Nonlocal models with a finite range of nonlocal interactions

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ABSTRACT

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Nonlocal phenomena are ubiquitous in nature. The nonlocal models investigated in this thesis use integration in replace of differentiation and provide alternatives to the classical partial differential equations. The nonlocal interaction kernels in the models are assumed to be as general as possible and usually involve finite range of nonlocal interactions. Such settings on one hand allow us to connect nonlocal models with the existing classical models through various asymptotic limits of the modeling parameter, and on the other hand enjoy practical significance especially for multiscale modeling and simulations.

To make connections with classical models at the discrete level, the central theme of the numerical analysis for nonlocal models in this thesis concerns with numerical schemes that are robust under the changes of modeling parameters, with mathematical analysis provided as theoretical foundations. Together with extensive discussions of linear nonlocal diffusion and nonlocal mechanics models, we also touch upon other topics such as high order nonlocal models, nonlinear nonlocal fracture models and coupling of models characterized by different scales.

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Chapter 1

Introduction

Nonlocal phenomena are ubiquitous in nature and nonlocal models have appeared naturally in various branches of physical, biological and social sciences. See [Andreu *et al.*, 2010; Bourgain *et al.*, 2001; Caffarelli and Silvestre, 2007; Coifman and Lafon, 2006; Du *et al.*, 2012; Felsinger *et al.*, 2013; Klafler and Sokolov, 2005; Gilboa and Osher, 2008; Metzler and Klafter, 2000; Silling, 2000; Tadmor, 2015] and references cited therein on related applications and mathematical analysis.

In recent years, there has been a great deal of interest in the nonlocal peridynamic (PD) continuum theory introduced first by Silling in [Silling, 2000]. PD models use integration in replace of differentiation to compute the force on a material particle by summing up interactions with other near-by particles. The models completely avoid the use of spatial derivatives and provide alternatives to the classical partial differential equation (PDE) based continuum mechanics, thus are found to be effective in dealing with spontaneous cracks formation and materials failure. A common feature of PD models is the introduction of the horizon parameter δ, which characterizes the range (radius) of nonlocal interactions [Du *et al.*, 2013a; Silling, 2000]. As $\delta \to 0$, nonlocal effect diminishes and the zero-horizon limit of nonlocal PD models becomes a classical local differential equation model when the latter is well-defined. Such limiting behavior provides connections and consistencies between nonlocal and local models which has immense practical significance especially for multiscale modeling and simulations.

On the other hand, there have been much interest in the study of anomalous diffusion which refers to diffusive processes that are not described by the Fick's first law [Neuman and Tartakovsky, 2009; Metzler and Klafter, 2000; Metzler and Klafter, 2004]. In particular, the fractional Laplacian operator is the infinitesimal generator of a Lévy process that is not Brownian motion. Our setting of nonlocal diffusion (ND) models has the fractional Laplacian and fractional derivative models as special cases. Inspired by the nonlocal PD model, the ND model inherits the nonlocal interaction parameter δ and are allowed to be connected with classical and fractional diffusion models through suitable chosen interaction kernels and parameter δ.

In the following sections of this chapter, we will take general reviews into the PD/ND models and give a summary of mathematical and numerical issues that will be discussed in this thesis. Chapter 2-6 contains numerical analysis for nonlocal models including finite difference, finite element and Fourier spectral analysis. The common theme of those chapters is to find robust numerical schemes, which we called asymptotically compatible schemes, that are insensitive to parameter change. Chapter 7-10 discuss mathematical analysis for nonlocal models which is much needed for providing insight to effective modeling and assurance to convergent simulations. Chapter 11-12 deals with multiscale modeling related to nonlocal models.

1.1 The nonlocal peridynamic theory for fracture mechanics

The peridynamic (PD) system of equations of motion, for a bond-based material [Silling, 2000], is given by

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{B_\delta(\mathbf{x})} \mathbf{f}(\mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t), \mathbf{x}' - \mathbf{x}) d\mathbf{x}' + \mathbf{b}(\mathbf{x},t)\,, \tag{1.1}$$

where the body has a mass density $\rho(\mathbf{x})$ and occupies the bounded domain $\Omega \subset \mathbb{R}^d$. The vector valued function $\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi})$ is a pairwise force density function that contains all constitutive relations. The function $\mathbf{u}$ is the displacement filed and $\mathbf{b}$ is a given loading force density function. $B_\delta(\mathbf{x})$ denotes the ball of radius δ centered at $\mathbf{x}$ and the parameter $\delta > 0$ is called the horizon and specifies the extent of the nonlocal interaction. Each pair $(\mathbf{x}',\mathbf{x})$ is called a bond and the integral goes over all the bonds around a material point thus the theory is bond based.

In common practice, the pairwise force density function $\mathbf{f}$ for a micro-elastic material is given in the form of

$$\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \omega_\delta(|\boldsymbol{\xi}|) \frac{|\boldsymbol{\eta} + \boldsymbol{\xi}| - |\boldsymbol{\xi}|}{|\boldsymbol{\xi}|} \frac{\boldsymbol{\eta} + \boldsymbol{\xi}}{|\boldsymbol{\eta} + \boldsymbol{\xi}|}, \tag{1.2}$$

where the force $\mathbf{f}$ is linearly dependent on the relative elongation $(|\boldsymbol{\eta} + \boldsymbol{\xi}| - |\boldsymbol{\xi}|)/|\boldsymbol{\xi}|$. In addition, to model the dynamic crack propagation, a "memory" term is often added to the force density function which records whether the bond has broken or not based on the history of the relative elongation (see more discussions in chapter 9).

A large amount of simulations have done within the formulations of peridynamic theory. They are turned out to be very successful in modeling materials with crack nucleation and propagation [Askari *et al.*, 2008; Bobaru and Duangpanya, 2010; Ha and Bobaru, 2010; Hu *et al.*, 2012]. Figure 1.1 shows our recent work [Du *et al.*, Preprint 2017b] on the simulations of crack branching in a brittle material. The material is set with a pre-notch in the center. After applying tensile loading on the upper and lower side of the boundary, we observe the crack branching.

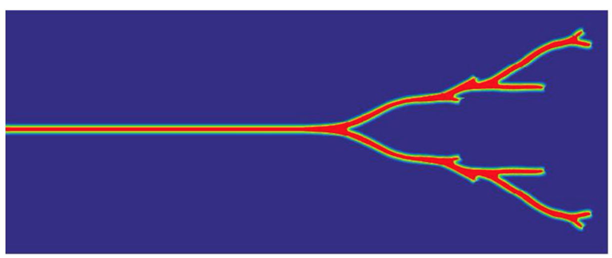

Figure 1.1: Crack branches in a brittle material: simulations done by using the PD theory.

While nonlocal models have been successful alternatives to local models based on PDEs, the mathematical and numerical issues related to such models are much more challenging largely due to the nonlocality and nonlinearity involved. To our main purpose of better understandings of nonlocality in this thesis, we will be mostly focusing on mathematical and numerical issues of linearized models and will also show some preliminary results on the nonlinear peridynamic models.

When the relative displacement $|\boldsymbol{\eta}| = |\mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t)|$ is uniformly small, we arrive at the linearized force density function corresponding to (1.2):

$$\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \omega_\delta(|\boldsymbol{\xi}|)\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^3}\boldsymbol{\eta}\,.$$

Thus the linearized PD system of isotropic bond-based materials takes the form

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{B_\delta(\mathbf{x})} \mathbb{C}_\delta(\mathbf{x}'-\mathbf{x})(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t))d\mathbf{x}' + \mathbf{b}(\mathbf{x},t)\,, \tag{1.3}$$

where the micromodulus tensor $\mathbb{C}_\delta(\boldsymbol{\xi})$ is given by

$$\mathbb{C}_\delta(\boldsymbol{\xi}) = \gamma_\delta(|\boldsymbol{\xi}|)\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\,.$$

Here we denote $\gamma_\delta(|\boldsymbol{\xi}|) = \omega_\delta(|\boldsymbol{\xi}|)/|\boldsymbol{\xi}|$ and use it as the kernel function in the rest of this thesis.

The nonlocal peridynamics model presented here is naturally connected to the classical linear elasticity theory with nonlocality shrinking to zero.

Connection with linear elasticity. As $\delta \to 0$, we expect (1.3) going to the linear elasticity equation. Indeed, [Mengesha and Du, 2014a] shows the rigorous results of the convergence of weak solutions of the steady-state linear PD equation to the weak solution of Navier equations of linear elasticity with Poisson ratio 1/4. To recover linear elasticity theory with general Poisson ratios, one would go to the "state-based" PD theory [Silling *et al.*, 2007], and the related analysis results can be found in [Mengesha and Du, 2014c].

Now we conclude this section with a few remarks on some recurrent issues in this thesis including the choices kernel functions and the suitable boundary condition for nonlocal models.

The kernel function. The kernel γ_δ in this thesis is assumed to be a nonnegative radial function compactly supported in $B_\delta(\mathbf{0})$ (the ball of radius δ centered at the origin). In addition, we always assume that γ_δ has finite second moment, namely

$$\int_{B_\delta(\mathbf{0})} |\boldsymbol{\xi}|^2\gamma_\delta(|\boldsymbol{\xi}|)d\boldsymbol{\xi} < \infty\,. \tag{1.4}$$

Sometimes we need to compare the nonlocal equation with its local limit as $\delta \to 0$. In such cases, we would assume γ_δ to be a rescaled kernel:

$$\gamma_\delta(|\boldsymbol{\xi}|) = \frac{1}{\delta^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{\xi}|}{\delta}) \,, \tag{1.5}$$

for some radial kernel $\hat{\gamma}$ compactly supported on $B_1(\mathbf{0})$. Intuitively speaking, such rescaling allows $|\boldsymbol{\xi}|^2 \gamma_\delta(|\boldsymbol{\xi}|)$ to be approximating the Dirac-Delta measure in the zero δ limit such that local models are recovered.

Boundary condition. Boundary conditions for nonlocal models are not in the conventional sense. The Dirichlet boundary condition for equations (1.1) and (1.3) should be imposed as a *volumetric constraint* [Du *et al.*, 2012] on a δ-neighborhood of the domain Ω:

$$\Omega_\delta := \{x \in \mathbb{R}^d \backslash \Omega : \mathrm{dist}(x, \partial\Omega) \leq \delta\} \,. \tag{1.6}$$

This is in contrast to local problems where conditions are prescribed on $\partial\Omega$. The Neumann boundary condition is one that needs more extensive discussions and is beyond the scope of this thesis. See [Tao *et al.*, Submitted 2016] for related discussions on Neumann boundary conditions.

1.2 Nonlocal diffusion and related models

The nonlocal diffusion (ND) equation with Dirichlet type volume constraint takes the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathcal{L}_\delta u = b & \text{on } \Omega,\ t > 0 \\ u(\mathbf{x}, 0) = u_0 & \text{on } \Omega \cup \Omega_\delta \\ u(\mathbf{x}, t) = g & \text{on } \Omega_\delta,\ t > 0 \,, \end{cases} \tag{1.7}$$

for a bounded, open domain $\Omega \subset \mathbb{R}^d$ and its nonlocal *interaction domain* Ω_δ is defined as (1.6). $\mathcal{L}_\delta$ is a nonlocal elliptic operator in the form of

$$\mathcal{L}_\delta u(\mathbf{x}) = \int_{\mathbb{R}^d} (u(\mathbf{y}) - u(\mathbf{x})) \gamma_\delta(|\mathbf{y} - \mathbf{x}|) d\mathbf{y} \,, \tag{1.8}$$

where the kernel γ_δ is radial and nonnegative with $\mathrm{supp}(\gamma_\delta) \subset B_\delta(\mathbf{0})$. Moreover, γ_δ has finite second order moment as in (1.4).

Intuitively, taking $\gamma_\delta(|\mathbf{y}-\mathbf{x}|) = \Delta_{\mathbf{y}}\delta(\mathbf{y}-\mathbf{x})$, with $\delta(\cdot)$ denoting the Dirac delta measure, then $\mathcal{L}_\delta = \Delta$ in the distributional sense. We note also that the operator can be written as $\mathcal{L}_\delta = \mathcal{D}\gamma_\delta\mathcal{D}^*$ where $\mathcal{D}$ and $\mathcal{D}^*$ are the basic nonlocal divergence operator and its dual defined in a nonlocal vector calculus [Du *et al.*, 2013a; Du *et al.*, 2012]. Such a formulation draws a natural analogy between the nonlocal operator and local second order elliptic differential operator $\nabla \cdot (\mathbf{C} \cdot \nabla)$.

Now we give some remarks on how nonlocal diffusion is connected with the other topics.

Connection with local diffusion. Taking γ_δ to be the rescaled kernel in (1.5), we expect that (1.7) goes back to the classical local diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - C\Delta u = b \quad \text{on } \Omega,\ t > 0 \\ \quad u(\mathbf{x}, 0) = u_0 \quad \text{on } \Omega \\ \quad u(\mathbf{x}, t) = g \quad \text{on } \partial\Omega,\ t > 0\,. \end{cases} \tag{1.9}$$

Such results have been rigorously investigated in [Du *et al.*, 2012; Mengesha and Du, 2014a].

Connection with fractional diffusion Taking γ_δ to be in the form of a fractional type kernel, namely,

$$\gamma_\delta(|\mathbf{y}-\mathbf{x}|) = \begin{cases} \dfrac{C_{d,\alpha}}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}} & \mathbf{y} \in B_\delta(\mathbf{x}) \\ 0 & \mathbf{y} \in \mathbb{R}^d \backslash B_\delta(\mathbf{x})\,, \end{cases} \tag{1.10}$$

we see that it is related to the fractional diffusion equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^\alpha u = b \quad \text{on } \Omega,\ t > 0 \\ \quad u(\mathbf{x}, 0) = u_0 \quad \text{on } \mathbb{R}^d \\ \quad u(\mathbf{x}, t) = g \quad \text{on } \mathbb{R}^d \backslash \Omega,\ t > 0\,. \end{cases} \tag{1.11}$$

through taking $\delta \to \infty$ [Tian *et al.*, 2016]. We refer to [Ros-Oton, 2015] for a survey of properties of fractional diffusion operators.

Stochastic counterpart. The nonlocal diffusion operator in the form (1.8) arise naturally in the study of stochastic processes with jumps, and more precisely in Lévy processes. A Lévy process is a stochastic process with independent and stationary increments. It is an extension of the concept of Brownian motion, where the paths of the trajectories can involve jumps. Figure 1.2 shows the sample path of Brownian motion versus that of a Lévy process.

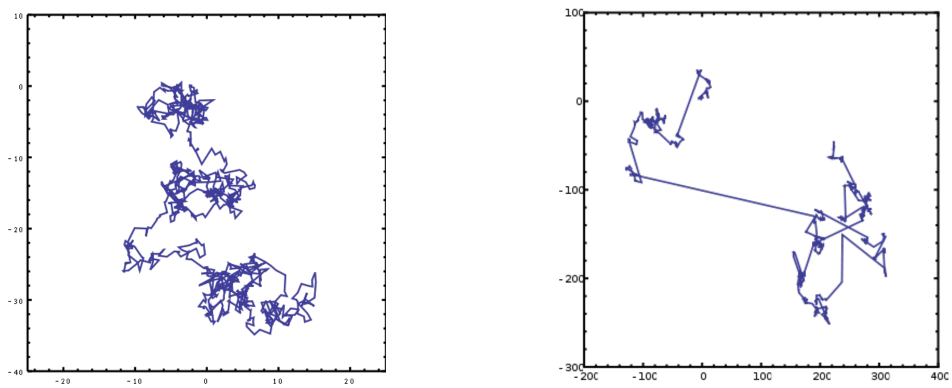

Figure 1.2: Left: trajectory of Brownian motion. Right: trajectory of Lévy process with jump.

By the Lévy-Khintchine Formula, the infinitesimal generator of any Lévy processes can be written as

$$\mathcal{L}u(\mathbf{x}) = \sum a_{ij}\partial_{ij}u + \sum b_j \partial_j u_j + \int_{\mathbb{R}^d} (u(\mathbf{x}+\boldsymbol{\xi}) - u(\mathbf{x}) - \boldsymbol{\xi}\cdot\nabla u(\mathbf{x})\chi_{B_1}(\boldsymbol{\xi}))d\nu(\boldsymbol{\xi})\,,$$

where ν is the Lévy measure that satisfies $\int_{\mathbb{R}^d}\min\{1,|\boldsymbol{\xi}|^2\}d\nu(\boldsymbol{\xi}) < \infty$. Now suppose that the process has no local diffusion or drift part and in addition the process is symmetric and that ν is absolutely continuous, then $\mathcal{L}$ takes the form

$$\mathcal{L}u(\mathbf{x}) = \int_{\mathbb{R}^d} (u(\mathbf{x}+\boldsymbol{\xi}) - u(\mathbf{x}))K(\boldsymbol{\xi})d\boldsymbol{\xi}\,.$$

Our nonlocal operator (1.8) corresponds to this case with K being the kernel function γ_δ.

1.3 Overview of the mathematical and numerical issues discussed in the thesis.

As a common feature of our nonlocal models, the horizon parameter δ characterizes the range of nonlocal interactions and allows us to make connections to classical models as δ

going to some asymptotic limits. A natural and important question to ask is how such limiting behaviors can be preserved in various discrete approximations. This is a critical issue in the applications of PD like models to problems involving possibly different scales, given the popularity and practicality to perform PD simulations with a coupled horizon δ and mesh spacing h. Chapter 2 compares some standard numerical methods for PD/ND models and shows that some convergent numerical methods for a fixed δ may approximate the wrong local limit when the ratio of δ and h is kept constant, while convergence to the desired local limit can also be established for some other discretizations. To keep the discussion relatively simple, the results presented in chapter 2 have been confined to one-dimensional models. Still, they have clearly exposed the risks involved in some popular practices for dealing with nonlocal models and exemplified the need for more comprehensive numerical analysis of the relevant issues. Chapter 3 introduces asymptotically compatible schemes and the corresponding abstract mathematical framework for their rigorous numerical analysis with respect to certain classes of parametrized problems and their asymptotic limits. The framework can be successfully applied to nonlocal models and their various asymptotic limits, followed by extensive discussions in chapter 4-5. Other numerical approaches toward nonlocal models are also discussed in chapter 6, one of those being the Fourier approach that can help us obtain convergence rates in terms of both δ and h. Related results for part I are published in [Tian and Du, 2013; Tian and Du, 2014a; Du and Tian, 2014; Tian and Du, 2015a; Du *et al.*, 2016].

In comparison to studies of PDEs and local models, mathematical analysis of nonlocal models is still at the nascent stage. The analysis works presented in part II are largely needed for effective modeling and assurance to convergent simulations. Chapter 7 contains various extensions of the seminal work of Bourgain-Brezis-Mironescu [Bourgain *et al.*, 2001] that are useful in a number of applications to nonlocal models. While the original result of Bourgain-Brezis-Mironescu involves a sequence of nonlocal interaction kernels approaching to a limit kernel given by the Dirac-Delta measure, our first extension allows for more general limits which can be seen as a new characterizations of nonlocal energy spaces. The work is interesting as a tool for developing nonconforming DG schemes presented in chapter 3. Second, we study a class of nonlocal operators that may be seen as high order generalizations of the nonlocal diffusion operators. With high order nonlocal function spaces defined as analogs of Sobolev spaces that involve higher derivatives, we touch upon characterizations of Sobolev properties of those spaces, including the various embedding and compactness properties. Analog to the classical counterpart, nonlocal versions of Poincaré inequalities are also presented, including a version valid for more general nonlocal kernels and a second one for more specialized kernels that leads to sharper control on the Poincaré constant. In addition, we can naturally consider nonlocal Gagliardo-Nirenberg type inequalities for norms associated with the class of nonlocal spaces. Such results, in their general forms that go beyond the conventional forms related to fractional Sobolev spaces, have not appeared in the literature before to the best of our knowledge. These results offer extensions to some previous works given initially in [Bourgain *et al.*, 2001; Ponce, 2004] and more recently in [Felsinger *et al.*, 2013; Mengesha and Du, 2013; Mengesha and Du, 2014a]. Moreover, localizations of the related nonlocal spaces also offer nonlocal characterizations of high order Sobolev spaces in the spirit of Bourgain-Brezis-Mironescu are provided. Related results for chapter 7 are published in [Tian and Du, 2015a; Tian and Du, 2015b].

Chapter 8 involves generalization and improvement of classical trace theorem for Sobolev spaces to nonlocal function space. While the classical result of Sobolev spaces that any H^1 function has a well-defined $H^{1/2}$ trace on the boundary of a domain with sufficient regularity, we improve the result from the H^1 function space to nonlocal function space with interaction kernels defined heterogeneously with a special localization feature on the boundary. The result is a refinement of the classical results since the boundary trace can be attained without imposing regularity of the function in the interior of the domain. The work is a foundation for multiscale nonlocal models with spatially variant interactions. Chapter 9 involves analysis for a nonlinear PD model with a memory term that describes the irreversibility of bond breaking. We establish a new result on the existence and uniqueness of the the model with a properly defined bond-breaking rule. The result is a first step towards mathematical analysis of the peridynamic model that has "memory" on its bond. Related results for chapter 8 and 9 are found in [Tian and Du, 2016; Du *et al.*, Preprint 2017b].

Part III discusses multiscale models for materials with fractures or defects involve local interaction where classical models work well and nonlocal interaction where defects display. Two types of coupling methods are discussed in chapter 11 and 12. The first idea [Tian and Du, 2016] comes from peridynamics model with a heterogeneous nonlocal interaction $\delta(x)$. By allowing a smooth change of $\delta(x)$ from nonzero to zero, we effectively have a seamless coupling of nonlocal and local models. Another idea is borrowed from the quasicontiumm method in the atomistic-to-continuum coupling which leads to a way to get a well-posed model without any ghost force [Du *et al.*, Preprint 2017a].

Part I

Asymptotic compatible schemes for nonlocal models

Chapter 2

Comparison of different schemes

In this chapter, we compare two classes of popular numerical schemes including finite difference and finite element methods for linear steady-state nonlocal diffusion model with Dirichlet volume constraint. To serve out purpose of illustration, we will be focusing on the one-dimensional problem in this chapter. Additional complications arising in practical implementations due to high dimensionality, nonuniform and unstructured mesh will be addressed in chapters 3-5. Discussions of multiscale modelings that involve different scales of nonlocal interactions can be found in chapters 8, 11 and 12.

Without loss of generality, we take $\Omega = (0,1)$, and its nonlocal volume constrained boundary $\Omega_\delta = (-\delta, 0) \cup (1, 1+\delta)$. By adopting the notations introduced in chapter 1, our main subject of interest is to solve the nonlocal equation

$$\begin{cases} -\mathcal{L}_\delta u = f_\delta \quad \text{on } \Omega \, , \\ \qquad u = 0 \quad \text{on } \Omega_\delta \, , \end{cases} \tag{2.1}$$

where $\mathcal{L}_\delta$ is defined by (1.8). To connect with the local limit, we assume that

$$\int_0^\delta s^2 \gamma_\delta(s) ds = C_\delta \to C > 0 \, , \;\; \|f_\delta - f\|_\infty \to 0 \, , \quad \text{as } \; \delta \to 0 \, . \tag{2.2}$$

Then the solutions of (2.1) converge to the solution of two-point boundary value problem:

$$\begin{cases} -Cu'' = f \quad \text{on } \Omega \, , \\ \qquad u = 0 \quad \text{on } \partial\Omega = \{0\} \cup \{1\} \, . \end{cases} \tag{2.3}$$

The main theme of this chapter is to elucidate how such limiting behaviors can or cannot be preserved in various discrete approximations.

2.1 Numerical schemes

In this section two classes of discrete schemes for (2.1) are introduced, including a new class of difference/quadrature schemes which preserves the discrete maximum principle (DMP), and conforming finite element Galerkin approximations with piecewise constant and piecewise linear finite element spaces. We also make a comparison of resulting linear systems (nonlocal stiffness matrices) generated by those methods. It is standard textbook material that for the local problem (2.3) on a uniform mesh, the simple second order centre finite difference, the standard finite volume and the continuous piecewise linear finite element discretization generate same linear systems of equations for (2.3), it is thus interesting to check if there are such similarities in the nonlocal case. As our study shows, while the linear systems are more complicated in a nonlocal world, some similarities can still be drawn.

Throughout this section, we consider a uniform mesh (grid). For a positive integer N, we set $h = 1/(N+1)$ and let $\delta = rh + \delta_0$ for a nonnegative integer $r < N$ and $\delta_0 \in [0, h)$. We now introduce grid points on $\Omega \cup \Omega_\delta$ as $\{x_i = ih\}_{i \in \Omega_N}$ where the index set is defined by $\Omega_N = \{-r, ..., 0, 1, ..., N+r+1\}$. Denote $I_j = ((j-1)h, jh)$ for $1 \le j \le r$, and $I_{r+1} = (rh, \delta)$. We also define the standard piecewise constant basis functions by

$$\phi_i^0(x) = \begin{cases} 1 & \text{for } x \in (x_{i-1}, x_i), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } i \in \Omega_N, \tag{2.4}$$

and the standard continuous piecewise linear *hat* basis functions by

$$\phi_i^1(x) = \begin{cases} (x - x_{i-1})/h & \text{for } x \in (x_{i-1}, x_i), \\ (x_{i+1} - x)/h & \text{for } x \in [x_i, x_{i+1}), \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } i \in \Omega_N. \tag{2.5}$$

2.1.1 Quadrature based finite difference discretization.

For a parameter $\alpha \in [0, 2]$, we first use the symmetry of γ_δ to write the nonlocal operator $\mathcal{L}_\delta$ as

$$\begin{aligned} \mathcal{L}_\delta u(x) &= \int_{-\delta}^{\delta} (u(x+s) - u(x))\gamma_\delta(s)ds \\ &= \int_0^{\delta} (u(x-s) - 2u(x) + u(x+s))\gamma_\delta(s)ds \\ &= \int_0^{\delta} \frac{u(x-s) - 2u(x) + u(x+s)}{s^\alpha} s^\alpha \gamma_\delta(s)ds\,. \end{aligned} \tag{2.6}$$

This simple reformulation is revealing: given a non-negative kernel γ_δ, the nonlocal operator $\mathcal{L}_\delta$ is in fact a weighted average of second order difference operators. The fact that the average is taken by integrating over $(0, \delta)$ means that rather than a finite difference, we in fact have a *continuum of differences* if γ_δ is supported over a continuum region (or rather it is more regular than a singular measure such as a finite combination of Dirac-delta measures and possibly distributional derivatives).

With (2.6), it is then intuitively clear that we may approximate the *continuum difference* represented by $\mathcal{L}_\delta$ by discrete finite differences through various quadrature approximations (thus the name *quadrature based finite difference* discretization). To first give a simple class of quadrature based difference approximation for $\mathcal{L}_\delta$, we consider a class of discrete operators $\mathcal{L}_{\delta,0}^h$, parametrized by a constant $\alpha \in [0, 2]$ given by

$$\mathcal{L}_{\delta,0}^h u_i = \sum_{m=1}^{r+1} \frac{u_{i-m} - 2u_i + u_{i+m}}{(mh)^\alpha} \int_{I_m} s^\alpha \gamma_\delta(s)ds\,. \quad i = 1, ...N\,, \quad \alpha \in [0, 2]\,, \tag{2.7}$$

where $\{u_i\}$ are approximations of $\{u(x_i)\}$.

We note that (2.7) is well-defined only for integrable $s^\alpha \gamma_\delta(s)$. If $\gamma_\delta(s)$ is itself integrable as in many existing studies, then we may take any α in $[0, 2]$, otherwise, restrictions on α may be required, for example, if only the finiteness of second moment of the kernel is assumed, then it leaves $\alpha = 2$ as the only feasible choice for all kernels having finite second order moment.

Since (2.7) represents a simple Riemann sum like quadrature of the integral (2.6), the quadrature error is of order $O(h)$ with a fixed horizon δ. We may consider other higher order quadratures like the trapezoidal and Simpson's rules for the integral (2.6), with which we can expect higher orders of accuracy. Equivalently, we may work with other reformulations of (2.6). For example, since continuous piecewise linear basis functions $\{\phi_i^1\}$ share the property that $\sum_{j=0}^{r+1}\phi_j^1(x)=1$, for $x\in(0,\delta)$, we have

$$\begin{aligned}\mathcal{L}_\delta u(x) &= \int_0^\delta \frac{u(x-s)-2u(x)+u(x+s)}{s^\alpha}s^\alpha\gamma_\delta(s)ds\\ &= \sum_{m=0}^{r+1}\int_0^\delta \frac{u(x-s)-2u(x)+u(x+s)}{s^\alpha}\phi_m^1(s)s^\alpha\gamma_\delta(s)ds\,.\end{aligned} \tag{2.8}$$

Then, by utilizing that $(u(x-s)-2u(x)+u(x+s))/s^\alpha\to 0$ as $s\to 0$ for smooth u and $\alpha\in[0,2)$, we arrive at the following

$$\begin{aligned}\mathcal{L}_{\delta,1}^h u_i = &\sum_{m=1}^{r}\frac{u_{i-m}-2u_i+u_{i+m}}{(mh)^\alpha}\int_{I_m\cup I_{m+1}}\phi_m^1(s)s^\alpha\gamma_\delta(s)ds\\ &+\frac{u_{i-r-1}-2u_i+u_{i+r+1}}{(r+1)^\alpha h^\alpha}\int_{I_{r+1}}\phi_{r+1}^1(s)s^\alpha\gamma_\delta(s)ds\,,\quad i=1,...N\,.\end{aligned} \tag{2.9}$$

As to be demonstrated later, the class of schemes (2.9) is more accurate than that corresponding to (2.7) and it is also closely related to finite element approximations. The nonlocal stiffness matrices for above schemes are denoted by $\mathbb{A}_{D,0}^\alpha$ and $\mathbb{A}_{D,1}^\alpha$ for (2.7) and (2.9) respectively, where the subscript $\{D,1\}$ refers to the use of piecewise linear interpolation in the difference approximation (in contrast to the subscript $\{D,0\}$ used for (2.7) which corresponds to piecewise constant interpolation) while the superscript refers to the value of α used in (2.8). If $\delta_0=0$, that is $\delta=rh$ for an integer r, then the integral over I_{r+1} in above schemes should be taken as zero.

With the above defined discrete nonlocal difference operators, the proposed quadrature based finite difference schemes of (2.1) are

$$\begin{cases}-\mathcal{L}^h u_i = f_\delta(x_i) & i\in\{1,...,N\}\\ u_i=0 & i\in\{-r,...,-1,0\}\cup\{N+1,...,N+r+1\}\end{cases} \tag{2.10}$$

where $\mathcal{L}^h$ can be either $\mathcal{L}_{\delta,0}^h$ or $\mathcal{L}_{\delta,1}^h$ as defined above. Let $\boldsymbol{U}$ be a column vector with entries $\{u_i\}_{i=1}^N$, and $\boldsymbol{b}$ be that with entries $\{b_i=f_\delta(x_i)\}_{i=1}^N$, we may rewrite the corresponding linear systems as

$$\mathbb{A}_{D,0}^\alpha\boldsymbol{U}=\boldsymbol{b}\quad\text{and}\quad\mathbb{A}_{D,1}^\alpha\boldsymbol{U}=\boldsymbol{b} \tag{2.11}$$

respectively, where $\{\mathbb{A}^{\alpha}_{D,0},\ \alpha \in [0,2]\}$ are nonlocal stiffness matrices associated with (2.7) while $\{\mathbb{A}^{\alpha}_{D,1},\ \alpha \in [0,2)\}$ are for (2.9). To avoid technical complications in numerical analysis, we only work with the cases $\alpha = 0$ and 1 for schemes defined by (2.9) in the sequel though more general α are considered for that defined by (2.7).

2.1.2 Finite element discretization

The natural energy space associated with (2.1) is

$$\mathcal{S}_\delta = \{u \in L^2(\Omega \cup \Omega_\delta) \mid \int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} \gamma_\delta(|y-x|)(u(y)-u(x))^2 dydx < \infty \mid u = 0 \text{ in } \Omega_\delta\}\,.$$

Denote the bilinear form on $\mathcal{S}_\delta \times \mathcal{S}_\delta$ by

$$B(u,v) := \tfrac{1}{2}\textstyle\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta}(u(y)-u(x))(v(y)-v(x))\gamma_\delta(|y-x|)dydx\ , \tag{2.12}$$

then, the weak formulation of (2.1) is given by: finding $u \in \mathcal{S}_\delta$ such that $\forall v \in \mathcal{S}_\delta$,

$$B(u,v) = (f_\delta, v)_\Omega. \tag{2.13}$$

Let $\mathcal{S}^h_\delta \subset \mathcal{S}_\delta$ be a family of finite element spaces corresponding to a uniform mesh $\{x_i\}$ parameterized by the mesh size h, as described earlier, with $\{\phi^k_i\}_{i=1}^{N_h}$ being the nodal basis. Let $u_h \in \mathcal{S}^h_\delta$ be the Galerkin approximation of u given by

$$B(u_h, v_h) = (f_\delta, v_h)_\Omega \quad \forall v_h \in \mathcal{S}^h_\delta\ . \tag{2.14}$$

Now suppose $u_h = \sum_{i=1}^{N_h} u_i \phi^k_i(x)$, we pay particular attention to the cases $k = 0$ and 1 with the case $k = 0$ corresponding to piecewise constant basis functions (2.4) (if the energy space admits such functions, which is guaranteed if $\gamma_\delta(r)$ has finite first order moment), and the case $k = 1$ corresponding to standard continuous piecewise linear elements with *hat* basis functions given by (2.5) (which works for $\gamma_\delta(r)$ that has finite second order moment).

Similar to difference approximations, let $\boldsymbol{U}$ be the column vector composed of the nodal values $\{u_i\}_{i=1}^{N_h}$, and $\boldsymbol{b}^k$ being the vector with entries $\{(f_\delta, \phi^k_i)_\Omega/h\}_{i=1}^{N_h}$ which represent the weighted average of f_δ around x_i. Then (2.14) gives linear systems $\mathbb{A}_{E,k}\boldsymbol{U} = \boldsymbol{b}^k$ with $\{\mathbb{A}_{E,k}\}_{k=0,1}$ being the nonlocal stiffness matrices for the finite element approximation. Entries of $\mathbb{A}_{E,k}$ are given by $\{B(\phi^k_i, \phi^k_j)/h\}$. The scaling factor h is needed to have the discrete system being consistent to that obtained from finite difference approximations. Indeed, one has $(f_\delta, \phi_i^k)_\Omega/h - f_\delta(x_i) \to 0$ as $h \to 0$.

We will examine how properties of $\mathbb{A}_{E,k}$ are affected by choices of the nonlocal kernels and discrete spaces and compare with $\mathbb{A}^\alpha_{D,0}$ and $\mathbb{A}^\alpha_{D,1}$ generated by the quadrature based finite difference methods.

2.1.3 Nonlocal stiffness matrices

When comparing finite difference methods with finite element methods, we distinguish two cases for simplicity: in one case the horizon is no more than h while $\delta = rh$ for some integer $r > 1$ in the second case. We let $\{a_{ij}\}$ represent entries of nonlocal stiffness matrices. Their specific forms are provided as a reference.

Case 1: $\delta \le h$, first, entries of $\mathbb{A}^\alpha_{D,0}$ are given by:

$$a_{ij} = \begin{cases} \dfrac{2}{h^\alpha}\displaystyle\int_0^\delta s^\alpha \gamma_\delta(s)ds, & i = j \\ -\dfrac{1}{h^\alpha}\displaystyle\int_0^\delta s^\alpha \gamma_\delta(s)ds, & |i-j| = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.15}$$

As for entries of $\mathbb{A}^\alpha_{D,1}$, we only discuss the cases $\alpha = 0$ or 1 and we note that

$$\mathbb{A}^\alpha_{D,1} = \mathbb{A}^{1+\alpha}_{D,0} \quad \text{for} \quad \alpha = 0,\, 1 \tag{2.16}$$

so that their expressions can be found in (2.15). Next, for entries of $\mathbb{A}_{E,0}$, we have

$$\mathbb{A}_{E,0} = \mathbb{A}^0_{D,1} = \mathbb{A}^1_{D,0}\,. \tag{2.17}$$

Finally, entries of $\mathbb{A}_{E,1}$ are given by:

$$a_{ij} = \begin{cases} \dfrac{2}{h^2}\displaystyle\int_0^\delta s^2 \gamma_\delta(s)ds - \dfrac{1}{h^3}\displaystyle\int_0^\delta s^3 \gamma_\delta(s)ds, & i = j \\ \dfrac{2}{3h^3}\displaystyle\int_0^\delta s^3 \gamma_\delta(s)ds - \dfrac{1}{h^2}\displaystyle\int_0^\delta s^2 \gamma_\delta(s)ds, & |i-j| = 1 \\ -\dfrac{1}{6h^3}\displaystyle\int_0^\delta s^3 \gamma_\delta(s)ds, & |i-j| = 2 \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

Case 2: $\delta = rh$, and let $m = |j-i|$, entries of $\mathbb{A}^{\alpha}_{D,0}$ are given by:

$$a_{ij} = \begin{cases} \dfrac{2}{h^\alpha}\displaystyle\sum_{l=1}^{r}\int_{I_l}\frac{s^\alpha}{l^\alpha}\gamma_\delta(s)ds, & m=0 \\ -\dfrac{1}{h^\alpha}\displaystyle\int_{I_m}\frac{s^\alpha}{m^\alpha}\gamma_\delta(s)ds, & 1\le m\le r \\ 0, & \text{otherwise.} \end{cases} \tag{2.19}$$

Then, entries of $\mathbb{A}^{\alpha}_{D,1}$ for $\alpha = 0$ or 1 are given by:

$$a_{ij} = \begin{cases} 2\displaystyle\sum_{k=1}^{r-1}\int_{I_k\cup I_{k+1}}\frac{\phi^1_k(s)}{(kh)^\alpha}s^\alpha\gamma_\delta(s)ds + 2\int_{I_r}\frac{\phi^1_r(s)}{(rh)^\alpha}s^\alpha\gamma_\delta(s)ds & m=0 \\ -\dfrac{1}{(mh)^\alpha}\displaystyle\int_{I_m\cup I_{m+1}}\phi^1_m(s)s^\alpha\gamma_\delta(s)ds, & 1\le m\le r-1 \\ -\dfrac{1}{(mh)^\alpha}\displaystyle\int_{I_r}\phi^1_r(s)s^\alpha\gamma_\delta(s)ds, & m=r \\ 0, & \text{otherwise.} \end{cases} \tag{2.20}$$

Next, for entries of $\mathbb{A}_{E,0}$, we have

$$\mathbb{A}_{E,0} = \mathbb{A}^0_{D,1}\,. \tag{2.21}$$

Finally, entries of $\mathbb{A}_{E,1}$ are given by

$$\begin{aligned} a_{ii} = &\int_0^h\Big(-\frac{s^3}{h^3}+\frac{2s^2}{h^2}\Big)\gamma_\delta(s)ds + \int_h^{2h}\Big(\frac{s^3}{3h^3}-\frac{2s^2}{h^2}+\frac{4s}{h}-\frac{4}{3}\Big)\gamma_\delta(s)ds \\ &+\frac{4}{3}\int_{2h}^{\delta}\gamma_\delta(s)ds, \end{aligned}$$

and for $m = |j-i| = 1$,

$$\begin{aligned} a_{ij} = &\int_0^h\Big(\frac{2s^3}{3h^3}-\frac{s^2}{h^2}\Big)\gamma_\delta(s)ds + \int_h^{2h}\Big(-\frac{s^3}{2h^3}+\frac{5s^2}{2h^2}-\frac{7s}{2h}+\frac{7}{6}\Big)\gamma_\delta(s)ds \\ &+\int_{2h}^{3h}\Big(\frac{s^3}{6h^3}-\frac{3s^2}{2h^2}+\frac{9s}{2h}-\frac{25}{6}\Big)\gamma_\delta(s)ds + \frac{1}{3}\int_{3h}^{\delta}\gamma_\delta(s)ds. \end{aligned}$$

The expressions for a_{ij} with $2 \le m = |j-i| \le r+1$ are more involved:

$$\begin{aligned} a_{ij} = &\int_{I_{m-1}}\Big(\frac{-s^3}{6h^3}+\frac{(m-2)s^2}{2h^2}-\frac{(m^2-4m+4)s}{2h}+\frac{m^3-6m^2+12m-8}{6}\Big)\gamma_\delta(s)ds \\ &+\int_{I_m}\Big(\frac{s^3}{2h^3}-\frac{(3m-2)s^2}{2h^2}+\frac{(3m^2-4m)s}{2h}-\frac{3m^3-6m^2+4}{6}\Big)\gamma_\delta(s)ds \\ &+\int_{I_{m+1}}\Big(\frac{-s^3}{2h^3}+\frac{(3m+2)s^2}{2h^2}-\frac{(3m^2+4m)s}{2h}+\frac{3m^3+6m^2-4}{6}\Big)\gamma_\delta(s)ds \\ &+\int_{I_{m+2}}\Big(\frac{s^3}{6h^3}-\frac{(m+2)s^2}{2h^2}+\frac{(m^2+4m+4)s}{2h}-\frac{m^3+6m^2+12m+8}{6}\Big)\gamma_\delta(s)ds. \end{aligned}$$

We note that for $m > r+1$, integrals over I_m are set to be zero in the above expressions.

Now the observed properties of the nonlocal stiffness matrices are given in the following.

Lemma 2.1.1. *For quadrature based finite difference schemes* (2.10) *with operators in* (2.7) *and* (2.9)*, and the finite element discretization* (2.14) *with either piecewise constant or continuous linear elements, the nonlocal stiffness matrices are all symmetric, positive definite matrices.*

Proof. The symmetry is obvious. The positive definiteness of nonlocal stiffness matrices of the finite element methods is a consequence of the coercivity of the bilinear form B [Du *et al.*, 2012; Mengesha and Du, 2013]. Meanwhile, $\mathbb{A}^{\alpha}_{D,0}$ and $\mathbb{A}^{\alpha}_{D,1}$ can all be viewed as nonnegative linear combinations of the stiffness matrices associated with the 2nd order center difference approximations which thus must remain positive definite. □

Lemma 2.1.2. *For quadrature based finite difference schemes* (2.10) *with operators in* (2.7) *and* (2.9)*, and finite element discretization* (2.14)*,*

1) the nonlocal stiffness matrices $\mathbb{A}^{\alpha}_{D,0}$*,* $\mathbb{A}^{\alpha}_{D,1}$*, and* $\mathbb{A}_{E,0} = \mathbb{A}^{0}_{D,1}$ *are all* M*-matrices;*

2) for the case $\delta \in (0, h]$*,* $\mathbb{A}^{\alpha}_{D,0}$*,* $\mathbb{A}^{\alpha}_{D,1}$*, and* $\mathbb{A}_{E,0}$ *are all scalar multiples of the tridiagonal matrix* $\mathbb{A}$ *associated with the second order centeral difference operator with* $2/h^2$ *as diagonal entries and* $-1/h^2$ *as off-diagonal entries, in particular,*

$$\mathbb{A}^{\alpha}_{D,0} = h^{2-\alpha}\left(\int_0^{\delta} s^{\alpha}\gamma_{\delta}(s)ds\right)\mathbb{A}\,,\ \ \mathbb{A}^{1}_{D,1} = \mathbb{A}^{2}_{D,0}\ \ \textit{and}\ \ \mathbb{A}_{E,0} = \mathbb{A}^{0}_{D,1} = \mathbb{A}^{1}_{D,0}\,. \tag{2.22}$$

Thus, $cond(\mathbb{A}^{\alpha}_{D,0}) = cond(\mathbb{A}^{0}_{D,1}) = cond(\mathbb{A}^{1}_{D,1}) = cond(\mathbb{A}_{E,0}) = \sin^{-2}(\frac{h\pi}{2}) = O(h^{-2})$.

Proof. Direct inspection shows that $\mathbb{A}^{\alpha}_{D,0}$ and $\mathbb{A}_{E,0}$ have positive diagonal and non-positive off-diagonal entries. Moreover, any of their row sums is either zero or is the negative of some partial sum of of the off-diagonal entries, hence we get M-matrices. For the case $\delta \leq h$, the conclusions in 2) are obvious following from the well-known spectral estimates of the tridiagonal matrix $\mathbb{A}$. □

Lemma 2.1.3. *For the finite element nonlocal stiffness matrix* $\mathbb{A}_{E,1}$*, we have*

1) any off-diagonal entry not adjacent to the diagonal is non-positive;

2) $\mathbb{A}_{E,1}$ *is an* M*-matrix iff all entries adjacent to the diagonal are non-positive;*

3) for $\delta \le h$, $\mathbb{A}_{E,1}$ is an M-matrix.

4) for L^1 integrable kernel γ_δ with a given fixed $\delta > 0$, for small enough h, the entries adjacent to the diagonal are positive, thus $\mathbb{A}_{E,1}$ is not an M-matrix in this case.

Proof. We note that for $m = |i-j| \ge 2$, the basis functions ϕ_i^1 and ϕ_j^1 do not have overlapping support, thus the entries $\{a_{ij}\}$ of $\mathbb{A}_{E,1}$ satisfy

$$\begin{aligned} a_{ij} &= \frac{1}{2}\int_\Omega \int_{-\delta}^{\delta} (\phi_i^1(x+r) - \phi_i^1(x))(\phi_j^1(x+r) - \phi_j^1(x))\gamma_\delta(r)drdx \\ &= -\frac{1}{2}\int_\Omega \int_{-\delta}^{\delta} (\phi_i^1(x+r)\phi_j^1(x) + \phi_j^1(x+r)\phi_i^1(x))\gamma_\delta(r)drdx \le 0 \,, \end{aligned}$$

for $m \ge 2$. Moreover, $\mathbb{A}_{E,1}$ has positive diagonal entries, and any row sum of $\mathbb{A}_{E,1}$ is either zero or is the negative of some partial sum of the off-diagonal entries, we have 1) and thus 2). For 3), the entries adjacent to the diagonal are given by

$$\int_0^\delta \frac{s^2}{3h^3}(2s-3h)\gamma_\delta(s)ds \le \int_0^\delta \frac{s^2}{h^3}(s-h)\gamma_\delta(s)ds \le 0$$

with $\delta \le h$. Thus, $\mathbb{A}_{E,1}$ is an M-matrix. On the other hand, for 4), based on the explicit expressions of the entries of stiffness matrix given earlier, we note that for $|i-j| = 1$, a_{ij} contains the term $\frac{1}{3}\int_{3h}^\delta \gamma_\delta(s)ds$. With L^1 integrable kernel γ_δ and a fixed $\delta > 0$, as $h \to 0$, this term converges to a positive constant $\frac{1}{3}\int_0^\delta \gamma_\delta(s)ds > 0$. Meanwhile, it is easy to see that all the other terms are bounded in absolute value by a constant multiple of $\int_0^{3h} \gamma_\delta(s)ds$ which goes to 0 as $h \to 0$. So for small h, the sign of a_{ij} is strictly positive, and this implies 4). □

Comparing different nonlocal stiffness matrices, a few remarks are in order. First, we know that for the classical Poisson equation (2.3), standard centeral finite difference, finite volume and continuous piecewise finite element discretizations lead to an identical stiffness matrix (given by the tridiagonal $\mathbb{A}$), with the difference being the right hand side vectors corresponding to $\boldsymbol{b}$, $\boldsymbol{b}^0$ and $\boldsymbol{b}^1$ respectively. Such a feature no longer holds in general for nonlocal models and we encounter generally different nonlocal stiffness matrices. However, the nonlocal stiffness matrix $\mathbb{A}_{E,0}$ generated by the piecewise constant finite element and the matrix $\mathbb{A}_{D,1}^0$ generated by a quadrature finite difference scheme do remain the same in both cases considered above. Secondly, for $\delta \le h$, we see that the condition numbers of nonlocal stiffness matrices are on the order of $O(h^{-2})$ uniformly with respect to δ, which is consistent with the bounds derived in [Aksoylu and Mengesha, 2010; Du *et al.*, 2012; Zhou and Du, 2010]. The limit when $\delta \to 0$ is examined more closely later.

2.1.4 Discrete maximum principle

Since the kernels γ_δ of the nonlocal operators $\mathcal{L}_\delta$ are assumed to be nonnegative and symmetric, it is easy to see that the nonlocal equation (2.1) satisfies maximum principle. Based on earlier discussions on the nonlocal stiffness matrices, we can investigate if such a property is preserved in the discrete schemes. We note that the discrete maximum principle can be readily used to establish the stability of the discrete schemes. As it turns out, our finite difference discretization and finite element discretization using piecewise constant basis always preserve discrete maximum principle (DMP). Meanwhile, for finite element discretization under piecewise linear function basis, the discrete maximum principle may not always hold for general kernels.

Theorem 2.1.4. *The quadrature based difference schemes* (2.10) *with operators in* (2.7) *and* (2.9)*, and the finite element discretization* (2.14) *with piecewise constants always satisfy the DMP:*

$$f_\delta = f_\delta(x) \leq 0 \ \ \textit{for} \ \ x \in \Omega \Rightarrow u_i \leq \max(u_j, j \in \mathcal{I}_b) \quad (i = 1, ..., N)$$
$$\textit{or} \quad f_\delta = f_\delta(x) \geq 0 \ \ \textit{for} \ \ x \in \Omega \Rightarrow u_i \geq \min(u_j, j \in \mathcal{I}_b) \quad (i = 1, ..., N)$$

where $\mathcal{I}_b = \{-r, ..., -1, 0\} \cup \{N+1, ..., N+r+1\}$*. Moreover, the finite element discretization* (2.14) *with piecewise linear basis also satisfies the DMP for* $\delta \leq h$*.*

Proof. These properties follow directly from lemmas 2.1.1 and 2.1.2 which imply that $(\mathbb{A}^\alpha_{D,0})^{-1}$, $(\mathbb{A}^\alpha_{D,1})^{-1}$ and $(\mathbb{A}_{E,0})^{-1}$ are non-negative matrices. Similarly, by lemma 2.1.3 $(\mathbb{A}_{E,1})^{-1}$ is non-negative when $\delta \leq h$. The DMP then follows. □

A consequence of the DMP is the stability of finite difference approximations.

Theorem 2.1.5. *For the quadrature based finite difference discretization* (2.10) *with* $\mathcal{L}^h$ *defined by operators in* (2.7) *and* (2.9)*,* $\|(\mathcal{L}^h)^{-1}\|_\infty$ *is bounded uniformly in* h *for a given* δ*. The* h*-independent bound of* $\|(\mathcal{L}^h)^{-1}\|_\infty$ *is also uniform in* δ *as* $\delta \to 0$*.*

Proof. By the results of lemma 2.1.1 and theorem 2.1.4, we have that $(-\mathcal{L}^h)^{-1}$ is nonnegative. Let us consider first the operator corresponding to (2.7). Define

$$C^r_{m,\alpha}(h) = \sum_{j=1}^{r+1} \frac{2(jh)^{2m+2-\alpha}}{(2m+2)!} \int_{I_j} s^\alpha \gamma_\delta(s) ds\,, \quad \forall\, m \geq 0\,, \tag{2.23}$$

then we have $C^r_{0,\alpha}(h) \geq C_\delta$ and

$$-\mathcal{L}^h_{\delta,0} \tilde{w}^h_\delta = (1,1,...,1)^T\,, \quad \text{where } \tilde{w}^h_\delta = \frac{1}{C^r_{0,\alpha}(h)} x(1-x) + \frac{\delta(1+\delta)}{C^r_{0,\alpha}(h)}\,,$$

and $\tilde{w}^h_\delta(x_i) \geq 0$ for $i \in \{-r,...,-1,0\} \cup \{N+1,...,N+r+1\}$. By the DMP,

$$\|(-\mathcal{L}^h_{\delta,0})^{-1}\|_\infty \leq \|\tilde{w}^h_\delta\|_\infty \leq \frac{1+4\delta(1+\delta)}{4C^r_{0,\alpha}(h)} \leq \frac{1+4\delta(1+\delta)}{4C_\delta}\,.$$

so $\|(-\mathcal{L}^h_{\delta,0})^{-1}\|_\infty$ is uniformly bounded in h. Similarly, we may define

$$\begin{aligned} C^r_\alpha(h) = & \sum_{m=1}^{r} \int_{I_m \cup I_{m+1}} (mh)^{2-\alpha} \phi^1_m(s) s^\alpha \gamma_\delta(s) ds \\ & + \frac{1}{2} \int_{I_{r+1}} ((r+1)h)^{2-\alpha} \phi^1_{j+1}(s) s^\alpha \gamma_\delta(s) ds \\ = & \, C_\delta + \int_0^\delta (\mathcal{I}_h(s^{2-\alpha}) - s^{2-\alpha}) s^\alpha \gamma_\delta(s) ds \geq C_\delta \end{aligned} \tag{2.24}$$

where $\mathcal{I}_h(s^{2-\alpha})$ is the piecewise linear interplant of $s^{2-\alpha}$ with respect to the mesh. By replacing $C^r_{0,\alpha}(h)$ with $C^r_\alpha(h)$ in the definition of $\tilde{w}^h_\delta$, we see that $\|(-\mathcal{L}^h_{\delta,1})^{-1}\|_\infty$ is also uniformly bounded in h for fixed δ. Moreover, since $C_\delta \to C$ as $\delta \to 0$, we also have the bound being independent of δ as $\delta \to 0$.. □

The uniform bounds above give the L^∞ stability of nonlocal quadrature based finite difference approximations and the piecewise constant finite element approximation, which is needed later in convergence analysis. We note that for the finite element discretization (2.14) with continuous piecewise linear basis, the DMP does not hold in general as the corresponding nonlocal stiffness matrix fails to be an M-matrix.

2.1.5 Local limits of discrete schemes

To link with the later analysis on the behavior of discrete schemes as the horizon goes to zero, we are interested in analyzing the limit $\delta \to 0$ for fixed h. In this case, all nonlocal stiffness matrices are of the forms studied in the case 1 of section 2.1.3. Thus, except for the finite element piecewise linear approximations, all other schemes give scalar multiples of the tridiagonal matrix $\mathbb{A}$ as shown in (2.22) of lemma 2.1.2. These scalars are given by either C_δ or for $\alpha \in [0, 2)$,

$$h^{2-\alpha}\left(\int_0^\delta s^\alpha \gamma_\delta(s)ds\right) = h^{2-\alpha}\frac{C_\delta \displaystyle\int_0^\delta s^\alpha \gamma_\delta(s)ds}{\displaystyle\int_0^\delta s^2 \gamma_\delta(s)ds} = h^{2-\alpha}\frac{C_\delta}{s_*^{2-\alpha}}, \quad \text{for some} \quad s_* \in (0, \delta),$$

which, as $\delta \to 0$, goes to infinity for fixed h, since $C_\delta \to C > 0$. Thus, we have

Theorem 2.1.6. *For quadrature based finite difference schemes* (2.10) *with operators in* (2.7) *and* (2.9)*, and finite element discretization* (2.14)*, as $\delta \to 0$, we have*
1) $\mathbb{A}^2_{D,0} = \mathbb{A}^1_{D,1} = C_\delta \mathbb{A} \to C\mathbb{A}$ so the corresponding discrete schemes converge to the standard 2nd order centeal finite difference approximation of the local limit. Similarly, $\mathbb{A}_{E,1} \to C\mathbb{A}$ so the corresponding discrete schemes converge to the standard continuous piecewise linear approximation of the local limit;
2) the solutions of (2.10) *with operators in* (2.7) *for $\alpha \in [0, 2)$ and in* (2.9) *for $\alpha = 0$, together with piecewise constant element solutions of* (2.14)*, all converge to zero.*

We note also that the only difference between various local limits of the schemes considered in 1) of theorem 2.1.6 are the right hand side vectors given by either point wise values or weighted averages using finite element basis functions as weights. The above theorem indicates the limits of those discrete schemes proposed for nonlocal problems may not always yield convergent discrete schemes of the correct continuum local limit if we fix the mesh size while letting the horizon $\delta \to 0$.

2.2 Convergence analysis

In terms of convergence studies, we are concerned with several different limiting processes in this section, namely: 1) limit of nonlocal continuum models as $\delta \to 0$; 2) limit of discrete schemes for nonlocal models as $\delta \to 0$ for a fixed h; 3) limit of discrete schemes for nonlocal models with a fixed δ as $h \to 0$; and 4) limit of the discrete schemes for the nonlocal models with both $\delta \to 0$ and $h \to 0$. While the first of these limiting processes has been studied in the literature, see for instance [Du *et al.*, 2012; Mengesha and Du, 2014a; Du and Zhou, 2011; Zhou and Du, 2010], we provide analysis here on the other three limits. We note that much of the numerical analysis presented here deal with problems whose exact solutions are considered to be smooth, though one distinct advantage of the nonlocal model is in admitting non-smooth solutions. We remark that by considering smooth solutions, we are able to derive many analytical results more directly with less technical jargon. Much of the conclusions presented here can be extended suitably to cases involving non-smooth solutions using more careful functional analytical tools. We leave such analysis to chapter 3-5 while focusing on getting the main messages across here with relatively simple analysis.

2.2.1 Finite difference discretization for fixed δ.

In this section, we show the convergence of discrete schemes presented earlier to nonlocal model (2.1) as $h \to 0$ with δ and $\gamma_\delta(s)$ being given. Let us consider the truncation error of the discrete operator $\mathcal{L}^h$. We begin with the following lemma.

Lemma 2.2.1. *Suppose that a function $G = G(x)$ has a bounded derivative G' on $[0, \delta]$ and a function g is nonnegative and integrable on $[0, \delta]$, then*

$$\int_0^\delta G(s)g(s)ds = \sum_{j=1}^{r+1} G(jh) \int_{I_j} g(s)ds + O(h)\,, \quad \text{as} \quad h \to 0\,. \tag{2.25}$$

Proof. It is easy to see that as $h \to 0$,

$$\left| \int_0^\delta G(s)g(s)ds - \sum_{j=1}^{r+1} G(jh) \int_{I_j} g(s)ds \right| \le \sum_{j=1}^{r+1} \int_{I_j} |G(s) - G(jh)| g(s)ds \le ch \int_0^\delta g(s)ds$$

for some constant c depending on $\|G'\|_\infty$ which gives (2.25). □

For the quadrature based finite difference discretization given by (2.7), we may take the G and g in (2.25) as

$$G(s) = [u(x_i + s) + u(x_i - s) - 2u(x_i)]/s^\alpha, \quad g(s) = s^\alpha \gamma_\delta(s) \quad i = 1, ..., N\,. \tag{2.26}$$

Using Taylor expansion, it is easy to see that if u' is uniformly Holder continuous with exponent α, then for $\alpha \le 1$, we have $G'(s)$ uniformly bounded. Moreover, if $u^{(3)}$ is bounded, then for $\alpha = 2$, we also have $G'(s)$ uniformly bounded. Then by lemma 2.2.1, the consistency (truncation) error of the quadrature based finite difference discretization defined by operators in (2.7) is $O(h)$ if $s^\alpha \gamma_\delta(s)$ is in L^1.

Theorem 2.2.2. *Let the solution u of the nonlocal problem be smooth, and either u' is uniformly Holder continuous with exponent $\alpha \leq 1$, $s^\alpha \gamma_\delta(s)$ is bounded in L^1, or $u^{(3)}$ is uniformly bounded and $\alpha = 2$, then for the quadrature based finite difference discretization* (2.10) *with operator defined by* (2.7), *the consistency error satisfies*

$$\max_{1 \leq i \leq N} |\mathcal{L}_\delta u(x_i) - \mathcal{L}^h_{\delta,0} u(x_i)| = O(h) \,, \qquad \text{as} \;\; h \to 0 \,. \tag{2.27}$$

So the error of the difference solution is also order $O(h)$, i.e., $\|u_i - u(x_i)\|_\infty = O(h)$.

Proof. The result follows from the truncation error analysis and the stability given in theorem 2.1.5. □

We see that the discretization based on (2.7) is a first order method. This is not surprising as the difference scheme is obtained via a simple Riemann sum. To improve the accuracy of convergence, we may consider other quadratures for (2.25). The scheme using (2.9) is one way to improve the accuracy. Instead of the Riemann sum using the piecewise constant approximation of G, it is based on a trapezoidal rule using a piecewise linear interpolation, denoted by $\mathcal{I}_h G$. The following lemma discusses the higher order of accuracy if we approximate G by $\mathcal{I}_h G$.

Lemma 2.2.3. *Suppose that G'' is bounded on $[0, \delta]$, g is nonnegative and integrable on $[0, \delta]$, then the discretization*

$$\int_0^\delta G(s) g(s) ds = \sum_{j=1}^{r+1} \int_{I_j} \mathcal{I}_h G(s) g(s) ds + O(h^2) \,. \tag{2.28}$$

Proof. Direct calculation shows

$$\left| \int_0^\delta G(s) g(s) ds - \sum_{j=1}^{r} \int_{I_j} \mathcal{I}_h G(s) g(s) ds \right| \leq \sum_{j=1}^{r+1} \int_{I_j} \frac{h^2}{2} \max_s |G''(s)| g(s) ds = O(h^2).$$

□

Now consider again G and g given by (2.26), we get the schemes corresponding to (2.9). Moreover, for $\alpha = 0$ or 1 with $u^{(2+\alpha)}$ uniformly bounded and $s^{1+\alpha}\gamma_\delta(s)$ bounded in L^1, these schemes are second order accurate approximations that satisfy the DMP and thus are also numerical stable. Notice that in order to a well-defined numerical scheme, we need $G(0) = 0$ and we also need the smoothness of $G(s)$ in s as $s \to 0$, both of which are valid for $\alpha = 0$ or 1. We provide the following theorem related to schemes (2.9) whose proof is similar to that of (2.2.2) and is omitted.

Theorem 2.2.4. *For the quadrature based difference schemes* (2.10) *with operators in* (2.9)*, if* $u^{(2+\alpha)}$ *are uniformly bounded and* $s^{1+\alpha}\gamma_\delta(s)$ *are bounded in* L^1 *corresponding to* $\alpha = 0$ *or* 1*, then we have the nonlocal stiffness matrices being* M*-matrices and the schemes convergent with error being* $O(h^2)$*, i.e.,* $\|u_i - u(x_i)\|_\infty = O(h^2)$*.*

For error analysis of finite element discretization, we expect that the piecewise constant approximation is of first order in L^2 with $\gamma_\delta(s)$ bounded in L^1, such results can be derived under much less regularity assumptions on the exact solution (say u is in H^1), while the continuous piecewise linear approximation is of first order in the energy space for more general kernels. We refer to [Du *et al.*, 2012] for details.

In some of the practical simulations of PD models, the grid size has been coupled with the horizon. While physical considerations might be behind such a coupling, an added benefit of making this choice is that the growth of interacting neighboring grid points or elements can be properly controlled in the numerical simulations. It is known that the nonlocal model (2.1) converges to the local model (2.3) as $\delta \to 0$ (see more general discussions in [Du *et al.*, 2012; Mengesha and Du, 2013]). Yet, we see that the local limits of discrete nonlocal schemes may or may not correspond to convergent discrete schemes of the correct local equation. Thus, a natural and important question is that when both h and δ approach zero, what is the limiting behavior of the numerical solution. Here, we consider two cases that $\delta \le h$ and $\delta = rh$ for a fixed integer $r > 1$ respectively. In both cases, when $h \to 0$, δ also tends to zero. We show that the limiting behavior of numerical approximations can be very complex and is very much dependent on the schemes used. Specifically, for our problem, we show that for the finite difference discretization (2.7), only the case of $\alpha = 2$ leads to the solution of the correct local equation in general. Similar conclusion can be drawn for (2.9) with $\alpha = 1$. Meanwhile, for finite element discretization, the case with continuous piecewise linear basis gives the correct local limit but not the case with piecewise constant basis. What is intriguing is that in all these cases, the limits often exist and they satisfy local differential equations that are different from the correct local limit. Practically speaking, this means that while numerical convergence might be observed, it is possible that a wrong limiting solution is obtained.

2.2.2 Finite difference discretization in local limit

First, let us examine the finite difference discretization (2.7). Given a smooth function u, we see that for finite difference discretization that :

$$\mathcal{L}^h u(x_i) = \sum_{m=0}^{\infty} C^r_{m,\alpha}(h) u^{(2m+2)}(x_i)\,, \tag{2.29}$$

where the coefficients are given by (2.23). While the limit of $\{C^r_{0,\alpha}(h)\}$ as $h \to 0$ has been examined for a given δ, some additional properties of $\{C^r_{m,\alpha}(h)\}$ are in order for more general cases.

Lemma 2.2.5. *Given the coefficients defined in* (2.23)*, we have*
1) $C^r_{m,\alpha}(h)$ *is a strictly decreasing function of* α*, in particular,*

$$C^r_{m,\alpha}(h) \geq C^r_{m,2}(h) = \sum_{j=1}^{r+1} \frac{2(jh)^{2m}}{(2m+2)!} \int_{I_j} s^2 \gamma_\delta(s) ds\,.$$

2) for $m \geq 1$*,*

$$C^r_{m,\alpha}(h) \leq \frac{\delta^{2m}}{(2m+2)!} C^r_{0,\alpha}(h) \leq \frac{((r+1)h)^{2m}}{(2m+2)!} C^r_{0,\alpha}(h)\,.$$

Proof. We note that 1) follows from $s \leq jh$ in I_j and 2) is a consequence of $jh \leq \delta$ for any j and $\delta \leq (r+1)h$. □

A consequence of the above lemma is that

Theorem 2.2.6. *Let* $r := \lfloor \delta/h \rfloor$ *be fixed and suppose that* $C^r_{0,\alpha}(h) \to C^r_\alpha$ *as* $h \to 0$ *for some constant* $C^r_\alpha \in (0,\infty)$*, then* $\mathcal{L}^h = \mathcal{L}^h_{\delta,0} u(x_i)$*, as defined in* (2.7)*, converges to* $C^r_\alpha u''(x_i)$ *as* $h \to 0$*. Moreover, if* $C^r_{0,\alpha}(h) - C^r_\alpha = O(h^\beta)$ *for some constant* $\beta > 0$*, then the truncation error is of order* $h^{\min(\beta,2)}$ *for a smooth solution* u*.*

Proof. Under the condition on $C^r_{0,\alpha}(h)$, we see that for $h \to 0$, $C^r_{0,\alpha}(h)$ is uniformly bounded, so that

$$\sum_{m=1}^{\infty} C^r_{m,\alpha}(h) u^{(2m+2)}(x_i) = O(h^2)\,,$$

based on 2) of lemma 2.2.5. This together with $C^r_{0,\alpha}(h) - C^r_\alpha = O(h^\beta)$ lead to the conclusion of the theorem. □

The above theorem further implies that

Corollary 2.2.7. *Let $r := \lfloor \delta/h \rfloor$ be fixed then as $h \to 0$ the difference approximation defined by* (2.7) *with $\alpha = 2$ converges to the solution of the local limit* (2.3)*. Consequently, the difference scheme with $\alpha = 2$ gives a convergent scheme in both nonlocal and local regimes.*

Proof. We note that by assumption of the kernel γ_δ, we have

$$C^r_{0,2}(h) = C_\delta = \int_0^\delta s^2 \gamma_\delta(s) ds \to C\,, \quad \text{as} \quad \delta \to 0\,.$$

The conclusion then follows from theorems 2.2.6 and 2.1.5. □

While the case $\alpha = 2$ offers a consistent approximation in both nonlocal and local regimes, we see that this fails to hold for more general $\alpha \neq 2$. Indeed, by 1) of lemma 2.2.5, for fixed r, we have $C^r_{0,\alpha}(h) > C^r_{0,2}(h)$. If the strict inequality holds in the limit $h \to 0$, we see that the difference approximation defined by (2.7) with $\alpha \neq 2$ would converge to the solution of a different local limit with a diffusion coefficient $C^r_{0,\alpha} > C$. Indeed, we illustrate such possibilities with a simple constant kernel $\gamma_\delta = 3\delta^{-3}$ which corresponds to the diffusion coefficient $C = 1$ in the local limit.

