Completed Symplectic Cohomology and Liouville Cobordisms

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ABSTRACT

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Symplectic cohomology is an algebraic invariant of filled symplectic cobordisms that encodes dynamical information. In this thesis we define a modified symplectic cohomology theory, called action-completed symplectic cohomology, that exhibits quantitative behavior. We illustrate the non-trivial nature of this invariant by computing it for annulus subbundles of line bundles over complex projective space. The proof relies on understanding the symplectic cohomology of the complex fibers and the quantum cohomology of the projective base. We connect this result to mirror symmetry and prove a non-vanishing result in the presence of Lagrangian submanifolds with non-vanishing Floer homology. The proof uses Lagrangian quantum cohomology in conjunction with a closed-open map.

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To my grandparents

He'd tackled analysis, complex and quasi,
He liked diff-E-Qs and he really liked Cauchy,
But reading Pogorolev still wasn't easy;
He packed up his bags and he left for Parisi.

The Parisian air is a balm for the weary
(And Burs-sur-Yvette isn't terribly dreary.)
He fluffed up his hair and could suddenly clearly
see neonate J-holomorphic curve theory.

Obstructing symplectic embeddings abound!
Rigidity, softness, and squeezing profound!
The mathematicians all gathered around
To understand where this new toy might redound.

Invariants leading to quantum homology,
Theories of Floer in symplectic topology,
Physics abuzz with the hum of tautology,
And Mikhail's left us to pursue biology.

Chapter 1

Introduction

1.1 A historical overview

This thesis is concerned primarily with *symplectic cohomology*, a variant of *Floer theory*. Floer theory traces its origins back to Mikhail Gromov's famous 1985 paper *Pseudo holomorphic curves in symplectic geometry* [34], which emerged accidentally out of Aleksei Pogorolev's work on the rigidity of surfaces. "Defeated by formulas", Gromov translated Pogorolev's analytic problems into geometric ones; the inspiration to examine Cauchy-Riemann equations on symplectic manifolds soon followed, and pseudoholomorphic curve theory was born [33].

Pseudoholomorphic curve theory was transformed in 1988 by Andreas Floer. In a quest to understand Hamiltonian dynamics Floer used pseudoholomorphic curves to build Morse-esque homologies on the loop spaces of symplectic manifolds (e.g. [23][24]). These "Floer theories" were applicable both to the study of periodic orbits of a dynamical system (or "closed strings") and the study of trajectories of a dynamical system with endpoints on Lagrangian submanifolds (the "open strings").

Lagrangian submanifolds are of immense importance in symplectic geometry. The "symplectic creed" of Weinstein states that *everything is a Lagrangian submanifold* (1981) [52]. Perhaps with this adage in mind, Kenji Fukaya introduced in 1993 an A_∞-category formed out of Lagrangian submanifolds and their Floer theories, now called the *Fukaya category* [30]. The influence of the Fukaya category has been felt far and wide, from new attacks on classical problems in symplectic geometry, such as Arnol'd's *nearby Lagrangian conjecture* [2], to finding mathematical formulations for the phenomenon of *mirror symmetry* occurring in theoretical physics [35].

Closed-string and open-string Floer theories use the same tools to study different aspects of the same dynamical system. It seems clear that they should be intimately related. This thesis thus falls under the wide umbrella of the following question.

QUESTION 1 *Do closed-string Floer theories detect the Fukaya category?*

Question 1 was first considered for closed symplectic manifolds in the 1994 ICM address *Homological algebra of mirror symmetry*, when Maxim Kontsevich conjectured that the Fukaya category is a categorification of quantum cohomology [35]. Specifically, Kontsevich conjectured that there is an isomorphism between the quantum cohomology of a compact manifold M (a closed-string theory) and the *Hochschild homology* of its Fukaya category:

$$QH^*(M) \cong HH_*(\mathcal{F}(M)). \tag{1}$$

Equation (1), enigmatic as it is, has fostered a multitude of interesting relationships between closed- and open-string theories. In particular, symplectic geometers have attempted to understand (1) in the context of manifolds with boundary, on which *symplectic cohomology*, written $SH^*(M)$, takes the place of quantum cohomology. Symplectic cohomology was first introduced by Kai Cieliebak, Andreas Floer, Helmut Hofer, and Chris Wysocki between 1994 and 1995 [17][18][25][26][27]. It entered mirror symmetry in 2002, when Paul Seidel defined a map, called the *closed-open* map [48]

$$\mathcal{CO}\colon SH^*(M) \longrightarrow HH^*(\mathcal{F}(M)). \tag{2}$$

In 2010 Mohammed Abouzaid defined the counterpart of (2), the *open-closed* map [1]

$$\mathcal{OC}\colon HH_*(\mathcal{F}(M)) \longrightarrow SH^*(M). \tag{3}$$

These maps have proved very useful in finding generating sets for the Fukaya category [1][4][45][50]. Indeed, as an example particularly relevant to this thesis, Alex Ritter and Ivan Smith used an extension of equation (3) to find generating families for the *wrapped Fukaya category* of negative line bundles over complex projective space (2016, [45]).

Equations (2) and (3) involve powerful machinery, but they have a simpler consequence: they induce maps between symplectic cohomology and the Floer cohomology of a Lagrangian submanifold L

$$HF^*(L,L) \xrightarrow{\mathcal{OC}} SH^*(M) \xrightarrow{\mathcal{CO}} HF^*(L,L). \tag{4}$$

The map $SH^*(M) \xrightarrow{\mathcal{CO}} HF^*(L,L)$ was studied by Seidel and Smith in [47] to show that an exact symplectic manifold M containing an exact Lagrangian L has non-vanishing symplectic cohomology (2010). In particular, they showed that the (non-zero) unit in $HF^*(L,L)$ is in the image of $\mathcal{CO}$. This result has further implications for closed-string theories on M.

Symplectic cohomology has a dual closed-string theory called symplectic homology, written $SH_*(M)$. If there is an exact Lagrangian $L \subset M$ then the non-vanishing of $SH^*(M)$ implies the non-vanishing of $SH_*(M)$. There is a natural map $\mathfrak{c}^*\colon SH_*(M) \longrightarrow SH^*(M)$ whose cone is isomorphic to yet another closed-string theory associated to the boundary ∂M of M, called *Rabinowitz Floer homology* and written $RFH^*(\partial M, M)$ (2010, [19]). In other words, there is a long-exact sequence

$$\cdots \longrightarrow SH_*(M) \xrightarrow{\mathfrak{c}^*} SH^*(M) \longrightarrow RFH^*(\partial M, M) \longrightarrow SH_{*+1}(M) \longrightarrow \cdots \tag{5}$$

In this setting, $\mathfrak{c}^*$ is never an isomorphism when $SH^*(M) \neq 0$, and so

$$SH^*(M) \neq 0 \iff SH_*(M) \neq 0 \iff RFH^*(\partial M, M) \neq 0. \tag{6}$$

The equivalences (6) were observed by Ritter in 2013 [44]. In particular,

ASSERTION 1 *If M is exact and contains an exact compact Lagrangian submanifold then* $RFH^*(\partial M, M) \neq 0$.

Symplectic geometers prefer the relative simplicity of exact symplectic manifolds, but there is nothing inherently exact about the above story. Yet Peter Albers and Jungsoo Kang produced a paper in 2017 showing that negative line bundles equipped with non-exact symplectic forms have uniformly-vanishing Rabinowitz Floer homology [9]. As noted above, some of these line bundles were studied by Ritter-Smith, and in these examples they found compact Lagrangian tori with non-vanishing Floer theory. It seemed that analogues of Assertion 1 would not hold beyond the exact setting. This thesis is devoted to reconciling the results of Albers-Kang with the findings of Ritter-Smith.

All manifolds we study can be completed to a manifold with a *conical end*, that is, outside a compact set the complete manifold is symplectomorphic to the product of an interval $(a, \infty) \subset \mathbb{R}$ with a contact manifold Σ. There is a key difference between exact symplectic manifolds and the monotone examples we will study: Liouville manifolds have a one-parameter family of diffeomorphisms identifying a hypersurface $\{r\} \times \Sigma$ with any other hypersurface $\{r'\} \times \Sigma$. This takes a non-displaceable Lagrangian submanifold contained in some $\{r\} \times \Sigma$ to a sibling non-displaceable Lagrangian submanifold in $\{r'\} \times \Sigma$. Monotone manifolds no longer have this property. For example, the Lagrangian tori shown by Ritter-Smith to split-generate the wrapped Fukaya category are contained in precisely one hypersurface; no other hypersurfaces seem to carry information about the wrapped Fukaya category.

QUESTION 2 *Is there a variant of Rabinowitz Floer homology that detects precisely when a hypersurface contains a non-trivial element of the Fukaya category?*

This thesis will partially answer Question 2. Along the way we will discover interesting behavior of the symplectic cohomology already appearing in the literature, discuss closed-open maps in the monotone setting, and finally circle back to the mirror-symmetric origins of our discussion.

1.2 Explanation and summary of results

1.2.1 Completing Floer chain complexes

The symplectic cohomology of a compact manifold M with contact boundary ∂M is, at the chain level, generated roughly by the singular cochains of M and the set of disks with boundary a positively-traversed Reeb orbit of ∂M. We set coefficients to be the Novikov field Λ, over which, for example, singular cohomology becomes the small quantum cohomology ring. We start with a toy example to illustrate the geometry of the construction. Consider the disk of radius R, $D_R \subset \mathbb{C}$, equipped with the standard symplectic form ω. Let $H\colon \mathrm{Int}(D_R) \longrightarrow \mathbb{R}$ be a monotone-increasing bounded function, dependent only on the radius, whose slope approaches infinity near ∂D_R. The non-constant one-periodic orbits of H are in bijection with Reeb orbits on ∂D_R and cluster near ∂D_R. Choose capping disks $\gamma^n_\pm\colon D^2 \to D_R$ for the two n-covered periodic orbits of H with starting point on the real line. The symplectic cochain complex of D_R is

$$SC^*(D_R) := \mathbb{K}\left\langle \{0\}, \gamma_+, \gamma_-, \gamma_+^2, \gamma_-^2, \cdots \right\rangle$$

for a fixed coefficient field $\mathbb{K}$. One can determine *a posteriori* the radius of the domain we have chosen by noting that

$$-\int_{D^2} (\gamma_\pm^n)^*\omega \approx -\pi R^2 n. \tag{7}$$

We want to turn this observation into a useful invariant.

With an appropriate choice of almost-complex structure, the integral (7) is increased by the differential on $SC^*(D_R)$. This defines a filtration $\{\mathcal{F}_a\}_{a\in\mathbb{R}}$ of $SC^*(D_R)$, and therefore a bidirected system formed from the quotient complexes

$$SC^*_{(a,b)}(D_R) := \mathcal{F}_a \big/ \mathcal{F}_b .$$

We define completed symplectic cohomology to be the limit over this system

$$\widehat{SH^*}(D_R) := H\left(\varinjlim_{a} \varprojlim_{b} SC^*_{(a,b)}(D_R) \right). \tag{8}$$

$\widehat{SH^*}(D_R)$ is the homology of formal sums of elements of $SC^*(D_R)$ whose symplectic area approaches negative infinity. Dualizing yields completed symplectic homology, $\widehat{SH}_*(D_R)$.

$SC^*(D_R)$ contains the singular cochains of D_R as a subcomplex (in this case, the rank-1 vector space generated by $\{0\}$), and the inclusion defines a map $QH^*(D_R) \longrightarrow \widehat{SH}_*(D_R)$. This map, and the dual map, define for all radii $R' < R$ the map $\mathfrak{c}^*$ of the long-exact sequence (5)

$$\begin{array}{ccc} \widehat{SH}_*(D_{R'}) & \xrightarrow{\mathfrak{c}^*} & \widehat{SH^*}(D_R) \\ \downarrow & & \uparrow \\ QH^*(D_{R'}, \partial D_{R'}) & \longrightarrow & QH^*(D_R) \end{array} \tag{9}$$

where the bottom map is a quantum-corrected version of the map coming from the long-exact sequence of a pair. The cone of $\mathfrak{c}^*$ is defined to be the completed symplectic cohomology of the annulus cobordism $A_{(R',R)} := (R', R) \times S^1 \subset \mathbb{C}$, $\widehat{SH^*}(A_{(R',R)})$.

Extend this construction to an exact or monotone manifold V with convex boundary contained inside another such manifold M. If M is monotone, symplectic cochains are generated by all possible capping disks of all periodic orbits, up to symplectic area. In this thesis we will assume that the complement $W := M \setminus V$ is exact. The following theorem is a consequence of the definitions and mimics equation (5).

THEOREM 1 *There is a long-exact sequence*

$$\cdots \longrightarrow \widehat{SH_i}(V;\Lambda) \longrightarrow \widehat{SH^i}(M;\Lambda) \longrightarrow \widehat{SH^i}(W;\Lambda) \longrightarrow \widehat{SH_{i+1}}(V;\Lambda) \longrightarrow \cdots$$

1.2.2 Computations for line bundles

The above example is trivial because all of the completed symplectic homology groups vanish. This is an illustration of the results in Appendix A, which show that the completed symplectic cohomology of an exact domain is isomorphic to the uncompleted symplectic cohomology with Novikov coefficients. However, we will use the construction of $\widehat{SH^*}(D_R)$ to examine the completed symplectic cohomology of the line bundles $E := \mathrm{Tot}\,(\mathcal{O}(-k) \longrightarrow \mathbb{C}P^m)$ with $1 \leq n \leq m$. E is an example of a *negative line bundle*, that is, a line bundle $E \xrightarrow{\rho} B$ over a symplectic manifold (B, ω) such that c_1^E is negatively proportional to $[\omega]$. In fact, it is shown in Oancea's paper [41] that a line bundle is negative if and only if it can be equipped with a Hermitian metric with curvature $\mathcal{F}$ satisfying $\mathcal{F}(v, Jv) < 0$ for all ω-compatible J and all $v \neq 0$. Such a Hermitian metric induces a connection form α satisfying $d\alpha = \rho^*\omega$, a radial coordinate r, and the symplectic form $d\alpha + d(\pi r^2 \alpha)$. With this structure, there are three important facts about E.

