero and two twists respectively). This is not unexpected; since the coarse lattice has a larger volume we expect the two pion energy at zero twists to be smaller on this lattice when m_π is the same on the two lattices. This is illustrated in Figure 121.

While neither of the two pion energies from the coarse calculation match the two pion energy of the extrapolation, we can perform an interpolation of the coarse lattice results to the latter two pion energy. This can be done in the same way as the extrapolation of Section 7.4.1 (see Eq. 7.26 and 7.29). The interpolated point is not too far from the zero twist point ($E_{\pi\pi} = 302.7(1.1)$ MeV compared to $E_{\pi\pi} = 288.0(5.0)$ MeV). We find

$$\text{Re}(A_2)^{\text{coarse, interpolated}} = 7.42(15) \times 10^{-9} \text{ GeV} \quad (7.35)$$

(error on lattice spacing of coarse lattice not propagated). This is directly comparable to the result of $\text{Re}(A_2)^{\text{fine, extrapolated}} = 7.79(24) \times 10^{-9} \text{ GeV}$ from Table 107, and we see that these two agree within their error bars. This gives us some confidence that errors due to finite volume and non-zero lattice spacing are under control since we were able to obtain results that agree from

two lattices with different volumes and lattice spacings.

7.4.3 Quenched 24^3 Lattices

We summarize the results for A_2 on the 24^3 quenched lattices presented in Section 7.3. We have results for light quark masses $m_l = 0.0023, 0.0047$, and 0.0071 , and strange quark masses $m_s = 0.046, 0.062, 0.078, 0.094$, and 0.11 . The three light quark masses correspond to pion masses of $m_\pi = 167.5(2.9)$ MeV, $214.9(3.4)$ MeV, and $252.8(4.0)$ MeV, and the different combinations of light and strange quark masses lead to several kaon masses in the range $435.6 \text{ MeV} \leq m_K \leq 661 \text{ MeV}$. The physical kaon mass is contained within the range, but the physical pion mass is below the range of pion masses, although not that far. Therefore, as in the previous section, we will be interpolating in kaon mass and extrapolating in pion mass.

For each combination of masses we have results for number of twists $n_{\text{tw}} = 0, 1, 2$, and 3 . This makes it possible to do an interpolation or extrapolation in n_{tw} to energy conserving kinematics (i.e. hold m_π and m_K constant while varying n_{tw} to change $E_{\pi\pi}$ until it equals m_K), or to any two pion energy one chooses. In fact, quark masses were chosen in such a way that several combinations of masses result in nearly energy conserving kinematics for some of the integer values of n_{tw} (see Tables 66, 67, 68, and 69).

The strategy we will follow is to perform the same analysis as Section 7.4.2 for each fixed n_{tw} , that is, fit quantities as a function of m_l and m_s for a fixed n_{tw} and extrapolate/interpolate to the physical kaon and pion masses

as well as to the kaon and pion masses of the 32^3 coarse lattices. For each n_{tw} we will then have A_2 extrapolated to physical/coarse masses but with an $E_{\pi\pi}$ (also determined by extrapolation) which is not equal to m_K in general. Then we will look at $E_{\pi\pi}^2$ as a function of n_{tw} and determine by interpolation the (non-integral) value of n_{tw} that corresponds to the two pion energy of the coarse lattice, and to energy-conserving kinematics for the physical decay. We will interpolate A_2 to this value of n_{tw} to get a value of A_2 at exactly the kinematics desired. The reason for fitting first with respect to quark masses and then looking at quantities as a function of n_{tw} is that A_2 and $E_{\pi\pi}^2$ will turn out to be quite linear in the quark masses, whereas they will be seen to deviate significantly from being linear in n_{tw} .

We first fit the meson mass to Eq. 7.24 and show the results in Figure 122 and Table 108. All combinations of quark masses are included in the fit. The determination of the quark masses that correspond to the meson masses of the coarse lattice is also shown in Figure 122, and these quark masses, as well as those corresponding to the physical meson masses, are given in Table 108. As in the previous section we do not propagate the errors on the lattice spacings of either the quenched or coarse lattices.

We next fit $E_{\pi\pi}$ as a function of m_l to Eq. 7.33 separately for each value of n_{tw} . We then extrapolate to the values of m_l corresponding to the physical and coarse lattice pion masses, as determined from the fit to the meson mass. This yields $E_{\pi\pi}$ for the coarse/physical pion mass at each n_{tw} . The fits and extrapolations are shown in Figure 123 and the results are tabulated in Table

109.

Finally we fit $\text{Re}(A_2)$ and $\text{Im}(A_2)$ as functions of m_l and m_s (Eq. 7.34) for each n_{tw} . (The error weighted averages of $\text{Re}(A_2)$ and $\text{Im}(A_2)$ over different time separations t_K are used). We then use the light and strange quark masses of Table 108 to find the values of $\text{Re}(A_2)$ and $\text{Im}(A_2)$ at each twist that correspond to the pion and kaon masses of the coarse lattice and to the physical pion and kaon masses. This will again be an extrapolation in m_l and an interpolation in m_s . The fits and extrapolations are shown in Figures 124 - 131, and the results are tabulated in Tables 110 and 111. Eq. 7.34 fits the data quite well for both $\text{Re}(A_2)$ and $\text{Im}(A_2)$.

We now look at $E_{\pi\pi}^2$, $\text{Re}(A_2)$, and $\text{Im}(A_2)$, that have been extrapolated to the pion and kaon masses of the coarse lattice, as a function of n_{tw} . These are shown in Figures 132, 133, and 134. (We do not show the same plots where quantities have been extrapolated to the physical masses, but they look similar). It can be seen that these plots are far from linear. However, the value of $E_{\pi\pi}^2$ on the coarse lattice (two twists) is very close to, but just above the value of $E_{\pi\pi}^2$ for one twist in Figure 132. $E_{\pi\pi}^2$ from the coarse lattice is indicated by a red circle. Thus we do a linear interpolation between $n_{\text{tw}} = 1$ and $n_{\text{tw}} = 2$, which is shown as a solid line in Figures 132, 133, and 134. From Figure 132 we can determine in this way the value of n_{tw} that gives rise to the desired two pion energy from the coarse lattice, and then we can interpolate to this value of n_{tw} in Figures 133 and 134 to find $\text{Re}(A_2)$ and $\text{Im}(A_2)$ at the masses and kinematics of the coarse lattices. A

similar procedure can be done for the physical masses and kinematics (energy conserving).

Final results for the physical masses and kinematics, and for the masses and kinematics of the coarse lattice, are tabulated in Table 112. The results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ extrapolated to the masses and kinematics of the coarse lattice disagree significantly with the actual results for these quantities obtained from the coarse lattice (compare the first column of Table 112 to the bottom row of Table 51). Recalling that these quantities are proportional to a^{-3} , we attribute this discrepancy mainly to inaccuracy in the determination of the lattice spacing, which used the ρ mass, on the 24^3 quenched lattices. (Furthermore, we used the hypothesis of Eq. 7.7 for the renormalization factors necessary to compute $\text{Im}(A_2)$ on the 32^3 coarse lattices, leading to an uncontrolled systematic in $\text{Im}(A_2)$). Table 112 also shows the factors by which our quenched calculation predicts that $\text{Re}(A_2)$ and $\text{Im}(A_2)$ should change from the masses and kinematics of the coarse lattice to the masses and kinematics of the physical decay. We see a change of 1.2% for $\text{Re}(A_2)$ and 0.8% for $\text{Im}(A_2)$. Since these percentage changes are dimensionless, they do not suffer from the inaccurate determination of the lattice spacing, and we can take them more seriously. It is certainly fine to take them as estimates of the systematic error of the coarse calculation due to not being exactly at the physical masses and kinematics.

7.5 Discussion of Errors

In this section we attempt to quantify the various sources of systematic error in the 32^3 coarse lattice calculation of $\text{Re}(A_2)$ and $\text{Im}(A_2)$, which we will take as our final result. The sources of systematic error are scaling violations, finite volume effects, partial quenching, our estimation of the derivative of the phase shift, unphysical meson masses and pion momentum, error in the NPR procedure used to calculate renormalization factors, and errors in the Wilson coefficients.

7.5.1 Scaling Violations

Scaling violations here refer to errors in quantities extracted from the lattice due to non-zero lattice spacing a . These errors might be estimated by using different quantities to set the scale (i.e. determine the lattice spacing), and examining the resulting change in the quantity of interest. This method of estimating finite lattice spacing errors will assign large errors to $\text{Re}(A_2)$ and $\text{Im}(A_2)$ because they are proportional to a^{-3} , the inverse lattice spacing cubed, as opposed to a^{-1} like particle masses and energies. We use three different quantities to set the scale on the 32^3 coarse lattice: m_Ω (the mass of the Ω), f_K (the decay constant of the kaon), and the r_0 parameter in the static quark potential. These are done in a combined analysis of several ensembles, and then each is used to calculate $\text{Re}(A_2)$ and $\text{Im}(A_2)$ in physical units. (Note that m_Ω was used to set the scale in all previous sections). The

results are given in Table 113. We use the standard deviation of the three quantities as an estimate of the systematic error due to scaling violations, and find that it is 7.5% of the average for both $\text{Re}(A_2)$ and $\text{Im}(A_2)$.

7.5.2 Finite Volume Effects

The Lellouch-Lüscher factor (see Section 5.3) accurately converts matrix elements in finite volume to infinite volume amplitudes provided that the spatial extent of the lattice is significantly larger than the Compton wavelength of the particles involved. In particular $m_\pi L$ must be significantly larger than 1. When this condition is not met then all quantities, including particle masses and energies, suffer from finite volume errors. For the 32^3 coarse lattice calculation $m_\pi L = 3.3$.

In order to estimate the systematic error due to these finite volume effects we use finite volume chiral perturbation theory. In the expressions for the $K \rightarrow \pi\pi$ matrix elements, the logarithmic terms at next-to-leading order are the lowest order terms that depend explicitly on the volume. The logarithmic terms result from diagrams with loops and the finite volume version is obtained by replacing integrals over momenta with sums over the discrete values of momentum allowed in finite volume. The difference between the sum and the integral is the finite volume correction at leading order. We estimate the percentage error due to finite volume as this correction over the leading order term. All low energy constants, besides the decay constant in the chiral limit, cancel in this ratio and thus we can get a numerical value at

the kaon and pion masses of the 32^3 coarse calculation. We find a value of 6.1% for the (27,1) matrix element and 6.7% for the (8,8) matrix elements. More details are given in Appendix B.

Table 52 shows the contributions to $\text{Re}(A_2)$ and $\text{Im}(A_2)$ from the different four quark operators on the 32^3 coarse lattices. We see that $\text{Re}(A_2)$ is dominated by the contribution from the (27,1) operator and $\text{Im}(A_2)$ is dominated by the contribution from the (8,8) mixed operator. Thus we take 6.1% and 6.7% as our estimate of systematic error in $\text{Re}(A_2)$ and $\text{Im}(A_2)$ respectively due to finite volume effects in the 32^3 coarse calculation.

7.5.3 Partial Quenching

In the 32^3 fine calculation we found that the results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ for three different sea light quark masses agreed with each other within error. Thus we hypothesize that $\text{Re}(A_2)$ and $\text{Im}(A_2)$ are approximately independent of sea light quark mass and estimate the largest possible deviation from this independence by the standard deviation of the different results for the different sea light quark masses. Specifically we look at Tables 105 and 106 and find the standard deviation among the “extrap to phys” results for different sea light quark masses. These are 3.5% and 1.7% of the mean for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ respectively. Although these results are for zero twists and represent a significantly energy non-conserving decay, we take these to be approximations to the deviations we might expect due to partial quenching of the light quark in the actual 32^3 coarse calculation. Thus we take these

values as our estimate of the systematic error due to partial quenching of the light quark in the 32^3 coarse calculation. We make no attempt to estimate the error due to partial quenching of the strange quark, but the amount of partial quenching in the strange quark mass is relatively small ($m_s^{\text{sea}} = 0.045$ compared to $m_s^{\text{val}} = 0.049$).

7.5.4 Derivative of the Phase Shift

We used a phenomenological fit to experimental data to find the derivative of the phase shift $\partial\delta_0/\partial k$ in the Lellouch-Lüscher factor for the 32^3 coarse calculation. We estimate the systematic error from this by approximating the derivative of the phase shift in a different way and taking the percent difference. Specifically, we take the slope of a straight line between the phase shift result for zero twists and the phase shift result for two twists on a plot of δ_0 vs. k . From Figure 109 we can see that this procedure will likely underestimate the derivative of the phase shift at the value of k corresponding to two twists. We find a difference of 1.5% in the results of this and the original procedure for the Lellouch-Lüscher factor, which we take as our estimate of the systematic error due to uncertainty in the derivative of the phase shift.

7.5.5 Unphysical Masses and Kinematics

In Section 7.4.3 we found from the 24^3 quenched calculation that the differences in the pion and kaon masses and in the two pion energies between

the 32^3 coarse calculation and the real world caused a 1.2% difference in $\text{Re}(A_2)$ and a 0.8% difference in $\text{Im}(A_2)$. We take these as our estimate of this systematic error in the 32^3 coarse calculation.

7.5.6 Non-Perturbative Renormalization

First we look at $Z_{(27,1),(27,1)}^{\text{RI-MOM}}$, the renormalization constant of Eq. 7.4 for the $(27,1)$ four quark operator. This is equal to $Z_{B_K} Z_A^2$, where Z_{B_K} is the renormalization constant for the quantity B_K that parametrizes $K^0 - \bar{K}^0$ mixing, and Z_A is the axial current renormalization factor. We first estimate the systematic error in Z_{B_K} , which has been analyzed on other ensembles in [33]. We follow this analysis of the systematic error in Z_{B_K} for the case of the 32^3 coarse lattices to the extent possible with available data. Statistical error, uncertainty in the lattice spacing, uncertainty due to infrared chiral symmetry breaking effects, and error due to truncation of the perturbation series in the matching to a continuum renormalization scheme are found as in [33], and we get 0.0073%, 0.10%, 0.15%, and 4.8% for these uncertainties in Z_{B_K} respectively [77]. Available data does not allow us to estimate uncertainty due to $O(4)$ symmetry breaking, or error due to fixed sea strange quark mass. Adding the uncertainties in quadrature we get 4.8% for our estimate of the total systematic error in Z_{B_K} .

For Z_A we have a statistical error of 0.10%, and we wish to estimate the systematic error by looking at the difference between Z_A/Z_A and Z_V/Z_V as in Appendix B of [69]. We look at the percent difference between these

two quantities in the chiral limit for the two lattice spacings provided in Table XLIII of [69] and extrapolate linearly in a^2 to the lattice spacing of the coarse lattices in this report. We find a percent difference of 3.7%. We add in quadrature with the statistical error to get a total systematic error on Z_A of 3.7%. We must add twice this value in quadrature to the 4.8% of the previous paragraph to find the systematic error in $Z_{(27,1),(27,1)}^{\text{RI-MOM}}$ since it is equal to $Z_{B_K} Z_A^2$. We thus obtain a value of 8.8% for the systematic error in $Z_{(27,1),(27,1)}^{\text{RI-MOM}}$.

Now we must give a systematic error on the renormalization constants for the (8,8) and (8,8) mixed four quark operators. We used the hypothesis of Eq. 7.7 to obtain values for these constants, and since the error in this hypothesis is uncontrolled, we take it to be the dominant systematic. In [16] we see that the actual renormalization constants for the (8,8) operators differ by at most 10% from this hypothesis. Since we have no reason to expect this much of an agreement in the present case, we will be conservative and take the systematic error in these renormalization constants to be 20%.

As mentioned in Section 7.5.2, $\text{Re}(A_2)$ is dominated by the contribution from the (27,1) operator and $\text{Im}(A_2)$ is dominated by the contribution from the (8,8) mixed operator. Thus we assign an 8.8% systematic error to $\text{Re}(A_2)$ and a 20% systematic error to $\text{Im}(A_2)$ from the renormalization constants.

7.5.7 Wilson Coefficients

The Wilson coefficients are calculated to next-to-leading order, meaning to order $O(\alpha_s(\mu_W))$, where μ_W is the weak scale. We take the truncation of the series at this order as the dominant source of systematic error in the Wilson coefficients. We would like to estimate this error by estimating the size of the $O(\alpha_s(\mu_W)^2)$ term, which we do by squaring the $O(\alpha_s(\mu_W))$ term.

To this end we use the results of Table XVIII of [57], which give Wilson coefficients calculated to leading order and to $O(\alpha_s(\mu_W))$ in the NDR scheme (we do not use the HV scheme). This table gives values for y_i and z_i , which are related to the Wilson coefficients that multiply the (27,1), (8,8), and (8,8) mixed operators by Eq. 4.33 - 4.35. Recall again that $\text{Re}(A_2)$ is dominated by the contribution from the (27,1) operator and $\text{Im}(A_2)$ is dominated by the contribution from the (8,8) mixed operator. Thus for $\text{Re}(A_2)$ we need only examine

$$\begin{aligned} \text{Re} \left[\frac{C_1(\mu) + C_2(\mu)}{3} + \frac{C_9(\mu) + C_{10}(\mu)}{2} \right] &\approx \text{Re} \left[\frac{C_1(\mu) + C_2(\mu)}{3} \right] \\ &\approx \frac{z_1(\mu) + z_2(\mu)}{3} \end{aligned} \quad (7.36)$$

where the approximations are made based on the numerical values of the Wilson coefficients in Table 45. For $\text{Im}(A_2)$ we need only examine

$$\text{Im} \left[\frac{C_8(\mu)}{2} \right] = \text{Im}(\tau) \frac{y_8(\mu)}{2}. \quad (7.37)$$

In summary, we need only look at the relative error of $z_1(\mu) + z_2(\mu)$ and $y_8(\mu)$ to find the related systematic error in $\text{Re}(A_2)$ and $\text{Im}(A_2)$ respectively. Using the method of squaring the $O(\alpha_s(\mu_W))$ term as outlined above, we find from the table of [57] that the relative error is no more than 1.1% for $z_1(\mu) + z_2(\mu)$ and no more than 0.6% for $y_8(\mu)$. Thus we quote systematic errors of 1.1% and 0.6% for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ respectively due to the Wilson coefficients.

7.5.8 Summary

We tabulate the estimates of the different sources of systematic error discussed above in Table 114. We add these in quadrature to arrive at a total systematic error of 13.7% for $\text{Re}(A_2)$ and 22.5% for $\text{Im}(A_2)$. For the dimensionless ratio $\text{Im}(A_2)/\text{Re}(A_2)$ there is no explicit dependence on the lattice spacing since the factors of a^{-3} cancel in the numerator and denominator, and this cancellation also occurs for the Lellouch-Lüscher factor. Thus we assign a systematic error for the ratio by omitting the error due to scaling violations and the derivative of the phase shift, and then adding the remaining relative errors on $\text{Re}(A_2)$ and $\text{Im}(A_2)$ in quadrature, and we arrive at a value of 24.0%.

8 Conclusion

We calculated $\text{Re}(A_2)$ and $\text{Im}(A_2)$ on the coarse lattices which had 2+1 dynamical flavors, dimensions $32^3 \times 64$, $L_s = 32$, and an inverse lattice spacing of $a^{-1} = 1.373(24)$ GeV. The calculation was performed for pion and kaon masses $m_\pi = 142.8(2.5)$ MeV and $m_K = 509.0(9.1)$ MeV and for two pion energies $E_{\pi\pi} = 288.0(5.0)$ MeV and $E_{\pi\pi} = 485.7(8.0)$ MeV, the latter corresponding to nearly energy conserving kinematics. We also performed calculations of $\text{Re}(A_2)$ and $\text{Im}(A_2)$ on the fine lattices with 2+1 dynamical flavors, dimensions $32^3 \times 64$, $L_s = 16$, and an inverse lattice spacing of $a^{-1} = 2.285(29)$ GeV, but obtained results only for periodic spatial boundary conditions so that the two pions had nearly zero momentum. Finally, calculations of $\text{Re}(A_2)$ and $\text{Im}(A_2)$ were done on the quenched lattices with dimensions $24^3 \times 64$, $L_s = 16$ and an inverse lattice spacing of $a^{-1} = 1.31(2)$ GeV. The calculations on the fine and quenched lattices were done for a variety of kaon and pion masses (and pion momenta for the quenched lattices), and served to give an idea of the dependence of A_2 on these parameters, and to aid in estimating the systematic error in the coarse lattice calculation. We note that it was possible to extrapolate the quenched results to physical

masses and kinematics, but in doing so we found a significant disagreement with the coarse results and with experiment, which we attribute largely to inaccuracy in determining the lattice spacing (which comes in as a factor of a^{-3} in A_2) of the quenched lattices using the ρ mass.

We quote the results from the coarse lattices with nearly energy conserving kinematics as our final results: $\text{Re}(A_2) = 1.461(87)_{\text{stat}}(200)_{\text{sys}} \times 10^{-8}$ GeV, $\text{Im}(A_2) = -8.67(45)_{\text{stat}}(1.95)_{\text{sys}} \times 10^{-13}$ GeV, and $\text{Im}(A_2)/\text{Re}(A_2) = -5.93(27)_{\text{stat}}(1.42)_{\text{sys}} \times 10^{-5}$. The central values and statistical errors are from Table 51, and the systematic errors are from Table 114. Note that for $\text{Im}(A_2)$ we used a rough hypothesis for the relevant renormalization constants since these were not yet available, and thus we quote a large systematic error for this quantity. Once the renormalization constants have been computed we will be able to quote a value with a systematic error that is about the same relative size as that of $\text{Re}(A_2)$. The result for $\text{Re}(A_2)$ is in good agreement with the experimental value of 1.484×10^{-8} GeV.

The result presented here is a significant milestone, being the first $K \rightarrow \pi\pi$ calculation with a realistically light pion mass and having other sources of error under control, notably: dynamical quarks are used, no chiral extrapolations are done, and the two pions are given the momentum of the physical decay without introducing too much statistical noise. A more difficult calculation of the $\Delta I = 1/2$ channel decay, which is underway [78], could be combined with our results to verify the experimentally observed $\Delta I = 1/2$ rule and to calculate ϵ' from first principles with the CP violating phase of

the CKM matrix as input.

Once the renormalization constants for $\text{Im}(A_2)$ have been calculated, our results can be further improved by repeating the calculation with different lattice spacings in order to take a continuum limit, and perhaps with different volumes in order to further investigate finite volume effects. The calculation could also be repeated with $m_l^{\text{sea}} = m_l^{\text{val}}$ so that there is no partial quenching of the light quark mass. The technique of reweighting could be used to accomplish this without having to generate new ensembles [79]. Finally, the derivative of the two pion phase shift could be calculated directly on the lattice by making use of more general spatial boundary conditions [80], rather than relying on experimental and phenomenological input as done in this report, however we showed that this is a small source of uncertainty.

Tables

Particle	Quark Content	Mass (MeV)
K^+	$u\bar{s}$	493.667(13)
K^0	$d\bar{s}$	497.648(22)
$\bar{K}^0$	$s\bar{d}$	497.648(22)
K^-	$s\bar{u}$	493.667(13)
π^+	$u\bar{d}$	139.57018(35)
π^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	134.9766(6)
π^-	$d\bar{u}$	139.57018(35)

Table 1: Quark contents and masses of the kaons and pions [20, 5]. These are the *valence quark contents*, meaning that they determine the quantum numbers of the particle state, but the state can also be thought to consist of *virtual* or *sea* quarks and gluons that create and annihilate each other and serve to keep the valence quarks bound together.

Correlator	Momentum (p)	A	E
Pion Wall-Point	0	$2.983(46) \times 10^7$	0.10400(37)
Pion Wall-Wall	0	$7.79(13) \times 10^9$	0.10400(38)
Pion Wall-Point	$\sqrt{2}\pi/L$	$4.114(90) \times 10^6$	0.1714(17)
Pion Wall-Wall	$\sqrt{2}\pi/L$	$1.017(19) \times 10^9$	0.1717(17)
Kaon Wall-Point	0	$1.025(18) \times 10^7$	0.3709(12)
Kaon Wall-Wall	0	$2.512(48) \times 10^9$	0.3706(13)
Kaon Wall-Point	$\sqrt{2}\pi/L$	$2.211(35) \times 10^6$	0.3935(12)
Kaon Wall-Wall	$\sqrt{2}\pi/L$	$5.18(10) \times 10^8$	0.3947(16)

Table 2: Results of the fits to the pion and kaon correlators on the coarse 32^3 lattice.

Correlator	Momentum (p)	$t_{\min}$	$t_{\max}$
Pion Wall-Point	0	5	63
Pion Wall-Wall	0	5	63
Pion Wall-Point	$\sqrt{2}\pi/L$	5	63
Pion Wall-Wall	$\sqrt{2}\pi/L$	5	35
Kaon Wall-Point	0	9	63
Kaon Wall-Wall	0	5	28
Kaon Wall-Point	$\sqrt{2}\pi/L$	7	45
Kaon Wall-Wall	$\sqrt{2}\pi/L$	5	31

Table 3: Fitting ranges for the pion and kaon correlators on the coarse 32^3 lattice.

Correlator	m	p	$\sqrt{m^2 + p^2}$	$E(p)$
Pion Wall-Point	0.10400(37)	$\sqrt{2}\pi/L$	0.17347(22)	0.1714(17)
Pion Wall-Wall	0.10400(38)	$\sqrt{2}\pi/L$	0.17347(23)	0.1717(17)
Kaon Wall-Point	0.3709(12)	$\sqrt{2}\pi/L$	0.3960(11)	0.3935(12)
Kaon Wall-Wall	0.3706(13)	$\sqrt{2}\pi/L$	0.3958(12)	0.3947(16)

Table 4: Check of the pion and kaon dispersion relation on the coarse 32^3 lattice.

Momentum (p)	A	E	C
0	$9.94(32) \times 10^{19}$	0.2100(10)	$3.43(23) \times 10^{14}$
$\sqrt{2}\pi/L$	$1.86(11) \times 10^{18}$	0.3687(61)	-

Table 5: Results of the fits to the two pion correlators on the coarse 32^3 lattice. The correlator with momentum $\sqrt{2}\pi/L$ was fit to Eq. 6.33 rather than Eq. 6.37 since the noise at larger times made it impossible to see the constant term in the latter.

Momentum (p)	$t_{\min}$	$t_{\max}$
0	5	63
$\sqrt{2}\pi/L$	5	16

Table 6: Fitting ranges for the two pion correlators on the coarse 32^3 lattice.

Momentum (p)	A	ΔE
0	1.6278(52)	0.00167(14)
$\sqrt{2}\pi/L$	1.407(32)	0.0101(26)

Table 7: Results of the fits to the ratio $C_{\pi\pi}/C_\pi^2$ on the coarse 32^3 lattice.

Momentum (p)	$t_{\min}$	$t_{\max}$
0	5	45
$\sqrt{2}\pi/L$	5	29

Table 8: Fitting ranges for the ratio $C_{\pi\pi}/C_\pi^2$ on the coarse 32^3 lattice.

Momentum (p)	$E_{\pi\pi}(p) =$ $\Delta E(p) + 2\sqrt{m_\pi^2 + p^2}$	$E_{\pi\pi}(p) =$ $\Delta E(p) + 2E_\pi(p)$
0	0.20968(76)	0.20968(76)
$\sqrt{2}\pi/L$	0.3571(25)	0.3536(37)

Table 9: Two pion energy on the coarse 32^3 lattice as obtained by fitting the ratio $C_{\pi\pi}/C_\pi^2$ for $\Delta E(p)$ and adding $2\sqrt{m_\pi^2 + p^2}$ or $2E_\pi(p)$. Values for $\Delta E(p)$ are found in Table 7, and values for $\sqrt{m_\pi^2 + p^2}$ and $E_\pi(p)$ are found in Table 4. Wall-wall pion correlators are used.

Weak Operator	Γ_A^μ	Γ_B^μ
$O_{(27,1)}^{\text{unmixed}}$	$\gamma^\mu(1 - \gamma^5)$	$\gamma^\mu(1 - \gamma^5)$
$O_{(8,8)}^{\text{unmixed}}$	$\gamma^\mu(1 - \gamma^5)$	$\gamma^\mu(1 + \gamma^5)$
$O_{(8,8)}^{\text{mixed}}$	$\gamma^\mu(1 - \gamma^5)$	$\gamma^\mu(1 + \gamma^5)$

Table 10: Gamma matrix structures for the different weak operators. (The mixed and unmixed version of the same operator have the same gamma matrix structure and only differ in how color and spin indices are contracted).