Let us examine a few special cases, first, we consider $\delta \leq h$. By the explicit construction given in (2.15), we get with $\alpha \neq 2$ that

$$C^r_{0,\alpha}(h) = \frac{3}{\alpha+1}\left(\frac{h}{\delta}\right)^{2-\alpha} \geq \frac{3}{\alpha+1} > 1\,.$$

We see that in this case, the finite difference approximation would either converge to zero (if $\delta/h \to 0$) or converge to a local equation with a diffusion coefficient strictly larger than

$C = 1$. Next, we consider the case $\delta = rh$ for some fixed integer $r > 1$. Then $C^r_{0,\alpha}(h)$ is a constant independent of h, that is,

$$C^r_{0,\alpha}(h) = \frac{3}{\alpha+1}\frac{1}{r^3}\sum_{j=1}^{r} j^{2-\alpha}(j^{\alpha+1} - (j-1)^{\alpha+1}) > C^r_{0,2}(h) = C = 1\,,$$

which again implies that the difference approximation defined by (2.7) would converge to the solution of a different local limit with a diffusion coefficient larger than $C = 1$. However, we see that, independent of h,

$$C^r_{0,\alpha}(h) \to C^r_{0,2}(h) = C = 1\,, \quad \text{as} \quad r \to \infty\,,$$

so we may expect the convergence of difference approximations to the solution of the correct local solution if the number of grid points increases faster than the horizon decreases, this is in agreement with related numerical experiments reported in [Bobaru *et al.*, 2009].

In summary, we see for the special constant kernel γ_δ, finite difference solutions given by (2.7) with $\alpha \neq 2$ have a convergent local limit, yet it might not be the solution of the correct local limit for the simple constant kernel as $h \to 0$ when either $\delta \leq h$ or δ/h is a fixed integer. Moreover, the limiting solutions can be quite different for schemes using different parameters (such as the constant α). Similar conclusions can be expected for other choices of nonlocal interaction kernels. In section 7 we will plot C^r_α for some special kernels and see clearly that C^r_α varies with α and r. Thus, we see the risk in using a mesh dependent horizon parameter in the numerical simulations of nonlocal models as the local limiting behavior might be rather unpredictable, even though the schemes can provide good convergence properties in the nonlocal regime.

Similar discussions can be made for finite difference solutions given by (2.9). Note that for $\delta \leq h$, based on (2.16), we have $\mathbb{A}^0_{D,1} = \mathbb{A}^1_{D,0}$ and $\mathbb{A}^1_{D,1} = \mathbb{A}^2_{D,0}$ thus the local limit is the same as that discussed above. In particular, we expect the convergence of (2.9) to the correct local limit for $\alpha = 1$ while to a different limit for $\alpha = 0$.

For $\delta = rh$ with some fixed integer $r > 1$, we evaluate the difference stencil for a smooth function u. Using Taylor expansion, the coefficient of the leading order term is then given by $-C^r_\alpha(h)$ as defined in (2.24). This implies:

Theorem 2.2.8. *For $\delta = rh$ with some fixed integer $r > 1$, solutions of the quadrature based finite difference scheme (2.10) with operators given in (2.9) converges to the solution of the local limit (2.3) as $h \to 0$ with the correct diffusion coefficient C if $\alpha = 1$. On the other hand, for $\alpha = 0$ and $C_0^r(h)$ as defined in (2.24), the difference solutions converge to the local limit solution of (2.3) with the diffusion coefficient*

$$\tilde{C}_0^r = \lim_{h\to 0} C_0^r(h) = C + \lim_{h\to 0}\int_0^{rh} (\mathcal{I}_h(s^2) - s^2)\gamma_\delta(s)ds \geq C \tag{2.30}$$

which equals to C if and only if

$$\lim_{h\to 0}\int_0^{rh} (\mathcal{I}_h(s^2) - s^2)\gamma_\delta(s)ds = 0\,. \tag{2.31}$$

Proof. By (2.24), we see that for $\alpha = 1$, $s = \mathcal{I}_h(s)$, so that the leading order coefficient is exactly $-C_\delta$ thus leading to convergence. On the other hand, if (2.31) holds, then for $\alpha = 0$, the leading order coefficient of the Taylor expansion goes to C as $h \to 0$. Moreover, the coefficients of the higher order terms are all going to zero uniformly, or rather the truncation error approaches to zero uniformly as $\delta \to 0$. Thus, by the theorem 2.1.5 we have the desired convergence result. □

We note that (2.31) does not hold in general. For example, take $\gamma_\delta = 3\delta^{-3}$ so that $C_\delta = C = 1$, then direct calculation shows that $C_0^r(h) = \tilde{C}_0^r = 1 + 1/(2r^2) > 1$. In fact, similar observations hold for the following popular choices of the kernel.

Lemma 2.2.9. *For $\delta = rh$ with fixed r, we have*
1) given a non-increasing function $\gamma_\delta = \gamma_\delta(s)$ in $[0, \delta)$, $\tilde{C}_0^r \geq (1 + 1/(8r^3))C_\delta$.
2) if $\gamma_\delta(s) = \gamma_1(s/\delta)/(\delta s^2)$ for some nonnegative function $\gamma_1(s)$ on $(0, 1)$ with

$$\int_0^{1/(2r)} \gamma_1(s)ds = \tau, \quad \text{and} \quad \int_0^1 \gamma_1(s)ds = 1\,,$$

for some $\tau > 0$, then $C = 1$ and $\tilde{C}_0^r \geq 1 + \tau$.

Proof. For 1), we notice that

$$\int_0^{rh} (\mathcal{I}_h(s^2) - s^2)\gamma_\delta(s)ds \geq \int_0^{h/2} (\mathcal{I}_h(s^2) - s^2)\gamma_\delta(s)ds \geq \int_0^{h/2} s^2\gamma_\delta(s)ds$$
$$= \frac{1}{8r^3}\int_0^{\delta} t^2\gamma_\delta(\frac{t}{2r})dt \geq \frac{1}{8r^3}\int_0^{\delta} t^2\gamma_\delta(t)dt = \frac{C_\delta}{8r^3}\,.$$

For 2), by the scaling it is easy to see $C = 1$. Similar as the above, we have

$$\int_0^{rh} (\mathcal{I}_h(s^2) - s^2)\gamma_\delta(s)ds \geq \int_0^{h/2} s^2\gamma_\delta(s)ds = \int_0^{h/2} \frac{1}{\delta}\gamma_1(\frac{s}{\delta})ds = \tau > 0\,.$$

□

We see that (2.31) does not hold for kernels considered in the above lemma and the corresponding local limits have coefficients different from the desired one.

2.2.3 Finite element discretization in local limit

Now we consider finite element methods. Numerical experiments in [Chen and Gunzburger, 2011] indicate that when the ratio of horizon radius and the grid size is fixed as a constant, piecewise constant approximations fail to converge in general, while piecewise linear approximations converge to the desired limit. We now provide some in-depth analysis. First, the piecewise constant element solution with $\delta \leq h$ would converge to the local second order equation with a diffusion coefficient

$$\tilde{C} = \lim_{\delta \leq h \to 0} h \int_0^{\delta} s\gamma_\delta(s)ds \geq \lim_{\delta \to 0} C_\delta = C\,.$$

Thus, similar to the finite difference with $\alpha = 1$, we generally do not expect convergence to the solution of the correct local equation.

Theorem 2.2.10. *For the case $\delta \leq h$, the piecewise constant finite element approximation converges to the solution of the local limit* (2.3) *as $h \to 0$ with the diffusion coefficient $\tilde{C} \geq C$.*

For the case where $\delta = rh$ with a fixed integer $r > 1$, by the equivalence of the nonlocal stiffness matrices between the piecewise constant finite element approximation and that for the difference approximation (2.9) with $\alpha = 0$, we get:

Theorem 2.2.11. *In the case $\delta = rh$ for some fixed integer $r > 1$, the piecewise constant finite element approximation converges to the solution of the local limit* (2.3) *as $h \to 0$ with the correct diffusion coefficient C iff* (2.31) *holds.*

Next, for the continuous piecewise linear element case with $\delta \le h$, we see from the nonlocal stiffness matrix that it is a combination of the standard three-point second order centeral difference with a five-point fourth order finite difference of the form:

$$C_\delta \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} - \frac{h}{6}\int_0^\delta s^3\gamma_\delta(s)ds \left(\frac{u(x_{j+2}) - 4u(x_{j+1}) + 6u(x_j) - 4u(x_{j-1}) + u(x_{j-2})}{h^4}\right) .$$

Since $0 \le \int_0^\delta hs^3\gamma_\delta(s)ds \le h\delta C_\delta \to 0$, as $\delta, h \to 0$, we see that the correct local limit is always assured. This gives the following theorem.

Theorem 2.2.12. *For the case $\delta \le h$, the continuous piecewise linear finite element approximation gives a consistent difference approximation to the local limit* (2.3) *with the correct diffusion coefficient C as $h \to 0$.*

Finally, for the piecewise linear finite element scheme and $\delta = rh$ with some fixed integer $r > 1$, we may examine the difference stencil:

$$a_{ii}u(x_i) + \sum_{m=1}^{r+1}(a_{i,i+m}u(x_{i+m}) + a_{i,i-m}u(x_{i-m})) ,$$

with $(a_{ij}) = \mathbb{A}_{E,1}$ being entries of the nonlocal stiffness matrix $\mathbb{A}_{E,1}$. Again, to leading order of expansion, by the symmetry of $\mathbb{A}_{E,1}$, we get the coefficient of $u''(x_i)$ as

$$\begin{aligned}\sum_{m=1}^{r+1} a_{i,i+m}(mh)^2 &= \frac{1}{h}\sum_{m=1}^{r+1} B(\phi^1_{i+m}, \phi^1_i)(mh)^2 = \frac{1}{2h}B(\mathcal{I}_h((x-x_i)^2), \phi^1_i) \\ &= -C_\delta + \frac{1}{2h}B(\mathcal{I}_h((x-x_i)^2) - (x-x_i)^2, \phi^1_i) .\end{aligned} \tag{2.32}$$

where $\mathcal{I}_h$ is the continuous piecewise linear interpolation operator with respect to the mesh and we have used the observation that

$$B((x-x_i)^2, \phi^1_i) = (-\mathcal{L}_\delta(x-x_i)^2, \phi^1_i) = -2C_\delta(1, \phi^1_i) = -2hC_\delta .$$

Let $w(x) = (x-x_i)^2$, w^h be the piecewise linear finite element approximation of w with its values outside Ω matching with w. Then, $B(\mathcal{I}_h(w) - w, \phi^1_i) = B(\mathcal{I}_h(w) - w^h, \phi^1_i)$. So the term remains to be estimated is equivalent to $B(\mathcal{I}_h(w) - w^h, \phi^1_i)$. We first recall that for the 1d second order differential operator, it is a well-known textbook fact that the continuous piecewise linear finite element solution is precisely the linear interpolant. This property is not expected to remain true in general in our nonlocal setting. However, we have a perhaps surprising property which meets our needs here.

Lemma 2.2.13. *For the bilinear form defined in* (2.12), *we have*

$$B(u, \phi_i^1) = B(\mathcal{I}_h^1 u, \phi_i^1), \quad \forall \;\; i \tag{2.33}$$

for any quadratic function u.

Proof. Since B is bilinear, we only need to prove the equation for $u(x) = x^2$. Now we know that the error function $e_u = u - \mathcal{I}_h^1 u$ is piecewise quadratic of the form:

$$e_u(x) = u(x) - \mathcal{I}_h^1 u(x) = (x - x_{j-1})(x - x_j)\,, \quad \forall \, x \in [x_{j-1}, x_j],$$

Thus, on a uniform mesh, e_u is periodic with period h, and symmetric with respect to each mesh grid x_j and the mid-point of the mesh element $x_j + h/2$. Then,

$$\begin{aligned} B(e_u, \phi_i^1) &= -(\mathcal{L}_\delta e_u, \phi_i^1) \\ &= -\int_{x_{i-1}}^{x_i+1} \phi_i^1(x)\Big(\int_0^\delta (e_u(x+s) + e_u(x-s) - 2e_u(x))\gamma_\delta(s)ds\Big)dx \\ &= -\int_0^\delta \gamma_\delta(s)\Big(\int_{x_{i-1}}^{x_i+1} \phi_i^1(x)(e_u(x+s) + e_u(x-s) - 2e_u(x))dx\Big)ds\,. \end{aligned} \tag{2.34}$$

We now show that

$$\beta(s) = \int_{x_{i-1}}^{x_i} \phi_i^1(x)(e_u(x+s) + e_u(x-s))dx$$

is independent of s. Indeed, since e_u is h-periodic, we have

$$\beta(s) = \int_0^h \frac{x}{h}(e_u(x+s) + e_u(x-s))dx\,.$$

Now, we first do a change of variable $y = -x$ in the second integral to get

$$\beta(s) = \int_0^h \frac{x}{h} e_u(x+s)dx + \int_0^{-h} \frac{y}{h} e_u(-y-s)dy = \int_0^h \frac{x}{h} e_u(x+s)dx - \int_{-h}^0 \frac{y}{h} e_u(y+s)dy$$

where in the second integral of the second equality, we have used the fact that e_u is even. Apply a change of variable $y = x - h$ to the last integral, we get for any s,

$$\beta(s) = \int_0^h \frac{x}{h} e_u(x+s)dx + \int_0^h \frac{h-x}{h} e_u(x+s)dx = \int_0^h e_u(x+s)dx = \int_0^h e_u(x)dx,$$

where we have used the h-periodicity of e_u. This implies $\beta(s) = \beta(0)$. Consequently,

$$\int_{x_{i-1}}^{x_{i+1}} \phi_i^1(x)(e_u(x+s) + e_u(x-s) - 2e_u(x))dx = 0 \; .$$

In turn, this proves $B(e_u, \phi_i^1) = 0$ and the lemma. □

The lemma implies that the continuous piecewise linear finite element solution to the nonlocal problem with a quadratic polynomial solution is the same as the piecewise linear interplant of the solution, in other words, the pointwise errors at the mesh points are identically zero, which gives a super-close result for nonlocal problems.

We now have the following theorem.

Theorem 2.2.14. *For the case $\delta = rh$ with a fixed integer $r > 1$, the continuous piecewise linear finite element approximation provides a consistent difference approximation to the local limit* (2.3) *as $h \to 0$ with the correct diffusion coefficient C. Moreover, for problems with sufficiently smooth solutions, the order of truncation error is $O(h^2)$.*

Proof. Using (2.32) and the lemma 2.2.13, we see the consistency to the correct local limit as $h \to 0$ is always true for continuous piecewise linear elements. Moreover, the coefficients of higher order terms, like for $u^{(2k)}(x_i)$ are all bounded by constant multiples of $\delta^{2k}/(2k)!$. Thus, for $\delta = rh$ with a fixed integer $r > 1$, we get that the truncation error is $O(h^2)$. □

The lack of discrete maximum principle for piecewise linear finite element schemes when $\delta > h$ means that we do not yet have an L^∞ stability result (being either uniform or δ-dependent) in this case. However, this perhaps is a technicality. First, using ideas given in [Mengesha and Du, 2014a], we may prove that for kernels of the form $\gamma_\delta(s) = \gamma_1(s/\delta)/(\delta^3)$ with γ_1 being a non-negative and non-increasing density function defined on $(0, 1)$ and strictly positive near the origin, the finite element method is stable in L^2 uniformly as h and δ go to zero simultaneously. Combining with consistency estimates, we may thus deduce the uniform L^2 convergence of continuous piecewise linear finite element solutions to the solution of the desirable (correct) local limit. More discussions on finite element schemes in higher dimensions and for more general meshes can be found in chapter 3.

2.3 Numerical results

We now report results of numerical experiments which substantiate the analysis given earlier and offer quantitative pictures to the behavior of numerical solutions in particular in the local limit. The orders of convergence are also examined. We choose both integrable and non-integrable kernels in the following examples which all give $C_\delta = C = 1$ for all δ. In order to get simpler benchmark solutions, we calculate the right hand side of (2.1) based on an exact solution $u(x) = x^2(1-x^2)$ and specify the nonlocal constraints on Ω_I to match with $u(x)$. This naturally leads to a δ-dependent right hand side $f_\delta = f_\delta(x)$. Moreover, with u independent of δ, we get that it is also the exact solution of the local problem

$$\begin{cases} -u''(x) = f(x) = 12x^2 - 2, \quad \text{in } (0,1), \\ u(0) = u(1) = 0\ . \end{cases} \tag{2.35}$$

We solve the nonlocal problem on a uniform mesh and take δ to be a constant multiple of h and reduce h to check convergence properties of the different schemes. We choose two popular examples of kernel functions in our discussion.

2.3.1 Example 1

We first take a constant box potential $\gamma_\delta(s) = 3\delta^{-3}$ for $s \in [0,\delta)$ which leads to $f_\delta(x) = f(x) + 6\delta^2/5$. Since f_δ is of order $O(\delta^2) = O(h^2)$ away from $f(x)$ for any fixed r, so that this δ dependence in f_δ would not affect the convergence behavior up to the second order. For difference methods corresponding to (2.7), we get $C^r_{0,2}(h) \equiv C = 1$ and

$$C^r_{0,\alpha}(h) = \frac{3}{\delta^3}\sum_{j=1}^{r}(jh)^{2-\alpha}\int_{(j-1)h}^{jh} s^\alpha ds \equiv \frac{3}{r^3(\alpha+1)}\sum_{j=1}^{r} j^{2-\alpha}[j^{\alpha+1} - (j-1)^{\alpha+1}] = C^r_\alpha.$$

We plot C^r_α, for several different values of r as a function of α in Fig. 2.1. It shows that C^r_α is strictly decreasing with respect to α and only when $\alpha = 2$ we obtain the right coefficient $C = 1$. And as expected, C^r_α approaches 1 as $r \to \infty$.

As an illustration we choose $r = 3$ and refine the mesh with decreasing h. While we know that $C^3_2 = C = 1$, direct calculation shows that $C^3_0 = \frac{14}{9}$ and $C^3_1 = \frac{11}{9}$. Then only when $\alpha = 2$ the numerical solutions converge to the correct local limit $u(x) = x^2(1-x^2)$. Table 2.1 shows errors and convergence rates of solutions of different difference schemes corresponding

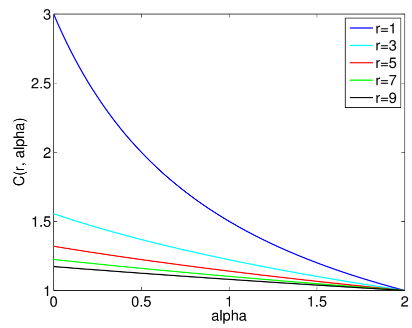

Figure 2.1: C_α^r with $\gamma_\delta(s) = 3\delta^{-3}$.

to (2.7) to limiting solutions $x^2(1-x^2)/C_\alpha^3$ for $\alpha = 0, 1, 2$ respectively. Notice that to make the form of the exact solution analytically simple or, when computing discrete solutions to the nonlocal equation for schemes parametrized by α, we impose the corresponding values of $x^2(1-x^2)/C_\alpha^3$ on Ω_I so that the exact solution to the corresponding local limiting problem matches with $x^2(1-x^2)/C_\alpha^3$ on Ω as well. From the data in the table, we see that the convergence rates are all $O(h^2)$ as proved in earlier analysis. Obviously, if we measure the errors between numerical solutions for $\alpha \neq 2$ against the correct limit $u(x) = x^2(1-x^2)$, we would not observe any convergence. Thus, we do not present such data here.

	$\alpha = 0$		$\alpha = 1$		$\alpha = 2$	
h	L^∞error	Rate	L^∞error	Rate	L^∞error	Rate
2^{-3}	2.40×10^{-2}	$--$	3.29×10^{-2}	$--$	4.17×10^{-2}	$--$
2^{-4}	5.07×10^{-3}	2.24	6.77×10^{-3}	2.28	8.47×10^{-3}	2.30
2^{-5}	1.18×10^{-3}	2.10	1.57×10^{-3}	2.11	1.96×10^{-3}	2.11
2^{-6}	2.85×10^{-4}	2.05	3.77×10^{-4}	2.06	4.70×10^{-4}	2.06
2^{-7}	6.99×10^{-5}	2.03	9.25×10^{-5}	2.03	1.15×10^{-4}	2.03
2^{-8}	1.73×10^{-5}	2.01	2.29×10^{-5}	2.01	2.85×10^{-5}	2.01
2^{-9}	4.31×10^{-6}	2.01	5.70×10^{-6}	2.01	7.10×10^{-6}	2.01
2^{-10}	1.08×10^{-6}	2.00	1.42×10^{-6}	2.00	1.77×10^{-6}	2.00

Table 2.1: L^∞ errors and convergence rates of finite difference method using (2.7) for fixed $r = 3$ and $\gamma_\delta(s) = 3\delta^{-3}$ to solutions $x^2(1-x^2)/C_\alpha^3$ with $\alpha = 0, 1, 2$.

For finite element methods with piecewise constant basis, we expect to get different local limits for different values of the fixed parameter r with diffusion coefficients being $\tilde{C}_0^r = 1 + 1/(2r^2)$ as calculated before. This explains similar numerical results in [Chen and Gunzburger, 2011] that piecewise constant finite element methods fail to converge if the ratio of δ and h is fixed. Instead, they converge to $u(x)/\tilde{C}_0^r$. Taking the example of $r = 3$, we have $\tilde{C}_0^3 = \frac{19}{18}$. Table 2.2 shows errors and convergence rates to the local limits of the piecewise constant and piecewise linear finite element approximations with a fixed $r = 3$ while refining mesh with a decreasing h. The errors are measured against the wrong local limit $18x^2(1 - x^2)/19$ for the piecewise constant element case while the correct local limit is $x^2(1 - x^2)$. Note that when solving nonlocal equations, the values of the solution on Ω_δ are again taken to match with exact solutions associated with corresponding values of α.

	p.w constant		p.w linear	
h	L^∞error	Rate	L^∞error	Rate
2^{-3}	2.65×10^{-2}	$--$	3.50×10^{-2}	$--$
2^{-4}	5.80×10^{-3}	2.19	7.34×10^{-3}	2.25
2^{-5}	1.36×10^{-3}	2.09	1.69×10^{-3}	2.12
2^{-6}	3.31×10^{-4}	2.04	4.06×10^{-4}	2.06
2^{-7}	8.15×10^{-5}	2.02	9.95×10^{-5}	2.03
2^{-8}	2.02×10^{-5}	2.01	2.46×10^{-5}	2.01
2^{-9}	5.04×10^{-6}	2.01	6.12×10^{-6}	2.01
2^{-10}	1.26×10^{-6}	2.00	1.53×10^{-6}	2.00

Table 2.2: L^∞ errors and convergence rates of finite element method for fixed $r = 3$ and $\gamma_\delta(s) = 3\delta^{-3}$ to solutions $18x^2(1 - x^2)/19$ and $x^2(1 - x^2)$ respectively.

2.3.2 Example 2

We now repeat the experiments done in example 1 for a non-integrable kernel $\gamma_\delta(s) = \frac{2}{\delta^2 s}$. The corresponding $f_\delta(x) = f(x) + \delta^2$ which is again $O(h^2)$ away from $f(x)$. The constrained values on Ω_δ are treated the same way as in the above example so as to keep exact solutions to be a multiple of $x^2(1-x^2)$. For this kernel, when the scheme corresponding to (2.7) is used, we need to take $\alpha > 0$ so only numerical solutions for $\alpha = 1$ and $\alpha = 2$ are computed. As before we see that $C_\delta = C = 1$ and $C_2^r(h) \equiv 1 = C_2^r = 1$, and for any $\alpha > 0$, the limiting diffusion coefficient is given by

$$C_\alpha^r(h) = \frac{2}{\delta^2}\sum_{j=1}^{r}(jh)^{2-\alpha}\int_{(j-1)h}^{jh} s^{\alpha-1}ds \equiv \frac{2}{r^2}\sum_{j=1}^{r} j^{2-\alpha}[j^\alpha - (j-1)^\alpha] = C_\alpha^r \geq 1\,.$$

For schemes corresponding to (2.9) with $\alpha = 0$ and ones using piecewise constant finite element approximations, the diffusion coefficients are given by $\tilde{C}_0^r$ in (2.30). We plot in Fig. 2.2 the diffusion coefficients C_α^r and also $\tilde{C}_0^r$ for different values of r.

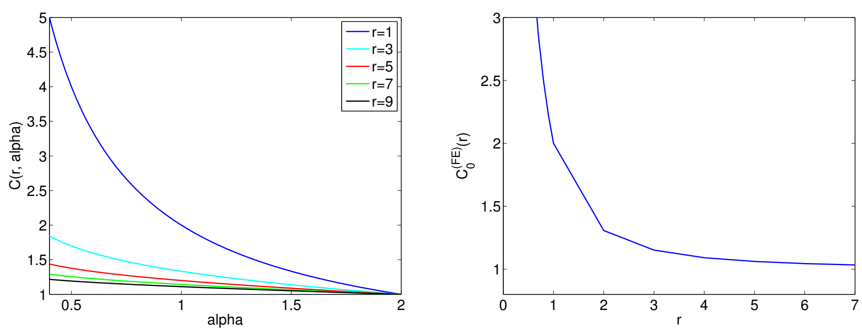

Figure 2.2: C_α^r (left) and $\tilde{C}_0^r$ (right) with $\gamma_\delta(s) = \frac{2}{\delta^2 s}$.

By choosing $r = 3$, we have $C_2^3 = 1$, and $C_1^3 = \frac{4}{3}$, so numerical solutions of finite difference methods corresponding to (2.7) for $\alpha = 1$ are expected to converge to $3x^2(1-x^2)/4$ instead of $u(x) = x^2(1-x^2)$ which is the correct local limit when $\alpha = 2$ is used. Similarly, $4831x^2(1-x^2)/5562$ and $x^2(1-x^2)$ give the limiting exact solutions to the piecewise constant and piecewise linear finite element solutions respectively. As in the previous example, these different limits are used when measuring errors and convergence rates of numerical solutions reported in Tables 2.3 and 2.4.

2.3.3 Discussions

The numerical experiments here are mostly confined to smooth solutions. When local limits are considered, the results are also limited to cases where δ/h remains bounded. We leave experiments involving singular solutions and more general limits as h and δ go to zero

	$\alpha = 1$		$\alpha = 2$	
h	L^∞error	Rate	L^∞error	Rate
2^{-3}	2.22×10^{-2}	$--$	2.80×10^{-2}	$--$
2^{-4}	4.91×10^{-3}	2.18	6.10×10^{-3}	2.20
2^{-5}	1.16×10^{-3}	2.10	1.43×10^{-3}	2.10
2^{-6}	2.82×10^{-4}	2.04	3.45×10^{-4}	2.05
2^{-7}	6.96×10^{-5}	2.01	8.49×10^{-5}	2.02
2^{-8}	1.73×10^{-5}	2.01	2.10×10^{-5}	2.01
2^{-9}	4.30×10^{-6}	2.00	5.24×10^{-6}	2.01
2^{-10}	1.07×10^{-6}	2.00	1.31×10^{-6}	2.00

Table 2.3: L^∞ errors and convergence rates of finite difference method using (2.7) for fixed $r = 3$ and $\gamma_\delta(s) = \frac{2}{\delta^2 s}$ to solutions $3x^2(1-x^2)/4$ and $x^2(1-x^2)$ respectively.

	p.w constant		p.w linear	
h	L^∞error	Rate	L^∞error	Rate
2^{-3}	1.75×10^{-2}	$--$	3.50×10^{-2}	$--$
2^{-4}	4.00×10^{-3}	2.13	7.34×10^{-3}	2.25
2^{-5}	9.59×10^{-4}	2.06	1.69×10^{-3}	2.12
2^{-6}	2.35×10^{-4}	2.03	4.06×10^{-4}	2.06
2^{-7}	5.81×10^{-5}	2.01	9.95×10^{-5}	2.03
2^{-8}	1.45×10^{-5}	2.01	2.46×10^{-5}	2.01
2^{-9}	3.61×10^{-6}	2.00	6.12×10^{-6}	2.01
2^{-10}	9.03×10^{-7}	2.00	1.53×10^{-6}	2.00

Table 2.4: L^∞ errors and convergence rates of finite element method for fixed $r = 3$ and $\gamma_\delta(s) = \frac{2}{\delta^2 s}$ to solutions $4831x^2(1-x^2)/5562$ and $x^2(1-x^2)$ respectively.

to future works. Moreover, we note that for the linear scalar problem under consideration and on a uniform mesh, the different local limits identified in our analysis differ from the desired correct limit only by a multiplicative constant factor. This could be corrected with a proper scaling which has been a remedy developed heuristically in some applications. We caution that this simple strategy may not work for more complex models involving heterogeneities, nonlinearities and complicated meshes.

2.4 Conclusion

We discussed in this chapter two classes of methods, namely quadrature based finite difference and conforming finite element, for discretizing nonlocal diffusion and peridynamic models. There are representatives from both classes of methods which can be applied to problems with very general nonlocal interaction kernels having a finite second order moment. We attempted to address several numerical analysis issues whose local analogues have become standard textbook materials. For instance, we identified cases where the nonlocal stiffness matrices generated by the two different classes of methods coincide. We also established discrete maximum principles for most of the schemes except for continuous piecewise linear finite element approximations. These methods are all convergent when applied to the nonlocal problem with a fixed horizon δ as shown in section 2.2. However, they behave differently when local limits are considered. The local limits of the discrete schemes for fixed mesh size h and vanishing horizon δ was discussed in section 2.1.5, indicating that only for a few selected schemes their local limits remain convergent discrete schemes for the limiting local differential equation. These selected schemes include the special quadrature based finite difference schemes (2.10) with operators in (2.7) with $\alpha = 2$ and (2.9) with $\alpha = 1$, as well as the continuous piecewise linear element scheme. Moreover, for precisely these same schemes, the results of sections 2.2.2 and 2.2.3 demonstrate that they all converge to the expected solution in both nonlocal setting and in the local limit (as δ and h both approach zero either with $\delta \leq h$ or $\delta = hr$ with a fixed integer $r > 1$). At the same time, numerical solutions based on other schemes discussed in the paper would likely converge to the wrong local limit. Our findings help clarifying various observations based on numerical experiments reported in the literature and improve our understanding of the potential risks involved when discretization parameters are tied with modeling parameters. Meanwhile, we also discovered some unexpected properties, such as the super-close result for piecewise linear finite element approximations of nonlocal problems with quadratic polynomial solutions.

While the study in this chapter is based on a simple one-dimensional linear model for the sake of offering insight without being impeded by tedious calculations, the surprising findings motivate us to look into the more general cases (see chapter 3-5) of robust numerical schemes of nonlocal models involving high dimensionality, nonuniform and unstructured mesh.

Chapter 3

An abstract framework of asymptotically compatible schemes

In this chapter, we introduce the notion of asymptotically compatible schemes and propose an abstract framework for their numerical analysis when they are applied to a special class of parametrized problems. This framework was first introduced in [Tian and Du, 2014b]. Practical implications of this framework are discussed in chapters 4-5.

3.1 Notation and assumptions

We first introduce notations and state the main assumptions in the section. The assumptions are given in the order of (infinite-dimensional) function spaces, then bilinear forms, followed by induced linear operators and finally the approximations.

We begin by considering a decreasing family of Hilbert spaces $\{\mathcal{T}_\sigma, \sigma \in [0,\infty]\}$ over $\mathbb{R}$ in the sense that $\mathcal{T}_{\sigma_2}$ is a dense subspace of $\mathcal{T}_{\sigma_1}$, for any $0 \leq \sigma_1 \leq \sigma_2 \leq \infty$.

Let $(\cdot,\cdot)_{\mathcal{T}_\sigma}$ and $\|\cdot\|_{\mathcal{T}_\sigma}$ denote the corresponding inner product and norm on $\mathcal{T}_\sigma$ and denote the dual space of $\mathcal{T}_\sigma$ by $\mathcal{T}_{-\sigma} = \mathcal{T}_\sigma^*$.

We note that both spaces $\mathcal{T}_0$ and $\mathcal{T}_\infty$ are of particular interests to our discussions here. Indeed, we identify the dual space of $\mathcal{T}_0$ with itself $\mathcal{T}_0^* = \mathcal{T}_0$. A typical example is given by the standard L^2 function space in applications we consider later. Moreover, we assume that $\mathcal{T}_0$ serves as the pivot space between $\mathcal{T}_{-\sigma}$ and $\mathcal{T}_\sigma$ so that a realization of the duality pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{T}_{-\sigma}$ and $\mathcal{T}_\sigma$ is given as the extension of the inner product on $\mathcal{T}_0$. In other words, for any $\sigma \in [0,\infty]$, and for $w \in \mathcal{T}_0 \subseteq \mathcal{T}_{-\sigma}$,

$$\langle w, v\rangle = (w,v)_{\mathcal{T}_0}\,, \quad \forall v \in \mathcal{T}_\sigma\,, \tag{3.1}$$

Thus, we do not specify any subscript related to σ to distinguish the duality pairing.

Assumptions given above on the spaces $\{\mathcal{T}_\sigma\}$ are very generic so far. To discuss the special class of variational problems defined on spaces $\{\mathcal{T}_\sigma\}$, we state the following assumptions which are crucial to the problems under consideration.

Assumption 3.1.1. *The spaces $\{\mathcal{T}_\sigma\}$ are assumed to satisfy the properties below.*
i) Uniform embedding *property: there are positive constants M_1 and M_2, independent of $\sigma \in (0,\infty)$, such that*

$$M_1\|u\|_{\mathcal{T}_0} \le \|u\|_{\mathcal{T}_\sigma}\,, \quad \forall u \in \mathcal{T}_\sigma \quad \textit{and} \quad \|u\|_{\mathcal{T}_\sigma} \le M_2\|u\|_{\mathcal{T}_\infty}\,, \quad \forall u \in \mathcal{T}_\infty\,.$$

ii) Asymptotically compact embedding *property: for any sequence $(u_n \in \mathcal{T}_n)$, if there is a constant $C > 0$ independent of n such that*

$$\|u_n\|_{\mathcal{T}_n} \le C\,, \quad \forall n\,,$$

then the sequence (u_n) is relatively compact in $\mathcal{T}_0$ and each limit point is in $\mathcal{T}_\infty$.

With spaces $\{\mathcal{T}_\sigma\}$ given, we now consider some parametrized bilinear forms.

Assumption 3.1.2. *Let $a_\sigma\colon \mathcal{T}_\sigma \times \mathcal{T}_\sigma \to \mathbb{R}$ be a symmetric bilinear form, $\sigma \in [0,\infty]$.*
i) a_σ is bounded*: there exists a constant $C_2 > 0$ such that*

$$a_\sigma(u,v) \le C_2\|u\|_{\mathcal{T}_\sigma}\|v\|_{\mathcal{T}_\sigma}\,, \quad \forall u,v \in \mathcal{T}_\sigma\,.$$

ii) a_σ is coercive*: there exists a constant $C_1 > 0$ such that*

$$a_\sigma(u,u) \ge C_1\|u\|^2_{\mathcal{T}_\sigma}\,, \quad \forall u \in \mathcal{T}_\sigma\,.$$

Moreover, we assume that C_1 and C_2 are constants independent of σ.

Given the above assumption on the bilinear forms, for any $\sigma \in [0,\infty]$, we see that $a_\sigma(u,\cdot)$ defines a bounded linear functional for any $u \in \mathcal{T}_\sigma$. Moreover, by the Lax-Milgram theorem, it induces naturally a bounded linear operator, denoted by $\mathcal{A}_\sigma$, from $\mathcal{T}_\sigma$ to its dual $\mathcal{T}_{-\sigma}$, with a bounded inverse $\mathcal{A}_\sigma^{-1} : \mathcal{T}_{-\sigma} \to \mathcal{T}_\sigma$. In other words, using the notations given above, we have

$$\langle \mathcal{A}_\sigma u, v \rangle = a_\sigma(u, v) \,, \quad \forall u, v \in \mathcal{T}_\sigma \,. \tag{3.2}$$

By the symmetry of a_σ, we easily see that $\mathcal{A}_\sigma$ is self-adjoint,

$$\langle \mathcal{A}_\sigma u, v \rangle = \langle u, \mathcal{A}_\sigma v \rangle \,, \quad \forall u, v \in \mathcal{T}_\sigma \,. \tag{3.3}$$

and thus positive definite. We next give some assumptions on $\mathcal{A}_\sigma$.

Assumption 3.1.3. *For $\mathcal{A}_\sigma$ defined by* (3.2)*, we assume the following.*

i) A subspace $\mathcal{T}_*$ is dense in $\mathcal{T}_\infty$, and also dense in any $\mathcal{T}_\sigma$ with $\sigma \geq 0$, such that

$$\mathcal{A}_\sigma u \in \mathcal{T}_0 \,, \quad \forall u \in \mathcal{T}_* \,.$$

ii) $\mathcal{A}_\infty$ is the limit of $\mathcal{A}_\sigma$ in $\mathcal{T}_*$ in the sense that

$$\lim_{\sigma \to \infty} \| \mathcal{A}_\sigma u - \mathcal{A}_\infty u \|_{\mathcal{T}_{-\sigma}} = 0 \,, \quad \forall u \in \mathcal{T}_* \,. \tag{3.4}$$

Since we are concerned with numerical approximations of the variational problems associated with the operators $\{\mathcal{A}_\sigma\}$ for $\sigma \in [0, \infty]$, we consider a family of closed subspaces $\{W_{\sigma,h} \subset \mathcal{T}_\sigma\}$ parametrized by an additional real parameter $h \in (0, h_0]$. The fact that we take $W_{\sigma,h}$ as a subspace of $\mathcal{T}_\sigma$ implies that we are effectively adopting a standard, internal or equivalently conforming type, Galerkin approximation approach.

Although in practice $W_{\sigma,h}$ is always finite dimensional with h being the corresponding meshing parameter, it is not necessary to make such an assumption here for the theoretical analysis. Moreover, while $\{\mathcal{T}_\sigma\}$ is a decreasing family, for each $h > 0$, the family $\{W_{\sigma,h}\}$ does not have to be. All we need here are that $W_{\sigma,h}$ is closed in $\mathcal{T}_\sigma$ for each given σ and h, and some basic assumptions on the approximation properties of $W_{\sigma,h}$ to $\mathcal{T}_\sigma$ as $h \to 0$ as stated below. The first part of these assumptions ensures the convergence of approximations to $\mathcal{T}_\sigma$ as $h \to 0$ for each σ, while the second part is concerned with the limiting behavior as both $h \to 0$ and $\sigma \to \infty$ at the same time.

Assumption 3.1.4. *Assume that the family of subspaces $\{W_{\sigma,h} \subset \mathcal{T}_\sigma\}$ parametrized by $\sigma \in [0, \infty]$ and $h \in (0, h_0]$, satisfies the following properties.*

i) For each $\sigma \in [0, \infty]$*, the family* $\{W_{\sigma,h}, h \in (0, h_0]\}$ *is dense in* $\mathcal{T}_\sigma$ *in the sense that,* $\forall v \in \mathcal{T}_\sigma$*, there exists a sequence* $\{v_n \in W_{\sigma,h_n}\}$ *with* $h_n \to 0$ *as* $n \to \infty$*, such that*

$$\|v - v_n\|_{\mathcal{T}_\sigma} \to 0 \quad as \quad n \to \infty\,. \tag{3.5}$$

ii) $\{W_{\sigma,h}, \sigma \in [0, \infty), h \in (0, h_0]\}$ *is* asymptotically dense *in* $\mathcal{T}_\infty$*, i.e.,* $\forall v \in \mathcal{T}_\infty$*, there exists a sequence* $\{v_n \in W_{\sigma_n,h_n}\}_{h_n \to 0,\, \sigma_n \to \infty}$ *as* $n \to \infty$*, such that*

$$\|v - v_n\|_{\mathcal{T}_\infty} \to 0 \quad as \quad n \to \infty\,. \tag{3.6}$$

3.2 The parametrized variational problems and their approximations

Consider a family of parametrized variational problems defined by: given $f \in \mathcal{T}_{-\sigma}$,

$$\text{find } u_\sigma \in \mathcal{T}_\sigma \text{ such that } \; a_\sigma(u_\sigma, v) = \langle f, v \rangle \quad \forall v \in \mathcal{T}_\sigma\,, \tag{3.7}$$

for $\sigma \in [0, \infty]$. The approximation to u_σ in a subspace $W_{\sigma,h}$ is defined by:

$$\text{find } u_{\sigma,h} \in W_{\sigma,h} \text{ such that } \; a_\sigma(u_{\sigma,h}, v) = \langle f, v \rangle \quad \forall v \in W_{\sigma,h}\,. \tag{3.8}$$

The existence and uniqueness of u_σ and $u_{\sigma,h}$ follow from assumptions made earlier. We may also express equations (3.7) and (3.8) in strong forms as

$$\mathcal{A}_\sigma u_\sigma = f, \tag{3.9}$$

$$\mathcal{A}_\sigma^h u_{\sigma,h} = \pi_\sigma^h f, \tag{3.10}$$

where π_σ^h is the L^2 projection operator onto the subspace $W_{\sigma,h}$ and $\mathcal{A}_\sigma^h : W_{\sigma,h} \to W_{\sigma,h}^*$ is the operator induced by the bilinear form a_σ in $W_{\sigma,h}$ (or the solution operator of (3.8) in the specified subspace).

We are interested in establishing an abstract framework to analyze the various limits of $\{u_{\sigma,h}\}$ as we take limits in the parameters. We first state a convergence result for the solutions of the parametrized variational problems as $\sigma \to \infty$.

Theorem 3.2.1 (Convergence of variational solutions as $\sigma \to \infty$)**.** *Given the assumptions on the family of spaces and the bilinear forms and operators, we have*

$$\|u_\sigma - u_\infty\|_{\mathcal{T}_0} \to 0 \quad \text{as } \sigma \to \infty\,.$$

Proof. By (3.7) and the assumptions, we have

$$C_1 \|u_\sigma\|^2_{\mathcal{T}_\sigma} \le a_\sigma(u_\sigma, u_\sigma) = \langle f, u_\sigma \rangle \le \|f\|_{\mathcal{T}_{-\sigma}} \|u_\sigma\|_{\mathcal{T}_\sigma}\ ,$$

which leads to the uniform boundedness of $\{u_\sigma \in \mathcal{T}_\sigma\}$ and thus by the asymptotically compact embedding property, we get the convergence of a subsequence of $\{u_\sigma\}$ in $\mathcal{T}_0$ to a limit point $u_* \in \mathcal{T}_\infty$. For notational convenience, we use the same $\{u_\sigma\}$ to denote the subsequence. Now, taking $v \in \mathcal{T}_* \subset \mathcal{T}_\infty$, we have

$$\begin{aligned}\langle f - \mathcal{A}_\infty u_*, v\rangle &= \langle \mathcal{A}_\sigma u_\sigma - \mathcal{A}_\infty u_*, v\rangle = \langle u_\sigma, \mathcal{A}_\sigma v\rangle - \langle u_*, \mathcal{A}_\infty v\rangle \\ &= \langle u_\sigma, \mathcal{A}_\sigma v - \mathcal{A}_\infty v\rangle + (u_\sigma - u_*, \mathcal{A}_\infty v)_{\mathcal{T}_0}\ .\end{aligned}$$

We know that as $\sigma \to \infty$, u_σ is uniformly bounded in $\mathcal{T}_\sigma$ and thus in $\mathcal{T}_0$ so is $\mathcal{A}_\infty v$ in $\mathcal{T}_0$. Moreover, we have that $u_\sigma - u_*$ goes to zero in $\mathcal{T}_0$ by the choice of u_* and that $\|\mathcal{A}_\sigma v - \mathcal{A}_\infty v\|_{\mathcal{T}_{-\sigma}} \to 0$ by the assumption (3.4). Together with the uniform boundedness of $\|u_\sigma\|_{\mathcal{T}_\sigma}$ and the assumption $\mathcal{A}_\infty v \in \mathcal{T}_0$, we thus arrive at

$$\langle f - \mathcal{A}_\infty u_*, v\rangle \to 0 \qquad \text{as } \ \sigma \to \infty\,.$$

Moreover, since $\mathcal{T}_*$ is dense in $\mathcal{T}_\infty$, we see that u_* is the unique solution u_∞ of $\mathcal{A}_\infty u_\infty = f$ and the convergence of the whole sequence also follows. □

Next, we consider the convergence of approximations as $h \to 0$ for a given σ.

Theorem 3.2.2 (Convergence with a fixed $\sigma \in [0, \infty]$ as $h \to 0$)**.** *For any given $\sigma \in [0, \infty]$, let u_σ and $u_{\sigma,h}$ be defined by* (3.7) *and* (3.8)*. Given the assumptions on the approximate spaces and the approximate bilinear forms, there exists a constant $C > 0$, independent of h such that,*

$$\|u_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} \le C \inf_{v_{\sigma,h} \in W_{\sigma,h}} \|v_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as } h \to 0\,.$$

Proof. The proof is similar to the standard best approximation property of the Ritz-Galerkin approximation. Given $\sigma \in [0, \infty]$, for any $v_{\sigma,h} \in W_{\sigma,h}$,

$$\begin{aligned} C_1 \|u_\sigma - u_{\sigma,h}\|_{\mathcal{T}_\sigma}^2 &\le a_\sigma(u_\sigma - u_{\sigma,h}, u_\sigma - u_{\sigma,h}) = a_\sigma(u_\sigma - u_{\sigma,h}, u_\sigma) \\ &= a_\sigma(u_\sigma - u_{\sigma,h}, u_\sigma - v_{\sigma,h}) \le C_2 \|u_\sigma - u_{\sigma,h}\|_{\mathcal{T}_\sigma} \|u_\sigma - v_{\sigma,h}\|_{\mathcal{T}_\sigma} . \end{aligned}$$

So we have

$$\|u_\sigma - u_{\sigma,h}\|_{\mathcal{T}_\sigma} \le C \inf_{v_{\sigma,h} \in W_{\sigma,h}} \|u_\sigma - v_{\sigma,h}\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as } h \to 0 .$$

This proves the theorem. □

We now move on to an analog of theorem 3.2.1 for approximate problems, that is, we consider the convergence as $\sigma \to \infty$ but for a fixed $h > 0$. For this, we need a few additional assumptions on the approximation spaces.

Theorem 3.2.3 (Convergence of approximate solutions with $h > 0$ as $\sigma \to \infty$)**.** *Given the assumptions on the family of spaces, bilinear forms, operators and approximate spaces, and assume in addition that for a given $h > 0$, we have*

i). Limit of approximate spaces:

$$W_{\infty,h} = \mathcal{T}_\infty \cap \big(\bigcap_{\sigma \ge 0} W_{\sigma,h} \big) . \tag{3.11}$$

ii). Approximation property of bilinear forms:

$$\lim_{\sigma \to \infty} a_\sigma(u_h, v_h) = a_\infty(u_h, v_h) , \quad \forall u_h, v_h \in W_{\infty,h}. \tag{3.12}$$

iii). A strengthened continuity property: for any sequence $(w_{\sigma,h} \in W_{\sigma,h})$ with uniformly bounded $(\|w_{\sigma,h}\|_{\mathcal{T}_\sigma})$ and satisfying $w_{\sigma,h} \to 0$ in $\mathcal{T}_0$ as $\sigma \to \infty$, we have

$$\lim_{\sigma \to \infty} a_\sigma(w_{\sigma,h}, v_h) = 0 , \quad \forall v_h \in W_{\infty,h}. \tag{3.13}$$

Then, for the approximate solutions $u_{\sigma,h}$ of (3.8) *with $\sigma \in (0, \infty)$, we have*

$$\|u_{\sigma,h} - u_{\infty,h}\|_{\mathcal{T}_0} \to 0 \quad \textit{as } \sigma \to \infty . \tag{3.14}$$

Proof. Similar to the proof of theorem 3.2.1, we have

$$C_1 \|u_{\sigma,h}\|_{\mathcal{T}_\sigma}^2 \le a_\sigma(u_{\sigma,h}, u_{\sigma,h}) = \langle f, u_{\sigma,h} \rangle \le \|f\|_{\mathcal{T}_{-\sigma}} \|u_{\sigma,h}\|_{\mathcal{T}_\sigma} ,$$

which leads to the uniform boundedness of $\{u_{\sigma,h} \in \mathcal{T}_\sigma\}$ and thus by the asymptotically compact embedding property, we get the convergence of a subsequence in $\mathcal{T}_0$ to a limit point $u_{*,h} \in \mathcal{T}_\infty$. By the assumption (3.11), we have necessarily that $u_{*,h} \in W_{\infty,h}$. Using again the same $\{u_{\sigma,h}\}$ to denote the subsequence and taking $v_h \in W_{\infty,h} \subset W_{\sigma,h}$,

$$\begin{aligned}\langle f, v_h\rangle - a_\infty(u_{*,h}, v_h) &= a_\sigma(u_{\sigma,h}, v_h) - a_\infty(u_{*,h}, v_h)\\ &= a_\sigma(u_{\sigma,h}, v_h) - a_\sigma(u_{*,h}, v_h) + a_\sigma(u_{*,h}, v_h) - a_\infty(u_{*,h}, v_h)\\ &= a_\sigma(u_{\sigma,h} - u_{*,h}, v_h) + \big[a_\sigma(u_{*,h}, v_h) - a_\infty(u_{*,h}, v_h)\big] = \mathrm{I}_1 + \mathrm{I}_2\,.\end{aligned}$$

Now, to estimate the first term, we let $w_{\sigma,h} = u_{\sigma,h} - u_{*,h} \in W_{\sigma,h}$ and apply the strengthened continuity property of a_σ to get $\mathrm{I}_1 \to 0$. Assumption (3.12) implies that $\mathrm{I}_2 \to 0$. Thus, $u_{*,h}$ is the unique solution of (3.8) with $\sigma = \infty$ and the unique limit point of the whole sequence $\{u_{\sigma,h}\}$. The theorem thus follows. □

3.3 Asymptotically compatible schemes

While we have the convergence of $\{u_{\sigma,h}\}$ for a given σ as $h \to 0$, as well as the convergence of $\{u_\sigma\}$ to u_∞ and $\{u_{\sigma,h}\}$ to $u_{\infty,h}$ as $\sigma \to \infty$, we are also interested in the behavior also as both $\sigma \to \infty$ and $h \to 0$. We summarize this in Figure 3.1.

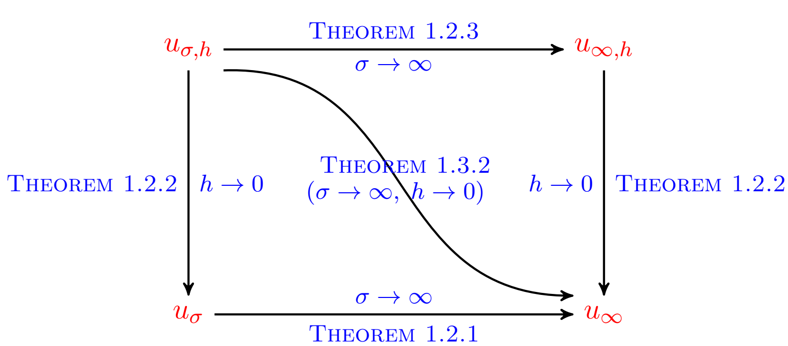

Figure 3.1: A diagram for asymptomatically compatible schemes and convergence results.

Definition 3.3.1. *The family of convergent approximations $\{u_{\sigma,h}\}$ defined by (3.8) is said to be asymptotically compatible (AC) to the solution u_∞ defined by (3.7) with $\sigma = \infty$, if for any sequence $\sigma_n \to \infty$ and $h_n \to 0$, we have $\|u_{\sigma_n,h_n} - u_\infty\|_{\mathcal{T}_0} \to 0$.*

Note that since u_{σ_n,h_n} and u_∞ may live in different spaces, the space $\mathcal{T}_0$ is the most natural space that contains all the elements involved.

Theorem 3.3.2 (Asymptotically compatibility)**.** *Under the assumptions 3.1.1-3.1.4, the family of approximations is asymptotically compatible.*

Proof. The first step is again similar to that in the proof of theorems 3.2.1 and 3.2.3, that is, we can get $\|u_{\sigma,h}\|_{\mathcal{T}_\sigma}$ being uniformly bounded by some constant,

$$\|u_{\sigma,h}\|_{\mathcal{T}_\sigma} \le C\,. \tag{3.15}$$

Then for any sequences $\{\sigma_n\}$ and $\{h_n\}$, where $\sigma_n \to \infty$, $h_n \to 0$, the sequence $(u_{\sigma_n,h_n})_n$ is relatively compact in $\mathcal{T}_0$, and any limit point u_* of the convergent subsequence in $\mathcal{T}_0$, still denoted by $(u_n = u_{\sigma_n,h_n})$ without loss of generality, is in $\mathcal{T}_\infty$. Let us show that u_* solves (3.7) with $\sigma = \infty$ and therefore is unique so that the entire sequence actually converges to the unique solution $u_* = u_\infty$. That is, for $\|u_* - u_n\|_{\mathcal{T}_0} \to 0$ as $n \to \infty$, we need to prove for every $v \in \mathcal{T}_*$, u_* satisfies (3.7).

By the asymptotically dense property (3.6) of $W_{\sigma,h}$ in $\mathcal{T}_\infty$, we can choose $v_n \in W_{\sigma_n,h_n}$ such that $\|v - v_n\|_{\mathcal{T}_\infty} \to 0$. Then we have the following equation,

$$\begin{aligned} a_\infty(u_*, v) - \langle f, v\rangle &= [a_\infty(u_*, v) - a_\infty(u_n, v)] + [a_\infty(u_n, v) - a_{\sigma_n}(u_n, v)] \\ &\quad +[a_{\sigma_n}(u_n, v) - a_{\sigma_n}(u_n, v_n)] + [\langle f, v_n\rangle - \langle f, v\rangle] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\,. \end{aligned} \tag{3.16}$$

We now show that as $n \to \infty$, all four terms vanish. Now for the first part, since the operator a_∞ is symmetric and $\mathcal{A}_\infty v \in \mathcal{T}_0$, we can rewrite I as

$$|\mathrm{I}| = |a_\infty(u_* - u_n, v)| = |(\mathcal{A}_\infty v, u_* - u_n)_{\mathcal{T}_0}| \le \|\mathcal{A}_\infty v\|_{\mathcal{T}_0}\|u_* - u_n\|_{\mathcal{T}_0} \to 0\,.$$

Similarly we can rewrite the second part and use assumption 3.1.3 to obtain

$$|\mathrm{II}| = |(\mathcal{A}_\infty v - \mathcal{A}_{\sigma_n} v, u_n)_{\mathcal{T}_0}| \le \|\mathcal{A}_\infty v - \mathcal{A}_{\sigma_n} v\|_{\mathcal{T}_{-\sigma_n}}\|u_n\|_{\mathcal{T}_{\sigma_n}} \le C\|\mathcal{A}_\infty v - \mathcal{A}_{\sigma_n} v\|_{\mathcal{T}_{-\sigma_n}} \to 0.$$

We then use the bound on the bilinear form a_σ and the uniform embedding to get

$$\begin{aligned} |\mathrm{III}| &= |a_{\sigma_n}(u_n, v - v_n)| \le C_2\|u_n\|_{\mathcal{T}_{\sigma_n}}\|v - v_n\|_{\mathcal{T}_{\sigma_n}} \\ &\le C_2 C\|v - v_n\|_{\mathcal{T}_{\sigma_n}} \le C_2 C M_2\|v - v_n\|_{\mathcal{T}_\infty} \to 0\,. \end{aligned}$$

Finally the last term can be estimated by

$$|\mathrm{IV}| \leq \|f\|_{\mathcal{T}_{-\sigma}} \cdot \|v - v_n\|_{\mathcal{T}_\sigma} \leq M_2 \|f\|_{\mathcal{T}_{-\sigma}} \|v - v_n\|_{\mathcal{T}_\infty} \to 0.$$

This shows that u_* solves (3.7) which completes the proof of the theorem.

Here, we note that we assume f is bounded under $\mathcal{T}_{-\sigma}$ norm for any $\sigma > 0$. This is particularly true for $f \in \mathcal{T}_0$. In this case, in the estimation of IV, we do not need $\|v - v_n\|_{\mathcal{T}_\infty} \to 0$ but only $\|v - v_n\|_{\mathcal{T}_0} \to 0$ is enough. □

3.4 Conclusion

In this chapter, asymptotically compatible schemes and the corresponding abstract mathematical framework are introduced for their rigorous numerical analysis with respect to certain classes of parametrized problems and their asymptotic limits. This allows us to go beyond the discussion on approximations of nonlocal models to establish a more general mathematical theory with a much broader perspective. Indeed, many classical problems change type with a parameter. For instance, the vanishing viscosity limit of the Navier-Stokes equations to the Euler equations, the convergence of phase field models to their sharp interface limits, as well as the linear elasticity problem as the Lamé constant tends to infinity, etc. All these problems share a common feature that properties of the underlying equations change significantly in the limit process, so that it is not at all obvious what numerical methods may be effective for a vast range of parameter values and in some limiting cases. It is interesting and challenging to develop numerical methods that behave as desired while taking limits of the problems, and we consider such methods here which are named as asymptotically compatible schemes. While it is perhaps impossible to develop a theory that would encompass problems of many different types, our attempt to develop an abstract framework, may offer new insights into the study of other problems involving both a modeling parameter and a discretization parameter.

The consequences of the abstract framework, which will be considered chapter 4-5, is the identification of asymptotically compatible finite element methods for the robust discretization of nonlocal diffusion and peridynamic models.

Chapter 4

Nonlocal diffusion model

This chapter contains studies of finite element discretization of nonlocal diffusion model and its various asymptotic limits. We first consider conforming finite element method for nonlocal diffusion model and its relation with local diffusion model (as $\delta \to 0$) and fractional diffusion model (as $\delta \to \infty$) in sections 4.1 and 4.2. Then a nonconforming DG scheme for nonlocal diffusion model is studied in section 4.3. The framework of asymptotically compatible schemes developed in chapter 3 is the frequently used to look into the fundamental difference of the subtle different choices of finite element spaces.

4.1 Nonlocal diffusion and its relation with classical diffusion

4.1.1 Introduction

In this section, we consider finite element discretization of nonlocal diffusion model and its local limit as $\delta \to 0$. The key message is that as long as the finite element space contains piecewise linear functions, the Galerkin finite element approximation is always asymptotically compatible. In addition, for piecewise constant finite element, whenever applicable, it is shown that a correct local limit solution can also be obtained as long as the discretization (mesh) parameter decreases faster than the modeling (horizon) parameter does. Numerical examples are also shown to compensate for the lack of analysis on the order of convergence.

We adopt notations introduced in chapter 1. Consider the steady-state nonlocal diffusion problem with volumetric constraint,

$$\begin{cases} -\mathcal{L}_\delta u = f & \text{on } \Omega\,, \\ \quad u = 0 & \text{on } \Omega_\delta \end{cases} \tag{4.1}$$

where Ω is an open and bounded set in $\mathbb{R}^d$. $\mathcal{L}_\delta$ and Ω_δ are defined by (1.8) and (1.6). We assume further in this section that the kernel function γ_δ satisfies

$$\gamma_\delta(|\boldsymbol{\xi}|) = \frac{1}{\delta^{d+2}}\hat{\gamma}(\frac{|\boldsymbol{\xi}|}{\delta}), \quad \text{with } \frac{1}{2}\int_{B_1(\mathbf{0})} |\boldsymbol{\xi}|^2 \hat{\gamma}(|\boldsymbol{\xi}|)d\boldsymbol{\xi} = d\,.$$

With $\mathcal{L}_0 = \Delta$, the nonlocal equation (4.1) is a generalization of the classical problem

$$\begin{cases} -\mathcal{L}_0 u = f & \text{on } \Omega\,, \\ \quad u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.2}$$

The natural energy space associated with (4.1) are:

$$\mathcal{S}_\delta = \left\{ u \in L^2(\Omega\cup\Omega_\delta) : \int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x})-u(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y} < \infty,\ u|_{\Omega_\delta} = 0 \right\} \tag{4.3}$$

for $\delta \in (0,1]$. It is clear that $\mathcal{S}_\delta$ is a subspace of $L^2(\Omega_\delta)$ with an inner product $(\cdot,\cdot)_{\mathcal{S}_\delta}$ defined as

$$(u,v)_{\mathcal{S}_\delta} = \int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x})-u(\mathbf{y}))(v(\mathbf{x})-v(\mathbf{y}))d\mathbf{x}d\mathbf{y},$$

and $\|\cdot\|_{\mathcal{S}_\delta}$ the associated norm. Let $\mathcal{S}_0 = H_0^1(\Omega)$ with an inner product and norm

$$(u,v)_{\mathcal{S}_0} = \int_\Omega \nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})d\mathbf{x}, \quad \|u\|_{\mathcal{S}_0} = \big(\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x}\big)^{1/2}.$$

We note that $\|\cdot\|_{\mathcal{S}_\delta}$ are usually semi-norms, but for $\{\mathcal{S}_\delta\}$, they are equivalent to full norms as demonstrated by the Poincaré inequality given later, just as on $H_0^1(\Omega)$, $|u|_{H^1(\Omega)}$ and $\|u\|_{H^1(\Omega)}$ are equivalent. It can be shown that $\mathcal{S}_\delta$ is also the completion of $C_0^\infty(\Omega)$ in $L^2(\Omega\cup\Omega_\delta)$ under the norm $\|\cdot\|_{\mathcal{S}_\delta}$ (see [Mengesha and Du, 2013]).

In order to apply the framework given in chapter 3, it is convenient to have functions in the different spaces $\{\mathcal{S}_\delta, \delta\in[0,1]\}$ be specified in a common spatial domain, say $\Omega_w = \Omega\cup\Omega_1$, we thus make all functions in $\mathcal{S}_\delta$ to be equivalent to themselves with zero extension outside Ω and norms defined by

$$\left\{\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x})-u(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y}\right\}^{1/2},$$

and

$$\left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x})-u(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} \right\}^{1/2}$$

are also equivalent for such functions, independently of δ. These equivalence properties will be implicitly used throughout the chapter unless otherwise noted.

Now we present weak formulations for the nonlocal (and local) diffusion models. Define a family of bilinear forms $\{b_\delta\}$ by:

$$b_\delta(u,v) = \begin{cases} \displaystyle\int_{\Omega\cup\Omega_\delta} \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{y}-\mathbf{x}|)(u(\mathbf{y})-u(\mathbf{x}))(v(\mathbf{y})-v(\mathbf{x})) d\mathbf{y} d\mathbf{x} \quad (\delta > 0) \\ \displaystyle\int_\Omega \nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x}) \quad (\delta = 0)\ . \end{cases} \tag{4.4}$$

for $u, v \in \mathcal{S}_\delta$. Then the weak formulations of (4.1) and (4.2) are

$$\text{find } \ u_\delta \in \mathcal{S}_\delta \text{ such that } b_\delta(u_\delta, v) = (f,v)_{L^2} \quad \forall v \in \mathcal{S}_\delta\ . \tag{4.5}$$

Now for each δ, we introduce the finite element spaces $\{V_{\delta,h}\} \subset \mathcal{S}_\delta$ associated with the triangulation $\tau_h = \{K\}$ of the domain $\Omega \cup \Omega_\delta$ (or Ω_w). We set

$$V_{\delta,h} := \{v \in \mathcal{S}_\delta : v|_K \in P(K) \quad \forall K \in \tau_h\}$$

where $P(K) = \mathcal{P}_p(K)$ is the space of polynomials on $K \in \tau_h$ of degree less or equal than p. Again, for different δ, in order to have the finite element functions defined on a common spatial domain, we also assume, as in the case for $\mathcal{S}_\delta$, that any element in $V_{\delta,h}$ automatically vanishes outside Ω. As $h \to 0$, $\{V_{\delta,h}\}$ is dense in $\mathcal{S}_\delta$, i.e., for any $v \in \mathcal{S}_\delta$, there exists a sequence $(v_h \in V_{\delta,h})$ such that

$$\|v_h - v\|_{\mathcal{S}_\delta} \to 0 \quad \text{as} \quad h \to 0\,. \tag{4.6}$$

These properties are easily satisfied by standard finite element spaces.

The Galerkin approximation is to replace $\mathcal{S}_\delta$ xby $V_{\delta,h}$ in (4.5), namely,

$$\text{find } \ u_{\delta,h} \in V_{\delta,h} \text{ such that } b_\delta(u_{\delta,h}, v) = (f,v)_{L^2} \quad \forall v \in V_{\delta,h}\ . \tag{4.7}$$

4.1.2 Asymptotically compatible schemes

To apply the abstract framework of AC schemes to the nonlocal diffusion model, we define $\mathcal{T}_\sigma$ in the context of chapter 3 as

$$\mathcal{T}_\sigma = \begin{cases} \mathcal{S}_{1/\sigma} & \text{for } \sigma \in [1, \infty], \\ L_0^2(\Omega) & \text{for } \sigma = 0, \\ \mathcal{T}_1 & \text{for } \sigma \in (0, 1)\,. \end{cases} \tag{4.8}$$

where $L_0^2(\Omega)$ contains all elements in $L^2(\Omega)$ which vanish outside Ω. We define $\mathcal{T}_\sigma$ for $\sigma \in (0,1)$ the same as $\mathcal{T}_1$, since this would not affect the limiting behavior as $\sigma \to \infty$, or equivalently, $\delta \to 0$. Indeed, we are interested in approximations of both nonlocal problems with a finite horizon parameter and their local limits.

For the family of spaces, we need to verify the assumptions made in the earlier section. First, let us state a simple lemma.