FACT 1 (THEOREM: CHO-OH [21]) *E contains a monotone Lagrangian torus L in the circle bundle of radius $1/\sqrt{\pi(1+m-k)}$ such that the Floer cohomology of L, twisted by an appropriate line bundle, is non-vanishing.*

FACT 2 (THEOREM: RITTER-SMITH [45]) *There exist $1+m-k$ choices of local systems $\{\gamma_i\}_{i=1}^{1+m-k}$ on L such that the pairs $\{(L, \gamma_i)\}$ split-generate the wrapped Fukaya category of E.*

These imply the third fact.

FACT 3 $SH^*(E) \neq 0$. *In fact, Ritter computed in [43] that $SH^*(E)$ has rank $1+m-k$.*

Facts 1, 2, and 3 lend the following Theorem significance.

THEOREM 2 *Let* $E = \mathrm{Tot}\left(\mathcal{O}(-k) \xrightarrow{\rho} \mathbb{C}P^m\right)$ *be a negative line bundle with* $1 \le k \le m$*, and equip* E *with a Hermitian metric that induces an angular form* α *satisfying* $d\alpha = \rho^*\omega_{FS}$*, a radial coordinate* r*, and a symplectic form* $\Omega = d\alpha + d(k\pi r^2\alpha)$*. Let* $A_{(R_1,R_2)}$ *be the annulus subbundle between radii* R_1 *and* R_2*. Then*

$$\widehat{SH^*}(A_{(R_1,R_2)};\Lambda) \cong \begin{cases} SH^*(E;\Lambda) & R_1 < \frac{1}{\sqrt{\pi(1+m-k)}} \le R_2 \\ 0 & \textit{else} \end{cases}.$$

Thus, for these examples, $\widehat{SH^*}$ detects precisely when an annulus subbundle contains the generator of the wrapped Fukaya category. For nested annuli $A_{(r',r)} \subset A_{(R',R)}$ there is a morphism of Λ-modules

$$\widehat{SH^*}(A_{(R',R)}) \longrightarrow \widehat{SH^*}(A_{(r',r)})$$

that is an isomorphism if $\widehat{SH^*}(A_{(r',r)}) \neq 0$. Taking the limit over all annuli containing a circle bundle Σ of radius $\mathfrak{r}$, we define a candidate Rabinowitz Floer homology

$$\widehat{RFH^*}(\Sigma) := \varinjlim_{R'<\mathfrak{r}<R} \widehat{SH^*}(A_{(R',R)}).$$

$\widehat{RFH^*}(\Sigma)$ is non-vanishing precisely when $\mathfrak{r} = 1/\sqrt{\pi(1+m-k)}$, yielding an experiential answer to Question 2.

As mentioned above, the proof of Theorem 2 relies on the computation $\widehat{SH^*}(D_R) = 0$. This follows the method engineered by Albers-Kang in [9], where they showed that Rabinowitz Floer homology vanishes for certain circle bundles. Let D_RE be the disk subbundle of radius R in E. There is a model of $SC^*(D_RE)$ that decomposes (as a vector space) as

$$\bigoplus_{i=1}^{m+1} SC^*(D_R)_i$$

The diagonal component ∂_0 of the differential ∂ is the component-wise differential on each $SC^*(D_R)_i$. We will show that any cocycle $\mathbf{X_1}$ in $SC^*(D_RE)$ is a sum of cocycles in each $SC^*(D_R)_i$. As $SC^*(D_R)_i$ vanishes, $\mathbf{X_1}$ is in the image of ∂_0. Suppose $\mathbf{X_1} = \partial_0(\mathbf{Y_1})$. Then the differential applied to $\mathbf{Y_1}$ defines some cocycle $\mathbf{X_2}$, with $\partial(\mathbf{Y_1}) = \mathbf{X_1} + \mathbf{X_2}$. $\mathbf{X_2}$ is again a sum of cocycles in each $SC^*(D_R)_i$, and so there exists some $\mathbf{Y_2}$ with $\mathbf{X_2} = \partial_0(\mathbf{Y_2})$. We can iterate this procedure to find a sequence

$\{\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \cdots\}$ with telescoping sum

$$\partial\left(\sum_{i=1}^{\infty} \mathbf{Y}\right) = \mathbf{X}_1.$$

This inductive procedure is illustrated in Figure 1.1 and explained again, both heuristically and technically, in Section 4.1. We will show that the sum $\sum_{i=1}^{\infty} \mathbf{Y}$ is an element of the completed chain complex if and only if $R < 1/\sqrt{\pi(1+m-k)}$. The computation for $\widehat{SH}^*(D_R)$, the dual computation for $\widehat{SH}_*(D_R)$, and the long-exact sequence appearing in Theorem 1 will together prove Theorem 2.

The difference between this setup and the exact case is in the symplectic area of the generating disks. The completion (8) allows formal sums of disks whose area approaches negative infinity. In the exact setting, any infinite formal sum of discs will have positive area (as the periods of the bounding Reeb orbits are positive). In the monotone setting however, the area of the capping disk is not dictated solely by the boundary behavior. For example, the capping disk of a period-ℓ Reeb orbit on the boundary of $D_R E$ is the sum of the fiber disk and some a-multiple of a sphere representing $a \in H_2(\mathbb{C}P^m) \cong \mathbb{Z}$. The fiber disk has area $\pi R^2 \ell$ and the a-multiple of the sphere has area a. Thus, the area of the capping disk is $a + \pi R^2 \ell$. A formal sum of these disks *a priori* have areas with unpredictable limiting behavior, depending entirely on the weights a and the radius R.

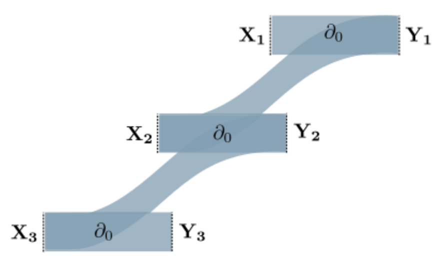


Figure 1.1: Trying to form a coboundary

1.2.3 Proving non-vanishing

The vanishing result in Theorem 2 is proved using very special properties of the line bundle E. We do not expect to be able to prove general vanishing results in the absence of elements of the Fukaya category. However, using the machinery of Lagrangian quantum cohomology developed by Biran-Cornea in [12][13][14] in conjunction with the techniques of Seidel-Smith in [47], we prove the following Theorem.

THEOREM 3 *Let* M *be a monotone symplectic manifold and* $W \subset M$ *a Liouville cobordism. Suppose that* W *contains a compact, oriented monotone Lagrangian* L*. If* L *admits a flat line bundle* E_γ *such that the Floer homology* $HF^*(L, E_\gamma) \neq 0$*, then*

$$\widehat{SH^*}(W; \Lambda) \neq 0.$$

If Λ *is defined over a coefficient field of characteristic not equal to two, we also require the Lagrangian to be spin.*

We construct a non-zero map from $\widehat{SH^*}(M)$ to the twisted Lagrangian quantum cohomology $QH^*(L, E_\gamma)$ by counting "half-tubes": Floer solutions on the half-cylinder $[0, \infty) \times S^1$ with boundary on L and limit a periodic orbit appearing as a generator of $\widehat{SC^*}(M)$. The unit of $QH^*(L, E_\gamma)$ is in the image of this map: it is represented by a half-tube satisfying a Floer equation of data (H_s, J), where $H_s = 0$ for $s >> 0$. In particular, this half-tube represents the (quantum-corrected) unital component of the map $H^*(M) \longrightarrow H^*(L)$, factoring through $\widehat{SH^*}(M)$.

An action argument shows that the map $\widehat{SH^*}(M) \longrightarrow QH^*(L, E_\gamma)$ is zero on the image of $\mathfrak{c}^*$. This map therefore factors through a submodule of $\widehat{SH^*}(W)$, from which we conclude that $\widehat{SH^*}(W) \neq 0$.

1.3 Overview of contents

In Chapter 2 we define the action-completed symplectic cohomology of a Liouville cobordism W with monotone filling, denoted by $\widehat{SH^*}(W; \Lambda)$. Rather than work with a single Hamiltonian with infinite growth, we will build a cochain complex as the homotopy colimit of a family of Hamiltonians. This is a less intuitive approach, but it simplifies the technical framework without diluting the behavior of the resulting homology theory. After recalling Hamiltonian Floer theory we define action-completed symplectic cohomology and homology, as well as action-completed Rabinowitz Floer homology. We then define the completed symplectic cohomology of a Liouville cobordism.

In Chapter 3 we prove that $\widehat{SH^*}(W; \Lambda) \neq 0$ whenever W contains a compact, oriented, monotone Lagrangian with non-vanishing Floer theory. We recall the definition of Lagrangian quantum cohomology, then use this definition to prove the result.

In Chapter 4 we show that the converse is true for monotone negative line bundles over projective space: if W is an annulus subbundle then $\widehat{SH^*}(W; \Lambda) \neq 0$ if and only if there exists such a Lagrangian L in W. We use this result to examine closed-mirror symmetry predictions for these spaces. In Chapter 5 we illustrate the phenomenon of Chapter 4 by finding an explicit chain model for $\widehat{SH}^*(W;\Lambda)$ for W is an annulus subbundle in the blow-up of $\mathbb{C}^2$ at a point. Finally, in the Appendix we relate our work to the work in the literature concerned with Liouville domains.

Chapter 2

Action-completed symplectic cohomology

2.1 Hamiltonian Floer theory on monotone manifolds

Let M be a compact symplectic manifold of dimension $2n$, equipped with symplectic form ω. Under favourable conditions one can define the Floer theory of M as a homology theory on the loop space $\mathcal{L}M$ of M. In this chapter we assume three conditions that, in conjunction, prove exceptionally favorable. The first condition requires M to be monotone: there exists a constant $c > 0$ satisfying

$$c_1^{TM} = c[\omega].$$

The second condition requires the boundary of M to be contact. Thus, there is a one-form λ defined near the boundary of M satisfying $d\lambda = \omega$, and such that $\lambda|_{\partial M}$ is a contact form on ∂M. The final condition requires the boundary orientation of ∂M to match the contact orientation induced by $\lambda|_{\partial M}$. This is equivalent to asking that the Liouville flow X_λ defined by $\omega(X_\lambda, -) = \lambda$ points outwards along the boundary.

Given a suitable Hamiltonian function $H : M \times S^1 \longrightarrow \mathbb{R}$ (defined in Section 2.2) , one can define a Floer cohomology theory as follows. Define the set of closed orbits of H to be

$$\mathcal{P}(H) = \left\{ x \in \mathcal{C}^\infty(S^1, M) \;\middle|\; \dot{x} = X_H(x) \right\}.$$

Choose a basepoint β for each connected component of $\mathcal{L}M$. Then $\mathcal{P}(H)$ decomposes as a direct sum

$$\mathcal{P}(H) = \bigoplus_{[\beta]} \mathcal{P}_\beta(H),$$

where $\mathcal{P}_\beta(H) = \left\{x \in \mathcal{P}(H) \middle| [x] = [\beta]\right\}$. For each $x \in \mathcal{P}_\beta(H)$ choose a path $\tilde{x}$ from x to β.

Fix a coefficient ring $\mathbb{K}$. Define a ring over a formal variable T by

$$\Gamma = \left\{ \sum_{j=0}^{n} a_j T^{k_j} \middle| a_j \in \mathbb{K}, k_j \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

We will define the structure of a cochain complex on the set $\Gamma\langle\mathcal{P}(H)\rangle$. Let o_x be the orientation line associated to x (see section 1.4 in [3] for a detailed account of orientation lines). Define

$$CF^*(H;\Gamma) = \bigoplus_{x\in\mathcal{P}(H)} \Gamma \otimes_{\mathbb{Z}} o_x. \tag{10}$$

The Conley-Zehnder index μ_{CZ} gives $\mathcal{P}(H)$ a well-defined $\mathbb{Z}/2\mathbb{Z}$ - grading, and we grade Γ trivially by setting $|T| = 0$. Elements $\zeta_x \in o_x$, for $x \in \mathcal{P}(H)$, are then graded by $|\zeta_x| = \mu_{CZ}(x) \in \mathbb{Z}/2\mathbb{Z}$. If $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$ one can replace each orientation line o_x in (10) by the corresponding periodic orbit x. See [46] for details on the Conley-Zehnder index.

The negative flow of X_λ defines a collar neighborhood $[-\epsilon, 0] \times \partial M$ of the boundary of M, on which $\omega_{(r,x)} = e^r dr \wedge \lambda_x + e^r d\lambda_x$ (where r is the coordinate on $[-\epsilon, 0]$). Let J be an almost-complex structure on M that is *cylindrical* on the collar neighborhood of ∂M. Recall that a cylindrical almost-complex structure satisfies

$$e^r dr = J^*\lambda.$$

We will always choose our cylindrical almost-complex structures to be ω-compatible.

Let $w : \mathbb{R} \times S^1 \longrightarrow M$ satisfy Floer's equation

$$\frac{\partial w}{\partial s} + J\left(\frac{\partial w}{\partial t} - X_H\right) = 0, \tag{11}$$

where the cylinder $\mathbb{R} \times S^1$ has coordinates (s, t). Associated to such maps is the *energy*, defined by

$$E(w) = \frac{1}{2}\int_{\mathbb{R}\times S^1} \|dw - X_H \otimes dt\|^2 ds \wedge dt. \tag{12}$$

If the energy of w is finite, $w(s,\cdot)$ converges asymptotically in s to periodic orbits of X_H. For any two periodic orbits x_- and x_+, define $\mathcal{M}^0(x_-,x_+)$ to be the space of rigid solutions $w(s,t)$ of (11) satisfying $\lim_{s\to\pm\infty} u(s,\cdot) = x_\pm(\cdot)$. Each such w has an $\mathbb{R}$-action of translation in the s-direction, and modding out by this $\mathbb{R}$-action produces a compact zero-dimensional moduli space $\hat{\mathcal{M}}^0(x_-,x_+)$. Each w further induces an isomorphism $d_w : o_{x_+} \longrightarrow o_{x_-}$ of orientation lines (see Lemma 1.5.4 of [3]). We denote by $\widetilde{x_-}\#w\#(-\widetilde{x_+})$ the element of $\pi_2(M)$ formed by gluing w to $\widetilde{x_-}$ and $\widetilde{x_+}$, the latter with reversed orientation. For $A \in \pi_2(M)$, we will use the shorthand

$$\omega(A) := \int_{S^2} A^*\omega.$$

Equip $CF^*(H;\Gamma)$ with the differential ∂^{fl} given on generators by

$$\partial^{fl}\big|_{o_{x_+}} = \sum_{\substack{x_-\in\mathcal{P}(H)\\ w\in\hat{\mathcal{M}}^0(x_-,x_+)}} T^{\omega(\widetilde{x_-}\#w\#(-\widetilde{x_+}))}\cdot d_w.$$

As M is monotone, ∂^{fl} is well-defined. Extending the differential T-linearly yields the cochain complex $CF^*(H;\Gamma)$.