Weak Operator	p	t_K	$A_q \times 10^{18}$
(27,1)	0	20	6.86(21)
(27,1)	0	24	6.25(17)
(27,1)	0	28	6.24(19)
(27,1)	0	32	6.30(23)
(27,1)	0	36	6.56(26)
(27,1)	0	40	6.48(24)
(27,1)	0	44	6.33(21)
(8,8)	0	20	323.0(8.2)
(8,8)	0	24	313.6(9.0)
(8,8)	0	28	310.7(6.4)
(8,8)	0	32	318.6(8.9)
(8,8)	0	36	328.7(8.6)
(8,8)	0	40	330.4(7.7)
(8,8)	0	44	314.3(8.0)
(8,8) mixed	0	20	1047(27)
(8,8) mixed	0	24	1015(29)
(8,8) mixed	0	28	1005(21)
(8,8) mixed	0	32	1031(29)
(8,8) mixed	0	36	1063(28)
(8,8) mixed	0	40	1069(25)
(8,8) mixed	0	44	1014(26)

Table 11: Results of fitting the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) on the coarse 32^3 lattices. In this table we show results for 0 momentum. The two pions are at $t_\pi = 0$ and the kaon is at t_K , so that t_K is also the time separation between the two pions and the kaon.

Weak Operator	p	t_K	$A_q \times 10^{18}$
(27,1)	$\sqrt{2}\pi/L$	20	60.3(4.9)
(27,1)	$\sqrt{2}\pi/L$	24	60.5(3.8)
(27,1)	$\sqrt{2}\pi/L$	28	67.6(5.1)
(27,1)	$\sqrt{2}\pi/L$	32	53.6(8.3)
(27,1)	$\sqrt{2}\pi/L$	36	47.9(9.4)
(27,1)	$\sqrt{2}\pi/L$	40	79(14)
(27,1)	$\sqrt{2}\pi/L$	44	48(14)
(8,8)	$\sqrt{2}\pi/L$	20	1186(59)
(8,8)	$\sqrt{2}\pi/L$	24	1294(71)
(8,8)	$\sqrt{2}\pi/L$	28	1267(83)
(8,8)	$\sqrt{2}\pi/L$	32	1295(96)
(8,8)	$\sqrt{2}\pi/L$	36	1040(150)
(8,8)	$\sqrt{2}\pi/L$	40	1320(180)
(8,8)	$\sqrt{2}\pi/L$	44	860(140)
(8,8) mixed	$\sqrt{2}\pi/L$	20	4000(200)
(8,8) mixed	$\sqrt{2}\pi/L$	24	4330(230)
(8,8) mixed	$\sqrt{2}\pi/L$	28	4190(280)
(8,8) mixed	$\sqrt{2}\pi/L$	32	4300(320)
(8,8) mixed	$\sqrt{2}\pi/L$	36	3550(480)
(8,8) mixed	$\sqrt{2}\pi/L$	40	4390(590)
(8,8) mixed	$\sqrt{2}\pi/L$	44	2890(450)

Table 12: Results fitting the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) on the coarse 32^3 lattices. In this table we show results for $p = \sqrt{2}\pi/L$ (two twists). The two pions are at $t_\pi = 0$ and the kaon is at t_K , so that t_K is also the time separation between the two pions and the kaon.

m_l^{sea}	Configurations Analyzed (MD time units)
0.004	280, 320, 970, 1000, 1010, 1050, 1080, 1090, 1120, 1130, 1150, 1160, 1170, 1190, 1200, 1210, 1230, 1240, 1250, 1270, 1280, 1290, 1310 - 1650, 1670, 1680, 1690, 1710 - 1790, 1810, 1830, 1850, 1860, 1870, 1890, 1900, 1910, 1930, 1940, 1950, 1970 - 3300
0.006	530, 570, 610, 650, 690, 730, 770, 810, 850, 890, 930, 970, 1010, 1050, 1090, 1130, 1170, 1210, 1250, 1290, 1330, 1370, 1410, 1450, 1490, 1530, 1570, 1610, 1650, 1690 - 3610
0.008	260 - 2780

Table 13: List of configurations analyzed for the fine 32^3 ensembles with sea light quark masses $m_l^{\text{sea}}=0.004$, 0.006 , and 0.008 . Where ranges are given, for example 1310 - 1650, it refers to configurations in this range counting by units of 10.

m_l^{sea}	m_l^{val}	$A \times 10^{-9}$	E
0.004	0.002	18.71(22)	0.09791(37)
0.004	0.004	14.29(14)	0.12728(34)
0.004	0.006	12.28(11)	0.15086(32)
0.004	0.008	11.078(94)	0.17117(31)
0.004	0.025	7.880(59)	0.29348(25)
0.004	0.03	7.537(56)	0.32175(24)
0.006	0.002	19.40(24)	0.09943(38)
0.006	0.004	14.48(14)	0.12831(33)
0.006	0.006	12.32(11)	0.15163(31)
0.006	0.008	11.053(89)	0.17178(30)
0.006	0.025	7.756(56)	0.29369(26)
0.006	0.03	7.408(54)	0.32191(26)
0.008	0.002	19.47(24)	0.09989(37)
0.008	0.004	14.56(14)	0.12891(33)
0.008	0.006	12.38(10)	0.15234(31)
0.008	0.008	11.102(87)	0.17255(30)
0.008	0.025	7.744(50)	0.29442(27)
0.008	0.03	7.388(47)	0.32261(26)

Table 14: Results of the fits to the pion correlators on the fine 32^3 lattices. We show results for several different sea and valence light quark masses, as given in the first two columns.

m_l^{val}	m_s^{val}	$A \times 10^{-9}$	E
0.002	0.004	16.03(17)	0.11365(35)
0.002	0.006	14.40(15)	0.12744(35)
0.002	0.008	13.26(13)	0.13985(36)
0.002	0.025	9.373(98)	0.21864(46)
0.002	0.03	8.875(95)	0.23704(49)
0.004	0.006	13.14(12)	0.13961(33)
0.004	0.008	12.29(11)	0.15094(33)
0.004	0.025	9.089(82)	0.22571(37)
0.004	0.03	8.645(79)	0.24358(39)
0.006	0.008	11.61(10)	0.16136(32)
0.006	0.025	8.903(75)	0.23274(33)
0.006	0.03	8.502(72)	0.25013(34)
0.008	0.025	8.751(71)	0.23966(31)
0.008	0.03	8.386(69)	0.25662(31)
0.025	0.03	7.694(57)	0.30787(24)

Table 15: Results of the fits to the kaon correlators on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices. We show results for several different valence light and strange quark masses, as given in the first two columns.

m_l^{val}	m_s^{val}	$A \times 10^{-9}$	E
0.002	0.004	16.42(18)	0.11491(35)
0.002	0.006	14.65(15)	0.12855(35)
0.002	0.008	13.43(13)	0.14085(34)
0.002	0.025	9.328(93)	0.21905(40)
0.002	0.03	8.812(91)	0.23737(43)
0.004	0.006	13.25(12)	0.14051(32)
0.004	0.008	12.35(11)	0.15174(32)
0.004	0.025	9.001(78)	0.22604(35)
0.004	0.03	8.543(75)	0.24383(36)
0.006	0.008	11.616(97)	0.16204(31)
0.006	0.025	8.803(71)	0.23304(32)
0.006	0.03	8.391(69)	0.25037(33)
0.008	0.025	8.646(68)	0.23996(31)
0.008	0.03	8.273(66)	0.25687(32)
0.025	0.03	7.568(55)	0.30806(26)

Table 16: Results of the fits to the kaon correlators on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices. We show results for several different valence light and strange quark masses, as given in the first two columns.

m_l^{val}	m_s^{val}	$A \times 10^{-9}$	E
0.002	0.004	16.50(18)	0.11544(35)
0.002	0.006	14.73(15)	0.12914(34)
0.002	0.008	13.51(13)	0.14148(34)
0.002	0.025	9.420(87)	0.22010(41)
0.002	0.03	8.913(83)	0.23857(44)
0.004	0.006	13.32(12)	0.14116(32)
0.004	0.008	12.41(11)	0.15244(31)
0.004	0.025	9.081(73)	0.22716(35)
0.004	0.03	8.632(70)	0.24509(37)
0.006	0.008	11.673(95)	0.16278(30)
0.006	0.025	8.863(67)	0.23411(32)
0.006	0.03	8.457(64)	0.25155(33)
0.008	0.025	8.688(63)	0.24095(30)
0.008	0.03	8.318(61)	0.25794(31)
0.025	0.03	7.552(49)	0.30878(27)

Table 17: Results of the fits to the kaon correlators on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices. We show results for several different valence light and strange quark masses, as given in the first two columns.

m_l^{sea}	m_l^{val}	$A \times 10^{-19}$	E	C
0.004	0.002	55.0(1.4)	0.20044(84)	$4.33(24) \times 10^{15}$
0.004	0.004	32.85(66)	0.25895(73)	$6.04(30) \times 10^{13}$
0.004	0.006	24.46(44)	0.30600(67)	$2.24(11) \times 10^{12}$
0.004	0.008	19.97(34)	0.34647(63)	$1.384(72) \times 10^{11}$
0.004	0.025	9.92(15)	0.58994(50)	$1.159(53) \times 10^4$
0.004	0.03	9.02(13)	0.64626(48)	$2.87(13) \times 10^2$
0.006	0.002	59.7(1.6)	0.20352(85)	$4.07(27) \times 10^{15}$
0.006	0.004	33.79(67)	0.26074(69)	$5.68(34) \times 10^{13}$
0.006	0.006	24.61(43)	0.30717(64)	$2.12(12) \times 10^{12}$
0.006	0.008	19.87(33)	0.34732(62)	$1.314(74) \times 10^{11}$
0.006	0.025	9.63(15)	0.59028(55)	$1.103(54) \times 10^4$
0.006	0.03	8.73(13)	0.64653(54)	$2.70(12) \times 10^2$
0.008	0.002	60.2(1.6)	0.20415(86)	$3.90(22) \times 10^{15}$
0.008	0.004	34.20(67)	0.26178(71)	$5.38(28) \times 10^{13}$
0.008	0.006	24.87(43)	0.30843(66)	$1.964(98) \times 10^{12}$
0.008	0.008	20.03(32)	0.34869(63)	$1.200(58) \times 10^{11}$
0.008	0.025	9.56(12)	0.59149(56)	$1.012(45) \times 10^4$
0.008	0.03	8.65(11)	0.64769(55)	$2.53(11) \times 10^2$

Table 18: Results of the fits to the two pion correlators on the fine 32^3 lattices. We show results for several different sea and valence light quark masses, as given in the first two columns.

m_l^{val}	m_s^{val}	$A_q \times 10^{19}$ $m_l^{\text{sea}} = 0.004$	$A_q \times 10^{19}$ $m_l^{\text{sea}} = 0.006$	$A_q \times 10^{19}$ $m_l^{\text{sea}} = 0.008$
0.002	0.002	3.615(89)	3.426(83)	3.547(74)
0.002	0.004	3.972(97)	3.787(90)	3.884(78)
0.002	0.006	4.26(11)	4.088(97)	4.160(83)
0.002	0.008	4.52(12)	4.36(10)	4.402(88)
0.002	0.025	6.03(20)	6.14(18)	5.91(15)
0.002	0.03	6.39(22)	6.57(20)	6.26(17)
0.004	0.004	6.55(14)	6.50(14)	6.501(98)
0.004	0.006	6.87(14)	6.86(14)	6.83(10)
0.004	0.008	7.16(15)	7.19(15)	7.12(11)
0.004	0.025	9.02(22)	9.37(21)	9.03(15)
0.004	0.03	9.46(24)	9.90(23)	9.49(17)
0.006	0.006	9.38(18)	9.54(18)	9.40(12)
0.006	0.008	9.68(18)	9.89(18)	9.72(13)
0.006	0.025	11.73(24)	12.26(23)	11.84(17)
0.006	0.03	12.23(26)	12.84(25)	12.34(19)
0.008	0.008	12.19(22)	12.57(22)	12.29(15)
0.008	0.025	14.37(26)	15.01(27)	14.53(19)
0.008	0.03	14.91(28)	15.60(28)	15.06(21)
0.025	0.025	35.93(51)	36.57(53)	36.51(44)
0.025	0.03	36.48(52)	37.08(53)	37.10(45)
0.03	0.03	42.76(59)	43.47(59)	43.67(53)

Table 19: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (27,1) weak operator on the fine 32^3 lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for each of the three different sea light quark masses in each of the last three columns. The two pions always have zero momentum for this set of lattices.

m_l^{val}	m_s^{val}	$A_q \times 10^{17}$ $m^{\text{sea}} = 0.004$	$A_q \times 10^{17}$ $m^{\text{sea}} = 0.006$	$A_q \times 10^{17}$ $m^{\text{sea}} = 0.008$
0.002	0.002	2.375(47)	2.194(48)	2.278(45)
0.002	0.004	2.388(45)	2.231(45)	2.305(41)
0.002	0.006	2.387(44)	2.242(44)	2.306(39)
0.002	0.008	2.383(44)	2.247(43)	2.301(39)
0.002	0.025	2.371(52)	2.281(49)	2.274(43)
0.002	0.03	2.373(55)	2.295(52)	2.271(45)
0.004	0.004	2.336(41)	2.257(41)	2.300(35)
0.004	0.006	2.325(39)	2.265(40)	2.297(33)
0.004	0.008	2.315(39)	2.266(40)	2.289(32)
0.004	0.025	2.270(42)	2.268(40)	2.253(32)
0.004	0.03	2.264(43)	2.272(42)	2.248(34)
0.006	0.006	2.251(37)	2.237(39)	2.248(30)
0.006	0.008	2.241(36)	2.239(38)	2.241(29)
0.006	0.025	2.188(36)	2.223(35)	2.199(28)
0.006	0.03	2.179(37)	2.220(36)	2.191(28)
0.008	0.008	2.179(35)	2.208(37)	2.194(27)
0.008	0.025	2.125(33)	2.177(32)	2.148(25)
0.008	0.03	2.115(33)	2.169(32)	2.138(25)
0.025	0.025	1.847(25)	1.890(26)	1.880(21)
0.025	0.03	1.832(25)	1.874(25)	1.868(21)
0.03	0.03	1.780(24)	1.821(24)	1.822(21)

Table 20: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) weak operator on the fine 32^3 lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for each of the three different sea light quark masses in each of the last three columns. The two pions always have zero momentum for this set of lattices.

m_l^{val}	m_s^{val}	$A_q \times 10^{17}$ $m^{\text{sea}} = 0.004$	$A_q \times 10^{17}$ $m^{\text{sea}} = 0.006$	$A_q \times 10^{17}$ $m^{\text{sea}} = 0.008$
0.002	0.002	7.71(15)	7.14(16)	7.40(15)
0.002	0.004	7.77(15)	7.26(15)	7.50(13)
0.002	0.006	7.77(14)	7.31(14)	7.51(13)
0.002	0.008	7.76(14)	7.33(14)	7.50(13)
0.002	0.025	7.75(17)	7.47(16)	7.44(14)
0.002	0.03	7.76(18)	7.52(17)	7.43(15)
0.004	0.004	7.67(13)	7.42(14)	7.56(11)
0.004	0.006	7.64(13)	7.45(13)	7.55(11)
0.004	0.008	7.61(13)	7.46(13)	7.53(10)
0.004	0.025	7.51(14)	7.51(13)	7.46(11)
0.004	0.03	7.50(14)	7.53(14)	7.45(11)
0.006	0.006	7.47(12)	7.43(13)	7.46(10)
0.006	0.008	7.44(12)	7.44(13)	7.447(96)
0.006	0.025	7.32(12)	7.44(12)	7.363(92)
0.006	0.03	7.30(12)	7.44(12)	7.348(94)
0.008	0.008	7.31(12)	7.41(12)	7.362(91)
0.008	0.025	7.19(11)	7.37(11)	7.270(84)
0.008	0.03	7.17(11)	7.35(11)	7.250(85)
0.025	0.025	6.820(93)	6.981(94)	6.951(77)
0.025	0.03	6.785(91)	6.938(92)	6.927(77)
0.03	0.03	6.766(90)	6.919(91)	6.935(79)

Table 21: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) mixed weak operator on the fine 32^3 lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for each of the three different sea light quark masses in each of the last three columns. The two pions always have zero momentum for this set of lattices.

Configurations Analyzed (MD time units)
32000, 32500, 33500 - 37500, 38500, 39000, 41000, 41500, 43000, 43500
45000, 45500, 47000, 47500, 49000, 49500, 50000, 51000, 51500, 52000
53000, 53500, 54000, 55000, 55500, 56000, 57000, 57500, 58000, 59000
60000 - 66000

Table 22: List of configurations analyzed for the quenched 24^3 ensemble. Where a range is given, for example 33500 - 37500, it means that configurations in this range counting by units of 500 were analyzed.

m_l	p	$A \times 10^{-8}$	E
0.0023	0	61.0(2.3)	0.1279(10)
0.0023	π/L	58.6(2.9)	0.1843(20)
0.0023	$\sqrt{2}\pi/L$	24.0(1.6)	0.2346(41)
0.0023	$\sqrt{3}\pi/L$	2.95(28)	0.2622(84)
0.0047	0	39.03(90)	0.16407(74)
0.0047	π/L	40.8(1.0)	0.2101(12)
0.0047	$\sqrt{2}\pi/L$	16.44(55)	0.2515(25)
0.0047	$\sqrt{3}\pi/L$	2.19(13)	0.2823(48)
0.0071	0	30.89(57)	0.19294(66)
0.0071	π/L	33.97(64)	0.23295(98)
0.0071	$\sqrt{2}\pi/L$	13.82(32)	0.2697(19)
0.0071	$\sqrt{3}\pi/L$	1.875(84)	0.2984(37)

Table 23: Results of the fits to the pion correlators on the quenched 24^3 lattices. We show results for several different light quark masses and momenta, as given in the first two columns.

m_l	m_s	$A \times 10^{-9}$	E
0.0023	0.0460	1.909(46)	0.3325(14)
0.0023	0.0620	1.723(47)	0.3806(18)
0.0023	0.0780	1.598(49)	0.4234(23)
0.0023	0.0940	1.502(50)	0.4623(27)
0.0023	0.1100	1.424(50)	0.4982(30)
0.0047	0.0460	1.768(31)	0.33879(98)
0.0047	0.0620	1.598(32)	0.3857(13)
0.0047	0.0780	1.482(34)	0.4276(16)
0.0047	0.0940	1.394(35)	0.4659(19)
0.0047	0.1100	1.323(36)	0.5014(21)
0.0071	0.0460	1.699(27)	0.34517(81)
0.0071	0.0620	1.541(27)	0.3910(10)
0.0071	0.0780	1.430(28)	0.4321(13)
0.0071	0.0940	1.347(30)	0.4699(15)
0.0071	0.1100	1.279(31)	0.5049(17)

Table 24: Results of the fits to the zero momentum kaon correlators on the quenched 24^3 lattices. We show results for several different light and strange quark masses as given in the first two columns.

p	$A \times 10^{-8}$	E
0	17.23(47)	0.3806(18)
π/L	21.65(56)	0.4022(19)
$\sqrt{2}\pi/L$	9.16(25)	0.4224(23)
$\sqrt{3}\pi/L$	1.319(54)	0.4432(39)

Table 25: Results of the fits to the kaon correlators with $m_l=0.0023$ and $m_s=0.062$ for different momenta, on the quenched 24^3 lattices. This is the only quark mass combination for which non-zero momentum kaon correlators were analyzed.

p	$t_{\min}$	$t_{\max}$
0	5	63
π/L	5	63
$\sqrt{2}\pi/L$	5	30
$\sqrt{3}\pi/L$	5	30

Table 26: Fit ranges for the pion correlators on the quenched 24^3 lattices. These correspond to the results of Table 23. Fit ranges are kept the same for the different light quark masses $m_l=0.0023$, 0.0047 , and 0.0071 .

p	$t_{\min}$	$t_{\max}$
0	5	63
π/L	5	39
$\sqrt{2}\pi/L$	5	26
$\sqrt{3}\pi/L$	5	22

Table 27: Fit ranges for the kaon correlators with $m_l=0.0023$ and $m_s=0.062$ on the quenched 24^3 lattices. These correspond to the results of Table 25.

m_l	m_π	p	$\sqrt{m_\pi^2 + p^2}$	$E_\pi(p)$
0.0023	0.1279(10)	π/L	0.18300(70)	0.1843(20)
0.0023	0.1279(10)	$\sqrt{2}\pi/L$	0.22500(57)	0.2346(41)
0.0023	0.1279(10)	$\sqrt{3}\pi/L$	0.26031(49)	0.2622(84)
0.0047	0.16407(74)	π/L	0.20989(58)	0.2101(12)
0.0047	0.16407(74)	$\sqrt{2}\pi/L$	0.24736(49)	0.2515(25)
0.0047	0.16407(74)	$\sqrt{3}\pi/L$	0.27986(44)	0.2823(48)
0.0071	0.19294(66)	π/L	0.23315(55)	0.23295(98)
0.0071	0.19294(66)	$\sqrt{2}\pi/L$	0.26739(48)	0.2697(19)
0.0071	0.19294(66)	$\sqrt{3}\pi/L$	0.29771(43)	0.2984(37)

Table 28: Check of the dispersion relation (Eq. 6.34) for the pion correlators on the quenched 24^3 lattices. For each light quark mass we take the pion mass as computed from the zero momentum correlator (column 1) and compute the expected pion energy from the dispersion relation (column 4) for each non-zero momentum (column 3). Column 4 is to be compared with column 5, where the latter is the pion energy computed directly from the non-zero momentum correlator. We see a 2.3σ disagreement between the two columns for $m_l=0.0023$ and $p = \sqrt{2}\pi/L$, and a 1.7σ disagreement for $m_l=0.0047$ and $p = \sqrt{2}\pi/L$, but otherwise the values in columns 4 and 5 agree within error. The data in columns 2 and 5 are taken from Table 23.

m_K	p	$\sqrt{m_K^2 + p^2}$	$E_K(p)$
0.3806(18)	π/L	0.4025(17)	0.4022(19)
0.3806(18)	$\sqrt{2}\pi/L$	0.4232(16)	0.4224(23)
0.3806(18)	$\sqrt{3}\pi/L$	0.4430(16)	0.4432(39)

Table 29: Check of the dispersion relation (Eq. 6.34) for the kaon correlators with $m_l=0.0023$ and $m_s=0.062$ on the quenched 24^3 lattices. We take the kaon mass as computed from the zero momentum correlator (column 1) and compute the expected kaon energy from the dispersion relation (column 3) for each non-zero momentum (column 2). Column 3 is to be compared with column 4, where the latter is the kaon energy computed directly from the non-zero momentum correlator. Indeed the values in columns 3 and 4 all agree within error. The data in columns 1 and 4 are taken from Table 25.

m_l	p	$A \times 10^{-18}$	E	C
0.0023	0	59.2(4.9)	0.2598(20)	$1.11(19) \times 10^{13}$
0.0023	π/L	12.1(1.8)	0.3790(72)	—
0.0023	$\sqrt{2}\pi/L$	8.2(1.6)	0.491(16)	—
0.0023	$\sqrt{3}\pi/L$	3.2(1.3)	0.496(69)	—
0.0047	0	24.3(1.1)	0.3316(14)	$3.97(47) \times 10^{10}$
0.0047	π/L	5.83(35)	0.4292(30)	—
0.0047	$\sqrt{2}\pi/L$	3.61(33)	0.5188(79)	—
0.0047	$\sqrt{3}\pi/L$	1.89(39)	0.561(36)	—
0.0071	0	15.34(55)	0.3894(12)	$6.21(67) \times 10^8$
0.0071	π/L	4.11(16)	0.4749(20)	—
0.0071	$\sqrt{2}\pi/L$	2.61(16)	0.5552(56)	—
0.0071	$\sqrt{3}\pi/L$	1.51(23)	0.607(25)	—

Table 30: Results of the fits to the two pion correlators on the quenched 24^3 lattices. We show results for several different light quark masses and momenta, as given in the first two columns. For non-zero momenta we used a single exponential fit (Eq. 6.67) rather than the more complicated fit form of Eq. 6.37, so there are no C values for non-zero momenta.

p	$t_{\min}$	$t_{\max}$
0	5	63
π/L	5	39
$\sqrt{2}\pi/L$	5	17
$\sqrt{3}\pi/L$	5	10

Table 31: Fit ranges for the two pion correlators on the quenched 24^3 lattices. These correspond to the results of Table 30. Fit ranges are kept the same for the different light quark masses $m_l = 0.0023$, 0.0047 , and 0.0071 .

m_l	m_s	$A_q \times 10^{17}$ $\Delta t_K = 20$	$A_q \times 10^{17}$ $\Delta t_K = 24$	$A_q \times 10^{17}$ $\Delta t_K = 28$	$A_q \times 10^{17}$ $\Delta t_K = 32$
0.0023	0.0460	1.268(58)	1.341(67)	1.214(77)	1.378(60)
0.0023	0.0620	1.424(66)	1.506(78)	1.362(87)	1.569(72)
0.0023	0.0780	1.563(73)	1.651(88)	1.497(95)	1.734(84)
0.0023	0.0940	1.691(80)	1.783(98)	1.62(10)	1.881(94)
0.0023	0.1100	1.808(86)	1.90(11)	1.74(11)	2.02(10)
0.0047	0.0460	2.143(59)	2.315(80)	2.116(82)	2.284(69)
0.0047	0.0620	2.359(68)	2.562(92)	2.338(95)	2.542(82)
0.0047	0.0780	2.550(77)	2.78(10)	2.54(11)	2.767(97)
0.0047	0.0940	2.722(86)	2.98(12)	2.73(12)	2.97(11)
0.0047	0.1100	2.880(94)	3.16(13)	2.90(13)	3.15(12)
0.0071	0.0460	2.902(68)	3.126(92)	2.894(93)	3.052(76)
0.0071	0.0620	3.159(78)	3.42(11)	3.17(11)	3.347(90)
0.0071	0.0780	3.385(88)	3.68(12)	3.41(12)	3.60(11)
0.0071	0.0940	3.589(98)	3.91(13)	3.64(13)	3.83(12)
0.0071	0.1100	3.78(11)	4.13(14)	3.85(14)	4.04(13)

Table 32: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (27,1) weak operator with zero momentum on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{17}$ $\Delta t_K = 20$	$A_q \times 10^{17}$ $\Delta t_K = 24$	$A_q \times 10^{17}$ $\Delta t_K = 28$	$A_q \times 10^{17}$ $\Delta t_K = 32$
0.0023	0.0460	5.19(45)	5.84(48)	5.01(43)	5.68(45)
0.0023	0.0620	5.69(54)	6.44(54)	5.51(50)	6.50(60)
0.0023	0.0780	6.12(63)	6.98(60)	5.98(58)	7.21(75)
0.0023	0.0940	6.49(71)	7.49(66)	6.43(66)	7.82(92)
0.0023	0.1100	6.81(79)	7.95(72)	6.85(74)	8.4(1.1)
0.0047	0.0460	7.56(31)	8.41(37)	7.62(38)	8.00(32)
0.0047	0.0620	8.19(37)	9.22(43)	8.30(45)	8.80(39)
0.0047	0.0780	8.76(42)	9.95(48)	8.94(51)	9.46(46)
0.0047	0.0940	9.28(48)	10.62(54)	9.55(57)	10.00(54)
0.0047	0.1100	9.76(53)	11.25(59)	10.14(63)	10.46(64)
0.0071	0.0460	9.32(29)	10.11(35)	9.43(38)	9.68(28)
0.0071	0.0620	10.02(34)	10.96(40)	10.21(43)	10.47(33)
0.0071	0.0780	10.66(38)	11.73(46)	10.93(49)	11.12(38)
0.0071	0.0940	11.25(42)	12.43(51)	11.61(54)	11.66(44)
0.0071	0.1100	11.79(47)	13.08(56)	12.26(59)	12.13(51)

Table 33: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (27,1) weak operator with momentum $p = \pi/L$ (one twist) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{16}$ $\Delta t_K = 20$	$A_q \times 10^{16}$ $\Delta t_K = 24$	$A_q \times 10^{16}$ $\Delta t_K = 28$	$A_q \times 10^{16}$ $\Delta t_K = 32$
0.0023	0.0460	1.43(17)	1.63(18)	1.41(23)	1.55(33)
0.0023	0.0620	1.53(21)	1.82(22)	1.55(30)	1.77(48)
0.0023	0.0780	1.63(25)	1.98(28)	1.70(38)	2.01(67)
0.0023	0.0940	1.71(28)	2.12(37)	1.86(48)	2.26(85)
0.0023	0.1100	1.79(32)	2.27(47)	2.02(60)	2.5(1.0)
0.0047	0.0460	2.19(13)	2.37(14)	2.27(17)	2.43(17)
0.0047	0.0620	2.34(16)	2.55(17)	2.43(22)	2.66(23)
0.0047	0.0780	2.48(19)	2.71(20)	2.59(28)	2.84(31)
0.0047	0.0940	2.61(21)	2.86(24)	2.75(34)	2.97(38)
0.0047	0.1100	2.72(24)	2.99(28)	2.91(40)	3.07(46)
0.0071	0.0460	2.64(12)	2.78(12)	2.69(16)	2.85(11)
0.0071	0.0620	2.81(14)	2.97(15)	2.85(20)	3.07(15)
0.0071	0.0780	2.97(16)	3.13(17)	3.01(23)	3.24(19)
0.0071	0.0940	3.11(18)	3.29(20)	3.17(27)	3.37(24)
0.0071	0.1100	3.24(20)	3.43(23)	3.33(31)	3.46(30)