Lemma 4.1.1. *For $\alpha \in (0,2]$ and a kernel γ_δ satisfying $|\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) \in L^1(\mathbb{R}^d)$, we have a constant C depending only on Ω such that*

$$\|u\|^2_{\mathcal{S}_\delta} \le C\Big(\int |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi}\Big) \|u\|^2_{H^{\alpha/2}(\Omega)} \quad \forall\, u \in H^{\alpha/2}(\Omega) \cap L_0^2(\Omega)\,. \tag{4.9}$$

Proof. We consider the zero extension of functions in $H^{\alpha/2}(\Omega) \cap L_0^2(\Omega)$ to $\mathbb{R}^d$, so that there exists a constant $C = C(\Omega)$, independent of α, such that

$$\|u\|_{H^{\alpha/2}(\mathbb{R}^d)} \le C\|u\|_{H^{\alpha/2}(\Omega)} \quad \forall u \in H^{\alpha/2}(\Omega) \cap L_0^2(\Omega)\,,$$

where we denote the extension of u by the same notation. The lemma is then a consequence of the following:

$$\int_{\mathbb{R}^d} |u(\mathbf{x} + \boldsymbol{\xi}) - u(\mathbf{x})|^2 d\mathbf{x} \le C|\boldsymbol{\xi}|^\alpha \|u\|^2_{H^{\alpha/2}(\mathbb{R}^d)}\,.$$

To see the above, we have by the Fourier transform that

$$|u|^2_{H^{\alpha/2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\mathbf{k}|^\alpha \hat{u}^2(\mathbf{k}) d\mathbf{k}\,, \quad \int_{\mathbb{R}^d} |u(\mathbf{x} + \boldsymbol{\xi}) - u(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |e^{i\mathbf{k}\cdot\boldsymbol{\xi}} - 1|^2 \hat{u}^2(\mathbf{k}) d\mathbf{k}\,.$$

So the desired inequality follows from an elementary inequality $|e^{i\mathbf{k}\cdot\boldsymbol{\xi}} - 1|^2 \le 2|\boldsymbol{\xi}\cdot\mathbf{k}|^\alpha$ for $\alpha \in (0,2]$. Indeed, we have $|e^{ir} - 1| \le 2 \le 2|r|^{\alpha/2}$ for $|r| \ge 1$, while for $|r| \le 1$, we have

$|e^{ir}-1| \leq |r|$ and $|r|^{\alpha/2}$ is decreasing in α for $\alpha \in (0,2]$. Hence, we get

$$\int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{y}-\mathbf{x}|)(u(\mathbf{y})-u(\mathbf{x}))^2 d\mathbf{y}d\mathbf{x} \leq \int_{\mathbb{R}^d} \gamma_\delta(|\boldsymbol{\xi}|) \int_{\mathbb{R}^d} (u(\mathbf{x}+\boldsymbol{\xi})-u(\mathbf{x}))^2 d\mathbf{x}d\boldsymbol{\xi}$$
$$\leq C\|u\|_{H^{\alpha/2}(\Omega)} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|)d\boldsymbol{\xi}$$

which leads to the lemma. □

By applying the above to functions in $\mathcal{S}_0 = \mathcal{T}_\infty$ for the case of $\alpha = 2$, we have the uniform embedding of $\mathcal{T}_\infty$ in $\mathcal{T}_\sigma$ since for the kernel γ_δ has bounded second moment.

To verify Assumption 3.1.1 for $\{\mathcal{T}_\sigma\}$, it remains to apply a uniform Poincaré-type inequality for the uniform embedding of $\mathcal{T}_\sigma$ in $\mathcal{T}_0$.

Lemma 4.1.2 (Uniform Poincaré inequality)**.** *There exists $C > 0$ independent of δ such that for all $\delta \in (0,1]$,*

$$\|u\|^2_{L^2(\Omega_\delta)} \leq C\|u\|^2_{\mathcal{S}_\delta}\,, \quad \forall u \in \mathcal{S}_\delta\,. \tag{4.10}$$

The above is a special case of [Mengesha and Du, 2014a, Proposition 5.3] for scalar valued functions (see [Mengesha and Du, 2014a] for the proof). Also from [Mengesha and Du, 2013], we know that $\mathcal{S}_\delta$ is complete thus a Hilbert space.

To check the Assumption 3.1.1 ii), we need a compactness lemma which can be found in [Bourgain *et al.*, 2001, Theorem 4] and [Ponce, 2004, Theorem 1.2, 1.3].

Lemma 4.1.3. *Suppose $u_n \in \mathcal{S}_{\delta_n}$ with $\delta_n \to 0$. If*

$$\sup_n \int_\Omega \int_{B_{\delta_n}(\mathbf{x})} \gamma_{\delta_n}(|\mathbf{x}-\mathbf{y}|)(u_n(\mathbf{x})-u_n(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y} \leq \infty\,,$$

then u_n is precompact in $L^2_0(\Omega)$. Moreover, any limit point $u \in \mathcal{S}_0$.

We note that, in establishing the above lemmas 4.1.2 and 4.1.3, an argument of [Bourgain *et al.*, 2001] can be often used which requires that $r^{d+1}\gamma_\delta(r)$ is non-increasing. It has been noted that by techniques introduced in [Andreu *et al.*, 2008], the results remain true under a less restrictive condition where $\gamma_\delta(r)$ is assumed to be non-increasing. Moreover, [Ponce, 2004] proves an even more general argument that works for $d \geq 2$ without the assumption on γ_δ being non-increasing. Related discussions on these issues can be found in [Mengesha and Du, 2013; Mengesha and Du, 2014a].

We next move to the bilinear forms. Note that b_δ is exactly the inner product defined on $\mathcal{S}_\delta$, so Assumption 3.1.2 is naturally satisfied with $C_1 = C_2 = 1$.

Assumption 3.1.3 is about the convergence of the operator $\mathcal{L}_\delta$ which has been shown in many works such as [Du *et al.*, 2013a; Mengesha and Du, 2014a]. We state here as a proposition without proof. It is a pointwise convergence result of the nonlocal integral operator $\mathcal{L}_\delta$ (interpreted in the principal value sense in general [Mengesha and Du, 2014a]) to the Laplacian.

Proposition 4.1.4. *For all $v \in C_c^\infty(\Omega)$, and all $\mathbf{x} \in \Omega$, we have*

$$\mathcal{L}_\delta v(\mathbf{x}) \longrightarrow \Delta v(\mathbf{x}) \quad \text{as } \delta \to 0. \tag{4.11}$$

Moreover, there exists a constant $C = C(d, v)$ such that

$$\sup_{\delta \in (0,1)} \sup_{\mathbf{x} \in \Omega} |\mathcal{L}_\delta v(\mathbf{x})| \le C\,. \tag{4.12}$$

With pointwise convergence and uniform boundedness estimate of $\mathcal{L}_\delta v$, Assumption 3.1.3 is obviously true by the bounded convergence theorem. This is stated in the following lemma, which is a stronger result than what Assumption 3.1.3 ii) requires.

Lemma 4.1.5. $\forall v \in C_c^\infty(\Omega)$,

$$\|\mathcal{L}_\delta v - \mathcal{L}_0 v\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } \delta \to 0\,.$$

As for Assumption 3.1.4, (4.6) ensures that $V_{\delta,h}$ satisfies i). To check ii), for convenience, we define a special family of spaces $\hat{V}_{\delta,h}$.

Definition 4.1.6. *Let $\hat{V}_{\delta,h} \subset V_{0,h} \subset \mathcal{S}_0$ be the continuous piecewise linear finite element space that corresponds to the same mesh τ_h with $V_{\delta,h}$.*

The following lemma is simply a re-statement of a simple fact that continuous piecewise linear subspace of H_0^1 approximates the whole space as mesh size goes to zero.

Lemma 4.1.7. *The family $\{\hat{V}_{\delta,h}\}$ is asymptotically dense in $\mathcal{S}_0$, that is, it satisfies Assumption 3.1.4 ii).*

Now we see that if $\hat{V}_{\delta,h} \subset V_{\delta,h}$, then $V_{\delta,h}$ also satisfies Assumption 3.1.4 ii).

With all assumptions 3.1.1-3.1.4 verified, the following theorem offers a remedy for developing asymptotically compatible schemes when one wants to solve nonlocal diffusion equations.

Theorem 4.1.8. *Let u_δ and $u_{\delta,h}$ be solutions of* (4.5) *and* (4.7) *respectively, and $\hat{V}_{\delta,h}$ is defined in Definition 4.1.6. If $\hat{V}_{\delta,h} \subset V_{\delta,h}$, then $\|u_{\delta,h} - u_0\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$, $h \to 0$.*

Proof. Taking $\mathcal{T}_\sigma := \mathcal{S}_{1/\sigma}$, $a_\sigma := b_{1/\sigma}$, $\mathcal{A}_\sigma := \mathcal{L}_{1/\sigma}$, and $W_{\sigma,h} := V_{1/\sigma,h}$, we see that the above theorem follows from Theorem 3.3.2, since in the above discussions we have verified all the assumptions 3.1.1-3.1.4 for this case. □

In short, we see that if the finite element spaces contain a continuous finite element subspace which have desired approximation properties in $\mathcal{S}_0$, then the corresponding discretization is asymptotically compatible. This is particularly true for any continuous or discontinuous finite element spaces containing at least the subspace of continuous piecewise linear elements.

We now examine the local limit of discrete solutions on a fixed mesh, following the discussions in theorem 3.2.3. To verify the additional assumptions required, we state a couple of technical results.

For a given triangulation τ_h, we define the space

$$\mathcal{V}_h := \{v \in C(\overline{\Omega_\delta}) : \ v|_K \in C^\infty(\bar{K}),\, K \in \tau_h,\ v|_{\Omega_{\mathcal{I}_\delta}} = 0\}\,.$$

Again, functions in $\mathcal{V}_h$ are set to vanish outside Ω. Then, we have the convergence of the bilinear forms on the subspace $\mathcal{V}_h$.

Lemma 4.1.9. *For any $u, v \in \mathcal{V}_h$, as $\delta \to 0$, we have*

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} \to 0\,.$$

Consequently, for any $u_h, v_h \in V_{0,h}$,

$$\lim_{\delta\to 0} b_\delta(u_h, v_h) = b_0(u_h, v_h)\,.$$

Proof. First, we note that

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} = \sum_{K\in\tau_h} \int_K u\mathcal{L}_\delta v - \sum_{K\in\tau_h} \int_K \nabla u \cdot \nabla v\,.$$

Now, for any mesh element $K \in \tau_h$, integration by parts on each K gives

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} = \sum_{K\in\tau_h} \int_K u(\mathcal{L}_\delta v + \Delta v) - \sum_{e\in\mathcal{E}_h^0} \int_e u[\![\nabla v]\!]_e$$

where $\mathcal{E}_h^0$ is the set of internal edges of τ_h and $[\![\nabla v]\!]_e$ is the the jump of the the vector on the edge e. For the first term, using [Du *et al.*, 2013b, Theorem 3.7] which remains valid for the kernel under consideration here, we have

$$\int_K u(\Delta v + \mathcal{L}_\delta v) \to \frac{1}{2} \sum_{e\in edge(K)} \int_e u[\![\nabla v]\!]_e \quad \text{as } \delta \to 0\,.$$

Summing over $K \in \tau_h$, we get

$$\sum_{K\in\tau_h} \int_K u(\Delta v + \mathcal{L}_\delta v) \to \sum_{e\in\mathcal{E}_h^0} \int_e u[\![\nabla v]\!]_e\,.$$

Thus we have $(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} \to (\nabla u, \nabla v)_{L^2(\Omega)}$ and the lemma follows. □

We now consider a simple case when $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$ which means that all functions in $V_{\delta,h}$ are in $H^1(\Omega)$, continuous over Ω and vanishes outside Ω. In this case, we state an inverse inequality.

Lemma 4.1.10. *For $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, there exists a constant $C > 0$, independent of δ such that for any $u_h \in V_{\delta,h}$,*

$$\|u_h\|_{\mathcal{S}_\delta} \le C\|u_h\|_{L^2(\Omega)}\,.$$

Proof. Since $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, we first invoke the standard inverse inequality for finite element functions in H^1 to get

$$\|u_h\|_{\mathcal{S}_0} \le C\|u_h\|_{L^2(\Omega)}\,, \quad \forall u_h \in V_{\delta,h}$$

for some generic constant $C > 0$ that only depends on the triangulation and the finite element basis. The lemma then follows from the uniform embedding of $\mathcal{S}_\delta$ in $\mathcal{S}_0$. □

Theorem 4.1.11. *Suppose $u_{\delta,h}$ and $u_{0,h}$ are discrete solutions as defined in (4.7) with $\delta > 0$ and $\delta = 0$ respectively. If $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, then for each fixed h and τ_h,*

$$\|u_{\delta,h} - u_{0,h}\|_{\mathcal{S}_0} \to 0 \text{ as } \delta \to 0 \,.$$

Proof. We first check the additional conditions assumed in the Theorem 3.2.3. Obviously the condition (3.11) holds since $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$. As for the condition (3.12), that is the convergence of the approximate bilinear forms on the finite element spaces, we get it from lemma 4.1.9, Next, combining the continuity of the bilinear form $b_\delta(\cdot,\cdot)$ and the uniform bound of $\|\cdot\|_{\mathcal{S}_\delta}$ by $\|\cdot\|_{\mathcal{S}_0}$ with the uniform inverse inequality given in the lemma 4.1.10, we get

$$|b_\delta(w_{\delta,h}, v_h)| \le \|w_{\delta,h}\|_{\mathcal{S}_\delta} \|v_h\|_{\mathcal{S}_\delta} \le C \|w_{\delta,h}\|_{L^2(\Omega)} \|v_h\|_{\mathcal{S}_0}$$

for some constant $C > 0$, independent of δ. The condition (3.13), that is the strengthened continuity of the approximate bilinear form, thus follows. Theorem 3.2.3 then implies that $\|u_{\delta,h} - u_{0,h}\|_{L^2(\Omega)} \to 0$ and the result of the above theorem follows again from the standard inverse inequality. □

The above theorem shows that if the finite element spaces are taken to be the same conforming finite elements for nonlocal problems and their local limit, then the discrete nonlocal solutions also converges to the local discrete solution in the same space as $\delta \to 0$. The following results give an extension: it only requires that all continuous piecewise linear functions form a subspace of the finite element space.

Theorem 4.1.12. *For fixed h and τ_h, let $u_{\delta,h}$ and $u_{0,h}$ be discrete solutions as defined in (4.7) with $\delta > 0$ and $\delta = 0$ respectively. Assume further that $V_{\delta,h} \subset \mathcal{S}_\delta$ is a finite element space that contains all continuous piecewise linear functions. Moreover, $V_{0,h} = \mathcal{S}_0 \cap (\bigcap_{\delta>0} V_{\delta,h})$. Then, for fixed h and τ_h, we have $\|u_{\delta,h} - u_{0,h}\|_{L^2} \to 0$ as $\delta \to 0$.*

Proof. We only need to show that $\forall v_h \in V_{0,h}$,

$$b_0(u_{*,h}, v_h) = (f, v_h)$$

where $u_{*,h}$ is a limit point of $u_{\delta,h}$, i.e, $\|u_{\delta,h} - u_{*,h}\|_{L^2} \to 0$. Consider

$$\begin{aligned}(f, v_h) - b_0(u_{*,h}, v_h) &= b_\delta(u_{\delta,h}, v_h) - b_0(u_{*,h}, v_h) \\ &= b_\delta(u_{\delta,h} - u_{*,h}, v_h) + \big[b_\delta(u_{*,h}, v_h) - b_0(u_{*,h}, v_h)\big] = \mathrm{I}_1 + \mathrm{I}_2 \,.\end{aligned}$$

First, $\mathrm{I}_2 \to 0$ comes from Lemma 4.1.9. As for I_1, let $w_{\delta,h} := u_{\delta,h} - u_{*,h}$, we now prove that $\mathrm{I}_1 = b_\delta(w_{\delta,h}, v_h) \to 0$ (notice that $w_{\delta,h} \notin \mathcal{S}_0$, the technique used in the proof of Theorem 4.1.11 does not apply).

Since $w_{\delta,h}$ and v_h are smooth on each element $K \subset \tau_h$, we will prove the result on each $K \subset \tau_h$. Also, we define Γ_K for each $K \subset \tau_h$ by

$$\Gamma_K := \{\mathbf{x} \notin K \mid \mathrm{dist}(\mathbf{x}, K) \leq \delta\}\,.$$

Then

$$b_\delta(w_{\delta,h}, v_h) = \sum_{K \in \tau_h} \int_K \int_{K \cup \Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x}\,.$$

By [Bourgain *et al.*, 2001, Theorem 1], for smooth $w_{\delta,h}$ and v_h restricted on K,

$$\begin{aligned}\int_K \int_K \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \\ \leq \|w_{\delta,h}\|_{\mathcal{S}_\delta(K)} \|v_h\|_{\mathcal{S}_\delta(K)} \leq C\|w_{\delta,h}\|_{H^1(K)}\|v_h\|_{H^1(K)}\,.\end{aligned}$$

Now by the norm equivalence of finite dimensional spaces,

$$\|w_{\delta,h}\|_{H^1(K)} \leq C\|w_{\delta,h}\|_{L^2(K)} \to 0 \quad \text{as } \delta \to 0\,,$$

so

$$\int_K \int_K \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \to 0;.$$

For the second term,

$$\begin{aligned}&\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \\ &\leq 2\|w_{\delta,h}\|_{L^\infty} \int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})|v_h(\mathbf{x}') - v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x}\,.\end{aligned}$$

Now by the norm equivalence of finite dimensional spaces,

$$\|w_{\delta,h}\|_{L^\infty} \leq C\|w_{\delta,h}\|_{L^2} \to 0\,,$$

it remains to prove that for any $v_h \in V_{0,h}$,

$$\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})|v_h(\mathbf{x}') - v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x}$$

is bounded uniformly in δ.

Since v_h is piecewise smooth for $\mathbf{x} \in K$ and $\mathbf{x}' \in \Gamma_K$ respectively, we use $\mathbf{s}$ to denote the intersection of ∂K and the line between $\mathbf{x}'$ and $\mathbf{x}$. By Taylor expansion, we have

$$v_h(\mathbf{x}') = v_h(\mathbf{x}) + \nabla v_h(\mathbf{x}) \cdot (\mathbf{s} - \mathbf{x}) + \nabla v_{\Gamma_K}(\mathbf{s}) \cdot (\mathbf{x}' - \mathbf{s}) + o(\delta)\,.$$

Denote $K_{out} := K \cap B_\delta(\partial K)$ (the latter being a δ neighborhood of ∂K, then

$$\begin{aligned}
&\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x}) |v_h(\mathbf{x}') - v_h(\mathbf{x})| d\mathbf{x}' d\mathbf{x} \\
&= \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x}) |\nabla v_h(\mathbf{x}) \cdot (\mathbf{s} - \mathbf{x}) + \nabla v_{\Gamma_K}(\mathbf{s}) \cdot (\mathbf{x}' - \mathbf{s})| d\mathbf{x}' d\mathbf{x} \\
&\quad + \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta \cdot o(\delta) d\mathbf{x}' d\mathbf{x} \\
&\le 2\|\nabla v\|_{L^\infty} \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x}) |\mathbf{x}' - \mathbf{x}| d\mathbf{x}' d\mathbf{x} + \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta \cdot o(\delta) d\mathbf{x}' d\mathbf{x}\,.
\end{aligned}$$

Now it is easy to see that the second term on the above right hand side tends to zero as $\delta \to 0$. For the first term, following the proof of [Du *et al.*, 2013b, Theorem 3.7, in particular equation (3.35)], we have

$$2 \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x}) |\mathbf{x}' - \mathbf{x}| d\mathbf{x}' d\mathbf{x} \le \int_{B_\delta(0)} \gamma_\delta(\mathbf{z}) |\mathbf{z}|^2 d\mathbf{z} (\sum_{e \in K} |e|)$$

which is bounded uniformly in δ under the assumption on the kernel γ_δ.

In summary, we now have proved that for each $K \in \tau_h$,

$$\int_K \int_{K \cup \Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x})) d\mathbf{x}' d\mathbf{x} \to 0\,,$$

which leads to $\mathrm{I}_1 \to 0$ and thus completes the proof. □

We note that the above theorem implies that as long as all piecewise continuous linear elements are included, the finite element spaces for nonlocal problems may not be conforming subspaces of the local limit problem, but can still solutions that converge to the conforming local finite element solution.

4.1.3 A case of conditional asymptotic stability

In chapter 2, it is shown that for piecewise constant finite element approximations to the nonlocal diffusion models are not AC, in particular, if δ is taken to be proportional to h, then as $h \to 0$, the discrete solutions may converge to the wrong limit. It is interesting from a practical point of view to provide some constructive remedies to avoid such undesirable effects. This is the purpose of the discussion here. We show that as long as the condition $h = o(\delta)$ is met as $\delta \to 0$, then we are also able to obtain the correct local limit even for discontinuous piecewise constant finite element approximations when they are of conforming type.

Theorem 4.1.13. *Let u_δ, $u_{\delta,h}$ be solutions of* (4.5) *and* (4.7)*. If $V_{\delta,h}$ is the piecewise constant space, then $\|u_{\delta,h} - u_0\|_{L^2} \to 0$ if $h = o(\delta)$ as $\delta \to 0$.*

Proof. We revisit the proof of Theorem 3.3.2. Recall that $a_\infty(u, v) - (f, v)$ is split into four parts. Without Assumption 3.1.4 ii), three of the four terms are not affected. We need to prove that III $\to 0$ if $\sigma_n \cdot h_n \to 0$ as $n \to \infty$. In fact, by lemma 4.1.1,

$$\text{III} \le C\|v - v_n\|_{\mathcal{S}_{\sigma_n}} \le C\|v - v_n\|_{H^{\alpha/2}(\Omega)} \left(\int |\boldsymbol{\xi}|^\alpha \gamma_{\delta_n}(|\boldsymbol{\xi}|) d\boldsymbol{\xi} \right)^{1/2}$$

where $v_n \in V_{\delta_n,h_n} = W_{\sigma_n,h_n}$. A direct calculation shows

$$\int |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = \delta^{\alpha-2} \int_{B(0,1)} |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = C\delta^{\alpha-2} .$$

So, III $\le C\sigma_n^{1-\alpha/2}\|v - v_n\|_{H^{\alpha/2}(\Omega)}$ for $\alpha \in [0, 1]$. Now, by taking v_n as the piecewise constant L^2-orthogonal projection of $v \in \mathcal{S}_0$ onto V_{δ_n,h_n}, we have [Belgacem and Brenner, 2001, (1.3)]

$$\|v - v_n\|_{H^{\alpha/2}(\Omega)} \le C h_n^{1-\alpha/2}\|v\|_{H^1(\Omega)} .$$

Thus, III $\le C(\sigma_n \cdot h_n)^{1-\alpha/2}\|v\|_{H^1(\Omega)} \to 0$ as $n \to \infty$ which completes the proof. □

4.1.4 Numerical experiments

Here, we report numerical results which validate our analysis and provide results on the order of convergence that cannot be seen from our convergence theorems. We use discontinuous piecewise linear finite element to solve a 1d nonlocal problem $-\mathcal{L}_\delta u = f$ on $(0, 1)$ with the nonlocal constraint $u = 0$ outside $(0, 1)$ and the nonlocal operator given by

$$\mathcal{L}_\delta u = 2 \int_{-\delta}^{\delta} \gamma_\delta(s)(u(x + s) - u(x))ds.$$

A special kernel is chosen to be $\gamma_\delta(s) = \delta^{-2}s^{-1}$ in our numerical examples.

We choose a relatively smooth function as u_0 given by a fourth-order B-spline:

$$u_0(x) = \frac{1}{h^5}\begin{cases} 0 & x < 0 \\ \dfrac{x^4}{120}, & 0 \le x < 0.2 \\ -\dfrac{x^4}{30} + \dfrac{x^3}{30} - \dfrac{x^2}{100} + \dfrac{x}{750} - \dfrac{1}{15000}, & 0.2 \le x < 0.4 \\ \dfrac{x^4}{20} - \dfrac{x^3}{10} + \dfrac{7x^2}{100} - \dfrac{x}{50} + \dfrac{31}{15000}, & 0.4 \le x < 0.5 \end{cases}$$

with u_0 symmetric (even) with respect to $x = 0.5$. Its graph is shown in the Figure 4.1.

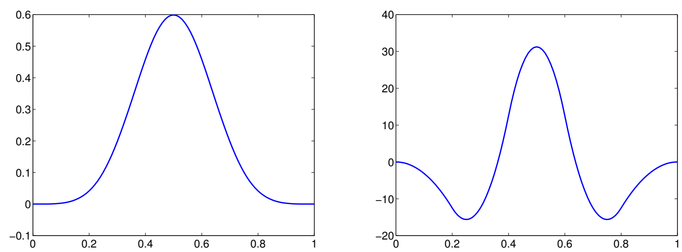

Figure 4.1: Graph of $u_0(x)$ and its second order derivative.

We then calculate analytically $f := -u_0''$ and solve nonlocal problems on a uniform mesh using a discontinuous piecewise linear finite element space. The corresponding point wise errors $e(x) = u_{\delta,h}(x) - u_0(x)$ are plotted in Figures 4.2-4.3-4.4 for the three cases respectively. Note that the red dots are highlighted to show errors at nodal points. Qualitatively, one may observe some common features in these plots: first, while the errors are generally discontinuous at the nodal points given the use of discontinuous finite element functions, the magnitude of discontinuity diminishes as $\delta \to 0$, leading to a continuous (and conforming) approximation to the local limit solution as predicted by the theory; secondly, the error profiles, in particular, the maximum and minimum envelopes of the errors, are all nicely correlated with the second derivatives of u_0 showing in the Figure 4.1. While this does not follow from our analytical framework here, this is consistent with the errors of typical piecewise linear interpolations and may not also tie this with the more detailed truncation error analysis given in chapter 2. Meanwhile, the error plots also show different oscillation patterns of the errors inside the mesh intervals in comparison with those at nodal points for the three cases. A possible explanation is that oscillations are related to discretization errors which become more pronounced with smaller δ due to the reduction of modeling errors (between nonlocal and local equations).

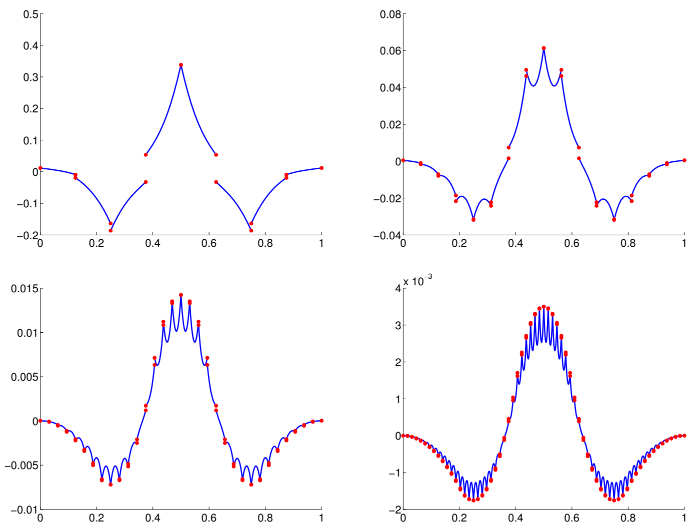

Figure 4.2: Pointwise error $u_{\delta,h}(x) - u_0(x)$ with $r = \frac{\delta}{h} = 3$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.

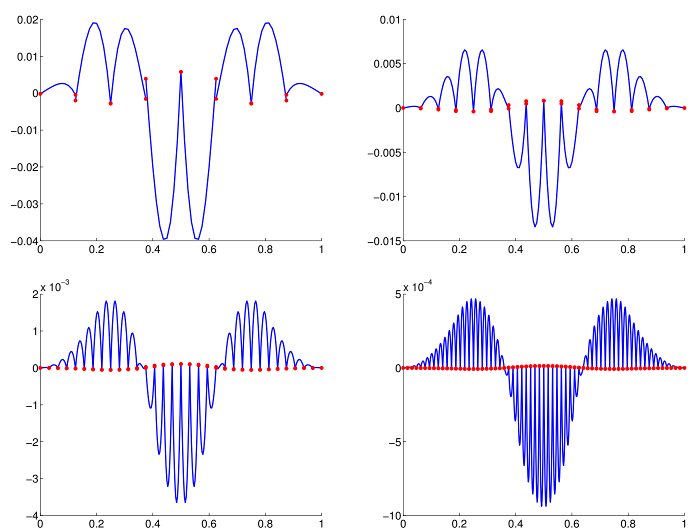

Figure 4.3: Pointwise error $u_{\delta,h}(x) - u_0(x)$ with $\delta = h^2$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.

To be more quantitate, the L^2 error of the function values $\|e\|_0$ and the piecewise first order derivatives $\|e\|_1$ are computed along with the mesh weighted discrete ℓ^2 errors of functions values and the first order derivatives at mid-points of mesh intervals (denoted by $\|\bar{e}\|_0$ and $\|\bar{e}\|_1$ respectively). Tables 4.1-4.3 provide errors and convergence orders (given

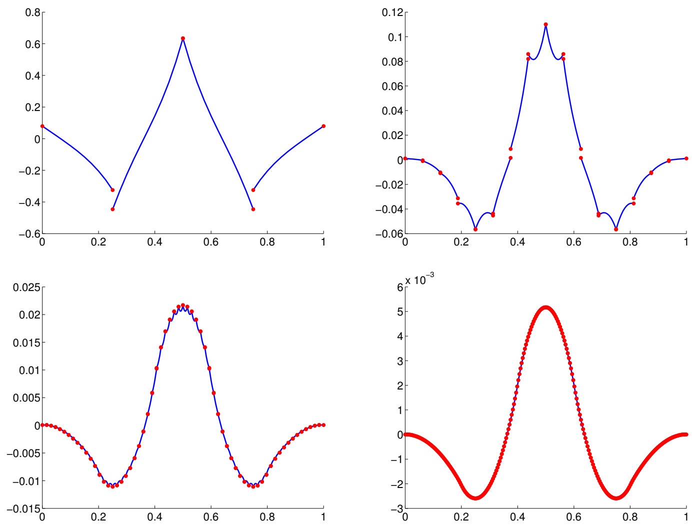

Figure 4.4: Pointwise error $u_{\delta,h}(x) - u_0(x)$ with $\delta = \sqrt{h}$ and $h = 2^{-k}$, $k = 3, 4, 6, 8$.

inside parenthesis) in different norms and with different relations between δ and h.

Table 4.1 shows the errors and convergence rates when δ/h is fixed as a constant as the mesh is refined with a decreasing h. The errors are measured against the exact solution u_0, the local limit. The L^2 convergence rate for function values is of second order and that for piecewise first order derivatives is of first order. Given that we expect the modeling error (that is, the difference between solutions of the nonlocal and local models) is of the order δ^2 (see chapter 2), we see that the orders of the numerical approximation errors are consistent with the optimal orders predicted by standard approximation theory.

Table 4.1: Errors and convergence rates for fixed $r := \delta/h = 3$.

h	$\|e\|_0$	$\|e\|_1$	$\|\bar{e}\|_0$	$\|\bar{e}\|_1$
2^{-3}	$1.15 \times 10^{-1}(-)$	$1.50 \times 10^{0}(-)$	$1.50 \times 10^{-1}(-)$	$1.32 \times 10^{0}(-)$
2^{-4}	$2.25 \times 10^{-2}(2.4)$	$3.79 \times 10^{-1}(2.0)$	$2.79 \times 10^{-2}(2.4)$	$2.58 \times 10^{-1}(2.3)$
2^{-5}	$5.26 \times 10^{-3}(2.1)$	$1.42 \times 10^{-1}(1.4)$	$6.43 \times 10^{-3}(2.1)$	$5.81 \times 10^{-2}(2.2)$
2^{-6}	$1.29 \times 10^{-3}(2.0)$	$6.54 \times 10^{-2}(1.1)$	$1.58 \times 10^{-3}(2.0)$	$1.42 \times 10^{-2}(2.0)$
2^{-7}	$3.22 \times 10^{-4}(2.0)$	$3.20 \times 10^{-2}(1.0)$	$3.92 \times 10^{-4}(2.0)$	$3.51 \times 10^{-3}(2.0)$

For Table 4.2, we let $\delta = h^2$ when refining the mesh. This is the case where δ decreases faster than h while they both go to zero. We find that the L^2 convergence orders stay the same as the above case, though the errors are smaller than the other cases, for a given mesh of the same size.

Table 4.2: Errors and convergence rates for $\delta = h^2$.

h	$\|e\|_0$	$\|e\|_1$	$\|\bar{e}\|_0$	$\|\bar{e}\|_1$
2^{-3}	$1.67 \times 10^{-2}(-)$	$4.94 \times 10^{-1}(-)$	$2.79 \times 10^{-3}(-)$	$8.19 \times 10^{-2}(-)$
2^{-4}	$4.62 \times 10^{-3}(1.9)$	$2.52 \times 10^{-1}(1.0)$	$3.79 \times 10^{-4}(3.0)$	$2.53 \times 10^{-2}(1.8)$
2^{-5}	$1.21 \times 10^{-3}(1.9)$	$1.27 \times 10^{-1}(1.0)$	$4.76 \times 10^{-5}(3.0)$	$6.40 \times 10^{-3}(2.0)$
2^{-6}	$3.08 \times 10^{-4}(2.0)$	$6.34 \times 10^{-2}(1.0)$	$5.96 \times 10^{-6}(3.0)$	$1.63 \times 10^{-3}(2.0)$

On the other hand, in the results for $\delta = \sqrt{h}$ listed in Table 4.3, since δ decreases slower than h, while the correct local limit is obtained as predicted by our theory, the L^2 convergence order for function values drops to 1st order. A possible explanation is that the modeling error dominates and it is of the order $O(\delta^2) = O(h)$.

Data in these tables on the discrete ℓ^2 norms show similar patterns in convergence order as the continuous error norms in Tables 4.1 and 4.3, but some superconvergence order can be observed in Table 4.2 for discrete norms. For analysis of super convergence properties for nonlocal equations, we refer to some related findings in chapter 2. In addition, we note that with the same mesh spacing, say $h = 2^{-6}$, the errors decrease as δ changes from $O(\sqrt{h})$ to $O(h)$ and $O(h^2)$, a reasonable and desirable behavior showing the efficiency of localization (small horizon) if the objective is to capture the local limit when the latter is well defined.

Table 4.3: Errors and convergence rates for $\delta = \sqrt{h}$.

h	$\|e\|_0$	$\|e\|_1$	$\|\bar{e}\|_0$	$\|\bar{e}\|_1$
2^{-2}	$2.35 \times 10^{-1}(-)$	$3.32 \times 10^{0}(-)$	$4.22 \times 10^{-1}(-)$	$2.74 \times 10^{0}(-)$
2^{-4}	$4.31 \times 10^{-2}(1.2)$	$5.94 \times 10^{-1}(1.2)$	$4.94 \times 10^{-2}(1.5)$	$5.12 \times 10^{-1}(1.2)$
2^{-6}	$9.54 \times 10^{-3}(1.1)$	$1.29 \times 10^{-1}(1.1)$	$9.84 \times 10^{-3}(1.2)$	$1.11 \times 10^{-1}(1.1)$
2^{-8}	$2.31 \times 10^{-3}(1.0)$	$3.11 \times 10^{-2}(1.0)$	$2.31 \times 10^{-3}(1.0)$	$2.66 \times 10^{-2}(1.0)$

4.1.5 Conclusion

In this section, we considered Dirichlet type nonlocal constrained value problems associated with a scalar nonlocal diffusion equation and showed using the framework developed in chapter 3 that any finite element discretization that contains piecewise linear functions provides an asymptotically compatible scheme and thus is a robust discretization to both the nonlocal problems and the local limit. The convergence of approximations to the correct solutions and models is assured independent of the relations between the horizon parameter δ and the discretization parameter h as shown in the diagram 3.1. Among various studies of numerical methods and their asymptotic behavior with a parameter approaching to a limit (ranging from uniformly convergent schemes for singularly perturbed problems [Roos *et al.*, 1996], numerical viscosity solutions of conservation laws [Chen *et al.*, 1993] to asymptotically preserving schemes for kinetic equations [Jin, 1999]), perhaps the analysis in [Guermond and Kanschat, 2010] offers the closest resemblance to the work here in spirit. In [Guermond and Kanschat, 2010], the approximations to the zero mean free path $\epsilon \to 0$ limit or diffusive limit of radiative transport models have been studied. The model studied in the paper share similar features as the nonlocal models considered here in that the parametrized problems may have singular solutions but they approach to a more regular solution of the diffusive limit.

We note also that chapter 2 has examples that exposed possible risks in using piecewise constant finite element for nonlocal problem when the horizon is proportional to the mesh size, section 4.1.3 provided new remedy to deal with the issue by showing that piecewise constant finite element for the nonlocal diffusion problem, when conforming, would be a conditionally asymptotically compatible discretization, under the natural condition that $h = o(\delta)$ which has been pointed out in some of the simulations works [Bobaru *et al.*, 2009; Chen and Gunzburger, 2011].

In addition, to compensate for the lack of analysis on the order of convergence, we carried out numerical experiments of a 1d nonlocal diffusion equation discretized with conforming discontinuous piecewise linear finite elements. The discontinuous linear finite element solutions of the nonlocal problem converge to the solution of the correct local differential problem as predicted no matter how δ varies with h, but the convergence rates show dependence on the choices of δ and h. The convergence and superconvergence orders observed lead to interesting theoretical issues to be studied further along with the development of possible post-processing techniques [Cockburn *et al.*, 2003] to improve the order of convergence especially for derivatives and stress variables when singular behaviors are likely to be present in practice.

4.2 Nonlocal diffusion and its relation with fractional diffusion

4.2.1 Introduction

We consider the fractional Laplacian operator as a special case of the linear nonlocal diffusion operator. A class of conforming Galerkin finite approximations of nonlocal diffusion equation is developed and proved to be asymptotically compatible schemes through the framework established in chapter 3.

The fractional diffusion problem, as modeled using the fractional Laplacian operator, is defined as

$$\begin{cases} (-\Delta)^\alpha u = f & \text{on } \Omega \\ u = 0 & \text{on } \mathbb{R}^d \backslash \Omega \end{cases} \tag{4.13}$$

Here, the fractional Laplacian $(-\Delta)^\alpha$ with $0 < \alpha < 1$ is the pseudo-differential operator with symbol $|\boldsymbol{\xi}|^{2\alpha}$, that is [Stein, 1970],

$$\mathcal{F}[(-\Delta)^\alpha u](\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2\alpha}\mathcal{F}[u](\boldsymbol{\xi})\,. \tag{4.14}$$

It can be shown that an equivalent definition the operator is by given by

$$(-\Delta)^\alpha u(\mathbf{x}) = C_{d,\alpha}\int_{\mathbb{R}^d}\frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}d\mathbf{y}\,, \tag{4.15}$$

where $C_{d,\alpha}$ is a constant related to the dimension d and the fractional order α.

The problem (4.13) involves global interactions in $\mathbb{R}^d$. Computationally, it is convenient to restrict the interaction to a smaller neighborhood. Thus we see the fractional Laplacian as the limit of the nonlocal diffusion operator with horizon going to infinity. Now to avoid confusions with the zero limit of the horizon δ that we considered earlier, we use λ in replace of δ where it reminds us of the concern with limit when λ goes to infinity in the rest of the section. We are then led to a finite domain approximate problem given by

$$\begin{cases} -\mathcal{L}_\lambda u(\mathbf{x}) = -\int_{\mathbb{R}^d} (u(\mathbf{y}) - u(\mathbf{x}))\gamma_\lambda(|\mathbf{y}-\mathbf{x}|)d\mathbf{y} = f \quad \text{on } \Omega \\ \qquad u = 0 \quad \text{on } \Omega_\lambda\,, \end{cases} \tag{4.16}$$

where

$$\gamma_\lambda(|\mathbf{y}-\mathbf{x}|) = \begin{cases} \dfrac{C_{d,\alpha}}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}} & \mathbf{y} \in B_\lambda(\mathbf{x}) \\ 0 & \mathbf{y} \in \mathbb{R}^d \backslash B_\lambda(\mathbf{x})\,. \end{cases} \tag{4.17}$$

The natural energy space associated with (4.16) are once again defined as (4.3). Here, we use $\mathcal{T}_\lambda$ in replace of $\mathcal{S}_\delta$ to denote the nonlocal energy space, namely

$$\mathcal{T}_\lambda = \left\{ u \in L^2(\Omega \cup \Omega_\lambda) : \int_{\Omega\cup\Omega_\lambda} \int_{\Omega\cup\Omega_\lambda} \gamma_\lambda(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y} < \infty,\ u|_{\Omega_\lambda} = 0 \right\}.$$

It is clear that $\mathcal{T}_\lambda$ is a subspace of $L^2(\Omega \cup \Omega_\lambda)$ with an inner product $(\cdot,\cdot)_{\mathcal{T}_\lambda}$ defined as

$$(u,v)_{\mathcal{T}_\lambda} = \int_{\Omega\cup\Omega_\lambda} \int_{\Omega\cup\Omega_\lambda} \gamma_\lambda(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))d\mathbf{x}d\mathbf{y},$$

and $\|\cdot\|_{\mathcal{T}_\lambda}$ the associated norm. Let $\mathcal{T}_\infty = H^\alpha_\Omega(\mathbb{R}^d) := \{u \in H^\alpha(\mathbb{R}^d) : u|_{\mathbb{R}\backslash\Omega} = 0\}$ with inner product and norm

$$(u,v)_{\mathcal{T}_\infty} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{C_{d,\alpha}}{2|\mathbf{x}-\mathbf{y}|^{d+2\alpha}} (u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))d\mathbf{x}d\mathbf{y}$$
$$\|u\|_{\mathcal{T}_\infty} = \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{C_{d,\alpha}}{2|\mathbf{x}-\mathbf{y}|^{n+2\alpha}} (u(\mathbf{x}) - u(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y}\Big)^{1/2}.$$

We note once again that $\|\cdot\|_{\mathcal{T}_\lambda}$ as defined above are usually semi-norms, but for the spaces $\{\mathcal{T}_\lambda\}$, they are equivalent to full norms as demonstrated by the Poincare inequality (see Lemma 4.1.2).

In the next section by Lemma 4.2.1 , we will show that $\mathcal{T}_\lambda$ and $\mathcal{T}_\infty$ are equivalent. Here, since we need $f \in (\mathcal{T}_\lambda)^*$ (the dual of $\mathcal{T}_\lambda$), the weak formulations are: given $f \in (H^\alpha_\Omega(\mathbb{R}^d))^*$,

$$\text{find } \ u_\lambda \in \mathcal{T}_\lambda \text{ such that } a_\lambda(u_\lambda, v) = \langle f, v\rangle \quad \forall v \in \mathcal{T}_\lambda\ , \tag{4.18}$$

where $a_\lambda(\cdot,\cdot) := (\cdot,\cdot)_{\mathcal{T}_\lambda}$.

Now for each λ, we introduce the finite element spaces $\{W_{\lambda,h}\} \subset \mathcal{T}_\lambda$ associated with the triangulation $\tau_h = \{K\}$ of the domain $\Omega \cup \Omega_\lambda$. We set

$$W_{\lambda,h} := \{v \in \mathcal{T}_\lambda : v|_K \in P(K) \quad \forall K \in \tau_h\}$$

where $P(K) = \mathcal{P}_p(K)$ is the space of polynomials on $K \in \tau_h$ of degree less or equal than p.

As $h \to 0$, $\{W_{\lambda,h}\}$ is dense in $\mathcal{T}_\lambda$, i.e., for any $v \in \mathcal{T}_\lambda$, there exists a sequence $(v_h \in W_{\lambda,h})$ such that

$$\|v_h - v\|_{\mathcal{T}_\lambda} \to 0 \quad \text{as} \quad h \to 0\,. \tag{4.19}$$

These properties are easily satisfied by standard finite element spaces.

The Galerkin approximation is to replace $\mathcal{T}_\infty$ by $W_{\lambda,h}$ in (4.18), namely, given $f \in (H^\alpha_\Omega(\mathbb{R}^d))^*$,

$$\text{find } \; u_{\lambda,h} \in W_{\lambda,h} \text{ such that } a_\lambda(u_{\lambda,h}, v) = \langle f, v\rangle \quad \forall v \in W_{\lambda,h}\,. \tag{4.20}$$

4.2.2 Asymptotically compatible schemes

Now, let $\mathcal{T}_0$ be the space of L^2 functions on $\mathbb{R}^d$ with zero values on $\mathbb{R}^d\backslash\Omega$. To apply the abstract framework to the nonlocal diffusion model, we need to verify the assumptions made in the earlier section. First, it is clear that $\|u\|_{\mathcal{T}_\lambda} \leq \|u\|_{\mathcal{T}_\infty}$ for $\lambda > 0$ just by definition of the norms. Then, for $\lambda = 1$, through the Poincaré type inequality (Lemma 4.1.2),

$$\|u\|_{L^2(\Omega)} \leq C\|u\|_{\mathcal{T}_1} \quad \forall u \in \mathcal{T}_1\,, \tag{4.21}$$

we have $\|u\|_{\mathcal{T}_0} \leq C\|u\|_{\mathcal{T}_\lambda}$ for $\lambda \geq 1$. Since we only consider the asymptotic behavior when $\lambda \to \infty$, we can ignore the case $\lambda < 1$ and Assumption 3.1.1 i) is verified.

To check the Assumption 3.1.1 ii), we first state a lemma here. It establishes the particular relationship of the space $\mathcal{T}_\lambda$ and $\mathcal{T}_\infty$ in our example here.

Lemma 4.2.1. *The spaces $\{\mathcal{T}_\lambda\}_{\lambda\geq 1}$ and $\mathcal{T}_\infty$ defined in Section 2.1 are equivalent, i.e.,*

$$C_1\|u\|_{\mathcal{T}_\infty} \leq \|u\|_{\mathcal{T}_\lambda} \leq C_2\|u\|_{\mathcal{T}_\infty} \quad \text{for } \lambda \geq 1\,, \tag{4.22}$$

where C_1 and C_2 are constants independent of λ. Moreover, $\|\cdot\|_{\mathcal{T}_\lambda}$ goes to $\|\cdot\|_{\mathcal{T}_\infty}$ as $\lambda \to \infty$, i.e.,

$$\|u\|_{\mathcal{T}_\lambda} \to \|u\|_{\mathcal{T}_\infty} \quad \forall u \in \mathcal{T}_\infty\,. \tag{4.23}$$

Proof. First, for the norm equivalence, we only have to proof the left-hand part, since $\|u\|_{\mathcal{T}_\lambda} \leq \|u\|_{\mathcal{T}_\infty}$ is obvious. Now, for $\lambda = 1$, in [Du *et al.*, 2012], it is proved that $\mathcal{T}_1$ is equivalent to the fractional order Sobolev space $H^{\alpha}_{\Omega}(\Omega \cup \Omega_1)$ [1]. so

$$\|u\|_{H^{\alpha}_{\Omega}(\Omega\cup\Omega_1)} \le C\|u\|_{\mathcal{T}_1}\,. \tag{4.24}$$

In addition, [Webb, 2012] showed that $H^{\alpha}_{\Omega}(\Omega \cup \Omega_1)$ is equivalent to $H^{\alpha}_{\Omega}(\mathbb{R}^d)(=: \mathcal{T}_\infty)$. So we have $\|u\|_{\mathcal{T}_\infty} \le \tilde{C}\|u\|_{\mathcal{T}_1} \le \tilde{C}\|u\|_{\mathcal{T}_\lambda}$, for $\lambda \ge 1$. This completes the proof of (4.22).

Now that $\gamma_\lambda(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{y})-u(\mathbf{x}))^2$ converges to $\frac{C_{d,\alpha}}{2}(u(\mathbf{y})-u(\mathbf{x}))^2/|\mathbf{x}-\mathbf{y}|^{d+2\alpha}$ pointwise as $\lambda \to \infty$, $\|u\|_{\mathcal{T}_\lambda} \to \|u\|_{\mathcal{T}_\infty}$ by dominated convergence theorem.

□

Now we are ready to proof the Assumption 3.1.1 ii).

Lemma 4.2.2. *Suppose $u_n \in \mathcal{T}_{\lambda_n}$ with $\lambda_n \to \infty$. If*

$$\sup_n \|u_n\|_{\mathcal{T}_{\lambda_n}} < \infty\,, \tag{4.25}$$

then u_n is precompact in $L^2_0(\Omega)$. Moreover, any limit point $u \in \mathcal{T}_\infty$.

Proof. By Lemma 4.2.1, (4.25) implies $\sup_n \|u_n\|_{\mathcal{T}_\infty} < \infty$. So the lemma is true since H^α is compactly embedded L^2. □

We next move to the bilinear forms. Note that a_δ is exactly the inner product defined on $\mathcal{T}_\lambda$, so Assumption 3.1.2 is naturally satisfied with $C_1 = C_2 = 1$.

Assumption 3.1.3 is about the convergence of the operator $\mathcal{L}_\lambda$ to the fractional Laplacian $(-\Delta)^\alpha$ on a dense subspace of $\mathcal{T}_\infty$ which we state here as a lemma.

Lemma 4.2.3. *For all $w \in C^\infty_c(\Omega)$ with zero extension outside Ω, we have*

$$\mathcal{L}_\lambda w \in L^2_0(\Omega \cup \Omega_\lambda) \tag{4.26}$$

and

$$\|\mathcal{L}_\lambda w - (-\Delta)^\alpha w\|_{L^2(\mathbb{R}^d)} \longrightarrow 0 \quad \text{as } \lambda \to \infty\,.$$

[1] $H^{\alpha}_{\Omega}(\Omega\cup\Omega_1) := \{w \in H^\alpha(\Omega\cup\Omega_1) : w|_{\Omega_1} = 0\}$, where for a general domain $\tilde{\Omega}$ the space $H^\alpha(\tilde{\Omega})$ is defined by $H^\alpha(\tilde{\Omega}) := \{w \in L^2(\tilde{\Omega}) : \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{(w(\mathbf{y})-w(\mathbf{x}))^2}{|\mathbf{x}-\mathbf{y}|^{n+2\alpha}} d\mathbf{y}d\mathbf{x} < \infty\}$

Proof. Fix any $w \in C_c^\infty(\Omega)$, we know that $(-\Delta)^\alpha w \in L^2(\mathbb{R}^d)$. Also it is not difficult to see that $\mathcal{L}_\lambda w \in L_0^2(\Omega \cup \Omega_\lambda)$(functions in $L^2(\Omega \cup \Omega_\lambda)$ with zero extension outside $\Omega \cup \Omega_\lambda$), since $\mathcal{L}_\lambda w(\mathbf{x})$ is bounded. In addition , it is easy to see that $\mathcal{L}_\lambda w(\mathbf{x})$ converges to $(-\Delta)^\alpha w(\mathbf{x})$ pointwise and uniformly because

$$|\mathcal{L}_\lambda w(\mathbf{x}) - (-\Delta)^\alpha w(\mathbf{x})| \leq C\|w\|_\infty \int_{\mathbb{R}^d \setminus B_\lambda(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}|^{n+2\alpha}} d\mathbf{y} \to 0 \,. \tag{4.27}$$

Now that both $\mathcal{L}_\lambda w$ and $(-\Delta)^\alpha w$ are in L^2, we conclude that $\mathcal{L}_\lambda w$ converges to $(-\Delta)^\alpha w$ in L^2 norm by dominated convergence theorem. □

As for Assumption 3.1.4, (4.19) ensures that $W_{\lambda,h}$ satisfies i). But ii) needs to be checked for different values of α. For $\alpha < 1/2$, discontinuous piecewise polynomial functions are contained in the space $H_\Omega^\alpha(\mathbb{R}^d)$. But for $\alpha \geq 1/2$, they are not in $H_\Omega^\alpha(\mathbb{R}^d)$. Thus making the asymptotically dense subspaces different for the two cases.

Now, before we go into the proof of convergence of discrete solutions, we give the following theorem for the completeness of this paper. It states the existence and uniqueness of solutions to nonlocal diffusion equation and fractional Laplacian equation and the convergence of the solution as $\lambda \to \infty$.

Theorem 4.2.4. *There exists a unique u_λ to the solution of* (4.18) *for $\lambda \in (0,\infty]$. Moreover,*

$$\|u_\lambda - u_\infty\|_{H^\alpha} \to 0 \,. \tag{4.28}$$

Proof. The existence and uniqueness come from Lax-Milgram thoerem. Now, as a direct application of Theorem 3.2.1, we know that $\|u_\lambda - u_\infty\|_{L^2} \to 0$ as $\lambda \to \infty$. In addition, we know from Lemma 4.2.2 that u_λ converges to u_∞ weakly in $\mathcal{T}_\infty$. Now since u_λ and u_∞ are solutions of (4.18), we have

$$a_\lambda(u_\lambda, u_\lambda) = \langle f, u_\lambda \rangle \to \langle f, u_\infty \rangle = a_\infty(u_\infty, u_\infty) \,, \tag{4.29}$$

which is equivalent to $\|u_\lambda\|_{\mathcal{T}_\lambda} \to \|u_\infty\|_{\mathcal{T}_\infty}$ as $\lambda \to \infty$. Now we know that $u_\lambda|_{\Omega^c} = 0$, then for λ large enough,

$$\begin{aligned}
\|u_\lambda\|^2_{\mathcal{T}_\infty} - \|u_\lambda\|^2_{\mathcal{T}_\lambda} &= \frac{C_{d,\alpha}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_\lambda(\mathbf{x})}\frac{(u_\lambda(\mathbf{y})-u_\lambda(\mathbf{x}))^2}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}d\mathbf{y}d\mathbf{x} \\
&= \frac{C_{d,\alpha}}{2}\int_\Omega u_\lambda^2(\mathbf{x})\int_{\mathbb{R}^d\setminus B_\lambda(\mathbf{x})}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}d\mathbf{y}d\mathbf{x} \\
&= C\|u_\lambda\|_{L^2}\int_{\mathbb{R}^d\setminus B_\lambda(\mathbf{0})}\frac{1}{|\mathbf{z}|^{d+2\alpha}}d\mathbf{z}\to 0\ .
\end{aligned} \tag{4.30}$$

Then we have $\|u_\lambda\|_{\mathcal{T}_\infty}\to\|u_\infty\|_{\mathcal{T}_\infty}$. This implies $\|u_\lambda - u_\infty\|_{\mathcal{T}_\infty}\to 0$ because

$$\begin{aligned}
\|u_\lambda - u_\infty\|^2_{\mathcal{T}_\infty} &= \|u_\lambda\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2(u_\lambda, u_\infty)_{\mathcal{T}_\infty} \\
&= \|u_\lambda\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2\langle u_\lambda, (-\Delta)^\alpha u_\infty\rangle \\
&\to 2\|u_\infty\|^2_{\mathcal{T}_\infty} - 2\|u_\infty\|^2_{\mathcal{T}_\infty} = 0\ .
\end{aligned} \tag{4.31}$$

□

4.2.2.1 The case when $0<\alpha<1/2$

In this section, we always assume that the parameter α associated with the fractional Laplacian is always less than 1/2.

Definition 4.2.5. *Let $V^0_{\lambda,h}\subset W_{\infty,h}\subset\mathcal{T}_\infty$ be the discontinuous piecewise constant finite element space that corresponds to the same mesh τ_h with $W_{\lambda,h}$.*

The following lemma is simply a re-statement of a simple fact that discontinuous piecewise constant subspace of $\mathcal{T}_\infty$ approximates the whole space as mesh size goes to zero.

Lemma 4.2.6. *The family $\{V^0_{\lambda,h}\}$ is asymptotically dense in $\mathcal{T}_\infty$, that is, it satisfies Assumption 3.1.4 ii).*

Now we see that if $V^0_{\lambda,h}\subset W_{\lambda,h}$, then $W_{\lambda,h}$ also satisfies Assumption 3.1.4 ii).

With all assumptions 3.1.1-3.1.4 verified, the following theorem provides a criterion for asymptotically compatible schemes for solving nonlocal diffusion equation as an approximation of fractional Laplacian. It is basically a direct application of Theorem 3.3.2. However, for our special example here, we can prove convergence in the energy norm than just the L^2 norm.

Theorem 4.2.7. *Let u_λ and $u_{\lambda,h}$ be solutions of* (4.18) *and* (4.20) *respectively with* $0 < \alpha < 1/2$, *and $V^0_{\lambda,h}$ is defined in Definition 4.2.5. If $V^0_{\lambda,h} \subset W_{\lambda,h}$, then $\|u_{\lambda,h} - u_\infty\|_{H^\alpha} \to 0$ as $\lambda \to \infty$, $h \to 0$.*

Proof. First, as a direct application of Theorem 3.3.2, we know that $\|u_{\lambda_n,h_n} - u_\infty\|_{L^2} \to 0$ as $n \to \infty$ for any sequence $\lambda_n \to \infty$ and $h_n \to 0$. Now the H^α convergence is essentially the same as the proof of Theorem 4.2.4. Specifically, since u_{λ_n,h_n} converges to u_∞ weakly in $\mathcal{T}_\infty$. and they are solutions of (4.20) and (4.18) respectively, we have

$$a_{\lambda_n}(u_{\lambda_n,h_n}, u_{\lambda_n,h_n}) = \langle f, u_{\lambda_n,h_n}\rangle \to \langle f, u_\infty\rangle = a_\infty(u_\infty, u_\infty)\,, \tag{4.32}$$

which is equivalent to $\|u_{\lambda_n,h_n}\|_{\mathcal{T}_{\lambda_n}} \to \|u_\infty\|_{\mathcal{T}_\infty}$ as $n \to \infty$. Then we can show $\|u_{\lambda_n,h_n}\|_{\mathcal{T}_\infty} \to \|u_\infty\|_{\mathcal{T}_\infty}$ as in the proof of Theorem 4.2.4, and this implies $\|u_{\lambda_n,h_n} - u_\infty\|_{\mathcal{T}_\infty} \to 0$ because

$$\begin{aligned}\|u_{\lambda_n,h_n} - u_\infty\|^2_{\mathcal{T}_\infty} &= \|u_{\lambda_n,h_n}\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2(u_{\lambda_n,h_n}, u_\infty)_{\mathcal{T}_\infty} \\ &= \|u_{\lambda_n,h_n}\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2\langle u_{\lambda_n,h_n}, (-\Delta)^\alpha u_\infty\rangle \\ &\to 2\|u_\infty\|^2_{\mathcal{T}_\infty} - 2\|u_\infty\|^2_{\mathcal{T}_\infty} = 0\,.\end{aligned} \tag{4.33}$$

□

In short, for $\alpha < 1/2$, since discontinuous piecewise constant functions are dense in H^α, we could apply any finite element method on the nonlocal diffusion equation to get an asymptotically compatible scheme for the fractional Laplacian, this is, no matter how λ and h changes, the solution of (4.20) always approximate (4.18) properly as $\lambda \to \infty$ and $h \to 0$.

4.2.2.2 The case when $1/2 \le \alpha < 1$

In this section, we always assume that the parameter α associated with the fractional Laplacian is always between 1/2 and 1.

Definition 4.2.8. *Let $V^1_{\lambda,h} \subset W_{\infty,h} \subset \mathcal{T}_\infty$ be the continuous piecewise linear finite element space that corresponds to the same mesh τ_h with $W_{\lambda,h}$.*

The following lemma is simply a re-statement of a simple fact that continuous piecewise linear subspace of $\mathcal{T}_\infty$ approximates the whole space as mesh size goes to zero.

Lemma 4.2.9. *The family $\{V^1_{\lambda,h}\}$ is asymptotically dense in $\mathcal{T}_\infty$, that is, it satisfies Assumption 3.1.4 ii).*

Now we see that if $V^1_{\lambda,h} \subset W_{\lambda,h}$, then $W_{\lambda,h}$ also satisfies Assumption 3.1.4 ii).

With all assumptions 3.1.1-3.1.4 verified, the following theorem provides a criterion for asymptotically compatible schemes for solving nonlocal diffusion equation as an approximation of fractional Laplacian. Again, here we have a stronger version than Theorem 3.3.2.

Theorem 4.2.10. *Let u_λ and $u_{\lambda,h}$ be solutions of* (4.18) *and* (4.20) *respectively with $1/2 \leq \alpha < 1$, and $V^1_{\lambda,h}$ is defined in Definition 4.2.8. If $V^1_{\lambda,h} \subset W_{\lambda,h} \subset \mathcal{T}_\lambda$, then $\|u_{\lambda,h} - u_\infty\|_{H^\alpha} \to 0$ as $\lambda \to \infty$, $h \to 0$.*

Proof. The proof is essentially the same as Theorem 4.2.7 □

In short, for $\alpha \geq 1/2$, if the finite element spaces contain a continuous finite element subspace which have desired approximation properties in $\mathcal{T}_\infty$, then the corresponding discretization is asymptotically compatible. Now that we know the space $\mathcal{T}_\lambda$ is equivalent to $\mathcal{T}_\infty$ for $\lambda > 0$, the conforming elements of $\mathcal{T}_\lambda$ can also only be continuous functions. That means as long as we choose conforming method to solve (4.18), asymptotic compatibility is assured.

4.2.3 Conclusion

In this section, our central goal was to obtain convergence results for approximations, due to both domain truncation and spatial discretization, of solutions of the fractional Laplacian problem (4.13). We considered the fractional Laplacian operator as a limit of the linear nonlocal diffusion operator developed in [Du *et al.*, 2013a; Du *et al.*, 2012], motivated by the work of [D'Elia and Gunzburger, 2013], in which careful a prior error estimates on the solution, and their numerical experiments were carried out when one of λ and h is assumed to be sufficiently small. However, It is not clear from [D'Elia and Gunzburger, 2013] that whether correct solution to the fractional Laplacian is ensured when λ and h changes simultaneously (since some estimates depend on λ). Here, we eliminate such concerns by showing theoretically that any conforming Galerkin finite element methods for the nonlocal equation are asymptotic compatible no matter how the nonlocal interaction parameter λ changes with discretization parameter h. Moreover, the convergence of $u_{\lambda,h}$ to u in H^α norm is given under minimal regularity assumption (only $\mathcal{L}_\lambda u \in H^{-\alpha}$) as well as general geometric meshes for conforming Galerkin finite element approximation of nonlocal equations, which is another major contribution.

In this section, we only considered conforming methods for nonlocal equations which may or may not allow discontinuous finite element methods, depending on the value of α. There may be possible extensions to nonconforming methods even for $\alpha \geq 1/2$, such as those that will be presented in the next section of this chapter. It is also feasible to consider problems involving inhomogeneous nonlocal Dirichlet volumetric constraints or different types of nonlocal boundary conditions.

4.3 Nonconforming discontinuous Galerkin methods for nonlocal diffusion

4.3.1 Introduction

We present in this section a nonconforming discontinuous Galerkin finite element scheme for nonlocal diffusion models defined on a bounded spatial domain in any given space dimension. In terms of Galerkin type approximations, most of the existing analysis for nonlocal diffusion have focused on the conforming ones and very few studies have been carried out for nonconforming discontinuous Galerkin (DG) finite element discretizations.

For classical local models such as variational problems associated with second order elliptic equations, a conforming or internal approximation refers to the use of finite element space which is a subspace of the continuum solution space. Nonconforming methods use finite element spaces that are no longer the subspace of the continuum solutions and are often based on weak forms that are different from the original ones. DG methods are mostly nonconforming for second order elliptic equations in this regard. We adopt the same notion here. DG is an important class of numerical methods for solving classical PDEs that can provide stable and high-order accurate approximations and can easily handle complex geometries and irregular meshes. The study of DG methods began in 1970's and has been an active research area since then, see for example [Arnold *et al.*, 2002; Cockburn *et al.*, 2011; Cockburn and Shu, 1998]. Since PD/ND based models may allow potentially more singular solutions and more complex structures in comparison with the corresponding local PDE models, DG methods may have added advantages with their generality and flexibility to deal with complexities and substructures in practical nonlocal modeling. While there are on-going studies of conforming DG methods for PD/ND models (see discussions in chapter 2 and chapter 4), the development and analysis of nonconforming DG methods have largely lacked behind. Indeed, the one-dimensional fractional operators considered in [Cifani *et al.*, 2011] are low order ones (less than first order to be precise) so that discontinuous elements remain in the natural domain of the operators and are thus conforming. Among other works in the literature, the study given in [Mustapha and McLean, 2013] focused on DG for time-fractional differential equations. In [Deng and Hesthaven, 2013], DG methods were developed for a class of fractional differential equations based on a special decomposition of some fractional differential operators in one space dimension. Similar idea was adopted in [Xu and Hesthaven, 2014]. Since such decompositions are not readily available for the more general and higher dimensional nonlocal operators considered in this section, we adopt a new strategy for our model problems which is thus the first of its kind in the literature. The new approach is based on modifications of the nonlocal interaction kernels near their singularities and allows straightforward implementation in any space dimension. Furthermore, it maintains the same variational setting as that of the original problem without introducing a saddle-point formulation. More general discrete function spaces such as those represented by radial basis functions, reproducing kernel spaces, partition of unity and other generalized/extended finite element basis can also be used [Belytschko *et al.*, 2009; Bessa *et al.*, 2014; Bond *et al.*, 2013; Buhmann, 2000; Liu *et al.*, 1996; Strouboulis *et al.*, 2000]. The convergence of our approach can be rigorously analyzed under minimal regularity assumptions on the underly problems. The latter is of interests as PD/ND type of models have often been introduced to model phenomena that may exhibit singularities and heterogeneities so that convergence and robustness of numerical approximations in the presence of singular solutions are of much concern in applications.

To introduce a nonconforming DG discretization of PD/ND models, our main idea is to utilize the recent study on the asymptotic compatible (AC) schemes for nonlocal models described in chapter 3. Such a framework motivates us to modify the nonlocal interaction kernels so that nonconforming discontinuous finite elements can be constructed via conforming discretization of the modified problem. The modification introduces an additional modeling parameter, roughly speaking a cut-off level, and as it goes to infinity, we recover the original problem. The challenge is to establish the convergence of the discrete numerical schemes as the mesh is refined, either irrespectively how the cut-off level is taken to infinity or under some conditions. Hence, we see naturally the relevance of the current work with the study of AC schemes. It then remains to verify the properties required by the general theoretical framework for AC schemes. To accomplish the latter, nevertheless, there is a substantial amount of work, such as establishing new compactness results in the modified nonlocal energy spaces. To this end, we need an extension of the compactness result of Bourgain-Brezis-Mironescu [Bourgain *et al.*, 2001]. This part of the theoretical is put to chapter 7 for detailed mathematical study.

In the subsequent of this section, we will always consider the model equation (4.1) with a fixed parameter δ. To this end, we omit the index δ and write our nonlocal diffusion problem just as

$$\begin{cases} -\mathcal{L}u(\mathbf{x}) = -\displaystyle\int_{\Omega\cup\Omega_{\mathcal{I}}} (u(\mathbf{y}) - u(\mathbf{x}))\gamma(|\mathbf{y}-\mathbf{x}|)d\mathbf{y} = f \quad \text{on } \Omega\,, \\ \qquad u = 0 \quad \text{on } \Omega_{\mathcal{I}} \end{cases} \tag{4.34}$$

where $\Omega_{\mathcal{I}}$ denotes the interaction domain of Ω which depends on the support of the kernel γ.

We assume that the kernel γ a radial and nonnegative function with a bounded second order moment. Moreover, we assume that for some $\epsilon_0 > 0$

$$\int_{|\mathbf{x}|<\epsilon} |\mathbf{x}|\gamma(|\mathbf{x}|)d\mathbf{x} = \infty\,, \quad \forall \epsilon \in (0, \epsilon_0]\,. \tag{4.35}$$

This is to guarantee that the associated energy space $\mathcal{S}$ does not accept discontinuous functions, thus are DG scheme would be nonconforming. One may check that if $\gamma = \gamma(r)$ has a singularity of the type $1/r^{d+2s}$ for r at the origin where the exponent s belongs to $(1/2, 1)$, which is a typical kernel for the standard Sobolev space H^s, then (4.35) is satisfied.