Define an action functional $\mathcal{A}_H$ on $\{T^\alpha x\}_{\alpha\in\mathbb{R},x\in\mathcal{P}(H)}$ by

$$\mathcal{A}_H(T^\alpha x) = \alpha - \int_{S^1\times[0,1]} \tilde{x}^*\omega + \int_{S^1} H(x(t))dt$$

and set $\mathcal{A}_H(T^\alpha\zeta_x) = \mathcal{A}_H(T^\alpha x)$ for any $\zeta_x \in o_x$. A standard computation shows that the differential increases $\mathcal{A}_H$. Thus, the subsets

$$CF^*_a(H;\Gamma) := \mathbb{K}\left\langle\left\{T^\alpha\zeta_x \;\middle|\; \alpha\in\mathbb{R};\ \zeta_x\in o_x;\ x\in\mathcal{P}(H);\ \mathcal{A}_H(T^\alpha\zeta_x) > a\right\}\right\rangle$$

are subcomplexes and form a filtration of $CF^*(H;\Gamma)$.

For $a < b$ define the quotient complex

$$CF^*_{(a,b)}(H;\Gamma) := CF^*_a(H;\Gamma)\Big/CF^*_b(H;\Gamma).$$

There are natural chain maps

$$CF^*_{(a,b)}(H;\Gamma) \hookrightarrow CF^*_{(a',b)}(H;\Gamma) \text{ and } CF^*_{(a,b)}(H;\Gamma) \twoheadrightarrow CF^*_{(a,b')}(H;\Gamma) \tag{13}$$

whenever $a' \leq a$ or $b' \leq b$, given by, respectively, inclusion and projection. Following the example of [16], we will use this quotient complex and the natural maps of (13) to define a Novikov-type completion of different Floer homology theories on open manifolds.

REMARK 1) The (a, b)-filtered complex is independent of lifts $x \mapsto \tilde{x}$, as choosing a different lift corresponds to rescaling x by some power of T.

2.2 Symplectic cohomology

Hamiltonian Floer theory is not invariant under the choice of Hamiltonian when working on manifolds with boundary. To rectify this, one usually takes a colimit over the Floer homologies of all suitable Hamiltonians. The resulting homology theory captures information about the singular cohomology of M and the positively-traversed Reeb orbits of various contact hypersurfaces in the conical completion of M.

Our construction of symplectic cohomology mimics the homotopy construction of [45], which is based on the telescope constructions in [5]. We use a more restricted class of Hamiltonians than is usual in the literature; in particular, we will require that the Reeb orbits captured in our cohomology theory cluster near ∂M. This will define a Floer cohomology of M (as opposed to its conical completion) that we will show displays, under completion-by-action, surprising behavior.

Choose $\epsilon_M > 0$. The Liouville flow near the boundary of M enables us to smoothly attach $[0, \epsilon_M) \times \partial M$ to M via ∂M. Define the enlarged manifold

$$\widetilde{M} = M \cup_{\partial M} [0, \epsilon_M) \times \partial M.$$

Choose any $\mathcal{C}^2$-small function $\mathcal{H} : M \times S^1 \longrightarrow \mathbb{R}$ with non-degenerate, constant time-one orbits. Choose a sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ that is monotone decreasing, bounded above by ϵ_M, and converges to 0. Choose a family of Hamiltonians $Ad(M) = \{H^{\tau_i} : \widetilde{M} \times S^1 \longrightarrow \mathbb{R}\}_{i \in \mathbb{Z}}$, that we call *admissible*, such that

1. $H^{\tau_i}|_{M\times S^1} = \mathcal{H}$ for all i,

2. $H^{\tau_i} \geq H^{\tau_j}$ whenever $i \geq j$,

3. H^{τ_i} is $\mathcal{C}^2$ close to a radially-dependent function $h^{\tau_i}(e^r)$ on $[0, \epsilon_M) \times \partial M$ for some function $h^{\tau_i} : \mathbb{R}_+ \longrightarrow \mathbb{R}$,

4. h^{τ_i} is linear of slope τ_i on $(\epsilon_{|i|}, \epsilon_M)$,

5. $\tau_i > 0$ if and only if $i \geq 0$,

6. $|H^{\tau_i}|$ is universally bounded on one-periodic orbits, and

7. the one-periodic orbits of H^{τ_i} are non-degenerate.

Finally, require that τ_0 be smaller than the smallest period of a positive Reeb orbit on ∂M. See Figure 2.1 for a cartoon of the elements of $Ad(M)$.

REMARK 2) Item 3 merits a discussion. We require the one-periodic orbits of each H_n to be non-degenerate to avoid defining chain complexes through *cascades*. However, later proofs require knowing a fair amount about the behavior of Floer solutions. This is most easily achieved if the Hamiltonians are radially-dependent around one-periodic orbits, which then leaves the periodic orbits only transversely non-degenerate. A result by Bourgeois-Oancea, which we recall as Theorem 4, says that there is a bijection between an isolated cascade defined through radially-dependent Floer data (h, J) and an isolated Floer trajectory defined through non-degenerate Floer data that is $\mathcal{C}^2$-close to (h, J). We will therefore prove results for Floer solutions of the radially-dependent Floer data, then use Theorem 4 to transfer these proofs to results about the non-degenerate Floer data $\{H^{\tau_i}\}$. We discuss Theorem 4 at the end of this section.

Define $Ad_+(M)$ to be the non-negatively-indexed Hamiltonians and $Ad_-(M)$ to be the negatively-indexed Hamiltonians.

REMARK 3) Instead of attaching $[0, \epsilon_M) \times \partial M$ to M, we could have attached the entire positive symplectization $[0, \infty) \times \partial M$, and extended each H^{τ_i} linearly to define elements of $Ad(M)$ on this completed manifold. The Floer theory of H^{τ_i} is well-defined in this setting. Condition (4) and the maximum principle ensure that Floer trajectories of H^{τ_i} of finite energy, in particular the trajectories used to define the differential, do not exit $\widetilde{M}$. All of the data used to define $CF^*(H^{\tau_i}; \Gamma)$ therefore

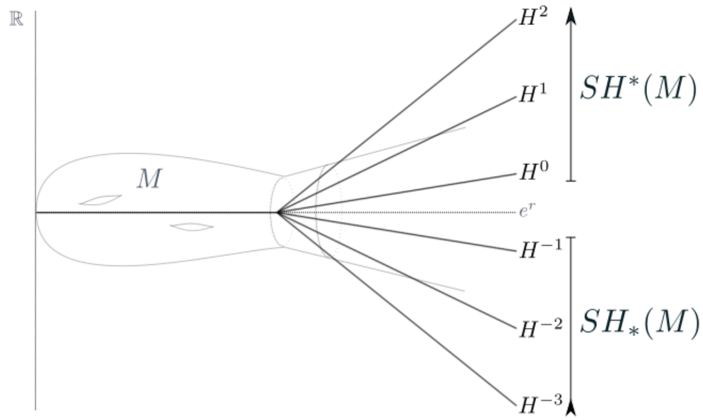


Figure 2.1: The family of Hamiltonians defining completed symplectic homology and cohomology, and completed Rabinowitz Floer cohomology

lives in $\widetilde{M}$, and so we can "do Floer theory" on $\widetilde{M}$ instead of on the completed manifold. In this paper we will only define Floer theory on manifolds of the form $\widetilde{M}$, and never on the full completed manifold.

There are continuation maps $c^i : CF^*(H^{\tau_i};\Gamma) \longrightarrow CF^*(H^{\tau_{i+1}};\Gamma)$ for each i induced by a monotone-decreasing homotopy from $H^{\tau_{i+1}}$ to H^{τ_i}. Again by Condition (4) and the maximum principle each c^i is well-defined. These maps respect the action filtration, thereby inducing continuation maps

$$c^i : CF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow CF^*_{(a,b)}(H^{\tau_{i+1}};\Gamma).$$

This leads to a directed system

$$\cdots \xrightarrow{c^{-2}} CF^*_{(a,b)}(H^{\tau_{-1}};\Gamma) \xrightarrow{c^{-1}} CF^*_{(a,b)}(H^{\tau_0};\Gamma) \xrightarrow{c^0} CF^*_{(a,b)}(H^{\tau_1};\Gamma) \xrightarrow{c^1} \cdots$$

The non-negatively-indexed continuation maps induce a chain map

$$\{c^i - \mathrm{id}\} : \bigoplus_{i=0}^{\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow \bigoplus_{i=0}^{\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma)$$

defined componentwise. The cone of this map is a cochain complex

$$SC^*_{(a,b)}(M;\Gamma) := \bigoplus_{i=0}^{\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma) \oplus \bigoplus_{i=0}^{\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma)[1]$$

with differential given by

$$\delta^* = \left\{ \begin{pmatrix} \partial^{fl} & c^i - id \\ 0 & \partial^{fl}[1] \end{pmatrix} \right\}.$$

For ease of notation let θ be a formal variable of degree $|\theta| = -1$ satisfying $\theta^2 = 0$. Rewrite the symplectic chain complex as

$$SC^*_{(a,b)}(M;\Gamma) = \bigoplus_{i=0}^{\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma)[\theta]$$

with differential

$$\delta^*|_{o_x + o_y\theta} = \partial^{fl}|_{o_x} + c^i|_{o_y} - id_{o_y} + (\partial^{fl}|_{o_y})\theta. \tag{14}$$

The maps between filtered chain complexes in equation (13) extend componentwise to chain maps

$$SC^*_{(a,b)}(M;\Gamma) \hookrightarrow SC^*_{(a',b)}(M;\Gamma) \text{ and } SC^*_{(a,b)}(M;\Gamma) \twoheadrightarrow SC^*_{(a,b')}(M;\Gamma)$$

that defines a bi-directed system. Define the *completed symplectic cochains* to be the limit over this bi-directed system, and denote it by

$$\widehat{SC^*}(M;\Gamma) = \varinjlim_{a} \varprojlim_{b} SC^*_{(a,b)}(M;\Gamma).$$

Note that, under our conventions, the limits take a to negative infinity and b to positive infinity.

REMARK 4) By Theorem 5.6 in [29], taking the limits in the opposite order creates an isomorphic complex. (Also see [16] for an application of this theorem to Morse Theory.)

Completed symplectic cohomology is the homology of this complex, and is denoted by $\widehat{SH^*}(M;\Gamma)$.

REMARK 5) By Condition (6) on $Ad(M)$, the action of a one-period orbit of any admissible Hamiltonian is close to the negative symplectic area of its chosen lift. We can therefore write the elements of $\widehat{SC^*}(M;\Gamma)$ directly as $a + b\theta$, where a and b are sums of the form

$$\left\{ \sum_{j=0}^{\infty} a_j T^{k_j} \cdot \zeta_j \,\middle|\, a_j \in \mathbb{K};\, k_j \in \mathbb{R};\, \zeta_j \in o_{x_j} \text{ for some } x_j \in \bigcup_{H \in Ad(M)} \mathcal{P}(H);\, \lim_{j\to\infty} -\omega(\tilde{x}_j) + k_j = \infty \right\}.$$

Thus, the completed symplectic cochain complex agrees with the complex formed by taking a

Novikov-type completion. In particular, it is a module over the *universal Novikov ring over* $\mathbb{K}$, defined by

$$\Lambda := \left\{ \sum_{j=1}^{\infty} a_j T^{k_j} \middle| a_j \in \mathbb{K}; k_j \in \mathbb{R}; \lim_{j\to\infty} k_j = \infty \right\}. \tag{15}$$

Since the differential respects the Λ action, we will henceforth take coefficients of completed complexes in Λ, working with $\widehat{SC^*}(M;\Lambda) := \widehat{SC^*}(M;\Gamma)$.

2.2.1 Morse-Bott Floer theory and cascades

We follow the exposition in [15]. To set up Theorem 4, let H be a Hamiltonian, linear at infinity, whose one-periodic orbits are either constant or correspond to Reeb orbits of the contact boundary of M. For each non-constant orbit $x \in \mathcal{P}(H)$ choose a generic perfect Morse function $f_x : S^1 \longrightarrow \mathbb{R}$. A choice of ω-compatible almost-complex structure J defines *cascades*: tuples $\mathbf{u} = (c_m, u_m, c_{m-1}, u_{m-1}, \dots, u_1, c_0)$ associated to a sequence of orbits $x_m, x_{m-1}, \dots, x_0$ with $x_{m-1}, \dots, x_1$ non-constant, such that

1. $c_i \in \operatorname{im}(x_i)$
2. u_i is a finite-energy Floer solution corresponding to the Floer data (H, J),
3. $\lim_{s\to\infty} u_i(s, 0)$ is in the stable manifold of c_i (or $c_i = \lim_{s\to\infty} u_i(s,t)$ if x_i is constant), and
4. c_i is in the stable manifold of $\lim_{s\to-\infty} u_{i+1}(s, 0)$ (or $c_i = \lim_{s\to-\infty} u_{i+1}(s,t)$ if x_i is constant).

Choose capping discs $\tilde{x}_i$ for x_0 and x_m. The union $\tilde{x}_m \# - u_m \# - u_{m-1} \# \dots \# - u_1 \# - \tilde{x}_0$ represents a homology class $A \in H_2(M)$. Let p and q be constant orbits of H or critical points of some functions f_x and $f_{x'}$. The moduli space $\widehat{\mathcal{M}}^A_m(q,p)$ is the space of tuples $(c_m, u_m, c_{m-1}, u_{m-1}, \dots, u_1, c_0)$ representing class A such that c_m is in the stable manifold of p (or equal to a constant orbit p) and c_0 is in the stable manifold of q (or equal to a constant orbit q). Each component of $\widehat{\mathcal{M}}^A(q,p)$ carries an $\mathbb{R}^m$ action, induced by the $\mathbb{R}$-actions on each Floer trajectory. By Proposition 3.2 in [15],

$$\mathcal{M}^A(q,p) := \bigsqcup_{m\geq 1} \widehat{\mathcal{M}}^A_m(q,p) \,/\mathbb{R}^m$$

is a manifold.