Table 34: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (27,1) weak operator with momentum $p = \sqrt{2}\pi/L$ (two twists) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{16}$ $\Delta t_K = 20$	$A_q \times 10^{16}$ $\Delta t_K = 24$	$A_q \times 10^{16}$ $\Delta t_K = 28$	$A_q \times 10^{16}$ $\Delta t_K = 32$
0.0023	0.0460	3.2(1.3)	2.7(1.0)	2.2(1.2)	3.2(1.5)
0.0023	0.0620	3.3(1.6)	2.9(1.3)	2.5(1.7)	3.9(1.7)
0.0023	0.0780	3.4(1.8)	3.1(1.7)	2.9(2.2)	4.4(2.0)
0.0023	0.0940	3.4(2.1)	3.3(2.2)	3.3(2.7)	4.8(2.3)
0.0023	0.1100	3.4(2.4)	3.3(2.7)	3.8(3.3)	5.0(2.8)
0.0047	0.0460	6.42(85)	5.63(90)	5.88(92)	5.8(1.1)
0.0047	0.0620	6.94(97)	6.0(1.1)	6.5(1.2)	6.5(1.4)
0.0047	0.0780	7.4(1.1)	6.3(1.3)	7.2(1.4)	7.2(1.7)
0.0047	0.0940	7.8(1.2)	6.5(1.6)	7.8(1.7)	7.9(2.1)
0.0047	0.1100	8.2(1.4)	6.6(2.0)	8.5(2.0)	8.4(2.6)
0.0071	0.0460	7.61(68)	7.18(80)	7.47(80)	7.32(81)
0.0071	0.0620	8.11(75)	7.58(92)	8.11(97)	8.0(1.0)
0.0071	0.0780	8.57(84)	7.9(1.1)	8.8(1.1)	8.7(1.3)
0.0071	0.0940	9.00(93)	8.3(1.2)	9.4(1.3)	9.4(1.6)
0.0071	0.1100	9.4(1.0)	8.6(1.4)	10.1(1.5)	10.0(1.9)

Table 35: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (27,1) weak operator with momentum $p = \sqrt{3}\pi/L$ (three twists) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{16}$ $\Delta t_K = 20$	$A_q \times 10^{16}$ $\Delta t_K = 24$	$A_q \times 10^{16}$ $\Delta t_K = 28$	$A_q \times 10^{16}$ $\Delta t_K = 32$
0.0023	0.0460	4.25(18)	4.33(24)	3.96(22)	4.38(19)
0.0023	0.0620	4.28(18)	4.37(24)	3.97(22)	4.46(21)
0.0023	0.0780	4.30(18)	4.40(25)	3.98(22)	4.52(22)
0.0023	0.0940	4.31(19)	4.43(25)	4.00(23)	4.56(23)
0.0023	0.1100	4.33(19)	4.46(25)	4.02(23)	4.59(25)
0.0047	0.0460	4.88(12)	5.06(16)	4.68(16)	5.02(16)
0.0047	0.0620	4.89(12)	5.09(16)	4.68(17)	5.07(17)
0.0047	0.0780	4.90(12)	5.13(17)	4.69(18)	5.12(18)
0.0047	0.0940	4.91(12)	5.16(18)	4.70(18)	5.15(19)
0.0047	0.1100	4.92(12)	5.19(18)	4.71(19)	5.17(20)
0.0071	0.0460	5.01(11)	5.21(14)	4.87(14)	5.15(14)
0.0071	0.0620	5.01(11)	5.24(14)	4.88(15)	5.19(15)
0.0071	0.0780	5.01(11)	5.26(15)	4.88(16)	5.22(16)
0.0071	0.0940	5.01(11)	5.29(15)	4.89(16)	5.24(16)
0.0071	0.1100	5.01(11)	5.31(16)	4.90(17)	5.26(17)

Table 36: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) weak operator with zero momentum on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{16}$ $\Delta t_K = 20$	$A_q \times 10^{16}$ $\Delta t_K = 24$	$A_q \times 10^{16}$ $\Delta t_K = 28$	$A_q \times 10^{16}$ $\Delta t_K = 32$
0.0023	0.0460	8.33(40)	8.70(75)	7.97(50)	8.45(49)
0.0023	0.0620	8.31(43)	8.71(80)	7.99(56)	8.60(54)
0.0023	0.0780	8.29(47)	8.71(84)	8.03(61)	8.70(61)
0.0023	0.0940	8.27(51)	8.72(87)	8.08(66)	8.78(68)
0.0023	0.1100	8.26(54)	8.73(90)	8.13(69)	8.84(77)
0.0047	0.0460	9.96(27)	10.41(46)	9.60(41)	10.02(38)
0.0047	0.0620	9.93(29)	10.42(48)	9.58(45)	10.08(41)
0.0047	0.0780	9.91(31)	10.43(51)	9.58(49)	10.10(45)
0.0047	0.0940	9.89(33)	10.45(53)	9.58(53)	10.11(49)
0.0047	0.1100	9.88(35)	10.47(55)	9.60(55)	10.12(53)
0.0071	0.0460	10.33(24)	10.77(36)	10.08(37)	10.45(31)
0.0071	0.0620	10.29(25)	10.76(37)	10.05(40)	10.48(33)
0.0071	0.0780	10.26(27)	10.76(38)	10.03(42)	10.49(35)
0.0071	0.0940	10.23(28)	10.77(40)	10.03(44)	10.49(38)
0.0071	0.1100	10.22(29)	10.78(41)	10.04(46)	10.49(40)

Table 37: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) weak operator with momentum $p = \pi/L$ (one twist) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{15}$ $\Delta t_K = 20$	$A_q \times 10^{15}$ $\Delta t_K = 24$	$A_q \times 10^{15}$ $\Delta t_K = 28$	$A_q \times 10^{15}$ $\Delta t_K = 32$
0.0023	0.0460	1.65(14)	1.67(18)	1.45(22)	1.77(30)
0.0023	0.0620	1.62(18)	1.66(22)	1.42(27)	1.92(39)
0.0023	0.0780	1.59(22)	1.66(26)	1.41(32)	2.04(48)
0.0023	0.0940	1.56(26)	1.66(30)	1.43(36)	2.16(56)
0.0023	0.1100	1.53(29)	1.67(34)	1.47(41)	2.27(64)
0.0047	0.0460	2.042(88)	2.11(11)	1.93(15)	2.15(15)
0.0047	0.0620	2.01(10)	2.09(12)	1.91(18)	2.22(17)
0.0047	0.0780	1.97(12)	2.08(14)	1.90(20)	2.26(20)
0.0047	0.0940	1.94(14)	2.08(16)	1.89(23)	2.30(23)
0.0047	0.1100	1.92(15)	2.08(18)	1.89(25)	2.33(26)
0.0071	0.0460	2.141(70)	2.203(95)	2.07(12)	2.248(89)
0.0071	0.0620	2.107(77)	2.188(99)	2.05(13)	2.28(10)
0.0071	0.0780	2.076(85)	2.18(11)	2.04(15)	2.30(12)
0.0071	0.0940	2.050(93)	2.18(11)	2.03(16)	2.32(14)
0.0071	0.1100	2.03(10)	2.17(12)	2.03(17)	2.33(16)

Table 38: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) weak operator with momentum $p = \sqrt{2}\pi/L$ (two twists) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{15}$ $\Delta t_K = 20$	$A_q \times 10^{15}$ $\Delta t_K = 24$	$A_q \times 10^{15}$ $\Delta t_K = 28$	$A_q \times 10^{15}$ $\Delta t_K = 32$
0.0023	0.0460	3.26(92)	2.81(69)	2.6(1.5)	3.9(1.3)
0.0023	0.0620	3.2(1.1)	2.99(84)	2.9(1.9)	4.3(2.0)
0.0023	0.0780	3.0(1.2)	3.2(1.0)	3.3(2.8)	4.5(2.5)
0.0023	0.0940	2.9(1.4)	3.3(1.2)	4.3(3.5)	4.2(3.0)
0.0023	0.1100	2.7(1.5)	3.4(1.4)	4.9(4.1)	3.6(3.3)
0.0047	0.0460	4.29(53)	4.32(45)	3.83(74)	5.00(81)
0.0047	0.0620	4.20(59)	4.39(53)	3.89(85)	5.22(97)
0.0047	0.0780	4.12(66)	4.47(61)	4.00(97)	5.4(1.2)
0.0047	0.0940	4.05(75)	4.52(69)	4.2(1.1)	5.6(1.4)
0.0047	0.1100	4.00(84)	4.56(79)	4.4(1.3)	5.8(1.7)
0.0071	0.0460	4.62(37)	4.77(34)	4.21(46)	5.11(52)
0.0071	0.0620	4.54(40)	4.79(38)	4.19(53)	5.19(61)
0.0071	0.0780	4.47(44)	4.81(42)	4.23(60)	5.26(73)
0.0071	0.0940	4.41(48)	4.82(47)	4.31(67)	5.34(87)
0.0071	0.1100	4.37(53)	4.82(52)	4.39(76)	5.5(1.0)

Table 39: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) weak operator with momentum $p = \sqrt{3}\pi/L$ (three twists) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{15}$ $\Delta t_K = 20$	$A_q \times 10^{15}$ $\Delta t_K = 24$	$A_q \times 10^{15}$ $\Delta t_K = 28$	$A_q \times 10^{15}$ $\Delta t_K = 32$
0.0023	0.0460	1.416(60)	1.450(80)	1.325(74)	1.464(62)
0.0023	0.0620	1.428(61)	1.466(81)	1.331(74)	1.494(67)
0.0023	0.0780	1.436(63)	1.480(83)	1.339(75)	1.515(71)
0.0023	0.0940	1.443(64)	1.493(84)	1.347(76)	1.532(76)
0.0023	0.1100	1.450(65)	1.505(84)	1.356(77)	1.546(80)
0.0047	0.0460	1.648(39)	1.713(53)	1.585(55)	1.699(55)
0.0047	0.0620	1.657(40)	1.730(55)	1.591(57)	1.724(59)
0.0047	0.0780	1.664(40)	1.746(58)	1.598(60)	1.742(62)
0.0047	0.0940	1.670(41)	1.760(60)	1.606(62)	1.757(64)
0.0047	0.1100	1.675(42)	1.773(63)	1.614(64)	1.770(66)
0.0071	0.0460	1.712(36)	1.784(46)	1.669(49)	1.763(49)
0.0071	0.0620	1.719(36)	1.799(48)	1.676(51)	1.783(51)
0.0071	0.0780	1.724(37)	1.813(50)	1.684(54)	1.799(54)
0.0071	0.0940	1.729(38)	1.826(52)	1.691(56)	1.811(56)
0.0071	0.1100	1.733(39)	1.838(55)	1.699(57)	1.821(58)

Table 40: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) mixed weak operator with zero momentum on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{15}$ $\Delta t_K = 20$	$A_q \times 10^{15}$ $\Delta t_K = 24$	$A_q \times 10^{15}$ $\Delta t_K = 28$	$A_q \times 10^{15}$ $\Delta t_K = 32$
0.0023	0.0460	2.86(14)	3.00(26)	2.75(17)	2.96(16)
0.0023	0.0620	2.86(15)	3.01(28)	2.77(19)	3.03(18)
0.0023	0.0780	2.86(16)	3.03(29)	2.79(21)	3.09(20)
0.0023	0.0940	2.86(18)	3.04(31)	2.82(23)	3.14(23)
0.0023	0.1100	2.86(19)	3.06(32)	2.86(24)	3.20(26)
0.0047	0.0460	3.458(93)	3.63(16)	3.35(14)	3.51(13)
0.0047	0.0620	3.46(10)	3.65(17)	3.36(16)	3.55(14)
0.0047	0.0780	3.47(11)	3.67(18)	3.37(18)	3.58(16)
0.0047	0.0940	3.47(11)	3.70(19)	3.38(19)	3.61(17)
0.0047	0.1100	3.47(12)	3.72(20)	3.40(20)	3.63(19)
0.0071	0.0460	3.631(84)	3.80(13)	3.55(13)	3.69(11)
0.0071	0.0620	3.632(88)	3.82(13)	3.56(14)	3.72(12)
0.0071	0.0780	3.633(92)	3.84(14)	3.57(15)	3.74(12)
0.0071	0.0940	3.637(97)	3.85(14)	3.58(16)	3.76(13)
0.0071	0.1100	3.64(10)	3.87(15)	3.59(17)	3.77(14)

Table 41: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) mixed weak operator with momentum $p = \pi/L$ (one twist) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{15}$ $\Delta t_K = 20$	$A_q \times 10^{15}$ $\Delta t_K = 24$	$A_q \times 10^{15}$ $\Delta t_K = 28$	$A_q \times 10^{15}$ $\Delta t_K = 32$
0.0023	0.0460	5.79(47)	5.78(63)	5.01(70)	6.1(1.0)
0.0023	0.0620	5.73(57)	5.77(73)	4.86(84)	6.5(1.3)
0.0023	0.0780	5.64(69)	5.83(86)	4.78(98)	6.9(1.6)
0.0023	0.0940	5.55(82)	5.93(99)	4.8(1.1)	7.2(1.9)
0.0023	0.1100	5.47(95)	6.1(1.1)	4.8(1.3)	7.5(2.2)
0.0047	0.0460	7.26(31)	7.43(39)	6.87(49)	7.69(52)
0.0047	0.0620	7.18(35)	7.41(43)	6.78(59)	7.94(62)
0.0047	0.0780	7.09(41)	7.43(49)	6.71(68)	8.11(74)
0.0047	0.0940	7.01(47)	7.46(56)	6.64(77)	8.22(85)
0.0047	0.1100	6.94(53)	7.51(63)	6.59(86)	8.29(96)
0.0071	0.0460	7.70(26)	7.91(33)	7.46(41)	8.15(32)
0.0071	0.0620	7.62(28)	7.90(35)	7.42(46)	8.31(37)
0.0071	0.0780	7.55(31)	7.92(37)	7.38(52)	8.39(44)
0.0071	0.0940	7.49(34)	7.94(41)	7.34(57)	8.44(51)
0.0071	0.1100	7.43(37)	7.99(45)	7.30(62)	8.46(58)

Table 42: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) mixed weak operator with momentum $p = \sqrt{2}\pi/L$ (two twists) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

m_l	m_s	$A_q \times 10^{15}$ $\Delta t_K = 20$	$A_q \times 10^{15}$ $\Delta t_K = 24$	$A_q \times 10^{15}$ $\Delta t_K = 28$	$A_q \times 10^{15}$ $\Delta t_K = 32$
0.0023	0.0460	11.8(3.2)	9.0(2.4)	9.3(5.8)	12.8(5.3)
0.0023	0.0620	11.6(3.7)	9.2(3.0)	11.7(6.8)	13.5(7.9)
0.0023	0.0780	11.3(4.2)	9.4(3.7)	13.6(7.6)	15.4(9.4)
0.0023	0.0940	10.8(4.7)	9.6(4.5)	15.6(8.9)	16(11)
0.0023	0.1100	10.3(5.3)	9.7(5.3)	17(10)	15(13)
0.0047	0.0460	15.9(1.9)	15.4(1.5)	13.9(2.3)	17.7(3.1)
0.0047	0.0620	15.8(2.1)	15.5(1.8)	13.9(2.7)	18.3(3.8)
0.0047	0.0780	15.6(2.3)	15.7(2.1)	14.1(3.0)	18.6(4.5)
0.0047	0.0940	15.5(2.6)	15.8(2.4)	14.4(3.5)	18.8(5.4)
0.0047	0.1100	15.4(2.9)	15.8(2.7)	14.6(4.0)	18.9(6.4)
0.0071	0.0460	17.3(1.3)	17.5(1.2)	15.5(1.6)	19.0(1.9)
0.0071	0.0620	17.1(1.4)	17.5(1.3)	15.3(1.8)	19.3(2.2)
0.0071	0.0780	17.0(1.6)	17.6(1.5)	15.2(2.0)	19.5(2.6)
0.0071	0.0940	16.9(1.7)	17.7(1.6)	15.3(2.3)	19.6(3.1)
0.0071	0.1100	16.8(1.9)	17.7(1.8)	15.3(2.6)	19.9(3.7)

Table 43: Results of the fits to the quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ to a constant (A_q) for the (8,8) mixed weak operator with momentum $p = \sqrt{3}\pi/L$ (three twists) on the quenched 24^3 lattices. We have results for several different light and strange quark masses, as given in the first two columns. For each light and strange quark mass we show results for four different separations Δt_K between the kaon and two pions in the last four columns. Since we are still measuring the same quantity, the values for the different separations should agree with each other.

Number of twists	0	2
p (lattice units)	0	$\sqrt{2}\pi/L = 0.13884$
p (MeV)	0	190.7(3.3)
$E_{\pi\pi}$ (lattice units)	0.20968(76)	0.3536(37)
$E_{\pi\pi}$ (MeV)	288.0(5.0)	485.7(8.0)
k (lattice units)	0.01322(56)	0.1430(23)
k (MeV)	18.15(81)	196.4(3.7)
q	0.0673(28)	0.728(12)
δ_0 (degrees)	-0.332(41)	-6.0(3.3)

Table 44: Two pion energies in physical units and s wave phase shifts from the 32^3 coarse lattices. Results are given for $p = 0$ (zero twists) and $p = \sqrt{2}\pi/L$ (two twists). The quantity $p = \sqrt{n_{\text{tw}}}\pi/L$ is the momentum of a single meson with the given number of twists n_{tw} , whereas k is determined from the two pion energy through $E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$. Due to the interaction of the two pions k and p are not equal, although they are very close in value. The difference between the two is measurable and is what allows us to calculate the phase shift on the lattice using the Lüscher relation.

i	$y_i(2 \text{ GeV})$	$z_i(2 \text{ GeV})$	$\text{Re}(C_i(2 \text{ GeV}))$	$\text{Im}(C_i(2 \text{ GeV}))$
1	3.6927×10^{-16}	-0.264162	-0.264162	-2.19167×10^{-19}
2	-2.99278×10^{-16}	1.12578	1.12578	1.77625×10^{-19}
7	-0.000198084	6.37236×10^{-5}	6.3433×10^{-5}	1.17566×10^{-7}
8	0.000977582	-9.24209×10^{-5}	-9.09867×10^{-5}	-5.80208×10^{-7}
9	-0.0106803	1.8324×10^{-5}	2.65511×10^{-6}	6.33892×10^{-6}
10	0.0027158	5.70894×10^{-5}	6.10737×10^{-5}	-1.61187×10^{-6}

Table 45: Wilson coefficients in the RI-MOM scheme needed for the 32^3 coarse and fine lattice calculations, linearly interpolated to $\mu = 2 \text{ GeV}$ from values at $\mu = 1.92 \text{ GeV}$ and $\mu = 2.15 \text{ GeV}$ which were computed in [15]. Here $C_i(\mu) = z_i(\mu) + \tau y_i(\mu)$ where the value of τ is found in Table 48.

J	$\text{Re}(C_J(2 \text{ GeV}))$	$\text{Im}(C_J(2 \text{ GeV}))$
(27,1)	0.287238	2.36353×10^{-6}
(8,8)	3.17165×10^{-5}	5.87829×10^{-8}
(8,8) mixed	-4.54933×10^{-5}	-2.90104×10^{-7}

Table 46: Wilson coefficients in the RI-MOM scheme for the (27,1), (8,8), and (8,8) mixed operators calculated at $\mu = 2 \text{ GeV}$. These are computed from the $C_i(\mu)$ of Table 45 using Eq. 4.33 - 4.35, and are to be used to obtain final results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ using Eq. A.50.

Number of twists	0	2
q	0.0673(28)	0.728(12)
$q\partial\phi/\partial q$	0.0169(20)	3.637(83)
$k\partial\delta_0/\partial k$	-0.00579(71)	-0.2125(34)
$\sqrt{\text{LL factor}}$	44.44(22)	36.94(11)
$\sqrt{\frac{\text{LL factor}}{\text{LL free field}}}$	0.9616(31)	0.948(12)

Table 47: Lellouch-Lüscher factor for the 32^3 coarse lattices. We also show two terms, $q\partial\phi/\partial q$ and $k\partial\delta_0/\partial k$, that are added together to form one of the factors in the Lellouch-Lüscher factor to see how much each contributes. Note that for two twists we obtain $\partial\delta_0/\partial k$ from the phenomenological fit to the two pion scattering experimental data, and thus the error quoted for $k\partial\delta_0/\partial k$ is only due to k itself.

Quantity	Value
G_F	$1.16637 \times 10^{-5} \text{ GeV}^{-2}$
V_{ud}	0.97428
$V_{us}^* = V_{us}$	0.2253
$\tau = -V_{td}V_{ts}^*/(V_{ud}V_{us}^*)$	$0.0014671 - 0.00059351i$

Table 48: Physical constants needed to calculate A_2 . Values are taken from the PDG [5].

Weak Operator	p	t_K	$ \mathcal{M} $
(27,1)	0	20	0.00342(10)
(27,1)	0	24	0.003117(97)
(27,1)	0	28	0.003108(87)
(27,1)	0	32	0.00314(12)
(8,8)	0	20	0.1610(38)
(8,8)	0	24	0.1563(45)
(8,8)	0	28	0.1548(33)
(8,8)	0	32	0.1588(48)
(8,8) mixed	0	20	0.522(12)
(8,8) mixed	0	24	0.506(14)
(8,8) mixed	0	28	0.501(10)
(8,8) mixed	0	32	0.514(15)

Table 49: Lattice matrix elements with zero twists ($p = 0$) for the 32^3 coarse lattices. We show results for the different weak operators and different separations t_K between the kaon and two pions.

Weak Operator	p	t_K	$ \mathcal{M} $
(27,1)	$\sqrt{2}\pi/L$	20	0.00387(28)
(27,1)	$\sqrt{2}\pi/L$	24	0.00388(26)
(27,1)	$\sqrt{2}\pi/L$	28	0.00434(31)
(27,1)	$\sqrt{2}\pi/L$	32	0.00344(55)
(8,8)	$\sqrt{2}\pi/L$	20	0.0761(33)
(8,8)	$\sqrt{2}\pi/L$	24	0.0830(51)
(8,8)	$\sqrt{2}\pi/L$	28	0.0813(53)
(8,8)	$\sqrt{2}\pi/L$	32	0.0831(62)
(8,8) mixed	$\sqrt{2}\pi/L$	20	0.257(11)
(8,8) mixed	$\sqrt{2}\pi/L$	24	0.278(17)
(8,8) mixed	$\sqrt{2}\pi/L$	28	0.268(18)
(8,8) mixed	$\sqrt{2}\pi/L$	32	0.276(21)

Table 50: Lattice matrix elements with two twists ($p = \sqrt{2}\pi/L$) for the 32^3 coarse lattices. We show results for the different weak operators and different separations t_K between the kaon and two pions.

p	t_K	$\text{Re}(A_2) \times 10^8 \text{ (GeV)}$	$\text{Im}(A_2) \times 10^{13} \text{ (GeV)}$
0	20	0.750(44)	-11.01(63)
0	24	0.683(39)	-10.71(62)
0	28	0.682(40)	-10.60(63)
0	32	0.688(45)	-10.89(69)
0	Avg.	0.699(39)	-10.78(61)
$\sqrt{2}\pi/L$	20	1.43(12)	-8.36(54)
$\sqrt{2}\pi/L$	24	1.43(11)	-9.13(65)
$\sqrt{2}\pi/L$	28	1.60(14)	-8.64(63)
$\sqrt{2}\pi/L$	32	1.27(20)	-9.20(81)
$\sqrt{2}\pi/L$	Avg.	1.461(87)	-8.67(45)

Table 51: Results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ for different separations t_K between the kaon and two pions, on the 32^3 coarse lattices. We show results for both $p = 0$ and $p = \sqrt{2}\pi/L$ (zero and two twists). Results for different t_K should agree with each other, since the same quantity is being measured. We therefore take an error weighted average of the results for different t_K , which is labeled “Avg.” in the table.

p	Weak Operator	$\text{Re}(A_2) \times 10^8 \text{ (GeV)}$	$\text{Im}(A_2) \times 10^{13} \text{ (GeV)}$
0	(27,1)	0.713(40)	0.587(33)
0	(8,8)	0.00410(23)	0.759(43)
0	(8,8) mixed	-0.0190(11)	-12.13(68)
0	Tot.	0.699(39)	-10.78(61)
$\sqrt{2}\pi/L$	(27,1)	1.474(88)	1.213(72)
$\sqrt{2}\pi/L$	(8,8)	0.00344(17)	0.637(32)
$\sqrt{2}\pi/L$	(8,8) mixed	-0.01650(84)	-10.52(53)
$\sqrt{2}\pi/L$	Tot.	1.461(87)	-8.67(45)

Table 52: Contributions to $\text{Re}(A_2)$ and $\text{Im}(A_2)$ from the different weak operators for the 32^3 coarse lattices. We show results for both $p = 0$ and $p = \sqrt{2}\pi/L$ (zero and two twists).

m_l^{sea}	m_l^{val}	m_π (MeV)
0.004	0.002	223.7(2.9)
0.004	0.004	290.8(3.7)
0.004	0.006	344.7(4.4)
0.004	0.008	391.1(5.0)
0.004	0.025	670.6(8.4)
0.004	0.03	735.2(9.2)
0.006	0.002	227.2(3.0)
0.006	0.004	293.2(3.7)
0.006	0.006	346.5(4.4)
0.006	0.008	392.5(5.0)
0.006	0.025	671.1(8.4)
0.006	0.03	735.5(9.2)
0.008	0.002	228.2(3.0)
0.008	0.004	294.6(3.8)
0.008	0.006	348.1(4.4)
0.008	0.008	394.3(5.0)
0.008	0.025	672.7(8.4)
0.008	0.03	737.2(9.3)

Table 53: Pion mass in MeV on the fine 32^3 lattices. We show results for several different sea and valence light quark masses, as given in the first two columns.

m_l^{val}	m_s^{val}	m_K (MeV) $m_l^{\text{sea}} = 0.004$	m_K (MeV) $m_l^{\text{sea}} = 0.006$	m_K (MeV) $m_l^{\text{sea}} = 0.008$
0.002	0.004	259.7(3.4)	262.6(3.4)	263.8(3.4)
0.002	0.006	291.2(3.7)	293.7(3.8)	295.1(3.8)
0.002	0.008	319.5(4.1)	321.8(4.1)	323.3(4.1)
0.002	0.025	499.6(6.3)	500.5(6.3)	502.9(6.4)
0.002	0.03	541.6(6.9)	542.4(6.9)	545.1(6.9)
0.004	0.006	319.0(4.1)	321.1(4.1)	322.5(4.1)
0.004	0.008	344.9(4.4)	346.7(4.4)	348.3(4.4)
0.004	0.025	515.7(6.5)	516.5(6.5)	519.0(6.6)
0.004	0.03	556.6(7.0)	557.1(7.0)	560.0(7.1)
0.006	0.008	368.7(4.7)	370.2(4.7)	371.9(4.7)
0.006	0.025	531.8(6.7)	532.5(6.7)	534.9(6.7)
0.006	0.03	571.5(7.2)	572.1(7.2)	574.8(7.2)
0.008	0.025	547.6(6.9)	548.3(6.9)	550.6(6.9)
0.008	0.03	586.4(7.4)	586.9(7.4)	589.4(7.4)
0.025	0.03	703.5(8.8)	703.9(8.8)	705.5(8.9)

Table 54: Kaon mass in MeV on the fine 32^3 lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for the three different sea light quark masses $m_l^{\text{sea}}=0.004$, 0.006, and 0.008 in each of the last three columns respectively.