Modified kernels. For the nonlocal problem with a kernel γ satisfying (4.35), we introduce a sequence of modified kernels $\{\gamma_n\}$. For each n, γ_n is the cutoff of the original kernel γ, that is, for any $r \geq 0$,

$$\gamma_n(r) := \begin{cases} \gamma(r), & \text{if } \gamma(r) \leq n, \\ n, & \text{if } \gamma(r) > n\,. \end{cases} \tag{4.36}$$

First we note that there are many other ways to define the cutoff of γ which may alter how γ_n converges to γ as n goes to infinity and/or how the modified problem associated with γ_n can be solved efficiently (see for example related discussions on conditional convergence in section 3 where the relation between n and the discretization parameter h plays an important role). The essential requirement on the construction of γ_n is that the resulting modified kernel becomes integrable (for given n) and it converges to the original kernel in the pointwise sense. We adopt the particular form (4.36) here for simplicity.

Once a modified kernel γ_n is introduced, the corresponding modified model equation is defined as

$$\begin{cases} -\mathcal{L}_n u = f & \text{on } \Omega\,, \\ \qquad u = 0 & \text{on } \Omega_{\mathcal{I}} \end{cases} \tag{4.37}$$

where the modified nonlocal operator is given by

$$\mathcal{L}_n u(\mathbf{x}) = \int_\Omega (u(\mathbf{y}) - u(\mathbf{x}))\gamma_n(|\mathbf{x} - \mathbf{y}|)d\mathbf{y}\,.$$

The associated energy space $\mathcal{S}_n$ and bilinear form b_n are then defined similarly with respect to $\mathcal{L}_n$ and γ_n. A weak formulation for the modified problem (4.37) is

$$\text{find } \; u_n \in \mathcal{S}_n \text{ such that } b_n(u_n, v) = (f, v)_{L^2} \quad \forall v \in \mathcal{S}_n\,. \tag{4.38}$$

It is not hard to see that as $n \to \infty$, the modified problem reduces to the original model. This statement can be made rigorous by showing that u_n goes to u in the $L^2(\Omega)$ space.

4.3.2 Discontinuous Galerkin approximation

Now by the definition of the kernel cut-off, for each n, γ_n is a bounded kernel (and thus integrable due to the compact support). This means that $\mathcal{S}_n$ is essentially the space of L^2 functions vanishing outside Ω. Thus we may use discontinuous finite element spaces for standard conforming Galerkin approximation of the modified problem (4.38). To this end, we introduce the finite element spaces $\{V_{n,h}\} \subset \mathcal{S}_n$ associated with a conforming triangulation $\tau_h = \{K\}$ of the domain Ω which also conforms with the domain Ω'. For each n, we choose $V_{n,h}$ to be the space of piecewise polynomials,

$$V_{n,h} := \{v \in \mathcal{S}_n : v|_K \in P(K), \quad \forall K \in \tau_h\}$$

where $P(K) = \mathcal{P}_p(K)$ is the space of polynomials on $K \in \tau_h$ of degree less or equal than p (a given non-negative integer). Note that with this definition, elements in $V_{n,h}$ automatically satisfy the volumetric constraint imposed on $\mathcal{S}_n$ and only nodal basis corresponding to interior nodes of Ω' are taken. As $h \to 0$, $\{V_{n,h}\}$ is dense in $\mathcal{S}_n$, i.e., for any $v \in \mathcal{S}_n$, there exists a sequence $\{v_h \in V_{n,h}\}$ such that

$$\|v_h - v\|_{\mathcal{S}_n} \to 0 \quad \text{as} \quad h \to 0\,. \tag{4.39}$$

The Galerkin approximation is to replace $\mathcal{S}_n$ by $V_{n,h}$ in (4.38), namely,

$$\text{Find } \; u_{n,h} \in V_{n,h} \text{ such that } b_n(u_{n,h}, v) = (f, v)_{L^2} \quad \forall v \in V_{n,h}\,. \tag{4.40}$$

While (4.40) is a conforming Galerkin approximation of the modified problem (4.37), it may also be viewed as a nonconforming DG scheme for the original problem (1.7) when h, the discretization parameter, goes to zero and at the same time, n, the cut-off level of the modified kernel, tends to infinity. Hence, the discrete problem (4.40) contains, besides the meshing parameter h, an additional parameter given by the cut-off level n in the modified kernel. This is neither a drawback nor a feature pertaining only to nonlocal problems, since it is often the case that DG approximations to standard second order elliptic equations also contain extra parameters.

Further, we comment that in the DG formulation, it is not essential to require that the discrete approximations are made of piecewise polynomials. In fact, more general discrete function spaces such as those represented by reproducing kernel spaces, radial basis functions, partition of unity and other generalized/extended finite element basis [Belytschko *et al.*, 2009; Bessa *et al.*, 2014; Bond *et al.*, 2013; Buhmann, 2000; Liu *et al.*, 1996; Strouboulis *et al.*, 2000] can also be used . We note that the key is to have the basis functions satisfying the volumetric constraints so that they conform with the modified energy spaces. To study the convergence of the above nonconforming DG scheme to the model problem (1.7) as $h \to 0$ and $n \to \infty$ simultaneously, our main approach is to utilize the the framework of asymptotically compatible schemes to paramterized problems developed in chapter 3.

4.3.3 Convergence Analysis

Our goal in this section is to prove that the DG approximation $u_{n,h}$ defined by (4.40) converges to the solution of the original nonlocal problem (1.7). In order to apply the framework developed chapter 3, a crucial ingredient is a compactness result that will be established in Theorem 7.1.1.

4.3.3.1 Properties of energy spaces and bilinear forms

The following inequality shows that $\mathcal{S}_n$ is uniformly continuously embedded in L^2.

Lemma 4.3.1 (Uniform Poincaré inequality)**.** *There exists $C > 0$ independent of n such that for all n,*

$$\|u\|_{L^2}^2 \le C\|u\|_{\mathcal{S}_n}^2 \,, \quad \forall u \in \mathcal{S}_n \,. \tag{4.41}$$

Proof. First, we will apply a nonlocal Poincaré inequality just for $n = 1$ that has been already developed in the literature [Du *et al.*, 2013a; Mengesha and Du, 2014a], that is, there exists $C_1 > 0$ such that $\|u\|_{L^2}^2 \le C_1\|u\|_{\mathcal{S}_1}^2$, for all $u \in \mathcal{S}_1$. Now by the definition of the modified kernels, we know that γ_n is a sequence of nonnegative kernels that is increasing in n, which means that $\|u\|_{\mathcal{S}_1} \le \|u\|_{\mathcal{S}_n}$ for every $n \ge 1$. So it is easy to see that we have the uniform Poincaré inequality. □

Let us make a remark. By the definition of the kernels, it is easy to prove the desired uniform Poincaré inequality. If we just assume that γ_n approaches γ but do not have $\gamma_n \le \gamma_{n+1}$ in general, we can still prove the above result by using the compactness result developed in the above subsection.

The proof of the above lemma also brings up a fact that, since $0 \le \gamma_n \le \gamma$ for every n, we have for every $u \in \mathcal{S}(\Omega)$,

$$\int\int \gamma_n(u(\mathbf{y}) - u(\mathbf{x}))^2 d\mathbf{y}d\mathbf{x} \le \int\int \gamma(u(\mathbf{y}) - u(\mathbf{x}))^2 d\mathbf{y}d\mathbf{x} \,.$$

This simply implies that $\mathcal{S}$ is continuously embedded in $\mathcal{S}_n$, which is stated in the following lemma.

Lemma 4.3.2. *For all n,*

$$\|u\|_{\mathcal{S}_n}^2 \leq \|u\|_{\mathcal{S}}^2 , \quad \forall u \in \mathcal{S}(\Omega) . \tag{4.42}$$

Combining Theorem 7.1.1 and Lemma 4.3.1, we also have the asymptotic compactness. Together with the above, we have verified all the Assumption 3.1.1 associated with the energy spaces that are needed.

As for Assumption 3.1.2 of bilinear forms, i.e., the uniform boundedness and coercivity, we note that b_n is exactly the inner product defined on $\mathcal{S}_n$, so these properties are naturally satisfied.

For Assumption 3.1.3, we need to establish the convergence, in a suitable sense, of the operator $\mathcal{L}_n$ to $\mathcal{L}$ in a dense subspace of $\mathcal{S}$. First, the density of $C_c^\infty(\Omega)$ in $\mathcal{S}$ follows from the definition of the space. Moreover, $\mathcal{L}_n u$ is well-defined as an L^2 function in this dense subspace. This is given next.

Lemma 4.3.3. *Let $\mathcal{L}_n$ and $\mathcal{L}$ be defined in this section. For and $w \in C_c^\infty(\Omega)$ with zero extension outside Ω, we have as $n \to \infty$,*

$$\mathcal{L}_n w \in L^2(\Omega), \tag{4.43}$$

and

$$\|\mathcal{L}_n w - \mathcal{L}w\|_{L^2} \longrightarrow 0 . \tag{4.44}$$

Proof. First, since the modified kernel is integrable for each fixed n, we see easily that $\mathcal{L}_n w \in L^2(\Omega)$. Also, since

$$\begin{aligned}|(\mathcal{L}w)(\mathbf{x})| &= |\int \gamma(\mathbf{z})(w(\mathbf{x}+\mathbf{z}) - w(\mathbf{x}))d\mathbf{z}| = |\int \gamma(\mathbf{z})\Big(\nabla w(\mathbf{x}) \cdot \mathbf{z} + \frac{1}{2}\langle D^2 w(\boldsymbol{\xi})\mathbf{z}, \mathbf{z}\rangle\Big)\mathbf{z}| \\ &\leq \frac{1}{2}\|D^2 w\|_\infty \int |\mathbf{z}|^2 \gamma(\mathbf{z}) d\mathbf{z} \leq C , \quad \forall x \in \Omega ,\end{aligned}$$

we get $\mathcal{L}w \in L^2(\Omega)$. Next, by evaluating $(\mathcal{L}_n w - \mathcal{L}w)(\mathbf{x})$ directly:

$$\begin{aligned}|(\mathcal{L}_n w - \mathcal{L}w)(\mathbf{x})| &= |\int (\gamma - \gamma_n)(\mathbf{z})(w(\mathbf{x}+\mathbf{z}) - w(\mathbf{x}))d\mathbf{z}| \\ &= |\int (\gamma - \gamma_n)(\mathbf{z})\Big(\nabla w(\mathbf{x}) \cdot \mathbf{z} + \frac{1}{2}\langle D^2 w(\boldsymbol{\xi})\mathbf{z}, \mathbf{z}\rangle\Big)\mathbf{z}| \\ &\leq \frac{1}{2}\|D^2 w\|_\infty \int |\mathbf{z}|^2 (\gamma - \gamma_n)(\mathbf{z}) d\mathbf{z} \to 0 , \quad \text{as } n \to \infty ,\end{aligned}$$

where the last convergence uses dominated convergence theorem. Now for $w \in C_c^\infty(\Omega')$, since $\mathcal{L}_n w$ and $\mathcal{L}w$ are both in L^2, we conclude that $\mathcal{L}_n w$ converges to $\mathcal{L}w$ in L^2 norm by dominated convergence theorem. □

4.3.3.2 Unconditional convergence of nonconforming DG

We now prove the convergence of nonconforming DG approximations in this subsection. To apply the framework of asymptotically compatible schemes in chapter 3, or more specifically, it remains to examine the properties of the approximation spaces $\{V_{n,h}\}$ as specified by Assumption 3.1.4. The first one is that $\{V_{n,h}\}$ should approximate $\mathcal{S}_n$ properly with each fixed n, as $h \to 0$, which can be ensured by (4.39). The second one is the asymptotically dense property. For convenience, let us define the following special family of spaces $\hat{V}_h$.

Definition 4.3.4. *Let $\hat{V}_h \subset \mathcal{S}$ be the continuous piecewise linear finite element space that corresponds to the same mesh τ_h with $V_{n,h}$.*

Since we know that $\{\hat{V}_h\}$ is a sequence of approximation spaces to $\mathcal{S}$, it is easy to see that if $\hat{V}_h \subset V_{n,h}$, then $\{V_{n,h}\}$ is asymptotically dense in $\mathcal{S}$ in the sense of assumptions 3.1.4 which we state here as a lemma.

Lemma 4.3.5. *The family $\{V_{n,h}\}$ is asymptotically dense in $\mathcal{S}$ if $\hat{V}_h \subset V_{n,h}$, that is, $\forall v \in \mathcal{S}$, there exists a sequence $\{v_n \in V_{n,h_n}\}_{n\to\infty, h_n\to 0}$ such that*

$$\|v_n - v\|_{\mathcal{S}} \to 0 \quad \text{as } n \to \infty\,.$$

By now, we have verified all properties needed to obtain an asymptotically compatible scheme, the following theorem is a restatement of the Theorem 3.3.2, but with the assumptions removed since they have already been verified.

Theorem 4.3.6. *Let u and $u_{n,h}$ be solutions of (4.34) and (4.40) respectively, and $\hat{V}_h$ be given in Definition 4.3.4. If $\hat{V}_h \subset V_{n,h}$, then $\|u_{n,h} - u\|_{L^2(\Omega)} \to 0$ as $n \to \infty$ and $h \to 0$.*

To recap, we see that if the finite element spaces (which may contain discontinuous elements and are nonconforming) contain some conforming (thus continuous) finite element subspaces which approximate $\mathcal{S}$ properly, then we can take any sequence $\{h_n\}$ with $h_n \to 0$, such that u_{n,h_n} always converges to u. This is what we call the unconditional convergence of DG scheme. The assumptions put on the finite element spaces are particularly true for any discontinuous finite element spaces containing at least the subspace of continuous piecewise linear elements. It means that the nonconforming DG method for $\mathcal{L}$ with space mesh size h_n produces approximate solutions that converge to u as long as the finite element spaces contain continuous piecewise linear elements regardless how we let $h_n \to 0$ and $n \to \infty$.

4.3.3.3 Conditional convergence of nonconforming piecewise constant DG

So far we see that discontinuous Galerkin approximation containing continuous piecewise linear elements works well for our nonlocal problem. But in practice, it may be very desirable to use piecewise constant elements due to their simplicity despite the fact that the piecewise constant subspaces are clearly not asymptotically dense in $\mathcal{S}$. In the following, we will develop a conditional convergence result for the nonconforming discontinuous piecewise constant finite element Galerkin approximation.

Going through the proof of theorem 3.3.2, we note that, in the absence of the asymptotically dense property, there is a need to establish $\|v - v_n\|_{\mathcal{S}_n} \to 0$ by some other techniques where $v \in \mathcal{S}$ and $v_n \in V_{n,h_n}$ for some h_n. One idea is to bound $\|v - v_n\|_{\mathcal{S}_n}$ by

$$\|v - v_n\|_{L^2} \cdot (\int \gamma_n(|\mathbf{x}|) d\mathbf{x})^{1/2}$$

where the first factor $\|v - v_n\|_{L^2}$ tends to zero as $h_n \to 0$, and the second factor involving the integral of $\gamma_n(|\mathbf{x}|)$ tends to infinity as $n \to \infty$. So if h_n tends to zero fast enough, it is possible that $\|v - v_n\|_{L^2}$ dominates the growth of the integral factor so that the resulting product still goes to zero. Thus, we next estimate the above two factors to see how h_n should be related to n so that the discontinuous piecewise constant finite element approximation is assured to be convergent.

For the purpose of simple illustration, let us specialize to the following special type of kernels, which have appeared in [Du *et al.*, 2012], to make the estimation simpler: we assume that there exist $s \in (1/2, 1)$ and positive constants γ_* and γ^* such that

$$\frac{\gamma_*}{|\mathbf{y} - \mathbf{x}|^{d+2s}} \le \gamma(|\mathbf{x} - \mathbf{y}|) \le \frac{\gamma^*}{|\mathbf{y} - \mathbf{x}|^{d+2s}} \quad \text{for } \mathbf{x}, \mathbf{y} \in \Omega \, . \tag{4.45}$$

We note that for this special class of kernels, $\|u\|_{\mathcal{S}}$ is equivalent to the fractional H^s norm. Meanwhile, we also adopt the following modification of the kernel

$$\gamma_n(r) := \begin{cases} \gamma(r), & \text{for } r \geq 1/n, \\ \gamma(1/n), & \text{for } 0 < r < 1/n \,, \end{cases} \tag{4.46}$$

which makes the computation of the integral factor simpler but it is in essence the same as (4.36) with a relabeling.

For the above definition of γ and γ_n, it is not difficult to calculate that

$$\int \gamma_n(|\mathbf{x}|)d\mathbf{x} = O(n^{2s}) \;\Rightarrow \left(\int \gamma_n(|\mathbf{x}|)d\mathbf{x} \right)^{1/2} = O(n^s)\,.$$

The following theorem shows that as long as the condition $h = o(1/n)$ is met as $n \to \infty$, the approximate solution $u_{n,h}$ converges to the solution of the original nonlocal problem u even for discontinuous piecewise constant finite element approximations.

Theorem 4.3.7. *Let u and $u_{n,h}$ be solutions of* (4.34) *and* (4.40)*, and γ and γ_n defined by* (4.45) *and* (4.46) *respectively. If $V_{n,h}$ is the piecewise constant space, then $\|u_{n,h} - u\|_{L^2} \to 0$ if $h = o(1/n)$ as $n \to \infty$.*

Proof. The proof the theorem follows from the proof of Theorem 4.3.6 except for the III part of the four parts that $b(u, v) - (f, v)$ is splited into. We only need to prove that $\|v - v_n\|_{\mathcal{S}_n} \to 0$ if $h_n \cdot n \to 0$ as $n \to \infty$.

First, it is not difficult to see the following estimate

$$\|v - v_n\|_{\mathcal{S}_n} \leq C\|v - v_n\|_{L^2(\Omega)} \left(\int \gamma_n(|\boldsymbol{\xi}|)d\boldsymbol{\xi} \right)^{1/2} ,$$

where $v \in \mathcal{S}$ and $v_n \in V_{n,h_n}$. Now by definitions (4.45) and (4.46), a direct calculation shows

$$\left(\int \gamma_n(|\boldsymbol{\xi}|)d\boldsymbol{\xi} \right)^{1/2} = Cn^s\,, \quad s \in (\frac{1}{2}, 1)\,.$$

Next, by taking v_n as the piecewise constant L^2-orthogonal projection of $v \in \mathcal{S}$ onto V_{n,h_n}, we have

$$\|v - v_n\|_{L^2(\Omega)} \leq Ch_n^s\|v\|_{H^s(\Omega)}\,,$$

a result that can be found in many literatures, see for example, [Ciarlet, 2013, Corollary 3.4]. Here $\|v\|_{H^s(\Omega)}$ makes sense because $\|v\|_{H^s(\Omega)}$ is equivalent to $\|v\|_{\mathcal{S}}$ for the special class of kernels (4.45). Thus,

$$\|v - v_n\|_{\mathcal{S}_n} \leq C(n \cdot h_n)^s \|v\|_{H^s(\Omega)} \to 0$$

as $n \to \infty$ which completes the proof. □

The above theorem shows that for γ_n defined by (4.46), h should decrease faster than $1/n$ to make sure that the approximate solution converges to the true solution of the desired nonlocal problem. Basically, it suggests that the cut-off of the original kernel γ should go slow enough to assure the convergence, and a cutoff slower than (4.46) would be sufficient.

4.3.4 Numerical experiments

We now report results of numerical experiments which validate our analysis and provide results on the orders of convergence that are not implied by the convergence theorems. We use discontinuous piecewise constant and piecewise linear finite element to solve the one dimensional scalar nonlocal problem

$$\begin{cases} -2\displaystyle\int_{-\delta_0}^{\delta_0} \gamma(s)(u(x+s) - u(x))ds = f(x) \quad \text{on } (0,1)\,, \\ \qquad\qquad\qquad\qquad u(x) = g(x) \quad \text{on } (\delta_0, 0) \cup (1, 1+\delta_0) \end{cases}$$

where δ_0 is a fixed number. A special kernel is chosen to be $\gamma(s) = \frac{1}{4}\delta_0^{-1/2}|s|^{-5/2}$ in our numerical examples. Notice that discontinuous elements are nonconforming for this singular kernel. Our benchmark problem is chosen to have $u(x) = -x^2(1-x)^2$ as the exact solution. This means to provide an example that offer indicative performance of the numerical schemes without the complication of geometry associated with high dimensional problems. The simple and smooth solution allows us to more easily reveal, through numerical experiments, the possible rate of convergence that has not been addressed in the numerical analysis. Through Taylor expansion of the nonlocal operator we obtain

$$-\frac{1}{2\delta_0^{1/2}} \int_{-\delta_0}^{\delta_0} \frac{u(x+s) - u(x)}{|s|^{5/2}} ds = -u''(x) - \frac{1}{60} u''''(x)\delta_0^2 - \ldots$$

which can be used to determine the right hand side $f(x) = 12x^2 - 12x + 2 + \frac{2}{5}\delta_0^2$. The modified kernels are defined by (4.46).

4.3.4.1 Discontinuous piecewise linear finite element spaces

We solve the modified nonlocal problems on a uniform mesh using a discontinuous piecewise linear finite element spaces. The corresponding point wise errors $e(x) = u_{n,h}(x) - u(x)$ are plotted in Figures 4.5-4.8 for the cases $n = 1/\sqrt{h}, 1/h, 1/h^2, 1/h^4$ respectively. The red dots are highlighted to show errors at nodal points.

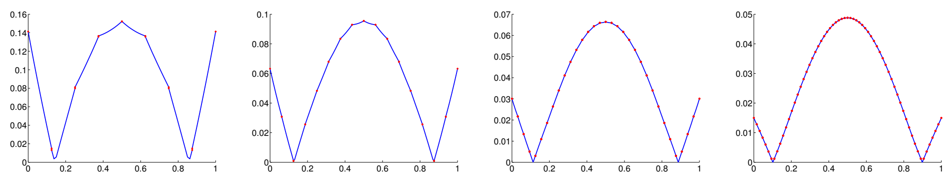

Figure 4.5: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/\sqrt{h}$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.

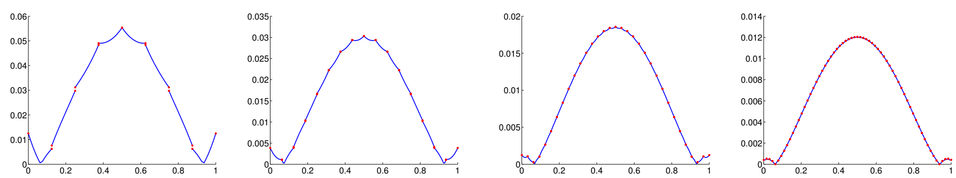

Figure 4.6: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/h$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.

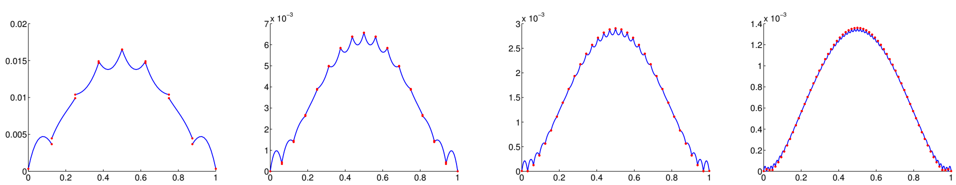

Figure 4.7: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/h^2$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.

We observe convergence in all these cases as predicted by our theorem. However, the convergence rates varies with the relations of n and h. In general, these plots show a faster convergence for n to increase faster. Quantitatively, Table 4.4 shows the L^2 errors and convergence rates (given inside parenthesis) for different relations between n and h as the mesh is refined with a decreasing h. Errors are measured against the exact solution $u(x) = -x^2(1-x)^2$ to the nonlocal problem on the interval $(0, 1)$. In particular, the L^2 convergence rate for discontinuous piecewise linear finite element method is of first order for the case $n = 1/h^2$ and is of second order for the case $n = 1/h^4$.

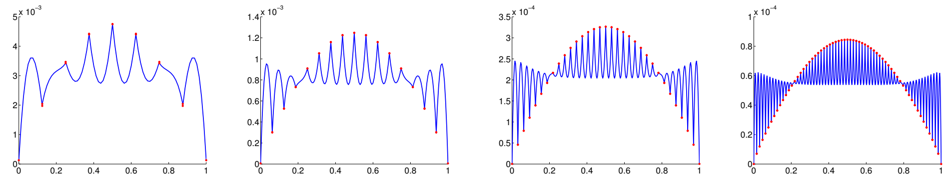

Figure 4.8: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/h^4$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.

Table 4.4: L^2 errors and convergence rates for nonconforming piecewise linear DG.

h	$n = 1/\sqrt{h}$	$n = 1/h$	$n = 1/h^2$	$n = 1/h^4$
2^{-3}	$1.01 \times 10^{-1}(-)$	$3.38 \times 10^{-2}(-)$	$1.03 \times 10^{-2}(-)$	$3.13 \times 10^{-3}(-)$
2^{-4}	$6.11 \times 10^{-2}(0.74)$	$1.88 \times 10^{-2}(0.84)$	$4.11 \times 10^{-3}(1.33)$	$8.31 \times 10^{-4}(1.91)$
2^{-5}	$4.17 \times 10^{-2}(0.55)$	$1.17 \times 10^{-2}(0.69)$	$1.83 \times 10^{-3}(1.17)$	$2.19 \times 10^{-4}(1.92)$
2^{-6}	$3.06 \times 10^{-2}(0.45)$	$7.60 \times 10^{-3}(0.73)$	$8.59 \times 10^{-4}(1.09)$	$5.68 \times 10^{-5}(1.95)$
2^{-7}	$2.32 \times 10^{-2}(0.40)$	$5.10 \times 10^{-3}(0.47)$	$4.16 \times 10^{-4}(1.05)$	$1.46 \times 10^{-5}(1.96)$
2^{-8}	$1.81 \times 10^{-2}(0.36)$	$3.48 \times 10^{-3}(0.55)$	$2.04 \times 10^{-4}(1.03)$	$3.71 \times 10^{-6}(1.98)$
2^{-9}	$1.43 \times 10^{-2}(0.34)$	$2.40 \times 10^{-3}(0.54)$	$1.01 \times 10^{-4}(1.01)$	$9.40 \times 10^{-7}(1.98)$

4.3.4.2 Discontinuous piecewise constant finite element spaces

Discontinuous piecewise constant finite element is not always a convergent scheme. We also predict its convergence under the condition that h decreases faster than $1/n$. Figures 4.9-4.12 show the point wise errors $e(x) = u_{n,h}(x) - u(x)$ for the cases $n = 1/\sqrt{h}, 1/h, 1/h^2, 1/h^4$ respectively. Again, the red dots are highlighted to show errors at nodal points.

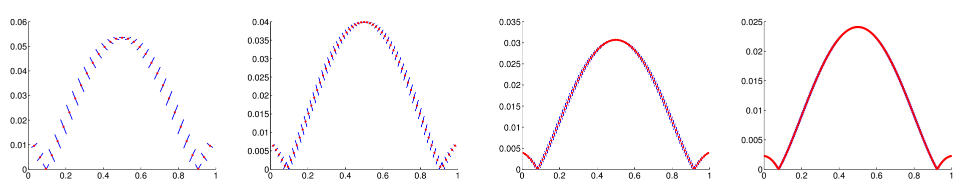

Figure 4.9: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/\sqrt{h}$ and $h = 2^{-k}$, $k = 5, 6, 7, 8$.

We observe convergence for the case $n = 1/\sqrt{h}$ which validates our conditional convergence theorem for piecewise constant finite element method. Our theorem did not cover the cases $n = 1/h, 1/h^2, 1/h^4$. In our numerical result, we observe convergence, albeit very slowly, for $n = 1/h$, which belongs to the borderline case of our analysis and is thus not totally surprising. However, $n = 1/h^2$ and $n = 1/h^4$ clearly violate the conditions and

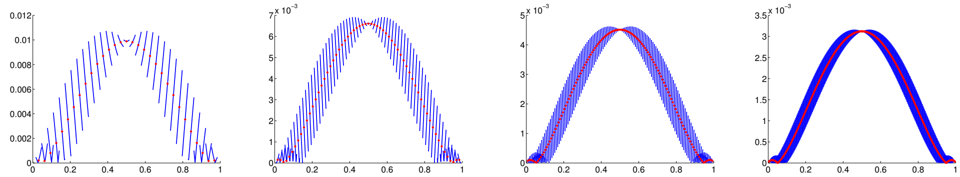

Figure 4.10: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/h$ and $h = 2^{-k}$, $k = 5, 6, 7, 8$.

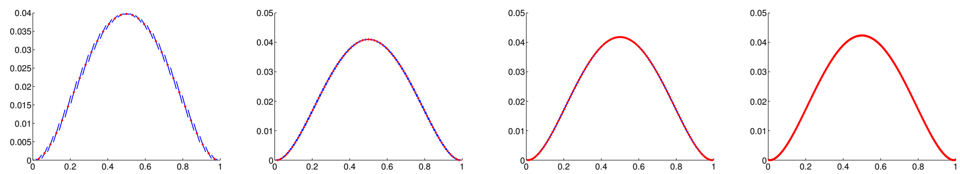

Figure 4.11: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/h^2$ and $h = 2^{-k}$, $k = 5, 6, 7, 8$.

they indeed lead to divergence (the errors do not get reduced as h gets smaller), indicating conditions on how fast n goes to infinity are necessary in practice.

In addition, table 4.5 shows the L^2 errors and convergence rates (given inside parenthesis) of piecewise constant finite element method for these cases as the mesh is refined with a decreasing h. Errors are measured against the exact solution $u(x) = -x^2(1-x)^2$ to the nonlocal problem on the interval $(0, 1)$. The convergence rate is approximately 0.3 for $n = 1/\sqrt{h}$, and 0.5 for $n = 1/h$ based on the numerical results.

4.3.5 Conclusion

In this section, we designed a new nonconforming DG scheme for homogeneous Dirichlet type nonlocal volumetrically constrained value problems associated with a nonlocal diffusion operator. Its convergence is rigorously investigated and remains valid with minimal assumptions on the underlying problems, the solution regularity, and the approximation spaces. We showed that any finite element discretization that contains piecewise linear functions provides an unconditionally convergent nonconforming scheme via a modified nonlocal in-

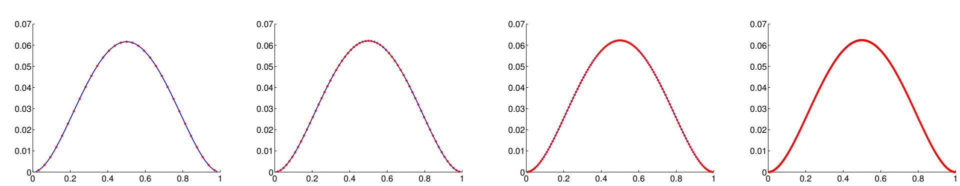

Figure 4.12: Pointwise error $u_{n,h}(x) - u(x)$ with $n = 1/h^4$ and $h = 2^{-k}$, $k = 5, 6, 7, 8$.

Table 4.5: L^2 errors and convergence rates for nonconforming piecewise constant DG.

h	$n = 1/\sqrt{h}$	$n = 1/h$	$n = 1/h^2$	$n = 1/h^4$
2^{-5}	$3.36 \times 10^{-2}(-)$	$6.29 \times 10^{-3}(-)$	$2.53 \times 10^{-2}(-)$	$3.93 \times 10^{-2}(-)$
2^{-6}	$2.51 \times 10^{-2}(0.42)$	$4.19 \times 10^{-3}(0.59)$	$2.61 \times 10^{-2}(-0.04)$	$3.96 \times 10^{-2}(-0.01)$
2^{-7}	$1.93 \times 10^{-2}(0.38)$	$2.86 \times 10^{-3}(0.55)$	$2.66 \times 10^{-2}(-0.03)$	$3.97 \times 10^{-2}(-0.00)$
2^{-8}	$1.52 \times 10^{-2}(0.34)$	$1.98 \times 10^{-3}(0.53)$	$2.70 \times 10^{-2}(-0.02)$	$3.98 \times 10^{-2}(-0.00)$
2^{-9}	$1.21 \times 10^{-2}(0.33)$	$1.38 \times 10^{-3}(0.52)$	$2.72 \times 10^{-2}(-0.01)$	$3.98 \times 10^{-2}(0.00)$

teraction kernel, that is the convergence is assured as long as the discretization parameter h vanishes and the cutoff parameter n goes to infinity. On the other hand, convergence of discontinuous piecewise constant finite elements can also be established subject to a condition of $hn = o(1)$ for a class of nonlocal kernels. Intuitively, the rationale behind the condition is that the cutoff should slow enough such that discontinuous piecewise constant finite element methods could be convergent. The numerical analysis is interesting as it also provides an extension to a compactness lemma previously established in [Bourgain *et al.*, 2001].

In addition, to compensate for the lack of analysis on the order of convergence, we carried out numerical experiments of a one dimensional scalar nonlocal diffusion equation discretized with nonconforming discontinuous piecewise linear and piecewise constant finite elements. The discontinuous linear finite element solutions of the nonlocal problem converge to the solution of the correct nonlocal solution as predicted no matter how n varies with h as theoretically predicted, but the convergence rates show dependence on the choices of n and h. In general, the convergence rate increases as n increases faster. In particular, second order convergence for a particular choice of $n = 1/h^4$ is observed. The discontinuous constant finite element solutions are conditionally convergent. In theory, the convergence is assured for $h = o(1/n)$, but the convergence becomes slower as n increases slower. In our particular 1d example, we also observe convergence for $n = 1/h$, but not when $n = 1/h^2, 1/h^4$.

Rigorous studies of the convergence order remains an interesting subject that will be pursued in the future. Similarly, extensions to nonlocal systems may also be feasible as the general framework of chapter 3 on asymptotically compatible schemes have been applied to systems of nonlocal models previously. Such a work would require further extension of the compactness result proved here, much like those given in [Mengesha and Du, 2013; Mengesha and Du, 2014c] for systems associated with a sequence of kernels approaching to Dirac-Delta measures. Connections of the nonconforming DG methods to their local limits as the nonlocal horizon vanishes is another interesting issue both in theory and in practice.

Chapter 5

Nonlocal mechanics model

For the bond-based and state-based peridynamic models and their local limits with shrinking nonlocality, we can also use the framework of chapter 3 to verify the asymptotic compatibility of a finite element scheme. The theoretical foundations that allow us the verify the assumptions posed in chapter can be found [Mengesha and Du, 2014a; Mengesha and Du, 2014c].

In this chapter, we take the state-based peridynamic model as an illustration since it is a more general model than the bond-based model. A state-based peridynamic model was presented in [Silling *et al.*, 2007] as an generalization of bond-based PD models. We refer to [Mengesha and Du, 2014c] for the mathematical analysis. Given the similarity with the nonlocal diffusion models in applying the abstract framework, we omit most of the technical details but emphasize on filling in the necessary ingredients (and references) for verifying all the needed assumptions.

5.1 The state-based peridynamic models

Using the same notations as for the nonlocal diffusion model, we present the peridynamic model as in [Mengesha and Du, 2014c] for a constitutively linear, isotropic solid undergoing deformation. For simplicity, we omit mechanical descriptions and define directly the corresponding bilinear form:

$$B_\delta(\mathbf{u},\mathbf{v}) := \int_{\Omega\cup\Omega_\delta}\left(\left(k(\mathbf{x}) - \frac{\alpha(\mathbf{x})\,m(\mathbf{x})}{d^2}\right)\mathrm{Tr}(\mathcal{D}^*_\omega\mathbf{u})(\mathbf{x})\mathrm{Tr}(\mathcal{D}^*_\omega\mathbf{v})(\mathbf{x}) + \alpha(\mathbf{x})\int_{\Omega\cup\Omega_\delta}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\mathrm{Tr}(\mathcal{D}^*\mathbf{v})(\mathbf{x}',\mathbf{x})d\mathbf{x}'\right)d\mathbf{x} \quad (5.1)$$

where $k(\mathbf{x})$ and $\alpha(\mathbf{x})$ are scalar functions that are closely related to the bulk and shear modulus of the material respectively and γ_δ is a kernel as defined for the nonlocal diffusion model given earlier. The function $m(\mathbf{x})$ is defined as

$$m(\mathbf{x}) = \int_{\Omega\cup\Omega_\delta}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)|\mathbf{x}'-\mathbf{x}|^2 d\mathbf{x}' \,.$$

$\mathrm{Tr}(\mathcal{D}^*)$ is the trace of the nonlocal gradient operator $\mathcal{D}^*$ defined in [Du *et al.*, 2013a]:

$$\mathcal{D}^*\mathbf{u}(\mathbf{x},\mathbf{y}) := (\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\otimes\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\,.$$

$\mathrm{Tr}(\mathcal{D}^*_\omega)$ is the trace of the nonlocal weighted gradient $\mathcal{D}^*_\omega$ defined in [Du *et al.*, 2013a]:

$$\mathcal{D}^*_\omega(\mathbf{u})(\mathbf{x}) := \int_{\Omega\cup\Omega_\delta}\mathcal{D}^*(\mathbf{u})(\mathbf{x},\mathbf{y})\omega(\mathbf{y},\mathbf{x})d\mathbf{y}, \quad \text{where} \;\; \omega(\mathbf{x}',\mathbf{x}) = \frac{d}{m(\mathbf{x})}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)|\mathbf{x}'-\mathbf{x}|.$$

The energy spaces, using the same notation for vector-valued function spaces as for the scalar nonlocal diffusion model, are given by

$$\mathcal{S}_\delta = \left\{\mathbf{u}\in L^2(\Omega\cup\Omega_\delta;\mathbb{R}^d) : \int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{x})}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)\left(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\right)^2 d\mathbf{x}'d\mathbf{x} < \infty, \mathbf{u}|_{\Omega_\delta} = 0\right\}$$

with an inner product

$$(\mathbf{u},\mathbf{v})_{\mathcal{S}_\delta} = \int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{x})}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\mathrm{Tr}(\mathcal{D}^*\mathbf{v})(\mathbf{x}',\mathbf{x})\,d\mathbf{x}'d\mathbf{x}$$

and an induced norm $\|\cdot\|_{\mathcal{S}_\delta}$ where the δ-dependence is due to the kernel γ_δ. Zero extensions to functions in $\mathcal{S}_\delta$ are again assumed as in the scalar case.

By the following uniform Poincaré-type inequality proved in [Mengesha and Du, 2014c, Proposition 3], we know that $(\cdot,\cdot)_{\mathcal{S}_\delta}$ is indeed a well-defined inner product which also induces a well-defined norm $\|\cdot\|_{\mathcal{S}_\delta}$.

Lemma 5.1.1 (Uniform Poincaré inequality)**.** *There exists a constant $C > 0$ independent of δ such that for all $\delta\in(0,1]$,*

$$\|\mathbf{u}\|^2_{L^2(\Omega\cup\Omega_\delta)} \le C\|\mathbf{u}\|^2_{\mathcal{S}_\delta}\,, \quad \forall\mathbf{u}\in\mathcal{S}_\delta\,.$$

Furthermore, by [Mengesha and Du, 2014c, Lemma 3], B_δ is a bounded and coercive bilinear operator on $\mathcal{S}_\delta$, i.e, there exist positive constants C_1 and C_2 independent of δ such that, $\forall \mathbf{u}, \mathbf{v} \in \mathcal{S}_\delta$,

$$B_\delta(\mathbf{u}, \mathbf{v}) \le C_2 \|\mathbf{u}\|_{\mathcal{S}_\delta} \|\mathbf{v}\|_{\mathcal{S}_\delta}, \quad \text{and} \quad B_\delta(\mathbf{u}, \mathbf{u}) \ge C_1 \|\mathbf{u}\|^2_{\mathcal{S}_\delta}\,.$$

Thus B_δ induces the nonlocal peridynamic Navier operator $\mathcal{L}_\delta : \mathcal{S}_\delta \to \mathcal{S}_\delta^*$ which is a bounded linear operator, uniformly in δ, defined by

$$B_\delta(\mathbf{u}, \mathbf{v}) = \langle \mathcal{L}_\delta(\mathbf{u}), \mathbf{v} \rangle \;\; \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{S}_\delta\,. \tag{5.2}$$

We also denote the space $\mathcal{S}_0$ to be

$$\mathcal{S}_0 = \left\{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \int_\Omega |(\nabla \mathbf{u} + \nabla \mathbf{u}^T)(\mathbf{x})|^2 d\mathbf{x} < \infty, \mathbf{u}|_{\partial\Omega} = 0 \right\} \tag{5.3}$$

equipped with a norm equivalent to $\|\cdot\|_{H_0^1(\Omega)}$.

Concerning the local limit of $\mathcal{L}_\delta$, we quote the following result [Mengesha and Du, 2014c, Thoerem 3].

Lemma 5.1.2. *Assume that $k(\mathbf{x})$ and $\alpha(\mathbf{x})$ are smooth functions (say, of the class C^1). Then for $\mathbf{w} \in C_c^\infty(\Omega; \mathbb{R}^d)$, $\mathcal{L}_\delta \mathbf{w}$ is uniformly bounded in $L^\infty(\Omega; \mathbb{R}^d)$, and*

$$\mathcal{L}_\delta \mathbf{w}(\mathbf{x}) \longrightarrow \mathcal{L}_0 \mathbf{w}(\mathbf{x}) \quad \text{as} \;\; \delta \to 0, \quad \forall\, \mathbf{x} \in \Omega,$$

where $\mathcal{L}_0$ is defined by $\mathcal{L}_0 \mathbf{w}(\mathbf{x}) = -\mathrm{div}(\mu(\mathbf{x}) \nabla \mathbf{w}(\mathbf{x})) - \nabla((\mu(\mathbf{x}) + \lambda(\mathbf{x}))\, \mathrm{div}\, \mathbf{w}(\mathbf{x}))$ with $\mu(\mathbf{x}) = \alpha(\mathbf{x})/[d(d+2)]$ and $\lambda(\mathbf{x}) = k(\mathbf{x}) - 2\alpha(\mathbf{x})/[d^2(d+2)]$.

For the given $\mathbf{w}$, combining the above pointwise convergence of $\mathcal{L}_\delta \mathbf{w}$ to $\mathcal{L}_0 \mathbf{w}$ with the uniform boundedness of $\mathcal{L}_\delta \mathbf{w}$, we get $\|\mathcal{L}_\delta \mathbf{w} - \mathcal{L}_0 \mathbf{w}\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$, a result stronger than what is needed later.

5.2 Asymptotically compatible finite element schemes for peridynamic models

As before, we define the spaces $\mathcal{T}_\sigma$ in Assumption 3.1.1 the same way as (4.8) except with $\{\mathcal{S}_\delta\}$ denoting vector valued function spaces associated with the state-based PD model.

We then define $\mathbf{u}_\delta$.

$$\text{Find } \mathbf{u}_\delta \in \mathcal{S}_\delta \text{ such that } B_\delta(\mathbf{u}_\delta, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall v \in \mathcal{S}_\delta \,. \tag{5.4}$$

Similarly, we define the limiting bilinear form on $\mathcal{S}_0$:

$$B_0(\mathbf{u}, \mathbf{v}) := \langle \mathcal{L}_0 \mathbf{u}, \mathbf{v} \rangle \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{S}_0 \,.$$

It is well-known that B_0 is bounded and coercive on $\mathcal{S}_0$ [Ciarlet, 1988; Duvaut and Lions, 1976]. We set $a_\sigma := B_{1/\sigma}$ and $\mathcal{A}_\sigma := \mathcal{L}_{1/\sigma}$ for $\sigma \in [1, \infty]$. Then one part of Assumption 3.1.1 i) is given by lemma 5.1.1 while the other is precisely [Mengesha and Du, 2014a, Lemma 2.2] restated as a lemma below.

Lemma 5.2.1. *There exists a constant $C > 0$ only depending on Ω such that*

$$\|\mathbf{u}\|^2_{\mathcal{S}_\delta} \leq C \Big(\int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2 \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} \Big) \|\mathbf{u}\|^2_{H^1(\Omega)} \,, \quad \forall \mathbf{u} \in H^1(\Omega) \cap \mathcal{S}_\delta \,.$$

Assumption 3.1.1 ii) is just [Mengesha and Du, 2014c, Lemma 7] which is restated below without proof.

Lemma 5.2.2. *Let $\mathbf{u}_\delta \in \mathcal{S}_\delta$ for $\delta > 0$. If $\sup_{\delta > 0} \|\mathbf{u}_\delta\|_{\mathcal{S}_\delta} < \infty$, then the sequence $(\mathbf{u}_\delta)$ is precompact in $L^2(\Omega; \mathbb{R}^d)$. Moreover, any limit point $\mathbf{u} \in \mathcal{S}_0$.*

Meanwhile, the discussions in the previous subsection easily lead to Assumption 3.1.2 and Assumption 3.1.3 in the present context.

For discrete approximations, as in the nonlocal diffusion case, let $\{V_{\delta,h}\} \subset \mathcal{S}_\delta$ denote a family of finite element subspaces where h characterizes the mesh size and for any $\mathbf{v} \in \mathcal{S}_\delta$, we have a family of elements $\{\mathbf{v}_h \in V_{\delta,h}\}$ such that $\|\mathbf{v}_h - \mathbf{v}\|_{\mathcal{S}_\delta} \to 0$ as $h \to 0$. Then, the Galerkin approximation is to replace $\mathcal{S}_\delta$ by $V_{\delta,h}$ in (5.4), namely,

$$\text{Find } \mathbf{u}_{\delta,h} \in V_{\delta,h} \text{ such that } B_\delta(\mathbf{u}_{\delta,h}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in V_{\delta,h} \,. \tag{5.5}$$

Clearly, $W_{\sigma,h} := V_{1/\sigma,h}$ satisfies Assumption 3.1.4 i). The Assumption 3.1.4 ii) is also satisfied with $\hat{V}_{\delta,h} \subset V_{\delta,h}$ and $\hat{V}_{\delta,h}$ being a vector valued version of the continuous piecewise linear element subspace which approximates $\mathcal{S}_0 = \mathcal{T}_\infty$ as $h \to 0$.

Now we are ready to state the convergence theorem on the finite element approximations of the linear state-based peridynamic model, as a direct consequence of Theorem 3.3.2. We skip the detailed proof.

Theorem 5.2.3. *Let* $\mathbf{u}_\delta$, $\mathbf{u}_{\delta,h}$ *be the solutions of* (5.4) *and* (5.5), *and* $\hat{V}_{\delta,h} \subset \mathcal{S}_\delta$ *is described as in the above. If* $\hat{V}_{\delta,h} \subset V_{\delta,h}$, *then* $\|\mathbf{u}_{\delta,h} - \mathbf{u}_0\|_{L^2} \to 0$ *as* $\delta \to 0$, $h \to 0$.

Consequently, we see also that for the state-based PD models, the asymptotic compatibility is preserved for conforming finite element approximations that contain continuous piecewise linear finite element subspaces.

By extending the convergence of the discrete linear forms from the nonlocal diffusion models to the state-based PD models, we can also get similar results on the convergence of the discrete solutions between the PD models and the local Navier equations as $\delta \to 0$ on a fixed mesh.

5.3 Conclusion

In this chapter, we considered the asymptotically compatible schemes for nonlocal state-based peridynamic system with Dirichlet type volumetric constraint using the framework developed in chapter 3. It is an extension to the result presented at section 4.1. We showed that for the vector valued case of the peridynamic system, similar results hold that any finite element discretization that contains piecewise linear functions provides an asymptotically compatible scheme. In other words, the convergence of approximations to the correct solutions and models is assured independent of the relations between the horizon parameter δ and the discretization parameter h as shown in the diagram 3.1. To verify the assumptions presented in chapter 3, we mainly resorted to the analysis results in [Mengesha and Du, 2014a; Mengesha and Du, 2014c], which are vector case generalizations of the work of [Bourgain *et al.*, 2001].

Finally, we note that our study is restricted to conforming approximations and linear problems. More studies are underway to extend them to other varieties of approximation methods including particle-based or meshfree methods and also to nonlinear and multiscale settings.

Chapter 6

Discussions and other related algorithmic works

6.1 Summary of the algorithmic works

In this part, we discussed asymptotically compatible (AC) schemes as robust algorithms for approximating nonlocal models and their various asymptotic limits. Begin from chapter 2, in which several popular schemes for a one-dimensional nonlocal diffusion model have been analyzed and compared, we clearly pointed out the potential risks involved in solving nonlocal problems when discretization parameters are tied with modeling parameters, thus motivating the study of AC schemes proposed in chapter 3. Section 4.1 and 4.2 made studies on the approximations of nonlocal diffusion models (associated with the nonlocal operator $\mathcal{L}_\delta$) and their local diffusion models (associated with the Laplacian operator $\mathcal{L}_0 = \Delta$ as $\delta \to 0$) and fractional diffusion models (associated with the fractional Laplacian operator $\mathcal{L}_\infty = -(-\Delta)^\alpha$ as $\delta \to \infty$), as illustrated in Fig. 6.1.

Furthermore, utilizing the framework of AC schemes in chapter 3, section 4.3 discussed a nonconforming discontinuous Galerkin finite element scheme for nonlocal diffusion models based on the idea of approximating nonlocal problem associated with a singular kernel by a sequence of nonlocal problems associated more regular kernels. From there, we have actually established a complete theory for the asymptotic compatibility of numerical schemes with respect to any limiting process $\delta \to \delta_* \in [0, \infty]$. Thus we can advocate asymptotically

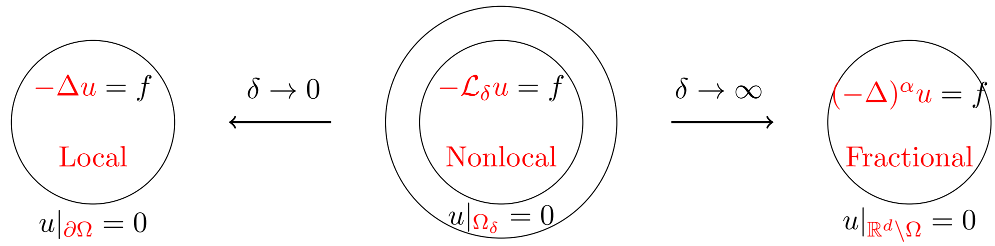

Figure 6.1: Different limits of nonlocal diffusion equations: partial differential equations as local limits ($\delta \to \infty$) and fractional Laplacian equations as global limits ($\lambda \to \infty$).

compatible schemes as robust algorithms for approximating local, nonlocal and fractional models in the sense that the convergence to the correct continuum limit is assured for any values of the parameter δ and in any of its limiting regimes as the numerical resolution is increased. Moreover, the framework of AC schemes can also be applied to the bond-based and state-based peridynamic systems as illustrated in chapter 5.

Until now, we have discussed quadrature based finite difference method and finite element method in solving nonlocal problems, other numerical methods such as Fourier spectral method are also possible as discussed in [Du and Yang, 2016]. Moreover, note that the framework in chapter 3 does not provide us with error estimates in terms of both the discretization parameter and modeling parameter, we thus discuss here the Fourier analysis approach to obtain error estimates as an example.

6.2 Fourier analysis for error estimates

We developed the variational framework to identify asymptotically compatible schemes in chapter 3 and saw its powerfulness in the previous chapters. However, the variational approach does not give us error estimate in terms of both δ and h. To compensate the lack of error estimate, we use Fourier analysis for a problem defined on a one-dimensional periodic cell $\Omega = (0, 1)$ in this section.

We consider again the nonlocal operator $\mathcal{L}_\delta$ given by (1.8). We adopt the quadrature collocation scheme (2.9) with $\alpha = 1$, namely,

$$\begin{aligned} -\mathcal{L}_{\delta,1}^h u_i = &-\sum_{m=1}^{r} \frac{u_{i-m} - 2u_i + u_{i+m}}{(mh)} \int_{I_m \cup I_{m+1}} \phi_m^1(s) s\gamma_\delta(s) ds \\ &- \frac{(u_{i-r-1} - 2u_i + u_{i+r+1})}{(r+1)h} \int_{I_{r+1}} \phi_{r+1}^1(s) s\gamma_\delta(s) ds = f_i \,, \end{aligned} \tag{6.1}$$

where $I_j = ((j-1)h, jh)$ for $1 \le j \le r$, and $I_{r+1} = (rh, \delta)$.

Let us use $\mathbb{A}_\delta^h$ to denote the quadrature collocation matrix associated with horizon δ and mesh spacing h and let $\mathbb{A}^h$ correspond to the coefficient (stiffness) matrix associated with the discretization of local operator. The stiffness matrix $\mathbb{A}_\delta^h$ is a symmetric Toeplitz matrix with first row

$$\mathbb{A}_\delta^h(1,:) = -(b_0, b_1, \cdots, b_{r+1}, 0, \cdots, 0, b_{r+1}, b_r, \cdots, b_1)\,,$$

where

$$-b_m := \begin{cases} \frac{2}{h}\sum_{k=1}^{r} \int_{I_k \cup I_{k+1}} \frac{\phi_k^1(s)}{k} s\gamma_\delta(s) ds + \frac{4}{h}\int_{I_{r+1}} \frac{\phi_{r+1}^1(s)}{r+1} s\gamma_\delta(s) ds, & m = 0 \\ -\frac{1}{mh}\int_{I_m \cup I_{m+1}} \phi_m^1(s) s\gamma_\delta(s) ds, & 1 \le m \le r \\ -\frac{1}{(r+1)h}\int_{I_{r+1}} \phi_{r+1}^1(s) s\gamma_\delta(s) ds, & m = r+1 \\ 0, & \text{otherwise.} \end{cases} \tag{6.2}$$

Suppose the number of grid points is $2N$ on the interval $[0, 1]$ with spacing $h = \frac{1}{2N}$. Now for a periodic array $\{u_j\}$ defined on the grid, we use the *discrete Fourier transform*: for $j = 1, 2, \cdots, 2N$, and $n = -N+1, \cdots, 1, \cdots, N$,

$$u_j = \frac{1}{2\pi}\sum_{n=-N+1}^{N} e^{i2\pi n x_j}\hat{u}_n \qquad \text{and} \qquad \hat{u}_n = h\sum_{j=1}^{2N} e^{-i2\pi n x_j} u_j\,.$$

Let us take $\delta = rh$ for an integer $r \ge 1$ for simplicity right now. Let $U_\delta^N = (u_1, u_2, ..., u_N)^T$ and $F_N = (f_1, f_2, ..., f_N)^T$. For any $\eta \in \mathbb{R}$, we take $\|\cdot\|_\eta$ as the discrete H^η norm and $|\cdot|_\eta$ as the discrete semi-H^η norm defined by

$$|U_\delta^N|_\eta^2 = h\sum_{j=1}^{2N} |n|^{2\eta}|u_j|^2 = \frac{1}{2\pi}\sum_{\substack{n=-N+1 \\ n\neq 0}}^{N} |n|^{2\eta}|\hat{u}_n|^2 \quad \text{and} \quad \|U_\delta^N\|_\eta^2 = |U_\delta^N|_\eta^2 + \|U_\delta^N\|_0^2$$

with $\|\cdot\|_0$ being the standard L^2 norm.

First, $\{(e^{i2\pi n x_1}, e^{i2\pi n x_2}, \cdots, e^{i2\pi n x_N})^T\}$ forms an eigenbasis of the discrete operators $\mathbb{A}_\delta^h$ and $\mathbb{A}^h$ with eigenvalues given respectively by

$$\lambda_\delta^h(n) = 2\sum_{j=1}^{r} b_j(1-\cos(2\pi n j h)), \quad \lambda^h(n) = \frac{2}{h^2}(1-\cos(2\pi n h)) \,.$$

for integer n between 1 and N where b_j's are given as in (6.2).

Now denote u_0 and U_0^N to be the solution to the local PDE and its discrete approximation respectively. Our is to estimate the error of $\|U_\delta^N - u_0\|_0$. We approach it through the triangle inequality,

$$\|U_\delta^N - u_0\|_0 \le \|U_\delta^N - U_0^N\|_0 + \|U_0^N - u_0\|_0 \,.$$

Since $\|U_0^N - u_0\|_0$ can be obtained by a standard error estimate for a local PDE problem, we only need to get a uniform (with respect to δ) error estimate for the term $\|U_\delta^N - U_0^N\|_0$ in order to verify the asymptotic compatibility.

Theorem 6.2.1. *Let $\mathbb{A}_\delta^h U_\delta^N = F_N$ and $\mathbb{A}^h U_0^N = F_N$, then*

$$\|U_\delta^N - U_0^N\|_0 \le C\delta^2 \|F_N\|_0 \,, \tag{6.3}$$

for a constant C independent of δ, N and F_N.

Proof. It is easy to see that $U_\delta^N - U_0^N = ((\mathbb{A}_\delta^h)^{-1} - (\mathbb{A}^h)^{-1})F_N$. So,

$$\|U_\delta^N - U_0^N\|_0^2 = \frac{1}{2\pi} \sum_{\substack{n=-N+1 \\ n\neq 0}}^{N} \left|\frac{1}{\lambda_\delta^h(n)} - \frac{1}{\lambda^h(n)}\right|^2 |\hat{f}_n|^2 \,.$$

Thus, (6.3) follows from

$$\frac{1}{\delta^2}\left|\frac{1}{\lambda_\delta^h(n)} - \frac{1}{\lambda^h(n)}\right| \le C \qquad \text{for } 1 \le |n| \le N \,, \tag{6.4}$$

which is shown in the Lemma 6.2.2. □

Lemma 6.2.2. *There exists a positive constant C independent of N, h and δ, such that*

$$\frac{1}{\delta^2}\left|\frac{1}{\lambda_\delta^h(n)} - \frac{1}{\lambda^h(n)}\right| \le C \qquad \text{for } 1 \le |n| \le N \,.$$

Proof. First, $\mathbb{A}_\delta^h$ is exactly $\mathbb{A}^h$ when $\delta \le h$ (Chapter 2), so we only show the result for $\delta > h$.

For $\delta|n| \le 1/2$, by $\theta^2 - \theta^4/12 \le 2(1-\cos(\theta)) \le \theta^2$ and the fact that

$$\sum_{j=1}^{r} b_j (jh)^2 = 1 \quad \text{and} \quad \sum_{j=1}^{r} b_j (jh)^4 \le \delta^2 \sum_{j=1}^{r} b_j (jh)^2 = \delta^2 ,$$

we can show that

$$0 < (2\pi n\delta)^2 - \frac{(2\pi n\delta)^4}{12} \le \delta^2 \lambda_\delta^h(n) \le (2\pi n\delta)^2 ,$$

$$0 < (2\pi n\delta)^2 - \frac{(2\pi n\delta)^4}{12} \le (2\pi n\delta)^2 - \frac{(2\pi n)^4 \delta^2 h^2}{12} \le \delta^2 \lambda^h(n) \le (2\pi n\delta)^2 .$$

Therefore,

$$\frac{1}{\delta^2}\left| \frac{1}{\lambda_\delta^h(n)} - \frac{1}{\lambda^h(n)} \right| \le \frac{1}{(2\pi n\delta)^2 - \frac{(2\pi n\delta)^4}{12}} - \frac{1}{(2\pi n\delta)^2} \le \frac{1}{12-\pi^2} .$$

Now when $\delta|n| > 1/2$, let us show that both $\frac{1}{\delta^2 \lambda_\delta^h(n)}$ and $\frac{1}{\delta^2 \lambda^h(n)}$ are uniformly bounded above. For the former, using $|n|h \le Nh = 1/2$, the bound follows since we have

$$\delta^2 \lambda^h(n) \ge (2\pi n\delta)^2 (1 - \frac{(2\pi nh)^2}{12}) \ge \pi^2 (1 - \frac{\pi^2}{12}) .$$

Considering $\delta^2 \lambda_\delta^h(n)$, we split the sum in the expression for $\lambda_\delta^h(n)$ into two parts:

$$\begin{aligned} \delta^2 \sum_{j=1}^{4njh \le 1} b_j (1 - \cos(2\pi njh)) &\ge ((2\pi n\delta)^2 - \frac{(2\pi n)^4 \delta^2}{12(4n)^2}) \sum_{j=1}^{4njh \le 1} b_j (jh)^2 \\ &\ge \pi^2 (1 - \frac{\pi^2}{48}) \sum_{j=1}^{4njh \le 1} b_j (jh)^2 . \end{aligned}$$

We can see that tthere exists a constant $c_0 > 0$ such that

$$\delta^2 \sum_{4njh>1}^{r} b_j (1 - \cos(2\pi njh)) \ge c_0 \delta^2 \sum_{4njh>1}^{r} b_j \ge c_0 \sum_{4njh>1}^{r} b_j (jh)^2 .$$

Therefore, the lemma is valid for all n. □

6.3 Future works and related problems

Many interesting topics that concern with asymptotically compatible schemes can be addressed in the future including different numerical methods and different models. For numerical methods, while the theory for finite element methods for nonlocal models have been understood quite well, the study of finite difference methods has been largely confined to one-dimensional setting. Besides, particle-based or meshfree methods can also be considered given their popularity in practice especially in simulating the high dimensional problems. More models can also be studied such as nonlocal convection-diffusion models [Du *et al.*, 2014], time-dependent equations [Deng and Hesthaven, 2013; Meerschaert *et al.*, 1999; Mustapha and McLean, 2013; Tadjeran *et al.*, 2006; Xu and Hesthaven, 2014; Yang *et al.*, 2011] as well as nonlinear and multiscale models that will be mentioned in the rest of this thesis.

To conclude, we give some interesting connections of the study of asymptotically compatible schemes to many other active research areas of applied mathematics. Just to give a few examples here. First, a class of numerical methods emerged in computational fluid dynamics are called RKPM/SPH [Du *et al.*, 2015]. These methods are closely related to issues we have studied since they often incorporate nonlocality into the approximation functions via a *smoothing length* (as in SPH for CFD) encoded in some weight kernel, then discretize the regularized model with the introduced nonlocality. Second, our problem has connection to continuum limit of various discrete problems such as those in graphs, networks and lattices, which often appear in data analysis or atomistic materials models. These problems encounters naturally a length scale h which is the space between two nearest points and a possible nonlocal interaction length scale which measures the number of interaction neighbors. A particular example is the recent analysis work [Trillos and Slepčev, 2016] concerning data clouds and graph partitioning that is reminiscent to the conditional convergence criterion for a discrete scheme that fails to be AC. Such an analysis considers the Γ-limit as the number of data points increases and concludes that the number of points has to increase faster than a certain rate related to the scale of the vicinity having nonlocal interactions. A less expected example is the connection to numerical solutions to nonlinear problems. Recently, in the field of numerical methods for Monge-Ampère equation, there are discussions about comparison of wide-stencil method (with more nonlocality involved) and narrow-stencil method [Oberman, 2008], similar to our discussion of conditionally AC schemes that have to deal with dense matrices and AC schemes that could end up with very sparse systems. Indeed, nonlocality is ubiquitous. Besides mechanics and materials science, models involving nonlocal features have also emerged in various other fields of applied mathematics, such as geometric descriptions of data sets [Bartholdi *et al.*, 2012; Coifman and Lafon, 2006] and mathematical biology [Massaccesi and Valdinoci, 2016]. The study of the various asymptotic limits of those models could also be related to what we have presented in this thesis.

Part II

Mathematical analysis for nonlocal models

Chapter 7

Extensions of Bourgain-Brezis-Mironescu theorem

The seminal work of Bourgain, Brezis and Mironescu [Bourgain *et al.*, 2001] explored a nonlocal characterization of the Sobolev space H^1 as the approximation of a sequence of nonlocal functionals living in spaces less regular than H^1 (e.g. the L^2 space) and the sequence is shown to have certain compactness asymptotically in the space L^2. The work provides an important tool for study the asymptotically compatible schemes for nonlocal diffusion in section 4.1 and is also extended to vector fields and applied on the nonlocal mechanics model in chapter 5. In this chapter, we extend the work to characterize more general spaces that can be used as theoretical foundation for more nonlocal problems. We will discuss new characterizations of nonlocal space that lies between H^1 and L^2 (see section 7.1) and high order Sobolev space H^k (see section 7.2) by sequences of nonlocal functionals that enjoy certain asymptotic compatibility. The first result is already used to study the nonconforming DG schemes for nonlocal diffusion model in section 4.3. The second result is useful in studying models that involve higher order nonlocal energies.

7.1 A new compactness result in spirit of Bourgain-Brezis-Mironescu

7.1.1 Introduction

We define as usual the nonlocal function space associated with a kernel γ_δ to be

$$\mathcal{S}_\delta = \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x})-u(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} < \infty \right\}.$$

The work of Bourgain-Brezis-Mironescu shows that if the kernel $|\mathbf{x}|^2\gamma_\delta(|\mathbf{x}|)$ approximates the Dirac-Delta measure as $\delta \to 0$, then the space $\mathcal{S}_\delta$ approximates the H^1 space and moreover, we expect asymptotic compactness of $\{\mathcal{S}_\delta\}_{\delta\to 0}$ into the L^2 space.

In this section, we extend the result to the more general case that $|\mathbf{x}|^2\gamma_\delta(|\mathbf{x}|)$ is approximating some function $|\mathbf{x}|^2\gamma(|\mathbf{x}|)$. Thus the space $\mathcal{S}_\delta$ is approximating another nonlocal function space $\mathcal{S}$ with kernel γ. Such extension allows for more general limits which can be seen as a new characterizations of nonlocal energy spaces.

The kernel γ are under the assumption that are nonnegative and non-increasing with finite second moment. Moreover, we assume in addition that

$$\lim_{\epsilon\to 0} \epsilon^2 \left(\int_{|\mathbf{x}|<\epsilon} |\mathbf{x}|^2 \gamma(|\mathbf{x}|) d\mathbf{x} \right)^{-1} = 0\,. \tag{7.1}$$

We note that by choosing $\gamma(|\mathbf{x}|)$ to be the special kernel $1/|\mathbf{x}|^{d+2s}$, the corresponding energy space $\mathcal{S}$ is equivalent to the fractional Sobolev space H^s. Thus, our result also recovers the analog of the compactness result in [Bourgain *et al.*, 2001] for standard fractional spaces which can be applicable to other studies of variational problems associated with fractional PDEs. We refer to [Mengesha and Du, 2013; Mengesha and Du, 2014a; Ponce, 2004] for more discussions on applications and generalizations of the compactness result of [Bourgain *et al.*, 2001] to nonlocal problems involving spaces of either scalar-valued or vector-valued functions.

The new compactness result is interesting by itself and is also a crucial theoretical establishment for applications to numerical analysis for steady-state nonlocal diffusion models (see chapter 4.3).

7.1.2 The compactness theorem

We now define γ_n to be a sequence of nonnegative and nonincreasing kernel functions that approximate γ in a monotone way, namely

$$\gamma(r) = \lim_{n\to\infty} \gamma_n(r) \quad \text{and } \gamma_n \le \gamma_{n+1}\,. \tag{7.2}$$

The assumption is essentially used to guarantee that the integrals of $|\mathbf{x}|^2\gamma_n(|\mathbf{x}|)$ also converges to the integrals of $|\mathbf{x}|^2\gamma(|\mathbf{x}|)$. Now we denote $\mathcal{S}_n$ to be the nonlocal function space corresponding to γ_n, then the following compactness result holds.