For each non-constant $x \in \mathcal{P}(H)$ choose a neighborhood U of the image of x and a bump function

$\rho : M \longrightarrow \mathbb{R}$ achieving a maximum value of 1 at every point in the image of x. Let τ_x be the period of the Reeb orbit underlying x. Symplectically parametrize $U \cong S^1 \times \mathbb{R}^k$ in coordinates (θ, z).

THEOREM 4 (BOURGEOIS-OANCEA: THEOREM 3.7 IN [15]) *For small enough $\delta > 0$ the time-dependent Hamiltonian*

$$H_\delta(t, \theta, z) := H - \delta \sum_{x \in \mathcal{P}(H)} \rho_x(\theta, z) \cdot f_x(t - \tau_x \theta)$$

has only non-degenerate periodic orbits, and these periodic orbits correspond bijectively to the union of the constant periodic orbits of H and the critical points of $\{f_x\}$.

Let x_p correspond to p and x_q to q under this bijection, and suppose $\mu(x_q) - 1 - 2c_1(A) = \mu(x_p)$. Let $\mathcal{M}^A(x_q, x_p; H_\delta)$ be the moduli space of Floer solutions corresponding to H_δ representing A. Then there exists $\delta_0 > 0$ such that

1. *H_δ is regular for all $\delta < \delta_0$,*

2. *and if*

$$\mathcal{M}^A_{(0,\delta_0)}(x_q, x_p) = \bigcup_{\delta \in (0,\delta_0)} \mathcal{M}^A(x_q, x_p; H_\delta)$$

is the moduli space formed by varying δ, then $\mathcal{M}^A_{(0,\delta_0)}(x_q, x_p)$ is a one-dimensional manifold whose components correspond bijectively to components of $\mathcal{M}^A(q, p)$.

While Theorem 4 is stated in [15] for exact symplectic manifolds, monotone symplectic manifolds satisfy the necessary energy bounds needed to carry the result over to setting.

REMARK 6) Suppose that $\{h^{\tau_n}\}_{n\in\mathbb{N}}$ and $\{h^{\tau_n}_s\}_{n\in\mathbb{N}}$ are the autonomous Hamiltonians used to define $Ad(M)$. There may not be a single "δ_0" such that Theorem 4 holds for every Hamiltonian $h^{\tau_n}_\delta$ with $n \in \mathbb{N}$ and $\delta < \delta_0$. To skirt this issue we can modify the definition of the symplectic cochain complex as follows. Choose a non-increasing sequence $\{\delta_n \geq 0\}_{n\in\mathbb{N}}$ such that Theorem 4 applies to all dimension-zero moduli spaces associated to the Floer data (h^{τ_n}, J), $(h^{\tau_{n+1}}, J)$, and $(h^{\tau_n}_s, J)$, and all parameters $\delta \leq \delta_n$. For notational simplicity, let $F_n = \sum_{x\in\mathcal{P}(h^{\tau_n})} \rho_x f_x$ be the perturbation term appearing in Theorem 4. Define a new countable family of admissible Hamiltonians

$H_{1/2} = h^{\tau_0} - \delta_0 F_0$

$H_{2/2} = h^{\tau_1} - \delta_0 F_1$

$H_{3/2} = h^{\tau_1} - \delta_1 F_1$

$H_{4/2} = h^{\tau_2} - \delta_1 F_2$

... and so forth. Let c^i be as before for $i \in \mathbb{N} + \frac{1}{2}$, and let c^i be the bijection of Theorem 4 for $i \in \mathbb{N}_{>0}$. Applying the telescope construction to this new data yields a complex that is homotopy equivalent to the one before and that can be canonically identified with a Morse-Bott cochain complex. We will suppress this modification from future notation.

2.3 Symplectic homology

Symplectic homology is defined analogously to symplectic cohomology. The negative continuation maps induce a chain map

$$\{c^i - \mathrm{id}\}\colon \prod_{i=-1}^{-\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow \prod_{i=-1}^{-\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma).$$

Define the (a,b)-truncated symplectic chains to be

$$SC_*^{(a,b)} := \prod_{i=-1}^{-\infty} CF^*_{(a,b)}(H^{\tau_i};\Gamma)[\theta],$$

with differential δ^* as in (14).

Akin to Section 2.2, the *completed symplectic chains* are defined to be

$$\widehat{SC}_*(M;\Lambda) := \varinjlim_a \varprojlim_b SC_*^{(a,b)}(M;\Gamma).$$

Completed symplectic homology is the homology of this complex, and is denoted by $\widehat{SH}_*(M;\Lambda)$.

REMARK 7) By Poincaré duality, there is a chain isomorphism

$$SC_*^{(a,b)}(M;\Gamma) \cong \prod_{i=-1}^{-\infty} CF_{-*}^{(-b,-a)}(-H^{\tau_i};\Gamma)[\zeta],$$

where $|\zeta| = 1$ and $\zeta^2 = 0$.

This implies the isomorphism

$$\widehat{SC}_*(M;\Lambda) \cong \left(\varinjlim_a \varprojlim_b \prod_{i=-1}^{-\infty} CF_{-*}^{(-b,-a)}(-H^{\tau_i};\Gamma)[\zeta] \right).$$

In particular, $\widehat{SC}_*(M;\Lambda)$ is chain-isomorphic to the dual complex of the completed symplectic cochain complex (after a shift in grading), and is thereby deserving of its name, despite the cohomological conventions used to define it.

2.4 Rabinowitz Floer cohomology

There is a map from (a, b)-truncated symplectic homology to (a, b)-truncated symplectic cohomology, given on chains by projecting onto $CF^*_{(a,b)}(H^{\tau_{-1}}; \Gamma)\theta$, applying the continuation map c^{-1}, and then including. Call this map $\mathfrak{c}$.

$$\begin{array}{ccc} SC_*^{(a,b)}(M;\Gamma) & \xrightarrow{\ \mathfrak{c}\ } & SC^*_{(a,b)}(M;\Gamma) \\ \Big\downarrow{\scriptstyle \pi} & & {\scriptstyle \iota}\Big\uparrow \\ CF^*_{(a,b)}(H^{\tau_{-1}};\Gamma)\theta & \xrightarrow{\ c^{-1}\ } & CF^*_{(a,b)}(H^{\tau_0};\Gamma) \end{array}$$

REMARK 8) One does not need to truncate by action; on monotone and exact domains the map $\mathfrak{c}$ extends to a map on the full complexes $SC_*(M) \longrightarrow SC^*(M)$. It was shown in [19] that the Rabinowitz Floer homology of the contact boundary of a Liouville domain is the cone of the induced map $\mathfrak{c}^* \colon SH_*(M) \longrightarrow SH^*(M)$. This motivates the following definition.

DEFINITION 1 Define the (a, b)-truncated Rabinowitz Floer cochain complex $RFC^*_{(a,b)}(M)$ to be the cone of $\mathfrak{c}$:

$$RFC^*_{(a,b)}(M) := \left(SC^*_{(a,b)}(M) \oplus SC_*^{(a,b)}(M)[1], \begin{pmatrix} \delta^* & \mathfrak{c} \\ 0 & \delta^*[1] \end{pmatrix} \right).$$

There is a triangle

$$\begin{array}{ccc} SC_*^{(a,b)}(M;\Gamma) & \xrightarrow{\qquad \mathfrak{c} \qquad} & SC^*_{(a,b)}(M;\Gamma) \\ & {\scriptstyle [-1]}\nwarrow \qquad \swarrow & \\ & RFC^*_{(a,b)}(M;\Gamma) & \end{array} \tag{16}$$

The inverse limit $\varprojlim_b$ is exact (the Mittag-Leffler condition is easily satisfied via surjection of the projection maps defining the limit). Clearly the limit $\varinjlim_a$ is exact. Applying the action-window limits to the triangle (16) creates a triangle of completed complexes.

$$\begin{array}{ccc} \widehat{SC}_*(M;\Lambda) & \xrightarrow{\quad c \quad} & \widehat{SC^*}(M;\Lambda) \\ & {\scriptstyle [-1]}\nwarrow \qquad \swarrow & \\ & \varinjlim_{a} \varprojlim_{b} RFC^*_{(a,b)}(M;\Gamma) & \end{array} \tag{17}$$

DEFINITION 2 The *completed Rabinowitz Floer cochain complex* is

$$\widehat{RFC^*}(M;\Lambda) := \varinjlim_{a} \varprojlim_{b} RFC^*_{(a,b)}(M;\Gamma)$$

Its homology is denoted by $\widehat{RFH^*}(M;\Lambda)$.

Note that applying homology to (17) yields the exact sequence

$$\cdots \longrightarrow \widehat{SH_i}(M;\Lambda) \longrightarrow \widehat{SH^i}(M;\Lambda) \longrightarrow \widehat{RFH^i}(M;\Lambda) \longrightarrow \widehat{SH_{i+1}}(M;\Lambda) \longrightarrow \cdots \tag{18}$$

REMARK 9) While we abuse language in calling our construction "completed Rabinowitz Floer homology", we expect that $\widehat{RFH^*}(M;\Lambda)$ is, after a degree adjustment, isomorphic to the Rabinowitz Floer homology found in the literature (defined for sphere bundles in negative line bundles). [9], [19], [28].

2.5 Symplectic cohomology of a Liouville cobordism

A *Liouville cobordism* is an exact symplectic manifold $(W, \omega = d\lambda)$ with contact boundary $(\partial W, \alpha = \lambda|_{\partial W})$. If the boundary orientation of a component $B \subset \partial W$ agrees with the orientation induced by $\alpha|_B$, we call B a *positive boundary component*. If the two orientations disagree, we say that B is a *negative boundary component*. In general, ∂W decomposes as the union of the positive boundary components $(\partial_+ W, \alpha_+ = \alpha|_{\partial_+ W})$ and negative boundary components$(\partial_- W, \alpha_- = \alpha|_{\partial_- W})$.

Suppose that M decomposes as the union of a Liouville cobordism W and a compact, monotone symplectic manifold V, glued along the boundary of V and the negative boundary of W. We will show that the map

$$\widehat{SH}_*(M;\Lambda) \xrightarrow{c^*} \widehat{SH^*}(M;\Lambda)$$

generalizes to a map

$$\widehat{SH}_*(V;\Lambda) \longrightarrow \widehat{SH}^*(M;\Lambda),$$

and we will define the completed symplectic cohomology of W analogously to completed Rabinowitz Floer cohomology. We first fix notation and technical conventions.

As in the previous sections, we will work over $\widetilde{M} = M \cup [0, \epsilon_M) \times \partial M$. If the flow $\Phi^t_{X_\lambda}(x)$ of X_λ is defined for all $t \in (T_1, T_2)$ and $x \in \partial_\pm W$, we identify the subdomain

$$\left\{\Phi^t_{X_\lambda}(x) \middle| t \in (T_1, T_2), x \in \partial_\pm W\right\}$$

with the subspace $(T_1, T_2) \times \partial_\pm W$ of the symplectization of $\partial_\pm W$. Let r be the coordinate on (T_1, T_2) and x the coordinate on $\partial_\pm W$. Under this identification, $\lambda_{r,x} = e^r(\alpha_\pm)_x$. Fix $R > 0$ such that Φ_{X_λ} is defined on $(-R, R) \times \partial_- W$ and $(-R, \epsilon_M) \times \partial_+ W$, and

$$\{(-R, R) \times \partial_- W\} \cap \{(-R, \epsilon_M) \times \partial_+ W\} = \varnothing\,.$$

Let $W_+ = (W \cup [0, \epsilon_M) \times \partial_+ W) \setminus [0, R) \times \partial_- W$

In the previous section we considered the set of admissible Hamiltonians $Ad(M) = Ad_+(M) \sqcup Ad_-(M)$. Leave the subfamily $Ad_+(M)$ unchanged and redefine $Ad_-(M)$ as follows. Choose $\epsilon_V \in (0, R)$. Let $\{\epsilon_i\}_{i \in \mathbb{Z}_{<0}}$ be a monotone decreasing sequence bounded above by R and converging to ϵ_V. Choose non-degenerate Hamiltonians inductively by requiring that H^{τ_i} satisfies the following conditions.

1. $H^{\tau_i}\big|_V = \mathcal{H}\big|_V$. To simplify later computations, assume $\mathcal{H}\big|_{\partial V} = 0$.

2. There exists $h^{\tau_i} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $H^{\tau_i}(r, x)$ is $\mathcal{C}^2$-close to $h^{\tau_i}(e^r)$ on $[0, R) \times \partial_- W$.

3. H^{τ_i} is convex on $(\epsilon_i, R) \times \partial_- W$ and concave on $(0, \epsilon_i) \times \partial_- W$ (adjust $\mathcal{H}$ if necessary).

4. h^{τ_i} is linear of slope τ_i on $\partial_- W \times [\epsilon_i - \epsilon_V, \epsilon_i]$,

5. After shifting by a constant, $H^{\tau_i}\big|_{((\epsilon_{i+1}, R) \times \partial_- W) \cup W_+} = H^{\tau_{i+1}}\big|_{((\epsilon_{i+1}, R) \times \partial_- W) \cup W_+}$. In particular, $H^{\tau_i}\big|_{W_+} = \mathcal{H}\big|_{W_+}$.

6. $H^{\tau_{i+1}} \geq H^{\tau_i}$ everywhere.

We denote the set of such Hamiltonians by $\mathcal{A}d_-(V, M)$ and let $\mathcal{A}d(V, M) = \mathcal{A}d_+(M) \sqcup \mathcal{A}d_-(V, M)$. See Figure 2.2 for a cartoon.

REMARK 10) Solutions of $\dot{x} = X_{H^{\tau_i}}(x)$ are partitioned by whether or not they live in $V \cup ([0, \epsilon_V] \times \partial_- W)$. We will see that conditions (2) – (5) ensure that solutions living "close to V" form a subcomplex of the Floer cochain complex of H^{τ_i} and that this subcomplex computes the symplectic homology of V. Conditions (5) and (6) enable continuation maps to respect these subcomplexes, and condition (1) bounds the action of constant orbits, so that they are all eventually accounted for under completion by action.