m_l^{sea}	m_l^{val}	$E_{\pi\pi}$ (lat. units)	$E_{\pi\pi}$ (MeV)	k (lat. units)	k (MeV)
0.004	0.002	0.20044(84)	458.0(6.1)	0.0214(10)	48.9(2.4)
0.004	0.004	0.25895(73)	591.7(7.6)	0.02374(80)	54.2(1.9)
0.004	0.006	0.30600(67)	699.2(8.9)	0.02546(72)	58.2(1.8)
0.004	0.008	0.34647(63)	792(10)	0.02666(69)	60.9(1.8)
0.004	0.025	0.58994(50)	1348(17)	0.02962(64)	67.7(1.7)
0.004	0.03	0.64626(48)	1477(19)	0.02985(64)	68.2(1.7)
0.006	0.002	0.20352(85)	465.0(6.1)	0.0216(10)	49.5(2.4)
0.006	0.004	0.26074(69)	595.8(7.6)	0.02306(78)	52.7(1.9)
0.006	0.006	0.30717(64)	701.9(8.9)	0.02443(70)	55.8(1.7)
0.006	0.008	0.34732(62)	794(10)	0.02550(66)	58.3(1.7)
0.006	0.025	0.59028(55)	1349(17)	0.02923(70)	66.8(1.8)
0.006	0.03	0.64653(54)	1477(19)	0.02956(73)	67.5(1.9)
0.008	0.002	0.20415(86)	466.5(6.2)	0.0210(10)	48.0(2.4)
0.008	0.004	0.26178(71)	598.1(7.7)	0.02267(80)	51.8(1.9)
0.008	0.006	0.30843(66)	704.7(9.0)	0.02397(71)	54.8(1.8)
0.008	0.008	0.34869(63)	797(10)	0.02497(65)	57.0(1.6)
0.008	0.025	0.59149(56)	1352(17)	0.02798(59)	63.9(1.6)
0.008	0.03	0.64769(55)	1480(19)	0.02820(62)	64.4(1.6)

Table 55: Two pion energies in physical units from the fine 32^3 lattices. We show results for several different sea and valence light quark masses, as given in the first two columns. All results on these lattices have $p = 0$ (zero twists).

m_l^{sea}	m_l^{val}	q	δ_0 (deg.)
0.004	0.002	0.1090(53)	-1.33(18)
0.004	0.004	0.1209(41)	-1.77(17)
0.004	0.006	0.1297(37)	-2.15(17)
0.004	0.008	0.1358(35)	-2.44(17)
0.004	0.025	0.1509(33)	-3.25(19)
0.004	0.03	0.1521(33)	-3.32(19)
0.006	0.002	0.1102(53)	-1.37(18)
0.006	0.004	0.1174(40)	-1.64(15)
0.006	0.006	0.1244(36)	-1.92(15)
0.006	0.008	0.1299(33)	-2.16(15)
0.006	0.025	0.1489(36)	-3.13(20)
0.006	0.03	0.1506(37)	-3.23(21)
0.008	0.002	0.1070(51)	-1.26(17)
0.008	0.004	0.1154(41)	-1.56(15)
0.008	0.006	0.1221(36)	-1.82(15)
0.008	0.008	0.1272(33)	-2.04(15)
0.008	0.025	0.1425(30)	-2.79(16)
0.008	0.03	0.1436(32)	-2.85(17)

Table 56: The s wave phase shifts from the fine 32^3 lattices. We show results for several different sea and valence light quark masses, as given in the first two columns. All results on these lattices have $p = 0$ (zero twists).

m_l^{sea}	m_l^{val}	q	$q\partial\phi/\partial q$	$k\partial\delta_0/\partial k$	$\sqrt{\frac{\text{LL factor}}{\text{LL free field}}}$
0.004	0.002	0.1090(53)	0.0652(85)	-0.0232(31)	0.9069(79)
0.004	0.004	0.1209(41)	0.0860(76)	-0.0309(29)	0.8885(64)
0.004	0.006	0.1297(37)	0.1034(77)	-0.0376(29)	0.8744(60)
0.004	0.008	0.1358(35)	0.1165(78)	-0.0426(30)	0.8644(58)
0.004	0.025	0.1509(33)	0.1522(82)	-0.0567(33)	0.8391(55)
0.004	0.03	0.1521(33)	0.1552(84)	-0.0579(34)	0.8371(56)
0.006	0.002	0.1102(53)	0.0672(86)	-0.0239(32)	0.9050(79)
0.006	0.004	0.1174(40)	0.0796(72)	-0.0285(27)	0.8940(62)
0.006	0.006	0.1244(36)	0.0928(70)	-0.0335(27)	0.8829(57)
0.006	0.008	0.1299(33)	0.1038(69)	-0.0377(27)	0.8741(54)
0.006	0.025	0.1489(36)	0.1472(88)	-0.0547(35)	0.8425(60)
0.006	0.03	0.1506(37)	0.1514(93)	-0.0564(37)	0.8397(63)
0.008	0.002	0.1070(51)	0.0621(80)	-0.0220(29)	0.9098(75)
0.008	0.004	0.1154(41)	0.0761(72)	-0.0272(27)	0.8971(63)
0.008	0.006	0.1221(36)	0.0883(69)	-0.0318(26)	0.8867(57)
0.008	0.008	0.1272(33)	0.0982(67)	-0.0356(26)	0.8785(53)
0.008	0.025	0.1425(30)	0.1318(71)	-0.0486(28)	0.8533(51)
0.008	0.03	0.1436(32)	0.1345(76)	-0.0497(30)	0.8514(53)

Table 57: Lellouch-Lüscher factor over free field normalization factor for the 32^3 fine lattices. We also show two terms, $q\partial\phi/\partial q$ and $k\partial\delta_0/\partial k$, that are added together to form one of the factors in the Lellouch-Lüscher factor to see how much each contributes. Results are given for the various valence and sea light quark masses as given in the first two columns. All of these results only depend on k and thus are independent of the valence strange quark mass.

m_l^{val}	m_s^{val}	$\sqrt{\text{LL factor}}$ $m_l^{\text{sea}} = 0.004$	$\sqrt{\text{LL factor}}$ $m_l^{\text{sea}} = 0.006$	$\sqrt{\text{LL factor}}$ $m_l^{\text{sea}} = 0.008$
0.002	0.002	20.59(17)	21.03(18)	21.25(17)
0.002	0.004	22.19(18)	22.61(19)	22.85(18)
0.002	0.006	23.49(19)	23.91(20)	24.17(19)
0.002	0.008	24.61(20)	25.03(21)	25.29(19)
0.002	0.025	30.77(24)	31.21(25)	31.55(23)
0.002	0.03	32.04(25)	32.49(26)	32.85(24)
0.004	0.004	29.72(21)	30.23(22)	30.52(21)
0.004	0.006	31.12(22)	31.63(23)	31.94(22)
0.004	0.008	32.36(23)	32.87(23)	33.19(22)
0.004	0.025	39.57(27)	40.12(27)	40.52(27)
0.004	0.03	41.11(28)	41.67(28)	42.09(28)
0.006	0.006	37.63(26)	38.24(25)	38.64(24)
0.006	0.008	38.91(27)	39.53(26)	39.94(25)
0.006	0.025	46.73(32)	47.40(30)	47.90(29)
0.006	0.03	48.45(33)	49.13(31)	49.66(30)
0.008	0.008	44.86(31)	45.56(29)	46.07(27)
0.008	0.025	53.08(36)	53.84(33)	54.44(32)
0.008	0.03	54.93(37)	55.71(35)	56.33(33)
0.025	0.025	97.09(65)	97.58(68)	99.14(57)
0.025	0.03	99.45(66)	99.94(70)	101.53(58)
0.03	0.03	111.10(74)	111.51(81)	113.39(68)

Table 58: Lellouch-Lüscher factor for the 32^3 fine lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for the three different sea light quark masses $m_l^{\text{sea}}=0.004$, 0.006, and 0.008 in each of the last three columns respectively.

m_l^{val}	m_s^{val}	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.004$	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.006$	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.008$
0.002	0.002	0.001159(30)	0.001166(31)	0.001215(30)
0.002	0.004	0.001179(31)	0.001186(31)	0.001224(29)
0.002	0.006	0.001199(32)	0.001209(31)	0.001239(30)
0.002	0.008	0.001219(33)	0.001234(32)	0.001256(30)
0.002	0.025	0.001369(47)	0.001448(44)	0.001407(40)
0.002	0.03	0.001411(52)	0.001507(49)	0.001450(44)
0.004	0.004	0.001420(31)	0.001439(32)	0.001450(26)
0.004	0.006	0.001428(32)	0.001452(32)	0.001457(26)
0.004	0.008	0.001438(32)	0.001468(32)	0.001467(26)
0.004	0.025	0.001558(41)	0.001633(39)	0.001592(31)
0.004	0.03	0.001594(43)	0.001682(41)	0.001630(33)
0.006	0.006	0.001625(33)	0.001661(33)	0.001649(26)
0.006	0.008	0.001632(33)	0.001672(33)	0.001656(26)
0.006	0.025	0.001731(38)	0.001805(37)	0.001757(28)
0.006	0.03	0.001763(40)	0.001846(40)	0.001790(30)
0.008	0.008	0.001814(34)	0.001862(35)	0.001833(26)
0.008	0.025	0.001900(37)	0.001968(38)	0.001916(28)
0.008	0.03	0.001929(38)	0.002001(40)	0.001943(29)
0.025	0.025	0.003176(50)	0.003161(53)	0.003141(42)
0.025	0.03	0.003187(50)	0.003166(53)	0.003152(42)
0.03	0.03	0.003525(55)	0.003496(56)	0.003491(47)

Table 59: Lattice matrix element $|\mathcal{M}|$ for the (27,1) weak operator on the 32^3 fine lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for the three different sea light quark masses $m_l^{\text{sea}} = 0.004$, 0.006 , and 0.008 in each of the last three columns respectively.

m_l^{val}	m_s^{val}	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.004$	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.006$	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.008$
0.002	0.002	0.0761(17)	0.0747(18)	0.0780(18)
0.002	0.004	0.0709(15)	0.0699(16)	0.0727(16)
0.002	0.006	0.0671(14)	0.0663(15)	0.0687(15)
0.002	0.008	0.0643(14)	0.0636(14)	0.0656(14)
0.002	0.025	0.0538(13)	0.0538(13)	0.0542(12)
0.002	0.03	0.0524(13)	0.0526(13)	0.0526(13)
0.004	0.004	0.05061(97)	0.0499(10)	0.05133(94)
0.004	0.006	0.04830(90)	0.04791(94)	0.04901(87)
0.004	0.008	0.04652(85)	0.04628(89)	0.04717(82)
0.004	0.025	0.03922(78)	0.03955(79)	0.03970(70)
0.004	0.03	0.03815(79)	0.03860(80)	0.03862(69)
0.006	0.006	0.03900(68)	0.03895(74)	0.03944(65)
0.006	0.008	0.03776(65)	0.03785(70)	0.03818(61)
0.006	0.025	0.03228(58)	0.03271(59)	0.03265(50)
0.006	0.03	0.03142(58)	0.03190(59)	0.03178(49)
0.008	0.008	0.03241(55)	0.03271(59)	0.03271(50)
0.008	0.025	0.02810(47)	0.02853(49)	0.02834(40)
0.008	0.03	0.02738(47)	0.02781(48)	0.02760(39)
0.025	0.025	0.01632(25)	0.01634(27)	0.01617(21)
0.025	0.03	0.01601(24)	0.01600(26)	0.01587(20)
0.03	0.03	0.01468(22)	0.01464(24)	0.01456(19)

Table 60: Lattice matrix element $|\mathcal{M}|$ for the (8,8) weak operator on the 32^3 fine lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for the three different sea light quark masses $m_l^{\text{sea}}=0.004$, 0.006, and 0.008 in each of the last three columns respectively.

m_l^{val}	m_s^{val}	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.004$	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.006$	$ \mathcal{M} $ $m_l^{\text{sea}} = 0.008$
0.002	0.002	0.2473(56)	0.2428(58)	0.2536(60)
0.002	0.004	0.2305(50)	0.2274(51)	0.2364(52)
0.002	0.006	0.2185(46)	0.2161(47)	0.2236(47)
0.002	0.008	0.2095(44)	0.2075(45)	0.2139(45)
0.002	0.025	0.1760(43)	0.1762(42)	0.1771(40)
0.002	0.03	0.1715(44)	0.1724(43)	0.1722(41)
0.004	0.004	0.1661(32)	0.1641(33)	0.1686(31)
0.004	0.006	0.1587(29)	0.1576(31)	0.1612(28)
0.004	0.008	0.1530(28)	0.1524(29)	0.1552(27)
0.004	0.025	0.1297(26)	0.1310(26)	0.1314(23)
0.004	0.03	0.1263(26)	0.1280(26)	0.1280(23)
0.006	0.006	0.1294(23)	0.1294(25)	0.1309(21)
0.006	0.008	0.1254(22)	0.1258(23)	0.1269(20)
0.006	0.025	0.1080(19)	0.1095(20)	0.1093(17)
0.006	0.03	0.1053(19)	0.1070(20)	0.1066(16)
0.008	0.008	0.1087(19)	0.1098(20)	0.1098(17)
0.008	0.025	0.0950(16)	0.0965(16)	0.0959(13)
0.008	0.03	0.0928(16)	0.0943(16)	0.0936(13)
0.025	0.025	0.06028(91)	0.06035(98)	0.05981(76)
0.025	0.03	0.05927(89)	0.05924(95)	0.05885(75)
0.03	0.03	0.05578(84)	0.05564(89)	0.05543(73)

Table 61: Lattice matrix element $|\mathcal{M}|$ for the (8,8) mixed weak operator on the 32^3 fine lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for the three different sea light quark masses $m_l^{\text{sea}} = 0.004$, 0.006 , and 0.008 in each of the last three columns respectively.

m_l^{val}	m_s^{val}	$\text{Re}(A_2) \times 10^8$ (GeV)	$\text{Re}(A_2) \times 10^8$ (GeV)	$\text{Re}(A_2) \times 10^8$ (GeV)
		$m_l^{\text{sea}} = 0.004$	$m_l^{\text{sea}} = 0.006$	$m_l^{\text{sea}} = 0.008$
0.002	0.002	0.662(30)	0.680(30)	0.716(32)
0.002	0.004	0.727(33)	0.745(33)	0.778(34)
0.002	0.006	0.784(36)	0.805(36)	0.834(36)
0.002	0.008	0.836(38)	0.861(38)	0.886(39)
0.002	0.025	1.181(60)	1.269(60)	1.245(58)
0.002	0.03	1.269(66)	1.375(67)	1.337(63)
0.004	0.004	1.185(51)	1.222(52)	1.243(51)
0.004	0.006	1.249(54)	1.291(55)	1.308(54)
0.004	0.008	1.309(56)	1.357(58)	1.369(56)
0.004	0.025	1.739(78)	1.849(80)	1.819(76)
0.004	0.03	1.850(84)	1.979(87)	1.936(82)
0.006	0.006	1.725(72)	1.793(75)	1.798(73)
0.006	0.008	1.792(75)	1.866(78)	1.867(75)
0.006	0.025	2.288(98)	2.42(10)	2.381(97)
0.006	0.03	2.42(10)	2.57(11)	2.51(10)
0.008	0.008	2.302(95)	2.401(99)	2.389(96)
0.008	0.025	2.86(12)	3.00(12)	2.96(12)
0.008	0.03	3.00(13)	3.16(13)	3.10(12)
0.025	0.025	8.77(36)	8.77(35)	8.86(35)
0.025	0.03	9.02(37)	9.00(36)	9.10(36)
0.03	0.03	11.14(45)	11.09(45)	11.26(45)

Table 62: Results for $\text{Re}(A_2)$ on the 32^3 fine lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for the three different sea light quark masses $m_l^{\text{sea}}=0.004$, 0.006, and 0.008 in each of the last three columns respectively.

m_l^{val}	m_s^{val}	$\text{Im}(A_2) \times 10^{13}$ (GeV)	$\text{Im}(A_2) \times 10^{13}$ (GeV)	$\text{Im}(A_2) \times 10^{13}$ (GeV)
		$m_l^{\text{sea}} = 0.004$	$m_l^{\text{sea}} = 0.006$	$m_l^{\text{sea}} = 0.008$
0.002	0.002	-13.90(60)	-13.92(61)	-14.69(65)
0.002	0.004	-13.90(60)	-13.96(60)	-14.67(63)
0.002	0.006	-13.91(60)	-13.99(60)	-14.64(62)
0.002	0.008	-13.93(60)	-14.02(60)	-14.61(62)
0.002	0.025	-14.39(64)	-14.55(63)	-14.82(64)
0.002	0.03	-14.54(65)	-14.76(65)	-14.94(65)
0.004	0.004	-13.03(55)	-13.06(55)	-13.58(56)
0.004	0.006	-12.98(54)	-13.08(55)	-13.53(56)
0.004	0.008	-12.97(54)	-13.10(54)	-13.49(55)
0.004	0.025	-13.14(55)	-13.39(56)	-13.62(56)
0.004	0.03	-13.22(56)	-13.51(57)	-13.70(56)
0.006	0.006	-12.40(51)	-12.57(52)	-12.88(53)
0.006	0.008	-12.38(51)	-12.59(52)	-12.85(52)
0.006	0.025	-12.45(51)	-12.76(53)	-12.91(52)
0.006	0.03	-12.50(52)	-12.82(53)	-12.96(52)
0.008	0.008	-11.96(49)	-12.24(51)	-12.40(50)
0.008	0.025	-11.99(49)	-12.30(50)	-12.41(50)
0.008	0.03	-12.02(49)	-12.33(50)	-12.43(50)
0.025	0.025	-9.51(39)	-9.61(39)	-9.66(39)
0.025	0.03	-9.43(38)	-9.51(38)	-9.59(38)
0.03	0.03	-8.56(35)	-8.63(35)	-8.72(35)

Table 63: Results for $\text{Im}(A_2)$ on the 32^3 fine lattices. We show results for several different valence light and strange quark masses, as given in the first two columns, and for the three different sea light quark masses $m_l^{\text{sea}}=0.004$, 0.006, and 0.008 in each of the last three columns respectively.

Weak Operator	$\text{Re}(A_2) \times 10^8 \text{ (GeV)}$	$\text{Im}(A_2) \times 10^{13} \text{ (GeV)}$
(27,1)	1.290(67)	1.061(55)
(8,8)	0.00556(25)	1.030(47)
(8,8) mixed	-0.0261(12)	-16.63(75)
Total	1.269(66)	-14.54(65)

Table 64: Contributions to $\text{Re}(A_2)$ and $\text{Im}(A_2)$ from the different weak operators for $m_l^{\text{sea}} = 0.004$, $m_l^{\text{val}} = 0.002$, and $m_s^{\text{val}} = 0.03$ on the 32^3 fine lattices.

m_l	m_π (MeV)
0.0023	167.5(2.9)
0.0047	214.9(3.4)
0.0071	252.8(4.0)

Table 65: Pion mass in MeV on the quenched 24^3 lattices for the three different light quark masses.

m_l	m_s	m_K (MeV)
0.0023	0.0460	435.6(6.9)
0.0023	0.0620	498.6(8.0)
0.0023	0.0780	554.7(9.0)
0.0023	0.0940	605.6(9.9)
0.0023	0.1100	653(11)
0.0047	0.0460	443.8(6.9)
0.0047	0.0620	505.2(7.9)
0.0047	0.0780	560.1(8.8)
0.0047	0.0940	610.3(9.6)
0.0047	0.1100	657(10)
0.0071	0.0460	452.2(7.0)
0.0071	0.0620	512.2(7.9)
0.0071	0.0780	566.1(8.8)
0.0071	0.0940	615.5(9.6)
0.0071	0.1100	661(10)

Table 66: Kaon mass in MeV on the quenched 24^3 lattices for the different light and strange quark masses as given in the first two columns.

Number of twists	0	1	2	3
p (lattice units)	0	0.13090	0.18512	0.22672
p (MeV)	0	171.5(2.6)	242.5(3.7)	297.0(4.5)
$E_{\pi\pi}$ (lattice units)	0.2598(20)	0.3790(72)	0.491(16)	0.496(69)
$E_{\pi\pi}$ (MeV)	340.3(5.8)	496(12)	643(24)	649(91)
k (lattice units)	0.0227(26)	0.1398(51)	0.2097(98)	0.212(40)
k (MeV)	29.7(3.4)	183.2(7.2)	275(13)	278(53)
q	0.0866(98)	0.534(19)	0.801(37)	0.81(15)
δ_0 (degrees)	-0.69(22)	-9.6(5.5)	-27(11)	12(32)

Table 67: Two pion energies in physical units and s wave phase shifts for $m_l = 0.0023$ on the 24^3 quenched lattices. The quantity $p = \sqrt{n_{\text{tw}}}\pi/L$ is the momentum of a single meson with the given number of twists n_{tw} , whereas k is determined from the two pion energy through $E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$. Due to the interaction of the two pions k and p are not equal, although they are very close in value. The difference between the two is measurable and is what allows us to calculate the phase shift on the lattice using the Lüscher relation. Note that for $n_{\text{tw}} = 3$ the difference is *not* measurable (the error on k is larger than the difference) and thus the phase shift is within error of 0.

Number of twists	0	1	2	3
p (lattice units)	0	0.13090	0.18512	0.22672
p (MeV)	0	171.5(2.6)	242.5(3.7)	297.0(4.5)
$E_{\pi\pi}$ (lattice units)	0.3316(14)	0.4292(30)	0.5188(79)	0.561(36)
$E_{\pi\pi}$ (MeV)	434.4(6.9)	562.3(9.4)	680(15)	735(48)
k (lattice units)	0.0241(13)	0.1383(25)	0.2010(53)	0.228(22)
k (MeV)	31.5(1.8)	181.2(4.3)	263.3(8.0)	298(29)
q	0.0919(49)	0.5284(95)	0.768(20)	0.870(84)
δ_0 (degrees)	-0.82(12)	-8.0(2.7)	-17.4(5.9)	-1(18)

Table 68: Two pion energies in physical units and s wave phase shifts for $m_l = 0.0047$ on the 24^3 quenched lattices. The quantity $p = \sqrt{n_{\text{tw}}}\pi/L$ is the momentum of a single meson with the given number of twists n_{tw} , whereas k is determined from the two pion energy through $E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$. Due to the interaction of the two pions k and p are not equal, although they are very close in value. The difference between the two is measurable and is what allows us to calculate the phase shift on the lattice using the Lüscher relation. Note that for $n_{\text{tw}} = 3$ the difference is *not* measurable (the error on k is larger than the difference) and thus the phase shift is within error of 0.

Number of twists	0	1	2	3
p (lattice units)	0	0.13090	0.18512	0.22672
p (MeV)	0	171.5(2.6)	242.5(3.7)	297.0(4.5)
$E_{\pi\pi}$ (lattice units)	0.3894(12)	0.4749(20)	0.5552(56)	0.607(25)
$E_{\pi\pi}$ (MeV)	510.1(8.0)	622.2(9.9)	727(13)	795(35)
k (lattice units)	0.0261(11)	0.1384(18)	0.1996(40)	0.234(16)
k (MeV)	34.2(1.6)	181.4(3.6)	261.5(6.6)	307(22)
q	0.0996(43)	0.5288(68)	0.762(15)	0.894(62)
δ_0 (degrees)	-1.03(12)	-8.1(1.9)	-15.9(4.5)	-6(13)

Table 69: Two pion energies in physical units and s wave phase shifts for $m_l = 0.0071$ on the 24^3 quenched lattices. The quantity $p = \sqrt{n_{\text{tw}}}\pi/L$ is the momentum of a single meson with the given number of twists n_{tw} , whereas k is determined from the two pion energy through $E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$. Due to the interaction of the two pions k and p are not equal, although they are very close in value. The difference between the two is measurable and is what allows us to calculate the phase shift on the lattice using the Lüscher relation. Note that for $n_{\text{tw}} = 3$ the difference is *not* measurable (the error on k is larger than the difference) and thus the phase shift is within error of 0.

J	$\text{Re}(C_J(1.44 \text{ GeV}))$	$\text{Im}(C_J(1.44 \text{ GeV}))$
(27,1)	0.275025	2.28503×10^{-6}
(8,8)	2.48636×10^{-5}	5.51968×10^{-8}
(8,8) mixed	-6.13442×10^{-6}	-3.50173×10^{-7}

Table 70: Wilson coefficients in the RI-MOM scheme for the (27,1), (8,8), and (8,8) mixed operators calculated at $\mu = 1.44 \text{ GeV}$. These are obtained from Tables 5 and 6 of [16], using τ from Table 48 and using Eq. 4.36.

	(27,1)	(8,8)	(8,8) mixed
(27,1)	0.8322(96)	0	0
(8,8)	0	0.8940(75)	-0.0561(71)
(8,8) mixed	0	-0.0795(58)	0.964(17)

Table 71: Renormalization factors in the RI-MOM scheme at the scale $\mu = 1.44 \text{ GeV}$ for the (27,1), (8,8), and (8,8) mixed operators. These were calculated in [16] on 16^3 lattices with the same action and coupling as the 24^3 quenched lattices in this work. This table is the same as Table 11 of [16].

m_l	A	B
0.0023	-0.133(29)	-0.70(25)
0.0047	-0.152(15)	-0.41(17)
0.0071	-0.177(14)	-0.32(14)

Table 72: Results on the quenched 24^3 quenched lattices for fit coefficients in the fit of δ_0 vs. q to the fit form Eq. 7.22. We show results for the three different light quark masses. Note that fit coefficients are in units of radians and not degrees.

m_l	p	q	$q\partial\phi/\partial q$	$k\partial\delta_0/\partial k$	$\sqrt{\frac{\text{LL factor}}{\text{LL free field}}}$
0.0023	0	0.0866(98)	0.035(11)	-0.0129(44)	0.921(21)
0.0023	π/L	0.534(19)	2.606(81)	-0.39(14)	0.859(56)
0.0023	$\sqrt{2}\pi/L$	0.801(37)	4.19(31)	-1.18(53)	0.771(82)
0.0023	$\sqrt{3}\pi/L$	0.81(15)	2.99(60)	-1.22(80)	0.82(27)
0.0047	0	0.0919(49)	0.0408(61)	-0.0149(23)	0.9204(92)
0.0047	π/L	0.5284(95)	2.583(39)	-0.262(86)	0.893(31)
0.0047	$\sqrt{2}\pi/L$	0.768(20)	3.93(15)	-0.67(28)	0.854(49)
0.0047	$\sqrt{3}\pi/L$	0.870(84)	3.22(33)	-0.94(42)	0.84(12)
0.0071	0	0.0996(43)	0.0510(60)	-0.0186(22)	0.9117(74)
0.0071	π/L	0.5288(68)	2.584(28)	-0.237(67)	0.897(23)
0.0071	$\sqrt{2}\pi/L$	0.762(15)	3.89(12)	-0.56(21)	0.872(38)
0.0071	$\sqrt{3}\pi/L$	0.894(62)	3.32(25)	-0.85(33)	0.836(85)

Table 73: Lellouch-Lüscher factor over free field normalization factor for the 24^3 quenched lattices. Results are given for the three different light quark masses and for 0, 1, 2, and 3 twists ($p = 0, \pi/L, \sqrt{2}\pi/L$, and $\sqrt{3}\pi/L$).

m_l	m_s	$\sqrt{\text{LL}}$ factor $p = 0$	$\sqrt{\text{LL}}$ factor $p = \pi/L$	$\sqrt{\text{LL}}$ factor $p = \sqrt{2}\pi/L$	$\sqrt{\text{LL}}$ factor $p = \sqrt{3}\pi/L$
0.0023	0.0460	32.43(75)	31.2(1.5)	25.7(1.9)	19.5(4.0)
0.0023	0.0620	34.70(79)	33.4(1.6)	27.5(2.1)	20.8(4.3)
0.0023	0.0780	36.60(83)	35.2(1.7)	29.0(2.2)	22.0(4.5)
0.0023	0.0940	38.24(87)	36.8(1.8)	30.3(2.3)	23.0(4.7)
0.0023	0.1100	39.70(91)	38.2(1.8)	31.4(2.3)	23.8(4.9)
0.0047	0.0460	41.78(52)	37.1(1.1)	30.3(1.3)	22.7(2.3)
0.0047	0.0620	44.57(54)	39.6(1.2)	32.3(1.4)	24.3(2.4)
0.0047	0.0780	46.93(57)	41.7(1.2)	34.1(1.5)	25.6(2.5)
0.0047	0.0940	48.99(59)	43.5(1.3)	35.5(1.5)	26.7(2.6)
0.0047	0.1100	50.82(62)	45.1(1.3)	36.9(1.6)	27.7(2.7)
0.0071	0.0460	49.05(49)	41.61(96)	33.4(1.2)	24.8(1.8)
0.0071	0.0620	52.20(52)	44.3(1.0)	35.6(1.2)	26.4(1.9)
0.0071	0.0780	54.88(55)	46.6(1.1)	37.4(1.3)	27.7(2.0)
0.0071	0.0940	57.23(57)	48.5(1.1)	39.0(1.3)	28.9(2.1)
0.0071	0.1100	59.32(59)	50.3(1.1)	40.4(1.4)	30.0(2.2)