Theorem 7.1.1. *Given the kernels γ_n, γ and the energy spaces $\mathcal{S}_n$, $\mathcal{S}$ as defined, suppose that $\{v_n\}$ is a bounded sequence in $L^2(\Omega)$ and the energy norms of $\{v_n \in \mathcal{S}_n\}$ have a uniform bound*

$$\sup_n \int_\Omega\int_\Omega \gamma_n(|\mathbf{y}-\mathbf{x}|)(v_n(\mathbf{y})-v_n(\mathbf{x}))^2 d\mathbf{y}d\mathbf{x} \le C_0,$$

then $\{v_n\}$ is relatively compact in $L^2(\Omega)$ and any of its limit point v is in $\mathcal{S}$ with

$$\int_\Omega\int_\Omega \gamma(|\mathbf{y}-\mathbf{x}|)(v(\mathbf{y})-v(\mathbf{x}))^2 d\mathbf{x}d\mathbf{y} \le C_0\,. \tag{7.3}$$

To prove the new compactness result, we need some estimates on the kernels.

Lemma 7.1.2. *Let γ satisfy* (7.1) *and γ_n be defined by* (7.2)*. Then,*

$$\lim_{\delta\to 0}\lim_{n\to\infty} \delta^2 \left(\int_{|\mathbf{x}|<\delta} |\mathbf{x}|^2\gamma_n(|\mathbf{x}|)d\mathbf{x}\right)^{-1} = 0\,. \tag{7.4}$$

Proof. By the assumptions of of γ_n, we have the pointwise convergence of $|\mathbf{x}|^2\gamma_n(|\mathbf{x}|)$ to $|\mathbf{x}|^2\gamma(|\mathbf{x}|)$ for $|\mathbf{x}| < \delta$ and the uniform bound of the second order moments. Thus, for each $\delta > 0$, there exists an integer n_δ such that

$$\int_{|\mathbf{x}|<\delta} |\mathbf{x}|^2\gamma_n(|\mathbf{x}|)d\mathbf{x} \ge \frac{1}{2}\int_{|\mathbf{x}|<\delta} |\mathbf{x}|^2\gamma(|\mathbf{x}|)d\mathbf{x}\,, \quad \text{for } n \ge n_\delta\,.$$

We get, by (7.1), the result in (7.4) which proves the lemma. □

The next inequality from [Bourgain *et al.*, 2001, Lemma 2] is quoted here without proof.

Lemma 7.1.3. *Let $g, G : (0, \delta) \to \mathbb{R}_+$. Assume $g(t) \leq g(t/2), t \in (0, \delta)$, and that G is non-increasing. Then, for some $C = C(N) > 0$,*

$$\int_0^\delta t^{N-1} g(t) G(t) \geq C\delta^{-N} \int_0^\delta t^{N-1} g(t) dt \int_0^\delta t^{N-1} G(t) dt.$$

Now we are ready to prove Theorem 7.1.1.

proof of Thoerem 7.1.1. The proof consists of two parts. The first part is to show that $\{v_n\}$ is relatively compact in L^2. The second part is showing that the limit point is in $\mathcal{S}$ satisfying (7.3).

To show the first part, following the remarks in [Bourgain *et al.*, 2001], we may assume without loss of generality that $\Omega = \mathbb{R}^d$ and $\mathrm{Supp}(v_n) \subset B$, the unit ball in $\mathbb{R}^d$. We can prove the compactness by applying a variant of Riesz-Fréchet-Kolmogorov theorem. First, set $\Phi_\delta := \frac{1}{|B_\delta|}\chi_{B_\delta}$, then $\{v_n\}$ is relatively compact in L^2 if and only if

$$\|v_n\|_{L^2} \leq C$$

and

$$\lim_{\delta \to 0}(\limsup_{n \to \infty} \|v_n - v_n * \Phi_\delta\|_{L^2}) = 0\,. \tag{7.5}$$

Denoting

$$\begin{aligned} F_n(\tau) &= \int_{\mathbf{w} \in S^{d-1}} \int_{\mathbb{R}^d} |(v_n(\mathbf{x} + \tau\mathbf{w}) - v_n(\mathbf{x}))|^2 d\mathbf{x} d\sigma \\ &= \frac{1}{\tau^{d-1}} \int_{|\mathbf{h}|=\tau} \int_{\mathbb{R}^d} |v_n(\mathbf{x} + \mathbf{h}) - v_n(\mathbf{x})|^2 d\mathbf{x} d\sigma\,, \end{aligned}$$

we can rewrite the energy norm as

$$\int_0^1 \tau^{d-1} F_n(\tau) \gamma_n(\tau) d\tau = \int_0^1 \tau^{d+1} \frac{F_n(\tau)}{\tau^2} \gamma_n(\tau) d\tau \leq C_0\,.$$

The function $F_n(\tau)$ has many properties as stated in [Bourgain *et al.*, 2001], in particular, we have

$$F_n(2\tau) \leq 2^2 F_n(\tau)\,.$$

Applying Lemma 7.1.3 with $g(\tau) = \tau^{-2} F_n(\tau)$, $G(\tau) = \gamma_n(\tau)$ and $N = d + 2$, we obtain

$$\delta^{-d-2} \int_0^\delta \tau^{d+1} \frac{F_n(\tau)}{\tau^2} d\tau \leq C \int_0^\delta \tau^{d+1} \frac{F_n(\tau)}{\tau^2} \gamma_n(\tau) d\tau \Big/ \int_{|\mathbf{x}|<\delta} |\mathbf{x}|^2 \gamma_n(|\mathbf{x}|) d\mathbf{x}\,.$$

By the assumption of the uniform boundedness of the energy norms,

$$\int_0^\delta \tau^{d-1} F_n(\tau)\gamma_n(\tau)d\tau$$

is a uniformly bounded quantity. Combining with Lemma 7.1.2, we have

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \delta^{-d}\int_0^\delta \tau^{d-1}F_n(\tau)d\tau = 0\,.$$

Then, as in [Bourgain *et al.*, 2001], (7.5) is true since we have

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^d} |v_n(\mathbf{x}) - (v_n * \Phi_\delta)(\mathbf{x})|^2 d\mathbf{x} = 0\,.$$

The relative compactness of $\{v_n\}$ is thus established.

Now we are ready to prove the second part of the theorem, namely, any limit point $v \in \mathcal{S}$ with (7.3) holds. We can assume, without loss of generality, that $v_n \to v$ in $L^2(\Omega)$. Then $v_n(\mathbf{x})$ converges to $v(\mathbf{x})$ pointwise for a.e. $\mathbf{x} \in \Omega$. As a consequence,

$$\gamma_n(|\mathbf{y}-\mathbf{x}|)(v_n(|\mathbf{y}|) - v_n(|\mathbf{x}|))^2 \to \gamma(|\mathbf{y}-\mathbf{x}|)(v(|\mathbf{y}|) - v(|\mathbf{x}|))^2 \tag{7.6}$$

up to a set of measure zero in $\Omega \times \Omega$. Then by Fatou's lemma we obtain

$$\begin{aligned}\int_\Omega\int_\Omega &\gamma(|\mathbf{y}-\mathbf{x}|)(v(|\mathbf{y}|) - v(|\mathbf{x}|))^2 d\mathbf{y}d\mathbf{x} \\ &\le \liminf_n \int_\Omega\int_\Omega \gamma_n(|\mathbf{y}-\mathbf{x}|)(v_n(|\mathbf{y}|) - v_n(|\mathbf{x}|))^2 d\mathbf{y}d\mathbf{x} \le C_0\,.\end{aligned}$$

□

We note that, in establishing the compactness result in [Bourgain *et al.*, 2001], $\hat{\gamma}(r) = r^{d+1}\gamma(r)$ is required to be non-increasing. Here we made variations in the proof so that it is applicable to a less restrictive condition on γ being non-increasing. This minor change is crucial to this study because we now can apply the result to much more general kernels with less singularity at zero. Such observations on more lenient conditions on the kernel have been noticed in the literature [Mengesha and Du, 2014c; Ponce, 2004]. Most notably, we point out that [Ponce, 2004] provided a very general argument for the original compactness result that works for $d \ge 2$ without the assumption on γ being non-increasing. Here, we do not go into such extensions as the result stated above is general enough for our objective.

7.1.3 Conclusion

The work of [Bourgain *et al.*, 2001] explored a new characterization of the Sobolev space H^1 as the approximation of a sequence of nonlocal function spaces and the sequence is shown to have certain compactness asymptotically in the space L^2. In this section, our extended result allows us to approximate a nonlocal function space with a sequence of less regular nonlocal spaces and achieve asymptotic compactness in the space of L^2. The analytical result in this section serves as a useful new tool to characterize nonlocal spaces such that results like the DG scheme developed in section 4.3 can be obtained. Finally, we note that this work considers only scalar-valued functions. Vector-valued generalizations that can be applied to the peridynamic system are also possible following the studies in [Mengesha and Du, 2014a; Mengesha and Du, 2014c].

7.2 High order nonlocal operators

7.2.1 Introduction

We are concerned with the following class of high order nonlocal energy functionals for a scalar function $u : \mathbb{R}^d \to R$,

$$E_n(u) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \gamma_n(|s|)|D_n^s[u](x)|^2 ds dx\,, \quad n \in \mathbb{N}\,, \tag{7.7}$$

where D_n^s denotes the n-th difference operator acting on any function u: $\mathbb{R}^d \to \mathbb{R}$ given by

$$D_n^s[u](x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} u(x + a_n^j s) \quad \text{where} \quad \binom{n}{j} = \frac{n!}{j!(n-j)!} \tag{7.8}$$

and

$$\begin{cases} a_n^j = \dfrac{n+1}{2} - j & \text{if } n \text{ is odd, or} \\ a_n^j = \dfrac{n}{2} - j & \text{if } n \text{ is even.} \end{cases} \tag{7.9}$$

The kernel functions $\{\gamma_n\}$ are assumed to satisfy

$$(K) \qquad \begin{cases} \gamma_n \text{ is nonnegative, compactly supported, } |x|^{2n}\gamma_n(|x|) \in L^1_{loc}(\mathbb{R}^d), \\ \text{and there exist a constant } \eta > 0, \text{ such that } B_\eta(0) \subset \text{Supp}\{\gamma_n(|\cdot|)\}\,. \end{cases}$$

for $n \in \mathbb{N}$.

With the kernels and difference operators defined, we can work on high order nonlocal function spaces given by

$$\mathcal{S}^{n,\gamma_n} = \{u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \gamma_n(|s|)|D_n^s[u](x)|^2 ds dx < \infty\}, \tag{7.10}$$

and its constrained space $\mathcal{S}_\Omega^{n,\gamma_n}$ defined to the closure of $C_c^\infty(\Omega)$ in $\mathcal{S}^{n,\gamma_n}$.

Our main effort is to investigate Sobolev type properties on the spaces $\mathcal{S}^{n,\gamma_n}$, including Poincaré inequality, Gagliardo-Nirenberg inequality, and compact embeddings. The main theorems are summarized below with details explained in the next few sections. Our theoretical results are applied to the peridynamic beams and plates model in the end.

Theorem 7.2.1 (1st nonlocal Poincaré inequality). *For $n \in \mathbb{Z}^+$ and kernel γ_n satisfying (K), there exists $C = C(n, \gamma_n, \Omega)$ such that*

$$\|u\|_{L^2} \leq C|u|_{\mathcal{S}^n} \quad \forall u \in \mathcal{S}_\Omega^n .$$

Theorem 7.2.2 (2nd nonlocal Poincaré inequality). *For $n \in \mathbb{Z}^+$ and kernels $\{\gamma_n\}$ satisfying (K), (7.19) and (7.20), there exists $C = C(n, \gamma_n, \gamma_{n+1}, \Omega)$ such that*

$$\|u\|_{\mathcal{S}^n} \leq C|u|_{\mathcal{S}^{n+1}} \quad \forall u \in \mathcal{S}_\Omega^{n+1} .$$

Theorem 7.2.3 (Nonlocal Gagliardo-Nirenberg inequality). *For nonnegative integers n, k_1 and k_2 with $k_1 + k_2 > 0$, suppose that γ_n satisfies (K), and $\alpha = k_1/(k_1 + k_2)$, the following nonlocal Gagliardo-Nirenberg type inequality holds*

$$|u|_{\mathcal{S}^n} \leq |u|_{\mathcal{S}^{(n-k_1)}}^{1-\alpha} |u|_{\mathcal{S}^{(n+k_2)}}^{\alpha} ,$$

where $\mathcal{S}^{(n-k_1)} = \mathcal{S}^{n-k_1,\gamma_{(n-k_1)}}$ and $\mathcal{S}^{(n+k_2)} = \mathcal{S}^{n+k_2,\gamma_{(n+k_2)}}$ with properly chosen kernels that, in particular, are given by $\gamma_{(n-k_1)} = (\gamma_n)^{1-k_1/n}$ and $\gamma_{(n+k_2)} = (\gamma_n)^{1+k_2/n}$.

Theorem 7.2.4 (Compact embedding). *Suppose that the kernels $\gamma_1, \gamma_n, \gamma_{n+1}$ satisfy (7.19), (7.20) and (7.26). Let $\mathcal{F}$ be a bounded set in $\mathcal{S}_\Omega^n$. If*

$$|u|_{\mathcal{S}^{n+1}} \leq C_0 \quad \forall u \in \mathcal{F} ,$$

then $\mathcal{F}$ is precompact in $\mathcal{S}_\Omega^n$ and any of its limit point is in $\mathcal{S}_\Omega^{n+1}$ with a norm bounded by C_0.

Theorem 7.2.5 (A variant of compact embedding theorem)**.** *Suppose that the kernels* $\gamma_1, \gamma_n, \gamma_{n+1}$ *satisfy assumptions* (7.19)-(7.20)*. If* (u_k) *is a bounded sequence in* $\mathcal{S}^n_\Omega$*, and*

$$|u_k|_{\mathcal{S}^{n+1}} \to 0 \quad \text{as } k \to 0\,,$$

then (u_k) *is relatively compact in* $\mathcal{S}^n_\Omega$ *and any of its limit point* u *is in* $\mathcal{S}^{n+1}_\Omega$ *with* $|u|_{\mathcal{S}^{n+1}} = 0$*.*

Now we consider a fixed integer n, and study the a family of kernels γ_n^δ parametrized by δ that characterizes the nonlocal interaction length. Suppose that γ_n satisfies (K) and $\text{Supp}\{\gamma_n\} \subset B_1(\mathbf{0})$. Then the rescaled kernel γ_n^δ is defined by

$$\gamma_n^\delta(|s|) = \frac{1}{|\delta|^{d+2n}}\gamma_n(\frac{|s|}{\delta}) \quad \text{for } s \in B_\delta(\mathbf{0}) \tag{7.11}$$

which satisfies

$$\int |s|^{2n}\gamma_n^\delta(|s|)ds = \int |s|^{2n}\gamma_n(|s|)ds\,.$$

We are also concerned with the asymptotic compactness property as $\delta \to 0$ in spirit of the work by [Bourgain *et al.*, 2001].

Theorem 7.2.6 (Asymptotic compactness)**.** *Suppose that* $\{u_k\}$ *is a bounded sequence in* $L^2(\Omega)$ *with zero extension outside* Ω*. If*

$$\sup_k \int_{\Omega\cup\Omega_{\delta_k}} \int_{B_{\delta_k}(\mathbf{0})} \gamma_n^{\delta_k}(|s|)|D_n^s[u_k](x)|^2 ds dx < \infty\,,$$

then $\{u_k\}$ *is relatively compact in* $L^2(\Omega)$*. Moreover, any limit point* $u \in H^n_0(\Omega)$*.*

7.2.2 Function spaces and operators

For the function space defined by (7.10), we observe firstly that $\mathcal{S}^{n,\gamma_n}$ is a subspace of $L^2(\mathbb{R}^d)$. Moreover, for $0<\alpha<1$, by taking $\gamma_2(|s|) = c_{N,\alpha}|s|^{-d-2\alpha}$ with a suitable positive constant $c_{d,\alpha}$, the space associated with $\mathcal{S}^{2,\gamma_2}$ corresponds to the usual fractional Sobolev space H^α. For simplicity, whenever there is no notational confusion, we use $\mathcal{S}^n$ to denote $\mathcal{S}^{n,\gamma_n}$. Let the bilinear form $((\cdot,\cdot))_{\mathcal{S}^n} : \mathcal{S}^n \times \mathcal{S}^n \to \mathbb{R}$ be defined by

$$((u,v))_{\mathcal{S}^n} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \gamma_n(|s|)\Big(D_n^s[u](x)\Big)\Big(D_n^s[v](x)\Big)dsdx\,.$$

Then the space $(\mathcal{S}^n, (\cdot,\cdot)_{\mathcal{S}^n})$ is a real inner product space with $(\cdot,\cdot)_{\mathcal{S}^n}$ defined as

$$(u, v)_{\mathcal{S}^n} = (u, v)_{L^2} + ((u, v))_{\mathcal{S}^n} \, .$$

Now we use $|u|_{\mathcal{S}^n}$ to denote the semi-norm $\sqrt{((u,u))_{\mathcal{S}^n}}$. Then $\mathcal{S}^n$ is equipped with a norm $\|\cdot\|_{\mathcal{S}^n}$ given by

$$\|u\|^2_{\mathcal{S}^n} = \|u\|^2_{L^2} + |u|^2_{\mathcal{S}^n}$$

Our main goal in this subsection is to show that $\mathcal{S}^n$ is a Hilbert space.

Theorem 7.2.7. *For $n \in \mathbb{Z}^+$, assume that γ_n satisfies (K). Then $(\mathcal{S}^n, (\cdot,\cdot)_{\mathcal{S}^n})$ is a Hilbert space.*

Before proving the theorem, we need to establish some technical observations. First for difference operators defined by (7.8), we can interpret a higher order difference operator by the composition of lower order difference operators. This simple fact together with two other basic equalities are given in a lemma that will be useful in subsequent calculations. The proof of the lemma is rudimentary and we refer to [Tian and Du, 2015b] for details.

Lemma 7.2.8. *For $n \geq 2$, $s \in \mathbb{R}^d$, D^s_n satisfies the following:*

$$D^s_n = D^s_1 \circ D^s_{n-1} \ \text{ if } n \text{ is odd, or } \ D^s_n = -D^{-s}_1 \circ D^s_{n-1} \ \text{ if } n \text{ is even.} \tag{7.12}$$

$$\Big|\sum_{j=0}^{n} (-1)^j \binom{n}{j} e^{i\xi \cdot a^j_n s}\Big|^2 = (2 - 2\cos(\xi \cdot s))^n \, , \tag{7.13}$$

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (a^j_n)^n = n! \, . \tag{7.14}$$

Next, we consider a simple property of the difference operators which may be viewed as an analog of the integration by parts formula. Similar formulae have been discussed in many earlier works such as [Du *et al.*, 2012; Du *et al.*, 2013a]. For convenience, we drop the domain of integration in the integral whenever there is no ambiguity, in particular, when it is an integral over the whole space.

Lemma 7.2.9 (Integration by parts formula)**.** *The following is true for functions $u, v \in L^2(\mathbb{R}^d)$ and $\rho \in L^1(\mathbb{R}^d)$.*

$$\iint \rho(|s|) u(x) D^s_1[v](x) ds dx = \iint \rho(|s|) D^{-s}_1[u](x) v(x) ds dx \, .$$

Proof. It is very easy to check the following equalities.

$$\begin{aligned}\iint \rho(|s|)u(x)D_1^s[v](x)dsdx &= \iint \rho(|s|)u(x)(v(x+s)-v(x))dsdx \\ &= \iint \rho(|s|)(u(x-s)-u(x))v(x)dsdx \\ &= \iint \rho(|s|)D_1^{-s}[u](x)v(x)dsdx\,.\end{aligned}$$

□

Another lemma is on characterizing the norm $|\cdot|_{\mathcal{S}^n}$ by the Fourier transform.

Lemma 7.2.10. *Suppose u is a function defined on $\mathbb{R}^d$, and $\mathscr{F}$ is denoted as the Fourier transform of u, then for any $n \geq 1$*

$$\iint \gamma_n(|s|)|D_n^s[u](x)|^2 dsdx = \iint \gamma_n(|s|)(2-2\cos(\xi\cdot s))^n|\hat{u}(\xi)|^2 d\xi ds\,. \tag{7.15}$$

Proof. By Plancherel Formula, we have

$$\begin{aligned}\iint \gamma_n(|s|)|D_n^s[u](x)|^2 dsdx &= \iint \gamma_n(|s|)|D_n^s[u](x)|^2 dxds \\ &= \int_{\mathbb{R}^d} \gamma_n(|s|)\|D_n^s[u](\cdot)\|^2_{L^2(R^N)}ds \\ &= \int_{\mathbb{R}^d} \gamma_n(|s|)\|\mathscr{F}\Big(D_n^s[u](\cdot)\Big)\|^2_{L^2(R^N)}ds\,.\end{aligned}$$

Now by the definition of D_n^s, we obtain

$$\int_{\mathbb{R}^d} \gamma_n(|s|)\|\mathscr{F}\Big(D_n^s[u](\cdot)\Big)\|^2_{L^2(R^N)}ds = \iint \gamma_n(|s|)\Big|\sum_{j=0}^n (-1)^j\binom{n}{j}e^{i\xi\cdot a_n^j s}\Big|^2|\hat{u}(\xi)|^2 d\xi ds\,.$$

Hence, by Lemma 7.2.8,

$$\Big|\sum_{j=0}^n (-1)^j\binom{n}{j}e^{i\xi\cdot a_n^j s}\Big|^2 = (2-2\cos(\xi\cdot s))^n,$$

we thus get the desired result. □

Now we show that the inner product space $\mathcal{S}^n = \mathcal{S}^{n,\gamma_n}$ is complete, thus a Hilbert space. Analogous results for the special case of $n=1$, but for more general vector fields, can be found in, for example, [Mengesha and Du, 2013; Mengesha and Du, 2014a].

Proof of Theorem 7.2.7. It suffices to check that the space is complete under the norm $\|\cdot\|_{\mathcal{S}^n}$. Given a Cauchy sequence $\{u_k\} \in \mathcal{S}^n$, it is also Cauchy in $L^2(\mathbb{R}^d)$. So $\{u_k\}$, up to a subsequence, converges to some $u \in L^2(\mathbb{R}^d)$. We show that $|u_k - u|_{\mathcal{S}^n} \to 0$ as $k \to \infty$.

For any $\epsilon > 0$, we may choose M large enough such that $|u_k - u_m|^2_{\mathcal{S}^n} \le \epsilon^2$ for any $k, m \ge M$. We then follow similar techniques as that in [Mengesha and Du, 2013; Mengesha and Du, 2014a] to define a cut-off of kernel γ_n by $\gamma_n^\tau(r) = \gamma_n(r)\chi_{[\tau,\infty)}(r)$ to make γ_n^τ integrable for any given $\tau > 0$, and

$$\iint \gamma_n^\tau(|s|)(D_n^s[u_k - u_m](x))^2 dsdx \le |u_k - u_m|^2_{\mathcal{S}^n} \,.$$

We now first claim that, for any $w, v \in \mathcal{S}^n$,

$$\begin{aligned} &\iint \gamma_n^\tau(|s|) D_n^s[w](x) D_n^s[v](x) dsdx = \\ &(-1)^n \iint \gamma_n^\tau(|s|) D_{2n}^s[w](x) v(x) dsdx \,. \end{aligned} \tag{7.16}$$

In fact, by Lemma 7.2.8, $D_n^s[v]$ can be decomposed into $D_1^s \circ D_{n-1}^s[v]$ or $-D_1^{-s} \circ D_{n-1}^s[v]$. Then using the integration by parts formula in Lemma 7.2.9, we may throw the first order difference operator to the term involving $D_n^s[w](x)$ and apply Lemma 7.2.8 again to get $-D_{n+1}^s[w](x)$. Repeating this procedure we can finally get (7.16).

Subsequently, we get

$$\begin{aligned} &\iint \gamma_n^\tau(|s|)(D_n^s[u_k - u_m](x))^2 dsdx = \\ &(-1)^n \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u_m](x) ds \Big) (u_k - u_m)(x) dx \,. \end{aligned}$$

Since $u_m \to u$ in $L^2(\mathbb{R}^d)$, we have for a fixed k, and all $x \in \mathbb{R}^d$,

$$\lim_{m\to\infty} \int_{\mathbb{R}^d} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u_m](x) ds = \int_{\mathbb{R}^d} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u](x) ds \,.$$

Therefore by dominated convergence theorem,

$$\begin{aligned} &(-1)^n \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u](x) ds \Big) (u_k - u)(x) dx \\ &= \lim_{m\to\infty} (-1)^n \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u_m](x) ds \Big) (u_k - u_m)(x) dx \le \epsilon^2 \,. \end{aligned}$$

for $k \ge M$. Now we can apply equation (7.16) again and obtain

$$\iint \gamma_n^\tau(|s|)(D_n^s[u_k - u](x))^2 dsdx \le \epsilon^2 \,.$$

In the end, by letting $\tau \to 0$ and applying Fatou's lemma, we have

$$|u_k - u|_{\mathcal{S}^n} = \iint \gamma_n(|s|)(D_n^s[u_k - u](x))^2 dsdx \le \epsilon^2 \,,$$

which completes the proof. □

For $\Omega \subset \mathbb{R}^d$, we let $\mathcal{S}^n_\Omega = \mathcal{S}^{n,\gamma_n}_\Omega$ denote the closure of $C^\infty_c(\Omega)$ in $\mathcal{S}^n$, i.e.,

$$\mathcal{S}^n_\Omega = \{u \in L^2(\mathbb{R}^d) : u_k \to u \text{ in } \mathcal{S}^n \text{ as } k \to \infty \text{ for some } u_k \in C^\infty_c(\Omega)\}.$$

It is important to note that, for the later discussion on problems defined in a bounded domain, if $u \in \mathcal{S}^n_\Omega$ and $supp(\gamma_n) \subset B_\delta(\mathbf{0})$, then

$$\iint \gamma_n(|s|)|D^s_n[u](x)|^2 dsdx = \int_{\Omega\cup\Omega_\delta} \int_{B_\delta(\mathbf{0})} \gamma_n(|s|)|D^s_n[u](x)|^2 dsdx$$

where $\Omega_\delta := \{x \in \mathbb{R}^d \backslash \overline{\Omega} : dist(x, \partial\Omega) < \delta\}$.

We may see that $|\cdot|_{\mathcal{S}^n}$ is indeed a norm as demonstrated in the following lemma.

Lemma 7.2.11. *Suppose $u \in \mathcal{S}^n_\Omega$ with γ_n satisfying (K) and $supp(\gamma_n) \subset B_\delta(\mathbf{0})$, and*

$$\iint \gamma_n(|s|)|D^s_n[u](x)|^2 dsdx = 0\,,$$

then $u = 0$.

Proof. The conditions of the lemma imply that

$$D^s_n[u](x) = 0 \quad \text{for a.e } x \in \Omega \cup \Omega_\delta \text{ and } s \in B_\delta(\mathbf{0})\,.$$

So $u = u(x)$ must be a polynomial of degree $(n-1)$ almost everywhere in $\Omega \cup \Omega_\delta$. Now since $u|_{\Omega_\delta} = 0$ by assumption, we have $u(x) \equiv 0$ for a.e $x \in \Omega \cup \Omega_\delta$. □

The nonlocal operators. It is easy to see that $\mathcal{S}^n_\Omega$ is also a Hilbert space with respect to the same inner product, we can define naturally via the Riesz representation theorem a linear operator $\mathcal{L}_n$ from $\mathcal{S}^n_\Omega$ to its dual $(\mathcal{S}^n_\Omega)^*$ by

$$\langle \mathcal{L}_n u, v\rangle = ((u, v))_{\mathcal{S}^n}\,, \quad \forall u, v \in \mathcal{S}^n_\Omega, . \tag{7.17}$$

First, $\mathcal{L}_n$ is a bounded linear operator on $\mathcal{S}^n_\Omega$. Next, if in addition to (K), we also have $\gamma_n = \gamma_n(|x|) \in L^1_{loc}(\mathbb{R}^d)$, then we have already seen from equation (7.16) that

$$\mathcal{L}_n u(x) = (-1)^n \int_{\mathbb{R}^d} \gamma_n(|s|) D^s_{2n}[u](x) ds\,, \quad a.e.\ x \in \Omega\,.$$

However, $\mathcal{L}_n$ is an unbounded operator if $\gamma_n(|\cdot|) \notin L^1_{loc}(\mathbb{R}^d)$. In this case, we proceed in the way suggested by [Mengesha and Du, 2014a]. By introducing the sequence of operator

$$\mathcal{L}_n^\tau u(x) = (-1)^n \int_{\mathbb{R}^d} \gamma_n^\tau(|s|) D_{2n}^s[u](x) ds \,,$$

for $\gamma_n^\tau(|s|)$ defined earlier, we can show that $\mathcal{L}_n^\tau u \to \mathcal{L}_n u$, where $\mathcal{L}_n u$ is defined as

$$\mathcal{L}_n u(x) = \text{(P.V.)}\ (-1)^n \int_{\mathbb{R}^d} \gamma_n(|s|) D_{2n}^s[u](x) ds \,. \tag{7.18}$$

7.2.3 Sobolev type inequalities

In this section, we show several nonlocal Sobolev type inequalities, including Poincaré type and Gagliardo-Nirenberg type inequalities. First, we prove two kinds of nonlocal Poincaré type inequalities. The first kind says that for every function u of the space $\mathcal{S}^n_\Omega$, the L^2 norm of u can be bounded in terms of $|u|_{\mathcal{S}^n}$.

Theorem 7.2.12 (1st nonlocal Poincaré inequality)**.** *For $n \in \mathbb{Z}^+$ and kernel γ_n satisfying (K), there exists $C = C(n, \gamma_n, \Omega)$ such that*

$$\|u\|_{L^2} \leq C|u|_{\mathcal{S}^n} \quad \forall u \in \mathcal{S}^n_\Omega \,.$$

The second kind of nonlocal Poincaré type inequalities extends the results above and shows that a lower order norm (say, the n-th order norm $|u|_{\mathcal{S}^n}$) can be bounded by a higher order norm (say, $|u|_{\mathcal{S}^{n+1}}$) for any function u in the space defined by the latter (that is, $\mathcal{S}^{n+1}_\Omega$). Obviously, this cannot be true for arbitrary kernel functions γ_n and γ_{n+1}. Hence, besides the assumption that $\gamma_1, \gamma_n, \gamma_{n+1}$ are kernels satisfying (K) respectively, we assume further that,

$$\gamma_k \text{ is non-increasing, } \operatorname{supp}\{\gamma_k(|\cdot|)\} \subset B_1(\mathbf{0}) \;\text{ for } k\text{=}1,\, n,\; \gamma_{n+1} = \gamma_1\, \gamma_n \,, \tag{7.19}$$

and there is a constant C such that

$$\mathcal{I}_n(\xi)\, \mathcal{I}_1(\xi) \leq C\, \mathcal{I}_{n+1}(\xi) \,,\ \forall \xi \in \mathbb{R}^d \,, \tag{7.20}$$

where

$$\mathcal{I}_k(\xi) = \int \gamma_k(|s|)(1 - \cos(\xi \cdot s))^k ds \,,\ \forall k \,. \tag{7.21}$$

We remark that the requirement (7.20) might not appear very intuitive at the first sight, it actually can be satisfied by a large class of kernels. For instance, if

$$\gamma_n(|s|) = (\gamma_1)^n(|s|)\,, \quad \text{and} \quad \gamma_{n+1}(|s|) = (\gamma_1)^{n+1}(|s|)\,, \tag{7.22}$$

then (7.20) is true. This is a simple consequence of the fact that for $A, B : X \to \mathbb{R}$ and μ a positive measure on X, if

$$(A(x) - A(y))(B(x) - B(y)) \geq 0, \quad \forall x, y \in X$$

then

$$\int_X AB d\mu \geq \frac{1}{\mu(X)} \int_X A d\mu \int_X B d\mu\,.$$

We may apply $A(x) = (B(x))^n$ with $B(x) = \gamma_1(|x|)(1 - \cos(\xi \cdot x))$ and μ the Lebesgue measure, then we see that (7.20) holds with C being the volume of the unit ball in $\mathbb{R}^d$.

To present another class of examples different from the above, we first offer an alternative characterization of (7.20) in the following lemma whose proof is left to the appendix.

Lemma 7.2.13 (Another characterization of (7.20)). *Assume that* γ_1, γ_n *and* $\gamma_{n+1} = \gamma_1\gamma_n$ *are kernels that satisfy* (K) *respectively and without loss of generality that* $B_\eta(\mathbf{0}) \subset Supp\{\gamma_k(|\cdot|)\} \subset B_1(\mathbf{0})$ *for all* k. *In addition, assume that the following properties are satisfied.*

i) $$\int_{|s|<\epsilon} \gamma_n(|s|)|s|^{2n} ds \int_{|s|<\epsilon} \gamma_1(|s|)|s|^2 ds \leq C \int_{|s|<\epsilon} \gamma_{n+1}(|s|)|s|^{2n+2} ds$$

for any $\epsilon \leq 1$ *with* C *independent of* ϵ.

ii) $$\left(\int_{|s|<\epsilon} \gamma_k(|s|)|s|^{2k} ds\right)\left(\int_{\epsilon<|s|<1} \gamma_k(|s|) ds\right)^{-1} \leq C\epsilon^{2k}\,, \quad k = 1, n,$$

for any $\epsilon \leq \eta$ *with* C *independent of* ϵ.

Then we have (7.20) *satisfied.*

Let us now check that if γ_1 and γ_n satisfy the following

$$m_1|x|^{-\beta_1} \leq \gamma_1(|x|) \leq M_1|x|^{\beta_1} \quad \text{and} \quad m_2|x|^{-\beta_2} \leq \gamma_n(|x|) \leq M_2|x|^{\beta_2}$$

for some $\beta_1 \in [0, d+2)$ and $\beta_2 \in [0, d+2n)$, then (7.20) holds. Indeed, by direct integration, we get the condition i) of Lemma 7.2.13 since

$$\int_{|s|<\epsilon} \gamma_n(|s|)|s|^{2n} ds \int_{|s|<\epsilon} \gamma_1(|s|)|s|^2 ds \le CM_1M_2\epsilon^{2d+2n+2-\beta_2-\beta_1} \,,$$

and

$$\int_{|s|<\epsilon} \gamma_n(|s|)\gamma_1(|s|)|s|^{2n+2} ds \ge Cm_1m_2\epsilon^{d+2n+2-\beta_1-\beta_2} \,.$$

As for the condition ii) in Lemma 7.2.13, we show the case for $k = 1$ as an illustration. For $\beta_1 \le d$,

$$\Big(\int_{|s|<\epsilon} \gamma_1(|s|)|s|^2 ds\Big)\Big/\Big(\int_{\epsilon<|s|<1} \gamma_1(|s|)ds\Big) \le CM_1\epsilon^{d+2-\beta_1}\Big/\Big(\int_{\epsilon<|s|<1} \gamma_1(|s|)ds\Big) \le C\epsilon^2 \,,$$

and for $d < \beta_1 < d+2$,

$$\begin{aligned}\Big(\int_{|s|<\epsilon} \gamma_1(|s|)|s|^2 ds\Big)\Big/\Big(\int_{\epsilon<|s|<1} \gamma_1(|s|)ds\Big) &\le C\frac{M_1}{m_1}\epsilon^{d+2-\beta_1}/(\epsilon^{d-\beta_1}-1) \\ &= C\frac{M_1}{m_1}\epsilon^2/(1-\epsilon^{\beta_1-d}) \le C\epsilon^2 \,, \quad \text{for } \epsilon \le 1/2 \,.\end{aligned}$$

We now present the second nonlocal Poincaré inequality whose proof is presented in the next section as it relies on a compactness result given there.

Theorem 7.2.14 (2nd nonlocal Poincaré inequality)**.** *For $n \in \mathbb{Z}^+$ and kernels $\{\gamma_n\}$ satisfying (K), (7.19) and (7.20), there exists $C = C(n, \gamma_n, \gamma_{n+1}, \Omega)$ such that*

$$\|u\|_{\mathcal{S}^n} \le C|u|_{\mathcal{S}^{n+1}} \quad \forall u \in \mathcal{S}^{n+1}_\Omega \,.$$

The final result of this section is a Gagliardo-Nirenberg inequality stated below.

Theorem 7.2.15 (Nonlocal Gagliardo-Nirenberg inequality)**.** *For nonnegative integers n, k_1 and k_2 with $k_1 + k_2 > 0$, suppose that γ_n satisfies (K), and $\alpha = k_1/(k_1+k_2)$, the following nonlocal Gagliardo-Nirenberg type inequality holds*

$$|u|_{\mathcal{S}^n} \le |u|^{1-\alpha}_{\mathcal{S}^{(n-k_1)}} |u|^{\alpha}_{\mathcal{S}^{(n+k_2)}} \,, \tag{7.23}$$

where $\mathcal{S}^{(n-k_1)} = \mathcal{S}^{n-k_1,\gamma_{(n-k_1)}}$ and $\mathcal{S}^{(n+k_2)} = \mathcal{S}^{n+k_2,\gamma_{(n+k_2)}}$ with properly chosen kernels that, in particular, are given by $\gamma_{(n-k_1)} = (\gamma_n)^{1-k_1/n}$ and $\gamma_{(n+k_2)} = (\gamma_n)^{1+k_2/n}$.

We note that for $n \in \mathbb{N}$, if $k_1 = 0$, $k_2 = 1$, then the above reduces to theorem 7.2.14 with the simple case of (7.22) satisfied.

Proof of Theorem 7.2.12 Variants of the case $n = 1$ can be found in many existing papers, say for example [Du *et al.*, 2012; Mengesha and Du, 2013]. Our proof follows a similar path.

Suppose that the conclusion of the theorem is false, then we can find a sequence $\{u_k \in \mathcal{S}^n_\Omega\}$ such that $\|u_k\|_{L^2} = 1$ and $|u_k|_{\mathcal{S}^n} \to 0$ as $k \to \infty$. This leads to the existence of a weak limit $u \in L^2$ such that $u_k \rightharpoonup u$ in L^2.

Step 1. We show that u is in fact 0. Suppose $\{\phi_\epsilon\}$ are standard mollifiers, then

$$\begin{aligned} |\phi_\epsilon * u_k|^2_{S^n} &= \iint \gamma_n(|s|)\, |D^s_n[\phi_\epsilon * u_k](x)|^2 \, dsdx \\ &= \iint \gamma_n(|s|) \left| \int D^s_n[u_k](x-y)\phi_\epsilon(y)dy \right|^2 dsdx \\ &\leq \int \phi_\epsilon(y) \Big(\iint \gamma_n(|s|)|D^s_n[u_k](x-y)|^2 dsdx \Big) dy \\ &= \int \phi_\epsilon(y) \Big(\iint \gamma_n(|s|)|D^s_n[u_k](x)|^2 dsdx \Big) dy, \end{aligned}$$

where all integrals are over $\mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{R}^d$ respectively and the first inequality follows from the Jensen's inequality. This then leads to

$$|\phi_\epsilon * u_k|_{S^n} \leq |u_k|_{S^n} \, . \tag{7.24}$$

Now $u_k \rightharpoonup u$ in L^2 implies $\phi_\epsilon * u_k \to \phi_\epsilon * u$ strongly in L^2. So $\phi_\epsilon * u_k \to \phi_\epsilon * u$ almost everywhere as $k \to \infty$. Applying Fatou's lemma to (7.24), we get, for any fixed $\epsilon > 0$,

$$\begin{aligned} \iint \gamma_n(|s|)|D^s_n[\phi_\epsilon * u](x)|^2 dsdx &\leq \liminf_k \iint \gamma_n(|s|)|D^s_n[\phi_\epsilon * u_k](x)|^2 dsdx \\ &\leq \liminf_k |u_k|^2_{S^n} = 0\, . \end{aligned}$$

With $\phi_\epsilon * u \to u$ pointwise, by applying Fatou's lemma again, we get

$$\iint \gamma_n(|s|)|D^s_n[u](x)|^2 dsdx \leq \liminf_\epsilon \iint \gamma_n(|s|)|D^s_n[\phi_\epsilon * u](x)|^2 dsdx = 0\, .$$

In addition, since $u_k|_{\mathbb{R}^d \setminus \overline{\Omega}} = 0$ for any k and $u_k \rightharpoonup u$, we have $u|_{\mathbb{R}^d \setminus \overline{\Omega}} = 0$. This is $u = 0$ by lemma 7.2.11.

Step 2. We next show that $u_k \to u$ strongly in L^2. First, for some $M > 0$, we define

$\bar{\gamma}_n(|x|) = \min\{M, \gamma_n(|x|)\}$. Then, with $b_n^j := (-1)^j \binom{n}{j}$, we have

$$\begin{aligned}
|u_k|^2_{\mathcal{S}^n} &\geq \iint \bar{\gamma}_n(|s|)|D_n^s[u_k](x)|^2 dsdx = \iint \bar{\gamma}_n(|s|)\Big|\sum_{j=0}^n b_n^j u_k(x + a_n^j s)\Big|^2 dsdx \\
&= \sum_{j=0}^n (b_n^j)^2 \iint \bar{\gamma}_n(|s|) u_k^2(x + a_n^j s) dsdx \\
&\quad +2\sum_{i\neq j} b_n^i b_n^j \iint \bar{\gamma}_n(|s|) u_k(x + a_n^i s) u_k(x + a_n^j s) dsdx \\
&= \sum_{j=0}^n (b_n^j)^2 \iint \bar{\gamma}_n(|s|) u_k^2(x) dsdx \\
&\quad +2\sum_{i\neq j} b_n^i b_n^j \int \Big(\int \bar{\gamma}_n(|s|) u_k(x + (a_n^i - a_n^j)s) ds\Big) u_k(x) dx \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Now the first term

$$\mathrm{I} = \sum_{j=0}^n (b_n^j)^2 \int \bar{\gamma}_n(|s|) ds \int u_k^2(x) dx \to c\|u_k\|^2_{L^2}, \quad \text{as } k \to \infty$$

for a constant $c > 0$. The second term

$$\begin{aligned}
\mathrm{II} &= 2\sum_{i\neq j} b_n^i b_n^j \int \Big(\int \bar{\gamma}_n(|s|) u_k(x + (a_n^i - a_n^j)s) ds\Big) u_k(x) dx \\
&= 2\sum_{i\neq j} b_n^i b_n^j \int F_k(x) u_k(x) dx,
\end{aligned}$$

where

$$\begin{aligned}
F_k(x) &= \int_{\mathbb{R}^d} \bar{\gamma}_n(|s|) u_k(x + (a_n^i - a_n^j)s) ds = \frac{1}{a_n^i - a_n^j} \int_{\mathbb{R}^d} \bar{\gamma}_n\Big(\Big|\frac{y - x}{a_n^i - a_n^j}\Big|\Big) u_k(y) dy \\
&= \frac{1}{a_n^i - a_n^j} \int_{\Omega} \bar{\gamma}_n\Big(\Big|\frac{y - x}{a_n^i - a_n^j}\Big|\Big) u_k(y) dy,
\end{aligned}$$

which can be seen as the action of a Hilbert-Schmidt operator since $\bar{\gamma}_n \leq M$. So

$$F_k = F_k(x) \to F = F(x) = \frac{1}{a_n^i - a_n^j} \int_{\Omega} \bar{\gamma}_n\Big(\Big|\frac{y - x}{a_n^i - a_n^j}\Big|\Big) u(y) dy = 0$$

strongly in L^2. Now that $F_k \to 0$, and $u_k \rightharpoonup u$, we have

$$\lim_k \int F_k(x) u_k(x) dx = 0$$

which implies $\mathrm{II} \to 0$. Thus $\|u_k\|_{L^2} \to 0$ which is a contradiction to $\|u_k\|_{L^2} = 1$. □

Proof of Theorem 7.2.15 First, by applying Lemma 7.2.10, we only need to show the following inequality in order to have (7.23).

$$\begin{aligned}&\iint \gamma_n(|s|)(2-2\cos(\xi\cdot s))^n|\hat{u}(\xi)|^2 d\xi ds\\ &\leq \Big(\iint \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi\cdot s))^{n-k_1}|\hat{u}(\xi)|^2 d\xi ds\Big)^{1-\alpha}\\ &\quad\cdot\Big(\iint \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi\cdot s))^{n+k_2}|\hat{u}(\xi)|^2 d\xi ds\Big)^{\alpha}\end{aligned} \tag{7.25}$$

where $\hat{u}$ denotes the Fourier transform of u. Now let $n_1=(1-\alpha)(n-k_1)$ and $n_2=\alpha(n+k_2)$, then $n=n_1+n_2$. By applying Hölder's inequality, we have

$$\begin{aligned}\int \gamma_n(|s|)(2-2\cos(\xi\cdot s))^n ds &= \int (\gamma_n(|s|))^{n_1/n+n_2/n}(2-2\cos(\xi\cdot s))^{n_1+n_2}ds\\ &\leq \Big(\int \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi\cdot s))^{n-k_1}ds\Big)^{1-\alpha}\\ &\quad\cdot\Big(\int \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi\cdot s))^{n+k_2}ds\Big)^{\alpha}.\end{aligned}$$

Then, by splitting $|\hat{u}(\xi)|^2$ and using Hölder's inequality with respect to the integral in ξ, we have

$$\begin{aligned}&\iint \gamma_n(|s|)(2-2\cos(\xi\cdot s))^n|\hat{u}(\xi)|^2 ds d\xi\\ &\leq \int \Big(\int \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi\cdot s))^{n-k_1}ds|\hat{u}(\xi)|^2\Big)^{1-\alpha}\\ &\quad\cdot\Big(\int \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi\cdot s))^{n+k_2}ds|\hat{u}(\xi)|^2\Big)^{\alpha}d\xi\\ &\leq \Big(\iint \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi\cdot s))^{n-k_1}|\hat{u}(\xi)|^2 ds d\xi\Big)^{1-\alpha}\\ &\quad\cdot\Big(\iint \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi\cdot s))^{n+k_2}|\hat{u}(\xi)|^2 ds d\xi\Big)^{\alpha}.\end{aligned}$$

Thus (7.25) is true. □

7.2.4 Compact embeddings

The nonlocal Poincaré type inequalities imply continuous embedding between spaces. In many applications, a stronger compact embedding result is necessary. Here we give conditions so that such compactness holds.

Let γ_1 satisfy (K). In addition, as in [Bourgain *et al.*, 2001], we assume that

$$\lim_{\epsilon\to 0}\epsilon^2\Big(\int_{|x|<\epsilon}|x|^2\gamma_1(|x|)dx\Big)^{-1}=0\,. \tag{7.26}$$

We note that if $\gamma_1(|x|)$ has a singularity of the type $1/|x|^{N+2s}$ for x at the origin with exponent $s \in (0,1)$, which is a typical kernel for the standard Sobolev space H^s, then the assumption (7.26) is satisfied.

Theorem 7.2.16. *Suppose that the kernels $\gamma_1, \gamma_n, \gamma_{n+1}$ satisfy* (7.19), (7.20) *and* (7.26). *Let $\mathcal{F}$ be a bounded set in $\mathcal{S}^n_\Omega$. If*

$$|u|_{\mathcal{S}^{n+1}} \le C_0 \quad \forall u \in \mathcal{F}\,, \tag{7.27}$$

then $\mathcal{F}$ is precompact in $\mathcal{S}^n_\Omega$ and any of its limit point is in $\mathcal{S}^{n+1}_\Omega$ with a norm bounded by C_0.

First, we quote here a result from [Bourgain *et al.*, 2001, Lemma 2] as a lemma.

Lemma 7.2.17. *Let $g, h : (0,\delta) \to \mathbb{R}_+$. Assume $g(t) \le g(t/2), t \in (0,\delta)$, and that $h = h(t)$ is non-increasing. Then, for some $N \in \mathbb{N}$ and $C = C(N) > 0$,*

$$\int_0^\delta t^{N-1} g(t)h(t)dt \ge C\delta^{-N} \int_0^\delta t^{N-1} g(t)dt \int_0^\delta t^{N-1}h(t)dt.$$

Proof of Thoerem 7.2.16 Step 1. Suppose (ϕ_ϵ) are standard mollifiers defined as $\phi_\epsilon(x) = \epsilon^{-N}\phi(x/\epsilon)$ with integrals equal to 1, $\|\phi\|_\infty \le C$ and $\mathrm{Supp}\{\phi_\epsilon\} \subset B_\epsilon(\mathbf{0})$. We claim that for any $\epsilon > 0$, there exists $\sigma = \sigma(\epsilon)$ such that

$$|\phi_\epsilon * u - u|_{\mathcal{S}^n} \le \sigma \quad \forall u \in \mathcal{F}\,.$$

Indeed,

$$\begin{aligned}
|\phi_\epsilon * u - u|_{\mathcal{S}^n} &= \iint \gamma_n(|s|)|D^s_n[\phi_\epsilon * u - u](x)|^2 dsdx \\
&= \iint \gamma_n(|s|)|(\phi_\epsilon * D^s_n[u] - D^s_n[u])(x)|^2 dsdx \\
&= \iint \gamma_n(|s|)\left| \int_{\mathbb{R}^d} (D^s_n[u](x-y) - D^s_n[u](x))\phi_\epsilon(y)dy \right|^2 dsdx \\
&\le \int_{\mathbb{R}^d} \iint \gamma_n(|s|)\phi_\epsilon(y)|D^s_n[u](x-y) - D^s_n[u](x)|^2 dydsdx
\end{aligned}$$

by Jensen's inequality. Similarly to the proof of Lemma 7.2.10,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\iint \gamma_n(|s|)\phi_\epsilon(y)|D^s_n[u](x-y) - D^s_n[u](x)|^2 dyds \right) dx \\
&= \iint \left\| \sqrt{\gamma_n(|s|)\phi_\epsilon(y)}\mathscr{F}\left(D^s_n[u](\cdot - y) - D^s_n[u](\cdot)\right) \right\|^2_{L^2(R^N)} dyds \\
&= \iint \left(\int_{\mathbb{R}^d} \gamma_n(|s|)\phi_\epsilon(y)(2 - 2\cos(\xi\cdot y))(2 - 2\cos(\xi \cdot s))^n |\hat{u}(\xi)|^2 d\xi \right) dyds\,.
\end{aligned}$$

Combining the above two equalities and using $\|\phi_\epsilon\|_\infty \le C\epsilon^{-d}$, we get

$$\begin{aligned} &|\phi_\epsilon * u - u|_{\mathcal{S}^n} \\ &\le C\epsilon^{-d} \int_{|y|<\epsilon} \iint \gamma_n(|s|)(1-\cos(\xi\cdot s))^n |\hat{u}(\xi)|^2 (1-\cos(\xi\cdot y)) d\xi ds dy \\ &= C\epsilon^{-d} \int_0^\epsilon t^{d-1} G(t) dt \end{aligned} \tag{7.28}$$

where $G(t)$ is defined as

$$G(t) = \int_{\omega\in S^{d-1}} \iint \gamma_n(|s|)(1-\cos(\xi\cdot s))^n |\hat{u}(\xi)|^2 (1-\cos(t\xi\cdot w)) d\xi ds d\omega\,.$$

Using the fact that $1-\cos(2a) \le 2^2(1-\cos(a))$ for any $a\in\mathbb{R}$, we have

$$G(2t) \le 2^2 G(t)\,.$$

Applying Lemma 7.2.17 with $g(t) = t^{-2}G(t)$, $h(t) = \gamma_1(t)$, and $N = d+2$, we get

$$\frac{C}{\epsilon^{d+2}} \int_0^\epsilon t^{d-1} G(t) dt \le C\Big(\int_0^\epsilon t^{d-1} G(t)\gamma_1(t) dt\Big) / \Big(\int_{|x|<\epsilon} |x|^2 \gamma_1(|x|) dx\Big)\,. \tag{7.29}$$

For the right hand side, we have that, without loss of generality, for $\epsilon < 1$,

$$\begin{aligned} &\int_0^\epsilon t^{d-1} G(t)\gamma_1(t) dt \\ &= \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \mathcal{I}_n(\xi) \big(\int_{|s|<\epsilon} \gamma_1(|s|)(1-\cos(\xi\cdot s)) ds\big) d\xi \\ &\le \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \mathcal{I}_n(\xi) \big(\int_{|s|<1} \gamma_1(|s|)(1-\cos(\xi\cdot s)) ds\big) d\xi \\ &= \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \mathcal{I}_n(\xi)\, \mathcal{I}_1(\xi) d\xi \end{aligned}$$

where $\mathcal{I}_n(\xi)$ and $\mathcal{I}_1(\xi)$ are defined as in (7.21).

With the assumptions (7.19)-(7.20), we have by Lemma 7.2.10 that

$$\begin{aligned} \int_0^\epsilon t^{d-1} G(t)\gamma_1(t) dt &\le C \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \mathcal{I}_{n+1}(\xi) \mathcal{I}_1(\xi) d\xi \\ &= \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \int_{|s|<1} \gamma_{n+1}(|s|)(1-\cos(\xi\cdot s))^{n+1} ds d\xi \\ &\le C \iint \gamma_{n+1}(|s|) |D^s_{n+1}[u](x)|^2 ds dx\,. \end{aligned} \tag{7.30}$$

Combining (7.28), (7.29) and (7.30), we obtain

$$|\phi_\epsilon * u - u|_{\mathcal{S}^n} \le C\epsilon^2 \big(\int_{|x|<\epsilon} |x|^2 \gamma_1(|x|) dx\big)^{-1} \to 0 \quad \text{as } \epsilon \to 0$$

by condition (7.26)

Step 2. In order to use Arzela-Ascoli, we first claim that $\{\phi_\epsilon * u\}_{u\in\mathcal{F}}$ are uniformly bounded and equicontinuous, i.e.,

$$\|\phi_\epsilon * u\|_{L^\infty(R^N)} \le C_\epsilon \|u\|_{L^2(\mathbb{R}^d)}$$

and

$$|\phi_\epsilon * u(x) - \phi_\epsilon * u(y)| \le C_\epsilon \|u\|_{L^2(\mathbb{R}^d)} |x-y|$$

where C_ϵ only depends on ϵ. The first inequality follows from Hölder's inequality with $C_\epsilon = \|\phi_\epsilon\|_{L^2}$. For the second inequality, since $\|\nabla(\phi_\epsilon * u)\|_{L^\infty} = \|(\nabla\phi_\epsilon) * u\|_{L^\infty} \le \|\nabla\phi_\epsilon\|_{L^2}\|u\|_{L^2}$, the inequality is true with $C_\epsilon = \|\nabla\phi_\epsilon\|_{L^2}$. Now by Theorem 7.2.12 and the bound (7.27), we see that $\{\phi_\epsilon * u\}_{u\in\mathcal{F}}$ are uniformly bounded and equicontinuous as claimed. So $\phi_\epsilon * u$ has a uniformly convergent subsequence by Arzela-Ascoli. Moreover, since $|\phi_\epsilon * u|_{\mathcal{S}^n} \le |u|_{\mathcal{S}^n} \le C$ by (7.24), the convergence is also true in $\mathcal{S}^n_\Omega$ by dominated convergence theorem. So $\{\phi_\epsilon * u\}_{u\in\mathcal{F}}$ is precompact in $\mathcal{S}^n_\Omega$ for any $\epsilon > 0$.

Step 3. We now combine Steps 1 and 2 to show that $\mathcal{F}$ is also precompact in $\mathcal{S}^n_\Omega$.

Indeed, $\forall\epsilon > 0$, by Step 1, there exists $\sigma > 0$, such that

$$|\phi_\epsilon * u - u|_{\mathcal{S}^n} \le \sigma \quad \forall u \in \mathcal{F}\,.$$

So for $u, g \in \mathcal{F}$ with $|\phi_\epsilon * u - \phi_\epsilon * g|_{\mathcal{S}^n} \le \epsilon$, we have

$$|u-g|_{\mathcal{S}^n} \le 2\sigma + \epsilon$$

by triangle inequality. Now for any $\lambda > 0$, we could choose ϵ small enough such that $\epsilon + 2\sigma < \lambda$. Since $\{\phi_\epsilon * u\}_{u\in\mathcal{F}}$ is precompact in $\mathcal{S}^n_\Omega$, there exists a finite ϵ-cover $\{\phi_\epsilon * u_1, \phi_\epsilon * u_2, ..., \phi_\epsilon * u_k\}$ of $\{\phi_\epsilon * u\}_{u\in\mathcal{F}}$. Then it immediately follows that $\{u_1, u_2, ..., u_k\}$ is a λ-cover of $\mathcal{F}$, which means that $\mathcal{F}$ is precompact in $\mathcal{S}^n_\Omega$.

Step 4. Let us verify that the limit point of $\mathcal{F}$ is in $\mathcal{S}^{n+1}_\Omega$. Suppose without loss of generality that $\{u_k\} \subset \mathcal{F}$ and $u_k \to u$ in $\mathcal{S}^n_\Omega$, then we need to show $|u|_{\mathcal{S}^{n+1}} \le C_0$. By Theorem 7.2.12 we know that u_k converges to u strongly in L^2. So $u_k(x) \to u(x)$ pointwise up to a set of measure zero. Then by Fatou's lemma

$$|u|^2_{\mathcal{S}^{n+1}} \le \liminf_k |u_k|^2_{\mathcal{S}^{n+1}} \le C_0^2\,.$$

□

Another compactness result To satisfy the assumption (7.26), the kernel γ_1 has to have certain singularity at zero in general, in particular, γ_1 cannot be integrable. In the latter case, we have the following variant of Theorem 7.2.16.

Theorem 7.2.18 (A variant of Thoerem 7.2.16)**.** *Suppose that the kernels* $\gamma_1, \gamma_n, \gamma_{n+1}$ *satisfy assumptions* (7.19)-(7.20)*. If* (u_k) *is a bounded sequence in* $\mathcal{S}^n_\Omega$*, and*

$$|u_k|_{\mathcal{S}^{n+1}} \to 0 \quad \text{as } k \to 0\,, \tag{7.31}$$

then (u_k) *is relatively compact in* $\mathcal{S}^n_\Omega$ *and any of its limit point* u *is in* $\mathcal{S}^{n+1}_\Omega$ *with* $|u|_{\mathcal{S}^{n+1}} = 0$*.*

Proof. Following Step 1 of proof of Thoerem 7.2.16, we have

$$\|\phi_\epsilon * u_k - u_k\|_{\mathcal{S}^n} \le C\epsilon^2 \Big(\int_{|x|<\epsilon} |x|^2 \gamma_1(|x|) dx \Big)^{-1} \|u_k\|_{\mathcal{S}^{n+1}} \quad \forall k\,. \tag{7.32}$$

Now since $|u_k|_{\mathcal{S}^{n+1}} \to 0$ as $k \to \infty$, (7.32) reduces to

$$|\phi_\epsilon * u_k - u_k|_{\mathcal{S}^n} \to 0 \quad \text{as } k \to \infty \quad \forall \epsilon > 0. \tag{7.33}$$

Then similarly as Step 2 of proof of Thoerem 7.2.16, we can show $(\phi_\epsilon * u_k)_k$ is relatively compact in $\mathcal{S}^n_\Omega$ for any $\epsilon > 0$. Therefore (u_k) is also relatively compact in $\mathcal{S}^n_\Omega$ by (7.33).

Finally, suppose $u_k \to u$ in $\mathcal{S}^n_\Omega$ without loss of generality. Similarly as Step 4 of Theorem 7.2.16, we have

$$|u|^2_{\mathcal{S}^{n+1}} \le \liminf_k |u_k|^2_{\mathcal{S}^{n+1}} = 0\,.$$

□

Proof of Theorem 7.2.14 The nonlocal Poincaré type inequality in Theorem 7.2.14 is a corollary of Theorem 7.2.18.

Assume the opposite, then there exists a sequence (u_k) such that $|u_k|_{\mathcal{S}^n} = 1$ and $|u_k|_{\mathcal{S}^{n+1}} \to 0$. Then by Theorem 7.2.18, (u_k) is relatively compact in $\mathcal{S}^n_\Omega$. Suppose the limit of u_k (up to a subsequence) in $\mathcal{S}^n_\Omega$ is u. Then on one hand, $|u|_{\mathcal{S}^n} = 1$. But on the other hand, $|u|_{\mathcal{S}^{n+1}} = 0$ by Theorem 7.2.18, which implies $u = 0$ by Lemma 7.2.11, so a contradiction to $|u|_{\mathcal{S}^n} = 1$. □

7.2.5 Limiting properties for vanishing nonlocality

In this section, we consider a fixed integer n, and study the a family of kernels γ_n^δ parametrized by δ that characterizes the nonlocal interaction length. Suppose that γ_n satisfies (K) and $\mathrm{Supp}\{\gamma_n\} \subset B_1(\mathbf{0})$. Then the rescaled kernel γ_n^δ given by (7.11) satisfies

$$\int |s|^{2n}\gamma_n^\delta(|s|)ds = \int |s|^{2n}\gamma_n(|s|)ds\,.$$

To study the limiting behavior as $\delta \to 0$, we first give a continuous embedding property.

Lemma 7.2.19. *Assume the kernel γ_n satisfies (K) with $Supp\{\gamma_n\} \subset B_1(\mathbf{0})$ and $u \in H^n(\Omega\cup\Omega_1)$. Then*

$$\int_{\Omega\cup\Omega_1}\int_{B_1(\mathbf{0})}\gamma_n(|s|)|D_n^s[u](x)|^2 dsdx \le C\Big(\int |s|^{2n}\gamma_n(|s|)ds\Big)\|u\|_{H^n}^2\,,$$

where C depends only on n and $\Omega\cup\Omega_1$ and H^n is the standard Sobolev space.

Proof. First, by standard extension we may always assume that $u \in H^n(\mathbb{R}^d)$. Then, by the multivariate version of Talyor's theorem, we have

$$D_n^s[u](x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{|\alpha|=n}\frac{|\alpha|}{\alpha!}(a_n^j s)^\alpha\int_0^1(1-t)^{n-1}D^\alpha u(x+ta_n^j s)dt\,,$$

where the multi-index notation is used, namely, for $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_d)$

$$|\alpha| = \alpha_1+\ldots+\alpha_d,\quad \alpha! = \alpha_1!\ldots\alpha_d!,\quad s^\alpha = s_1^{\alpha_1}\ldots s_d^{\alpha_d}\,.$$

Therefore we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d}|D_n^s[u](x)|^2dx &\le c_1(n)\int_{\mathbb{R}^d}|s|^{2n}\sum_{j=0}^{n}\sum_{|\alpha|=n}\Big(\int_0^1(1-t)^{n-1}D^\alpha u(x+ta_n^j s)dt\Big)^2dx\\
&\le c_1(n)\int_{\mathbb{R}^d}|s|^{2n}\sum_{j=0}^{n}\sum_{|\alpha|=n}\int_0^1\big(D^\alpha u(x+ta_n^j s)\big)^2dtdx\\
&= c_1(n)|s|^{2n}\sum_{j=0}^{n}\sum_{|\alpha|=n}\int_0^1\int_{\mathbb{R}^d}\big(D^\alpha u(x+ta_n^j s)\big)^2dxdt\\
&= c_1(n)(n+1)|s|^{2n}\sum_{|\alpha|=n}\int_{\mathbb{R}^d}\big(D^\alpha u(x)\big)^2dx\\
&\le c_1(n)(n+1)|s|^{2n}\|u\|_{H^n(\mathbb{R})} \le C(n,\Omega\cup\Omega_\delta)|s|^{2n}\|u\|^2_{H^n(\Omega\cup\Omega_\delta)}
\end{aligned}$$

where $c_1(n) = \sum_{j=0}^{n} \sum_{|\alpha|=n} \big((-1)^j \binom{n}{j} (a_n^j)^n \frac{|\alpha|}{\alpha!}\big)^2$. This implies that

$$\iint \gamma_n(|s|)|D_n^s[u](x)|^2 dsdx \leq C\Big(\int |s|^{2n}\gamma_n(|s|)ds\Big)\|u\|_{H^n}^2 .$$

□

Lemma 7.2.20. *Suppose that γ_n^δ are rescaled kernels defined as* (7.11)*, then for $u \in H_0^n(\Omega \cup \Omega_1)$,*

$$\lim_{\delta\to 0}\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})} \gamma_n^\delta(|s|)|D_n^s[u](x)|^2 dsdx = \sum_{|\alpha|=n} M(n,\alpha)\int_\Omega (D^\alpha u(x))^2 dx \tag{7.34}$$

where

$$0 < M(n,\alpha) = \int s^{2\alpha}\gamma_n(|s|)ds \sum_{\substack{|\beta|=|\theta|=n,\\ \beta+\theta=2\alpha}} \frac{(n!)^2}{\beta!\theta!}, \quad D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$$

with multi-indices $\alpha = (\alpha_1, \dots, \alpha_d), \beta = (\beta_1, \dots, \beta_d)$, and $\theta = (\theta_1, \dots, \theta_d)$ used.

Proof. First let $U_\delta(x,s) = (\gamma_n^\delta(|s|))^{1/2}|D_n^s[u](x)|$, then we have to prove that

$$\lim_{\delta\to 0}\|U_\delta\|_{L^2}^2 = \sum_{|\alpha|=n} M(n,\alpha)\int_\Omega (D^\alpha u(x))^2 dx .$$

By Lemma 7.2.19, we have for any $u, v \in H_0^n(\Omega\cup\Omega_1)$,

$$|\|U_\delta\|_{L^2} - \|V_\delta\|_{L^2}| \leq \|U_\delta - V_\delta\|_{L^2} \leq C\|u-v\|_{H^n} .$$

Therefore it suffices to prove the result for u in the dense subset $C_c^\infty(\Omega\cup\Omega_1)$. In this case, we obtain by Taylor expansion that

$$D_n^s[u](x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{|\beta|=n}\frac{(a_n^j s)^\beta}{\beta!}D^\beta u(x) + O(|s|^{n+1}) .$$

Then the higher order terms can be dropped since

$$\int \gamma_n^\delta(|s|)|s|^{2n+1} \to 0.$$

So

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})}\gamma_n^\delta(|s|)|D_n^s[u](x)|^2dsdx\\
=&\Big(\sum_{j=0}^{n}(-1)^j\binom{n}{j}(a_n^j)^n\Big)^2\lim_{\delta\to 0}\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})}\gamma_n^\delta(|s|)\Big(\sum_{|\beta|=n}\frac{s^\beta}{\beta!}D^\beta u(x)\Big)^2dsdx\\
=&(n!)^2\lim_{\delta\to 0}\sum_{|\beta|=|\theta|=n}\frac{1}{\beta!\theta!}\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})}\gamma_n^\delta(|s|)s^{\beta+\theta}D^\beta u(x)D^\theta u(x)dsdx\\
=&\sum_{|\beta|=|\theta|=n}\frac{(n!)^2}{\beta!\theta!}\int s^{\beta+\theta}\gamma_n(|s|)ds\int_\Omega D^\beta u(x)D^\theta u(x)dx\,.
\end{aligned}$$

where equation (7.14) is used.

Now, for the summation in the last term, if there exists some index i such that $\beta_i+\theta_i$ is odd, then the integral of $s^{\beta+\theta}\gamma_n(|s|)$ becomes zero, which implies that the summation is over all β,θ with $|\beta|=|\theta|=n$ and $\beta+\theta=2\alpha$, for some $|\alpha|=n$. In addition, since $u\in C_c^\infty(\Omega\cup\Omega_1)$, using integration by parts, we have

$$\int_\Omega D^\beta u(x)D^\theta u(x)dx=\int_\Omega(D^{\frac{\beta+\theta}{2}}u(x))^2dx\,.$$

Combining the above results we get the expression as the righthand side of (7.34). □

In the following we choose a sequence of kernels $\{\gamma_n^{\delta_k}\}$, where $\delta_k\to 0$, and study the compactness property as $k\to\infty$.