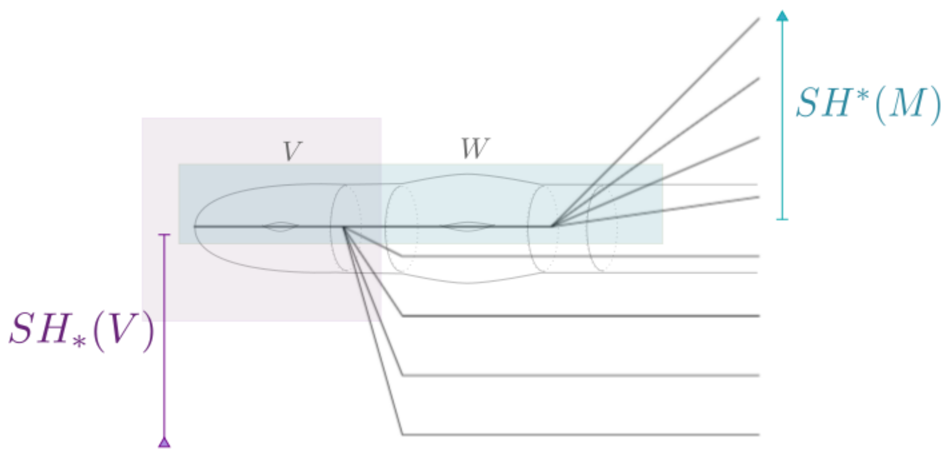


Figure 2.2: The family of Hamiltonians $\mathcal{A}d(V, M)$ used to define the completed symplectic cohomology of a Liouville cobordism W with monotone filling V.

To see these conditions in play, let

$$CF^*_{V,(a,b)}(H^{\tau_i}; \Gamma) = \left\langle T^\alpha \zeta_x \in CF^*_{(a,b)}(H^{\tau_i}; \Gamma) \;\middle|\; \zeta_x \in \mathfrak{o}_x;\ x \subset V \cup [0, \epsilon_V] \times \partial_- W \right\rangle.$$

LEMMA 1 *For* $H^{\tau_i} \in \mathcal{A}d_-(V, M)$, *the subset* $CF^*_{V,(a,b)}(H^{\tau_i}; \Gamma)$ *is a subcomplex of* $CF^*_{(a,b)}(H^{\tau_i}; \Gamma)$.

PROOF: We show that if H^{τ_i} is instead an autonomous Hamiltonian, there are no solutions of Floer's equation (11) with positive limit a periodic orbit in $V \cup [0, \epsilon_V] \times \partial_- W$ and either

1. negative limit a non-constant orbit in $W_+ \cup (\epsilon_V, R] \times \partial_- W$, or

2. negative limit a constant orbit in $W_+ \cup (\epsilon_V, R] \times \partial_- W$.

After invoking Theorem 4 this proves the result.

Assume for contradiction that $u(s,t)$ is a solution of Floer's equation with positive end an orbit $x(t)$ in $V \cup [0, \epsilon_V] \times \partial_- W$ and negative end an orbit $y(t)$ in $(\epsilon_V, R] \times \partial_- W \cup W_+$.

1. This can be found in [20]. Assume that $y(t)$ is a non-constant orbit. By the construction of H^{τ_i}, $y(t) \subset (\epsilon_i, R) \times \partial_- W$. Since $h(r)$ is convex in this region, the proof of Proposition 5 in [15] shows that $u(s,t)$ "rises above" $y(t)$. In other words, if $y(t) \subset \{r_1\} \times \partial_- W$, there exists $(s_1, t_1) \in \mathbb{R} \times S^1$ and $r_2 \in (r_1, R)$ such that $u(s_1, t_1) \subset \{r_2\} \times \partial_- W$. The integrated maximum principle then applies to reach a contradiction. We recall this final argument, which we learned from [3].

   Let $\rho \colon [-R, R] \times \partial_- W \longrightarrow \mathbb{R}$ be projection onto the r-coordinate, and choose $r_3 \in (r_1, r_2)$ so that r_3 is a regular value of $\rho \circ u$. Consider the surface

$$\Sigma = u^{-1}\left([r_3, R] \times \partial_- W \cup W_+\right).$$

   Define $v \colon \Sigma \longrightarrow \widetilde{M}$ by $v = u|_\Sigma$. As v is a solution to Floer's equation (11), the energy defined in equation (12) may be rewritten as

$$\begin{aligned} E^{top}(v) &= \frac{1}{2}\int_\Sigma \omega(\partial_s v, J\partial_s v) + \omega\left(\partial_t v - X_{H^{\tau_i}}, J(\partial_t v - X_{H^{\tau_i}})\right) ds \wedge dt \\ &= \frac{1}{2}\int_\Sigma \omega(\partial_s v, \partial_t v - X_{H^{\tau_i}}) + \omega(\partial_t v - X_{H^{\tau_i}}, -\partial_s v) ds \wedge dt \\ &= \frac{1}{2}\int_\Sigma 2\omega(\partial_s v, \partial_t v) - 2\omega(\partial_s v, X_{H^{\tau_i}}) ds \wedge dt \\ &= \int_\Sigma v^*\omega - v^* dH^{\tau_i} \otimes dt. \end{aligned}$$

   By Stokes theorem this is equivalent to

$$E^{top}(v) = \int_{\partial\Sigma} v^*\lambda - H^{\tau_i}(v(s,t))dt. \tag{19}$$

   As Σ is a collection of bounded regions of $\mathbb{C}^*$,

$$\int_{\partial\Sigma} dt = 0,$$

so that, in particular,

$$\int_{\partial\Sigma} h^{\tau_i}(e^r)dt = h^{\tau_i}(e^{r_3})\int_{\partial\Sigma} dt = 0$$

and

$$\int_{\partial\Sigma} \lambda_r(X_{H^{\tau_i}})dt = \int_{\partial\Sigma} e^r(h^{\tau_i})'(e^r)dt = e^{r_3}(h^{\tau_i})'(e^{r_3})\int_{\partial\Sigma} dt = 0.$$

Thus, trivially,

$$\int_{\partial\Sigma} H^{\tau_i}(v(t))dt = \int_{\partial\Sigma} \lambda(X_{H^{\tau_i}})dt.$$

Using this equality, rewrite the energy as

$$E^{top}(v) = \int_{\partial\Sigma} v^*\lambda - \lambda(X_{H^{\tau_i}}) \otimes dt \tag{20}$$

$$= \int_{\partial\Sigma} \lambda(dv - X_{H^{\tau_i}} \otimes dt). \tag{21}$$

A solution $v(s,t)$ of Floer's equation satisfies $(dv - X_{H^{\tau_i}} \otimes dt)^{(0,1)} = 0$. As J is conical and H^{τ_i} is radially-dependent, $JX_{H^{\tau_i}}$ is proportional to ∂_r on $\{r_3\} \times \partial_- W$. Thus, $\lambda\big|_{\{r_3\}\times\partial_- W}$ vanishes on $JX_{H^{\tau_i}}\big|_{\{r_3\}\times\partial_- W}$. These observations imply that equation (6) can be written as

$$\begin{aligned} E^{top}(v) &= \int_{\partial\Sigma} -\lambda J(dv - X_{H^{\tau_i}} \otimes dt)j \\ &= \int_{\partial\Sigma} -e^r dr \circ dv \circ j \\ &= \int_{\partial\Sigma} -e^r d(r \circ v) \circ j. \end{aligned}$$

A properly-oriented boundary vector ζ on $\partial\Sigma$ implies that $j\zeta$ points inwards. Since $r \circ v$ achieves its minimum on $\partial\Sigma$, $d(r \circ v)(j\zeta) \geq 0$. The energy thus satisfies

$$E(v) \leq 0.$$

However, by definition, $E(v) \geq 0$, and so $E(v) = 0$. Unpacking the properties of Floer solutions, this condition is only satisfied if v is constant in s. We reach a contradiction: v cannot, in fact, exist.

2. Assume that $y(t)$ is a constant orbit. Thus, $y \in W_+$. Assume without loss of generality that ϵ_i is a regular value of $\rho \circ u$, and let $\Sigma = u^{-1}([\epsilon_i, R] \times \partial_- W \cup W_+)$. Note that

$H((\epsilon_i, x)) = \lambda_{(\epsilon_i,x)}(X_{H^{\tau_i}}) + \sigma$ for some constant $\sigma > 0$. While Equation (19) still holds, Σ now decomposes a priori as a collection of bounded regions in $\mathbb{C}^*$ and one unbounded region, which we call $\partial_+\Sigma$. The previous computation shows that, in fact, the bounded regions do not exist. Choose $\mathfrak{s} \in \mathbb{R}$ such that $u|_{\{\mathfrak{s}\}\times S^1} \in \Sigma$ and $|\lambda(\partial_t u(\mathfrak{s}, t))| < \sigma$ for all t. The latter condition is possible because $y(t)$ is a constant orbit to which the curves $u(s, \cdot)$ converge smoothly, and so $\lim_{s\to-\infty} \lambda(\partial_t u(s, t)) = \lambda(\partial_t y(t)) = 0$. The curves $\partial_+\Sigma$ and $\{\mathfrak{s}\} \times S^1$ bound a region in $\mathbb{R}\times S^1$, which we call Σ'. Let $v = u|_{\Sigma'}$. Note that the boundary orientation of $\{\mathfrak{s}\}\times S^1$ in Σ' is induced by $-dt$, so that

$$\int_{\partial_+\Sigma} dt = \int_{\{\mathfrak{s}\}\times S^1} dt = 1.$$

By assumption, $u|_{\{\mathfrak{s}\}\times S^1} \subset [\epsilon_i, R] \times \partial_+ W \cup W_+$, a region on which H^{τ_i} is negative. Thus, $\max_{t\in S^1} H(u(\mathfrak{s}, t)) < 0$. Applying a computation similar to the computation above, we find that

$$\begin{aligned} 0 \le E(v) &= \int_{\partial_+\Sigma} -e^{\epsilon_i} d(r\circ v)\circ j - \int_{\partial_+\Sigma} \sigma dt - \int_{\{\mathfrak{s}\}\times S^1} v^*\lambda + \int_{\{\mathfrak{s}\}\times S^1} v^* H^{\tau_i} dt \\ &< -\sigma + \sigma + \max_{t\in S^1} H(u(\mathfrak{s}, t)) \\ &< 0. \end{aligned}$$

A contradiction is again reached.

□

We have shown that $CF^*_{V,(a,b)}(H^{\tau_i}; \Gamma)$ is a subcomplex of $CF^*_{(a,b)}(H^{\tau_i}; \Gamma)$, but, recalling the definition of symplectic chains, we actually want to find continuation maps $\{c^i\}$ so that

$$\prod_{i=-1}^{-\infty} CF^*_{V,(a,b)}(H^{\tau_i}; \Gamma)[\theta] \qquad \text{is a subcomplex of} \qquad \prod_{i=-1}^{-\infty} CF^*_{(a,b)}(H^{\tau_i}; \Gamma)[\theta]$$

when equipped with the differential δ^* of equation (14).

Due to condition (6) on elements of $Ad(V, M)$, there exists a constant $\kappa_i > 0$ such that

$$H^{\tau_i}|_{(\partial_- W\times(\epsilon_{i+1},R))\cup W_+} + \kappa_i = H^{\tau_{i+1}}|_{(\partial_- W\times(\epsilon_{i+1},R))\cup W_+}$$

Let $\chi(s)$ be a bump function that is 1 when s is very negative and 0 when s is very positive. Let $\{H_s \colon \widetilde{M} \times \mathbb{R} \longrightarrow \mathbb{R}\}_{s \in \mathbb{R}}$ be an $\mathbb{R}$-family of Hamiltonians, monotone decreasing in s, such that

$$H_s\big|_{((\epsilon_{i+1},R)\times\partial_-W)\cup W_+} = H^{\tau_i}\big|_{((\epsilon_{i+1},R)\times\partial_-W)\cup W_+} + \kappa_i \cdot \chi(s).$$

After choosing a suitable almost-complex structure, H_s induces a continuation map

$$c^i \colon CF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow CF^*_{(a,b)}(H^{\tau_{i+1}};\Gamma).$$

LEMMA 2 *The continuation map* c^i *restricts to a map* $c^i \colon CF^*_{V,(a,b)}(H^{\tau_i};\Gamma) \longrightarrow CF^*_{V,(a,b)}(H^{\tau_{i+1}};\Gamma)$. .

PROOF: Let $y \subset (\epsilon_V, R] \times \partial_- W \cup W_+$ be a one-periodic orbit of $H^{\tau_{i+1}}$. Choose a neighborhood U of y inside $(\epsilon_{i+1}, R) \times \partial_- W \cup W_+$. By construction, X_{H_s} is independent of s on U. The proof of Lemma 1 now applies verbatim, being only concerned with the behavior of trajectories in the part of $\widetilde{M}$ on which H_s is s-independent.

□

Define

$$\widehat{SC_*^{V,(a,b)}}(M;\Gamma) := \prod_{i=-1}^{-\infty} CF^*_{V,(a,b)}(H^{\tau_i};\Gamma)[\theta]. \tag{22}$$

and denote the action-completion of $\widehat{SC_*^{V,(a,b)}}(M;\Gamma)$ by $\widehat{SC_*^V}(M;\Lambda)$.

LEMMA 3 *There is a chain isomorphism*

$$\widehat{SC_*}(V;\Lambda) \cong \widehat{SC_*^V}(M;\Lambda).$$

PROOF: Abuse notation slightly, and let $CF^*_{(a,b)}(H^{\tau_i}\big|_V;\Gamma)$ be the Floer complex associated to the restricted Hamiltonian $H^{\tau_i} \colon V \cup [0,\epsilon_V) \times \partial V \longrightarrow \mathbb{R}$. The elements of $CF^*_{(a,b)}(H^{\tau_i}\big|_V;\Gamma)$ are in clear bijection with the elements of $CF^*_{V,(a,b)}(H^{\tau_i};\Gamma)$. Furthermore, by the proof of Lemma 1, the differentials are canonically identified. Similarly, continuation maps are canonically identified, yielding a chain isomorphism

$$\prod_{i=-1}^{-\infty} CF^*_{(a,b)}(H^{\tau_i}|_V;\Gamma)[\theta] \cong \prod_{i=-1}^{-\infty} CF^*_{V,(a,b)}(H^{\tau_i};\Gamma)[\theta].$$

After taking action limits, the left-hand side agrees with the completed symplectic chain complex of V.