Table 74: Lellouch-Lüscher factor for the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different momenta/number of twists in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.00426(18)	0.00451(18)	0.00408(19)	0.00463(23)
0.0023	0.0620	0.00455(20)	0.00481(20)	0.00435(21)	0.00501(27)
0.0023	0.0780	0.00481(23)	0.00508(22)	0.00460(23)	0.00533(30)
0.0023	0.0940	0.00504(25)	0.00532(25)	0.00484(26)	0.00561(33)
0.0023	0.1100	0.00525(27)	0.00553(27)	0.00506(29)	0.00586(35)
0.0047	0.0460	0.00444(14)	0.00480(16)	0.00439(15)	0.00473(14)
0.0047	0.0620	0.00465(15)	0.00505(18)	0.00461(16)	0.00501(16)
0.0047	0.0780	0.00484(16)	0.00528(19)	0.00482(18)	0.00525(19)
0.0047	0.0940	0.00501(17)	0.00549(21)	0.00502(20)	0.00546(22)
0.0047	0.1100	0.00516(18)	0.00567(23)	0.00520(22)	0.00565(24)
0.0071	0.0460	0.00469(13)	0.00505(15)	0.00467(13)	0.00493(12)
0.0071	0.0620	0.00486(14)	0.00526(16)	0.00487(15)	0.00515(13)
0.0071	0.0780	0.00501(15)	0.00545(17)	0.00505(16)	0.00534(16)
0.0071	0.0940	0.00516(15)	0.00563(19)	0.00523(17)	0.00551(18)
0.0071	0.1100	0.00529(16)	0.00578(20)	0.00539(19)	0.00566(20)

Table 75: Lattice matrix elements for the (27,1) operator with zero twists ($p = 0$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.00558(45)	0.00628(51)	0.00539(41)	0.00610(56)
0.0023	0.0620	0.00581(53)	0.00658(54)	0.00564(50)	0.00665(73)
0.0023	0.0780	0.00602(60)	0.00687(57)	0.00589(60)	0.00710(91)
0.0023	0.0940	0.00619(66)	0.00714(60)	0.00613(70)	0.0075(11)
0.0023	0.1100	0.00633(72)	0.00739(63)	0.00636(79)	0.0078(13)
0.0047	0.0460	0.00543(24)	0.00604(29)	0.00547(23)	0.00574(23)
0.0047	0.0620	0.00559(27)	0.00629(32)	0.00567(25)	0.00601(28)
0.0047	0.0780	0.00576(30)	0.00654(35)	0.00587(29)	0.00622(34)
0.0047	0.0940	0.00591(33)	0.00677(38)	0.00609(32)	0.00637(40)
0.0047	0.1100	0.00606(36)	0.00699(41)	0.00629(35)	0.00650(47)
0.0071	0.0460	0.00550(19)	0.00597(22)	0.00557(18)	0.00572(16)
0.0071	0.0620	0.00564(21)	0.00617(24)	0.00574(20)	0.00589(19)
0.0071	0.0780	0.00578(23)	0.00636(26)	0.00593(22)	0.00603(22)
0.0071	0.0940	0.00591(25)	0.00654(29)	0.00611(24)	0.00613(25)
0.0071	0.1100	0.00604(27)	0.00670(31)	0.00628(26)	0.00621(29)

Table 76: Lattice matrix elements for the (27,1) operator with one twist ($p = \pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.00632(67)	0.00721(96)	0.00624(99)	0.0068(15)
0.0023	0.0620	0.00643(75)	0.0076(11)	0.0065(13)	0.0074(21)
0.0023	0.0780	0.00658(84)	0.0080(13)	0.0069(16)	0.0081(28)
0.0023	0.0940	0.00671(95)	0.0083(16)	0.0073(20)	0.0089(35)
0.0023	0.1100	0.0068(11)	0.0087(19)	0.0077(24)	0.0095(41)
0.0047	0.0460	0.00619(40)	0.00670(51)	0.00640(46)	0.00686(55)
0.0047	0.0620	0.00630(46)	0.00686(57)	0.00652(57)	0.00714(71)
0.0047	0.0780	0.00642(52)	0.00702(63)	0.00669(69)	0.00735(90)
0.0047	0.0940	0.00654(57)	0.00717(71)	0.00689(83)	0.0075(11)
0.0047	0.1100	0.00665(62)	0.00732(80)	0.00711(98)	0.0075(13)
0.0071	0.0460	0.00621(31)	0.00654(34)	0.00632(35)	0.00670(33)
0.0071	0.0620	0.00630(35)	0.00665(38)	0.00639(41)	0.00687(40)
0.0071	0.0780	0.00641(38)	0.00677(42)	0.00650(47)	0.00699(50)
0.0071	0.0940	0.00652(41)	0.00688(47)	0.00664(54)	0.00705(60)
0.0071	0.1100	0.00662(45)	0.00700(52)	0.00680(62)	0.00706(71)

Table 77: Lattice matrix elements for the (27,1) operator with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.0044(22)	0.0037(16)	0.0031(19)	0.0044(24)
0.0023	0.0620	0.0044(24)	0.0038(19)	0.0033(24)	0.0051(26)
0.0023	0.0780	0.0043(27)	0.0039(22)	0.0037(30)	0.0056(28)
0.0023	0.0940	0.0042(29)	0.0040(27)	0.0041(35)	0.0059(31)
0.0023	0.1100	0.0041(32)	0.0040(32)	0.0045(41)	0.0059(35)
0.0047	0.0460	0.0066(12)	0.0057(11)	0.0060(11)	0.0059(13)
0.0047	0.0620	0.0067(13)	0.0058(12)	0.0063(13)	0.0063(15)
0.0047	0.0780	0.0069(14)	0.0059(13)	0.0067(15)	0.0067(18)
0.0047	0.0940	0.0071(15)	0.0059(15)	0.0071(17)	0.0071(21)
0.0047	0.1100	0.0072(17)	0.0058(17)	0.0075(19)	0.0074(24)
0.0071	0.0460	0.00682(81)	0.00644(82)	0.00670(83)	0.00656(82)
0.0071	0.0620	0.00693(86)	0.00648(87)	0.00693(94)	0.00683(94)
0.0071	0.0780	0.00705(92)	0.00654(93)	0.0072(10)	0.0072(11)
0.0071	0.0940	0.00719(99)	0.0066(10)	0.0075(12)	0.0075(13)
0.0071	0.1100	0.0073(11)	0.0067(11)	0.0078(13)	0.0078(15)

Table 78: Lattice matrix elements for the (27,1) operator with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.1429(57)	0.1457(53)	0.1331(58)	0.1473(55)
0.0023	0.0620	0.1366(55)	0.1396(53)	0.1268(58)	0.1425(62)
0.0023	0.0780	0.1321(55)	0.1354(54)	0.1225(61)	0.1389(69)
0.0023	0.0940	0.1286(56)	0.1322(56)	0.1193(64)	0.1359(75)
0.0023	0.1100	0.1257(56)	0.1296(59)	0.1167(66)	0.1334(79)
0.0047	0.0460	0.1010(29)	0.1048(30)	0.0970(30)	0.1040(27)
0.0047	0.0620	0.0964(28)	0.1004(29)	0.0923(30)	0.1000(28)
0.0047	0.0780	0.0930(27)	0.0973(30)	0.0890(31)	0.0971(30)
0.0047	0.0940	0.0903(27)	0.0949(31)	0.0865(33)	0.0947(32)
0.0047	0.1100	0.0882(27)	0.0930(32)	0.0845(34)	0.0927(34)
0.0071	0.0460	0.0808(21)	0.0841(22)	0.0787(21)	0.0831(19)
0.0071	0.0620	0.0770(20)	0.0805(21)	0.0750(21)	0.0798(20)
0.0071	0.0780	0.0742(20)	0.0779(21)	0.0723(21)	0.0773(21)
0.0071	0.0940	0.0720(19)	0.0760(22)	0.0703(22)	0.0753(22)
0.0071	0.1100	0.0702(19)	0.0743(22)	0.0686(23)	0.0737(23)

Table 79: Lattice matrix elements for the (8,8) operator with zero twists ($p = 0$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.0896(58)	0.0936(58)	0.0857(59)	0.0909(64)
0.0023	0.0620	0.0849(55)	0.0890(58)	0.0816(66)	0.0879(70)
0.0023	0.0780	0.0816(53)	0.0857(59)	0.0790(71)	0.0856(77)
0.0023	0.0940	0.0789(52)	0.0831(61)	0.0770(74)	0.0837(85)
0.0023	0.1100	0.0767(52)	0.0811(63)	0.0755(75)	0.0822(92)
0.0047	0.0460	0.0715(25)	0.0748(28)	0.0689(25)	0.0719(21)
0.0047	0.0620	0.0678(24)	0.0711(28)	0.0654(27)	0.0688(23)
0.0047	0.0780	0.0651(24)	0.0686(28)	0.0629(29)	0.0664(25)
0.0047	0.0940	0.0631(24)	0.0666(29)	0.0611(31)	0.0645(28)
0.0047	0.1100	0.0614(25)	0.0650(30)	0.0596(32)	0.0628(31)
0.0071	0.0460	0.0610(17)	0.0636(18)	0.0595(17)	0.0617(13)
0.0071	0.0620	0.0579(17)	0.0605(18)	0.0565(18)	0.0590(14)
0.0071	0.0780	0.0556(17)	0.0583(17)	0.0544(19)	0.0569(15)
0.0071	0.0940	0.0538(17)	0.0566(18)	0.0527(20)	0.0552(16)
0.0071	0.1100	0.0523(17)	0.0552(18)	0.0514(20)	0.0537(18)

Table 80: Lattice matrix elements for the (8,8) operator with one twist ($p = \pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.0729(76)	0.0737(86)	0.0640(87)	0.078(15)
0.0023	0.0620	0.0681(79)	0.0696(90)	0.059(10)	0.080(18)
0.0023	0.0780	0.0643(85)	0.0670(99)	0.057(12)	0.082(22)
0.0023	0.0940	0.0610(94)	0.065(11)	0.056(13)	0.085(24)
0.0023	0.1100	0.058(10)	0.064(13)	0.056(15)	0.087(27)
0.0047	0.0460	0.0577(31)	0.0595(39)	0.0546(38)	0.0608(46)
0.0047	0.0620	0.0540(31)	0.0562(38)	0.0514(44)	0.0595(53)
0.0047	0.0780	0.0511(33)	0.0539(39)	0.0491(50)	0.0585(60)
0.0047	0.0940	0.0488(35)	0.0522(43)	0.0474(55)	0.0577(67)
0.0047	0.1100	0.0468(38)	0.0508(47)	0.0461(60)	0.0571(73)
0.0071	0.0460	0.0504(18)	0.0518(24)	0.0486(24)	0.0529(22)
0.0071	0.0620	0.0472(18)	0.0490(23)	0.0459(26)	0.0511(25)
0.0071	0.0780	0.0448(18)	0.0470(23)	0.0440(29)	0.0497(29)
0.0071	0.0940	0.0429(19)	0.0456(25)	0.0426(31)	0.0485(33)
0.0071	0.1100	0.0414(20)	0.0444(26)	0.0414(34)	0.0475(37)

Table 81: Lattice matrix elements for the (8,8) operator with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.045(16)	0.0388(85)	0.036(23)	0.054(22)
0.0023	0.0620	0.042(16)	0.0392(98)	0.039(28)	0.056(29)
0.0023	0.0780	0.038(17)	0.040(12)	0.042(38)	0.057(34)
0.0023	0.0940	0.035(18)	0.041(14)	0.052(46)	0.052(38)
0.0023	0.1100	0.033(20)	0.041(16)	0.059(52)	0.043(40)
0.0047	0.0460	0.0438(71)	0.0441(50)	0.0392(74)	0.051(10)
0.0047	0.0620	0.0408(72)	0.0426(51)	0.0377(84)	0.051(11)
0.0047	0.0780	0.0385(75)	0.0417(54)	0.0374(90)	0.051(13)
0.0047	0.0940	0.0368(80)	0.0410(58)	0.038(10)	0.051(15)
0.0047	0.1100	0.0354(86)	0.0402(63)	0.039(12)	0.051(17)
0.0071	0.0460	0.0415(44)	0.0428(38)	0.0377(45)	0.0458(58)
0.0071	0.0620	0.0388(44)	0.0409(37)	0.0358(47)	0.0443(63)
0.0071	0.0780	0.0368(46)	0.0396(38)	0.0348(50)	0.0433(70)
0.0071	0.0940	0.0352(48)	0.0385(39)	0.0344(55)	0.0427(79)
0.0071	0.1100	0.0340(50)	0.0375(41)	0.0341(60)	0.0425(90)

Table 82: Lattice matrix elements for the (8,8) operator with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.476(19)	0.488(18)	0.446(19)	0.492(18)
0.0023	0.0620	0.456(18)	0.468(18)	0.425(20)	0.477(20)
0.0023	0.0780	0.442(18)	0.455(18)	0.412(20)	0.466(23)
0.0023	0.0940	0.431(19)	0.445(19)	0.402(21)	0.457(25)
0.0023	0.1100	0.421(19)	0.437(20)	0.394(22)	0.449(26)
0.0047	0.0460	0.3414(96)	0.355(10)	0.328(10)	0.3522(89)
0.0047	0.0620	0.3264(93)	0.341(10)	0.314(10)	0.3397(95)
0.0047	0.0780	0.3156(92)	0.331(10)	0.303(11)	0.331(10)
0.0047	0.0940	0.3072(92)	0.324(11)	0.296(11)	0.323(11)
0.0047	0.1100	0.3003(93)	0.318(11)	0.289(11)	0.317(12)
0.0071	0.0460	0.2764(69)	0.2881(76)	0.2694(72)	0.2847(65)
0.0071	0.0620	0.2642(67)	0.2766(74)	0.2577(72)	0.2742(67)
0.0071	0.0780	0.2554(67)	0.2686(74)	0.2494(74)	0.2664(71)
0.0071	0.0940	0.2485(66)	0.2625(75)	0.2431(76)	0.2603(75)
0.0071	0.1100	0.2427(66)	0.2574(77)	0.2379(78)	0.2551(79)

Table 83: Lattice matrix elements for the (8,8) mixed operator with zero twists ($p = 0$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.307(20)	0.322(20)	0.296(20)	0.318(22)
0.0023	0.0620	0.292(19)	0.308(21)	0.283(22)	0.310(25)
0.0023	0.0780	0.281(18)	0.298(21)	0.275(24)	0.304(27)
0.0023	0.0940	0.272(18)	0.290(22)	0.269(25)	0.300(30)
0.0023	0.1100	0.265(18)	0.284(23)	0.266(26)	0.297(32)
0.0047	0.0460	0.2483(86)	0.2607(99)	0.2405(89)	0.2523(74)
0.0047	0.0620	0.2364(84)	0.2494(98)	0.2291(97)	0.2427(82)
0.0047	0.0780	0.2278(84)	0.242(10)	0.221(10)	0.2355(92)
0.0047	0.0940	0.2212(85)	0.236(10)	0.216(11)	0.230(10)
0.0047	0.1100	0.2158(86)	0.231(11)	0.211(12)	0.225(11)
0.0071	0.0460	0.2145(58)	0.2243(65)	0.2098(60)	0.2181(45)
0.0071	0.0620	0.2043(57)	0.2146(63)	0.2001(63)	0.2094(49)
0.0071	0.0780	0.1969(58)	0.2079(63)	0.1933(67)	0.2028(55)
0.0071	0.0940	0.1912(58)	0.2027(64)	0.1881(72)	0.1976(61)
0.0071	0.1100	0.1866(59)	0.1985(66)	0.1840(76)	0.1933(66)

Table 84: Lattice matrix elements for the (8,8) mixed operator with one twist ($p = \pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.256(27)	0.255(31)	0.222(27)	0.268(52)
0.0023	0.0620	0.240(27)	0.242(32)	0.204(31)	0.275(63)
0.0023	0.0780	0.228(29)	0.236(35)	0.193(35)	0.279(73)
0.0023	0.0940	0.218(32)	0.232(39)	0.186(40)	0.282(82)
0.0023	0.1100	0.209(35)	0.232(44)	0.183(45)	0.285(89)
0.0047	0.0460	0.205(11)	0.210(14)	0.194(13)	0.217(17)
0.0047	0.0620	0.193(11)	0.199(13)	0.182(14)	0.213(19)
0.0047	0.0780	0.184(12)	0.192(14)	0.174(16)	0.210(22)
0.0047	0.0940	0.176(13)	0.187(16)	0.167(18)	0.206(24)
0.0047	0.1100	0.170(13)	0.184(17)	0.161(20)	0.203(27)
0.0071	0.0460	0.1812(65)	0.1861(84)	0.1756(81)	0.1918(83)
0.0071	0.0620	0.1708(65)	0.1771(81)	0.1662(91)	0.1861(95)
0.0071	0.0780	0.1630(67)	0.1709(83)	0.159(10)	0.181(11)
0.0071	0.0940	0.1568(70)	0.1664(88)	0.154(11)	0.177(12)
0.0071	0.1100	0.1517(74)	0.1630(95)	0.149(12)	0.173(14)

Table 85: Lattice matrix elements for the (8,8) mixed operator with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$ \mathcal{M} $ $t_K = 20$	$ \mathcal{M} $ $t_K = 24$	$ \mathcal{M} $ $t_K = 28$	$ \mathcal{M} $ $t_K = 32$
0.0023	0.0460	0.162(56)	0.124(32)	0.129(87)	0.177(88)
0.0023	0.0620	0.152(57)	0.120(37)	0.153(97)	0.18(12)
0.0023	0.0780	0.142(60)	0.118(45)	0.17(11)	0.19(13)
0.0023	0.0940	0.132(64)	0.117(53)	0.19(12)	0.19(14)
0.0023	0.1100	0.123(68)	0.115(61)	0.21(14)	0.18(16)
0.0047	0.0460	0.162(27)	0.157(19)	0.142(24)	0.181(40)
0.0047	0.0620	0.153(27)	0.150(19)	0.135(26)	0.177(44)
0.0047	0.0780	0.146(28)	0.146(20)	0.132(29)	0.173(49)
0.0047	0.0940	0.141(29)	0.143(21)	0.130(33)	0.170(55)
0.0047	0.1100	0.136(31)	0.140(23)	0.129(37)	0.167(63)
0.0071	0.0460	0.155(16)	0.157(14)	0.139(16)	0.170(21)
0.0071	0.0620	0.146(16)	0.150(14)	0.130(16)	0.165(23)
0.0071	0.0780	0.140(17)	0.145(14)	0.125(17)	0.160(25)
0.0071	0.0940	0.135(18)	0.141(14)	0.122(19)	0.157(29)
0.0071	0.1100	0.131(19)	0.138(15)	0.119(21)	0.155(32)

Table 86: Lattice matrix elements for the (8,8) mixed operator with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of the last four columns.

m_l	m_s	$\text{Re}(A_2)$	$\text{Re}(A_2)$	$\text{Re}(A_2)$	$\text{Re}(A_2)$
		$\times 10^8 \text{ (GeV)}$	$\times 10^8 \text{ (GeV)}$	$\times 10^8 \text{ (GeV)}$	$\times 10^8 \text{ (GeV)}$
		$t_K = 20$	$t_K = 24$	$t_K = 28$	$t_K = 32$
0.0023	0.0460	1.089(77)	1.151(77)	1.043(78)	1.183(84)
0.0023	0.0620	1.243(88)	1.315(89)	1.189(89)	1.37(10)
0.0023	0.0780	1.386(99)	1.46(10)	1.33(10)	1.54(12)
0.0023	0.0940	1.52(11)	1.60(11)	1.46(11)	1.69(13)
0.0023	0.1100	1.64(12)	1.73(13)	1.58(12)	1.83(15)
0.0047	0.0460	1.461(84)	1.578(92)	1.443(87)	1.557(90)
0.0047	0.0620	1.632(95)	1.77(10)	1.617(99)	1.76(10)
0.0047	0.0780	1.79(10)	1.95(12)	1.78(11)	1.94(12)
0.0047	0.0940	1.93(11)	2.12(13)	1.94(12)	2.11(13)
0.0047	0.1100	2.07(12)	2.27(14)	2.08(13)	2.26(15)
0.0071	0.0460	1.81(10)	1.95(11)	1.81(10)	1.90(10)
0.0071	0.0620	2.00(11)	2.16(12)	2.00(11)	2.12(12)
0.0071	0.0780	2.17(12)	2.36(13)	2.18(13)	2.31(13)
0.0071	0.0940	2.33(13)	2.54(15)	2.36(14)	2.48(15)
0.0071	0.1100	2.47(14)	2.70(16)	2.52(15)	2.64(16)

Table 87: Results for $\text{Re}(A_2)$ with zero twists ($p = 0$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	Re(A_2) $\times 10^8$ (GeV) Ave.
0.0023	0.0460	1.116(70)
0.0023	0.0620	1.275(79)
0.0023	0.0780	1.421(89)
0.0023	0.0940	1.557(99)
0.0023	0.1100	1.68(11)
0.0047	0.0460	1.508(81)
0.0047	0.0620	1.691(91)
0.0047	0.0780	1.86(10)
0.0047	0.0940	2.01(11)
0.0047	0.1100	2.15(12)
0.0071	0.0460	1.865(97)
0.0071	0.0620	2.07(11)
0.0071	0.0780	2.25(12)
0.0071	0.0940	2.42(13)
0.0071	0.1100	2.57(14)

Table 88: Results for the error weighted average of $\text{Re}(A_2)$ with zero twists ($p = 0$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

m_l	m_s	Re(A_2)	Re(A_2)	Re(A_2)	Re(A_2)
		$\times 10^8$ (GeV)	$\times 10^8$ (GeV)	$\times 10^8$ (GeV)	$\times 10^8$ (GeV)
		$t_K = 20$	$t_K = 24$	$t_K = 28$	$t_K = 32$
0.0023	0.0460	1.94(18)	2.18(19)	1.87(15)	2.12(20)
0.0023	0.0620	2.16(21)	2.45(22)	2.10(19)	2.47(27)
0.0023	0.0780	2.36(25)	2.69(24)	2.31(23)	2.78(34)
0.0023	0.0940	2.54(29)	2.93(26)	2.51(28)	3.06(43)
0.0023	0.1100	2.69(32)	3.14(29)	2.71(32)	3.30(52)
0.0047	0.0460	2.24(15)	2.50(17)	2.26(15)	2.37(15)
0.0047	0.0620	2.46(17)	2.77(19)	2.50(17)	2.65(18)
0.0047	0.0780	2.67(19)	3.03(21)	2.73(19)	2.88(21)
0.0047	0.0940	2.86(21)	3.28(23)	2.95(21)	3.09(25)
0.0047	0.1100	3.04(23)	3.51(26)	3.16(23)	3.27(28)
0.0071	0.0460	2.55(15)	2.77(17)	2.58(16)	2.65(15)
0.0071	0.0620	2.78(17)	3.04(19)	2.83(18)	2.91(17)
0.0071	0.0780	3.00(19)	3.30(21)	3.07(19)	3.12(20)
0.0071	0.0940	3.20(20)	3.53(23)	3.30(21)	3.31(22)
0.0071	0.1100	3.39(22)	3.76(25)	3.52(23)	3.48(25)

Table 89: Results for $\text{Re}(A_2)$ with one twist ($p = \pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	Re(A_2) $\times 10^8$ (GeV) Ave.
0.0023	0.0460	2.00(14)
0.0023	0.0620	2.26(16)
0.0023	0.0780	2.50(18)
0.0023	0.0940	2.72(20)
0.0023	0.1100	2.92(22)
0.0047	0.0460	2.33(13)
0.0047	0.0620	2.58(15)
0.0047	0.0780	2.81(16)
0.0047	0.0940	3.03(18)
0.0047	0.1100	3.23(20)
0.0071	0.0460	2.63(14)
0.0071	0.0620	2.88(16)
0.0071	0.0780	3.11(17)
0.0071	0.0940	3.33(19)
0.0071	0.1100	3.53(20)

Table 90: Results for the error weighted average of $\text{Re}(A_2)$ with one twist ($p = \pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

m_l	m_s	Re(A_2)	Re(A_2)	Re(A_2)	Re(A_2)
		$\times 10^8$ (GeV)	$\times 10^8$ (GeV)	$\times 10^8$ (GeV)	$\times 10^8$ (GeV)
		$t_K = 20$	$t_K = 24$	$t_K = 28$	$t_K = 32$
0.0023	0.0460	2.55(27)	2.91(37)	2.52(39)	2.76(58)
0.0023	0.0620	2.78(33)	3.30(47)	2.81(53)	3.21(87)
0.0023	0.0780	3.00(39)	3.64(59)	3.13(71)	3.7(1.2)
0.0023	0.0940	3.20(46)	3.97(75)	3.47(91)	4.2(1.6)
0.0023	0.1100	3.38(53)	4.29(94)	3.8(1.1)	4.7(2.0)
0.0047	0.0460	2.96(22)	3.20(27)	3.06(26)	3.27(28)
0.0047	0.0620	3.21(25)	3.50(31)	3.32(32)	3.64(37)
0.0047	0.0780	3.44(29)	3.77(37)	3.59(39)	3.94(49)
0.0047	0.0940	3.66(33)	4.02(43)	3.86(48)	4.18(61)
0.0047	0.1100	3.87(37)	4.25(50)	4.13(58)	4.36(74)
0.0071	0.0460	3.27(21)	3.44(23)	3.33(25)	3.53(22)
0.0071	0.0620	3.53(24)	3.73(26)	3.58(29)	3.85(27)
0.0071	0.0780	3.78(27)	3.99(30)	3.83(34)	4.12(33)
0.0071	0.0940	4.01(30)	4.23(34)	4.08(39)	4.33(40)
0.0071	0.1100	4.22(33)	4.46(38)	4.33(44)	4.50(49)

Table 91: Results for $\text{Re}(A_2)$ with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	Re(A_2) $\times 10^8$ (GeV) Ave.
0.0023	0.0460	2.65(24)
0.0023	0.0620	2.94(28)
0.0023	0.0780	3.20(32)
0.0023	0.0940	3.45(37)
0.0023	0.1100	3.67(44)
0.0047	0.0460	3.09(20)
0.0047	0.0620	3.37(23)
0.0047	0.0780	3.62(26)
0.0047	0.0940	3.85(29)
0.0047	0.1100	4.06(33)
0.0071	0.0460	3.40(20)
0.0071	0.0620	3.67(22)
0.0071	0.0780	3.91(25)
0.0071	0.0940	4.14(28)
0.0071	0.1100	4.34(30)

Table 92: Results for the error weighted average of $\text{Re}(A_2)$ with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

m_l	m_s	$\text{Re}(A_2)$	$\text{Re}(A_2)$	$\text{Re}(A_2)$	$\text{Re}(A_2)$
		$\times 10^8 \text{ (GeV)}$	$\times 10^8 \text{ (GeV)}$	$\times 10^8 \text{ (GeV)}$	$\times 10^8 \text{ (GeV)}$
		$t_K = 20$	$t_K = 24$	$t_K = 28$	$t_K = 32$
0.0023	0.0460	1.91(80)	1.59(63)	1.34(69)	1.89(83)
0.0023	0.0620	2.03(96)	1.75(83)	1.6(1.0)	2.3(1.0)
0.0023	0.0780	2.1(1.1)	1.9(1.1)	1.8(1.3)	2.7(1.2)
0.0023	0.0940	2.1(1.3)	2.1(1.4)	2.1(1.7)	3.0(1.5)
0.0023	0.1100	2.2(1.6)	2.1(1.7)	2.4(2.1)	3.1(1.8)
0.0047	0.0460	3.32(55)	2.91(53)	3.04(53)	2.98(57)
0.0047	0.0620	3.64(62)	3.15(65)	3.42(66)	3.40(73)
0.0047	0.0780	3.93(71)	3.35(78)	3.82(82)	3.84(92)
0.0047	0.0940	4.20(81)	3.50(94)	4.22(98)	4.2(1.2)
0.0047	0.1100	4.44(92)	3.6(1.1)	4.6(1.1)	4.6(1.4)
0.0071	0.0460	3.77(46)	3.55(47)	3.70(49)	3.62(43)
0.0071	0.0620	4.07(51)	3.81(54)	4.07(59)	4.01(54)
0.0071	0.0780	4.36(56)	4.04(61)	4.45(69)	4.42(68)
0.0071	0.0940	4.63(62)	4.25(70)	4.84(79)	4.82(85)
0.0071	0.1100	4.89(69)	4.44(81)	5.23(90)	5.2(1.1)

Table 93: Results for $\text{Re}(A_2)$ with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	Re(A_2) $\times 10^8$ (GeV) Ave.
0.0023	0.0460	1.65(52)
0.0023	0.0620	1.90(59)
0.0023	0.0780	2.14(66)
0.0023	0.0940	2.31(75)
0.0023	0.1100	2.42(86)
0.0047	0.0460	3.07(46)
0.0047	0.0620	3.41(53)
0.0047	0.0780	3.74(60)
0.0047	0.0940	4.03(67)
0.0047	0.1100	4.30(74)
0.0071	0.0460	3.66(40)
0.0071	0.0620	3.99(46)
0.0071	0.0780	4.31(51)
0.0071	0.0940	4.61(56)
0.0071	0.1100	4.89(61)