Theorem 7.2.21 (Asymptotic compactness)**.** *Suppose that $\{u_k\}$ is a bounded sequence in $L^2(\Omega)$ with zero extension outside Ω. If*

$$\sup_k\int_{\Omega\cup\Omega_{\delta_k}}\int_{B_{\delta_k}(\mathbf{0})}\gamma_n^{\delta_k}(|s|)|D_n^s[u_k](x)|^2dsdx<\infty\,, \tag{7.35}$$

then $\{u_k\}$ is relatively compact in $L^2(\Omega)$. Moreover, any limit point $u\in H_0^n(\Omega)$.

Proof. We follow the proof of Theorem 7.2.16 but with a slight modification. Instead of comparing u_k with $\phi_\epsilon * u_k$, where ϕ_ϵ is the standard mollifier, we compare u_k with a combination of mollifications of u_k, in the L^2 norm, in order to get an upper bound in the form of (7.35).

As D_n^s is defined differently for n odd or even, different estimates are sought after for the two cases. Without any loss of generality, we will only prove the case n where n is an even number. The other case is essentially the same. Now we claim that

$$\lim_{\delta\to 0}\limsup_{k\to\infty}\Big\|\binom{n}{\frac{n}{2}}u_k-\sum_{j=0}^{\frac{n}{2}-1}2(-1)^{\frac{n}{2}-1-j}\binom{n}{j}\phi_{a_n^j\epsilon}*u_k\Big\|_{L^2}=0. \tag{7.36}$$

Indeed, we can write $u_k=\int u_k(x)\phi_\epsilon(s)ds$ and $\phi_{a_n^j\epsilon}*u_k=\int u(x+a_n^js)\phi_\epsilon(s)ds$. By equating $\binom{n}{j}$ with $\binom{n}{n-j}$ for $j=0,1,\cdots,\frac{n}{2}-1$, we can see that

$$\begin{aligned}\Big|\binom{n}{\frac{n}{2}}u_k-\sum_{j=0}^{\frac{n}{2}-1}2(-1)^{\frac{n}{2}-1-j}\binom{n}{j}\phi_{a_n^j\epsilon}*u_k\Big| &= \Big|\int\sum_{j=0}^{n}(-1)^j\binom{n}{j}u_k(x+a_n^js)\phi_\epsilon(s)ds\Big|\\ &= \Big|\int D_n^s[u_k](x)\phi_\epsilon(s)ds\Big|.\end{aligned}$$

Then by Jensen's inequality we have

$$\begin{aligned}\Big\|\binom{n}{\frac{n}{2}}u_k-\sum_{j=0}^{\frac{n}{2}-1}2(-1)^{\frac{n}{2}-1-j}\binom{n}{j}\phi_{a_n^j\epsilon}*u_k\Big\|_{L^2} &\le \frac{C}{\epsilon^d}\int_{|s|<\epsilon}\int|D_n^s[u_k](x)|^2dxds\\ &= \frac{C}{\epsilon^d}\int_0^\epsilon t^{d-1}G(t)dt\,,\end{aligned}$$

where

$$G(t)=\int_{\omega\in S^{N-1}}\int|D_n^{t\omega}[u_k](x)|^2dxd\omega\,.$$

Notice that $G(2t)\le 2^{2n}G(t)$. By applying Lemma 7.2.17 with $g(t)=t^{-2n}G(t)$, and $h(t)=\gamma_n^{\delta_k}(t)$, and $N=d+2n$, we get

$$\int_0^\epsilon t^{d-1}G(t)dt\le\frac{C\epsilon^{d+2n}}{\displaystyle\int_{|s|<\epsilon}|s|^{2n}\gamma_n^{\delta_k}(|s|)ds}\Big(\int_{|s|<\epsilon}\int\gamma_n^{\delta_k}(|s|)|D_n^s[u_k](x)|^2dxds\Big).$$

Now we see that, by (7.35) and the fact

$$\lim_{k\to\infty}\int_{|s|<\epsilon}|s|^{2n}\gamma_n^{\delta_k}(|s|)ds=\int|s|^{2n}\gamma_n(|s|)ds\,,$$

we conclude that (7.36) is true.

Similar to Step 2 of Theorem 7.2.16, we can show that the sequence

$$\Big\{\sum_{j=0}^{\frac{n}{2}-1}2(-1)^{\frac{n}{2}-1-j}\binom{n}{j}\phi_{a_n^j\epsilon}*u_k\Big\}_k$$

is uniformly bounded and equicontinuous thus relatively compact in L^2. And (7.36) implies that $\{u_k\}_k$ is also relatively compact in L^2.

Finally, suppose that $u_k \to u$ in L^2, we will show that $u \in H_0^n(\Omega)$. Suppose that

$$\int_{\Omega\cup\Omega_{\delta_k}} \int_{B_{\delta_k}(\mathbf{0})} \gamma_n^{\delta_k}(|s|)|D_n^s[u_k](x)|^2 ds dx \leq C_0^2 \,.$$

Then consider the mollification $\phi_\epsilon * u_k$, we also have

$$\iint \gamma_n^{\delta_k}(|s|)|D_n^s[\phi_\epsilon * u_k](x)|^2 ds dx \leq C_0^2 \,,$$

by Jensen's inequality. Observe that u_k vanishes outside Ω, so for each fixed ϵ, $\phi_\epsilon * u_k \to \phi_\epsilon * u$ in $C_c^\infty(\Omega \cup \Omega_\epsilon)$. So

$$\begin{aligned}\sum_{|\alpha|=n} M(n,\alpha)\int_{\Omega\cup\Omega_\epsilon}(D^\alpha(\phi_\epsilon * u)(x))^2 dx &= \lim_{k\to\infty}\iint \gamma_n^{\delta_k}(|s|)|D_n^s[\phi_\epsilon * u](x)|^2 ds dx \\ &= \lim_{k\to\infty}\iint \gamma_n^{\delta_k}(|s|)|D_n^s[\phi_\epsilon * u_k](x)|^2 ds dx \leq C_0^2 \,.\end{aligned}$$

where the first and second equalities are obtained by applying lemma 7.2.20 and lemma 7.2.19 respectively. Now let $\epsilon \to 0$, we have

$$\sum_{|\alpha|=n} M(n,\alpha)\int_\Omega (D^\alpha u(x))^2 dx \leq C_0^2 \,,$$

which implies that

$$\int_\Omega (D^\alpha u(x))^2 dx \quad \text{is bounded} \quad \forall \;\; \alpha = (\alpha_1, \ldots, \alpha_N) \;\; \text{with} \;\; |\alpha| = n \,.$$

Then by Gagliardo-Nirenberg interpolation inequalities, we conclude that all lower order derivatives are also bounded. Observe that the above estimates also work for any $\tilde{\Omega}$ that contains Ω (by viewing u_k and u as functions defined on $\tilde{\Omega}$ but vanish outside Ω, i.e., $u\,|_{\Omega^c} = 0$). Therefore $u \in H_0^n(\Omega)$. □

Finally, by applying the above results, we have a sharper version of the 1st nonlocal Poincaré inequality.

Theorem 7.2.22. *There exists δ_0 and $C(\delta_0)$ such that for all $\delta \in (0, \delta_0]$,*

$$\|u\|_{L^2} \leq C(\delta_0)|u|_{\mathcal{S}^{n,\gamma_n^\delta}} \,, \quad \forall v \in \mathcal{S}_\Omega^{n,\gamma_n^\delta}$$

Proof. Let

$$\frac{1}{A} = \inf\Big\{ \sum_{|\alpha|=n} M(n,\alpha) \int_{\Omega} (D^{\alpha}u(x))^2 dx : u \in H_0^n(\Omega), \|u\|_{L^2} = 1 \Big\}.$$

By standard local Poincaré inequalities, $0 < A < \infty$. We claim that for given ϵ, there exists some $\delta_0(\epsilon)$ such that for all $\delta < \delta_0$ the lemma holds with $C(\delta_0) = A + \epsilon$.

We prove it by contradiction. Suppose there exists a $C > A$, such that for all n, there exist $\delta_k \to 0$ and u_k with the property that

$$\|u_k\|_L^2 = 1 \text{ and } \int_{\Omega\cup\Omega_{\delta_k}} \int_{B_{\delta_k}(\mathbf{0})} \gamma_n^{\delta_k}(|s|) |D_n^s[u_k](x)|^2 ds dx \le \frac{1}{C},$$

then by lemma 7.2.20, u_k is relatively compact in L^2. Moreover, any limit point $u \in H_0^n(\Omega)$ and satisfies

$$\sum_{|\alpha|=n} M(n,\alpha) \int_{\Omega} (D^{\alpha}u(x))^2 dx \le \frac{1}{C}.$$

This contradicts to the assumption that A is the best Poincaré constant. □

7.2.6 Application to peridynamic beams and plates model

We consider an example to illustrate the application of our analytic framework. In [O'Grady and Foster, 2014a; O'Grady and Foster, 2014b], nonlocal peridynamic models for beams and plates have been developed which in the local limit recover the classical Euler-Bernoulli beam and Kirchhoff-Love plate. The well-posedness for the linear peridynamic beams and plates bending elasticity models, along with rigorous connections to their local limits, can be established using the theoretical results established above.

The variational problems Consider $u : \mathbb{R}^d \to \mathbb{R}$ to be the vertical displacement of a beam or plate, with $d = 1$ and $d = 2$ respectively. The total nonlocal bending energy proposed in [O'Grady and Foster, 2014a; O'Grady and Foster, 2014b] is defined by

$$W_\delta(u) = \frac{1}{2} \int_{\Omega\cup\Omega_\delta} \int_{B_\delta(\mathbf{0})} \omega_\delta(|\xi|) \Big(\frac{u(x+\xi) - 2u(x) + u(x-\xi)}{|\xi|} \Big)^2 d\xi dx, \tag{7.37}$$

where for consistency with discussions in the earlier sections, the notation ω_δ is used instead of the original notation appeared in [O'Grady and Foster, 2014a; O'Grady and Foster, 2014b].

Notice that the nonlocal bending energy (7.37) is exactly one half of the functional (7.7) with $n = 2$ and $\gamma_2(|\xi|)$ being replaced by $\frac{\omega_\delta(|\xi|)}{|\xi|^2}$. The existence of the minimizer of the energy (7.37) can be seen through the following formulation of the variational problem.

We associate it with a bilinear form $b_\delta(\cdot,\cdot) : \mathcal{S}_\Omega^{2,\gamma^\delta} \times \mathcal{S}_\Omega^{2,\gamma^\delta} \to \mathbb{R}$ through

$$b_\delta(u,v) = ((u,v))_{\mathcal{S}^{2,\gamma^\delta}} \,,$$

for any given $\delta > 0$, where $\gamma^\delta(|\xi|) = \omega_\delta(|\xi|)/|\xi|^2$ is supported on $B_\delta(\mathbf{0})$. We then consider the variational problem defined by: given $f \in L^2(\Omega)$,

$$\text{find } u_\delta \in \mathcal{S}_\Omega^{2,\gamma^\delta} \text{ such that } b_\delta(u_\delta, v) = (f,v)_{L^2} \quad \forall v \in \mathcal{S}_\Omega^{2,\gamma^\delta} \,. \tag{7.38}$$

As before, $b_\delta(\cdot,\cdot)$ induces a natural linear operator $\mathcal{L}^\delta : \mathcal{S}_\Omega^{2,\gamma^\delta} \to (\mathcal{S}_\Omega^{2,\gamma^\delta})^*$ via

$$\langle \mathcal{L}^\delta u, v\rangle = b_\delta(u,v) \qquad \text{for } u,v \in \mathcal{S}_\Omega^{2,\gamma^\delta}$$

which, based on Section 2.3, corresponds to equation (7.18) with $n = 2$. Therefore

$$\mathcal{L}^\delta u(x) = (\text{P.V.}) \int_{B_\delta(\mathbf{0})} \omega_\delta(|\xi|)/|\xi|^2 D_4^\xi[u](x) d\xi \quad \text{for } x \in \Omega \tag{7.39}$$

with

$$D_4^\xi[u](x) = u(x+2\xi) - 4u(x+\xi) + 6v(x) - 4u(x-\xi) + u(x+2\xi) \,.$$

The limiting behavior for vanishing nonlocality To study the limit as $\delta \to 0$, we first, similar to the practice in [O'Grady and Foster, 2014a; O'Grady and Foster, 2014b], assume that

$$m = \int_{B_\delta(0)} \omega_\delta(|\xi|)|\xi|^2 \,.$$

Now suppose that $u \in H^n$, by applying lemma 7.2.20 to the functional (7.37), namely, with $n = 2$ and $N = 1, 2$, we can get the limit energy functional. Notice that for $N = 1$, we have only one case $\alpha = \beta = \theta = 2$. For $N = 2$, the values of α, β and θ we can take are listed as follows,

- $\alpha = (2, 0)$, $\beta = \theta = (2, 0)$,
- $\alpha = (0, 2)$, $\beta = \theta = (0, 2)$,

- $\alpha = (1,1)$, $\beta = \theta = (1,1)$,
- $\alpha = (1,1)$, $\beta = (2,0)$, $\theta = (0,2)$,
- $\alpha = (1,1)$, $\beta = (0,2)$, $\theta = (2,0)$.

Hence, after collecting like terms, the limit local energy functional of (7.37) is of the form

$$W_0(u) = \frac{am}{2}\int_\Omega (u''(x))^2 dx$$

for $N = 1$, or, we have for $N = 2$,

$$W_0(u) = \frac{3am}{16}\int_\Omega (\Delta u(x))^2 dx\,.$$

Now define the associated bilinear form $b_0(\cdot,\cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$ as

$$b_0(u,v) = \begin{cases} m\displaystyle\int_\Omega u''(x)v''(x)dx\,, & \text{for } N = 1\,,\\ \dfrac{3m}{8}\displaystyle\int_\Omega \Big(\Delta u(x)\Delta v(x)\Big)dx\,, & \text{for } N = 2\,. \end{cases}$$

Then variational problem is defined by: given $f \in L^2(\Omega)$,

$$\text{find } u_0 \in H_0^2(\Omega) \text{ such that } b_0(u_0, v) = (f,v)_{L^2} \quad \forall v \in H_0^2(\Omega)\,. \tag{7.40}$$

Similarly, we can associate a linear operator $\mathcal{L}^0 : H_0^n(\Omega) \to (H_0^n(\Omega))^*$ through

$$\langle \mathcal{L}^0 u, v\rangle = b_0(u,v) \quad \text{ for } u, v \in H_0^2(\Omega)\,.$$

Now from integration by part, we know that for $u \in C^\infty$, $\mathcal{L}^0 u$ can be written as

$$\mathcal{L}^0 u(x) = \begin{cases} mu^{(4)}(x) & \text{for } N = 1\,;\\ \dfrac{3m}{8}\Delta^2 u(x) & \text{for } N = 2. \end{cases}$$

Lemma 7.2.23. *For all $u \in C_c^\infty(\Omega)$, and $x \in \Omega$, we have*

$$\mathcal{L}^\delta u(x) \to \mathcal{L}^0 u(x) \quad \text{ as } \delta \to 0\,.$$

Proof. Through Taylor expansion, it is straightforward to check the convergence. Here we only prove for $N = 2$. In this case,

$$\begin{aligned}
&\int_{B_\delta(\mathbf{0})} \omega_\delta(|\xi|)/|\xi|^2 D_4^\xi[u](x)d\xi \\
&= \sum_{j=0}^{4}(-1)^j \binom{4}{j} (a_4^j)^4 \sum_{|\alpha|=4} \frac{D^\alpha u(x)}{\alpha!}\Big(\int_{B_\delta(\mathbf{0})} \frac{\omega(|\xi|)}{|\xi|^2}\xi^\alpha d\xi + O\big(\int_{B_\delta(\mathbf{0})} \omega(|\xi|)|\xi|^3 d\xi\big)\Big) \\
&= 24 \sum_{|\alpha|=4} \frac{D^\alpha u(x)}{\alpha!}\Big(\int_{B_\delta(\mathbf{0})} \frac{\omega(|\xi|)}{|\xi|^2}\xi^\alpha d\xi + O\big(\int_{B_\delta(\mathbf{0})} \omega(|\xi|)|\xi|^3 d\xi\big)\Big)\,.
\end{aligned}$$

Notice that for $|\alpha| = 4$, the integrals of $\xi^\alpha \omega(|\xi|)/|\xi|^2$ over $B_\delta(\mathbf{0})$ are not zero only for $\alpha = (4,0), (0,4), (2,2)$. Then

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{B_\delta(\mathbf{0})} \omega(|\xi|)/|\xi|^2 D_4^\xi[u](x)d\xi \\
&= \int_0^\delta \omega(r)r^3 dr\Big(\frac{3\pi}{4}\frac{\partial^4 u}{\partial x_1^4} + \frac{3\pi}{4}\frac{\partial^4 u}{\partial x_2^4} + \frac{6\pi}{4}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}\Big) \\
&= \frac{3}{8}\Big(\int_{B_\delta(0)} \omega_\delta(|\xi|)|\xi|^2 d\xi\Big)\Delta^2 u(x) = \frac{3m}{8}\Delta^2 u(x)\,.
\end{aligned}$$

□

From the proof the above result, we see that the convergence is not only pointwise, but also uniform in Ω. So it is easy to see also that $\mathcal{L}^\delta u \to \mathcal{L}^0 u$ in $L^2(\Omega)$.

Theorem 7.2.24. *The variational problem* (7.38) *is well posed with a unique solution* $u_\delta \in \mathcal{S}_\Omega^{2,\gamma^\delta}$ *with a uniformly bounded norm* $\|u_\delta|_{\mathcal{S}_\Omega^{2,\gamma^\delta}}$*, independent of* $\delta > 0$*. Moreover,*

$$\|u_\delta - u_0\|_{L^2} \to 0 \qquad as\ \delta \to 0,$$

where u_0 *is the solution of the local variational problems*(7.40)*.*

Proof. First, for each given $\delta > 0$, by Lax-Milgram via the 1st nonlocal Poincaré inequality in Theorem 7.2.12, we have the existence of a unique solution $u_\delta \in \mathcal{S}_\Omega^{2,\gamma^\delta}$ to (7.38) and with a uniformly bounded norm $\|u_\delta|_{\mathcal{S}_\Omega^{2,\gamma^\delta}}$, independent of $\delta > 0$.

As for the local limit as $\delta \to 0$, we first have estimates for $|u_\delta|_{\mathcal{S}^{2,\gamma^\delta}}$ as follows.

$$|u_\delta|^2_{\mathcal{S}^{2,\gamma^\delta}} = b_\delta(u_\delta, u_\delta) = (f, u_\delta)_{L^2} \le \|f\|_{L^2}\|u\|_{L^2} \le C\|f\|_{L^2}|u_\delta|_{\mathcal{S}^{2,\gamma^\delta}}\,,$$

by the sharp Poincaré inequality, which implies that $|u_\delta|_{\mathcal{S}^{2,\gamma^\delta}}$ is uniformly bounded. Then $\{u_\delta\}$ is relatively compact in L^2 by Theorem 7.2.21, and each limit point $u \in H_0^n(\Omega)$. Now we only need to show that $u = u_0$. This is true because for any $v \in C_c^\infty(\Omega)$,

$$\begin{aligned}(f - \mathcal{L}^0 u, v)_{L^2} &= (\mathcal{L}^\delta u_\delta - \mathcal{L}^0 u, v)_{L^2} = (u_\delta, \mathcal{L}^\delta v)_{L^2} - (u_0, \mathcal{L}^0 v)_{L^2} \\ &= (u_\delta, \mathcal{L}^\delta v - \mathcal{L}^0 v)_{L^2} + (u_\delta - u, \mathcal{L}^0 v)_{L^2} \\ &\leq \|u_\delta\|_{L^2}\|\mathcal{L}^\delta v - \mathcal{L}^0 v\|_{L^2} + \|u_\delta - u\|_{L^2}\|\mathcal{L}^0 v\|_{L^2} \to 0\end{aligned}$$

as $\delta \to 0$. □

7.2.7 Conclusion

Our study has focused on generalizing analytical properties associated with the nonlocal diffusion operator to higher order nonlocal operators corresponding to, in the local limit, high order elliptic differential operators. Naturally, nonlocal extensions of local differential operators can be defined in various fashions that are different from the way given in this section. For example, one may take compositions of low order nonlocal operators directly to get high order ones, though such a formulation involves integrations in higher and higher dimensional spaces. Indeed, we note with much interest a recent work [Radu *et al.*, 2016] which has developed a nonlocal biharmonic operator as a square of the nonlocal diffusion (Laplacian) operator. In there, well-posedness and local limit to the conventional biharmonic operators subject to various type of boundary conditions. Furthermore, One may also consider combinations of nonlocal operators in these different forms to model various physical systems, a practice that was used in [Du *et al.*, 2013a; Mengesha and Du, 2014c]. Such studies may find more applications in the analysis of nonlocal, nonlinear systems and serve as the rigorous foundation to asymptotically compatible schemes in flavor of those presented in chapter 3.

Chapter 8

A new trace theorem for nonlocal models

In this chapter, we will show a new trace theorem on a class of nonlocal spaces. On a domain $\Omega \subset \mathbb{R}^d$, a trace operator T on a subset Γ of $\partial\Omega$ is defined as

$$Tu = u|_\Gamma \qquad \forall u \in C^1(\bar{\Omega})\,,$$

where $\bar{\Omega}$ is the closure of Ω. It is a classical result of Gagliardo [Gagliardo, 1957] that the linear operator T can be extended continuously as a map from $H^1(\Omega)$, the standard Sobolev space of L^2 functions with square integrable derivatives, to $H^{1/2}(\Gamma)$. We aim at extending this result to a class of nonlocal spaces that contain H^1 as a subspace. The continuity of the trace map from $H^{1/2}(\Gamma)$ to such nonlocal spaces are shown by Theorem 8.3.1 and Theorem 8.3.2 .

8.1 Motivations for a new trace theorem

Generically, nonlocal equations posed on a domain $\Omega \subset \mathbb{R}^d$ are complemented by nonlocal boundary conditions or more precisely, constraints on a some domain with nonzero d-dimensional volume, hence leading to so-called constrained value problems [Du *et al.*, 2012]. To avoid the use of such nonlocal constraints, the nonlocal operators need to be properly modified near the boundary which is often the case for fractional differential equations [Caffarelli and Silvestre, 2007]. In order to have well-defined nonlocal problems on Ω with Dirichlet type data on part of its boundary $\partial\Omega$ of codimension-1, study of the trace map becomes a necessity. More often the trace map is defined if functions under consideration enjoy suitable interior regularity. A consequence of a well-defined trace map with the trace belonging to a space similar to that for standard Sobolev spaces would allow a seamless coupling between a classical, local PDE (for instance the Poisson equation $-\Delta u = f$) on one side Ω_- of a codimension-1 interface Γ with a nonlocal equation (say the variational equation $-\mathcal{L}u = f$ associated with the nonlocal energy) on the other side Ω_+ of Γ, see Fig. 8.1 for an illustration (the circular domains depict domains of nonlocal interactions associated with a heterogeneously defined horizon parameter).

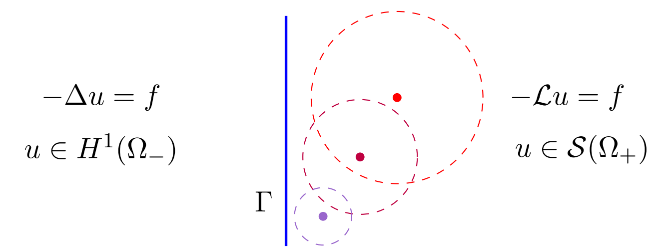

Figure 8.1: A PDE model (in Ω_-) is coupled with a nonlocal model (in Ω_+) using suitably defined boundary trace and transmission condition on Γ.

We note that the new nonlocal trace theorems can be viewed as extensions and refinement of their classical counterparts. Indeed, this can be appreciated from different perspectives.

First, the approach taken in this chapter provides one avenue to achieve sufficient regularity for defining the trace map without imposing extra regularity away from the boundary. More specifically, for a proper subdomain Ω' of Ω, with a positive distance away from $\partial\Omega$, it is easy to see that with the kernel given by (8.5), functions in $\mathcal{S}(\Omega)$ are generally not expected to have regularity better than $L^2(\Omega')$ over the subdomain Ω', or may be significantly less regular away from the boundary than H^1 functions. Yet, as elucidated in the introduction and rigorously established in the theorems, due to the shrinking horizon towards the boundary, there is enough regularity for these functions to have well-defined traces just on the boundary itself. Intuitively, this is a natural consequence of the localization of nonlocal interactions on the boundary.

On the other hand, it is well-known that [Bourgain *et al.*, 2001], for a translation invariant and radial kernel $\gamma(\mathbf{x},\mathbf{y}) = \tilde{\gamma}(|\mathbf{x}-\mathbf{y}|)$ with a finite second moment

$$\int_{\mathbb{R}^d} \tilde{\gamma}(|\mathbf{x}|)|\mathbf{x}|^2 d\mathbf{x} < \infty,$$

the nonlocal norm is bounded from above by a suitable multiple of the conventional H^1 norm and the Sobolev space H^1 is continuously imbedded in the corresponding nonlocal function space. It is natural to expect that such a result can be extended to the variable horizon case with a localization feature on the boundary so that the classical trace theorem of H^1 space becomes a direct consequence.

8.2 The nonlocal function spaces with heterogeneous nonlocal interaction

8.2.1 Definitions

On a domain $\Omega \subset \mathbb{R}^d$, the nonlocal function space $\mathcal{S}(\Omega)$ with heterogeneous nonlocal interaction is the completion of $C^1(\bar{\Omega})$ with respect to the norm

$$\|\cdot\|_{\mathcal{S}(\Omega)} = (\|\cdot\|^2_{L^2(\Omega)} + |\cdot|^2_{\mathcal{S}(\Omega)})^{1/2}, \tag{8.1}$$

with the associated nonlocal semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$ defined by

$$|u|^2_{\mathcal{S}(\Omega)} = \int_\Omega \int_{\Omega\cap\mathcal{H}(\boldsymbol{x})} \gamma(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x}\,, \tag{8.2}$$

corresponding to some nonlocal interaction kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$ given by

$$\gamma(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}\left(\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\delta(\boldsymbol{x})}\right) \tag{8.3}$$

where $\hat{\gamma} = \hat{\gamma}(s)$ is a non-increasing nonnegative function defined for $s \in (0,1)$ with a finite $d+1$ moment. The influence horizon $\delta = \delta(\mathbf{x})$ is a function defined on Ω that approaches zero when $\boldsymbol{x}$ approaches the boundary. A simple choice would be

$$\delta(\boldsymbol{x}) = \sigma\, \mathrm{dist}(x,\Gamma), \quad \boldsymbol{x} \in \Omega\,, \tag{8.4}$$

for some $\sigma > 0$ and $\Gamma \subset \partial\Omega$. For simplicity of illustration, a specific kernel where $\hat{\gamma}$ is a characteristic function on $(0,1)$ is used in most of the sections, namely,

$$\gamma(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{|\delta(\boldsymbol{x})|^{d+2}}\chi_{(0,1)}(|\boldsymbol{y}-\boldsymbol{x}|)\,. \tag{8.5}$$

More general discussions are put to section 8.4. The associated nonlocal neighborhood $\mathcal{H}(\boldsymbol{x})$ is defined by

$$\mathcal{H}(\boldsymbol{x}) := \{\boldsymbol{y}\in\Omega : |\boldsymbol{y}-\boldsymbol{x}| \le \delta(\boldsymbol{x})\}\,.$$

Before getting into the trace theorem, we first will look more closely at the nonlocal norm $\|\cdot\|_{\mathcal{S}(\Omega)}$ and derive some of its properties.

8.2.2 H^1 as a subspace of the nonlocal function space

First, we are going to show that the $H^1(\Omega)$ space is continuously embedded in $\mathcal{S}(\Omega)$.

Proposition 8.2.1. *For $\delta(\boldsymbol{x}) = \sigma\cdot dist(x,\Gamma)$ with $\sigma\in(0,1)$, the space $H^1(\Omega)$ is continuously imbedded in $\mathcal{S}(\Omega)$ and there exists a constant C depending only on σ and Ω such that*

$$\|u\|_{\mathcal{S}(\Omega)} \le C\|u\|_{H^1(\Omega)}\,, \quad \forall u\in H^1(\Omega)\,. \tag{8.6}$$

Moreover, C is independent of σ for σ small.

Proof. We begin with a proof of (8.6) for a smooth function $u\in C^1(\bar{\Omega})\cap H^1(\Omega)$.

Let the kernel γ be defined as (8.3). We can have a standard extension of u to $\mathbb{R}^d$ such that

$$\|u\|_{H^1(\mathbb{R}^d)} \le C_1\|u\|_{H^1(\Omega)}\,,$$

where C_1 only depends on Ω. Notice that for any $\boldsymbol{h}\in\mathbb{R}^d$,

$$|u(\boldsymbol{x}+\boldsymbol{h}) - u(\boldsymbol{x})|^2 = \Big|\int_0^1 \nabla u(\boldsymbol{x}+t\boldsymbol{h})\cdot\boldsymbol{h}dt\Big|^2 \le |\boldsymbol{h}|^2\int_0^1 |\nabla u(\boldsymbol{x}+t\boldsymbol{h})|^2dt\,.$$

So

$$\begin{aligned}
|u|^2_{\mathcal{S}(\Omega)} &\le \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<\delta(\boldsymbol{x})} \gamma(\mathbf{x},\mathbf{x}+\boldsymbol{h})|u(\boldsymbol{x}+\boldsymbol{h})-u(\boldsymbol{x})|^2 d\boldsymbol{h}d\boldsymbol{x}\\
&\le \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<\delta(\boldsymbol{x})} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}}\hat{\gamma}\Big(\frac{|\boldsymbol{h}|}{\delta(\boldsymbol{x})}\Big)|u(\boldsymbol{x}+\boldsymbol{h})-u(\boldsymbol{x})|^2 d\boldsymbol{h}d\boldsymbol{x}\\
&= \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<1} \frac{1}{|\delta(\boldsymbol{x})|^2}\hat{\gamma}(\boldsymbol{h})|u(\boldsymbol{x}+\delta(\boldsymbol{x})\boldsymbol{h})-u(\boldsymbol{x})|^2 d\boldsymbol{h}d\boldsymbol{x}\\
&\le \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<1} |\boldsymbol{h}|^2\hat{\gamma}(\boldsymbol{h})\int_0^1 |\nabla u(\boldsymbol{x}+t\delta(\boldsymbol{x})\boldsymbol{h})|^2 dtd\boldsymbol{h}d\boldsymbol{x}\,.
\end{aligned}$$

Let $\boldsymbol{y} = \boldsymbol{x} + t\delta(\boldsymbol{x})\boldsymbol{h}$, we see that

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = I + t\nabla\delta(\boldsymbol{x}) \otimes \boldsymbol{h}\,,$$

and its inverse are uniformly bounded everywhere if $\|\nabla\delta\| = \sigma < 1$. Moreover, the bounds are independent of σ if σ is small. Thus, there is a generic constant $C > 0$ such that

$$\begin{aligned}|u|^2_{\mathcal{S}(\Omega)} &\leq C(\int_{|\boldsymbol{h}|<1} |\boldsymbol{h}|^2 \hat{\gamma}(\boldsymbol{h}) d\boldsymbol{h})|u|^2_{H^1(\mathbb{R}^d)} \\ &\leq C\|u\|^2_{H^1(\Omega)}\,.\end{aligned}$$

The constant C may depend on Ω but is independent of σ for σ small.

Putting together, we have the inequality (8.6) for $u \in C^1(\bar{\Omega})$. Now invoking density argument, since $H^1(\Omega)$ is the completion of $C^1(\bar{\Omega})$ under the $\|\cdot\|_{H^1(\Omega)}$ norm, we easily see that (8.6) is true in $H^1(\Omega)$ and the continuous imbedding $H^1(\Omega)$ in the space $\mathcal{S}(\Omega)$ which is the completion of $C^1(\bar{\Omega})$ under a weaker norm. □

8.2.3 The nonlocal Hardy's inequality

The classical Hardy's inequality, see for instance [Davies, 1999], involves a bound on a weighted function norm by some norm of first order derivatives over the domain. In this part we focus on generalizing Hardy's type inequalities to the nonlocal spaces. Intuitively, our generalizations are derived by saying that the first derivative does not need to be well defined everywhere in the domain, but only at the place where the weighting factor blows up or when the nonlocal interactions are localized. The results are crucial in showing the new trace inequality and are also of interests on their own.

Proposition 8.2.2 (Nonlocal Hardy-type inequality). *Let $\Omega = (0, r)$ for some $r > 0$ and $u \in C^1(\bar{\Omega})$ with $u(0) = 0$, then we have*

$$\int_\Omega \frac{|u(x)|^2}{|x|^2} dx \leq C_{a,b} \int_\Omega \int_{ax}^{bx} \frac{|u(y) - u(x)|^2}{|x|^3} dy dx\,, \tag{8.7}$$

where $C_{a,b} = \frac{4(2+b+a)}{(b-a)(2-b-a)^2}$ with a and b satisfy $0 \leq a < b \leq 1$. In particular, this implies the Hardy-type inequality, for a constant $C > 0$ independent of Ω such that,

$$\int_\Omega \frac{|u(x)|^2}{dist(x, \Gamma)^2} dx \leq C|u|^2_{\mathcal{S}(\Omega)} \tag{8.8}$$

where $\Gamma = \{0\}$.

Proof. For any $x \in \Omega$ and $y \in \Omega$, let us write

$$u(x) = u(x) - u(y) + u(y)\,,$$

from which we get

$$|u(x)|^2 \le (1 + \frac{1}{\epsilon})|u(y) - u(x)|^2 + (1+\epsilon)|u(y)|^2\,,$$

where ϵ is a small number to be determined. Now integrating y over the interval (ax, bx), we get for $x \in \Omega$,

$$|u(x)|^2 \le \frac{1+1/\epsilon}{bx - ax}\int_{ax}^{bx} |u(y) - u(x)|^2 dy + \frac{1+\epsilon}{bx-ax}\int_{ax}^{bx} |u(y)|^2 dy\,,$$

so that

$$\begin{aligned}\int_\Omega \frac{|u(x)|^2}{|x|^2}dx &\le \frac{1+1/\epsilon}{b-a}\int_\Omega\int_{ax}^{bx}\frac{|u(y)-u(x)|^2}{|x|^3}dydx + \frac{1+\epsilon}{b-a}\int_\Omega\int_{ax}^{bx}\frac{|u(y)|^2}{|x|^3}dydx \\ &= \mathrm{I} + \mathrm{II}\,.\end{aligned}$$

The term I is our desired bound, and for the term II, since $u \in C^1(\bar\Omega)$ and $u(0) = 0$, we can use Fubini's theorem to change the order of integration to get

$$\begin{aligned}\mathrm{II} &= \frac{1+\epsilon}{b-a}\int_0^r\int_{ax}^{bx}\frac{|u(y)|^2}{|x|^3}dydx \\ &= \frac{1+\epsilon}{b-a}\int_0^{br}\int_{y/b}^{\max\{y/a,r\}}\frac{|u(y)|^2}{|x|^3}dxdy\,.\end{aligned}$$

Notice that $0 \le a < b \le 1$, we get

$$\begin{aligned}\mathrm{II} &\le \frac{1+\epsilon}{b-a}\int_0^r\int_{y/b}^{y/a}\frac{|u(y)|^2}{|x|^3}dxdy \\ &= \frac{1+\epsilon}{b-a}\cdot\frac{1}{2}(b^2 - a^2)\int_\Omega\frac{|u(y)|^2}{|y|^2}dy \\ &= \frac{(1+\epsilon)(b+a)}{2}\int_\Omega\frac{|u(x)|^2}{|x|^2}dx\,.\end{aligned}$$

Now since $b + a < 2$, we can pick $\epsilon = \frac{1}{b+a} - \frac{1}{2}$ to get

$$\begin{aligned}\int_\Omega\frac{|u(x)|^2}{|x|^2}dx &\le \frac{1+1/\epsilon}{b-a}\cdot\frac{2}{2-(1+\epsilon)(b+a)}\int_\Omega\int_{ax}^{bx}\frac{|u(y)-u(x)|^2}{|x|^3}dydx \\ &= C_{a,b}\int_\Omega\int_{ax}^{bx}\frac{|u(y)-u(x)|^2}{|x|^3}dydx\,.\end{aligned}$$

At last, inequality (8.8) follows straightforwardly from (8.7).

□

Although the Proposition 8.2.2 only shows the nonlocal Hardy-type inequality for the one dimensional case, it is not hard to see that the more general cases are also true.

Corollary 8.2.3 (Nonlocal Hardy's inequality in a multi-dimensional stripe domain)**.** *Let* $\Omega = (0, r) \times \mathbb{R}^{d-1}$ *(*$d \geq 2$*) and* $\Gamma = \{0\} \times \mathbb{R}^{d-1}$*. Assume that* $u \in C^1(\bar{\Omega})$ *and* $u(0, \bar{\boldsymbol{x}}) = 0$ *for* $\bar{\boldsymbol{x}} \in \mathbb{R}^{d-1}$*, then*

$$\int_\Omega \frac{|u(\boldsymbol{x})|^2}{dist(\boldsymbol{x}, \Gamma)^2} d\boldsymbol{x} \leq C|u|^2_{\mathcal{S}(\Omega)} \,. \tag{8.9}$$

Proof. Use Proposition 8.2.2, we have

$$\begin{aligned} \int_\Omega \frac{|u(\boldsymbol{x})|^2}{(\mathrm{dist}(\boldsymbol{x}, \Gamma))^2} d\boldsymbol{x} &= \int_{\mathbb{R}^{d-1}} \int_0^r \frac{|u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} dx_1 d\bar{\boldsymbol{x}} \\ &\leq \int_{\mathbb{R}^{d-1}} \int_0^r \int_{ax_1}^{bx_1} \frac{|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^3} dy_1 dx_1 d\bar{\boldsymbol{x}} \,, \end{aligned} \tag{8.10}$$

where the last term is bounded by $C|u|^2_{\mathcal{S}(\Omega)}$ by Lemma 8.2.7 that is shown later in this section. □

We note first that while Proposition 8.2.2 and Corollary 8.2.3 are shown for a specific kernel with influence horizon $\delta(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \Gamma)$, from the proof, we can see that the nonlocal Hardy's inequality also holds for any $\delta(\boldsymbol{x}) = \sigma \mathrm{dist}(\boldsymbol{x}, \Gamma)$ with constant C depending continuously on σ.Moreover, we will see in section 8.4 that C can be made a uniform constant with respect to σ for $\sigma \to 0$.

Secondly, although the nonlocal Hardy's inequality is presented only for a strip domain, it is not hard to see that it also holds for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. The procedure to establish such result is by the standard arguments of partition of unity which be avoided here. For readers that are interested in the proof, it follows a similar path as in the proof of Theorem 8.3.2 starting from the Theorem 8.3.1 which will come later.

Proposition 8.2.4 (Nonlocal Hardy's inequality)**.** *Given a bounded Lipshitz domain* Ω*, there exists a constant* $C > 0$ *such that if* $Tu = 0$ *on* $\partial\Omega$*, then*

$$\int_\Omega \frac{|u(\boldsymbol{x})|^2}{(dist(\boldsymbol{x}, \partial\Omega))^2} d\boldsymbol{x} \leq C|u|^2_{\mathcal{S}(\Omega)} \,. \tag{8.11}$$

8.2.4 The nonlocal partial derivatives

The part introduces some special quantities mimicking norms of directional derivatives that are particularly matched with our nonlocal setting. The study of such norms, however, are more involved technically than their local analog and require new techniques that, to our knowledge, have not beed used in the literature before.

The last integral in (8.10) involves weighted variations of the function u in its first component. In the same spirit of norms of directional derivatives in classical, local function spaces, we introduce the following definition as a nonlocal analog that refines our understanding of how the nonlocal norm $\|\cdot\|_{\mathcal{S}(\Omega)}$ provides control on the function variation in differential directions. This not only helps proving (8.10) but also plays important roles in proving the new trace theorems. For brevity of notation, $⨍$ is used to represent the integral average over the respective domain, that is, the integral over the domain divided by the volume of domain.

Definition 8.2.5. *On the domain $\Omega = (0, r) \times \mathbb{R}^{d-1}$, we define in the following two directional nonlocal semi-norms $|\cdot|_n$ and $|\cdot|_t$, standing for normal and tangential directions respectively with reference to the boundary segment $\Gamma = \{0\} \times \mathbb{R}^{d-1}$,*

$$|u|_n^2 = \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{ax_1}^{bx_1} \frac{|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 dx_1 d\bar{\boldsymbol{x}} \tag{8.12}$$

$$|u|_t^2 = \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|u(x_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \tag{8.13}$$

where $0 \le a < b \le 1$ and $0 < c < 1$ are constants.

***Remark* 8.2.6.** *To offer some insight, we make some heuristic comments. For a smooth function $u = u(\mathbf{x})$, we may approximately have, in an informal manner, that*

$$\begin{aligned}
|u|_n^2 &\approx \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{ax_1}^{bx_1} \frac{|y_1 - x_1|^2}{|x_1|^2} |u_{x_1}(x_1, \bar{\boldsymbol{x}})|^2 dy_1 dx_1 d\bar{\boldsymbol{x}} \\
&= C_n(a, b) \int_{\mathbb{R}^{d-1}} \int_0^r |u_{x_1}(x_1, \bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}} \\
|u|_t^2 &\approx \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|\nabla_{\bar{\boldsymbol{x}}} u(x_1, \bar{\boldsymbol{x}}) \cdot (\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \\
&= C_t(c, d) \int_{\mathbb{R}^{d-1}} \int_0^r |\nabla_{\bar{\boldsymbol{x}}} u(x_1, \bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}}
\end{aligned}$$

for some constants $C_n(a,b)$ and $C_t(c,d)$ that can be computed explicitly. This provides a hint that $|\cdot|_n$ and $|\cdot|_t$ may indeed mimic norms of directional directives. We thus see that it is reasonable to call $|\cdot|_n$ and $|\cdot|_t$ directional semi-norms. In comparion, we may also informally expand $|\cdot|^2_{\mathcal{S}(\Omega)}$ as

$$|u|^2_{\mathcal{S}(\Omega)} \sim \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{B_{x_1}(x_1,\bar{\boldsymbol{x}})\cap\Omega} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \approx C\|\nabla u\|^2_{L^2(\Omega)} .$$

More discussions to rigorously connect the nonlocal semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$ with $|\cdot|_{H^1(\Omega)}$ are to be made later in section 8.4.

As the norms of directional directives are obviously bounded by that of the total gradient, we may extend to the nonlocal case by establishing the following lemma saying that $|\cdot|_n$ and $|\cdot|_t$ are controlled by the original semi-norm $|\cdot|_{\mathcal{S}}$.

Lemma 8.2.7. *Let $\Omega = (0,r)\times\mathbb{R}^{d-1}$ for some $r>0$, a, b and c satisfy $0 \le a < b \le 1$, $0<c<1$ and $(a-1)^2 + c^2 \le 1$. Then there exists a constant C depending only on a, b and c such that for any $u \in \mathcal{S}(\Omega)$,*

$$|u|_n \le C|u|_{\mathcal{S}(\Omega)} , \tag{8.14}$$

$$|u|_t \le C|u|_{\mathcal{S}(\Omega)} . \tag{8.15}$$

Proof. First, let us briefly describe the idea of the proof. Instead of showing (8.14) and (8.15) directly, we show the following two inequalities instead.

$$|u|^2_n \le c_1|u|^2_t + C|u|^2_{\mathcal{S}} \tag{8.16}$$

$$|u|^2_t \le c_2|u|^2_n + C|u|^2_{\mathcal{S}} \tag{8.17}$$

where $c_1c_2 < 1$. We see that they immediately yield both (8.14) and (8.15). Moreover, by density argument, we can focus only on showing (8.16) and (8.17) for $u \in C^1(\bar{\Omega}) \cap \mathcal{S}(\Omega)$.

For any $(y_1,\bar{\boldsymbol{y}}) \in \Omega$, let us write

$$u(y_1,\bar{\boldsymbol{x}}) - u(x_1,\bar{\boldsymbol{x}}) = u(y_1,\bar{\boldsymbol{x}}) - u(y_1,\bar{\boldsymbol{y}}) + u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}}) ,$$

and we get the estimate

$$|u(y_1,\bar{\boldsymbol{x}}) - u(x_1,\bar{\boldsymbol{x}})|^2 \le (1+\epsilon)|u(y_1,\bar{\boldsymbol{x}}) - u(y_1,\bar{\boldsymbol{y}})|^2 + (1+\frac{1}{\epsilon})|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})|^2 ,$$

where ϵ is a small number to be determined. The relative positions of those points are depicted in Figure 8.2. The purple horizon dotted line shows the range of $(y_1, \bar{\boldsymbol{x}})$, the blue vertical dotted line for $(y_1, \bar{\boldsymbol{y}})$, and the red vertical dashed line for $(x_1, \bar{\boldsymbol{y}})$. The key to choose these positions is to make sure that $(y_1, \bar{\boldsymbol{y}})$ stands in the effective neighborhood of $(x_1, \bar{\boldsymbol{x}})$ bounded by the black dashed circle.

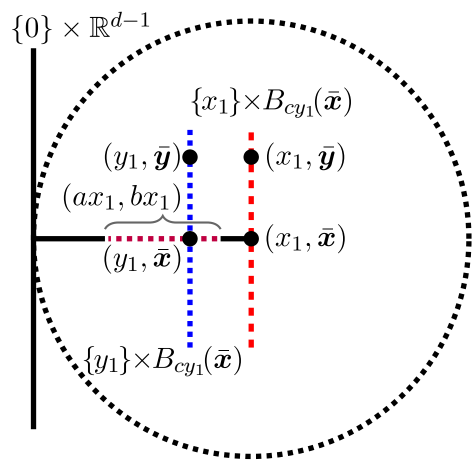

Figure 8.2: Depiction of geometry used in the proof of Lemma 8.2.7.

Now integrating $\bar{\boldsymbol{y}}$ over the ball $B_{cy_1}(\bar{\boldsymbol{x}})$ we have

$$\begin{aligned}|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2 \leq &(1+\epsilon) \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} |u(y_1, \bar{\boldsymbol{x}}) - u(y_1, \bar{\boldsymbol{y}})|^2 d\bar{\boldsymbol{y}} \\ &+ (1+1/\epsilon) \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} |u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2 d\bar{\boldsymbol{y}} \,.\end{aligned}$$

So, we have

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}} \int_0^r \fint_{ax_1}^{bx_1} \frac{|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 dx_1 d\bar{\boldsymbol{x}} \\ &\leq (1+\epsilon) \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{ax_1}^{bx_1} \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1, \bar{\boldsymbol{x}}) - u(y_1, \bar{\boldsymbol{y}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dy_1 dx_1 d\bar{\boldsymbol{x}} \\ &\quad + (1+1/\epsilon) \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{ax_1}^{bx_1} \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dy_1 dx_1 d\bar{\boldsymbol{x}} \\ &= \mathrm{I} + \mathrm{II} \,.\end{aligned} \tag{8.18}$$

It is easy to see that II is controlled by $|u|_{\mathcal{S}}^2$ since

$$\begin{aligned}\mathrm{II} &= (1+1/\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{ax_1}^{bx_1}\fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}}dy_1dx_1d\bar{\boldsymbol{x}} \\ &\le C(1+1/\epsilon)\int_\Omega \fint_{\mathcal{H}(\boldsymbol{x})\cap\Omega} \frac{|u(\boldsymbol{y})-u(\boldsymbol{x})|^2}{|x_1|^2} d\boldsymbol{y}d\boldsymbol{x}\,,\end{aligned}$$

where the last inequality is true since $\boldsymbol{y}\in\mathcal{H}(\boldsymbol{x})\cap\Omega$, a result we can see by using the assumption on a, b and c,

$$|\boldsymbol{y}-\boldsymbol{x}|^2 = (y_1-x_1)^2 + |\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2 \le (a-1)^2|x_1|^2 + c^2b^2|x_1|^2 \le |x_1|^2\,,$$

where $\bar{\boldsymbol{y}}\in B_{cy_1}(\bar{\boldsymbol{x}})$ and $ax_1\le y_1\le bx_1$ are used.

Now for I, by using Fubini's thoerem, we have

$$\begin{aligned}\mathrm{I} &= \frac{(1+\epsilon)}{b-a}\int_{\mathbb{R}^{d-1}}\int_0^r\int_{ax_1}^{bx_1}\fint_{B_{cy_1}(\bar{\boldsymbol{x}})}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{x}})|^2}{|x_1|^3}d\bar{\boldsymbol{y}}dy_1dx_1d\bar{\boldsymbol{x}} \\ &\le \frac{(1+\epsilon)}{b-a}\int_{\mathbb{R}^{d-1}}\int_0^r\Big(\int_{y_1/b}^{y_1/a}\fint_{B_{cy_1}(\bar{\boldsymbol{x}})}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{x}})|^2}{|x_1|^3}d\bar{\boldsymbol{y}}dx_1\Big)dy_1d\bar{\boldsymbol{x}} \\ &= \frac{(1+\epsilon)(b+a)}{2}\int_{\mathbb{R}^{d-1}}\int_0^r\fint_{B_{cy_1}(\bar{\boldsymbol{x}})}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{x}})|^2}{|y_1|^2}d\bar{\boldsymbol{y}}dy_1d\bar{\boldsymbol{x}} \\ &= \frac{(1+\epsilon)(b+a)}{2}\int_{\mathbb{R}^{d-1}}\int_0^r\fint_{B_{cx_1}(\bar{\boldsymbol{x}})}\frac{|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2}d\bar{\boldsymbol{y}}dx_1d\bar{\boldsymbol{x}}\end{aligned}$$

Together, (8.16) is proved for $c_1=(1+\epsilon)(b+a)/2$.

As for (8.17), we consider a point $(y_1,\bar{\boldsymbol{y}})\in\Omega$ (as depicted in Figure 8.2) so that $(y_1,\bar{\boldsymbol{y}})$ is in the effective neighborhood of $(x_1,\bar{\boldsymbol{x}})$. $|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2$ is estimated in a similar fashion by

$$|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2 \le (1+\epsilon)|u(x_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{y}})|^2 + (1+\frac{1}{\epsilon})|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2\,.$$

Integrating y_1 over the interval (ax_1,bx_1), we get

$$\begin{aligned}|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2 \le&(1+\epsilon)\fint_{ax_1}^{bx_1}|u(x_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{y}})|^2dy_1 \\ &+ (1+1/\epsilon)\fint_{ax_1}^{bx_1}|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2dy_1\,,\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}}dx_1 d\bar{\boldsymbol{x}} \\
&\le (1+\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(x_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}}dx_1 d\bar{\boldsymbol{x}} \\
&\quad + (1+1/\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}}dx_1 d\bar{\boldsymbol{x}} \\
&= \mathrm{III} + \mathrm{IV}\,.
\end{aligned}$$

The term IV is clearly controlled by $|u|^2_{\mathcal{S}}$,

$$\begin{aligned}
\mathrm{IV} &= (1+1/\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}}dx_1 d\bar{\boldsymbol{x}} \\
&\le C(1+1/\epsilon)\int_{\Omega}\fint_{\mathcal{H}(\boldsymbol{x})\cap\Omega} \frac{|u(\boldsymbol{y})-u(\boldsymbol{x})|^2}{|x_1|^2} d\boldsymbol{y}d\boldsymbol{x}\,,
\end{aligned}$$

where the last inequality is derived based on the observation that $\boldsymbol{y}\in\mathcal{H}(\boldsymbol{x})$:

$$|\boldsymbol{y}-\boldsymbol{x}|^2 = (y_1-x_1)^2 + |\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2 \le (a-1)^2|x_1|^2 + c^2|x_1|^2 \le |x_1|^2\,,$$

following assumptions on a, b and c.

For the term III, we use Fubini's theorem to get

$$\begin{aligned}
\mathrm{III} &= (1+\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}}dx_1 d\bar{\boldsymbol{x}} \\
&= (1+\epsilon)\int_0^r\int_{\mathbb{R}^{d-1}} \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}dx_1 \\
&= (1+\epsilon)\int_0^r\int_{\mathbb{R}^{d-1}} \fint_{B_{cx_1}(\bar{\boldsymbol{y}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{x}}d\bar{\boldsymbol{y}}dx_1 \\
&= (1+\epsilon)\int_0^r\int_{\mathbb{R}^{d-1}} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dx_1 d\bar{\boldsymbol{y}}dy_1 \\
&= (1+\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{x}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 dx_1 d\bar{\boldsymbol{x}}\,,
\end{aligned}$$

which implies (8.17) with $c_2 = (1+\epsilon)$. The product of c_1 and c_2 is

$$c_1c_2 = \frac{(1+\epsilon)^2(b+a)}{2}\,.$$

Since $b+a<2$, by choosing ϵ small enough such that $(1+\epsilon)^2(b+a)<2$, we have $c_1c_2<1$, so that (8.14) and (8.15) are true, and hence we have the lemma. □

8.3 The generalized trace theorem

We will show in this section the trace theorem on the function space $\mathcal{S}(\Omega)$. As in the classical case, we proceed first to a special stripe domain $\Omega = (0, r) \times \mathbb{R}^{d-1}$ with a portion of its boundary $\Gamma = \{0\} \times \mathbb{R}^{d-1}$ where r is any given positive constant. Then we use the technique of partition of unity to show the result on a general Lipschitz domain Ω.

Theorem 8.3.1 (Special trace theorem)**.** *For $\Omega = (0, r) \times \mathbb{R}^{d-1}$ and $\Gamma = \{0\} \times \mathbb{R}^{d-1}$, there exists a constant C depends only on d such that such that for any $u \in C^1(\bar{\Omega}) \cap \mathcal{S}(\Omega)$,*

$$\|u\|_{L^2(\Gamma)} \leq C\Big(r^{-1/2}\|u\|_{L^2(\Omega)} + r^{1/2}|u|_{\mathcal{S}(\Omega)}\Big)\,, \tag{8.19}$$

and for $d \geq 2$,

$$|u|_{H^{1/2}(\Gamma)} \leq C\Big(r^{-1}\|u\|_{L^2(\Omega)} + |u|_{\mathcal{S}(\Omega)}\Big)\,. \tag{8.20}$$

There are two remarks we want to make for equations (8.19) and (8.20). First, the r dependence of the imbedding coefficients we want to emphasize is specially applied to small r. For large r, the equations still hold true, but it is also important that we can get $\|u\|_{H^{1/2}(\Gamma)} \leq C\|u\|_{\mathcal{S}(\Omega)}$ where C is independent of $r \to \infty$. Second, the need of the L^2 term in equation (8.20) may not be obvious at first thought, since adding u with a constant will not change the $H^{1/2}$-semi norm. However, we can give the following example showing that it is not enough to bound $|\cdot|_{H^{1/2}(\Gamma)}$ with only $|\cdot|_{\mathcal{S}(\Omega)}$. We let $u(x_1, x_2) = \phi_a(x_2)$, where $\phi_a(\cdot)$ is a hat function defined by

$$\phi_a(x) = \begin{cases} (x+a)/a & \text{for } x \in (-a, 0), \\ (a-x)/a & \text{for } x \in [0, a), \\ 0 \quad \text{otherwise}. \end{cases}$$

In this case $\Omega = (0, r) \times \mathbb{R}$ and $\Gamma = \{0\} \times \mathbb{R}$. Then we can see that

$$\begin{aligned} |u|^2_{H^{1/2}(\Gamma)} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(0, y_2) - u(0, x_2)|^2}{|y_2 - x_2|^2} dy_2 dx_2 \\ &\geq \int_{-a/2}^{a/2}\int_{|y_2 - x_2| \geq 2a} \frac{|u(0, y_2) - u(0, x_2)|^2}{|y_2 - x_2|^2} dy_2 dx_2 \\ &\geq \frac{1}{4}\int_{-a/2}^{a/2}\int_{|y_2 - x_2| \geq 2a} \frac{1}{|y_2 - x_2|^2} dy_2 dx_2 = \frac{1}{4}\,. \end{aligned}$$

However, by the fact that $|u|^2_{\mathcal{S}(\Omega)} \leq C|u|^2_{H^1(\Omega)}$ which we will show in section 2, we see as follows that $|u|^2_{\mathcal{S}(\Omega)}$ alone cannot provide bound of $|u|^2_{H^{1/2}(\Gamma)}$ for sufficiently large a,

$$|u|^2_{\mathcal{S}(\Omega)} \leq C|u|^2_{H^1(\Omega)} = C\int_0^r \int_{\mathbb{R}} (\phi_a'(x_2))^2 dx_2 dx_1 = \frac{2Cr}{a}\,.$$

Once the special case is established, we know that the trace map T admits a continuous extension on $\mathcal{S}(\Omega)$, and that for any $u \in \mathcal{S}(\Omega)$. Furthermore, analogous to the case of classical Sobolev spaces, see for example [Ding, 1996], the above theorem has a more general version valid for Lipschitz domains Ω in $\mathbb{R}^d$ for $d \geq 2$, given as follows.

Theorem 8.3.2 (General trace theorem)**.** *Assume that Ω is a bounded simply connected Lipschitz domain in $\mathbb{R}^d$ $(d \geq 2)$ and $\Gamma = \partial\Omega$, then there exists a constant C depending only on Ω and Γ such that*

$$\|Tu\|_{H^{\frac{1}{2}}(\Gamma)} \leq C\|u\|_{\mathcal{S}(\Omega)}\,, \quad \forall u \in \mathcal{S}(\Omega)\,. \tag{8.21}$$

Proof of Theorem 8.3.1. We will only prove the result for the kernel defined in (8.5). The discussions for more general kernels are in section 8.4. Now let us show (8.19). For any $(x_1, \bar{\boldsymbol{x}}) \in (0,r) \times \mathbb{R}^{d-1}$, write

$$u(0,\bar{\boldsymbol{x}}) = u(x_1,\bar{\boldsymbol{x}}) - (u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}}))\,,$$

from which we have

$$u^2(0,\bar{\boldsymbol{x}}) \leq 2u^2(x_1,\bar{\boldsymbol{x}}) + 2(u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}}))^2.$$

Now by integrating x_1 over $(0,r)$, we obtain

$$u^2(0,\bar{\boldsymbol{x}}) \leq \frac{2}{r}\int_0^r u^2(x_1,\bar{\boldsymbol{x}})dx_1 + \frac{2}{r}\int_0^r |u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}})|^2 dx_1\,.$$

So, we get by Proposition 8.2.2 and Lemma 8.2.7 that

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} u^2(0,\bar{\boldsymbol{x}})d\bar{\boldsymbol{x}} &\leq \frac{2}{r}\|u\|^2_{L^2(\Omega)} + 2r\int_{\mathbb{R}^{d-1}}\int_0^r \frac{|u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}})|^2}{|x_1|^2}dx_1 d\bar{\boldsymbol{x}} \\ &\leq \frac{2}{r}\|u\|^2_{L^2(\Omega)} + 2r\int_{\mathbb{R}^{d-1}}\int_0^r\int_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{x}}) - u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^3}dy_1 dx_1 d\bar{\boldsymbol{x}} \\ &\leq C\Big(\frac{\|u\|^2_{L^2(\Omega)}}{r} + r\|u\|^2_{\mathcal{S}(\Omega)}\Big)\,.\end{aligned}$$

Let us show (8.20) next. First, for $d \geq 2$, by definition of $|u(0,\cdot)|^2_{H^{1/2}(\mathbb{R}^{d-1})}$,

$$\begin{aligned}|u(0,\cdot)|^2_{H^{1/2}(\mathbb{R}^{d-1})} &= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d} d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}} \\ &= \int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d} d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}} \\ &\quad + \int_{\mathbb{R}^{d-1}}\int_{B^c_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d} d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\,.\end{aligned}$$

Now the second part can be estimated by

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}}\int_{B^c_{r/2}(\bar{\boldsymbol{0}})} \frac{|u(0,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{h}}|^d} d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}} \\ &\leq \int_{B^c_{r/2}(\bar{\boldsymbol{0}})} \frac{1}{|\bar{\boldsymbol{h}}|^d}\int_{\mathbb{R}^{d-1}} \left(2u^2(0,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})+2u^2(0,\bar{\boldsymbol{x}})\right)d\bar{\boldsymbol{x}}d\bar{\boldsymbol{h}} \qquad (8.22)\\ &\leq \frac{C}{r}\|u(0,\cdot)\|^2_{L^2(\mathbb{R}^{d-1})}\,,\end{aligned}$$

where C is a constant depend only on d. Thus we only have to prove the following inequality to get (8.20),

$$\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d} d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}} \leq C|u|^2_{\mathcal{S}(\Omega)}\,. \qquad (8.23)$$

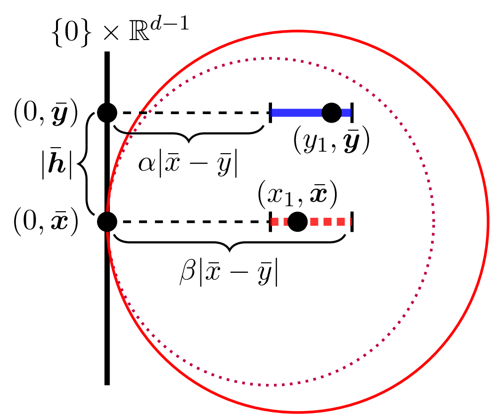

Figure 8.3: Depiction of geometry used in the proof of Theorem 8.3.1.

The idea is again to split the left-hand side into three parts that can be controlled by the right hand side.

As shown in Figure 8.3, we choose $(x_1,\bar{\boldsymbol{x}}),(y_1,\bar{\boldsymbol{y}})\in\Omega$ and rewrite

$$u(0,\bar{\boldsymbol{y}}) = u(0,\bar{\boldsymbol{y}}) - u(y_1,\bar{\boldsymbol{y}}) + u(y_1,\bar{\boldsymbol{y}})$$

$$u(0,\bar{\boldsymbol{x}}) = u(0,\bar{\boldsymbol{x}}) - u(x_1,\bar{\boldsymbol{x}}) + u(x_1,\bar{\boldsymbol{x}})\,.$$

Notice that the blue solid horizontal line and the red horizontal dashed line in Figure 8.3 show the possible positions of $(x_1, \bar{\boldsymbol{x}})$ and $(y_1, \bar{\boldsymbol{y}})$ respectively. The key is to determine the end points of these lines so that any $(y_1, \bar{\boldsymbol{y}})$ over the blue solid line should stand in the effective neighborhood (shown as red solid circle) of any $(x_1, \bar{\boldsymbol{x}})$ on the red horizontal dashed line, in particular, the left-most end point whose effective neighborhood is given by the dashed purple circle.

By splitting terms, we have

$$|u(0, \bar{\boldsymbol{y}}) - u(0, \bar{\boldsymbol{x}})|^2 \leq 3|u(0, \bar{\boldsymbol{y}}) - u(y_1, \bar{\boldsymbol{y}})|^2 + 3|u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2 + 3|u(x_1, \bar{\boldsymbol{x}}) - u(0, \bar{\boldsymbol{x}})|^2 .$$

Now let α and β be numbers to be determined and satisfy $1 < \alpha < \beta \leq 2$. Integrating both x_1 and y_1 in the interval $(\alpha|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|, \beta|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|)$, we have

$$\begin{aligned}
|u(0, \bar{\boldsymbol{y}}) - u(0, \bar{\boldsymbol{x}})|^2 \leq & \frac{3}{(\beta-\alpha)|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} |u(0, \bar{\boldsymbol{y}}) - u(y_1, \bar{\boldsymbol{y}})|^2 dy_1 \\
& + \frac{3}{(\beta-\alpha)|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} |u(x_1, \bar{\boldsymbol{x}}) - u(0, \bar{\boldsymbol{x}})|^2 dx_1 \\
& + \frac{3}{(\beta-\alpha)^2|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^2} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} |u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2 dy_1 dx_1 .
\end{aligned}$$

So our integral can controlled by three terms,

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \int_{B_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0, \bar{\boldsymbol{y}}) - u(0, \bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^d} d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} \\
&\quad \leq \frac{3}{\beta-\alpha} \int_{\mathbb{R}^{d-1}} \int_{B_{r/2}(\bar{\boldsymbol{x}})} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \frac{|u(y_1, \bar{\boldsymbol{y}}) - u(0, \bar{\boldsymbol{y}})|^2}{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^{d+1}} dy_1 d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} \\
&\qquad + \frac{3}{\beta-\alpha} \int_{\mathbb{R}^{d-1}} \int_{B_{r/2}(\bar{\boldsymbol{x}})} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \frac{|u(x_1, \bar{\boldsymbol{x}}) - u(0, \bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^{d+1}} dx_1 d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} \\
&\qquad + \frac{3}{(\beta-\alpha)^2} \int_{\mathbb{R}^{d-1}} \int_{B_{r/2}(\bar{\boldsymbol{x}})} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \frac{|u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^{d+2}} dy_1 dx_1 d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} \\
&\quad = \frac{6}{\beta-\alpha} \int_{\mathbb{R}^{d-1}} \int_{B_{r/2}(\bar{\boldsymbol{x}})} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \frac{|u(y_1, \bar{\boldsymbol{y}}) - u(0, \bar{\boldsymbol{y}})|^2}{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^{d+1}} dy_1 d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} \\
&\qquad + \frac{3}{(\beta-\alpha)^2} \int_{\mathbb{R}^{d-1}} \int_{B_{r/2}(\bar{\boldsymbol{x}})} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|} \frac{|u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^{d+2}} dy_1 dx_1 d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} \\
&\quad = \mathrm{I} + \mathrm{II} .
\end{aligned}$$

Let us first check that the term II is bounded by $C|u|^2_{\mathcal{S}(\Omega)}$. We take notice of Fubini's theorem and the fact that $\beta \le 2$ to get

$$\begin{aligned}\mathrm{II} &= \frac{3}{(\beta-\alpha)^2}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{h}}|^{d+2}}dy_1dx_1d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}}\\ &\le \frac{3\beta^{d+2}}{(\beta-\alpha)^2}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}}\end{aligned}$$

where $B_{r/2}(\mathbf{0})$ denotes the $d-1$ dimensional ball at the origin. The integral can be further estimated by

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}}\\ &=\int_{\mathbb{R}^{d-1}}\int_{S^{d-2}}\int_0^{r/2}\Big(\int_{\alpha h}^{\beta h}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1\Big)|J|dhdS^{d-2}d\bar{\boldsymbol{x}}\end{aligned}$$

where $|J|=|J(h)|$ is the volume element of $d-1$ dimensional ball and dS^{d-2} is the volume element of the $d-2$ dimensional unit sphere. After a change of order of integration, since $1<\alpha<\beta\le 2$, we end up with

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}}\int_{S^{d-2}}\Big(\int_0^{r/2}\int_{\alpha h}^{\beta h}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1|J|dh\Big)dS^{d-2}d\bar{\boldsymbol{x}}\\ &\le\int_{\mathbb{R}^{d-1}}\int_{S^{d-2}}\Big(\int_0^{r}\int_{\frac{x_1}{\beta}}^{\frac{x_1}{\alpha}}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1|J|dhdx_1\Big)dS^{d-2}d\bar{\boldsymbol{x}}\\ &=\int_{\mathbb{R}^{d-1}}\int_0^{r}\Big(\int_{S^{d-2}}\int_{\frac{x_1}{\beta}}^{\frac{x_1}{\alpha}}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1|J|dhdS^{d-2}\Big)dx_1d\bar{\boldsymbol{x}}\\ &\le\int_{\mathbb{R}^{d-1}}\int_0^{r}\Big(\int_{\frac{x_1}{\beta}\le|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|\le\frac{x_1}{\alpha}}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1d\bar{\boldsymbol{y}}\Big)dx_1d\bar{\boldsymbol{x}}\\ &\le\int_{\Omega}\int_{\mathcal{H}(\boldsymbol{x})}\frac{|u(\boldsymbol{y})-u(\boldsymbol{x})|^2}{|x_1|^{d+2}}d\boldsymbol{y}d\boldsymbol{x}\,.\end{aligned}$$

Note that the last inequality is true only if $\boldsymbol{y}=(y_1,\bar{\boldsymbol{y}})\in\mathcal{H}(\boldsymbol{x})$. Since $\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|\le y_1\le\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|$ and $x_1/\beta\le|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|\le x_1/\alpha$ implies that $\alpha x_1/\beta\le y_1\le\beta x_1/\alpha$, we have

$$|\boldsymbol{y}-\boldsymbol{x}|^2=(y_1-x_1)^2+|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2\le\max\{(1-\frac{\beta}{\alpha})^2+\frac{1}{\alpha^2},(1-\frac{\alpha}{\beta})^2+\frac{1}{\alpha^2}\}|x_1|^2\le|x_1|^2\,,$$

if we pick α and β such that

$$\max\left\{(1-\frac{\beta}{\alpha})^2+\frac{1}{\alpha^2},(1-\frac{\alpha}{\beta})^2+\frac{1}{\alpha^2}\right\}\le 1\,.$$

Then this in fact leaves us many chocies of α and β, for example, $\alpha=\frac{3}{2}$ and $\beta=2$ would work.