□

Choose a continuation map $c\colon CF^*_{(a,b)}(H^{\tau_{-1}};\Gamma) \longrightarrow CF^*_{(a,b)}(H^{\tau_0};\Gamma)$. This choice induces a chain map $c\colon CF^*_{V,(a,b)}(H^{\tau_{-1}};\Gamma) \longrightarrow CF^*_{(a,b)}(H^{\tau_0};\Gamma)$. Define a map $\mathfrak{c}$ by the commutative diagram

$$\begin{array}{ccc} SC^{V,(a,b)}_*(M;\Gamma) & \xrightarrow{\mathfrak{c}} & SC^*_{(a,b)}(M;\Gamma) \\ \downarrow \pi & & \uparrow \iota \\ CF^*_{V,(a,b)}(H^{\tau_{-1}};\Gamma)\theta & \xrightarrow{c} & CF^*_{(a,b)}(H^{\tau_0};\Gamma) \end{array} \tag{23}$$

As ∂^{fl} commutes with continuation maps, $\mathfrak{c}$ is a chain map.

DEFINITION 3 The (a,b)-truncated symplectic cochain complex of W is the cone of $\mathfrak{c}$. Denote it by

$$SC^*_{(a,b)}(W;\Gamma) := \mathrm{Cone}\left(\mathfrak{c}\colon SC^{V,(a,b)}_*(M;\Gamma) \longrightarrow SC^*_{(a,b)}(M;\Gamma)\right).$$

DEFINITION 4 The completed symplectic cochain complex of W is

$$\widehat{SC^*}(W;\Lambda) := \varinjlim_a \varprojlim_b SC^*_{(a,b)}(W;\Gamma)$$

The homology of this complex is denoted by $\widehat{SH^*}(W;\Lambda)$.

Analogously to the computations in Section 2.4, there is a long exact sequence

$$\cdots \longrightarrow \widehat{SH_n}(V;\Lambda) \xrightarrow{i} \widehat{SH^n}(M;\Lambda) \xrightarrow{q} \widehat{SH^n}(W;\Lambda) \longrightarrow \widehat{SH_{n+1}}(V;\Lambda) \longrightarrow \cdots \tag{24}$$

which shows

THEOREM 1 *There is a long-exact sequence*

$$\cdots \longrightarrow \widehat{SH_i}(V;\Lambda) \longrightarrow \widehat{SH^i}(M;\Lambda) \longrightarrow \widehat{SH^i}(W;\Lambda) \longrightarrow \widehat{SH_{i+1}}(V;\Lambda) \longrightarrow \cdots$$

REMARK 11) As M is monotone, the symplectic chain and cochain complexes are well-defined without truncating each Floer complex by action. Taking coefficients in an arbitrary ring R, denote these complexes by $SC_*(V;R)$ and $SC^*(M;R)$, respectively. The map $\mathfrak{c}$ is also well-defined without truncating by action; call the cone of $\mathfrak{c}$ the symplectic cochain complex of W, denoted by $SC^*(W;R)$. These three "uncompleted" complexes form a triangle analogous to (16), and taking homology results in a long exact sequence analogous to (24):

$$\cdots \longrightarrow SH_*(V) \xrightarrow{\mathfrak{c}^*} SH^*(M) \longrightarrow SH^*(W) \longrightarrow SH_{*+1}(V) \longrightarrow \cdots \tag{25}$$

Chapter 3

A non-vanishing theorem

Computing symplectic cohomology is quite difficult; it has only been computed (in the monotone case) for negative line bundles by Ritter in [43]. An easier line of inquiry is to ask, "is symplectic cohomology non-zero?" One method of answering this question affirmatively is to find a Lagrangian submanifold $L \subset M$ with non-vanishing Floer homology and show that there exists a map of unital rings $SH^*(M) \longrightarrow HF^*(L)$ from the symplectic cohomology of M to the Floer homology of L.

For example, equation (6.4) of [45] says that a monotone Lagrangian L contained in a monotone manifold M admits a map of unital rings

$$SH^*(M;\Lambda) \longrightarrow HF^*(L;\Lambda).$$

We will show that if, under suitable conditions, a Lagrangian L is contained in the Liouville cobordism $W \subset M$, then this map factors through $\widehat{SH}^*(W;\Lambda)$ via the map $q\colon \widehat{SH}^*(M;\Lambda) \longrightarrow \widehat{SH}^*(W;\Lambda)$ appearing in the long-exact sequence (24).

$$\begin{array}{ccc} \widehat{SH}^*(M;\Lambda) & \longrightarrow & HF^*(L;\Lambda) \\ & \searrow \quad \nearrow & \\ & \mathrm{Im}(q) & \\ & \cap & \\ & \widehat{SH}^*(W;\Lambda) & \end{array}$$

From this we will deduce the following theorem.

THEOREM 3 *Let* M *be a monotone symplectic manifold and* $W \subset M$ *a Liouville cobordism. Suppose that* W *contains a compact, oriented monotone Lagrangian* L. *If* L *admits a flat line bundle* E_γ *such that the Floer homology* $HF^*(L, E_\gamma) \neq 0$, *then*

$$\widehat{SH^*}(W; \Lambda) \neq 0.$$

If Λ *is defined over a coefficient field of characteristic not equal to two, we also require the Lagrangian to be spin.*

The conditions on L, orientability and monotonicity, control the behavior of Maslov discs. Recall that a Lagrangian $L \subset M$ is *monotone* if the area and the Maslov index of any J-holomorphic disc with boundary on L are positively proportional. That is, there exists a constant $c > 0$ associated to L such that for every J-holomorphic map $u\colon (D^2, \partial D^2) \longrightarrow (M, L)$ the symplectic area of u and the Maslov index $\mu(u)$ satisfy

$$\mu(u) = 2c \int_{D^2} u^* \omega. \tag{26}$$

If L is also orientable then the Maslov index of any such non-constant disc is at least two.

3.1 Lagrangian quantum cohomology

Fix a coefficient field $\mathbb{K}$. The Lagrangian Floer cohomology of a monotone Lagrangian submanifold L with coefficients in a flat line bundle E_γ is isomorphic to the Lagrangian quantum cohomology of L with coefficients twisted by $\gamma \in H^1(L)$, where the holonomies of E_γ are determined by γ. (This is stated in Section 2.4 of [12] and worked out in detail in [14] in the untwisted case.) We recall the definition of Lagrangian quantum cohomology.

Define a valuation on Λ by

$$val\colon \Lambda \longrightarrow \mathbb{R} \cup \{\infty\} \tag{27}$$

$$\sum_{n=1}^{\infty} c_n T^{k_n} = \begin{cases} \min\limits_{c_n \neq 0} k_n & \exists\, c_n \neq 0 \\ \infty & \text{else} \end{cases} \tag{28}$$

Let $U_\Lambda = val^{-1}(0)$, and fix $\gamma \in H^1(L, U_\Lambda)$. Fix a Morse-Smale pair (f, g) on L and a generic almost-complex structure J on M.

Let Φ_t be the flow of $-\nabla_g(f)$. For critical points x and y of f and an integer $\ell \geq 1$, let

$\mathcal{M}_\ell(x, y; f, g, J)$ be the moduli space of tuples $(u_1, \ldots, u_\ell)$, where

1. $u_i\colon (D^2, \partial D^2) \longrightarrow (M, L)$ is a non-constant J-holomorphic disc for all $1 \le i \le \ell$,
2. for every $1 \le i < \ell$ there exists a unique $t \in (-\infty, 0)$ such that $\Phi_t(u_{i+1}(1)) = u_i(-1)$, and
3. $u_1(1)$ lies in the unstable manifold of x and $u_\ell(-1)$ lies in the stable manifold of y.

Let $\mathrm{Aut}(D^2, \pm 1)$ be the automorphisms of the disc fixing -1 and 1, so that $\mathrm{Aut}(D^2, \pm 1)^\ell$ acts on $\mathcal{M}_\ell(x, y; f, g, J)$. Let $\mathcal{M}_0(x, y; f, g)$ be the moduli space of gradient flow lines of f with negative asymptotic limit x and positive asymptotic limit y. $\mathbb{R}$ acts on elements of $\mathcal{M}_0(x, y; f, g)$ by translation. Denote by $\mathcal{M}^0(x, y; f, g, J)$ the rigid elements of

$$\mathcal{M}_0(x, y; f, g) \,/\mathbb{R} \cup \bigcup_{\ell \ge 1} \mathcal{M}_\ell(x, y; f, g, J) \Big/ \mathrm{Aut}(D^2, \pm 1)^\ell .$$

REMARK 12) Transversality of the moduli spaces $\mathcal{M}_\ell(x, y; f, g, J)$ for generic triples (f, g, J) is not automatic. The discs may not be simple, or they may not be absolutely distinct. However, Biran-Cornea showed in [13] that somewhere-injectivity does not fail for dimension 0 and 1 strata of $\mathcal{M}_\ell(x, y; f, g, J) \Big/ \mathrm{Aut}(D^2, \pm 1)^\ell$. We can therefore use the moduli spaces $\mathcal{M}^0(x, y; f, g, J)$ to define a Floer homology theory, invariant up to generic choice of data (f, g, J).

Define a chain complex

$$CF^*(L, E_\gamma) := \bigoplus_{x \in \mathrm{Crit}(f)} \Lambda \cdot x.$$

A $\mathbb{Z}/2\mathbb{Z}$-grading on critical points is given by the Morse index, and we grade T by $|T| = 0$. The differential ∂ is given by

$$\partial(y) = \sum_{\substack{x \in \mathrm{Crit}(f) \\ \mathbf{u} = (u_1, \ldots, u_\ell) \\ \in \mathcal{M}^0(x, y; f, g, J)}} \pm T^{\omega([u_1] + \cdots + [u_\ell])} \langle \gamma, [\partial u_1] + \cdots + [\partial u_\ell] \rangle \cdot x.$$

The differential counts weighted 'pearly trajectories' between x and y (see Figure 3.1). The sign is determined by a choice of orientations on the unstable manifolds of critical points of f and a choice of spin structure on L. (See the Appendix of [12] for a careful discussion of orientations, in particular Section A.2, or [53] for orientations using orientation lines.) As shown in [12], ∂ is well-defined and squares to zero. The homology of $CF^*(L, E_\gamma)$ is the Lagrangian Floer homology $HF^*(L, E_\gamma)$.

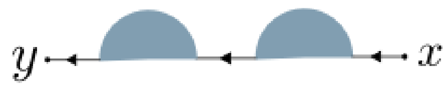


Figure 3.1: The Lagrangian quantum differential

Define the action $\mathcal{A}$ of an element $T^k x$, where $x \in \mathrm{Crit}(f)$ and $k \in \mathbb{R}$, to be $\mathcal{A}(T^k x) = k$. As J-holomorphic discs have non-negative area, the quantum differential increases $\mathcal{A}$. We may thus consider the subcomplex

$$CF^*_a(L, E_\gamma) := \left\langle T^k x \;\middle|\; k \in \mathbb{R}, , x \in \mathrm{Crit}(f), \mathcal{A}(T^k x) > a \right\rangle,$$

with cohomology denoted by $HF^*_a(L, E_\gamma)$, and the quotient complex

$$CF^*_{(a,b)}(L, E_\gamma) := {CF^*_a(L, E_\gamma)} \Big/ {CF^*_b(L, E_\gamma)},$$

with cohomology denoted by $HF^*_{(a,b)}(L, E_\gamma)$.

Let $\Lambda_{>0} = val^{-1}((0, \infty])$, where val is as defined in (27). As we will be working with action-truncated complexes, define

$$\Lambda_a = T^a \Lambda_{>0}$$

and let

$$\Lambda_{(a,b)} = {\Lambda_a} / {\Lambda_b}.$$

Note that, by definition, $\Lambda = \varinjlim_a \varprojlim_b \Lambda_{(a,b)}$, and

$$CF^*_{(a,b)}(L, E_\gamma) = \bigoplus_{x \in \mathrm{Crit}(f)} \Lambda_{(a,b)} \cdot x \tag{29}$$

LEMMA 4

$$\varinjlim_a \varprojlim_b HF^*_{(a,b)}(L, E_\gamma) \cong HF^*(L, E_\gamma) \cong H^*\left(\varinjlim_a \varprojlim_b CF^*_{(a,b)}(L, E_\gamma) \right).$$

PROOF: The first isomorphism is a canonical identification, analogous to Remark 5. To see the second isomorphism, note that the right-hand sum of (29) is finite and so commutes with both inverse and direct limits. Thus,

$$\varinjlim_{a}\varprojlim_{b} CF^*_{(a,b)}(L, E_\gamma) = \varinjlim_{a}\varprojlim_{b} \bigoplus_{x\in Crit(f)} \Lambda_{(a,b)}\cdot x \tag{30}$$

$$\cong \bigoplus_{x\in Crit(f)} \varinjlim_{a}\varprojlim_{b} \Lambda_{(a,b)}\cdot x \tag{31}$$

$$\cong \bigoplus_{x\in Crit(f)} \Lambda\cdot x \tag{32}$$

$$\cong CF^*(L, E_\gamma). \tag{33}$$

As the quantum differential is T-linear, the result follows. □

REMARK 13) The Lagrangian Floer cohomology of L, $HF^*(L, E_\gamma)$, is a unital ring. Suppose $HF^*(L, E_\gamma) \neq 0$. If f has a unique minimum m, then m represents the unit, and therefore survives in cohomology.

3.2 Proof of Theorem 3

We first define a map $SC^*_{(a,b)}(M;\Gamma) \longrightarrow CF^*_{(a,b)}(L, E_\gamma)$. This map will count 'half-cylinder' solutions to Floer's equation that rise asymptotically to generators of some $CF^*_{(a,b)}(H^{\tau_i};\Gamma)$ and whose boundary lies in L.

To these ends, fix a Hamiltonian H^{τ_i}. Recall the function $\mathcal{H}$ built into the definition of H^{τ_i} (see Section 2.2), and without loss of generality assume that $\mathcal{H} < 0$ in a neighborhood of L. Let $\{H^{\tau_i}_s\}_{s\in[0,\infty)}$ be a one-parameter family of Hamiltonians such that $H^{\tau_i}_s = H^{\tau_i}$ for $s >> 0$, and $H^{\tau_i}_s(x) = 0$ when both s is close to zero and x is close to L. Further assume that $H^{\tau_i}_s$ is monotone decreasing in s. These conditions will ensure that we create a well-defined chain map.