Table 94: Results for the error weighted average of $\text{Re}(A_2)$ with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 20$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 24$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 28$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 32$
0.0023	0.0460	-16.0(1.0)	-16.3(1.1)	-14.9(1.0)	-16.47(97)
0.0023	0.0620	-16.3(1.1)	-16.7(1.1)	-15.1(1.1)	-17.0(1.0)
0.0023	0.0780	-16.5(1.1)	-17.0(1.2)	-15.4(1.1)	-17.4(1.1)
0.0023	0.0940	-16.7(1.1)	-17.3(1.2)	-15.6(1.1)	-17.7(1.2)
0.0023	0.1100	-16.9(1.2)	-17.5(1.3)	-15.8(1.2)	-18.0(1.3)
0.0047	0.0460	-14.39(80)	-14.91(84)	-13.81(80)	-14.80(80)
0.0047	0.0620	-14.56(81)	-15.15(85)	-13.95(82)	-15.11(84)
0.0047	0.0780	-14.73(83)	-15.39(88)	-14.10(85)	-15.37(88)
0.0047	0.0940	-14.87(85)	-15.61(91)	-14.24(88)	-15.59(92)
0.0047	0.1100	-14.99(86)	-15.80(94)	-14.37(90)	-15.78(95)
0.0071	0.0460	-13.34(72)	-13.85(76)	-12.97(72)	-13.71(73)
0.0071	0.0620	-13.45(73)	-14.02(77)	-13.07(73)	-13.92(75)
0.0071	0.0780	-13.55(74)	-14.19(78)	-13.18(75)	-14.10(78)
0.0071	0.0940	-13.64(75)	-14.35(79)	-13.28(77)	-14.26(80)
0.0071	0.1100	-13.72(76)	-14.49(81)	-13.37(78)	-14.39(82)

Table 95: Results for $\text{Im}(A_2)$ with zero twists ($p = 0$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) Ave.
0.0023	0.0460	-15.99(94)
0.0023	0.0620	-16.30(96)
0.0023	0.0780	-16.6(1.0)
0.0023	0.0940	-16.8(1.0)
0.0023	0.1100	-17.0(1.1)
0.0047	0.0460	-14.50(76)
0.0047	0.0620	-14.71(77)
0.0047	0.0780	-14.91(79)
0.0047	0.0940	-15.08(81)
0.0047	0.1100	-15.22(83)
0.0071	0.0460	-13.47(69)
0.0071	0.0620	-13.62(70)
0.0071	0.0780	-13.76(71)
0.0071	0.0940	-13.88(72)
0.0071	0.1100	-13.99(73)

Table 96: Results for the error weighted average of $\text{Im}(A_2)$ with zero twists ($p = 0$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 20$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 24$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 28$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 32$
0.0023	0.0460	-13.24(95)	-13.8(1.2)	-12.74(91)	-13.63(95)
0.0023	0.0620	-13.31(97)	-13.9(1.2)	-12.9(1.0)	-14.0(1.1)
0.0023	0.0780	-13.4(1.0)	-14.0(1.3)	-13.1(1.2)	-14.3(1.2)
0.0023	0.0940	-13.4(1.0)	-14.1(1.4)	-13.3(1.3)	-14.6(1.4)
0.0023	0.1100	-13.5(1.1)	-14.2(1.5)	-13.5(1.4)	-14.9(1.6)
0.0047	0.0460	-12.41(73)	-12.92(83)	-11.95(73)	-12.54(71)
0.0047	0.0620	-12.45(75)	-13.00(85)	-11.98(77)	-12.70(75)
0.0047	0.0780	-12.50(77)	-13.09(88)	-12.04(82)	-12.84(80)
0.0047	0.0940	-12.54(79)	-13.18(92)	-12.09(88)	-12.97(86)
0.0047	0.1100	-12.57(81)	-13.25(95)	-12.15(93)	-13.09(92)
0.0071	0.0460	-11.72(67)	-12.18(73)	-11.40(67)	-11.89(66)
0.0071	0.0620	-11.73(68)	-12.23(73)	-11.40(69)	-11.98(68)
0.0071	0.0780	-11.74(69)	-12.29(74)	-11.42(71)	-12.07(70)
0.0071	0.0940	-11.76(70)	-12.35(76)	-11.44(75)	-12.15(73)
0.0071	0.1100	-11.77(71)	-12.40(77)	-11.46(78)	-12.22(76)

Table 97: Results for $\text{Im}(A_2)$ with one twist ($p = \pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) Ave.
0.0023	0.0460	-13.29(82)
0.0023	0.0620	-13.48(85)
0.0023	0.0780	-13.63(90)
0.0023	0.0940	-13.76(94)
0.0023	0.1100	-13.86(99)
0.0047	0.0460	-12.43(68)
0.0047	0.0620	-12.52(69)
0.0047	0.0780	-12.61(70)
0.0047	0.0940	-12.68(72)
0.0047	0.1100	-12.75(74)
0.0071	0.0460	-11.78(63)
0.0071	0.0620	-11.83(64)
0.0071	0.0780	-11.88(65)
0.0071	0.0940	-11.93(66)
0.0071	0.1100	-11.97(66)

Table 98: Results for the error weighted average of $\text{Im}(A_2)$ with one twist ($p = \pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 20$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 24$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 28$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 32$
0.0023	0.0460	-12.3(1.3)	-11.9(1.7)	-10.4(1.7)	-12.8(2.3)
0.0023	0.0620	-12.2(1.4)	-11.9(1.9)	-9.9(2.0)	-13.9(3.0)
0.0023	0.0780	-12.0(1.6)	-11.9(2.2)	-9.6(2.4)	-14.6(3.8)
0.0023	0.0940	-11.8(1.9)	-12.1(2.6)	-9.4(2.8)	-15.1(4.6)
0.0023	0.1100	-11.6(2.2)	-12.4(3.0)	-9.3(3.3)	-15.6(5.4)
0.0047	0.0460	-11.19(82)	-11.30(96)	-10.37(96)	-11.7(1.0)
0.0047	0.0620	-11.04(88)	-11.2(1.0)	-10.2(1.1)	-12.1(1.2)
0.0047	0.0780	-10.87(95)	-11.2(1.1)	-10.0(1.2)	-12.4(1.4)
0.0047	0.0940	-10.7(1.0)	-11.3(1.2)	-9.8(1.4)	-12.6(1.6)
0.0047	0.1100	-10.5(1.1)	-11.3(1.4)	-9.6(1.5)	-12.8(1.9)
0.0071	0.0460	-10.59(67)	-10.81(77)	-10.14(77)	-11.17(70)
0.0071	0.0620	-10.42(70)	-10.75(78)	-10.02(83)	-11.36(77)
0.0071	0.0780	-10.27(73)	-10.74(81)	-9.91(90)	-11.48(88)
0.0071	0.0940	-10.13(77)	-10.76(86)	-9.78(99)	-11.57(99)
0.0071	0.1100	-10.00(81)	-10.80(92)	-9.6(1.1)	-11.6(1.1)

Table 99: Results for $\text{Im}(A_2)$ with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) Ave.
0.0023	0.0460	-11.8(1.1)
0.0023	0.0620	-11.8(1.2)
0.0023	0.0780	-11.7(1.4)
0.0023	0.0940	-11.6(1.7)
0.0023	0.1100	-11.6(2.0)
0.0047	0.0460	-11.16(74)
0.0047	0.0620	-11.10(79)
0.0047	0.0780	-11.03(86)
0.0047	0.0940	-10.95(95)
0.0047	0.1100	-10.9(1.0)
0.0071	0.0460	-10.73(62)
0.0071	0.0620	-10.67(64)
0.0071	0.0780	-10.60(67)
0.0071	0.0940	-10.52(71)
0.0071	0.1100	-10.44(75)

Table 100: Results for the error weighted average of $\text{Im}(A_2)$ with two twists ($p = \sqrt{2}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 20$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 24$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 28$	$\text{Im}(A_2) \times$ 10^{13} (GeV) $t_K = 32$
0.0023	0.0460	-8.2(2.3)	-6.1(1.7)	-6.7(5.2)	-9.1(4.4)
0.0023	0.0620	-8.2(2.7)	-6.2(2.2)	-8.7(6.1)	-9.4(6.8)
0.0023	0.0780	-8.0(3.2)	-6.4(2.9)	-10.4(6.9)	-11.0(7.7)
0.0023	0.0940	-7.7(3.6)	-6.5(3.6)	-11.9(7.9)	-11.2(8.9)
0.0023	0.1100	-7.4(4.1)	-6.6(4.4)	-13.4(9.3)	-11(10)
0.0047	0.0460	-8.7(1.4)	-8.7(1.1)	-7.5(1.6)	-10.3(2.2)
0.0047	0.0620	-8.6(1.6)	-8.7(1.2)	-7.3(1.9)	-10.5(2.7)
0.0047	0.0780	-8.4(1.7)	-8.8(1.4)	-7.3(2.2)	-10.5(3.2)
0.0047	0.0940	-8.2(1.9)	-8.9(1.6)	-7.3(2.6)	-10.5(3.9)
0.0047	0.1100	-8.0(2.2)	-9.0(1.8)	-7.2(3.1)	-10.4(4.8)
0.0071	0.0460	-8.8(1.1)	-9.1(1.0)	-7.6(1.1)	-10.1(1.4)
0.0071	0.0620	-8.6(1.2)	-9.1(1.1)	-7.3(1.2)	-10.2(1.6)
0.0071	0.0780	-8.5(1.2)	-9.2(1.1)	-7.1(1.3)	-10.2(1.9)
0.0071	0.0940	-8.3(1.3)	-9.2(1.2)	-6.9(1.5)	-10.1(2.3)
0.0071	0.1100	-8.2(1.4)	-9.2(1.3)	-6.7(1.7)	-10.1(2.7)

Table 101: Results for $\text{Im}(A_2)$ with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns, and for the four different separations t_K between the kaon and two pions in each of columns 3 - 6.

m_l	m_s	$\text{Im}(A_2) \times$ 10^{13} (GeV) Ave.
0.0023	0.0460	-7.0(1.5)
0.0023	0.0620	-7.3(1.7)
0.0023	0.0780	-7.6(2.2)
0.0023	0.0940	-7.8(2.5)
0.0023	0.1100	-7.9(2.9)
0.0047	0.0460	-8.6(1.0)
0.0047	0.0620	-8.6(1.1)
0.0047	0.0780	-8.6(1.1)
0.0047	0.0940	-8.5(1.3)
0.0047	0.1100	-8.5(1.4)
0.0071	0.0460	-8.83(95)
0.0071	0.0620	-8.70(94)
0.0071	0.0780	-8.59(95)
0.0071	0.0940	-8.51(99)
0.0071	0.1100	-8.4(1.1)

Table 102: Results for the error weighted average of $\text{Im}(A_2)$ with three twists ($p = \sqrt{3}\pi/L$) on the 24^3 quenched lattices. We show results for the different light and strange quark masses, as given in the first two columns.

	$m_l^{\text{sea}} = 0.004$	$m_l^{\text{sea}} = 0.006$	$m_l^{\text{sea}} = 0.008$
B_0	1.6545(46)	1.6494(47)	1.6645(44)
B_1	0.002956(79)	0.003266(83)	0.003298(78)
χ^2	7.0(3.4)	9.9(3.5)	12.4(4.2)
$\chi^2/\text{d.o.f.}$	0.44(22)	0.62(22)	0.77(26)
m_l^{coarse}	0.000288(24)	0.000194(25)	0.000183(23)
m_s^{coarse}	0.027917(79)	0.027908(79)	0.027647(76)
m_l^{phys}	0.000235(24)	0.000141(25)	0.000131(23)
m_s^{phys}	0.026197(75)	0.026183(74)	0.025937(72)

Table 103: Results for B_0 and B_1 in the fit of pseudoscalar meson mass as a function of valence quark masses (Eq. 7.24) for different sea light quark masses on the 32^3 fine lattices. This fit is then used to determine the valence quark masses that would yield the pion and kaon masses of the 32^3 coarse lattices (m_l^{coarse} and m_s^{coarse}) and the valence quark masses that would yield the physical pion and kaon masses (m_l^{phys} and m_s^{phys}). All quantities are in the lattice units of the 32^3 fine lattices.

	$m_l^{\text{sea}} = 0.004$	$m_l^{\text{sea}} = 0.006$	$m_l^{\text{sea}} = 0.008$
A_0	13.317(49)	13.204(55)	13.322(49)
A_1	0.01364(36)	0.01508(37)	0.01512(37)
χ^2	0.75(36)	0.29(25)	0.51(30)
$\chi^2/\text{d.o.f.}$	0.149(72)	0.058(50)	0.103(59)
$E_{\pi\pi}^{\text{extrap to coarse}}$	0.13218(84)	0.13284(92)	0.13251(82)
$E_{\pi\pi}^{\text{extrap to phys}}$	0.12949(86)	0.13018(94)	0.12984(84)

Table 104: Results for A_0 and A_1 in the fit of $E_{\pi\pi}^2$ as a function of valence light quark masses (Eq. 7.33) for different sea light quark masses on the 32^3 fine lattices. Using the valence light quark masses of Table 103 we can then use the fit results to determine the $E_{\pi\pi}$ that would result on the fine lattices at zero twist at the pion mass from the 32^3 coarse calculation and at the physical pion mass ($E_{\pi\pi}^{\text{extrap to coarse}}$ and $E_{\pi\pi}^{\text{extrap to phys}}$ respectively). All quantities are in the lattice units of the 32^3 fine lattices.

	$m_l^{\text{sea}} = 0.004$	$m_l^{\text{sea}} = 0.006$	$m_l^{\text{sea}} = 0.008$
$C_0 \times 10^6$ (GeV)	2.830(59)	2.924(65)	2.886(60)
$C_1 \times 10^7$ (GeV)	2.30(11)	2.60(12)	2.39(11)
$C_2 \times 10^{10}$ (GeV)	2.9(3.4)	2.5(3.1)	6.0(3.4)
χ^2	0.469(87)	0.36(10)	0.63(16)
$\chi^2/\text{d.o.f.}$	0.094(17)	0.073(21)	0.126(32)
$\text{Re}(A_2)^{\text{extrap to coarse}}$	0.752(45)	0.808(42)	0.775(39)
$\times 10^8$ (GeV)			
$\text{Re}(A_2)^{\text{extrap to phys}}$	0.697(44)	0.748(41)	0.719(38)
$\times 10^8$ (GeV)			

Table 105: Results for C_0 , C_1 , and C_2 in the fit of $\text{Re}(A_2)$ as a function of valence quark masses (Eq. 7.34) for different sea light quark masses on the 32^3 fine lattices. Using the valence quark masses of Table 103 we can then use the fit results to determine the value of $\text{Re}(A_2)$ that would result on the fine lattices at zero twist at the meson masses from the 32^3 coarse calculation and at the physical meson masses ($\text{Re}(A_2)^{\text{extrap to coarse}}$ and $\text{Re}(A_2)^{\text{extrap to phys}}$ respectively).

	$m_l^{\text{sea}} = 0.004$	$m_l^{\text{sea}} = 0.006$	$m_l^{\text{sea}} = 0.008$
$C_0 \times 10^{11}$ (GeV)	3.68(36)	3.57(32)	3.71(33)
$C_1 \times 10^{12}$ (GeV)	-1.19(39)	-1.57(39)	-1.03(35)
$C_2 \times 10^{12}$ (GeV)	-1.451(36)	-1.463(33)	-1.501(34)
χ^2	4.9(1.4)	4.3(1.4)	4.5(1.6)
$\chi^2/\text{d.o.f.}$	0.98(28)	0.87(27)	0.91(32)
$\text{Im}(A_2)_{\text{extrap to coarse}}$ $\times 10^{13}$ (GeV)	-14.73(37)	-15.00(35)	-15.23(34)
$\text{Im}(A_2)_{\text{extrap to phys}}$ $\times 10^{13}$ (GeV)	-14.73(37)	-14.99(35)	-15.23(34)

Table 106: Same as Table 105 but for $\text{Im}(A_2)$. The fit is not very good for $\text{Im}(A_2)$ as the data shows visible curvature (see Figure 120), so these results are not used in further analyses.

	Extrap. to Coarse	Extrap. to Physical	Coarse/ Physical
m_π (MeV)	142.8	139.6	1.023
m_K (MeV)	509.0	493.7	1.031
$E_{\pi\pi}$ (MeV)	302.7(1.1)	296.6(1.2)	1.02058(18)
$\text{Re}(A_2) \times 10^8$ (GeV)	0.779(24)	0.722(24)	1.0789(22)

Table 107: Error weighted averages over different sea light quark masses of the results of extrapolations/interpolations on the 32^3 fine lattices. Results for each sea light quark mass are found in Tables 104 and 105. The last two rows are error weighted averages of the quantities indicated that were extrapolated to the kaon and pion masses of the first and second row. The first column is an extrapolation to the masses of the 32^3 coarse lattices, the second column an extrapolation to the physical masses, and the last column is the first divided by the second. Note that error on the fine lattice spacing is not propagated into values quoted in physical units (consistent with the rest of the extrapolation analysis).

B_0	2.124(16)
B_1	0.00692(28)
χ^2	11.1(6.8)
$\chi^2/\text{d.o.f.}$	0.69(43)
m_l^{coarse}	0.001169(62)
m_s^{coarse}	0.06665(45)
m_l^{phys}	0.001045(62)
m_s^{phys}	0.06256(42)

Table 108: Results for B_0 and B_1 in the fit of pseudoscalar meson mass as a function of valence quark masses (Eq. 7.24) on the 24^3 quenched lattices. This fit is then used to determine the valence quark masses that would yield the pion and kaon masses of the 32^3 coarse lattices (m_l^{coarse} and m_s^{coarse}) and the valence quark masses that would yield the physical pion and kaon masses (m_l^{phys} and m_s^{phys}). All quantities are in the lattice units of the 24^3 quenched lattices.

	$n_{\text{tw}} = 0$	$n_{\text{tw}} = 1$	$n_{\text{tw}} = 2$	$n_{\text{tw}} = 3$
A_0	17.53(16)	17.14(73)	14.9(2.3)	24.5(9.5)
A_1	0.0273(12)	0.1038(60)	0.202(18)	0.196(82)
χ^2	0.141(91)	0.009(56)	0.22(35)	0.022(72)
$\chi^2/\text{d.o.f.}$	0.141(91)	0.009(56)	0.22(35)	0.022(72)
$E_{\pi\pi}^{\text{extrap to coarse}}$	0.2187(18)	0.3519(78)	0.468(17)	0.474(74)
$E_{\pi\pi}^{\text{extrap to phys}}$	0.2137(19)	0.3489(80)	0.466(17)	0.470(76)

Table 109: Results for A_0 and A_1 in the fit of $E_{\pi\pi}^2$ as a function of light quark mass (Eq. 7.33) for different numbers of twists (n_{tw}) on the 24^3 quenched lattices. Using the light quark masses of Table 108 we can then use the fit results to determine the $E_{\pi\pi}$ that would result on the quenched lattices with n_{tw} twists at the pion mass from the 32^3 coarse calculation and at the physical pion mass ($E_{\pi\pi}^{\text{extrap to coarse}}$ and $E_{\pi\pi}^{\text{extrap to phys}}$ respectively). All quantities are in the lattice units of the 24^3 quenched lattices.

	$n_{\text{tw}} = 0$	$n_{\text{tw}} = 1$	$n_{\text{tw}} = 2$	$n_{\text{tw}} = 3$
$C_0 \times 10^7$ (GeV)	1.016(55)	1.43(11)	1.53(25)	1.78(54)
$C_1 \times 10^6$ (GeV)	1.697(89)	1.29(18)	1.45(38)	4.22(97)
$C_2 \times 10^8$ (GeV)	0.253(51)	1.08(12)	1.68(25)	-0.02(67)
χ^2	2.73(83)	0.32(18)	0.45(49)	3.0(2.0)
$\chi^2/\text{d.o.f.}$	0.227(69)	0.027(15)	0.037(41)	0.25(16)
$\text{Re}(A_2)^{\text{coarse}} \times 10^8$ (GeV)	1.128(54)	2.18(13)	2.87(27)	1.66(66)
$\text{Re}(A_2)^{\text{phys}} \times 10^8$ (GeV)	1.066(53)	2.10(13)	2.79(26)	1.53(66)

Table 110: Results for C_0 , C_1 , and C_2 in the fit of $\text{Re}(A_2)$ as a function of quark masses (Eq. 7.34) for different numbers of twists n_{tw} on the 24^3 quenched lattices. Using the quark masses of Table 108 we can then use the fit results to determine the value of $\text{Re}(A_2)$ that would result on the fine lattices at n_{tw} twists, at the meson masses from the 32^3 coarse calculation and at the physical meson masses ($\text{Re}(A_2)^{\text{coarse}}$ and $\text{Re}(A_2)^{\text{phys}}$ respectively).

	$n_{\text{tw}} = 0$	$n_{\text{tw}} = 1$	$n_{\text{tw}} = 2$	$n_{\text{tw}} = 3$
$C_0 \times 10^{12}$ (GeV)	-1.06(31)	-0.46(33)	0.44(74)	0.2(1.1)
$C_1 \times 10^{11}$ (GeV)	5.54(79)	3.39(91)	2.1(1.6)	-1.8(3.2)
$C_2 \times 10^{12}$ (GeV)	-1.689(64)	-1.392(66)	-1.24(12)	-0.75(19)
χ^2	1.4(1.0)	0.53(94)	0.09(37)	0.9(1.7)
$\chi^2/\text{d.o.f.}$	0.114(84)	0.044(78)	0.007(31)	0.08(14)
$\text{Im}(A_2)^{\text{coarse}} \times 10^{13}$ (GeV)	-16.94(63)	-13.83(62)	-11.9(1.2)	-7.6(1.8)
$\text{Im}(A_2)^{\text{phys}} \times 10^{13}$ (GeV)	-16.97(63)	-13.86(63)	-11.9(1.2)	-7.6(1.8)

Table 111: Same as Table 110 but for $\text{Im}(A_2)$.

	Extrap. to Coarse	Extrap. to Physical	Coarse/ Physical
m_π (MeV)	142.8	139.6	1.023
m_K (MeV)	509.0	493.7	1.031
$E_{\pi\pi}$ (MeV)	485.7	493.7	0.984
$\text{Re}(A_2) \times 10^8$ (GeV)	2.28(18)	2.25(18)	1.0123(84)
$\text{Im}(A_2) \times 10^{13}$ (GeV)	-13.55(85)	-13.44(84)	1.0081(71)

Table 112: Final results for $\text{Re}(A_2)$ and $\text{Im}(A_2)$ extrapolated to the masses and kinematics of the coarse lattice and of the physical decay on the 24^3 quenched lattices. The first three rows give the meson masses and two pion energy to which we extrapolated and the last two rows give results of the extrapolation for $\text{Re}(A_2)$ and $\text{Im}(A_2)$. In this case we *do* propagate the statistical error of the lattice spacing of the 24^3 quenched lattices, because we want to compare these values to the coarse calculation. However, we make no estimate of the *systematic* error of this lattice spacing, to which we ascribe the discrepancy between $\text{Re}(A_2)$, $\text{Im}(A_2)$ in this table and their values from the coarse lattice and from experiment (see Section 7.4.3). The last column of the table is the first column divided by the second.

Quantity Used to Set the Scale	a^{-1} (GeV)	$\text{Re}(A_2) \times 10^8$ (GeV)	$\text{Im}(A_2) \times 10^{13}$ (GeV)
m_Ω	1.374(24)	1.461(87)	-8.67(45)
f_K	1.342(22)	1.361(87)	-8.07(47)
r_0	1.410(29)	1.58(11)	-9.38(63)
Average	1.375(17)	1.467(75)	-8.71(39)
Std. Dev.	0.034(15)	0.110(50)	0.65(30)
Std. Dev./ Avg.	0.025(11)	0.075(34)	0.075(34)

Table 113: $\text{Re}(A_2)$ and $\text{Im}(A_2)$ calculated using m_Ω , f_K , and r_0 to determine the lattice spacing on the 32^3 coarse lattices. The standard deviation of these different determinations of $\text{Re}(A_2)$ and $\text{Im}(A_2)$ is an estimate of the error due to scaling violations in these quantities. Errors in brackets are statistical errors from both the lattice quantity and the lattice spacing. These errors are propagated using the superjackknife method.

Source of Error	Re(A_2)	Im(A_2)
Scaling Violations	7.5%	7.5%
Finite Volume Effects	6.1%	6.7%
Partial Quenching	3.5%	1.7%
Derivative of the Phase Shift	1.5%	1.5%
Unphysical Masses and Kinematics	1.2%	0.8%
Non-Perturbative Renormalization	8.8%	20%
Wilson Coefficients	1.1%	0.6%
Total	13.7%	22.5%

Table 114: Systematic error budget for the 32^3 coarse calculation. Estimates for the different sources of error discussed in Section 7.5 are shown for $\text{Re}(A_2)$ and $\text{Im}(A_2)$. The total is obtained by adding the errors in quadrature.

m_π	142.8 MeV
m_K	509.0 MeV
m_η	581.9 MeV
L	0.0233 MeV $^{-1}$
f	112.0 MeV

Table 115: Values used in the numerical estimate of finite volume corrections from chiral perturbation theory. The quantities m_π , m_K , and L are the values from the 32^3 coarse calculation, m_η is found from Eq. B.14, and f (the pseudoscalar decay constant in the chiral limit) is taken from [69].

n	$m(n)$	$\delta_\beta(\pi K \eta)$	$\delta_\beta(\pi K \pi)$	$\delta_\beta(K \pi \pi)$
1	6	0.328e-5	0.391e-2	0.920e-2
2	12	0.311e-7	0.113e-2	0.255e-2
3	8	0.362e-9	0.191e-3	0.417e-3
4	6	0.917e-11	0.470e-4	0.101e-3
5	24	0.189e-11	0.720e-4	0.153e-3
6	24	0.169e-12	0.307e-4	0.647e-4
7	0	0	0	0
8	12	-0.288e-14	0.349e-5	0.728e-5
9	30	0.423e-13	0.450e-5	0.935e-5
10	24	-0.336e-14	0.193e-5	0.401e-5
11	24	0.278e-13	0.108e-5	0.223e-5
12	8	-0.134e-13	0.205e-6	0.424e-6
13	24	-0.346e-13	0.362e-6	0.746e-6
14	48	-0.103e-12	0.435e-6	0.895e-6
15	0	0	0	0
Total	-	0.331e-5	0.539e-2	0.125e-1
$\Delta_L \beta = \delta_\beta/(4\pi^2)$	-	0.838e-7	0.137e-3	0.317e-3

Table 116: Terms contributing to the sum of Eq. B.33. In the header row of the table $\delta_\beta(ABC)$ stands for $\delta_\beta(m_A L, m_B L, m_C L)$.

Figures

Figure 1: Illustration of gauge links on a two-dimensional slice of the lattice. The combination of links that makes up a plaquette is also shown.

Figure 2: Schematic diagram showing that the OPE can be pictured as replacing the exchange of heavy virtual particles with a point-like interaction.

Figure 3: Diagrams in the full theory corresponding to the four quark operators of Eq. 4.7-4.16. (a) Current-current operators (Q_1 - Q_2). (b) QCD penguin operators (Q_3 - Q_6). (c) Electroweak penguin operators (Q_7 - Q_{10}). This figure is taken from [57].

Figure 4: Form of diagrams that contribute at lowest order to a change in $\pi\pi$ phase shift when the weak interaction is turned on. The external lines represent pions and the double line represents a kaon propagator. This figure is taken from [64].

Figure 5: Effective mass plot of the wall-point pion correlator with 0 momentum on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 6: Effective mass plot of the wall-wall pion correlator with 0 momentum on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 7: Effective mass plot of the wall-point pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 8: Effective mass plot of the wall-wall pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 9: Effective mass plot of the wall-point kaon correlator with 0 momentum on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 10: Effective mass plot of the wall-wall kaon correlator with 0 momentum on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 11: Effective mass plot of the wall-point kaon correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 12: Effective mass plot of the wall-wall kaon correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 13: Effective mass plot of the two pion correlator with 0 momentum on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 14: Effective mass plot of the two pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 15: Fit range dependence of the fitted $E_{\pi\pi}$ for the two pion correlator with $p = \sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. Specifically, we show how this quantity varies as a function of $t_{\max}$ of the fit range with $t_{\min}$ held fixed at 5.

Figure 16: Effective mass plot of the ratio $C_{\pi\pi}/C_{\pi}^2$ with zero momentum on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 17: Effective mass plot of the ratio $C_{\pi\pi}/C_\pi^2$ with momentum $p = \sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 18: Fit range dependence of the fitted $\Delta E(p)$ to the ratio $C_{\pi\pi}/C_\pi^2$ for momentum $p = \sqrt{2}\pi/L$ (two twists) on the coarse 32^3 lattices. Specifically, we show how this quantity varies as a function of $t_{\max}$ of the fit range with $t_{\min}$ held fixed at 5.