The term I is bounded by

$$\begin{aligned}
&\frac{6}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{h}}|^{d+1}}dx_1 d\bar{\boldsymbol{h}} d\bar{\boldsymbol{x}} \\
&= \frac{6C(d)}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_0^{r/2}\int_{\alpha h}^{\beta h}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|h|^{d+1}}h^{d-2}dx_1 dh d\bar{\boldsymbol{x}} \\
&\le \frac{6C(d)}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_0^{r}\Big(\int_{\frac{x_1}{\beta}}^{\frac{x_1}{\alpha}}\frac{1}{h^3}dh\Big)|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}} \\
&\le 3C(d)(\beta+\alpha)\int_{\mathbb{R}^{d-1}}\int_0^{r}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|x_1|^2}dx_1 d\bar{\boldsymbol{x}}\,.
\end{aligned}$$

By Proposition 8.2.2 and Lemma 8.2.7, we have

$$\int_{\mathbb{R}^{d-1}}\int_0^{r}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|x_1|^2}dx_1 d\bar{\boldsymbol{x}} \le C|u|^2_{\mathcal{S}(\Omega)}\,,$$

which completes the proof of theorem 8.3.1. □

□

***Remark* 8.3.3.** *The problem of relating boundary estimates and interior estimates appears often in the study of PDE boundary value problems, such as in Kellogg's theorem for deriving C^α regularity estimates up to the boundary with prescribed C^α data [Kellogg, 1931], and in deriving interior regularity estimates from the coincidence set for free boundary problems [Lin, 2016]. Indeed, the idea of relating boundary points to interior points in order to get an estimate of boundary from those in the interior leads to a popular approach to establish the classical trace theorem, see for example, [Leoni, 2009, chapter 15]. However, a new challenge in our work here in the nonlocal case, unlike the straightforward constructions in the classical case, is that the interior points need to be carefully chosen to make the nonlocal norm $\|u\|_{\mathcal{S}(\Omega)}$ coming into play. The lemma 8.2.7 provides us analogies of estimates on tangential and normal derivatives that are important to complete our derivation.*

Proof of Theorem 8.3.2. First, let us show that the theorem 8.3.2 is true when Ω is a special Lipschitz domain, namely, assume there exists a Lipschitz continuous function $\varphi:\mathbb{R}^{d-1}\to\mathbb{R}$ such that

$$\Omega=\{\boldsymbol{x}\in\mathbb{R}^d | x_1>\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}}\in\mathbb{R}^{d-1}\}\,,$$

and

$$\partial\Omega=\{\boldsymbol{x}\in\mathbb{R}^d | x_1=\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}}\in\mathbb{R}^{d-1}\}\,.$$

Then we can define two linear operators $G_\varphi : L^2(\Omega) \to L^2(\mathbb{R}^d_+)$, where $\mathbb{R}^d_+ = (0,\infty)\times\mathbb{R}^{d-1}$ and $D_\varphi : L^2(\partial\Omega) \to L^2(\mathbb{R}^{d-1})$ by: for $\boldsymbol{x} = (x_1, \bar{\boldsymbol{x}}) \in (0,\infty)\times\mathbb{R}^{d-1}$,

$$
\begin{aligned}
(G_\varphi u)(\boldsymbol{x}) &= u(x_1 + \varphi(\bar{\boldsymbol{x}}), \bar{\boldsymbol{x}}),\\
(D_\varphi u)(\bar{\boldsymbol{x}}) &= u(\varphi(\bar{\boldsymbol{x}}), \bar{\boldsymbol{x}}).
\end{aligned}
$$

It is known that D_φ is a bounded operator from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\mathbb{R}^{d-1})$, and its inverse D_φ^{-1} on the two spaces is also a bounded operator (see, for instance, Lemma 3 in [Ding, 1996]). The next step is to show that G_φ is a bounded operator from $\mathcal{S}(\Omega)$ to $\mathcal{S}(\mathbb{R}^d_+)$. We note that $\delta(\boldsymbol{x})$ used for the two spaces $\mathcal{S}(\Omega)$ and $\mathcal{S}(\mathbb{R}^d_+)$ may need to have different scalings, though this does not affect the purpose of proving the trace inequality.

$$
\begin{aligned}
\|G_\varphi u\|_{\mathcal{S}(\mathbb{R}^d_+)} &= \int_{\mathbb{R}^{d-1}}\int_0^\infty |u(x_1+\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}}\\
&\quad + \int_{\mathbb{R}^d_+}\int_{\mathcal{H}(\boldsymbol{x})} \frac{|u(y_1+\varphi(\bar{\boldsymbol{y}}),\bar{\boldsymbol{y}}) - u(x_1+\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}})|^2}{|\sigma_1\cdot x_1|^{d+2}} d\boldsymbol{y}d\boldsymbol{x} \\
&= \|u\|^2_{L^2(\Omega)} + \int_\Omega\int_{\mathcal{H}'(\boldsymbol{x})} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})|^2}{|\sigma_1\cdot(x_1-\varphi(\bar{\boldsymbol{x}}))|^{d+2}} d\boldsymbol{y}d\boldsymbol{x}
\end{aligned}
\tag{8.24}
$$

where $\mathcal{H}'(\boldsymbol{x}) = \{\boldsymbol{y}\in\Omega : |y_1 - x_1 - (\varphi(\bar{\boldsymbol{y}}) - \varphi(\bar{\boldsymbol{x}}))|^2 + |\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^2 \le \sigma_1^2\cdot|x_1 - \varphi(\bar{\boldsymbol{x}})|^2\}$. Now since $(x_1, \varphi(\bar{\boldsymbol{x}})) \in \partial\Omega$, we know that

$$
\operatorname{dist}(\boldsymbol{x},\partial\Omega) \le |x_1 - \varphi(\bar{\boldsymbol{x}})| \le K_1 \operatorname{dist}(\boldsymbol{x},\partial\Omega)\,,
$$

for some K_1 independent of $\boldsymbol{x}$. Then for any $\boldsymbol{y}\in\mathcal{H}'(\bar{\boldsymbol{x}})$,

$$
\begin{aligned}
|y_1 - x_1|^2 + |\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^2 &\le 2|y_1 - x_1 - (\varphi(\bar{\boldsymbol{y}}) - \varphi(\bar{\boldsymbol{x}}))|^2 + 2|(\varphi(\bar{\boldsymbol{y}}) - \varphi(\bar{\boldsymbol{x}}))|^2 + |\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^2\\
&\le \max\{2, 2M^2+1\}\big(|y_1 - x_1 - (\varphi(\bar{\boldsymbol{y}}) - \varphi(\bar{\boldsymbol{x}}))|^2 + |\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^2\big)\\
&\le (\sigma_1 K_2\cdot\operatorname{dist}(\boldsymbol{x},\partial\Omega))^2 =: (\sigma_2\cdot\operatorname{dist}(\boldsymbol{x},\partial\Omega))^2\,.
\end{aligned}
$$

This together with (8.24) implies that

$$
\|G_\varphi u\|_{\mathcal{S}(\mathbb{R}^d_+)} \le C\|u\|_{\mathcal{S}(\Omega)}\,,
$$

with $\delta(\boldsymbol{x})$ defined as $\sigma_1\operatorname{dist}(\boldsymbol{x},\partial\Omega)$ and $\sigma_2\operatorname{dist}(\boldsymbol{x},\partial\Omega)$ for $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}(\Omega)$ respectively, and σ_1, σ_2 satisfy $\sigma_2 = \sigma_1 K_2$. Taking into account the above observations and applying the special nonlocal trace theorem (8.3.1) already shown for a stripe domain, we have

$$\begin{aligned}\|Tu\|_{H^{\frac{1}{2}}(\partial\Omega)} &= \|D_\varphi^{-1}(D_\varphi Tu)\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\le C_1\|D_\varphi Tu\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \\ &= C_1\|(G_\varphi u)(0,\cdot)\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \\ &\le C_2\|G_\varphi u\|_{\mathcal{S}(\mathbb{R}^d_+)} \\ &\le C_3\|u\|_{\mathcal{S}(\Omega)}\,.\end{aligned}$$

Now for Ω, which is a bounded simply connected Lipschitz domain, there exists a finite number of pairs $\{B(\boldsymbol{x}_i, r_i), \varphi_i\}_{i=1}^N$ such that $\partial\Omega \subset \bigcup_{i=1}^N B(\boldsymbol{x}_i, r)$. Each φ_i is Lipschitz continuous, and we assume they have a uniform Lipschitz constant M. Now let $\{\zeta_i\}_{i=1}^N$ be a partition of unity of $\partial\Omega$, i.e.,

1. $\zeta_i \in C_c^\infty(B(\boldsymbol{x}_i, r_i))$, $1 \le i \le N$,
2. $0 \le \zeta_i \le 1$ and $\displaystyle\sum_{i=1}^N \zeta_i(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \partial\Omega$.

Then for $\boldsymbol{x} \in \partial\Omega$,

$$Tu(\boldsymbol{x}) = \sum_{i=1}^N T(\zeta_i u)(\boldsymbol{x})\,,$$

so

$$\|Tu\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \sum_{i=1}^N \|T(\zeta_i u)\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Now since $\zeta_i \in C_c^\infty(B(\boldsymbol{x}_i, r_i))$, we may assume without loss of generality that

$$\text{dist}(supp(\zeta_i), \partial B(\boldsymbol{x}_i, r_i)) \ge r_i - b, \quad \forall i = 1, 2, ...N$$

for some $b < r_i$. Then instead of considering the semi-$H^{\frac{1}{2}}(\partial\Omega)$ norm as integral over $\partial\Omega \times \partial\Omega$, we treat it as the integral over $\partial\Omega \times \partial\Omega \cap \{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}| \le b\}$, since the other part can be thrown into the $L^2(\partial\Omega)$ norm as we did before. Under this alternative definition of $H^{\frac{1}{2}}(\partial\Omega)$, we have

$$\|T(\zeta_i u)\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|T(\zeta_i u)\|_{H^{\frac{1}{2}}(\partial\Omega \cap B(\boldsymbol{x}_i, r_i))}\,.$$

Now since

$$\Omega \cap B(\boldsymbol{x}_i, r_i) = \{\boldsymbol{x} \in \mathbb{R}^d | x_1 > \varphi_i(\bar{\boldsymbol{x}})\} \cap B(\boldsymbol{x}_i, r_i),$$

$$\partial\Omega \cap B(\boldsymbol{x}_i, r_i) = \{\boldsymbol{x} \in \mathbb{R}^d | x_1 = \varphi_i(\bar{\boldsymbol{x}})\} \cap B(\boldsymbol{x}_i, r_i)\,,$$

we may apply a zero extension and consider $\zeta_i u$ as a function defined on $\{\boldsymbol{x} \in \mathbb{R}^d | x_1 > \varphi_i(\bar{\boldsymbol{x}})\}$. Hence the estimate in the beginning of this proof can be applied. Therefore

$$\begin{aligned}\|Tu\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq C_1 \sum_{i=1}^{N} \|T(\zeta_i u)\|_{H^{\frac{1}{2}}(\{\boldsymbol{x}\in\mathbb{R}^d | x_1 = \varphi_i(\bar{\boldsymbol{x}})\})} \\ &\leq C_2 \sum_{i=1}^{N} \|\zeta_i u\|_{\mathcal{S}(\{\boldsymbol{x}\in\mathbb{R}^d | x_1 > \varphi_i(\bar{\boldsymbol{x}})\})} \\ &\leq C_3 \sum_{i=1}^{N} \|\zeta_i u\|_{\mathcal{S}(\Omega\cap B(\boldsymbol{x}_i, r_i))} \\ &\leq C_4 \|u\|_{\mathcal{S}(\Omega)},\end{aligned}$$

where the last inequality is true because

$$\begin{aligned}&\int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} \int_{\Omega\cap B(\boldsymbol{x}_i,r)\cap\mathcal{H}(\boldsymbol{x})} \frac{(\zeta_i(\boldsymbol{y})u(\boldsymbol{y}) - \zeta_i(\boldsymbol{x})u(\boldsymbol{x}))^2}{|\delta(\boldsymbol{x})|^{d+2}} d\boldsymbol{y} d\boldsymbol{x} \\ &\leq 2 \int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} \int_{\Omega\cap B(\boldsymbol{x}_i,r)\cap\mathcal{H}(\boldsymbol{x})} \frac{\zeta_i^2(\boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 + u^2(\boldsymbol{x})(\zeta_i(\boldsymbol{y}) - \zeta_i(\boldsymbol{x}))^2}{|\delta(\boldsymbol{x})|^{d+2}} d\boldsymbol{y} d\boldsymbol{x} \\ &\leq 2 \left(\|\zeta_i\|_{C^0}^2 \|u\|_{\mathcal{S}(\Omega\cap B(\boldsymbol{x}_i,r))}^2 + \|\zeta_i\|_{C^1}^2 \int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} \int_{|\boldsymbol{y}-\boldsymbol{x}|\leq\delta(\boldsymbol{x})} u^2(\boldsymbol{x}) \frac{|\boldsymbol{y}-\boldsymbol{x}|^2}{|\delta(\boldsymbol{x})|^{d+2}} d\boldsymbol{y} d\boldsymbol{x} \right) \\ &\leq C \left(\|\zeta_i\|_{C^0}^2 \|u\|_{\mathcal{S}(\Omega\cap B(\boldsymbol{x}_i,r))}^2 + \|\zeta_i\|_{C^1}^2 \int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} u^2(\boldsymbol{x}) d\boldsymbol{x} \right).\end{aligned}$$

This completes the proof. □

8.4 Conclusion and discussions

We now make some further discussions on the main results given in the paper. As important as the role that Sobolev space plays in the study of partial differential equations, the mathematical theory of nonlocal space provides the essential tool towards rigorous analysis of nonlocal equations. When nonlocality is incorporated in the models, it could lead to more subtle definitions of suitable boundary value problems. Nonlocal equations on domains with boundary are often supplemented not by additional constraints on the codimension-1 boundary, but rather volumetric constraints [Du *et al.*, 2012]. We, however, are able to define new nonlocal spaces as presented here, allowing nonlocality to diminish when approaching the domain boundary. The resulting new nonlocal trace theorems can also be established for more general nonlocal interactions besides the special kernels investigated here, as demonstrated later in this section. We note that boundary value problems with local boundary conditions have also been widely studied for fractional differential equations which are also instances of nonlocal models [Caffarelli and Silvestre, 2007]. Indeed, classical fractional derivatives may also be seen as having a vanishing horizon near the boundary or a diminishing history dependence near the initial time [Allen *et al.*, 2016; Diethelm, 2015], however, their scaling features are completely different from our setting so that the boundary trace or initial value are sensible largely due to the sufficiently strong interior regularity for fractional derivatives, and not through the localization effect described in this chapter.

8.4.1 Trace theorems on portions of the domain boundary

In the classical, local case, we note that the trace inequality on $\partial\Omega$ automatically implies the same result for the trace on a subset Γ of $\partial\Omega$. This is not, however, as straightforward in the case for our nonlocal space whose definition involves the Γ dependent horizon, and thus the Γ dependent nonlocal kernel.

We expect similar results remain valid, as demonstrated by the special case given in Theorem 8.3.1, but careful investigations are needed for more general domains. A possible route is to consider first a special domain that is a (rectangular) section of the strip domain, for instance, $\Omega = (0, r) \times (a, b) \times \mathbb{R}^{d-2}$ and $\Gamma = \{0\} \times (a, b) \times \mathbb{R}^{d-2}$. By a suitable extension in the second variable from the interval (a, b) to the whole real line, we may first utilize the result in Theorem 8.3.1 for the whole strip domain to get the desired result on its subsection. One may then employ similar partition of unity techniques and domain transformations to more general domains and more general subset of their boundary.

8.4.2 More general kernels

We note that much of our discussion so far is focused on the choice that $\hat{\gamma}$ takes on a constant value over its support. There are a number of reasons behind this special choice. One is to avoid technical complication while keeping the essence of the issues to be investigated. But more importantly, this is out of the consideration that the nonlocal norm of u corresponding to this special case is among the weakest of nonlocal norms associated with popular kernels that have been used in the literature. For example, for a typical fractional power law kernel $\hat{\gamma}(s) = 1/s^\lambda$, for $\lambda \in [0, d+2)$ [Diethelm, 2015; Allen *et al.*, 2016], we have the fractional type kernels

$$\gamma^\lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{c_\lambda}{|\delta(\boldsymbol{x})|^{d+2-\lambda}} \cdot \frac{1}{|\boldsymbol{y} - \boldsymbol{x}|^\lambda} \quad \text{for } \boldsymbol{y} \in \mathcal{H}(\boldsymbol{x}), \quad \lambda \in [0, d+2)\,. \tag{8.25}$$

Notice that λ has to be less than $d+2$ to ensure that $\hat{\gamma}$ has a finite $d+2$ order moment so that all $C^1(\bar{\Omega})$ functions have finite nonlocal norms. For such kernels, it is easy to make the following comparison of norms.

Lemma 8.4.1. *For γ^λ defined in* (8.25)*,*

$$\int_\Omega \int_{\Omega \cap \mathcal{H}(\boldsymbol{x})} \gamma^0(\boldsymbol{x}, \boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x} \le C \int_\Omega \int_{\Omega \cap \mathcal{H}(\boldsymbol{x})} \gamma^\lambda(\boldsymbol{x}, \boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x}\,,$$

with $\lambda \in (0, d+2)$.

Proof. This is obvious since for $\boldsymbol{y} \in \mathcal{H}(\boldsymbol{x})$, i.e., $|\boldsymbol{y} - \boldsymbol{x}| \le \delta(\boldsymbol{x})$,

$$\frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \le \frac{1}{|\delta(\boldsymbol{x})|^{d+2-\lambda}} \cdot \frac{1}{|\boldsymbol{y} - \boldsymbol{x}|^\lambda}$$

for $\lambda \in (0, d+2)$. □

The lemma shows that $|u|_{\mathcal{S}(\Omega)}$ defined with $\lambda = 0$ indeed gives the weakest norm among ones corresponding to a large class of kernels either associated with (8.25) or are bounded below and above by such kernels. It is also possible to consider generalizing the choices of the variable horizon. For example, we are going to show that the nonlocal trace inequalities and Hardy-type inequalities proved previously also hold for $\delta(\boldsymbol{x}) = \sigma \text{dist}(\boldsymbol{x}, \Gamma)$ where $\sigma \in (0, \sigma_0]$ for $\sigma_0 > 0$. More importantly, the embedding constants in these inequalities only depend on σ_0. For this matter, we define some notations first.

$$|u|^2_{\delta(\boldsymbol{x}),r} = \int_{\Omega_r} \int_{\Omega_r \cap \mathcal{H}(\boldsymbol{x})} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}\Big(\frac{|\boldsymbol{y} - \boldsymbol{x}|}{\delta(\boldsymbol{x})}\Big)(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x}\,.$$

where $\Omega_r = (0, r) \times \mathbb{R}^{d-1}$. The next lemma shows that the smaller σ is, the larger the nonlocal norm we can get.

Lemma 8.4.2. *Let $\delta(\boldsymbol{x}) = \sigma\, dist(\boldsymbol{x}, \Gamma)$, where $\sigma \in [\frac{1}{2}, 1)$ and $\Gamma = \{0\} \times \mathbb{R}^{d-1}$, then there exists a constant C depending only on d such that the following inequality holds for any $r > 0$ and $\alpha \in (0, 1]$,*

$$|u|^2_{\delta(\boldsymbol{x}),r/2} \leq C\Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} |u|^2_{\alpha\delta(\boldsymbol{x}),r} \,. \tag{8.26}$$

Proof. First, $|u|_{\delta(\boldsymbol{x}),r/2}$ can be rewrite as

$$|u|^2_{\delta(\boldsymbol{x}),r/2} = \int_{\Omega_{r/2}} \int_{D_{\delta(\boldsymbol{x}),r/2}} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{s}|}{\delta(\boldsymbol{x})})(u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x}))^2 d\boldsymbol{s} d\boldsymbol{x} \,.$$

where $D_{\delta(\boldsymbol{x}),r/2} = \{\boldsymbol{s} \in \mathbb{R}^d : |\boldsymbol{s}| \leq \delta(\boldsymbol{x}), \boldsymbol{x}+\boldsymbol{s} \in \Omega_{r/2}, \text{ for some } \boldsymbol{x} \in \Omega_{r/2}\}$.

Now for any $n \in \mathbb{N}$, we decompose $u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x})$ into n parts,

$$u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x}) = \Big(u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x} + \frac{n-1}{n}\boldsymbol{s})\Big) + \cdots + \Big(u(\boldsymbol{x} + \frac{1}{n}\boldsymbol{s}) - u(\boldsymbol{x})\Big) \,.$$

By using the inequality $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$, we get

$$|u|^2_{\delta(\boldsymbol{x}),r/2} \leq n \sum_{i=1}^n \int_{\Omega_{r/2}} \int_{D_{\delta(\boldsymbol{x}),r/2}} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{s}|}{\delta(\boldsymbol{x})})(u(\boldsymbol{x} + \frac{i}{n}\boldsymbol{s}) - u(\boldsymbol{x} + \frac{i-1}{n}\boldsymbol{s}))^2 d\boldsymbol{s} d\boldsymbol{x} \,.$$

For each fixed i, let $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \frac{i-1}{n}\boldsymbol{s}$, then $\tilde{\boldsymbol{x}} \in \Omega_{r/2}$ as a result of $\boldsymbol{x}+\boldsymbol{s} \in \Omega_{r/2}$ and $\boldsymbol{x} \in \Omega_{r/2}$. Since $\delta(\boldsymbol{x}) = \sigma x_1$ and $|\boldsymbol{s}| \leq \delta(\boldsymbol{x})$, we have

$$(1-\sigma)\delta(\boldsymbol{x}) \leq \delta(\tilde{\boldsymbol{x}}) \leq (1+\sigma)\delta(\boldsymbol{x}) \,.$$

Then by the fact that $\delta(\boldsymbol{x}) \leq \delta(\tilde{\boldsymbol{x}})/(1-\sigma)$, $1/\delta(\boldsymbol{x}) \leq (1+\sigma)/\delta(\tilde{\boldsymbol{x}})$ and $\hat{\gamma}$ nonincreasing we have

$$\begin{aligned}
|u|^2_{\delta(\boldsymbol{x}),r/2} &\leq n^2 \int_{\Omega_{r/2}} \int_{|\boldsymbol{s}| \leq \frac{\delta(\tilde{\boldsymbol{x}})}{1-\sigma}} \frac{(1+\sigma)^{d+2}}{|\delta(\tilde{\boldsymbol{x}})|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{s}|}{\delta(\tilde{\boldsymbol{x}})/(1-\sigma)})(u(\tilde{\boldsymbol{x}} + \frac{1}{n}\boldsymbol{s}) - u(\tilde{\boldsymbol{x}}))^2 d\boldsymbol{s} d\tilde{\boldsymbol{x}} \\
&= n^{d+2} \int_{\Omega_{r/2}} \int_{n|\boldsymbol{s}| \leq \frac{\delta(\tilde{\boldsymbol{x}})}{1-\sigma}} \frac{(1+\sigma)^{d+2}}{|\delta(\tilde{\boldsymbol{x}})|^{d+2}} \hat{\gamma}(\frac{n|\boldsymbol{s}|}{\delta(\tilde{\boldsymbol{x}})/(1-\sigma)})(u(\tilde{\boldsymbol{x}} + \boldsymbol{s}) - u(\tilde{\boldsymbol{x}}))^2 d\boldsymbol{s} d\tilde{\boldsymbol{x}} \\
&= \Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} \int_{\Omega_{r/2}} \int_{|\boldsymbol{s}| \leq \frac{\delta(\boldsymbol{x})}{n(1-\sigma)}} \frac{1}{\big|\frac{\delta(\boldsymbol{x})}{n(1-\sigma)}\big|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{s}|}{\frac{\delta(\boldsymbol{x})}{n(1-\sigma)}})(u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x}))^2 d\boldsymbol{s} d\boldsymbol{x} \\
&\leq \Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} |u|^2_{\frac{\delta(\boldsymbol{x})}{n(1-\sigma)},r} \,,
\end{aligned}$$

where n is chosen as any number such that $n(1-\sigma) \geq 1$. This shows that (8.26) is true for any $\alpha = \frac{1}{n(1-\sigma)}$ with $n \in \mathbb{N}$ and $n(1-\sigma) \geq 1$. Now for a general $\alpha \in (0, 1]$, we can find a number $n \geq 1$ such that

$$\frac{1}{(n+1)(1-\sigma)} < \alpha \leq \frac{1}{n(1-\sigma)} \,.$$

Then it is easy to see that

$$|u|^2_{\delta(\boldsymbol{x}),r/2} \leq \left(\frac{1+\sigma}{1-\sigma}\right)^{d+2} |u|^2_{\frac{\delta(\boldsymbol{x})}{(n+1)(1-\sigma)},r} \leq \left(\frac{1+\sigma}{1-\sigma}\right)^{d+2} \left(\frac{n+1}{n}\right)^{d+2} |u|^2_{\alpha\delta(\boldsymbol{x}),r} .$$

So (8.26) is true with $C = 2^{d+2}$. □

Using this lemma, we arrive at the conclusion that our embedding results can be extend to any $\delta(\boldsymbol{x}) = \sigma \text{dist}(\boldsymbol{x}, \Gamma)$.

Proposition 8.4.3. *Theorem 8.3.1, theorem 8.3.2 and corollary 8.2.3 are true for influence horizon of the form $\delta(\boldsymbol{x}) = \sigma\, dist(\boldsymbol{x}, \Gamma)$ where $\sigma \in (0, \sigma_0]$ for some $\sigma_0 > 0$. Moreover, the embedding constant C depends only on Ω, Γ and σ_0.*

Proof. First we observe that theorem 8.3.2 follows completely from the theorem 8.3.1, so we only need to show the result for theorem 8.3.1 and corollary 8.2.3, using directly proposition 8.2.2 and lemma 8.2.7. It is not hard to see that the result holds for $\sigma \in [\frac{1}{2}, \sigma_0]$. Indeed, if we choose $a = \frac{1}{2}$, $b = 1$, $c = \frac{1}{2}$ in the proof of proposition 8.2.2 and lemma 8.2.7, and $\alpha = \frac{3}{2}$, $\beta = \frac{7}{4}$ in the proof of theorem 8.3.1, we see that the inequalities in these proofs hold with C depending on σ_0. Then for the rest that $\sigma \in (0, \frac{1}{2})$, the result is achieved by lemma 8.4.2. □

Moreover, the proportionality of the horizon on the distance to the boundary is only a specific choice that can be generalized. One instance is that $\delta(\boldsymbol{x})$ is proportional to $\text{dist}(\boldsymbol{x}, \Gamma)$ for $\boldsymbol{x}$ only on a boundary layer of finite positive width but remains constant elsewhere. A possible form of such a $\delta(\boldsymbol{x})$ might be

$$\delta(\boldsymbol{x}) = \min\{\sigma \text{dist}(x, \Gamma), \eta\},$$

for some $\eta > 0$ to be specified. Another possibility is to have $\delta(\boldsymbol{x})$ vanishes in some other nonlinear ways as $\boldsymbol{x}$ approaches the boundary. Similar results can be shown in these cases and they follow naturally from the fact that it is the nonlocal interaction in the boundary layer, rather than the interior of the domain, that provides the essential control on the $H^{\frac{1}{2}}$ trace.

The discussion on the general form of $\delta(\boldsymbol{x})$ is meaningful since it is important in many applications to note that the imbedding constant in (8.6) does not depend on σ, just like the constants appearing in the new nonlocal trace inequalities. For example, for the coupled PDE and nonlocal model depicted in Fig. 8.1, we may recover a coupled PDE models in the local limit as $\sigma \to 0$. This again implies that the nonlocal trace theorems are refinement and improvement of the classical trace theorems in $H^1(\Omega)$.

Chapter 9

Nonlinear nonlocal models with memory effect

While a large part of the existing mathematical work on peridynamics focuses on the linear model, a few recent works have touched upon the rigorous mathematical theory of nonlinear models [Emmrich and Puhst, 2013; Lipton, 2015; Mengesha and Du, 2016]. Still, these studies have not dealt with peridynamic models involving bond-breaking rules with no-healing, even though the latter has been used in many numerical simulations to describe the irreversibility of bond breaking, The well-posedness for such models remain an open question. In this chapter, we establish rigorous results on the existence and uniqueness of the nonlinear peridynamic model with a properly defined bond-breaking rule. The serves as a first step towards mathematical analysis of more general peridynamic models for fracture mechanics, in particular, those involving memory effects in bond forces. The model studied here is a variant of those studied in [Silling *et al.*, 2010; Ha and Bobaru, 2010]. However, modifications are introduced to the original models to preserve the necessary physics while providing more well-defined mathematical equations. From a dynamic system view point, it is natural to make such modifications in order to have a coherent dependence of the bond force on the history. Numerical simulations are carried out to show the effectiveness of the model for crack propagation for some test examples.

The rest of the chapter is organized as follows. In section 9.1, we present a previously studied peridynamics model with a bond-breaking rule, and introduce some necessary modifications to retain the same essential physics while making the rules more adapted to careful mathematical analysis. In section 9.2, we show that our modified model is well-posed by utilizing the ideas of of functional differential equations [Hale, 1971; Wu, 2012]. At last, we conclude the chapter in section 9.3.

9.1 Peridynamic models with bond-breaking

At present, studies of nonlocal peridynamic models that involve bond-breaking rules have been mostly limited to numerical simulations. Mathematically, they are nonlinear time-dependent differential integral equations with both spatial nonlocal and nonlinear interactions and memory and history dependence over time. Here, we describe the common practice in existing numerical simulations and the necessary reformulation that allows us to demonstrate that the nonlinear dynamic model is well-posed.

9.1.1 A common practice

First, we briefly recall the common practice to formulate a peridynamic model with bond-breaking rule for brittle fractures [Silling *et al.*, 2010; Ha and Bobaru, 2010]. Let $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ denote the displacement field and ρ be the constant density, the equations of motion is given by

$$\rho\ddot{\mathbf{u}}(t, \mathbf{x}) = \int_{B_\delta(\mathbf{x})} \mathbf{f}(t, \mathbf{u}(t, \hat{\mathbf{x}}) - \mathbf{u}(t, \mathbf{x}), \hat{\mathbf{x}} - \mathbf{x}) d\hat{\mathbf{x}} + \mathbf{b}(t, \mathbf{x}) \,,$$

where $\mathbf{b} = \mathbf{b}(t, \mathbf{x})$ denotes the body force, and $\mathbf{f}$ is the pairwise force density. The following model has often been used:

$$\mathbf{f}(t, \boldsymbol{\eta}(t), \boldsymbol{\xi}) = \begin{cases} \omega_\delta(|\boldsymbol{\xi}|) S(\boldsymbol{\eta}(t), \boldsymbol{\xi}) \mathbf{e}(\eta(t), \boldsymbol{\xi}) & \text{if } S(\boldsymbol{\eta}(s), \boldsymbol{\xi}) < S_c \text{ for all } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}$ are used to denote $\mathbf{u}(\hat{\mathbf{x}}, t) - \mathbf{u}(\mathbf{x}, t)$ and $\hat{\mathbf{x}} - \mathbf{x}$ respectively and $S_c > 0$ represents the critical value of bond breaking that is determined by the material. The unit vector $\mathbf{e}$ for bond direction and the bond relative elongation S are given by

$$\mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{\boldsymbol{\eta} + \boldsymbol{\xi}}{|\boldsymbol{\eta} + \boldsymbol{\xi}|} \quad \text{and} \quad S(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{|\boldsymbol{\eta} + \boldsymbol{\xi}| - |\boldsymbol{\xi}|}{|\boldsymbol{\xi}|} \,. \tag{9.1}$$

The kernel function ω_δ is assumed to be compactly supported, in particular

$$\omega_\delta(|\boldsymbol{\xi}|) = 0 \quad \text{if } |\boldsymbol{\xi}| > \delta\,.$$

with the constant $\delta > 0$ being the horizon parameter measuring the range of nonlocal interaction.

In this chapter, we may take δ either as a finite constant or let $\delta = \infty$. The study of the limiting case as $\delta \to 0$ is an interesting subject, which will be explored in the future.

9.1.2 A new mathematical formulation

Here, we reformulate the bond-breaking rule via a rigorously defined mathematical equation where the force density $\mathbf{f}$ is specified via a single scalar equation given by

$$\mathbf{f}(t, \boldsymbol{\eta}(t), \boldsymbol{\xi}) = \omega_\delta(|\boldsymbol{\xi}|) f(S(\boldsymbol{\eta}(s), \boldsymbol{\xi}))\mu(S^*(t, \boldsymbol{\eta}, \boldsymbol{\xi}))\mathbf{e}(\boldsymbol{\eta}(t), \boldsymbol{\xi}).$$

Several additional functions are introduced in the above force density formulation, which are intended to make the definition more precise. Here S^* is defined as

$$S^*(t, \boldsymbol{\eta}, \boldsymbol{\xi}) = \max_{0\le s\le t} S(\boldsymbol{\eta}(s), \boldsymbol{\xi})\,.$$

and f and μ are two scalar functions to be specified next in order to make the nonlocal peridynamic equation well-posed.

First, we remark that the equation in our simpler setting corresponds to the original PD system in the special case that the force density is $f(x) = \mu(x)x$ where $\mu(x) = \chi_{[-1,S_c)}(x)$. We note that S and S^* both cannot be less than -1 to avoid physical inconsistency.

We now introduce necessary modifications to f and μ in order to have desirable continuity of the force field. To this end, we define some constant parameters $-1 < S_1^- < S_0^- < 0 \le S_0^+ < S_1^+ < \infty$, and scalar functions $f, \mu \in C([-1,\infty])$ such that

$$f(x) = \begin{cases} x & \text{if } x \in (S_0^-, S_0^+) \\ 0 & \text{if } x \in (-1, S_1^-) \cup (S_1^+, \infty) \end{cases} \qquad \mu(x) = \begin{cases} 1 & \text{if } x \in (-1, S_0^+) \\ 0 & \text{if } x \in (S_1^+, \infty)\,. \end{cases}$$

An illustrative example of f and μ is An important observation is that f and μ are bounded and Lipschitz continuous.

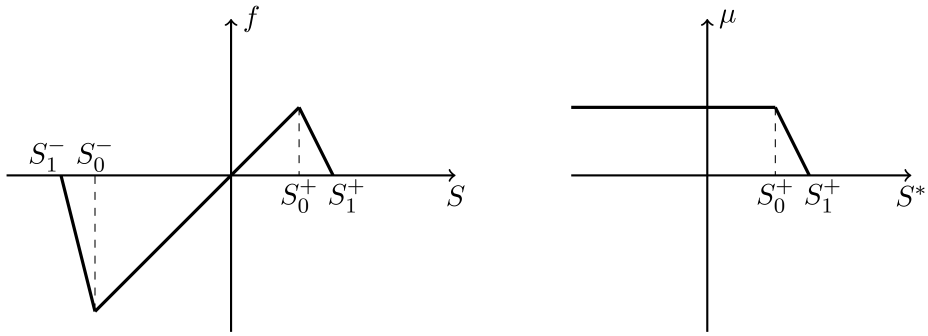

Figure 9.1: The functions f and μ.

***Remark* 9.1.1.** *We give some remarks on the choices of the parameters S_1^-, S_0^-, S_0^+ and S_1^+.*

1. *S_0^+ and S_1^+ can be any two positive constants that satisfy $(S_1^+ - S_0^+) > 0$. The critical stretch value S_c is in the interval $[S_0^+, S_1^+]$.*

2. *S_1^- and S_0^- can be any two negative constants that are larger than -1 and $(S_0^- - S_1^-) > 0$. These parameters are introduced to avoid the complication due to potential material penetration. Indeed, the use of the unit vector $\mathbf{e}$ for the force orientation can be problematic when it flips sign as two distinct material points collapse. From a practical point of view, most of the peridynamic based numerical simulations of crack initiation and growth for brittle materials are usually done by enforcing tensile loading such that the relative stretch does not reach a negative value very close to -1. In such cases, the modifications made for f never take effect so that the reformulated force field is in fact consistent with those employed in earlier experiments. It will be interesting to consider, in the future, the addition of contact forces to complement the modification of f when the relative stretch is near -1.*

9.2 Well-posedness of the new model

Assume $\Omega \subset \mathbb{R}^d$, we consider the equations on the domain Ω, namely, for $\mathbf{x} \in \Omega, t \in [0, T]$,

$$\begin{cases} \rho\ddot{\mathbf{u}}(t,\mathbf{x}) = \displaystyle\int_{\Omega\cup\Omega_{\mathcal{I}}} \omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|) f(S(t,\mathbf{x},\hat{\mathbf{x}},\mathbf{u}))\mu(S^*(t,\mathbf{x},\hat{\mathbf{x}},\mathbf{u}))\mathbf{e}(t,\mathbf{x},\hat{\mathbf{x}},\mathbf{u})d\hat{\mathbf{x}} + \mathbf{b}(t,\mathbf{x}) \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{w}(\mathbf{x}), \dot{\mathbf{u}}(0,\mathbf{x}) = \mathbf{v}(\mathbf{x}), \mathbf{u}(t,\cdot)|_{\Omega_{\mathcal{I}}} = 0\,, \end{cases} \tag{9.2}$$

where $\Omega_{\mathcal{I}}$ is the nonlocal interaction domain. We note that both Cauchy problem and boundary value problems can fit into equation (9.2). Specifically, we list the different types of problems in the following with different choices of Ω and $\Omega_{\mathcal{I}}$.

$$\begin{cases} \text{Cauchy problem: } \Omega = \mathbb{R}^d,\, \Omega_{\mathcal{I}} = \emptyset\,. \\ \text{Dirichlet boundary problem: } \Omega \text{ open and bounded, } \Omega_{\mathcal{I}} = \{x \in \mathbb{R}^d \backslash \Omega, \operatorname{dist}(x, \partial\Omega) < \delta\}\,. \\ \text{Neumann boundary problem: } \Omega \text{ open and bounded, } \Omega_{\mathcal{I}} = \emptyset\,. \end{cases}$$

Note that for simplicity of notations, the Dirichlet and Neumann boundary problems we considered here are both with homogenous boundary conditions. Non-homogenous boundary conditions are also possible and will not cause essential difficulty in the proofs. We will put discussion of those cases in section 9.3.

Let $X = (L_0^\infty(\Omega))^d$, where $L_0^\infty(\Omega)$ denotes functions in $L^\infty(\Omega)$ with zero value outside Ω. In the subsequent of this section we will develop existence and uniqueness of solution in the space $C^2([0,T], X)$.

First, we extend the initial condition identically to the interval $[-T, 0]$, thus any function with time input less than zero is treated as equal to time zero. Now consider the space $C([-T,0], X)$. For any $t \in [0,T]$, we use the notation $\mathbf{u}_t \in C([-T,0],X)$ to be given by $\mathbf{u}_t(\theta,\cdot) = \mathbf{u}(t+\theta,\cdot)$ for $\theta \in [-T, 0]$. This notation $\mathbf{u}_t$ is a common usage in the context of functional differential equations [Hale, 1971; Wu, 2012]. For derivatives with respect to t in this paper, we will always the dot notation $\dot{\mathbf{u}}$. Now consider $\mathbf{F} : [0,T] \times C([-T,0],X) \to X$ to be given by

$$\mathbf{F}(t,\boldsymbol{\phi}) = \int_{B_\delta(x)} \omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|) f(S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}))\mu(S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}))\mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi})d\hat{\mathbf{x}} + \mathbf{b}(t,\mathbf{x}) \tag{9.3}$$

where

$$\begin{cases} S(\mathbf{x},\hat{\mathbf{x}},\phi) = \dfrac{|\boldsymbol{\phi}(0,\hat{\mathbf{x}}) - \boldsymbol{\phi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}| - |\hat{\mathbf{x}} - \mathbf{x}|}{|\hat{\mathbf{x}} - \mathbf{x}|} \\ S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) = \max\limits_{-T\le t\le 0} \dfrac{|\boldsymbol{\phi}(t,\hat{\mathbf{x}}) - \boldsymbol{\phi}(t,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}| - |\hat{\mathbf{x}} - \mathbf{x}|}{|\hat{\mathbf{x}} - \mathbf{x}|} \\ \mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) = \dfrac{\boldsymbol{\phi}(0,\hat{\mathbf{x}}) - \boldsymbol{\phi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}}{|\boldsymbol{\phi}(0,\hat{\mathbf{x}}) - \boldsymbol{\phi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}|}\,. \end{cases} \tag{9.4}$$

Lemma 9.2.1. *Assume that* $\int \frac{\omega_\delta(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|} d\boldsymbol{\xi} < \infty$*, and* f *and* μ *are bounded and Lipschitz continuous functions, then* $\mathbf{F}$ *is Lipschitz continuous in its second variable, namely*

$$\|\mathbf{F}(t,\boldsymbol{\phi}) - \mathbf{F}(t,\boldsymbol{\psi})\|_X \le L\|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{C([-T,0],X)}\,. \tag{9.5}$$

Proof. From the boundedness and Lipschitz continuity of f and μ, we have

$$\begin{aligned} &\|\mathbf{F}(t,\boldsymbol{\phi}) - \mathbf{F}(t,\boldsymbol{\psi})\|_X \\ \le&\Big\| \int_{B_\delta(\mathbf{x})} \omega_\delta(|\hat{\mathbf{x}} - \mathbf{x}|)\Big(f(S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi})) - f(S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\Big)\mu(S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}))\mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi})d\hat{\mathbf{x}}\Big\|_X \\ &+ \Big\| \int_{B_\delta(\mathbf{x})} \omega_\delta(|\hat{\mathbf{x}} - \mathbf{x}|) f(S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi}))\Big(\mu(S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi})) - \mu(S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\Big)\mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi})d\hat{\mathbf{x}}\Big\|_X \\ &+ \Big\| \int_{B_\delta(\mathbf{x})} \omega_\delta(|\hat{\mathbf{x}} - \mathbf{x}|) f(S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi}))\mu(S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\Big(\mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) - \mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\Big)d\hat{\mathbf{x}}\Big\|_X \\ \le& C_1\Big\| \int_{B_\delta(\mathbf{x})} \omega_\delta(|\hat{\mathbf{x}} - \mathbf{x}|)\big|S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) - S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\big|d\hat{\mathbf{x}}\Big\|_{L^\infty(\Omega)} \\ &+ C_2\Big\| \int_{B_\delta(\mathbf{x})} \omega_\delta(|\hat{\mathbf{x}} - \mathbf{x}|)\big|S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) - S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\big|d\hat{\mathbf{x}}\Big\|_{L^\infty(\Omega)} \\ &+ \Big\| \int_{B_\delta(\mathbf{x})} \omega_\delta(|\hat{\mathbf{x}} - \mathbf{x}|) f(S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi}))\big|\mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) - \mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\big|d\hat{\mathbf{x}}\Big\|_{L^\infty(\Omega)} \\ \le& \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned}$$

Now for the first term, since

$$\begin{aligned} |S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) - S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})| &= \frac{\big||\boldsymbol{\phi}(0,\hat{\mathbf{x}}) - \boldsymbol{\phi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}| - |\boldsymbol{\psi}(0,\hat{\mathbf{x}}) - \boldsymbol{\psi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}|\big|}{|\hat{\mathbf{x}} - \mathbf{x}|} \\ &\le \frac{\big|\boldsymbol{\phi}(0,\hat{\mathbf{x}}) - \boldsymbol{\phi}(0,\mathbf{x}) - (\boldsymbol{\psi}(0,\hat{\mathbf{x}}) - \boldsymbol{\psi}(0,\mathbf{x}))\big|}{|\hat{\mathbf{x}} - \mathbf{x}|}\,, \end{aligned}$$

we have

$$\begin{aligned}
\mathrm{I} \leq & C_1 \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \int_{B_\delta(\mathbf{x})} \frac{\omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)}{|\hat{\mathbf{x}}-\mathbf{x}|} \big|(\boldsymbol{\phi}-\boldsymbol{\psi})(0,\hat{\mathbf{x}}) - (\boldsymbol{\phi}-\boldsymbol{\psi})(0,\mathbf{x})\big| d\hat{\mathbf{x}} \\
\leq & C_1 \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \Big\{ \operatorname*{ess\,sup}_{\mathbf{z}\in\Omega\cup\Omega_\delta} \big|(\boldsymbol{\phi}-\boldsymbol{\psi})(0,\mathbf{z}) - (\boldsymbol{\phi}-\boldsymbol{\psi})(0,\mathbf{x})\big| \int_{B_\delta(\mathbf{x})} \frac{\omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)}{|\hat{\mathbf{x}}-\mathbf{x}|} d\hat{\mathbf{x}} \Big\} \\
\leq & C \operatorname*{ess\,sup}_{x\in\Omega} \big|(\boldsymbol{\phi}-\boldsymbol{\psi})(0,\mathbf{x})\big| \leq \tilde{C} \|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{C([-T,0],X)} \,.
\end{aligned}$$

The second term can be estimated similarly by noticing that

$$|S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) - S^*(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})| = \max_{-T\leq t\leq 0} \frac{\big|(\boldsymbol{\phi}-\boldsymbol{\psi})(t,\hat{\mathbf{x}}) - (\boldsymbol{\phi}-\boldsymbol{\psi})(t,\mathbf{x})\big|}{|\hat{\mathbf{x}}-\mathbf{x}|} \,.$$

Thus

$$\begin{aligned}
\mathrm{II} \leq & C_2 \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \int_{B_\delta(x)} \frac{\omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)}{|\hat{\mathbf{x}}-\mathbf{x}|} \max_{-T\leq t\leq 0} \big|(\boldsymbol{\phi}-\boldsymbol{\psi})(t,\hat{\mathbf{x}}) - (\boldsymbol{\phi}-\boldsymbol{\psi})(t,\mathbf{x})\big| d\hat{\mathbf{x}} \\
\leq & C_2 \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \Big\{ \operatorname*{ess\,sup}_{\mathbf{z}\in\Omega\cup\Omega_\delta} \max_{-T\leq t\leq 0} \big|(\boldsymbol{\phi}-\boldsymbol{\psi})(t,\mathbf{z}) - (\boldsymbol{\phi}-\boldsymbol{\psi})(t,\mathbf{x})\big| \int_{B_\delta(\mathbf{x})} \frac{\omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)}{|\hat{\mathbf{x}}-\mathbf{x}|} d\hat{\mathbf{x}} \Big\} \\
\leq & C \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \max_{-T\leq t\leq 0} \big|(\boldsymbol{\phi}-\boldsymbol{\psi})(t,\mathbf{x})\big| = \tilde{C} \|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{C([-T,0],X)} \,.
\end{aligned}$$

Now we only need to show that $\mathrm{III} \leq \tilde{C}\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{C([-T,0],X)}$. First, if $S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi}) < S_1^-$, then by definition we have $f(S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})) = 0$. Since this case does not contribute to III, we can only consider the case where $S(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi}) \geq S_1^-$, which implies that

$$\frac{|\boldsymbol{\psi}(0,\hat{\mathbf{x}}) - \boldsymbol{\psi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}|}{|\hat{\mathbf{x}}-\mathbf{x}|} \geq 1 + S_1^- > 0 \,.$$

Now denote $\boldsymbol{\alpha} = \boldsymbol{\phi}(0,\hat{\mathbf{x}}) - \boldsymbol{\phi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}$, $\boldsymbol{\beta} = \boldsymbol{\psi}(0,\hat{\mathbf{x}}) - \boldsymbol{\psi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}$, then

$$\begin{aligned}
&\big|\mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\phi}) - \mathbf{e}(\mathbf{x},\hat{\mathbf{x}},\boldsymbol{\psi})\big| = \left|\frac{\boldsymbol{\alpha}|\boldsymbol{\beta}| - \boldsymbol{\beta}|\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}||\boldsymbol{\beta}|}\right| \leq \left|\frac{\boldsymbol{\alpha}(|\boldsymbol{\beta}| - |\boldsymbol{\alpha}|)}{|\boldsymbol{\alpha}||\boldsymbol{\beta}|}\right| + \left|\frac{|\boldsymbol{\alpha}|(\boldsymbol{\alpha}-\boldsymbol{\beta})}{|\boldsymbol{\alpha}||\boldsymbol{\beta}|}\right| \leq 2\frac{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}{|\boldsymbol{\beta}|} \\
&\leq 2\frac{|(\boldsymbol{\phi}-\boldsymbol{\psi})(0,\hat{\mathbf{x}}) - (\boldsymbol{\phi}-\boldsymbol{\psi})(0,\mathbf{x})|}{|\boldsymbol{\psi}(0,\hat{\mathbf{x}}) - \boldsymbol{\psi}(0,\mathbf{x}) + \hat{\mathbf{x}} - \mathbf{x}|} \leq \frac{2}{1+S_1^-} \frac{|(\boldsymbol{\phi}-\boldsymbol{\psi})(0,\hat{\mathbf{x}}) - (\boldsymbol{\phi}-\boldsymbol{\psi})(0,\mathbf{x})|}{|\hat{\mathbf{x}}-\mathbf{x}|} \,.
\end{aligned}$$

Therefore we get $\mathrm{III} \leq \tilde{C}\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{C([-T,0],X)}$ by similar arguments in case I. □

Notice now that by the definition of $\mathbf{F}$, the peridynamics system can be written into

$$\rho\ddot{\mathbf{u}}(t,\mathbf{x}) = \mathbf{F}(t,\mathbf{u}_t) \,.$$

We reduce the second order system to a first order one by denoting

$$U = \begin{bmatrix} \rho\mathbf{u} \\ \rho\dot{\mathbf{u}} \end{bmatrix}, \quad G(t,U_t) = \begin{bmatrix} \rho(\dot{\mathbf{u}})_t(0) \\ \mathbf{F}(t,\mathbf{u}_t) \end{bmatrix},$$

where $(\dot{\mathbf{u}})_t(\theta,\cdot) = \dot{\mathbf{u}}(t+\theta,\cdot)$ and $U_t = [\rho\mathbf{u}_t, \rho(\dot{\mathbf{u}})_t]^T$. Then the second order system is equivalent to

$$\dot{U} = G(t, U_t)\,.$$

Now denote the space $Y = (L_0^\infty(\Omega))^{d\times 2}$, then similarly as Lemma 9.2.1, we could show that

$$\|G(t,\Phi) - G(t,\Psi)\|_Y \leq L\|\Phi - \Psi\|_{C([-T,0],Y)}\,,$$

for some $L > 0$.

Theorem 9.2.2 (Well-posedness of the integral equation)**.** *Let $W \in Y$ and and $\mathbf{b}(t,\mathbf{x}) \in C([0,T],X)$, then there exists a unique solution $U \in C([0,T],Y)$ that satisfies the following initial value problem of the integral equation,*

$$\begin{cases} U(t) = \displaystyle\int_0^t G(s,U_s)ds + W \\ U_0(\theta) \equiv W \quad (-T \leq \theta \leq 0) \end{cases} . \tag{9.6}$$

Proof. We use Picard iteration. Define

$$U^0(t) \equiv W, \quad U^n(t) = \int_0^t G(s, U_s^{n-1})ds + W\,.$$

Then since G is continuous, there exists a M such that $\|G(s,U_s^{n-1})\|_Y \leq M$ for all $0 \leq s \leq t$. Let L be the Lipchitz constant for $G(s,\cdot)$, we have

$$\|U^1(t) - U^0(t)\|_Y \leq Mt$$

and in general,

$$\|U^n(t) - U^{n-1}(t)\|_Y \leq ML^{n-1}\frac{t^n}{n!}\,.$$

This implies that $\{U^n\}$ is a Cauchy sequence in $C([0,T],Y)$. The limit U satisfies the equation, since

$$\left\|U(t) - \int_0^t G(s,U_s)ds - W\right\|_Y \leq \|U - U^n\|_Y + \left\|\int_0^t G(s,U_s) - G(s,U_s^n)ds\right\|_Y \to 0\,.$$

For the uniqueness, suppose there is another solution $\tilde{U} \in C([0,T],Y)$, and

$$\|\tilde{U}(t) - U(t)\|_Y \leq \int_0^t \|G(s,\tilde{U}_s) - G(s,U_s)\|_Y ds \leq Ct\,.$$

Do the iteration for $\tilde{U}$ and U by n times we have

$$\|\tilde{U}(t) - U(t)\|_Y = \|(\tilde{U})^n(t) - U^n(t)\|_Y \leq CL^{n-1}\frac{t^n}{n!} \to 0\,.$$

□

Now we discuss the relation between the integral equation and the differential equation.

Lemma 9.2.3. *Let $W \in Y$ and $\mathbf{b}(t, \mathbf{x}) \in C([0, T], X)$. Then the solution $U \in C([0, T], Y)$ to the integral equation is the solution to the following delayed differential equation with $U \in C^1([0, T], Y)$.*

$$\begin{cases} \dfrac{dU}{dt} = G(t, U_t) \\ U_0(\theta) \equiv W \quad (-T \leq \theta \leq 0) \end{cases} \tag{9.7}$$

Proof. By definition, we have

$$\frac{U(t+h) - U(t)}{h} - G(t, U_t) = \frac{1}{h}\int_t^{t+h} G(s, U_s) - G(t, U_t)ds\,.$$

Since

$$\begin{aligned} \|G(s, U_s) - G(t, U_t)\|_Y &\leq \|G(s, U_s) - G(t, U_s)\|_Y + \|G(t, U_s) - G(t, U_t)\|_Y \\ &\leq C_1 + C_2\|U_s - U_t\|_{C([-T,0],Y)} \\ &\leq C_1 + \tilde{C}_2\|U\|_{C([0,T],Y)} \leq C \end{aligned}$$

then

$$\lim_{h\to 0}\left\|\frac{U(t+h) - U(t)}{h} - G(t, U_t)\right\|_Y \to 0\,.$$

So $\frac{dU}{dt} = G(t, U_t) \in C([0, T], Y)$. □

Combing the above arguments, we arrive at the following well-posedness of the peridynamic system.

Theorem 9.2.4. *Assume that $\int \frac{\omega_\delta(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}d\boldsymbol{\xi} < \infty$ and that f and μ are bounded and Lipschitz continuous. With initial data $\mathbf{w} \in X, \mathbf{v} \in X$ and forcing term $\mathbf{b}(t, \mathbf{x}) \in C([0, T], X)$, there exists a unique solution $u \in C^2([0, T], X)$ to the system* (9.2).

Proof. This is immediate from Theorem 9.2.2 and Lemma 9.2.3. □

Now assume that the external force does not change with respect to time, namely $\boldsymbol{b}(t,\mathbf{x}) = \boldsymbol{b}(\mathbf{x})$, we define the total energy of the system (9.2) to be

$$\begin{aligned} E(t) =& \frac{1}{2}\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} |\hat{\mathbf{x}}-\mathbf{x}|\omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)p(\mathcal{S}(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u}))\mu(\mathcal{S}^*(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u}))d\hat{\mathbf{x}}d\mathbf{x} \\ &+ \frac{1}{2}\int_\Omega \rho|\dot{\mathbf{u}}(t,\mathbf{x})|^2 d\mathbf{x} - \int_\Omega \boldsymbol{u}(t,\mathbf{x})\cdot\boldsymbol{b}(\mathbf{x})d\mathbf{x}\,. \end{aligned} \tag{9.8}$$

where $p(0)=0$ and $p'(x)=f(x)$. Then we can show that the total energy is decaying over time.

Theorem 9.2.5 (Energy decaying)**.** *The total energy of the system* (9.2) *is decreasing with time, namely,*

$$\frac{dE(t)}{dt} \leq 0\,.$$

Proof.

$$\begin{aligned} \frac{dE(t)}{dt} =& \int_\Omega \rho\dot{\mathbf{u}}(t,\mathbf{x})\cdot\ddot{\mathbf{u}}(t,\mathbf{x})\ d\mathbf{x} - \int_\Omega \dot{\mathbf{u}}(t,\mathbf{x})\cdot\boldsymbol{b}(\mathbf{x})\ d\mathbf{x} \\ &- \int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} \omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)f(\mathcal{S})\mu(\mathcal{S}^*)\mathbf{e}\cdot\dot{\mathbf{u}}(t,\mathbf{x})d\hat{\mathbf{x}}d\mathbf{x} \\ &+ \frac{1}{2}\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} |\hat{\mathbf{x}}-\mathbf{x}|\omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)p(\mathcal{S}(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u}))\frac{d\mu(\mathcal{S}^*(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u}))}{dt}d\hat{\mathbf{x}}d\mathbf{x} \\ =& \frac{1}{2}\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} |\hat{\mathbf{x}}-\mathbf{x}|\omega_\delta(|\hat{\mathbf{x}}-\mathbf{x}|)p(\mathcal{S}(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u}))\frac{d\mu(\mathcal{S}^*(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u}))}{dt}d\hat{\mathbf{x}}d\mathbf{x}\,. \end{aligned}$$

Now by the definition of $\mathcal{S}^*(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u})$, the maximum value of all $\mathcal{S}(s,\hat{\mathbf{x}},\mathbf{x},\mathbf{u})$ for $s\in[0,t]$, we know that $\mathcal{S}^*$ is increasing with time. Since μ is a nonincreasing function, we have $\frac{d\mu(\mathcal{S}^*(t,\hat{\mathbf{x}},\mathbf{x},\mathbf{u}))}{dt} \leq 0$. Thus the theorem is shown by noticing that $p(x)\geq 0$. □

9.3 Conclusion and discussions

In this chapter, we studied the well-posedness of nonlocal nonlinear peridynamic model with memory on its bond stretching. With nonlinearity involved, the bond-breaking model has considered as one of the most challenging peridynamic models. Our new model is a necessary modification of the conventional peridynamic model with bond-breaking in the literature under the consideration of mathematical validity, and it retains the same essential physics.

We focused our discussion in section 9.2 on Cauchy problems and homogeneous boundary value problems, but we could also treat non-homogeneuos boundary conditions. First, with non-homogeneous Dirichlet boundary conditions, we just need to modify the solution space X and the rest remains the same. Second, the (nonlocal) non-homogeneous Neumann boundary condition is equivalent to an extra body force around a δ-layer of the domain Ω (see discussions in [Tao *et al.*, Submitted 2016]). Such extra term added to the body force will cause no significant change in the arguments that follow equation (9.2).

Chapter 10

Discussions and other related analysis works

10.1 Summary of the analysis works

We presented in this part several analysis works that are needed in establishing the well-posedness of different nonlocal models and the convergence of numerical schemes. Chapter 7 contains various extensions of the seminal work of Bourgain-Brezis-Mironescu. While [Bourgain *et al.*, 2001] approximated the Sobolev space H^1 by a sequence of properly defined nonlocal spaces, we could characterize more spaces including the fractional Sobolev space and higher integer order Sobolev space. The results are useful in establishing the DG scheme for nonlocal diffusion equation in section 4.3 and the well-posedness of high order nonlocal models in [O'Grady and Foster, 2014b].

Chapter 8 contains generalization and improvement of classical trace theorem for Sobolev spaces to nonlocal function space. The importance of the generalized trace theorems manifests in at least two aspects. First, it helps to develop the well-posedness of nonlocal boundary value problems associated with nonlocal interactions having a varying horizon, in that the problem is well-posed with $H^{\frac{1}{2}}$ data assigned on the codimension-1 boundary. Second, given the growing interest in the modeling of coupled nonlocal and local problems, the idea of varying interaction length in nonlocal models provides a unique point of view towards the coupling process as depicted in Fig. 8.1. Instead of sewing nonlocal and local models together with a patch, we may glue the two together by assigning identical $H^{\frac{1}{2}}$ data on the interface. This is where a variable horizon model needs to be applied, and where the trace theorem established in chapter 8 comes in to provide the legitimacy of the coupling approach. Advantages of this approach are to be further explored, but for now we can see that it provides us with interesting new mathematical development.

While study of nonlocal models has been largely focused on linear models, chapter 9 is a first attempt to study the peridynamic system with memory effect, which appears in the original peridynamics formulation in describing dynamics of material fracture. Lots of simulations have done by using peridynamics especially with the bond-breaking rule. Rigorous mathematical analysis for such models, however, is quite behind. Start from the well-posedness result of the bond-breaking model showed in chapter 9, it is possible for us to characterize more mathematical properties of nonlinear nonlocal models in the future.

10.2 Future works and related problems

At last, we give some remarks about future works that related to the mathematical analysis of chapter 7-9.

For extensions BBM theorem In chapter 7, we have limited the study to the case of scalar fields. In the future, it is natural to further study the extension of nonlocal calculus of variations to high order operators and functionals defined for more general vector fields. Time-dependent and nonlinear problems can also be studied.

For trace theorems on nonlocal spaces In terms of further generalizations of the trace theorems presented in chapter 8, we note that although the results are only shown for the L^2 or the Hilbert space setting, it is not surprising that they can be generalized to the L^p and other more general Banach spaces. With the choices of more general kernels, one may also consider nonlocal extensions of trace results in fractional $W^{s,p}$ type spaces. Extensions of the notion of trace may also go beyond co-dimensional one manifolds to other more general subdomains or sets. Moreover, the position-dependent and heterogeneous feature in the nonlocal norms may be related for the study of more general Morrey, Campanato, Besov and Lizorkin-Triebel spaces, possibly of variable order and growth conditions, to obtain new type of spaces and the associated trace maps [Johnsen *et al.*, 2015; Nakamura *et al.*, 2016]. Interesting connections with the study of Sobolev and other function spaces on metric measure spaces may also be explored [Heinonen, 2017]. Another direction is to consider analogous results for spaces of vector fields such as those studied in [Mengesha and Du, 2014b; Mengesha and Du, 2015]. One may naturally investigate high order extensions as well, following the discussions of high order nonlocal spaces likes ones in section 7.2. Mathematically, one may also ask questions concerning optimal constants in the trace inequality, as in the classical case [Escobar, 1988].

In closing, the study of the nonlocal space in chapter 8 also motivates us to consider here the Hilbert space setting that is naturally associated with linear nonlocal equations, a first steps towards the understanding of more complex nonlinear models. One may investigate further regularity, multiscale analysis and homogenization issues associated with nonlocal problems with a heterogeneous choice of variable horizon and nonlocal interaction kernels. Having varying horizon allows one to harvest the flexibility in working with a wide range of nonlocal interactions so that more effective numerical simulations can be carried out, along the lines of asymptotically compatible schemes (chapter 3).

For the nonlinear peridynamic model with bond-breaking. The mathematical analysis in chapter 9 is based on ODE theories and the key relies on the Lipschitz continuity of the right-hand side defined in (9.3). To obtain the Lipschitz continuity, it is necessary to modify the original peridynamic bond-breaking model by adding four parameters $S_1^-, S_0^-, S_0^+, S_1^+$ that are used to smooth out the jumps. The original bond-breaking model then can be seen as the limit when $S_1^- \to -1, S_0^- \to -1$ and $S_0^+ \to S_c, S_1^+ \to S_c$. However, the limiting model does satisfy the Lipschitz continuity condition and the discussion of it is beyond the scope of this paper. In the subsequent work, the authors are trying to discuss the limiting model by means of theories of differential inclusion [Aubin and Cellina, 1984; Deimling, 1992], which are generalizations of ODE theories to include discontinuous right-hand side. The existence theory is possible to achieve by taking the right-hand side to be a multivalued map that satisfy certain continuity condition. The uniqueness theory, however, is a challenge to such model which we need to put further effort into.

Part III

Multiscale modeling

Chapter 11

Seamless coupling of nonlocal and local models

While nonlocal models are showing the effectiveness of modeling physical processes that may involve singular behaviors, they also increase the computational cost significantly compared to the conventional models based on PDEs. As a result, there have been studies emerged to find strategies that couple nonlocal models with the conventional local models so as to combine the accuracy of nonlocal models with the computational efficiency of PDEs.

In the following, we will propose an energy based method in section 11.1 for the coupling of nonlocal and local diffusion problems which uses the idea of variable horizon in chapter 8. Section 11.2 shows the well-posedness of coupled problem which can applied to multi-dimensional problems on general domains. Section 11.3 discusses the convergent finite element discretization of the model with conclusions in section 11.4.

11.1 A nonlocal-to-local coupling model based on variable horizon

Our method of seamless coupling of nonlocal and local models is based on using the spatially heterogenous nonlocal interaction $\delta(x)$. By allowing $\delta(x)$ to change over the space domain, we effectively get a coupling model within one framework, see Fig 8.1.

11.1.1 The energy space

We adopted the notations Ω_- and Ω_+ in chapter 8 to denote two open domains in $\mathbb{R}^d$ that satisfies $\overline{\Omega_-} \cap \overline{\Omega_+} = \Gamma \subset \mathbb{R}^{d-1}$ and $\mathrm{Int}(\overline{\Omega_-} \cup \overline{\Omega_+}) = \Omega$. Let $\mathcal{S}(\Omega_+)$ be the nonlocal space with heterogeneous localization on the boundary $\partial\Omega_+$ as in equations (8.1)-(8.4). Based on the trace theorem developed in chapter 8, we can define a proper energy space to be

$$\mathcal{W}(\Omega) = \{u \in H^1(\Omega_-) \cap \mathcal{S}(\Omega_+) | u_- = u_+ \text{ on } \Gamma, u = 0 \text{ on } \partial\Omega\},$$

where $u_-(x)$ and $u_+(x)$ are defined as $\lim_{y\to x, y\in\Omega_-} u(y)$ and $\lim_{y\to x, y\in\Omega_+} u(y)$ respectively. For $u \in \mathcal{W}(\Omega)$, the norm of u is defined as

$$\|u\|_{\mathcal{W}(\Omega)} = \|u\|_{H^1(\Omega_-)} + \|u\|_{\mathcal{S}(\Omega_+)}.$$

Our coupled nonlocal-to-local diffusion model comes from solving the minimization problem

$$\min_{u\in\mathcal{W}(\Omega)} \left\{\frac{1}{2}|u|^2_{H^1(\Omega_-)} + \frac{1}{2}|u|^2_{\mathcal{S}(\Omega_+)} - (f, u)_\Omega\right\}. \tag{11.1}$$

Now for problem (11.1) to be well-posed, we only need to show the Poincaré type inequality on the space $\mathcal{W}(\Omega)$. That would require us to obtain a Poincaré inequality on the nonlocal space.