For a periodic solution x of $X_{H^{\tau_i}}$ and generic one-parameter family of cylindrical almost-complex structures $\{J_s\}_{s\in[0,\infty)}$, let $\mathcal{M}(x, L; H^{\tau_i})$ be the moduli space of maps $w\colon [0,\infty)\times S^1 \longrightarrow M$ satisfying

1. $\lim_{s\to\infty} w(s,t) = x(t)$,
2. $w|_{\{0\}\times S^1} \in L$, and $(\star)$
3. $\partial_s(w) + J_s(\partial_t(w) - X_{H^{\tau_i}_s}) = 0.$

Fix a Morse-Smale pair (f, g) on L so that f has a unique minimum $\mathfrak{m}$. Let Φ_t be the flow of $-\nabla_g(f)$. For each integer $\ell \geq 1$ and $p \in \mathrm{Crit}(f)$, let $\mathcal{M}_\ell(p, x; H^{\tau_i})$ be the moduli space of tuples $(u_1, \ldots, u_\ell)$, where

1. $u_i : (D^2, \partial D^2) \longrightarrow (M, L)$ is a non-constant J-holomorphic disc for all $1 \leq i \leq \ell - 1$,

2. $u_\ell \in \mathcal{M}(x, L; H^{\tau_i})$,

3. $u_1(1)$ is in the unstable manifold of p, and

4. for every $1 \leq i < \ell$ there exists a unique $t \in (-\infty, 0)$ such that $\Phi_t(u_{i+1}(1)) = u_i(-1)$.

Let $\mathrm{Aut}(D^2, \pm 1)$ be the automorphisms of the disc fixing -1 and 1, so that $\mathrm{Aut}(D^2, \pm 1)^{\ell-1}$ acts on $\mathcal{M}_\ell(p, x; H^{\tau_i})$. Let

$$\mathcal{M}(p, x; H^{\tau_i}) = \bigcup_{\ell \geq 1} \mathcal{M}_\ell(p, x; H^{\tau_i}) \Big/ \mathrm{Aut}(D^2, \pm 1)^{\ell - 1}.$$

Denote by $\mathcal{M}^0(p, x; H^{\tau_i})$ the rigid elements of $\mathcal{M}(p, x; H^{\tau_i})$. Fix a class $\gamma \in H^1(L; U_\Lambda)$. Fix a spin structure on L and Morse orientations, so that any element $\mathbf{u} \in \mathcal{M}^0(p, x; H^{\tau_i})$ determines a map $d_{\mathbf{u}} : o_x \longrightarrow \pm 1$. Let

$$\omega(\mathbf{u}) = \omega(u_\ell \# (-\tilde{x})) + \omega(u_1) + \cdots + \omega(u_{\ell-1}) \tag{34}$$

and

$$[\partial \mathbf{u}] = [\partial u_1] + \cdots + [\partial u_{\ell-1}] + [u_\ell(0, \cdot)]. \tag{35}$$

Define a T-linear map

$$\begin{aligned} \iota^{\tau_i}_{(a,b)} : CF^*_{(a,b)}(H^{\tau_i}; \Gamma) &\longrightarrow CF^*_{(a,b)}(L, E_\gamma) \\ \zeta_x &\mapsto \sum_{p \in \mathrm{Crit}(f)} \sum_{\substack{\mathbf{u} = (u_1, \ldots, u_\ell) \\ \in \mathcal{M}^0(p, x; H^{\tau_i})}} d_{\mathbf{u}}(\zeta_x) T^{\omega(\mathbf{u})} \langle \gamma, [\partial \mathbf{u}] \rangle \cdot p. \end{aligned}$$

Geometrically, we are mapping a cylinder $\tilde{x} \in M$ to the sum of cylinders $-\tilde{x} \# w$ with boundary on L formed by gluing $\tilde{x}$ to the cylinder w, and adding in 'pearls' from pearly trajectories between $w(0, 1)$ and p. See Figure 3.2.

LEMMA 5 $\iota^{\tau_i}_{(a,b)}$ *is well-defined and descends to a map on homology.*

The techniques sketched to prove Lemma 5 appear in detail in [12] and [13] in the case $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$. They also appear in [53] for arbitrary $\mathbb{K}$ with a slight modification: the disc with one interior punc-

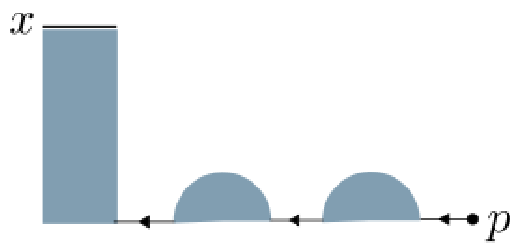


Figure 3.2: The map ι^{τ_i}

ture appearing in $\mathcal{M}(x, L)$ is replaced by a disk with one boundary puncture, and the Hamiltonians H_s^n are "turned off" away from the puncture, instead of at all points of the disc's boundary. The discussion of orientations in [53] does not depend on this modification, and so translates precisely to our setup. We therefore omit a discussion of orientations and refer to [53] for signs.

SKETCH OF PROOF: We first check that $\iota^{\tau_i}_{(a,b)}$ respects the action filtration. A standard computation shows that, if $(u_1, \dots, u_\ell) \in \mathcal{M}^0(p, x; H^{\tau_i})$, then the energy of u_ℓ is

$$E(u_\ell) = \omega(u_\ell \#(-\tilde{x})) - \mathcal{A}_{H^{\tau_i}}(x) + \int_{\mathbb{R}\times S^1} (\partial_s H_s^{\tau_i})(u_\ell)\, ds dt.$$

Since the energy is always non-negative,

$$\mathcal{A}_{H^{\tau_i}}(x) - \int_{\mathbb{R}\times S^1} (\partial_s H_s^{\tau_i})(u_\ell)\, ds dt \leq \omega(u_\ell \#(-\tilde{x})), \text{ so}$$
$$\mathcal{A}_{H^{\tau_i}}(x) \leq \omega(u_\ell \#(-\tilde{x}))$$

by the assumption that $H_s^{\tau_i}$ is monotone decreasing in s. Finally, $\gamma \in H^1(L; U_\Lambda)$ implies that $\langle \gamma, w(0, \cdot)\rangle \in U_\Lambda$, and each u_i is J-holomorphic, so $\omega(u_i) \geq 0$. It follows that

$$T^{\omega(u)}\langle \gamma, w(0, \cdot)\rangle \in \Lambda_{\omega(-\tilde{x}\# u_\ell)}.$$

Let $\mathcal{S}$ be a stratum of the set of maps $\{\mathbf{u} = (u_1, \dots, u_\ell) \in \mathcal{M}(p, x; H^{\tau_i})\}$ of fixed homology class A: $[-\tilde{x}\# u_\ell] + [u_1] + \cdots + [u_\ell - 1] = A \in H_2(M, L \cup \beta) \big/ \ker(\omega)$, where β is the fixed representative in $[x]$ (see section 2.1). Assume that $\dim(\mathcal{S}) \leq 1$, as these are the only strata contributing to the study of $\iota^{\tau_i}_{(a,b)}$. The transversality for pearly trajectories proved in Section 3 of [13] and the transversality for half-tubes discussed in [6] show that $\mathcal{S}$ is cut out transversely whenever the virtual dimension of $\mathcal{S}$ is less than or equal to 1, and thus, by regularity, whenever $\dim(\mathcal{S}) \leq 1$. We therefore only need to show that bubbling does not contribute to compactification.

There are six types of limit points that, a priori, contribute to the compactification of $\mathcal{S}$ (see Figure 3.3).

a) cylinder breaking contributing to $\iota^{\tau_i}_{(a,b)} \circ \partial^{fl}$,

b) pearly-trajectory breaking contributing to $\partial \circ \iota^{\tau_i}_{(a,b)}$

c) sphere-bubbling,

d) side-bubbling, where a disc bubbles off at a boundary point $u_i(q)$, where $q \neq \pm 1$ if $i < \ell$ and $q \neq 1$ if $i = \ell$,

e) disc bubbling at $q = u_i(\pm 1)$ (when $i < \ell$) or at $q = u_\ell(1)$, and

f) Morse-trajectory shrinking, where the trajectory between some u_i and u_{i+1} collapses, causing $u_i(-1)$ and $u_{i+1}(1)$ to collide.

It thus suffices to show that, if $\mathcal{S}$ has dimension less than 2, the sum contribution of types (c), (d), (e), and (f) is zero.

We first tackle (c) and (d). By monotonicity and orientability of L, the virtual dimension of side-bubbling or sphere-bubbling is at least 2. If either occurs in a limit, then the other component of the limit is a stratum $\mathcal{S}'$ of $\mathcal{M}(p, x; H^{\tau_i})$ of virtual dimension two less than the virtual dimension of $\mathcal{S}$. But by regularity this implies that $\dim(\mathcal{S}') \leq \dim(\mathcal{S}) - 2 < 0$. This shows that (c) and (d) cannot occur.

There is a canonical bijection between elements of type (e) and elements of type (f). An analysis of signs shows that an element of type (e) contributes with the opposite sign of its type (f) partner. See Section A.2.1 in [12] and Theorem 5.1 in [53] for a careful treatment of signs. Thus, counting the limit points of both types (e) and types (f) yields zero.

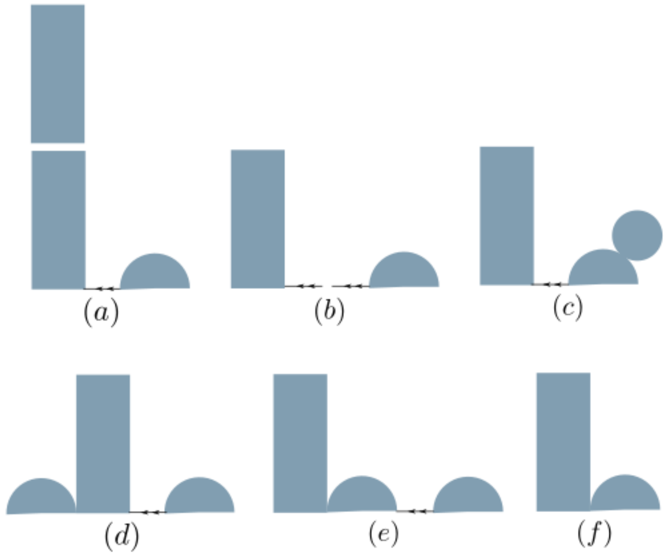


Figure 3.3: Degenerations of $\mathcal{S}$

If $\dim(\mathcal{S}) = 0$ then limits of types (a) and (b) cannot occur for index reasons, proving that $\iota^{\tau_i}_{(a,b)}$ is well-defined.

If $\dim(\mathcal{S}) = 1$ then the analysis of the boundary yields the equivalence

$$\iota^{\tau_i}_{(a,b)} \circ \partial^{fl}(x) = \partial \circ \iota^{\tau_i}_{(a,b)}(x), \tag{36}$$

as desired.

□

Let $(\iota^{\tau_i}_{(a,b)})^* \colon HF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow HF^*_{(a,b)}(L, E_\gamma)$ be the descent of $\iota^{\tau_i}_{(a,b)}$ to cohomology.

LEMMA 6 *The collection of maps* $\left\{(\iota^{\tau_i}_{(a,b)})^*\right\}_{i\in\mathbb{N}}$ *induces a map* $\varinjlim_i HF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow HF^*_{(a,b)}(L, E_\gamma)$.

PROOF: We must show that $(\iota^{\tau_{i+1}}_{(a,b)})^* \circ (c^i)^* = (\iota^{\tau_i}_{(a,b)})^*$. We will do this by finding a chain homotopy S such that

$$\iota^{\tau_{i+1}}_{(a,b)} \circ c^i - \iota^{\tau_i}_{(a,b)} = S \circ \partial^{fl} + \partial \circ S.$$

Let $x \in \mathcal{P}(H^{\tau_i})$. Choose a regular homotopy $\{H_s\}_{s\in\mathbb{R}}$ with $H_s = H^{\tau_i}$ when $s > 0$ and $H_s = H^{\tau_{i+1}}$ when $s < -1$.

Choose a generic smooth family of Hamiltonians $H\colon [0,\infty)\times\mathbb{R}\times S^1\times M \longrightarrow \mathbb{R}$ such that $H\big|_{\{\mathfrak{s}\}\times\{s\}\times S^1\times M}$ is equal to $H_s^{\tau_{i+1}}$ when $s<\mathfrak{s}-1$ and $H_s^{\tau_i}$ when $s>\mathfrak{s}$. Assume that, for $s\in[-1,0]$,

$$\lim_{\mathfrak{s}\to\infty}\left(H(\mathfrak{s},s+\mathfrak{s},t,x)-H_s(t,x)\right)=0.$$

Choose a generic $[0,\infty)$-family of domain-dependent cylindrical almost-complex structures $J_{s,\mathfrak{s}}$. Fix $p\in\mathrm{Crit}(f)$. Let $\mathcal{M}^1(p,x,A;H)$ be the one-dimensional strata of the space of tuples $(\mathfrak{s},(u_1,\dots,u_\ell)_{\mathfrak{s}})$, where $\mathfrak{s}\in[0,\infty)$ and $(u_1,\dots,u_\ell)_{\mathfrak{s}}\in\mathcal{M}^0(p,x,A;H_{\mathfrak{s}})$. Also require that $[-\tilde{x}\# u_\ell]_{\mathfrak{s}}+[u_1]_{\mathfrak{s}}+\cdots+[u_\ell]_{\mathfrak{s}}=A$ is a fixed class in $H_2(M,L\cup\beta)\big/\ker(\omega)$ for every $\mathfrak{s}$, where β is the fixed representative in $[x]$ (see section 2.1). This is a 1-dimensional manifold with a compactification given by cylinder-breaking and disc-bubbling. A priori, the (0-dimensional) boundary components of the compactification take one of six forms.

a) On the boundary $\mathfrak{s}=0$ appears the elements of the moduli space $\mathcal{M}^0(p,x,A;H^{\tau_i})$. This corresponds to the part of $\iota^{\tau_i}_{(a,b)}(\zeta_x)$ that contributes terms of action $\omega(A)$, twisted by γ.

b) In the limit $\mathfrak{s}\longrightarrow\infty$ appear elements of the product

$$\bigsqcup_{\substack{z\in\mathcal{P}(H^{\tau_{i+1}})\\ B\#C=A}}\mathcal{M}^0(p,z,B;H^{\tau_{i+1}})\times\mathcal{M}^0(z,x,C;H_s),$$

where $\mathcal{M}^0(z,x,C;H_s)$ is the space of index-0 Floer solutions between x and z induced by the family of Hamiltonians H_s, and contributing terms in '$T^{\omega(C)}o_z$' to $c^i(\zeta_x)$. This product contributes terms of action $\omega(A)$ to $\iota^{\tau_{i+1}}_{(a,b)}\circ c^i(\zeta_x)$, twisted by γ.

c) The moduli space can degenerate at an interior point $\mathfrak{s}\in(0,\infty)$, and near $s=\infty$, yielding elements of the form

$$\bigsqcup_{\substack{z\in\mathcal{P}(H^{\tau_i})\\ \mu(z)-\mu(x)=1\\ B\#C=A}}\mathcal{M}^{-1}(p,z,B;H^{\tau_i}_{\mathfrak{s}})\times\mathcal{M}^0(z,x,C;H^{\tau_i}),$$

where $\mathcal{M}^{-1}(p,z,B;H^{\tau_i}_{\mathfrak{s}})$ is the moduli space of rigid Floer/pearly trajectory amalgamates of virtual dimension -1 that can occur between z and p if $H_{\mathfrak{s}}$ is not regular (restricted, of course, to the relative homology class B).

d) The moduli space can degenerate at an interior point $\mathfrak{s}\in(0,\infty)$ and within a pearly trajectory, yielding elements of the form

$$\bigsqcup_{\substack{q\in \mathrm{Crit}(f)\\ B\# C=A\\ |p|-|q|-\mu(C)=0}} \mathcal{M}^0(p,q,C)\times \mathcal{M}^{-1}(q,x,B;H^{\tau_i}_{\mathfrak{s}}).$$

e) Finally, bubbling may occur. However, as in the proof of Lemma 5, the contribution of disc and sphere bubbling is zero.