Figure 19: Quark flow diagram of the $\Delta I = 3/2$ $K \rightarrow \pi\pi$ correlator computed in this report. The time t_K of the kaon source, t_π of the pion sink, and t of the weak operator, are indicated. The times t_K and t_π are fixed at the values shown for a given calculation, while the time t is varied.

Figure 20: The kaon source and pion sink can't be too close together or too far apart if we want to get a good signal for the $K \rightarrow \pi\pi$ correlator.

Figure 21: The correlators $C_{K \rightarrow \pi\pi}$ and $C'_{K \rightarrow \pi\pi}$. These are images of each other in the sense that the value of these correlators at the time slices marked a and a' should be the same in the average over configurations, but are statistically independent in that the values at these two time slices will not necessarily be exactly the same on a single configuration.

Figure 22: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with 0 momentum, the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. The effective mass is expected to have a plateau at the value $\Delta E_{K \rightarrow \pi\pi} = E_{\pi\pi} - m_K$. This value is calculated independently from the fits to the two point functions (m_K is the kaon wall-wall $p = 0$ value of E from Table 2 and $E_{\pi\pi}$ is taken from the right-most column of Table 9). The independently calculated value from the two point functions is shown as a solid red line on the plot and the error is indicated by two dashed red lines above and below the solid line. This should serve as a consistency check that the $K \rightarrow \pi\pi$ correlator in the plot is coupling to the same kaon and two pion states as the kaon and two pion correlators did.

Figure 23: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with 0 momentum, the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 24: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with 0 momentum, the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 25: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 26: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 27: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 28: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 20$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 29: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. This is the same as Figure 25, and is reproduced here in order to see more easily the variation of the correlator with increasing kaon-pion separation $t_K - t_\pi$.

Figure 30: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the $(27,1)$ weak operator, and $t_\pi = 0$, $t_K = 28$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 31: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 32$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 32: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 36$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 33: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 40$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 34: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 44$, on the coarse 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 22, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 35: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with 0 momentum, the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 36: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with 0 momentum, the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 37: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with 0 momentum, the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 38: Quotient of correlators $C_{K \rightarrow \pi\pi} / (C_K C_{\pi\pi})$ with $p = \sqrt{2}\pi/L$ (two twists), the (27,1) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 39: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 40: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with $p = \sqrt{2}\pi/L$ (two twists), the (8,8) mixed weak operator, and $t_\pi = 0$, $t_K = 24$, on the coarse 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 41: Effective mass plot of the pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 42: Effective mass plot of the pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 43: Effective mass plot of the pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 44: Effective mass plot of the kaon correlator with valence strange quark mass $m_s^{\text{val}} = 0.03$ and valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 45: Effective mass plot of the kaon correlator with valence strange quark mass $m_s^{\text{val}} = 0.03$ and valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 46: Effective mass plot of the kaon correlator with valence strange quark mass $m_s^{\text{val}} = 0.03$ and valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 47: Effective mass plot of the two pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.004$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 48: Effective mass plot of the two pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.006$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 49: Effective mass plot of the two pion correlator with valence light quark mass $m_l^{\text{val}} = 0.002$ on the $m_l^{\text{sea}} = 0.008$ fine 32^3 lattices. All correlators for the fine 32^3 lattices have zero momentum and are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 50: The two correlators that are folded together in Eq. 6.66. The folding is achieved in Eq. 6.66 by reflecting the bottom correlator in the dotted red line at $t = t' + N_t/2$ and then averaging the two correlators together. Reflecting the bottom correlator in the dotted red line results in a correlator that will be the same as the top correlator in the average over configurations. For example, the reflection maps the point a' to the position of the point a . However, the two correlators are statistically independent in that the values at a and a' will not necessarily be exactly the same on a single configuration.

Figure 51: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The effective mass is expected to have a plateau at the value $\Delta E_{K \rightarrow \pi\pi} = E_{\pi\pi} - m_K$, which is calculated independently from the fits to the two point functions (m_K is taken from Table 15 and $E_{\pi\pi}$ is taken from Table 18). The independently calculated value from the two point functions is shown as a solid red line on the plot and the error is indicated by two dashed red lines above and below the solid line. This should serve as a consistency check that the $K \rightarrow \pi\pi$ correlator in the plot is coupling to the same kaon and two pion states as the kaon and two pion correlators did.

Figure 52: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 53: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 54: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 55: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 56: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 57: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 58: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 59: Effective mass plot of the $K \rightarrow \pi\pi$ correlator with the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices. The pions have zero momentum as in all calculations on the fine 32^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 51, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 60: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 61: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 62: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (27,1) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 63: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 64: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 65: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 66: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.004$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 67: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.006$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 68: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ for the (8,8) mixed weak operator, $m_s^{\text{val}} = 0.03$, $m_l^{\text{val}} = 0.002$, and $m_l^{\text{sea}} = 0.008$ on the fine 32^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 69: Effective mass plot of the pion correlator with 0 momentum and $m_l=0.0023$ on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 70: Effective mass plot of the pion correlator with momentum π/L (one twist) and $m_l=0.0023$ on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 71: Effective mass plot of the pion correlator with momentum $\sqrt{2}\pi/L$ (two twists) and $m_l=0.0023$ on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 72: Effective mass plot of the pion correlator with momentum $\sqrt{3}\pi/L$ (three twists) and $m_l=0.0023$ on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 73: Effective mass plot of the kaon correlator with 0 momentum and $m_l=0.0023$, $m_s=0.062$, on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 74: Effective mass plot of the kaon correlator with momentum π/L (one twist) and $m_l=0.0023$, $m_s=0.062$, on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 75: Effective mass plot of the kaon correlator with momentum $\sqrt{2}\pi/L$ (two twists) and $m_l=0.0023$, $m_s=0.062$, on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 76: Effective mass plot of the kaon correlator with momentum $\sqrt{3}\pi/L$ (three twists) and $m_l=0.0023$, $m_s=0.062$, on the quenched 24^3 lattices. All correlators are wall-wall correlators. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 77: Effective mass plot of the two pion correlator with zero momentum and $m_l = 0.0023$ on the quenched 24^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 78: Effective mass plot of the two pion correlator with momentum $p = \pi/L$ (one twist) and $m_l = 0.0023$ on the quenched 24^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 79: Effective mass plot of the two pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists) and $m_l = 0.0023$ on the quenched 24^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 80: Effective mass plot of the two pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists) and $m_l=0.0023$ on the quenched 24^3 lattices. The fitted effective mass is shown as a solid red line and the error is indicated by two dashed red lines above and below the solid line.

Figure 81: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The effective mass is expected to have a plateau at the value $\Delta E_{K \rightarrow \pi\pi} = E_{\pi\pi} - m_K$, which is calculated independently from the fits to the two point functions (m_K is taken from the first row of Table 25 and $E_{\pi\pi}$ is taken from Table 30). The independently calculated value from the two point functions is shown as a solid red line on the plot and the error is indicated by two dashed red lines above and below the solid line. This should serve as a consistency check that the $K \rightarrow \pi\pi$ correlator in the plot is coupling to the same kaon and two pion states as the kaon and two pion correlators did.

Figure 82: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l = 0.0023$ and $m_s = 0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 83: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l = 0.0023$ and $m_s = 0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 84: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 85: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 86: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 87: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 88: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 89: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 90: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 91: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 92: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 93: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l = 0.0023$ and $m_s = 0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 20$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 94: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l = 0.0023$ and $m_s = 0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line. This is the same as Figure 82, and is reproduced here in order to see the effect of increasing Δt_K with other parameters held constant.

Figure 95: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l = 0.0023$ and $m_s = 0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 28$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 96: Effective mass plot of the $K \rightarrow \pi\pi$ pion correlator with momentum $p = \pi/L$ (one twist), $m_l = 0.0023$ and $m_s = 0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 32$ on the quenched 24^3 lattices. The quantity $\Delta E_{K \rightarrow \pi\pi}$, as described in Figure 81, is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 97: Quotient of correlators $C_{K \rightarrow \pi\pi} / (C_K C_{\pi\pi})$ with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (27,1) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 98: Quotient of correlators $C_{K \rightarrow \pi\pi} / (C_K C_{\pi\pi})$ with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the $(27,1)$ weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 99: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the $(27,1)$ weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 100: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the $(27,1)$ weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 101: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 102: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 103: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 104: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 105: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with zero momentum, $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 106: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \pi/L$ (one twist), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 107: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{2}\pi/L$ (two twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 108: Quotient of correlators $C_{K \rightarrow \pi\pi}/(C_K C_{\pi\pi})$ with momentum $p = \sqrt{3}\pi/L$ (three twists), $m_l=0.0023$ and $m_s=0.062$, the (8,8) mixed weak operator, and separation between the kaon and two pions of $\Delta t_K = 24$ on the quenched 24^3 lattices. It is fit to a constant which is indicated on the graph by a solid red line, and its error by dashed red lines above and below the solid line.

Figure 109: Plot of $I = 2$ two pion s wave phase shift (δ_0) as a function of k , where k is related to the two pion energy by $E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$. The results from zero and two twists are the two red points on the plot. For comparison we show several experimental data points [73, 74, 75] along with a phenomenological fit to the data [83] shown as a dashed curve. We show a close-up of the top left hand corner of this plot in Figure 110 so that the result from zero twists can be seen more clearly.

Figure 110: Zoom in on the top left corner of Figure 109 so that the result from zero twists can be seen more clearly. The phenomenological curve at the small values of k shown here uses input from chiral perturbation theory fits to a larger class of experimental data not shown on the plot [73]. The blue dashed line is the straight line $\delta_0(k) = a_0 k$, where a_0 is the scattering length determined from this chiral perturbation theory analysis. The blue dashed line thus corresponds to the leading term in a series expansion of δ_0 in powers of k . We can see that for $k \lesssim 20$ MeV the leading term is a good approximation to the phenomenological curve and both agree quite well with the result of this calculation.

Figure 111: Plot of $I = 2$ two pion s wave phase shift (δ_0) as a function of the dimensionless variable $q = kL/2\pi$ for $m_l = 0.0023$ on the quenched 24^3 lattices. Data points from zero, one, and two twists are shown, while the data point from three twists is outside of the area shown on the plot. (The phase shift for three twists has a very large error bar and is within error of 0). We show the fit Eq. 7.22 to the data (solid line) as well as just the linear term in the fit (dotted line). The linear term is dominant in the vicinity of the data point for zero twists, but the cubic term takes effect by the time we reach the non-zero twist data. The data point for three twists was not included in the fit since it is so noisy.

Figure 112: Plot of $I = 2$ two pion s wave phase shift (δ_0) as a function of the dimensionless variable $q = kL/2\pi$ for $m_l = 0.0047$ on the quenched 24^3 lattices. Data points from zero, one, two and three twists are shown. We show the fit Eq. 7.22 to the data (solid line) as well as just the linear term in the fit (dotted line). The linear term is dominant in the vicinity of the data point for zero twists, but the cubic term takes effect by the time we reach the non-zero twist data. The data point for three twists was not included in the fit since it is so noisy.

Figure 113: Plot of $I = 2$ two pion s wave phase shift (δ_0) as a function of the dimensionless variable $q = kL/2\pi$ for $m_l = 0.0071$ on the quenched 24^3 lattices. Data points from zero, one, two and three twists are shown. We show the fit Eq. 7.22 to the data (solid line) as well as just the linear term in the fit (dotted line). The linear term is dominant in the vicinity of the data point for zero twists, but the cubic term takes effect by the time we reach the non-zero twist data. The data point for three twists was not included in the fit since it is so noisy.

Figure 114: Plot of $E_{\pi\pi}^2$ vs. number of twists (n_{tw}) for the 32^3 coarse lattices, and extrapolation to energy conserving kinematics ($E_{\pi\pi}^2 = m_K^2$), in order to determine the value of n_{tw} corresponding to these kinematics.

Figure 115: Plot of $\text{Re}(A_2)$ vs. number of twists (n_{tw}) for the 32^3 coarse lattices, and extrapolation to energy conserving kinematics.

Figure 116: Plot of $\text{Im}(A_2)$ vs. number of twists (n_{tw}) for the 32^3 coarse lattices, and extrapolation to energy conserving kinematics.

Figure 117: Linear fit to pseudoscalar meson mass as a function of the sum of valence quark masses for $m_l^{\text{sea}} = 0.004$ on the 32^3 fine lattices. The blue square and red diamond correspond to the pion and kaon mass from the 32^3 coarse lattices, and allow us to extract the valence quark masses on the 32^3 fine lattices that would give rise to these pion and kaon masses. All quantities are in the lattice units of the 32^3 fine lattices.

Figure 118: Linear fit to $E^2_{\pi\pi}$ as a function of the light valence quark mass m_l^{val} , for $m_l^{\text{sea}} = 0.004$ on the 32^3 fine lattices. The red square is an extrapolation to the value of m_l^{val} that would yield the pion mass of the 32^3 coarse lattices, as determined from the fit shown in Figure 117. This extrapolation thus yields the two pion energy for zero twists at the pion mass of the coarse lattices. All quantities are in the lattice units of the 32^3 fine lattices.

Figure 119: Linear fit to $\text{Re}(A_2)$ as a function of the valence quark masses for $m_l^{\text{sea}} = 0.004$ on the 32^3 fine lattices. Valence light quark mass (m_l) appears on the x axis and data for valence strange quark masses $m_s = 0.025$ and 0.03 are shown in black and red respectively, along with the fit (lines of the same colors for the fit evaluated at the same value of m_s). The green dotted line is the fit evaluated at $m_s = 0.0279$, the strange quark mass corresponding to the meson masses of the 32^3 coarse lattices as determined from the fit of Figure 117. We then extrapolate along the green dotted line to $m_l = 0.000288$, the light quark mass corresponding to the meson masses of the coarse lattices, and the result is shown in green and surrounded by a circle. Thus we have found the value of $\text{Re}(A_2)$ corresponding to the pion and kaon masses of the coarse lattices.

Figure 120: Same as Figure 119, but for $\text{Im}(A_2)$. Here the data is *not* fit well by a linear form, unlike $\text{Re}(A_2)$, so we do not use the results of the extrapolation for any analyses.

Figure 121: Illustration showing that $E_{\pi\pi}$ will be lower in a larger volume at zero twist and with a fixed pion mass. The solid black line shows the actual $I = 2$, s wave two pion phase shift as a function of k , (for the purpose of this illustration we use the phenomenological curve in Figures 109 and 110). The red and green dashed lines show the function $-\phi(kL/(2\pi))$ (c.f. Eq. 5.60) with L the spatial extent in physical units of the 32^3 coarse and fine lattices respectively. The coarse lattice has a larger value of L , and thus the green curve is just the red curve stretched to the right. It can be seen from Eq. 5.59 (with $n = 0$) that the intersection point of a dashed line and the solid black line gives the values of k and δ_0 that will be obtained when measuring the two pion energy on the lattice corresponding to that dashed line. The two pion energy is given by $E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$, and thus we expect the coarse lattice to yield a smaller two pion energy at fixed m_π . We can see that in general, the larger the volume, the smaller the two pion energy we will obtain, and as $L \rightarrow \infty$, $E_{\pi\pi} \rightarrow 2m_\pi$.

Figure 122: Linear fit to pseudoscalar meson mass as a function of the sum of valence quark masses on the 24^3 quenched lattices. The blue square and red diamond correspond to the pion and kaon mass from the 32^3 coarse lattices, and allow us to extract the valence quark masses on the 24^3 quenched lattices that would give rise to these pion and kaon masses. All quantities are in the lattice units of the 24^3 quenched lattices.

Figure 123: Linear fit to $E^2_{\pi\pi}$ as a function of the light quark mass on the 24^3 quenched lattices. We show the data and fits for different numbers of twists, $n_{tw} = 0, 1, 2$, and 3 , on the same graph. The circles are extrapolations to the value of m_l that would yield the pion mass of the 32^3 coarse lattices, as determined from the fit shown in Figure 122. These extrapolations thus yield the two pion energies for $0, 1, 2$, and 3 twists at the pion mass of the coarse lattices. All quantities are in the lattice units of the 24^3 quenched lattices.

Figure 124: Linear fit to $\text{Re}(A_2)$ as a function of the quark masses for zero twists ($n_{\text{tw}} = 0$) on the 24^3 quenched lattices. Strange quark mass (m_s) appears on the x axis and data for light quark masses $m_l = 0.0023$, 0.0047 , and 0.0071 are shown in black, red, and green respectively, along with the fit (lines of the same colors for the fit evaluated at the same value of m_s). The blue dotted line is the fit evaluated at $m_l = 0.00117$, the light quark mass corresponding to the meson masses of the 32^3 coarse lattices as determined from the fit of Figure 122. We then interpolate along the blue dotted line to $m_s = 0.0667$, the strange quark mass corresponding to the meson masses of the coarse lattices, and the result is shown in blue and surrounded by a circle. Thus we have found the value of $\text{Re}(A_2)$ corresponding to the pion and kaon masses of the coarse lattices for $n_{\text{tw}} = 0$.

Figure 125: Same as Figure 124 but for one twist ($n_{\text{tw}} = 1$).

Figure 126: Same as Figure 124 but for two twists ($n_{\text{tw}} = 2$).

Figure 127: Same as Figure 124 but for three twists ($n_{\text{tw}} = 3$).

Figure 128: Same as Figure 124 but for $\text{Im}(A_2)$.

Figure 129: Same as Figure 128 but for one twist ($n_{\text{tw}} = 1$).

Figure 130: Same as Figure 128 but for two twists ($n_{\text{tw}} = 2$).

Figure 131: Same as Figure 128 but for three twists ($n_{\text{tw}} = 3$).

Figure 132: Plot of $E_{\pi\pi}^2$, extrapolated to the pion mass of the coarse lattice, as a function of n_{tw} for the 24^3 quenched lattices. The extrapolation from which this data was obtained is shown in Figure 123. We do a linear interpolation between the $n_{tw} = 1$ and $n_{tw} = 2$ points (solid black line) to the two pion energy of the coarse lattice (red circle). This tells us the value of n_{tw} corresponding to the kinematics of the coarse lattice, so that we can interpolate $\text{Re}(A_2)$ and $\text{Im}(A_2)$ to this point.

Figure 133: Plot of $\text{Re}(A_2)$, extrapolated to the pion and kaon masses of the coarse lattice, as a function of n_{tw} for the 24^3 quenched lattices. The extrapolations from which these data were obtained are shown in Figures 124 - 127. We do a linear interpolation between the $n_{\text{tw}} = 1$ and $n_{\text{tw}} = 2$ points (solid red line) to the value of n_{tw} (determined from Figure 132) corresponding to the two pion energy of the coarse lattice (red square).

Figure 134: Same as Figure 133 but for $\text{Im}(A_2)$.

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A Normalization Conventions for A_2

A.1 Relation to Decay Rates

The definition of A_2 is somewhat dependent on convention since A_2 is an amplitude, however decay rates are clearly not dependent on convention. Thus we show explicitly how $|A_2| \approx \text{Re}(A_2)$ ¹ is related to the decay rate of the process $K^+ \rightarrow \pi^+\pi^0$. This decay rate can be measured experimentally, and thus we use it to quote an “experimental value” of $\text{Re}(A_2)$.

A_2 is more directly related to the processes $K_s \rightarrow \pi^+\pi^-$ and $K_s \rightarrow \pi^0\pi^0$ as in Eq. 2.5. However, this equation implies that there can be interference between the complex amplitudes of the two processes, and one can only disentangle them to find $|A_2|$ and $|A_0|$ if one knows the values of the phase shifts δ_2 and δ_0 . The measurement of these phase shifts is significantly less accurate than the measurement of the decay rates. Furthermore, $K_s \rightarrow \pi^+\pi^-$ has Coulomb interaction between the two pions in the final state which affects the decay rate and is not accounted for in the lattice calculation of this report. Thus we will use the relation between $|A_2|$ and $K^+ \rightarrow \pi^+\pi^0$, which

¹Recall that this approximation is valid because CP violation is small. Its validity is confirmed by the lattice calculation of this report.

is a consequence of isospin symmetry², because no knowledge of phase shifts is required and because there is no Coulomb interaction between the π^+ and the π^0 .

We define the amplitude A_{+0} via

$$A_{+0} = \langle \pi^+(\mathbf{p})\pi^0(-\mathbf{p}) | \mathcal{L}_W(0) | K^+(0) \rangle \quad (\text{A.1})$$

(we choose not to separate out a phase in this definition). The normalization of states is given by Eq. 5.65 and 5.66, except that for pions that are not identical the second term in the square brackets in Eq. 5.66 no longer appears, i.e.

$$\langle \pi^+(\mathbf{p}_1)\pi^0(\mathbf{p}_2) | \pi^+(\mathbf{p}'_1)\pi^0(\mathbf{p}'_2) \rangle = 4E_1E_2(2\pi)^6\delta^3(\mathbf{p}_1 - \mathbf{p}'_1)\delta^3(\mathbf{p}_2 - \mathbf{p}'_2). \quad (\text{A.2})$$

The quantity A_{+0} is related directly to Γ_{+0} , the decay rate of the process $K^+ \rightarrow \pi^+\pi^0$, by

$$\Gamma_{+0} = \frac{1}{8\pi} |A_{+0}|^2 \frac{p_\pi}{m_K^2} \quad (\text{A.3})$$

where p_π is the magnitude of the pion momentum given by energy conservation

$$p_\pi = \sqrt{\frac{m_K^2}{4} - m_\pi^2}. \quad (\text{A.4})$$

We neglect the difference in mass between the charged and neutral mesons, in

²We assume isospin symmetry throughout because it is assumed by the lattice calculation, and any errors due to this approximation are much smaller than the errors in the lattice calculation.

accordance with isospin symmetry. For numerical values we use the charged meson masses. From experiment we have [5]

$$\Gamma_{+0} = 1.0984 \times 10^{-14} \text{ MeV}. \quad (\text{A.5})$$

The error in this measurement is less than 1%, which is much more accurate than the lattice calculation with which we are comparing, so we ignore it. Putting the value of Γ_{+0} into Eq. A.3 gives

$$|A_{+0}| = 1.818 \times 10^{-8} \text{ GeV}. \quad (\text{A.6})$$

The two pion isospin state $I = 2$, $m_I = 1$ is given by the *symmetrized* $\pi^+\pi^0$ state,

$$|\pi\pi_{m_I=1}^{I=2}(\mathbf{p}, -\mathbf{p})\rangle = \frac{1}{\sqrt{2}} [|\pi^+(\mathbf{p})\pi^0(-\mathbf{p})\rangle + |\pi^+(-\mathbf{p})\pi^0(\mathbf{p})\rangle]. \quad (\text{A.7})$$

Thus, using rotational symmetry in Eq. A.1, we have

$$A_{+0} = \frac{1}{2} [\langle \pi^+(\mathbf{p})\pi^0(-\mathbf{p}) | + \langle \pi^+(-\mathbf{p})\pi^0(\mathbf{p}) |] \mathcal{L}_W(0) | K^+(0) \rangle \quad (\text{A.8})$$

$$= \frac{1}{\sqrt{2}} \langle \pi\pi_{m_I=1}^{I=2}(\mathbf{p}, -\mathbf{p}) | \mathcal{L}_W(0) | K^+(0) \rangle. \quad (\text{A.9})$$

We can use the Wigner-Eckart theorem to write

$$\frac{\langle \pi\pi_{m_I=0}^{I=2}(\mathbf{p}, -\mathbf{p}) | \mathcal{L}_W(0) | K^0(0) \rangle}{\langle \pi\pi_{m_I=1}^{I=2}(\mathbf{p}, -\mathbf{p}) | \mathcal{L}_W(0) | K^+(0) \rangle} = \frac{\langle 2, 0 | \frac{3}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \rangle}{\langle 2, 1 | \frac{3}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \rangle} = \sqrt{\frac{2}{3}}. \quad (\text{A.10})$$

Therefore,

$$A_{+0} = \frac{\sqrt{3}}{2} \langle \pi \pi_{m_I=0}^{I=2}(\mathbf{p}, -\mathbf{p}) | \mathcal{L}_W(0) | K^0(0) \rangle \quad (\text{A.11})$$

$$= \sqrt{\frac{3}{2}} A_2 e^{i\delta_2} \quad (\text{A.12})$$

where we have used Eq. 2.5 in the last line.

Thus we have

$$\text{Re}(A_2) \approx |A_2| = \sqrt{\frac{2}{3}} |A_{+0}| = 1.484 \times 10^{-8} \text{ GeV} \quad (\text{A.13})$$

from experiment.

A.2 Calculating A_2 from the Lattice Matrix Elements

We now proceed to show precisely how A_2 can be extracted from the correlators that are calculated on the lattice described in Chapter 6. We first use the Lellouch-Lüscher relation, Eq. 5.76 to write

$$|A_2| = \frac{1}{\sqrt{2}} \sqrt{8\pi \left(q \frac{\partial \phi}{\partial q} + k \frac{\partial \delta_0}{\partial k} \right) \left(\frac{m_K E_{\pi\pi}^2}{k^3} \right)} |M_2|. \quad (\text{A.14})$$

Note that we used $A = \sqrt{2}A_2$ in Eq. 5.76 because of the different factors in front of the amplitude in Eq. 5.64 as opposed to Eq. 2.5. We also changed some factors of m_K into $E_{\pi\pi}$ since $m_K = E_{\pi\pi}$ when energy is conserved. The

finite volume matrix element is

$$M_2 = \langle \pi \pi_{m_I=0}^{I=2} | H_W | K^0 \rangle \quad (\text{A.15})$$

where the states have *unit norm*.

We wish to put the matrix element in terms of the four quark operators in the weak Hamiltonian. We have for the weak Hamiltonian on the lattice

$$H_W = a^3 \sum_{\mathbf{x}} \mathcal{L}_W(\mathbf{x}) \quad (\text{A.16})$$

$$= a^3 \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{JK} Z_{JK}(\mu) C_J(\mu) \sum_{\mathbf{x}} Q_K(\mathbf{x}) \quad (\text{A.17})$$

where we used Eq. 4.32 and Eq. 4.42 in the last line. Substituting back into Eq. A.15 we have

$$M_2 = a^3 \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{JK} Z_{JK}(\mu) C_J(\mu) \langle \pi \pi_{m_I=0}^{I=2} | \sum_{\mathbf{x}} Q_K^{\Delta I=3/2, \Delta m_I=1/2}(\mathbf{x}) | K^0 \rangle \quad (\text{A.18})$$

where we have emphasized the isospin changing properties of the four quark operators of Eq. 4.29 - 4.31.