11.2 Well-posedness of the coupling model

We will show a Poincaré type inequality on the nonlocal space with variable horizon, from which a Poincaré inequality on $\mathcal{W}(\Omega)$ is easily seen.

Theorem 11.2.1 (Poincaré inequality)**.** *Let $\mathcal{S}(\hat{\Omega})$ be the space defined in* (8.1)-(8.4) *for a given open domain $\hat{\Omega} \subset \mathbb{R}^d$. Assume that on the boundary $\hat{\Gamma} \subset \partial\hat{\Omega}$, we have $u = 0$, then the following inequality holds for C independent of u.*

$$\|u\|_{L^2(\hat{\Omega})} \le C|u|_{\mathcal{S}(\hat{\Omega})} \tag{11.2}$$

Proof. Suppose that equation (11.2) does not hold, then we can find a sequence $\{u_k \in \mathcal{S}(\hat{\Omega})\}$ such that $\|u_k\|_{L^2} = 1$ and $|u_k|_{\mathcal{S}(\hat{\Omega})} \to 0$ as $k \to \infty$. This leads to the existence of a weak limit $u \in L^2$ such that $u_k \rightharpoonup u$ in L^2.

Step 1. We show that u is in fact 0. We claim that $|\cdot|_{\mathcal{S}(\hat{\Omega})}$ is L^2-weakly lower semicontinuous, namely,

$$|u|_{\mathcal{S}(\hat{\Omega})} \leq \liminf_{k} |u_k|_{\mathcal{S}(\hat{\Omega})} \,. \tag{11.3}$$

In fact, since $|\cdot|_{\mathcal{S}(\hat{\Omega})}$ is a convex functional, then the weakly lower semicontinuity is equivalent to lower semicontinuity. So we only need to show that if $u_k \to u$ strongly in L^2, then (11.3) holds. In fact, from the assumption that $u_k \to u$ strongly in L^2, we can extract a subsequence of $\{u_k\}$ such that it converges to u pointwise up to a set of measure zero. Then (11.3) is true by applying Fatou's inequality. Now from (11.3) we have $|u|_{\mathcal{S}(\hat{\Omega})} = 0$ so that u equals a constant in $\hat{\Omega}$. From the boundary condition we know that u must be equal to 0.

Step 2. We next show that $u_k \to 0$ strongly in L^2, which results in a contradiction from the assumption. First, for the kernel γ defined in (8.3), we could replace it with the symmetric version $\frac{1}{2}(\gamma(\mathbf{x},\mathbf{y}) + \gamma(\mathbf{y},\mathbf{x}))$ and use the same notation γ as well. Next, for some $M > 0$, we define $\bar{\gamma} = \min\{M, \gamma\}$. Then we have

$$\begin{aligned}
|u_k|^2_{\mathcal{S}(\hat{\Omega})} &\geq \iint_{\hat{\Omega}\times\hat{\Omega}} \bar{\gamma}(x,y)|u_k(y) - u_k(x)|^2 dydx \\
&\geq 2\int_{\hat{\Omega}} \Big(\int_{\hat{\Omega}} \bar{\gamma}(x,y)dy\Big) u_k^2(x)dx - 2\int_{\hat{\Omega}} \Big(\int_{\hat{\Omega}} \bar{\gamma}(x,y)u_k(y)dy\Big) u_k(x)dx \\
&\geq 2C\|u_k\|_{L^2(\hat{\Omega})} - 2\int_{\hat{\Omega}} \Big(Ku_k(x)\Big) u_k(x)dx \,,
\end{aligned}$$

where

$$Ku_k(x) = \int_{\hat{\Omega}} \bar{\gamma}(x,y)u_k(y)dy$$

is a Hilbert-Schmidt operator. Now since $u_k \rightharpoonup 0$ by the first step, we have $Ku_k \to 0$ strongly in L^2. Thus

$$0 = \lim_{k\to\infty} |u_k|^2_{\mathcal{S}(\hat{\Omega})} \geq 2C \lim_{k\to\infty} \|u_k\|_{L^2(\hat{\Omega})} \,,$$

which implies that $u_k \to 0$ in L^2 and it is a contradiction to $\|u_k\|_{L^2(\hat{\Omega})} = 1$.

□

With Poincaré inequality established on $\mathcal{S}(\hat{\Omega})$ for any bounded and open domain $\hat{\Omega} \subset \mathbb{R}^d$, we can easily see that Poincaré inequality holds on $\mathcal{W}(\hat{\Omega})$, which shows that the minimization problem (11.1) is well-posed.

11.3 Numerical schemes

We consider finite element approximation of the coupled problem. For any $h > 0$, we introduce the finite element spaces $\{W_h\} \subset \mathcal{W}(\Omega)$ associated with the triangulation $\tau_h = \{K\}$ of the domain Ω. Let W_h be defined as

$$W_h := \{v \in \mathcal{W}(\Omega) \cap C(\overline{\Omega}) : v|_K \in \mathcal{P}_1(K) \quad \forall K \in \tau_h\}$$

where $\mathcal{P}_1(K)$ is the space of polynomials on $K \in \tau_h$ of degree less or equal than 1.

Now since by Proposition 8.2.1, we know that for any $\hat{\Omega} \subset \mathbb{R}^d$ the space $H^1(\hat{\Omega})$ is continuously embedded in the nonlocal space $\mathcal{S}(\hat{\Omega})$, then it is obvious that the space $H^1(\Omega)$ is continuously embedded in $\mathcal{W}(\Omega)$. Thus the sequence of spaces $\{W_h\}_h \to 0$ is dense in $\mathcal{W}(\Omega)$, namely, for any $v \in \mathcal{W}(\Omega)$, there exists a sequence $v_n \in W_{h_n}$ with $h_n \to 0$ as $n \to \infty$ such that

$$\|v_n - v\|_{\mathcal{W}(\Omega)} \to 0 \quad n \to \infty\,.$$

Then the Galerkin approximations of the variational problem (11.1) onto the spaces $\{W_h\}$ converge to the solution of (11.1) as $h \to 0$ by Céa's lemma.

11.4 Conclusion

In this chapter, we established a well-posed coupling strategy that combines nonlocal diffusion model with local classical diffusion model. The method is based on using nonlocal models with heterogeneous localization. Due to the trace theorems established in chapter 8, we can assign identical $d-1$ data on the surface where the nonlocal model meets the local model. The formulation of the coupling strategy is simple and suitable for multi-dimensional problems on any general geometry. Moreover, continuous finite element method with piecewise linear basis is shown to be a convergent scheme for the coupled model.

In the future, it is possible for us to develop asymptotically compatible finite element schemes for the coupled nonlocal-to-local diffusion model using the framework developed in chapter 3.

Chapter 12

A quasinonlocal coupling method for nonlocal and local diffusion models

In this chapter, we will investigate into another energy-based coupling method that combines the nonlocal diffusion and local diffusion problems. Borrowed from the quasi-nonlocal atomistic-to-continuum coupling methods for crystalline materials (see for example, [Shimokawa *et al.*, 2004; E *et al.*, 2006; Ming and Yang, 2009; Shapeev, 2012; Li and Luskin, 2011; Ortner and Zhang, 2014]), we call our coupling method the *quasi-nonlocal (QNL) coupling* of nonlocal and local diffusion. As a first step towards the QNL coupling, we will focus on one-dimensional problem to better illustrate the idea.

More specifically, in section 12.1 we first define the combined total energy from which the QNL operator is derived through energy variation, followed by the discussion of the concerned issue of patch-test consistency. Section 12.2 contains rigorous arguments of the well-posedness of the coupled problem. Section 12.3 further explores the modeling accuracy of the coupled method compared with the fully local diffusion equation in terms of small δ, in which the uniform first order accuracy in terms of δ is shown. Conclusion and discussions are put in section 12.4.

12.1 The quasinonlocal coupling

In this section, we formulate our idea of QNL coupling in a one-dimensional bar. Without loss of generality, we work on the domain $\Omega = (-1, 1)$ and it is used at all places in the remainder of this paper. We consider the nonlocal interaction region to be on the left side of the bar Ω and the local interaction region to be on the right side with a transition layer in the middle of width δ. Now that the domain Ω is composed of both nonlocal and local interaction regions, the Dirichlet boundary condition to impose should be considered as a mixture of nonlocal and local boundary conditions. Specifically, to the left of the bar Ω there is a nonlocal boundary $(-1-\delta, -1)$ and to the right of the bar a local boundary $\{1\}$. Thus we use $\Omega_\delta = (-1-\delta, -1) \cup \{1\}$ as the volumetric boundary domain in all further discussions.

12.1.1 The energy space

The QNL coupling method comes from energy variation of the total energy defined as

$$E_\delta^{\text{qnl}}(u) := \frac{1}{4} \iint_{x<0 \text{ or } y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))^2\, dydx + \frac{1}{2} \int_{x>0} |\nabla u(x)|^2 \omega_\delta(x)\, dx. \quad (12.1)$$

where the weight function ω_δ is given by

$$\omega_\delta(x) := \int_0^1 dt \int_0^{\frac{x}{t}} |s|^2 \gamma_\delta(|s|)\, ds. \quad (12.2)$$

We assume that γ_δ is a rescaled kernel given by (1.5) which satisfies $\frac{1}{2}\int_{-\delta}^{\delta} |s|^2 \gamma_\delta(|s|)\, ds = 1$. Then it is easy to see that $\omega_\delta(x)$ is a nondecreasing function on $[0, \infty)$ with $\omega_\delta(0) = 0$ and $\omega_\delta(x) = 1$ for $x \geq \delta$. Thus the total quasinonlocal energy has a transition from pure nonlocal to pure local through the interfacial region $(0, \delta)$. Further characterizations of the weight function $\omega_\delta(x)$ are put in the appendix.

The energy defined in (12.1) has a more intuitive interpretation from the geometric reconstruction formulation [E *et al.*, 2006; Li and Luskin, 2011; Li and Lu, 2016], which is equivalent to be written as

$$\begin{aligned} E_\delta^{\text{qnl}}(u) =& \frac{1}{4} \iint_{x<0 \text{ or } y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))^2\, dydx \quad (12.3)\\ &+ \frac{1}{4} \iint_{x>0 \text{ and } y>0} dydx\, \gamma_\delta(|y-x|) \cdot \int_0^1 dt |\nabla u(x+t(y-x))|^2 |y-x|^2. \end{aligned}$$

To better convey the idea of geometric reconstruction proposed in [Li and Lu, 2016], we first assume that $\Omega = \Omega_1 \sqcup \Omega_2$ is dominated by two different nonlocal kernels γ_{δ_1} and γ_{δ_2}, respectively. Next, we utilizes the interaction kernel γ_{δ_1} throughout the entire domain Ω, while in the subregion Ω_2, the displacement of bond $(u(y) - u(x))$ will be reconstructed so that it only involves x and y pairs that are closer in distance. More concretely, to link the interaction with kernel γ_{δ_2} to γ_{δ_1} where $\delta_1 = M\delta_2$, if a bond $\{x - y\}$ is completely contained in the subregion Ω_2, then the displacement of this bond $(u(y) - u(x))$ will be reconstructed by the following expression:

$$u(y) - u(x) \to \Big(u\big(x + \frac{j+1}{M}(y-x)\big) - u\big(x + \frac{j}{M}(y-x)\big)\Big) M, \quad \forall j = 0, \ldots, (M-1).$$

Hence, the bond interaction $\gamma_{\delta_1}(|y-x|)\,(u(y) - u(x))^2$ in Ω_2 is approximated by

$$\gamma_{\delta_1}(|y-x|)\frac{1}{M}\sum_{j=0}^{M-1}\Big(u\big(x + \frac{j+1}{M}(y-x)\big) - u\big(x + \frac{j}{M}(y-x)\big)\frac{\delta_1}{\delta_2}\Big)^2. \tag{12.4}$$

Note that if $|x-y| \le \delta_1$, the difference on the right is evaluated at points with distance at most $\frac{\delta_1}{M} = \delta_2$; thus effectively, the difference $u(y) - u(x)$ is reconstructed by a more local interaction (and hence the idea was referred to as the "geometric reconstruction" scheme in [E *et al.*, 2006]). In fact, if such reconstruction is adopted everywhere in the entire domain Ω, one will recover the fully nonlocal interactions with kernel γ_{δ_2} only. Notice that when $M = \frac{\delta_1}{\delta_2} \to \infty$, the summation in (12.4) becomes the Riemann sum of a path integral,

$$\frac{1}{M}\sum_{j=0}^{M-1}\Big(u\big(x + \frac{j+1}{M}(y-x)\big) - u\big(x + \frac{j}{M}(y-x)\big)\frac{\delta_1}{\delta_2}\Big)^2 \\ \to \int_0^1 |\nabla u(x + t(y-x))|^2 |y-x|^2 dt.$$

Therefore, the nonlocal bond interaction $\gamma_\delta(|y-x|)\,(u(y) - u(x))^2$ can be reconstructed by its local continuum approximation through a path integral:

$$\gamma_\delta(|y-x|) \cdot \int_0^1 dt |\nabla u(x + t(y-x))|^2 |y-x|^2. \tag{12.5}$$

Based on this approximation, we can derive the total coupling energy (12.3).

We will show now that the two ways of writing the quasinonlocal total energy are the same. From the expressions (12.1) and (12.3), it suffices to show that local contribution to the total energy is equivalent. The two different ways of writing the local contribution of the energy has their own advantages and we will adopt either definition at our convenience in the future.

Proposition 12.1.1. *We have an equivalent expression of local contribution to the total energy:*

$$E_\delta^{\mathrm{loc}}(u) = \frac{1}{4}\iint_{x>0 \ \text{and} \ y>0} dxdy\, \gamma_\delta(|y-x|)\cdot\int_0^1 dt|\nabla u(x+t(y-x))|^2|y-x|^2, \qquad (12.6)$$

where E_δ^{loc} *is defined as*

$$E_\delta^{\mathrm{loc}}(u) := \frac{1}{2}\int_{x>0}|\nabla u(x)|^2\omega_\delta(x)\,dx. \qquad (12.7)$$

Proof. We start with recasting the right hand side of (12.6)

$$\begin{aligned}
&\frac{1}{4}\iint_{x>0 \text{ and } y>0}\gamma_\delta(|y-x|)\cdot\int_0^1 dt|\nabla u(x+t(y-x))|^2|y-x|^2\\
&=\frac{1}{4}\int_0^1 dt\int_{x>0}dx\int_{z>(1-t)x}dz\gamma_\delta\left(\left|\frac{z-x}{t}\right|\right)|\nabla u(z)|^2\frac{1}{t^3}|z-x|^2\\
&=\frac{1}{4}\int_0^1 dt\int_{z>0}dz|\nabla u(z)|^2\int_{0<x<\frac{z}{1-t}}\gamma_\delta\left(\left|\frac{x-z}{t}\right|\right)\frac{1}{t^3}|x-z|^2\ dx\\
&=\frac{1}{4}\int_{z>0}dz|\nabla u(z)|^2\int_0^1 dt\int_{-\frac{z}{t}<s<\frac{z}{1-t}}\gamma_\delta\left(|s|\right)|s|^2\ ds\,.
\end{aligned}$$

Now since

$$\begin{aligned}
&\int_0^1 dt\int_{-\frac{z}{t}<s<\frac{z}{1-t}}|s|^2\gamma_\delta\left(|s|\right)\ ds\\
&=\int_0^1 dt\int_{-\frac{z}{t}<s<0}|s|^2\gamma_\delta\left(|s|\right)\ ds+\int_0^1 dt\int_{0<s<\frac{z}{1-t}}|s|^2\gamma_\delta\left(|s|\right)\ ds\\
&=\int_0^1 dt\int_{-\frac{z}{t}<s<0}|s|^2\gamma_\delta\left(|s|\right)\ ds+\int_0^1 dt\int_{0<s<\frac{z}{t}}|s|^2\gamma_\delta\left(|s|\right)\ ds\,,
\end{aligned}$$

we arrive at definition of E_δ^{loc} in (12.7) and the weight function ω_δ in (12.2). □

Now we give a technical lemma that characterizes the weight function and its derivatives which will be used quite often in the rest of this chapter.

Lemma 12.1.2. *By the definition of* ω_δ *in* (12.2)*, we have the following equations*

$$\omega_\delta(x) = \int_0^x s^2\gamma_\delta(|s|)ds + x\int_x^\infty s\gamma_\delta(|s|)ds \tag{12.8}$$

$$\omega'_\delta(x) = \int_x^\infty s\gamma_\delta(s)ds \tag{12.9}$$

Proof. For the first equation,

$$\begin{aligned}\omega_\delta(x) &= \int_0^1 dt \int_0^{\frac{x}{t}} s^2\gamma_\delta(|s|)ds \\ &= \int_0^x s^2\gamma_\delta(|s|)\int_0^1 dtds + \int_x^\infty s^2\gamma_\delta(|s|)\int_0^{\frac{x}{s}} dtds \\ &= \int_0^x s^2\gamma_\delta(|s|)ds + x\int_x^\infty s\gamma_\delta(|s|)ds\,.\end{aligned}$$

Then $\omega'_\delta(x)$ is obtained by taking derivatives of the expression. □

Naturally, we seek solutions in following the energy space defined as

$$\mathcal{S}_\delta^{\rm qnl}(\Omega) = \{u \in L^2(\Omega\cup\Omega_\delta) : E_\delta^{\rm qnl}(u) < \infty, u|_{\Omega_\delta} = 0\}. \tag{12.10}$$

First, it is a Hilbert space equipped with an inner product induced by the norm $\|\cdot\|_{\mathcal{S}_\delta^{\rm qnl}}$ defined by

$$\|u\|^2_{\mathcal{S}_\delta^{\rm qnl}} = \|u\|^2_{L^2} + |u|^2_{\mathcal{S}_\delta^{\rm qnl}}, \quad \text{where } |u|^2_{\mathcal{S}_\delta^{\rm qnl}} := 2E_\delta^{\rm qnl}(u)\,. \tag{12.11}$$

Second, Poincaré type inequality holds on the space $\mathcal{S}_\delta^{\rm qnl}(\Omega)$ that is crucial in showing the well-posedness of the variational problem.

Proposition 12.1.3 (Poincaré inequality)**.** *For* $u \in \mathcal{S}_\delta^{\rm qnl}(\Omega)$*, we have the following Poincaré type inequality,*

$$\|u\|_{L^2(\Omega)} \le C|u|_{\mathcal{S}_\delta^{\rm qnl}(\Omega)}\,. \tag{12.12}$$

Proof. From Proposition 12.2.2 which will be shown later in section 12.2, we know that the quasinonlocal energy $|u|_{\mathcal{S}_\delta^{\rm qnl}(\Omega)}$ is greater than a pure nonlocal energy defined on the entire domain Ω. Thus by the nonlocal Poincaré inequality established previously in many papers, e.g., [Du *et al.*, 2012; Mengesha and Du, 2014a], (12.12) is true. □

12.1.2 The QNL operator

We will derive the QNL operator denoted as $\mathcal{L}_\delta^{\text{qnl}}$ from energy variation. We take the first variation of $E_\delta^{\text{qnl}}(u)$ in (12.3) with any test function v that vanishes on $\partial\Omega$, and get

$$\begin{aligned}
&\langle dE_\delta^{\text{qnl}}(u), v\rangle \qquad (12.13)\\
&= \frac{1}{2} \iint\limits_{x<0 \text{ or } y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))\,(v(y)-v(x))\,dydx + \int\limits_{x>0} \omega_\delta(x)\nabla u(x)\nabla v(x)dx\\
&= - \iint\limits_{x<0 \text{ or } y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))\,v(x)dydx - \int\limits_{x>0} \nabla\cdot(\omega_\delta(x)\nabla u(x))v(x)dx,
\end{aligned}$$

where the last equality comes integration by parts and the fact that $\omega_\delta(0)=0$. The force formulism $\mathcal{L}_\delta^{\text{qnl}}u(x)$ is negative to the first variation of total energy, and it splits into three cases:

- Case I (nonlocal region): for $x<0$,

$$\mathcal{L}_\delta^{\text{qnl}}u(x) = \int_{y\in\mathbb{R}} \gamma_\delta(|y-x|)\,(u(y)-u(x))\,dy. \qquad (12.14)$$

- Case II (interfacial region): for $0<x\le\delta$,

$$\mathcal{L}_\delta^{\text{qnl}}u(x) = \int_{y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))\,dy + \nabla\cdot(\omega_\delta(x)\nabla u(x)) \qquad (12.15)$$

- Case III (local region): for $x\ge\delta$,

$$\mathcal{L}_\delta^{\text{qnl}}u(x) = \nabla\cdot(\omega_\delta(x)\nabla u(x)) = \Delta u(x)\,. \qquad (12.16)$$

12.1.3 Consistency at the interface

We will show in this part that the QNL coupling is consistent at the interface (in the language of atomistic-to-continuum coupling, it is free of ghost force), namely, for a linear displacement $u^{\text{lin}}(x) = Fx+a$, the force equals zero. For this matter, we only need to worry about the values of $\mathcal{L}_\delta^{\text{qnl}}u^{\text{lin}}$ in the interfacial region, since it is obvious zero in the pure nonlocal and local regions as given by case I and case III in (12.14) and (12.16). For a more general consideration that will also be useful in the next sections, we give the following lemma that involves the operator $\mathcal{L}_\delta^{\text{qnl}}$ acting on smooth functions in the interfacial region.

Lemma 12.1.4. *For any smooth function v,*

$$\mathcal{L}_\delta^{\text{qnl}} v(x) = a(x)\Delta v(x) + C\delta \|v\|_{C^3} \quad 0 \le x \le \delta\,, \tag{12.17}$$

where a is given by

$$a(x) = 1 - \frac{1}{2}\int_x^\delta s^2 \gamma_\delta(|s|)ds + x\int_x^\delta s\gamma_\delta(|s|)ds\,. \tag{12.18}$$

Proof. For $x \in (0,\delta)$, by the expressions of ω_δ and ω_δ' Lemma 12.1.2 in the appendix, we have

$$\begin{aligned}
\mathcal{L}_\delta^{\text{qnl}} v(x) &= \int_{y<0} \gamma_\delta(|y-x|)\,(v(y)-v(x))\,dy + \nabla\cdot(\omega_\delta(x)\nabla v(x)) \\
&= \int_{-\delta}^{-x} \gamma_\delta(s)\left(s\nabla v(x) + \frac{1}{2}s^2\Delta v(x) + O(|s|^3)\right) + \omega_\delta(x)\Delta v(x) + \omega_\delta'(x)\nabla v(x) \\
&= \frac{1}{2}\left(\int_x^\delta s^2\gamma_\delta(|s|)ds\right)\Delta v(x) + \omega_\delta(x)\Delta v(x) + O(\delta) \\
&= \left(1 - \frac{1}{2}\int_x^\delta s^2\gamma_\delta(|s|)ds + x\int_x^\delta s\gamma_\delta(|s|)ds\right)\Delta v(x) + O(\delta)\,.
\end{aligned}$$

□

***Remark* 12.1.5.** *We can further quantify $a(x)$ as follows.*

1. *One can show that $\frac{1}{2} \le a(x) \le \frac{3}{2}$ for $x \in (0,\delta)$ and $a(\delta) = 1$. Indeed,*

   $$a(x) \ge 1 - \frac{1}{2}\int_x^\delta s^2\gamma_\delta(|s|)ds \ge 1 - \frac{1}{2}\int_0^\delta s^2\gamma_\delta(|s|)ds = \frac{1}{2}\,,$$

   and

   $$a(x) \le 1 - \frac{1}{2}\int_x^\delta s^2\gamma_\delta(|s|)ds + \int_x^\delta s^2\gamma_\delta(|s|)ds \le 1 + \frac{1}{2}\int_0^\delta s^2\gamma_\delta(|s|)ds = \frac{3}{2}\,.$$

   As last, $a(\delta) = 1$ is obvious.

2. *We give an example of the coefficient $a(x)$ given by (12.18). Take $\gamma_\delta(s)$ to be*

   $$\gamma_\delta(s) = \begin{cases} \frac{2}{\delta^2|s|} & \text{if } |s| < \delta, \\ 0, & \text{if } |s| \ge \delta. \end{cases}$$

   Then $a(x)$ can be computed

   $$a(x) = 1 - \frac{1}{\delta^2}\int_x^\delta s ds + \frac{2x}{\delta^2}\int_x^\delta ds = 1 - \frac{\delta^2 - x^2}{2\delta^2} + \frac{2x(\delta - x)}{\delta^2} = \frac{1}{2} + \frac{2x}{\delta} - \frac{3x^2}{2\delta^2}\,.$$

We remark that although the equivalent strength of local diffusion $a(x)$ is not equal to a constant one for $0 \le x \le \delta$, we have

$$\int_0^\delta a(x)dx = \int_0^\delta \left(\frac{1}{2} + \frac{2x}{\delta} - \frac{3x^2}{2\delta^2}\right) dx = 1 \cdot \delta.$$

In other words, the spacial averaged diffusion for $0 \le x \le \delta$ is equal to one.

Lemma 12.1.4 shows the expansion of $\mathcal{L}_\delta^{\text{qnl}} v(x)$ in the interfacial region using with the second and higher derivatives of v. Thus it is obvious that for a linear function u^{lin}, $\mathcal{L}_\delta^{\text{qnl}} u^{\text{lin}} = 0$. In other words, the QNL coupling passes the patch-test.

Corollary 12.1.6 (Zero ghost force)**.** *For a linear function $u^{\text{lin}}(x) = Fx + a$,*

$$\mathcal{L}_\delta^{\text{qnl}} u^{\text{lin}} = 0\,.$$

Proof. This immediately follows from the Lemma 12.1.4 using (12.14), (12.15) and (12.16). □

12.2 Stability and well-posedness

We use the second first expression of total energy (12.1) to define the following bilinear form of the QNL coupling:

$$\begin{aligned} b_\delta^{\text{qnl}}(u, v) =& \frac{1}{2} \iint_{x,y\in\mathbb{R}, x<0 \text{ or } y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))\,(v(y)-v(x))\;dydx \\ &+ \int_{x>0} \nabla u(x) \cdot \nabla v(x) \omega_\delta(x)\,dx. \end{aligned} \tag{12.19}$$

We denote the local contribution as

$$b_\delta^{\text{loc}}(u, v) := \int_{x>0} \nabla u(x) \cdot \nabla v(x) \omega_\delta(x)\,dx. \tag{12.20}$$

Our next goal is to show that the bilinear form $b_\delta^{\text{qnl}}(\cdot,\cdot) : \mathcal{S}_\delta^{\text{qnl}}(\Omega) \times \mathcal{S}_\delta^{\text{qnl}}(\Omega) \to \mathbb{R}$ is bounded and coercive, thus the well-posedness of the variational problem can be followed. The boundedness of the bilinear norm is obvious since $\mathcal{S}_\delta^{\text{qnl}}(\Omega)$ is a Hilbert space and $b_\delta^{\text{qnl}}(\cdot,\cdot)$ is part of its inner product. The coercivity is from the Poincaré inequality (12.12), and the essential step is proved in Proposition 12.2.2. Firstly, in Lemma 12.2.1, we will show the lower bound of the local contribution of energy.

Lemma 12.2.1. *For $b_\delta^{\text{loc}}(u,v)$ defined in* (12.20)*, we have*

$$b_\delta^{\text{loc}}(u,u) \geq \frac{1}{2} \iint_{x>0 \ \textit{and} \ y>0} \gamma_\delta(|y-x|)\big(u(y)-u(x)\big)^2 \ dxdy. \tag{12.21}$$

Proof. The right hand side of (12.21) can be recast as

$$\begin{aligned}
&\frac{1}{2}\int_{x>0 \text{ and } y>0} \gamma_\delta(|y-x|)\big(u(y)-u(x)\big)^2 \ dxdy \\
&= \frac{1}{2}\int_{x>0} dx \int_{y>0} dy \gamma_\delta(|y-x|) \left[\int_{0<t<1} du\big(x+t(y-x)\big)\right]^2 \\
&= \frac{1}{2}\int_{x>0} dx \int_{y>0} dy \gamma_\delta(|y-x|) \left[\int_0^1 (y-x)\cdot\nabla u\big(x+t(y-x)\big)dt\right]^2 \\
&\leq \frac{1}{2}\int_{x>0} dx \int_{y>0} dy \gamma_\delta(|y-x|)(y-x)^2 \int_0^1 |\nabla u\big(x+t(y-x)\big)|^2 dt\,,
\end{aligned} \tag{12.22}$$

where the last expression is exactly $2E_\delta^{\text{loc}}(u) = b_\delta^{\text{loc}}(u,u)$ as shown in Proposition 12.1.1. □

Lemma 12.2.1 immediately leads to the stability property compared to the fully nonlocal bilinear operator.

Proposition 12.2.2. $b_\delta^{\text{qnl}}(u,v)$ *defined in* (12.20) *is coercive:*

$$b_{\text{qnl}}(u,u) \geq \frac{1}{2} \iint_{x,y\in\mathbb{R}} \gamma_\delta(|y-x|)\left(u(y)-u(x)\right)\left(v(y)-v(x)\right) \ dydx. \tag{12.23}$$

Proof. Recall the definition of $b_\delta^{\text{qnl}}(u,u)$ and use the conclusion of Lemma 12.2.1, we immediately get

$$\begin{aligned}
b_\delta^{\text{qnl}}(u,u) =& \frac{1}{2}\int_{x,y\in\mathbb{R}, x<0 \text{ or } y<0} \gamma_\delta(|y-x|)\left(u(y)-u(x)\right)\left(v(y)-v(x)\right) \ dxdy + b_\delta^{\text{loc}}(u,u) \\
\geq& \frac{1}{2}\int_{x,y\in\mathbb{R}, x<0 \text{ or } y<0} \gamma_\delta(|y-x|)\left(u(y)-u(x)\right)\left(v(y)-v(x)\right) \ dxdy \\
&+ \frac{1}{2}\int_{x>0 \text{ and } y>0} \gamma_\delta(|y-x|)\big(u(y)-u(x)\big)^2 \ dxdy
\end{aligned}$$

□

Now, with $b_\delta^{\text{qnl}}(\cdot,\cdot)$ being bounded and coercive, we are ready to give the theorem for the well-posedness of the QNL problem.

Theorem 12.2.3. *The QNL coupling is well-posed:*

$$\begin{cases} -\mathcal{L}_\delta^{\rm qnl} u_\delta^{\rm qnl} = f \quad \text{on } \Omega \\ \quad u_\delta^{\rm qnl}(x) = u_0(x) \quad \text{on } \Omega_\delta\,, \end{cases} \tag{12.24}$$

where $\mathcal{L}_\delta^{\rm qnl}$ *is defined in subsection 12.1.2.*

Proof. We could show the existence and uniqueness of the weak solution of (12.24) in the Hilbert space $\mathcal{S}_\delta^{\rm qnl}(\Omega)$ since the Poincaré inequality (12.12) holds. □

12.3 First order uniform convergence as $\delta \to 0$

We consider in this section the convergence of the steady-state problems as $\delta \to 0$. Namely, we will establish convergence of the solutions to the QNL problem (12.24) to the local differential equation

$$\begin{cases} -u_0''(x) = f(x) \quad x \in \Omega \\ u_0(-1) = u_0(1) = 0\,, \end{cases} \tag{12.25}$$

as $\delta \to 0$. Now denote the error $e_\delta(x) = u_\delta(x) - u_0(x)$. Although $u_0(x)$ is only defined on Ω, we could extend the definition of $u_0(x)$ to $\Omega \cup \Omega_\delta$. In this section we assume that u_0 can be smoothly extend to Ω_δ to avoid discussions of boundary effect. We suppose Dirichlet conditions are imposed such that $e_\delta(x) = 0$ for $x \in \Omega_\delta$.

Truncation error $T_\delta(x) = \mathcal{L}_\delta^{\rm qnl} u_0(x) - u_0''(x)$. According to the calculations in section 12.1.3, we know that $T_\delta(x) = T_\delta^1(x) + T_\delta^2(x)$, where $T_\delta^1(x) = O(\delta^2)\chi_{(-1,0)}(x)$ and $T_\delta^2(x) = O(1)\chi_{(0,\delta)}(x)$. Notice that from Lemma 12.1.4, for $x \in (0,\delta)$,

$$\begin{aligned} T_\delta^2(x) = \mathcal{L}_\delta^{\rm qnl} u_0(x) - u_0''(x) &= a(x)u_0''(x) - u_0''(x) + O(\delta) \\ &= \Big(2x\int_x^\delta s\gamma_\delta(s)ds - \int_x^\delta s^2\gamma_\delta(s)ds\Big)u_0''(x) + O(\delta) \\ &\le 3C^*\int_x^\delta s^2\gamma_\delta(s)ds + O(\delta)\,, \end{aligned}$$

where $C^* = \|u_0\|_{C^2}$. Now that $-\mathcal{L}_\delta^{\rm qnl} e_\delta(x) = -\mathcal{L}_\delta^{\rm qnl} u_\delta^{\rm qnl}(x) + \mathcal{L}_\delta^{\rm qnl} u_0(x) = T_\delta(x)$, we have $e_\delta(x) = (-\mathcal{L}_\delta^{\rm qnl})^{-1}T_\delta^1(x) + (-\mathcal{L}_\delta^{\rm qnl})^{-1}T_\delta^2(x) = e_\delta^1(x) + e_\delta^2(x)$, where $e_\delta^1(x)$ and $e_\delta^2(x)$ are defined as $(-\mathcal{L}_\delta^{\text{qnl}})^{-1}T_\delta^1(x)$ and $(-\mathcal{L}_\delta^{\text{qnl}})^{-1}T_\delta^2(x)$ respectively. We are going to show next that $|e_\delta^1(x)| = O(\delta^2)$ and $|e_\delta^2(x)| = O(\delta)$. Thus the total error is of order $O(\delta)$. The main ingredients are maximum principle and barrier functions.

Lemma 12.3.1 (Maximum principle)**.** *The operator $\mathcal{L}_\delta^{\text{qnl}}$ satisfies the maximum principle, namely, if $u \in C^2([-1,0]) \cap C^2([0,1])$, then $-\mathcal{L}_\delta^{\text{qnl}}u(x) \le 0$ in $(-1,0)\cup(0,1)$ implies that,*

$$\max_{x\in\Omega\cup\Omega_\delta} u(x) \le \max_{x\in\Omega_\delta} u(x).$$

Proof. First, from $-\mathcal{L}_\delta^{\text{qnl}}u(x) \le 0$ in $(0,1)$ we deduce from classical maximum principle that

$$\max_{x\in(0,1)} u(x) \le \max_{x\in\{0^+\}\cup\{1\}} u(x)\,, \tag{12.26}$$

where $u(0^+) = \lim_{x\to 0, x>0} u(x)$. Notice that in this case, for $x \in (0,\delta)$ the integral part in the definition of $\mathcal{L}_\delta^{\text{qnl}}$ would not cause extra difficulty in proving the above inequality. Second, from $-\mathcal{L}_\delta^{\text{qnl}}u(x) \le 0$ in $(-1,0)$ we could show

$$\max_{x\in(-1-\delta,\delta)} u(x) \le \max_{x\in(-1-\delta,-1]\cup[0,\delta)} u(x)\,. \tag{12.27}$$

The argument is the following. Assume the opposition is true, namely,

$$\max_{x\in(-1-\delta,\delta)} u(x) > \max_{x\in(-1-\delta,-1]\cup[0,\delta)} u(x),$$

then we could find $x^* \in (0,1)$ such that $u(x^*) = \max_{x\in(0,1)} u(x)$ and

$$-\mathcal{L}_\delta^{\text{qnl}}u(x^*) = -\int_{-\delta}^{\delta} \gamma_\delta(|s|)(u(x^*+s) - u(x^*))ds > 0,$$

which gives us a contradiction. So u has to satisfy (12.27).

Now combine the result of (12.26) and (12.27), we only need to show $u(0^+) \le \max_{x\in\Omega_\delta} u(x)$. Assume the opposite, namely $u(0^+) > u(x)$ for any $x \in [-1-\delta, 0)\cup(0,1]$. Then the discussion splits into two cases where u has a jump at 0 or not. First, we consider the case that u has a jump at 0, namely $u(0^+) > u(0^-)$. Then for sufficiently small $x > 0$, we have $\int_{y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))\,dy < 0$. Considering also that $u'(0^+) \le 0$ and $\omega_\delta(0^+) = 0$, we see that for small enough $x > 0$,

$$-\mathcal{L}_\delta^{\text{qnl}}u(x) = -\int_{y<0} \gamma_\delta(|y-x|)\,(u(y)-u(x))\,dy - \omega_\delta(x)u''(x) - \omega_\delta'(x)u'(x) > 0\,,$$

which gives us a contradiction.

Now we consider the case that $u(0^+) = u(0^-)$. We first show that if $-\mathcal{L}_\delta^{\text{qnl}} u(x) \leq -\epsilon$ for some small $\epsilon > 0$, the result is true. Since now $u \in C(\overline{\Omega \cup \Omega_\delta})$, we could choose $\eta > 0$ small enough such that for any $x \in (-\eta, 0)$, we could find a small $\delta_x > 0$ such that $u(x) \geq u(y)$ for all $y \in (-1-\delta, 1)\backslash(x-\delta_x, x+\delta_x)$. Notice that $\delta_x \to 0$ with $x \to 0$. Now by the assumption that $u \in C^2([-1, 0])$, we could find an extension function of u on $C^2([-1, \eta])$ denoted by $\tilde{u}$ such that $\tilde{u}(x) = u(x)$ for $x \in [-1, 0]$ and $\tilde{u}(x) \geq u(x)$ for $x \in [0, \eta]$. Then for all $x \in (-\eta, 0)$, we have

$$\begin{aligned} -\mathcal{L}_\delta^{\text{qnl}} u(x) &= -\int_{-\delta}^{\delta} \gamma_\delta(|s|)(u(x+s) - u(x))ds \\ &> -\int_{-\delta_x}^{\delta_x} \gamma_\delta(|s|)(u(x+s) - u(x))ds \\ &\geq -\int_{-\delta_x}^{\delta_x} \gamma_\delta(|s|)(\tilde{u}(x+s) - \tilde{u}(x))ds \end{aligned}$$

Now since $\tilde{u} \in C^2([-1, \eta])$, the last integral in the above line will converge to zero by letting $x \to 0$. Thus we will eventually get a contradiction with the assumption that $-\mathcal{L}_\delta^{\text{qnl}} u(x) \leq -\epsilon$. Now for the general case that $-\mathcal{L}_\delta^{\text{qnl}} u(x) \leq 0$, we use the same argument to $u(x) + \epsilon x^2$ and then let $\epsilon \to 0$ we conclude that $u(0^+) \leq \max_{x \in \Omega_\delta} u(x)$ and the lemma is proved. □

Theorem 12.3.2. *Suppose u_δ^{qnl} and u_0 are strong solutions to* (12.24) *and* (12.25) *respectively. Assume that $u_0 \in C^4(\overline{\Omega \cup \Omega_\delta})$, then*

$$\|u_\delta^{\text{qnl}}(x) - u_0(x)\|_{L^\infty(\Omega)} = O(\delta)\,.$$

Proof. We will construct barrier functions and then estimate $e_\delta^1(x)$ and $e_\delta^2(x)$ by utilizing the maximum principle. The first barrier function is a simple quadratic function. Take $\Phi_1(x) = -cx^2 + 4c$, then from the calculations in section 12.1.3 we know that $-\mathcal{L}_\delta^{\text{qnl}}(\delta^2 \Phi_1(x)) \geq c\delta^2$. Choose c large enough such that $c\delta^2 \geq T_\delta^1(x)$, then from the maximum principle we conclude that

$$\max_{x \in \Omega \cup \Omega_\delta} (e_\delta^1(x) - \delta^2 \Phi_1(x)) \leq \max_{x \in \Omega_\delta} (e_\delta^1(x) - \delta^2 \Phi_1(x)) \leq 0\,,$$

so we have $e_\delta^1(x) \leq \delta^2 \Phi_1(x) \leq 4c\delta^2$. Using the same arguments to $-e_\delta^1(x)$ we also have $-e_\delta^1(x) \leq 4c\delta^2$. Thus $|e_\delta^1(x)| = O(\delta^2)$.

The second barrier function $\Phi_2(x)$ is more carefully designed in order to get the estimate of $e_\delta^2(x)$. The key to to define $\Phi_2(x)$ in a way that it is C^2 and linear outside the interfacial region $(0, \delta)$. We define the barrier function $\Phi_2(x)$ to be

$$\Phi_2(x) = \begin{cases} \delta x + \delta + \delta^2 \quad x \in (-1-\delta, -\delta) \\ \dfrac{1}{8\delta^2}x^4 - \dfrac{3}{4}x^2 + \delta + \dfrac{5}{8}\delta^2 \quad x \in [-\delta, \delta] \\ -\delta x + \delta + \delta^2 \quad x \in [\delta, 1)\,. \end{cases} \tag{12.28}$$

One could check that $\Phi_2 \in C^2(\Omega) \cup C(\overline{\Omega \cup \Omega_\delta})$ and $-\mathcal{L}_\delta^{\mathrm{qnl}}(\Phi_2(x)) \geq 0$ for $x \in (-1, 1)$. In particular, for $x \in (0, \delta)$, after taking Taylor-expansion, we can write

$$\begin{aligned} -\mathcal{L}_\delta^{\mathrm{qnl}}(\Phi_2(x)) =& - a(x)\Phi_2''(x) - \int_{-\delta}^{-x} \gamma_\delta(s)\left(\frac{1}{6}s^3\Phi_2'''(x) + \frac{1}{24}s^4\Phi_2''''(x) + O(|s|^5)ds\right) \\ =& -\Big(1 - \frac{1}{2}\int_x^\delta s^2\gamma_\delta(s)ds + x\int_x^\delta s\gamma_\delta(s)ds\Big)\Phi_2''(x) \\ & -\frac{1}{6}\Big(\int_{-\delta}^{-x} s^3\gamma_\delta(s)ds\Big)\Phi_2'''(x) - \frac{1}{24}\Big(\int_{-\delta}^{-x} s^4\gamma_\delta(s)ds\Big)\Phi_2''''(x) \\ \geq& -\Big(\frac{1}{2}\int_{-\delta}^{x} s^2\gamma_\delta(s)ds + \int_x^\delta s^2\gamma_\delta(s)ds\Big)\Big(\frac{3}{2}\frac{x^2}{\delta^2} - \frac{3}{2}\Big) \\ & +\frac{x}{6}\Big(\int_x^\delta s^2\gamma_\delta(s)ds\Big)\frac{3x}{\delta^2} - \frac{\delta^2}{24}\Big(\int_x^\delta s^2\gamma_\delta(s)ds\Big)\frac{3}{\delta^2} \\ \geq& \frac{1}{2}\int_x^\delta s^2\gamma_\delta(s)ds\Big(\frac{11}{4} - 2\frac{x^2}{\delta^2}\Big) \geq \frac{3}{8}\int_x^\delta s^2\gamma_\delta(s)ds. \end{aligned}$$

Then we could take a $\tilde{c} > 0$ large enough such that $-\mathcal{L}_\delta^{\mathrm{qnl}}(\tilde{c}\Phi_2(x)) \geq T_\delta^2(x)$, then from the maximum principle we conclude that

$$\max_{x \in \Omega \cup \Omega_\delta}(e_\delta^2(x) - \tilde{c}\Phi_2(x)) \leq \max_{x \in \Omega_\delta}(e_\delta^2(x) - \tilde{c}\Phi_2(x)) \leq 0\,.$$

So we have $e_\delta^2(x) \leq \tilde{c}\Phi_2(x) \leq \tilde{c}(\delta + \frac{5}{8}\delta^2)$. Using the same arguments to $-e_\delta^2(x)$ we also have $-e_\delta^2(x) \leq \tilde{c}(\delta + \frac{5}{8}\delta^2)$. Thus $|e_\delta^2(x)| = O(\delta)$. □

12.4 Conclusion

In this section, we developed a quasinonlocal coupling method to study the local-to-nonlocal diffusion problem in one dimensional space in inspired by the quasicontinuum method in the atomistic modeling techiques. This new coupling framework removes interfacial inconsistency and maintains all physical properties at local continuum PDE levels, whereas little of existing coupling methods for local-to-nonlocal problems satisfy all of these properties. We proved the well-posedness of the coupling problem by a quasinonlocal version of the Poincaré inequality and showed the first order convergence of the quaisnonlocal model to the local model as $\delta \to 0$.

To conclude, we note that asymptotic compatible finite difference and finite element schemes for the coupled problem can be studied follow the work in chapter 2 and chapter 4. In the future, we may also consider multi-dimensional generalizations of the quasinonlocal coupling approach.

Part IV

Bibliography

Bibliography

[Aksoylu and Mengesha, 2010] Burak Aksoylu and Tadele Mengesha. Results on nonlocal boundary value problems. *Numerical Functional Analysis and Optimization*, 31(12):1301–1317, 2010.

[Allen *et al.*, 2016] Mark Allen, Luis Caffarelli, and Alexis Vasseur. A parabolic problem with a fractional time derivative. *Archive for Rational Mechanics and Analysis*, 221(2):603–630, 2016.

[Andreu *et al.*, 2008] F Andreu, JM Mazón, JD Rossi, and J Toledo. A nonlocal p-Laplacian evolution equation with Neumann boundary conditions. *Journal de Mathématiques Pures et Appliquées*, 90(2):201–227, 2008.

[Andreu *et al.*, 2010] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo. *Nonlocal Diffusion Problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2010.

[Arnold *et al.*, 2002] Douglas N Arnold, Franco Brezzi, Bernardo Cockburn, and L Donatella Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM journal on numerical analysis*, 39(5):1749–1779, 2002.

[Askari *et al.*, 2008] E Askari, F Bobaru, R B Lehoucq, M L Parks, S A Silling, and O Weckner. Peridynamics for multiscale materials modeling. *Journal of Physics: Conference Series*, 125:012078, 2008.

[Aubin and Cellina, 1984] Jean Pierre Aubin and A. Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*. Springer-Verlag New York, Inc., 1984.

[Bartholdi *et al.*, 2012] Laurent Bartholdi, Thomas Schick, Nat Smale, and Steve Smale. Hodge theory on metric spaces. *Foundations of Computational Mathematics*, 12(1):1–48, 2012.

[Belgacem and Brenner, 2001] Faker Ben Belgacem and Susanne C Brenner. Some nonstandard finite element estimates with applications to 3d Poisson and Signorini problems. *Electronic Transactions on Numerical Analysis*, 12:134–148, 2001.

[Belytschko *et al.*, 2009] Ted Belytschko, Robert Gracie, and Giulio Ventura. A review of extended/generalized finite element methods for material modeling. *Modelling and Simulation in Materials Science and Engineering*, 17(4):043001, 2009.

[Bessa *et al.*, 2014] MA Bessa, JT Foster, T Belytschko, and Wing Kam Liu. A meshfree unification: reproducing kernel peridynamics. *Computational Mechanics*, 53(6):1251–1264, 2014.

[Bobaru and Duangpanya, 2010] F. Bobaru and M. Duangpanya. The peridynamic formulation for transient heat conduction. *International Journal of Heat and Mass Transfer*, 53(19):4047–4059, 2010.

[Bobaru *et al.*, 2009] F. Bobaru, M. Yang, L.F. Alves, S.A. Silling, E. Askari, and J. Xu. Convergence, adaptive refinement, and scaling in 1d peridynamics. *International Journal for Numerical Methods in Engineering*, 77(6):852–877, 2009.

[Bond *et al.*, 2013] S.J. Bond, R. Lehoucq, and S. Rowe. A Galerkin radial basis function method for nonlocal diffusion. *Sandia National Laboratories, Technical report SAND 2013-10673P*, 2013.

[Bourgain *et al.*, 2001] Jean Bourgain, Haim Brezis, and Petru Mironescu. *Another look at Sobolev spaces*, pages 439–455. IOS Press, Amsterdam, 2001.

[Buhmann, 2000] Martin D Buhmann. Radial basis functions. *Acta Numerica 2000*, 9:1–38, 2000.

[Caffarelli and Silvestre, 2007] L. Caffarelli and L. Silvestre. An extension problem related to the fractional laplacian. *Communications in partial differential equations*, 32:1245–1260, 2007.

[Chen and Gunzburger, 2011] X. Chen and M. Gunzburger. Continuous and discontinuous finite element methods for a peridynamics model of mechanics. *Computer Methods in Applied Mechanics and Engineering*, 200(9-12):1237–1250, 2011.

[Chen *et al.*, 1993] G. Chen, Q. Du, and E. Tadmor. Spectral viscosity approximations to multidimensional scalar conservation laws. *Mathematics of Computation*, 61:629–643, 1993.

[Ciarlet, 1988] Philippe G Ciarlet. *Three-dimensional elasticity*, volume 1. Elsevier, 1988.

[Ciarlet, 2013] Patrick Ciarlet. Analysis of the Scott–Zhang interpolation in the fractional order Sobolev spaces. *Journal of Numerical Mathematics*, 21(3):173–180, 2013.

[Cifani *et al.*, 2011] Simone Cifani, Espen R Jakobsen, and Kenneth H Karlsen. The discontinuous Galerkin method for fractional degenerate convection-diffusion equations. *BIT Numerical Mathematics*, 51(4):809–844, 2011.

[Cockburn and Shu, 1998] Bernardo Cockburn and Chi-Wang Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM Journal on Numerical Analysis*, 35(6):2440–2463, 1998.

[Cockburn *et al.*, 2003] Bernardo Cockburn, Mitchell Luskin, Chi-Wang Shu, and Endre Süli. Enhanced accuracy by post-processing for finite element methods for hyperbolic equations. *Mathematics of Computation*, 72:577–606, 2003.

[Cockburn *et al.*, 2011] Bernardo Cockburn, George E Karniadakis, and Chi-Wang Shu. *Discontinuous Galerkin methods: theory, computation and applications.* Springer Publishing Company, Incorporated, 2011.

[Coifman and Lafon, 2006] Ronald R Coifman and Stéphane Lafon. Diffusion maps. *Applied and computational harmonic analysis*, 21(1):5–30, 2006.

[Davies, 1999] EB Davies. A review of hardy inequalities. In *The Maz'ya anniversary collection*, pages 55–67. Springer, 1999.

[Deimling, 1992] Klaus Deimling. *Multivalued differential equations*, volume 1. Walter de Gruyter, 1992.

[Deng and Hesthaven, 2013] WH Deng and Jan S Hesthaven. Local discontinuous Galerkin methods for fractional diffusion equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 47(06):1845–1864, 2013.

[Diethelm, 2015] K. Diethelm. The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type. *Springer*, 2015.

[Ding, 1996] Zhonghai Ding. A proof of the trace theorem of sobolev spaces on lipschitz domains. *Proceedings of the American Mathematical Society*, 124(2):591–600, 1996.

[Du and Tian, 2014] Qiang Du and Xiaochuan Tian. Asymptotically compatible schemes for peridynamics based on numerical quadratures. In *ASME 2014 International Mechanical Engineering Congress and Exposition*, pages V001T01A058–V001T01A058. American Society of Mechanical Engineers, 2014.

[Du and Yang, 2016] Qiang Du and Jiang. Yang. Asymptotic compatible fourier spectral approximations of nonlocal allen-cahn equations. *SIAM J. Numerical Analysis*, 54:1899–1919, 2016.

[Du and Zhou, 2011] Qiang Du and Kun Zhou. Mathematical analysis for the peridynamic nonlocal continuum theory. *Mathematical Modelling and Numerical Analysis*, 45:217–234, 2011.

[Du *et al.*, 2012] Q. Du, M. Gunzburger, RB Lehoucq, and K. Zhou. Analysis and approximation of nonlocal diffusion problems with volume constraints. *SIAM Review*, 56:676–696, 2012.

[Du *et al.*, 2013a] Q. Du, M. Gunzburger, RB Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. *Math. Mod. Meth. Appl. Sci.*, 23:493–540, 2013.

[Du *et al.*, 2013b] Q. Du, L. Ju, L. Tian, and K. Zhou. A posteriori error analysis of finite element method for linear nonlocal diffusion and peridynamic models. *Mathematics of Computation*, 82:1889–1922, 2013.

[Du *et al.*, 2014] Q. Du, Z. Huang, and R. Lehoucq. Nonlocal convection-diffusion volume-constrained problems and jump processes. *Disc. Cont. Dyn. Sys. B*, 19:373–389, 2014.

[Du *et al.*, 2015] Qiang Du, Richard Lehoucq, and Alexandre Tartakovsky. Integral approximations to classical diffusion and smoothed particle hydrodynamics. *Computer Methods in Applied Mechanics and Engineering*, 286:216–229, 2015.

[Du *et al.*, 2016] Qiang Du, Yunzhe Tao, Xiaochuan Tian, and Jiang Yang. Robust a posteriori stress analysis for approximations of nonlocal models via nonlocal gradients. *Compu. Meth. in Applied Mech Engineering*, 310:605–627, 2016.

[Du *et al.*, Preprint 2017a] Qiang Du, Xingjie Helen Li, Jianfeng Lu, and Xiaochuan Tian. A quasinonlocal coupling method for nonlocal and local diffusion models. Preprint, 2017.

[Du *et al.*, Preprint 2017b] Qiang Du, Yunzhe Tao, and Xiaochuan Tian. A peridynamic model of fracture mechanics with bond-breaking. Preprint, 2017.

[Duvaut and Lions, 1976] Georges Duvaut and Jacques Louis Lions. *Inequalities in mechanics and physics*. Springer-Verlag Berlin, 1976.

[D'Elia and Gunzburger, 2013] Marta D'Elia and Max Gunzburger. The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator. *Computers & Mathematics with Applications*, 66(7):1245–1260, 2013.

[E *et al.*, 2006] W. E, J. Lu, and J. Z. Yang. Uniform accuracy of the quasicontinuum method. *Phys. Rev. B*, 74(21):214115, 2006.

[Emmrich and Puhst, 2013] Etienne Emmrich and Dimitri Puhst. Well-posedness of the peridynamic model with lipschitz continuous pairwise force function. *Commun. Math. Sci*, 11(4):1039–1049, 2013.

[Escobar, 1988] J. Escobar. Sharp constant in a Sobolev trace inequality. *Indiana University Mathematics Journal*, 37(3):687–698, 1988.

[Felsinger *et al.*, 2013] Matthieu Felsinger, Moritz Kassmann, and Paul Voigt. The dirichlet problem for nonlocal operators. *Mathematische Zeitschrift*, 279:1–31, 2013.

[Gagliardo, 1957] Emilio Gagliardo. Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili. *Rendiconti del seminario matematico della università di Padova*, 27:284–305, 1957.

[Gilboa and Osher, 2008] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Modeling Simulation*, 7:1005–1028, 2008.

[Guermond and Kanschat, 2010] Jean-Luc Guermond and Guido Kanschat. Asymptotic analysis of upwind discontinuous Galerkin approximation of the radiative transport equation in the diffusive limit. *SIAM Journal on Numerical Analysis*, 48:53–78, 2010.

[Ha and Bobaru, 2010] Youn Doh Ha and Florin Bobaru. Studies of dynamic crack propagation and crack branching with peridynamics. *International Journal of Fracture*, 162(1-2):229–244, 2010.

[Hale, 1971] Jack K Hale. Functional differential equations. In *Analytic theory of differential equations*, pages 9–22. Springer, 1971.

[Heinonen, 2017] J. Heinonen. Nonsmooth calculus. *Bulletin of the American Mathematical Society*, 44(2):163–232, 2017.

[Hu *et al.*, 2012] Wenke Hu, Youn Doh Ha, and Florin Bobaru. Peridynamic model for dynamic fracture in unidirectional fiber-reinforced composites. *Computer Methods in Applied Mechanics and Engineering*, 217:247–261, 2012.

[Jin, 1999] Shi Jin. Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM Journal on Scientific Computing*, 21:441–454, 1999.

[Johnsen *et al.*, 2015] J. Johnsen, S. Hansen, and W. Sickel. Anisotropic Lizorkin-Triebel spaces with mixed norms–traces on smooth boundaries. *Mathematische Nachrichten*, 288:1327–1359, 2015.

[Kellogg, 1931] O. Kellogg. On the derivatives of harmonic functions on the boundary. *Transactions of the American Mathematical Society*, 33(2):486–510, 1931.

[Klafler and Sokolov, 2005] J. Klafler and I.M. Sokolov. Anomalous diffusion spreads its wings. *Physics world*, 18(8):29, 2005.

[Leoni, 2009] Giovanni Leoni. *A first course in Sobolev spaces*, volume 105. American Mathematical Society Providence, RI, 2009.

[Li and Lu, 2016] Xingjie Helen Li and Jianfeng Lu. Quasinonlocal coupling of nonlocal diffusions. 2016. preprint arXiv:1607.03940.

[Li and Luskin, 2011] Xingjie Helen Li and Mitchell Luskin. A generalized quasinonlocal atomistic-to-continuum coupling method with finite-range interaction. *IMA Journal of Numerical Analysis*, 32:373–393, 2011.

[Lin, 2016] F.-H. Lin. *Lectures on Elliptic Free Boundary Problems*, volume 4 of *Lectures on the analysis of nonlinear partial differential equations*. Higher Education Press and International Press, 2016.

[Lipton, 2015] Robert Lipton. Cohesive dynamics and brittle fracture. *Journal of Elasticity*, pages 1–49, 2015.

[Liu *et al.*, 1996] Wing Kam Liu, Y Chen, S Jun, JS Chen, T Belytschko, C Pan, RA Uras, and CT Chang. Overview and applications of the reproducing kernel particle methods. *Archives of Computational Methods in Engineering*, 3(1):3–80, 1996.

[Massaccesi and Valdinoci, 2016] Annalisa Massaccesi and Enrico Valdinoci. Is a nonlocal diffusion strategy convenient for biological populations in competition? *Journal of mathematical biology*, pages 1–35, 2016.

[Meerschaert *et al.*, 1999] M.M. Meerschaert, D.A. Benson, and B. Bäumer. Multidimensional advection and fractional dispersion. *Physical Review E*, 59(5):5026, 1999.

[Mengesha and Du, 2013] T. Mengesha and Q. Du. Analysis of a scalar nonlocal peridynamic model with a sign changing kernel. *Disc. Cont. Dyn. Sys, B*, 18(5):1415–1437, 2013.

[Mengesha and Du, 2014a] T. Mengesha and Q. Du. The bond-based peridynamic system with dirichlet type volume constraint. *Proc. of Royal Soc. Edinburgh, A*, 144:161–186, 2014.

[Mengesha and Du, 2014b] T. Mengesha and Q. Du. Nonlocal constrained value problems for a linear peridynamic Navier equation. *Journal of Elasticity*, 116:27–51, 2014.

[Mengesha and Du, 2014c] Tadele Mengesha and Qiang Du. Nonlocal constrained value problems for a linear peridynamic navier equation. *Journal of Elasticity*, 116:27–51, 2014.

[Mengesha and Du, 2015] Tadele Mengesha and Qiang Du. On the variational limit of a class of nonlocal functionals related to peridynamics. *Nonlinearity*, 28(11):3999, 2015.

[Mengesha and Du, 2016] Tadele Mengesha and Qiang Du. Characterization of function spaces of vector fields and an application in nonlinear peridynamics. *Nonlinear Analysis: Theory, Methods & Applications*, 140:82–111, 2016.

[Metzler and Klafter, 2000] R. Metzler and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339(1):1–77, 2000.

[Metzler and Klafter, 2004] Ralf Metzler and Joseph Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A: Mathematical and General*, 37(31):R161, 2004.

[Ming and Yang, 2009] Pingbing Ming and Jerry Zhijian Yang. Analysis of a one-dimensional nonlocal quasicontinuum method. *Multiscale Modeling and Simulation*, 7:1838–1875, 2009.

[Mustapha and McLean, 2013] Kassem Mustapha and William McLean. Superconvergence of a discontinuous Galerkin method for fractional diffusion and wave equations. *SIAM Journal on Numerical Analysis*, 51(1):491–515, 2013.

[Nakamura *et al.*, 2016] S. Nakamura, T. Noi, and Sawano Y. Generalized Morrey spaces and trace operator. *Science China Mathematics*, 59(2):281–336, 2016.

[Neuman and Tartakovsky, 2009] Shlomo P Neuman and Daniel M Tartakovsky. Perspective on theories of non-fickian transport in heterogeneous media. *Advances in Water Resources*, 32(5):670–680, 2009.

[Oberman, 2008] Adam M Oberman. Wide stencil finite difference schemes for the elliptic Monge-Ampère equation and functions of the eigenvalues of the Hessian. *Discrete Contin. Dyn. Syst. Ser. B*, 10(1):221–238, 2008.

[O'Grady and Foster, 2014a] James O'Grady and John Foster. Peridynamic beams: A non-ordinary, state-based model. *International Journal of Solids and Structures*, 51(18):3177–3183, 2014.

[O'Grady and Foster, 2014b] James O'Grady and John Foster. Peridynamic plates and flat shells: A non-ordinary, state-based model. *International Journal of Solids and Structures*, 51(25):4572–4579, 2014.

[Ortner and Zhang, 2014] Christoph Ortner and Lei Zhang. Energy-based atomistic-to-continuum coupling without ghost forces. *Computer Methods in Applied Mechanics and Engineering*, 279:29–45, 2014.

[Ponce, 2004] Augusto C Ponce. An estimate in the spirit of poincaré's inequality. *J. Eur. Math. Soc*, 6(1):1–15, 2004.

[Radu *et al.*, 2016] Petronela Radu, Daniel Toundykov, and Jeremy Trageser. A nonlocal biharmonic operator and its connection with the classical analogue. *preprint*, 2016.

[Roos *et al.*, 1996] Hans Görg Roos, Martin Stynes, and Lutz Tobiska. *Numerical Methods for Singularly Perturbed Differential Equations.: Convection-Diffusion and Flow Problems.*, volume 24. Springer, 1996.

[Ros-Oton, 2015] Xavier Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *arXiv preprint arXiv:1504.04099*, 2015.

[Shapeev, 2012] Alexander V. Shapeev. Consistent energy-based atomistic/continuum coupling for two-body potentials in one and two dimensions. *Multiscale Modeling and Simulation*, 9(3):905–932, 2012.

[Shimokawa *et al.*, 2004] T. Shimokawa, J. J. Mortensen, J. Schiotz, and K. W. Jacobsen. Matching conditions in the quasicontinuum method: Removal of the error introduced at the interface between the coarse-grained and fully atomistic region. *Phys. Rev. B*, 69(21):214104, 2004.

[Silling *et al.*, 2007] Stewart A Silling, M Epton, O Weckner, J Xu, and E Askari. Peridynamic states and constitutive modeling. *Journal of Elasticity*, 88(2):151–184, 2007.

[Silling *et al.*, 2010] S. Silling, O. Weckner, E. Askari, and F. Bobaru. Crack nucleation in a peridynamic solid. *International Journal of Fracture*, 162(1):219–227, 2010.

[Silling, 2000] S.A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48(1):175–209, 2000.

[Stein, 1970] Elias M Stein. *Singular integrals and differentiability properties of functions*, volume 2. Princeton university press, 1970.

[Strouboulis *et al.*, 2000] Theofanis Strouboulis, Ivo Babuška, and Kevin Copps. The design and analysis of the generalized finite element method. *Computer methods in applied mechanics and engineering*, 181(1):43–69, 2000.

[Tadjeran *et al.*, 2006] Charles Tadjeran, Mark M Meerschaert, and Hans-Peter Scheffler. A second-order accurate numerical approximation for the fractional diffusion equation. *Journal of Computational Physics*, 213:205–213, 2006.

[Tadmor, 2015] Eitan Tadmor. Mathematical aspects of self-organized dynamics consensus, emergence of leaders, and social hydrodynamics. *SIAM News*, 48, 2015.

[Tao *et al.*, Submitted 2016] Yunzhe Tao, Xiaochuan Tian, and Qiang Du. Nonlocal diffusion and peridynamic models with neumann type constraints and their numerical approximations. Submitted, 2016.

[Tian and Du, 2013] Xiaochuan Tian and Qiang Du. Analysis and comparison of different approximations to nonlocal diffusion and linear peridynamic equations. *SIAM J. Numerical Analysis*, 51:3458–3482, 2013.

[Tian and Du, 2014a] X. Tian and Q. Du. Asymptotically compatible schemes and applications to robust discretization of nonlocal models. *SIAM J. Numerical Analysis*, 52:1641–1665, 2014.

[Tian and Du, 2014b] Xiaochuan Tian and Qiang Du. Asymptotically compatible schemes and applications to robust discretization of nonlocal models. *SIAM J. Numerical Analysis*, 52:1641–1665, 2014.

[Tian and Du, 2015a] X. Tian and Q. Du. Nonconforming discontinuous galerkin methods for nonlocal variational problems. *SIAM J. Numerical Analysis*, 53:762781, 2015.

[Tian and Du, 2015b] Xiaochuan Tian and Qiang Du. A class of high order nonlocal operators. *Preprint*, -(-):–, 2015.

[Tian and Du, 2016] Xiaochuan Tian and Qiang Du. Trace theorems for some nonlocal function spaces with heterogeneous localization. *SIAM J. Mathematical Analysis*, page submitted, 2016.

[Tian *et al.*, 2016] Xiaochuan Tian, Qiang Du, and Max Gunzburger. Asymptotically compatible schemes for the approximation of fractional laplacian and related nonlocal diffusion problems on bounded domains. *Advances in Computational Mathematics*, pages 1–18, 2016.

[Trillos and Slepčev, 2016] Nicolás García Trillos and Dejan Slepčev. Continuum limit of total variation on point clouds. *Archive for Rational Mechanics and Analysis*, 220(1):193–241, 2016.

[Webb, 2012] Marcus Webb. *Analysis and Approximation of a Fractional Differential Equation*. PhD thesis, Master's Thesis, Oxford University, Oxford, 2012.

[Wu, 2012] Jianhong Wu. *Theory and applications of partial functional differential equations*, volume 119. Springer Science & Business Media, 2012.

[Xu and Hesthaven, 2014] Qinwu Xu and Jan S Hesthaven. Discontinuous Galerkin method for fractional convection-diffusion equations. *SIAM Journal on Numerical Analysis*, 52(1):405–423, 2014.

[Yang *et al.*, 2011] Qianqian Yang, Ian Turner, Fawang Liu, and Milos Ilic. Novel numerical methods for solving the time-space fractional diffusion equation in two dimensions. *SIAM Journal on Scientific Computing*, 33(3):1159–1180, 2011.

[Zhou and Du, 2010] K. Zhou and Q. Du. Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions. *SIAM J. Numerical Analysis*, 48(5):1759–1780, 2010.