We now wish to use the Wigner-Eckart theorem to convert the matrix element into one involving K^+ in the initial state, $\pi^+ \pi^+$ in the final state, and unphysical four quark operators for which the matrix element is non-zero. We do this because the three point functions we calculate on the lattice are for the process $K^+ \rightarrow \pi^+ \pi^+$ and involve the unphysical four quark operators

(see Eq. 6.42 - 6.44). We have

$$\frac{\langle \pi \pi_{m_I=0}^{I=2} | Q_K^{\Delta I=3/2, \Delta m_I=1/2}(\mathbf{x}) | K^0 \rangle}{\langle \pi \pi_{m_I=2}^{I=2} | Q_K^{\Delta I=3/2, \Delta m_I=3/2}(\mathbf{x}) | K^+ \rangle} = \frac{\langle 2, 0 | \frac{3}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \rangle}{\langle 2, 2 | \frac{3}{2}, \frac{3}{2}; \frac{1}{2}, \frac{1}{2} \rangle} = \frac{1}{\sqrt{2}}. \quad (\text{A.19})$$

The two pion isospin state $I = 2$, $m_I = 2$ is a just the $\pi^+ \pi^+$ state,

$$|\pi \pi_{m_I=2}^{I=2}\rangle = |\pi^+ \pi^+\rangle. \quad (\text{A.20})$$

Using Eq. A.19 and A.20 in Eq. A.18 we get

$$M_2 = a^3 \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{1}{\sqrt{2}} \sum_{JK} Z_{JK}(\mu) C_J(\mu) \langle \pi^+ \pi^+ | \sum_{\mathbf{x}} Q_K^{\Delta I=3/2, \Delta m_I=3/2}(\mathbf{x}) | K^+ \rangle. \quad (\text{A.21})$$

The generic form of the operators $Q_K^{\Delta I=3/2, \Delta m_I=1/2}$ can be seen in Eq. 4.29 - 4.31 to be

$$Q^{\Delta I=3/2, \Delta m_I=1/2} = \bar{s} \Gamma_{A\mu} d \bar{u} \Gamma_B^\mu u + \bar{s} \Gamma_{A\mu} u \bar{u} \Gamma_B^\mu d - \bar{s} \Gamma_{A\mu} d \bar{d} \Gamma_B^\mu d \quad (\text{A.22})$$

where color indices have been omitted because they are irrelevant here. Acting on this with the isospin raising operator³, we see that

$$Q^{\Delta I=3/2, \Delta m_I=3/2} = -\sqrt{3} O_W \quad (\text{A.23})$$

³We use the convention in which the operator u annihilates an up quark and $\bar{u}$ creates one, and similarly for d . Thus u, d transform under the complex conjugate representation of isospin while $\bar{u}, \bar{d}$ transform under the fundamental representation. Therefore, the raising operator turns u into $-d$ and $\bar{d}$ into $\bar{u}$.

where

$$O_W = \bar{s}\Gamma_{A\mu}d\bar{u}\Gamma_B^\mu d \quad (\text{A.24})$$

(as in Eq. 6.43 and 6.44). Substituting Eq. A.23 into Eq. A.21, and ignoring the overall minus sign in M_2 since Eq. A.14 only depends on its absolute value, we get

$$M_2 = a^3 \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{\frac{3}{2}} \sum_{JK} Z_{JK}(\mu) C_J(\mu) \langle \pi^+ \pi^+ | \sum_{\mathbf{x}} (O_W)_K(\mathbf{x}) | K^+ \rangle. \quad (\text{A.25})$$

The lattice calculation also involves dimensionless operators

$$\hat{O}_W = a^6 O_W. \quad (\text{A.26})$$

In terms of these, we have

$$M_2 = a^{-3} \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{\frac{3}{2}} \sum_{JK} Z_{JK}(\mu) C_J(\mu) \langle \pi^+ \pi^+ | \sum_{\mathbf{x}} (\hat{O}_W)_K(\mathbf{x}) | K^+ \rangle. \quad (\text{A.27})$$

The matrix element in Eq. A.27 is very close to what we can extract directly from the correlators of Chapter 6, the only complication being the different sources we used in the correlators. Thus we discuss the extraction of the matrix element from the correlators. The first one required to extract the matrix element is the zero momentum wall-wall kaon correlator, which

from Eq. 6.23 is

$$C_K(\mathbf{p} = 0, t, 0) = \langle 0 | O_{K^+}^{\text{wall}}(t) O_{K^+}^{\dagger \text{ wall}}(0) | 0 \rangle \quad (\text{A.28})$$

where

$$O_{K^+}^{\text{wall}} = \sum_{\mathbf{x}, \mathbf{y}} O_{K^+}(\mathbf{x}, \mathbf{y}, t) \quad (\text{A.29})$$

and the operator on the right hand side is defined in Eq. 6.25. Inserting a complete set of states into Eq. A.28, we find

$$C_K(\mathbf{p} = 0, t, 0) \sim \langle 0 | O_{K^+}^{\text{wall}}(t) | K^+ \rangle \langle K^+ | O_{K^+}^{\dagger \text{ wall}}(0) | 0 \rangle \quad t \gg 0 \quad (\text{A.30})$$

$$= Z_K Z_K^* e^{-m_K t} \quad (\text{A.31})$$

where

$$Z_K = \langle 0 | O_{K^+}^{\text{wall}}(0) | K^+ \rangle. \quad (\text{A.32})$$

Note that we will ignore around-the-world terms since they do not affect the constants out in front of the exponentials which are our interest in this discussion.

We proceed similarly for the two pion correlator with non-zero momentum in general. The correlator for a given pion momentum $\mathbf{p}$ is defined in Eq. 6.26, however, this correlator is averaged over all momenta with the same magnitude, which depends on the number of twists (e.g. Eq 6.17 for two twists). Each term in the average is the same by cubic symmetry, so that

the only effect of the averaging is to improve statistics, but here we choose to work with the averaged correlator for the sake of clarity. In fact, the average can be factored into two parts: one outer sum over different twist directions (e.g. x , y , and z for one twist), and one inner sum over the components of momentum in the twisted directions which can take on the value π/L or $-\pi/L$ (e.g. if the x direction is twisted then the sum is over the momenta in Eq. 6.15 or if the y and z directions are twisted then the sum is over the momenta in Eq. 6.16). Since all the terms in both sums are equal, we choose to write the two pion correlator as one of the terms in the outer sum, i.e. we fix the twist directions, but still average over the different possible momenta with components $\pm\pi/L$ in each twisted direction. The reason for doing this is that the cosine source can be written as such as sum.

Thus, following Eq. 6.26, we write the two pion correlator as

$$C_{\pi\pi}(\mathbf{p}, t, 0) = \frac{1}{2^{n_{\text{tw}}}} \sum_{\mathbf{p}'} \langle 0 | O_{\pi^+\pi^+}(\mathbf{p}', t) O_{\pi^+\pi^+}^{\dagger \cos}(\mathbf{p}, 0) | 0 \rangle \quad (\text{A.33})$$

where the twisted directions are fixed and the sum is over all momenta with components $\pm\pi/L$ in each twisted direction. There are $2^{n_{\text{tw}}}$ terms in this sum, hence the factor out in front. The interpolating operators are given by

$$O_{\pi^+\pi^+}(\mathbf{p}, t) = \sum_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}} e^{i\mathbf{p}\cdot\mathbf{w}} e^{-i\mathbf{p}\cdot\mathbf{y}} O_{\pi^+}(\mathbf{w}, \mathbf{x}, t) O_{\pi^+}(\mathbf{y}, \mathbf{z}, t) \quad (\text{A.34})$$

and

$$O_{\pi^+\pi^+}^{\text{cos}}(\mathbf{p}, t) = \sum_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}} s_{\text{cosine}}(\mathbf{w}; \mathbf{p}) s_{\text{cosine}}(\mathbf{y}; \mathbf{p}) O_{\pi^+}(\mathbf{w}, \mathbf{x}, t) O_{\pi^+}(\mathbf{y}, \mathbf{z}, t) \quad (\text{A.35})$$

where the operator on the right hand sides of Eq. A.34 and A.35 is defined in Eq. 6.24.

The cosine source can also be written in terms of a sum over all momenta with components $\pm\pi/L$ in each twisted direction:

$$s_{\text{cosine}}(\mathbf{x}; \mathbf{p}) = \cos(p_x x) \cos(p_y y) \cos(p_z z) \quad (\text{A.36})$$

$$= \frac{1}{2^{n_{\text{tw}}}} \sum_{\mathbf{p}'} e^{i\mathbf{p}' \cdot \mathbf{x}} \quad (\text{A.37})$$

where this time the factor of $1/2^{n_{\text{tw}}}$ just comes from the fact that $\cos(x) = (e^{ix} + e^{-ix})/2$. The product of two cosine sources in Eq. A.35 can then be written

$$s_{\text{cosine}}(\mathbf{w}; \mathbf{p}) s_{\text{cosine}}(\mathbf{y}; \mathbf{p}) = \frac{1}{2^{2n_{\text{tw}}}} \sum_{\mathbf{p}', \mathbf{p}''} e^{i\mathbf{p}' \cdot \mathbf{w}} e^{i\mathbf{p}'' \cdot \mathbf{y}} \quad (\text{A.38})$$

$$\rightarrow \frac{1}{2^{2n_{\text{tw}}}} \sum_{\mathbf{p}'} e^{i\mathbf{p}' \cdot \mathbf{w}} e^{-i\mathbf{p}' \cdot \mathbf{y}} \quad (\text{A.39})$$

where in the last line we have indicated that only terms with zero total momentum will survive in the correlator due to conservation of total momentum, and the fact that the two pion sink Eq. A.34 has zero total momentum. Putting this relation back into Eq. A.35 and Eq. A.33, we see that the two

pion correlator becomes

$$C_{\pi\pi}(\mathbf{p}, t, 0) = \frac{1}{2^{3n_{\text{tw}}}} \sum_{\mathbf{p}', \mathbf{p}''} \langle 0 | O_{\pi^+\pi^+}(\mathbf{p}', t) O_{\pi^+\pi^+}^\dagger(\mathbf{p}'', 0) | 0 \rangle \quad (\text{A.40})$$

Inserting a complete set of states we find

$$C_{\pi\pi}(\mathbf{p}, t, 0) \sim Z_{\pi\pi} Z_{\pi\pi}^* e^{-E_{\pi\pi} t} \quad t \gg 0 \quad (\text{A.41})$$

where

$$Z_{\pi\pi} = \langle 0 | \frac{1}{2^{3n_{\text{tw}}/2}} \sum_{\mathbf{p}'} O_{\pi^+\pi^+}(\mathbf{p}', 0) | \pi^+\pi^+ \rangle. \quad (\text{A.42})$$

Finally we look at the $K \rightarrow \pi\pi$ correlator. From Eq. 6.42,

$$C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, 0) = \langle 0 | T \left(O_{\pi^+\pi^+}^{\text{cos}}(\mathbf{p}, 0) \sum_{\mathbf{x}} \hat{O}_W(t, \mathbf{x}) O_{K^+}^\dagger(t_K) \right) | 0 \rangle \quad (\text{A.43})$$

(generically for each of the operators $(\hat{O}_W)_K$). Using Eq. A.39, and inserting a complete set of states on both sides of the weak operator, we find

$$C_{K \rightarrow \pi\pi}(\mathbf{p}, t_K, t, 0) \sim Z_K^* Z_{\pi\pi} \mathcal{M} e^{-E_{\pi\pi} t} e^{-m_K(t_K - t)} \quad 0 \ll t \ll t_K \quad (\text{A.44})$$

where Z_K and $Z_{\pi\pi}$ are given by Eq. A.32 and A.42, and

$$\mathcal{M} = \frac{1}{\sqrt{2^{n_{\text{tw}}}}} \langle \pi^+\pi^+ | \sum_{\mathbf{x}} \hat{O}_W(0, \mathbf{x}) | K^+ \rangle \quad (\text{A.45})$$

which we call the “lattice matrix element”.

Using Eq. A.45 in Eq. A.27 we arrive at our final expression for M_2 :

$$M_2 = a^{-3} \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{\frac{3}{2}} \sqrt{2^{n_{\text{tw}}}} \sum_{JK} Z_{JK}(\mu) C_J(\mu) \mathcal{M}_K. \quad (\text{A.46})$$

This can be substituted into Eq. A.14 to get an expression for $|A_2|$. However, we note that the lattice calculation is done in Euclidean time and therefore all correlators will be real and positive. Thus we can choose conventions in which Z_K , $Z_{\pi\pi}$, and $\mathcal{M}$ are also real and positive, meaning that the complex phase of A_2 comes *only* from the Wilson coefficients. Therefore we can remove the absolute values on both sides of Eq. A.14. This makes sense if one notes that the Wilson coefficients are only complex due to the CP violating phase in the CKM matrix, (which only enters in τ in Eq. 4.5), and thus A_2 will be purely real if the CP violating phase is 0, as it should be.

Substituting Eq. A.46 into Eq. A.14, and also using $q = kL/(2\pi)$ to put the Lellouch-Lüscher factor in terms of q rather than k , we get

$$\begin{aligned} A_2 = a^{-3} \frac{1}{\pi q^{3/2}} \sqrt{q \frac{\partial \phi}{\partial q} + k \frac{\partial \delta_0}{\partial k}} \sqrt{\hat{m}_K} \hat{E}_{\pi\pi} \hat{L}^{3/2} \\ \times G_F V_{ud} V_{us}^* \frac{\sqrt{3}}{2\sqrt{2}} \sqrt{2^{n_{\text{tw}}}} \sum_{JK} C_J(\mu) Z_{JK}(\mu) \mathcal{M}_K \end{aligned} \quad (\text{A.47})$$

(the hat denotes dimensionless lattice quantities, although the combination $\sqrt{m_K} E_{\pi\pi} L^{3/2}$ is dimensionless anyways, so either dimensionful or dimensionless quantities could be used here).

Finally, the CPS code which was used to run the lattice calculation has

a non-standard convention for propagators:

$$S^{\text{CPS}} = (5 - M_5)S. \quad (\text{A.48})$$

Based on the number of powers of the propagators that enter the correlators (Eq. 6.28, 6.30 - 6.31, and 6.47 - 6.50), we see that

$$\mathcal{M}^{\text{CPS}} = (5 - M_5)^2 \mathcal{M}. \quad (\text{A.49})$$

Substituting into Eq. A.47, we arrive at our final result

$$\begin{aligned} A_2 = & a^{-3} \frac{1}{\pi q^{3/2}} \sqrt{q \frac{\partial \phi}{\partial q} + k \frac{\partial \delta_0}{\partial k}} \sqrt{\hat{m}_K} \hat{E}_{\pi\pi} \hat{L}^{3/2} \\ & \times G_F V_{ud} V_{us}^* \frac{\sqrt{3}}{2\sqrt{2}} \sqrt{2^{n_{\text{tw}}}} \left(\frac{1}{5 - M_5} \right)^2 \sum_{JK} C_J(\mu) Z_{JK}(\mu) \mathcal{M}_K^{\text{CPS}}. \end{aligned} \quad (\text{A.50})$$

In summary, Eq. A.31, A.41, and A.44 show how the lattice matrix element $\mathcal{M}$ can be extracted from the correlators for each of the different four quark operators. This Euclidean time quantity is real and positive by definition. We can then use Eq. A.50 to compute the complex amplitude A_2 , where the complex phase comes in only through the Wilson coefficients. The factor of $1/(5 - M_5)^2$ in Eq. A.50 is necessary because of non-standard conventions for the propagators in the CPS code. Eq. A.50 is used, for example, to take the results of Table 49 and 50, and obtain the results of Table 51.

B Estimate of Finite Volume Systematic Error

B.1 Chiral Perturbation Theory Formulae

We estimate the size of the systematic error due to finite volume in the 32^3 coarse calculation, using finite volume chiral perturbation theory. We start from the leading analytic (labeled “LO”) and logarithmic (labeled “log”)

terms for $K \rightarrow \pi\pi$ matrix elements in SU(3) chiral perturbation theory:

$$\mathcal{M}_{\text{LO}}^{27} = \langle \pi^+ \pi^- | \mathcal{O}^{(27,1),3/2} | K^0 \rangle_{\text{LO}} = -\frac{4i\alpha_{27}}{f_K f_\pi^2} (m_K^2 - m_\pi^2) \quad (\text{B.1})$$

$$\begin{aligned} \mathcal{M}_{\text{log}}^{27} &= \langle \pi^+ \pi^- | \mathcal{O}^{(27,1),3/2} | K^0 \rangle_{\text{log}} \\ &= -\frac{4i\alpha_{27}}{f_K f_\pi^2} \frac{1}{f^2} \left[-\frac{1}{12} m_K^4 \left(1 - \frac{m_K^2}{m_\pi^2} \right) \beta(m_\pi^2, m_K^2, m_\eta^2) + m_K^2 \left(\frac{5}{4} \frac{m_K^4}{m_\pi^2} \right. \right. \\ &\quad \left. \left. - \frac{13}{4} m_K^2 + 2m_\pi^2 \right) \beta(m_\pi^2, m_K^2, m_\pi^2) + (m_K^4 - 3m_\pi^2 m_K^2 + 2m_\pi^4) \right. \\ &\quad \left. \times \beta(m_K^2, m_\pi^2, m_\pi^2) + \left(-\frac{1}{4} \frac{m_K^4}{m_\pi^2} - \frac{1}{12} m_K^2 + \frac{1}{3} m_\pi^2 \right) \ell(m_\eta^2) + \left(\frac{-m_K^4}{m_\pi^2} \right. \right. \\ &\quad \left. \left. - 4m_K^2 + 4m_\pi^2 \right) \ell(m_K^2) + \left(\frac{5}{4} \frac{m_K^4}{m_\pi^2} - \frac{45}{4} m_K^2 + 11m_\pi^2 \right) \ell(m_\pi^2) \right] \quad (\text{B.2}) \end{aligned}$$

$$\mathcal{M}_{\text{LO}}^{88} = \langle \pi^+ \pi^- | \mathcal{O}^{(8,8),3/2} | K^0 \rangle_{\text{LO}} = -\frac{4i\alpha_{88}}{f_K f_\pi^2} \quad (\text{B.3})$$

$$\begin{aligned} \mathcal{M}_{\text{log}}^{88} &= \langle \pi^+ \pi^- | \mathcal{O}^{(8,8),3/2} | K^0 \rangle_{\text{log}} \\ &= -\frac{4i\alpha_{88}}{f_K f_\pi^2} \frac{1}{f^2} \left[\left(\frac{5}{4} \frac{m_K^4}{m_\pi^2} - 2m_K^2 \right) \beta(m_\pi^2, m_K^2, m_\pi^2) + (m_K^2 - 2m_\pi^2) \right. \\ &\quad \left. \times \beta(m_K^2, m_\pi^2, m_\pi^2) + \frac{1}{4} \frac{m_K^4}{m_\pi^2} \beta(m_\pi^2, m_K^2, m_\eta^2) - \left(4 + \frac{1}{2} \frac{m_K^2}{m_\pi^2} \right) \ell(m_K^2) \right. \\ &\quad \left. + \left(\frac{5}{4} \frac{m_K^2}{m_\pi^2} - 8 \right) \ell(m_\pi^2) - \frac{3}{4} \frac{m_K^2}{m_\pi^2} \ell(m_\eta^2) \right] \quad (\text{B.4}) \end{aligned}$$

where α_{27} and α_{88} are low energy constants, f_K and f_π are the kaon and pion decay constants, f is the decay constant in the chiral limit, and

$$\ell(m^2) = \frac{m^2}{16\pi^2} \ln \left(\frac{m^2}{\Lambda_{\chi PT}^2} \right) \quad (\text{B.5})$$

$$\beta(q^2, m_1^2, m_2^2) = \frac{1}{16\pi^2} \int_0^1 dx \left\{ 1 + \ln \left[\frac{-x(1-x)q^2 + (1-x)m_1^2 + xm_2^2}{\Lambda_{\chi PT}^2} \right] \right\}. \quad (\text{B.6})$$

Here $\Lambda_{\chi PT}$ is the renormalization scale for chiral perturbation theory. Eq. B.2 was taken from [81] (equation C5 and adding in the logarithmic terms from $(m_K^2 - m_\pi^2)_{(1\text{-loop})}$ by hand) and Eq. B.4 was taken from [82] (equation E3).

The finite volume corrections to $\ell(m^2)$ and $\beta(q^2, m_1^2, m_2^2)$, which will be denoted by $\Delta_L \ell(m^2)$ and $\Delta_L \beta(q^2, m_1^2, m_2^2)$, are given by

$$\Delta_L \ell(m^2) = \frac{m^2}{16\pi^2} \delta_1(mL) \quad (\text{B.7})$$

and

$$\Delta_L \beta(q^2, m_1^2, m_2^2) = \frac{1}{4\pi^2} \delta_\beta(qL, m_1L, m_2L) \quad (\text{B.8})$$

where L is the spatial extent of the lattice, and

$$\delta_1(mL) = \frac{4}{mL} \sum_{\vec{n} \neq 0} \frac{K_1(|\vec{n}|mL)}{|\vec{n}|} \quad (\text{B.9})$$

$$\delta_\beta(qL, m_1L, m_2L) = \sum_{\vec{n} \neq 0} \int_0^\infty dk \frac{k \sin(k|\vec{n}|)}{|\vec{n}|} \frac{\omega_1 + \omega_2}{\omega_1 \omega_2 [(qL)^2 + (\omega_1 + \omega_2)^2]}. \quad (\text{B.10})$$

Here $\vec{n}$ is an integer vector and the sum is over all $\vec{n}$ except where all of the components are 0. Also, K_1 is a modified Bessel function, and ω_1 and ω_2 are given by

$$\omega_i = \sqrt{k^2 + (m_iL)^2}. \quad (\text{B.11})$$

Eq. B.7, B.8, B.9, and B.10 are from [82] equations 65, 71, 68, and 73 respectively.

No quantities other than $\ell(m^2)$ and $\beta(q^2, m_1^2, m_2^2)$ have a finite volume correction, so Eq. B.7 and B.8 can be used to find the finite volume correction to the matrix elements to leading logarithmic order. In particular, these corrections can be obtained if we replace β and ℓ with $\Delta_L \beta$ and $\Delta_L \ell$ in Eq. B.2 and B.4. We can estimate these corrections as a fraction of the actual matrix element if we then divide by the leading order analytic expression. Thus we get

$$\begin{aligned} \frac{\Delta_L \mathcal{M}_{\text{LO}}^{27}}{\mathcal{M}_{\text{LO}}^{27}} = & \frac{1}{f^2(m_K^2 - m_\pi^2)} \left[-\frac{1}{12} m_K^4 \left(1 - \frac{m_K^2}{m_\pi^2} \right) \Delta_L \beta(m_\pi^2, m_K^2, m_\eta^2) \right. \\ & + m_K^2 \left(\frac{5}{4} \frac{m_K^4}{m_\pi^2} - \frac{13}{4} m_K^2 + 2m_\pi^2 \right) \Delta_L \beta(m_\pi^2, m_K^2, m_\pi^2) + (m_K^4 \\ & - 3m_\pi^2 m_K^2 + 2m_\pi^4) \Delta_L \beta(m_K^2, m_\pi^2, m_\pi^2) + \left(-\frac{1}{4} \frac{m_K^4}{m_\pi^2} - \frac{1}{12} m_K^2 \right. \\ & + \frac{1}{3} m_\pi^2 \left. \right) \Delta_L \ell(m_\eta^2) + \left(\frac{-m_K^4}{m_\pi^2} - 4m_K^2 + 4m_\pi^2 \right) \Delta_L \ell(m_K^2) \\ & \left. + \left(\frac{5}{4} \frac{m_K^4}{m_\pi^2} - \frac{45}{4} m_K^2 + 11m_\pi^2 \right) \Delta_L \ell(m_\pi^2) \right] \quad (\text{B.12}) \end{aligned}$$

$$\begin{aligned} \frac{\Delta_L \mathcal{M}_{\text{LO}}^{88}}{\mathcal{M}_{\text{LO}}^{88}} = & \frac{1}{f^2} \left[\left(\frac{5}{4} \frac{m_K^4}{m_\pi^2} - 2m_K^2 \right) \Delta_L \beta(m_\pi^2, m_K^2, m_\pi^2) + (m_K^2 - 2m_\pi^2) \right. \\ & \times \Delta_L \beta(m_K^2, m_\pi^2, m_\pi^2) + \frac{1}{4} \frac{m_K^4}{m_\pi^2} \Delta_L \beta(m_\pi^2, m_K^2, m_\eta^2) - \left(4 + \frac{1}{2} \frac{m_K^2}{m_\pi^2} \right) \\ & \left. \times \Delta_L \ell(m_K^2) + \left(\frac{5}{4} \frac{m_K^2}{m_\pi^2} - 8 \right) \Delta_L \ell(m_\pi^2) - \frac{3}{4} \frac{m_K^2}{m_\pi^2} \Delta_L \ell(m_\eta^2) \right] \quad (\text{B.13}) \end{aligned}$$

for the fractional finite volume corrections to the matrix elements.

B.2 Numerical Results

We can obtain numerical estimates for the corrections of the previous section by using the pion and kaon mass from the 32^3 coarse lattice calculation, and the η mass from the LO ChPT relation (also known as the Gellmann-Okubo relation)

$$m_\eta^2 = \frac{1}{3} (4m_K^2 - m_\pi^2). \quad (\text{B.14})$$

Values used for these masses, as well as for L and f , the other two parameters needed in the corrections, are given in Table 115. With these values we obtain

$$\frac{\Delta_L \mathcal{M}_{\log}^{27}}{\mathcal{M}_{\text{LO}}^{27}} = 0.061285 \quad (\text{B.15})$$

$$\frac{\Delta_L \mathcal{M}_{\log}^{88}}{\mathcal{M}_{\text{LO}}^{88}} = 0.066642. \quad (\text{B.16})$$

Numerical values of the ℓ and β functions, and their finite volume corrections, are given in Section B.3.

In Eq. B.12 and B.13 we divided the finite volume corrections by the leading order ChPT expressions for the matrix elements. This raises the question of whether it makes a difference to divide by the leading order expressions or leading order *plus* leading log expressions. We can see how much of a correction the leading log expressions are to the leading order by examining the ratio of leading log terms to leading order terms, which is just

obtained by leaving out the Δ_L 's in Eq. B.12 and B.13. We find

$$\frac{\mathcal{M}_{\log}^{27}}{\mathcal{M}_{\text{LO}}^{27}} = 0.66698 + 0.28648i \quad (\text{B.17})$$

$$\frac{\mathcal{M}_{\log}^{88}}{\mathcal{M}_{\text{LO}}^{88}} = 0.45470 + 0.28648i \quad (\text{B.18})$$

where we have used $\Lambda_{\chi PT} = m_\rho = 770$ MeV. These corrections are quite significant. Thus if we divide the finite volume corrections by leading order plus leading log terms we get significantly smaller fractional corrections:

$$\frac{\Delta_L \mathcal{M}_{\log}^{27}}{|\mathcal{M}_{\text{LO}}^{27} + \mathcal{M}_{\log}^{27}|} = \frac{\Delta_L \mathcal{M}_{\log}^{27} / \mathcal{M}_{\text{LO}}^{27}}{|1 + \mathcal{M}_{\log}^{27} / \mathcal{M}_{\text{LO}}^{27}|} = 0.036233 \quad (\text{B.19})$$

$$\frac{\Delta_L \mathcal{M}_{\log}^{88}}{|\mathcal{M}_{\text{LO}}^{88} + \mathcal{M}_{\log}^{88}|} = \frac{\Delta_L \mathcal{M}_{\log}^{88} / \mathcal{M}_{\text{LO}}^{88}}{|1 + \mathcal{M}_{\log}^{88} / \mathcal{M}_{\text{LO}}^{88}|} = 0.044948. \quad (\text{B.20})$$

(The second equality in Eq. B.19 and B.20 shows how we can compute these quantities from the values already computed in Eq. B.15, B.16, B.17, and B.18).

We will be conservative and take the larger values of 6.1% and 6.7% for the finite volume corrections to the (27,1) and (8,8) operator respectively (Eq. B.15 and B.16) and use these as our estimate of systematic error due to finite volume effects. We also note, in closing, that partial quenching is not taken into account in these estimates since the analogous formulae have not been computed in partially quenched chiral perturbation theory. Nevertheless, we still expect the results of this section to be good estimates

of the systematic error since partial quenching effects have been found to be relatively small in general in this report.

B.3 Numerical Values of the Functions ℓ , β , $\Delta_L \ell$, and $\Delta_L \beta$

We give numerical values of the functions of Eq. B.5, B.6, B.7, and B.8, evaluated for the different arguments needed in the computation of the finite volume corrections and logarithmic terms. We use the input parameters of Table 115 and $\Lambda_{\chi PT} = m_\rho = 770$ MeV.

$$\beta(m_\pi^2, m_K^2, m_\eta^2) = 0.0019022 \quad (\text{B.21})$$

$$\beta(m_\pi^2, m_K^2, m_\pi^2) = -0.0040604 \quad (\text{B.22})$$

$$\beta(m_K^2, m_\pi^2, m_\pi^2) = -0.015294 + 0.016466i \quad (\text{B.23})$$

$$\ell(m_\pi^2) = -435.30 \text{ MeV}^2 \quad (\text{B.24})$$

$$\ell(m_K^2) = -1358.3 \text{ MeV}^2 \quad (\text{B.25})$$

$$\ell(m_\eta^2) = -1201.2 \text{ MeV}^2 \quad (\text{B.26})$$

$$\Delta_L \beta(m_\pi^2, m_K^2, m_\eta^2) = 8.3808 \times 10^{-8} \quad (\text{B.27})$$

$$\Delta_L \beta(m_\pi^2, m_K^2, m_\pi^2) = 1.3661 \times 10^{-4} \quad (\text{B.28})$$

$$\Delta_L \beta(m_K^2, m_\pi^2, m_\pi^2) = 3.1681 \times 10^{-4} \quad (\text{B.29})$$

$$\Delta_L \ell(m_\pi^2) = 35.045 \text{ MeV}^2 \quad (\text{B.30})$$

$$\Delta_L \ell(m_K^2) = 0.0088527 \text{ MeV}^2 \quad (\text{B.31})$$

$$\Delta_L \ell(m_\eta^2) = 0.0017230 \text{ MeV}^2. \quad (\text{B.32})$$

In order to evaluate the function $\Delta_L\beta$ one must evaluate the infinite sum of Eq. B.10. This sum can be written in the form

$$\delta_\beta(qL, m_1L, m_2L) = \sum_{n=1}^{\infty} m(n) \int_0^{\infty} dk \frac{k \sin(k\sqrt{n})}{\sqrt{n}} \frac{\omega_1 + \omega_2}{\omega_1\omega_2[(qL)^2 + (\omega_1 + \omega_2)^2]} \quad (\text{B.33})$$

where $m(n)$ is the number of integer vectors with a magnitude squared equal to n . In Table 116 we give numerical values of all of the terms of the sum used in the calculation of Eq. B.27, B.28, and B.29. In each case we calculate the first 15 terms of the sum, by which point the summand has fallen to smaller than 10^{-4} times its initial value.