Standard gluing techniques show that the degenerations of types (a) – (d) do indeed appear. In the notation of equations (34) and (35), define S on generators by

$$S\colon \bigoplus_{i\in\mathbb{N}} CF^*_{(a,b)}(H^{\tau_i};\Gamma)\longrightarrow CF^*_{(a,b)}(L,E_\gamma)$$

$$\zeta_x \mapsto \sum_{\substack{\mathfrak{s}\in(0,\infty)\\ B\in H_2(M,L)\\ q\in \mathrm{Crit}(f)}} \sum_{u\in\mathcal{M}^{-1}(q,x,B;H^{\tau_i}_{\mathfrak{s}})} d_u(\zeta_x)T^{\omega(u)}\langle\gamma,[\partial u]\rangle\cdot q$$

and extend T-linearly.

As the limiting degenerations at $\mathfrak{s}=\infty$ are regular it follows from Gromov compactness that there are finitely many degenerations of types (c), (d), and (e), and so S is well-defined.

Counting boundary components of type (c) yields the "$T^{\omega(A)}$" component of $S\circ\partial^{fl}$ and counting boundary components of type (d) yields the "$T^{\omega(A)}$" component of $\partial\circ S$. From this we deduce that

$$\iota^{\tau_{i+1}}_{(a,b)}\circ c^i-\iota^{\tau_i}_{(a,b)}=S\circ\partial^{fl}+\partial\circ S.$$

□

There is a surjective map

$$\Psi\colon \widehat{SH^*}(M;\Lambda)\longrightarrow \varinjlim_a \varprojlim_b \varinjlim_i HF^*_{(a,b)}(H^{\tau_i};\Gamma),$$

defined using the natural commutativity of direct limits with cohomology and the projection

$$H^*\left(\varprojlim_b SC^*_{(a,b)}(M;\Gamma)\right)\longrightarrow \varprojlim_b SH^*_{(a,b)}(M;\Gamma)$$

that appears in the Milnor exact sequence

$$0 \longrightarrow \varprojlim_b{}^1 SH^*_{(a,b)}(M) \longrightarrow H^*\left(\varprojlim_b SC^*_{(a,b)}(M;\Gamma)\right) \longrightarrow \varprojlim_b SH^*_{(a,b)}(M;\Gamma) \longrightarrow 0.$$

Under the equivalence $HF^*(L, E_\gamma) \cong \varinjlim_a \varprojlim_b HF^*_{(a,b)}(L, E_\gamma)$ proved in Lemma 4, let

$$\widehat{\iota^*}\colon \varinjlim_a \varprojlim_b \varinjlim_i HF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow HF^*(L, E_\gamma)$$

be the induced map formed by taking limits over the maps $\left\{\iota^{\tau_i}_{(a,b)}\right\}_{i,a,b}$. Define

$$\widehat{\mathcal{J}^*} = \widehat{\iota^*} \circ \Psi\colon \widehat{SH^*}(M;\Lambda) \longrightarrow HF^*(L, E_\gamma).$$

LEMMA 7 *The map $\widehat{\mathcal{J}^*}$ is non-vanishing.*

PROOF: As Ψ is a surjection, it suffices to prove that $\widehat{\iota^*}$ is non-vanishing.

As in [47], we build a representative of the unit of $\widehat{SH^*}(M;\Lambda)$ inside $HF^*(H^{\tau_0})$. Let H_s be an $\mathbb{R}$-family of Hamiltonians that is equal to H^{τ_0} when $s << 0$ and identically zero when $s >> 0$. Choose a generic $\mathbb{R}$-family of cylindrical almost-complex structures J_s. Let $\mathcal{M}^0(x)$ be the set of finite-energy rigid maps $w\colon \mathbb{R} \times S^1 \cup \{\infty\} \longrightarrow M$ satisfying $\partial_s w + J_s(\partial_t w - X_{H_s}) = 0$, and such that $\lim_{s\to-\infty} w(s,\cdot) = x(\cdot)$. Each w determines an element $\zeta^w_x \in o_x$. Let

$$Z = \sum_{\substack{x\in\mathcal{P}(H^{\tau_0})\\ w\in\mathcal{M}^0(x)}} T^{\omega(\tilde{x}\# w)} \cdot \zeta^w_x \in CF^*(H^{\tau_0};\Lambda).$$

The usual analysis of the boundary of a dimension-one moduli space of curves shows that Z is a well-defined cycle.

Recall that we chose the Morse function f on L to have a unique minimum $\mathfrak{m}$. For $p \in \mathrm{Crit}(f)$, let $\mathcal{M}^0(p, A)$ be the space of rigid pearly trajectories $\{(u_1, \dots, u_\ell)\}_{\ell\geq 1}$, defined in the same way as $\mathcal{M}^0(p, x, A; H^{\tau_0})$, but where u_ℓ is also now a (possibly constant) J-holomorphic disc with one interior marked point. If u_ℓ is not constant, the sequence $(u_1, \dots, u_\ell)$ is not rigid. If u_ℓ is constant and $p \neq \mathfrak{m}$ then either

1. there is a non-constant gradient trajectory $\beta(t)$ with $\beta(0) = \mathrm{Im}(u_\ell)$, in which case sliding the image of u_ℓ along $\beta(t)$ shows that $(u_1, \dots, u_\ell)$ is not rigid, or

2. $\mathrm{Im}(u_\ell) \in \mathrm{Crit}(f) \setminus \mathfrak{m}$. Then $\ell > 1$ and $\phi_t(u_\ell(1)) = u_{\ell-1}(-1)$ for all $t \in (-\infty, 0)$, contradicting the uniqueness assumption of Condition (4).

Thus,

$$\bigsqcup_{\substack{p \in \mathrm{Crit}(f) \\ A \in H_2(M,L)}} \mathcal{M}^0(p, A) = \{\mathfrak{m}\}.$$

Define $\iota^{\tau_0} \colon CF^*(H^{\tau_0};\Gamma) \longrightarrow CF^*(L, E_\gamma)$ analogously to $\iota^{\tau_0}_{(a,b)}$, but without truncating by action. As in the proof of Lemma 6,

$$\iota^{\tau_0}(Z) = \mathfrak{m} + \partial \circ S(Z)$$

for some chain homotopy S, and so $(\iota^{\tau_0})^*([Z]) = [\mathfrak{m}]$.

By assumption, $HF^*(L, E_\gamma)$ is a non-zero unital ring with unit represented by $\mathfrak{m}$ (see Remark 13). It follows that $(\iota^{\tau_0})^*(Z) \neq 0$.

As Z is a cycle, ι^{τ_0} descends to a non-trivial map $(\iota^{\tau_0})^*$ on homology. We will show that the non-vanishing of this map implies Lemma 7.

Let $(\widehat{\iota^{\tau_0}})^* = \varinjlim_a \varprojlim_b (\iota^{\tau_0}_{(a,b)})^*$. Analogously to Lemma 4 there is a quasi-isomorphism

$$\varinjlim_a \varprojlim_b HF^*_{(a,b)}(H^{\tau_0};\Gamma) \cong HF^*(H^{\tau_0};\Lambda).$$

This isomorphism induces a commutative diagram

$$\begin{array}{ccc}
\varinjlim_a \varprojlim_b HF^*_{(a,b)}(H^{\tau_0};\Gamma) & \xrightarrow{\simeq} & HF^*(H^{\tau_0};\Lambda) \\
\downarrow{\scriptstyle (\widehat{\iota^{\tau_0}})^*} & & \downarrow{\scriptstyle (\iota^{\tau_0})^*} \\
\varinjlim_a \varprojlim_b HF^*_{(a,b)}(L, E_\gamma) & \xrightarrow{\simeq} & HF^*(L, E_\gamma)
\end{array}$$

It follows that $(\widehat{\iota^{\tau_0}})^* \neq 0$.

The inclusion $CF^*_{(a,b)}(H^{\tau_0};\Gamma) \hookrightarrow \varinjlim_i CF^*_{(a,b)}(H^{\tau_i};\Gamma)$ induces a map

$$\varinjlim_a \varprojlim_b HF^*_{(a,b)}(H^{\tau_0};\Gamma) \longrightarrow \varinjlim_a \varprojlim_b \varinjlim_i HF^*_{(a,b)}(H^{\tau_i};\Gamma).$$

By the construction of $(\widehat{\iota^{\tau_0}})^*$, the following diagram commutes

$$\begin{array}{ccc}
\varinjlim_a \varprojlim_b HF^*_{(a,b)}(H^{\tau_0};\Gamma) & \longrightarrow & \varinjlim_a \varprojlim_b \varinjlim_i HF^*_{(a,b)}(H^{\tau_i};\Gamma) \\
& \searrow^{(\widehat{\iota^{\tau_0}})^*} & \downarrow \widehat{\iota}^* \\
& & HF^*(L,E_\gamma)
\end{array}$$

from which we deduce that $\widehat{\iota}^*$, and therefore $\widehat{\mathcal{J}}^*$, is non-vanishing.

$\square$

We are now in position to prove Theorem 3.

PROOF: The long-exact sequence (24) induces a short-exact sequence

$$0 \longrightarrow \widehat{SH}_*(V;\Lambda)/\mathrm{Ker}(i^*) \longrightarrow \widehat{SH}^*(M;\Lambda) \longrightarrow \mathrm{Im}(q^*) \longrightarrow 0$$

where $\mathrm{Im}(q^*) \subset \widehat{SH}^*(W;\Lambda)$.

We want to show that $\widehat{\mathcal{J}}^*$ factors through $\mathrm{Im}(q^*)$. By the universal property of quotients, it suffices to show that $\widehat{\mathcal{J}}^* \circ i^*$ is zero on $\widehat{SH}_*(V;\Lambda)$. Fix $A \in \mathbb{R}$. The Mittag-Leffler condition is trivially satisfied for each inverse system $\{SC_*^{(A,b)}(V;\Gamma)\}_b$ and $\{\Lambda_{(A,b)}\}_b$ (the maps defining the inverse systems are surjective). To each inverse system is therefore associated a Milnor $\lim^1$ short exact sequence. By naturality of this sequence, and as direct limits preserve exactness, there is a commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varinjlim_a \varprojlim_b {}^1 SH_*^{(a,b)}(V;\Lambda) & \longrightarrow & \widehat{SH}_*(V;\Lambda) & \longrightarrow & \varinjlim_a \varprojlim_b SH_*^{(a,b)}(V;\Lambda) & \longrightarrow & 0 \\
& & \downarrow i^* & & \downarrow i^* & & \downarrow i^* & & \\
0 & \longrightarrow & \varinjlim_a \varprojlim_b {}^1 SH^*_{(a,b)}(M;\Lambda) & \longrightarrow & \widehat{SH}^*(M;\Lambda) & \longrightarrow & \varinjlim_a \varprojlim_b SH^*_{(a,b)}(M;\Lambda) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \widehat{\mathcal{J}}^* & & \downarrow \widehat{\iota}^* & & \\
0 & \longrightarrow & 0 & \longrightarrow & HF^*(L,E_\gamma) & \xrightarrow{\simeq} & \varinjlim_a \varprojlim_b HF^*_{(a,b)}(L,E_\gamma) & \longrightarrow & 0
\end{array}$$

It thus suffices to show that the map $\varinjlim_a \varprojlim_b SH_*^{(a,b)}(V;\Lambda) \longrightarrow \varinjlim_a \varprojlim_b HF^*_{(a,b)}(L,E_\gamma)$ is zero. In particular, it suffices to show that each map $SH_*^{(a,b)}(V;\Lambda) \longrightarrow HF^*_{(a,b)}(L,E_\gamma)$ is zero.

Let $\mathbf{X} \in SH_*^{(a,b)}(V;\Lambda)$. A representative cochain of $\mathbf{X}$ is of the form

$$X := \{\zeta_i\} + \{\eta_i\}\theta \in \prod_{i=-1}^{-\infty} CF^*_{V,(a,b)}(H^{\tau_i};\Gamma)[\theta].$$

We will show that $\left(\iota_{(a,b)} \circ i\right)(X) = 0$.

Define a map $\iota^{\tau_i}_{(a,b)} \colon CF^*_{(a,b)}(H^{\tau_i};\Gamma) \longrightarrow CF^*_{(a,b)}(L, E_\gamma)$ by extending the construction of the maps defined in Section 3 to the Floer cochain complexes of negatively-indexed Hamiltonians. We choose Hamiltonians $H^{\tau_i}_s$ to define each $\iota^{\tau_i}_{(a,b)}$ so that $X_{H^{\tau_i}_s}$ agrees with $X_{H^{\tau_0}_s}$ on $W_+ \cup (\epsilon_{i+1}, R) \times \partial_- W$ for all s. In particular, $H^{\tau_i}_s(x)$ is constant when both x is close to L and s is close to 0, and $|H^{\tau_i}_s(q) - H^{\tau_i}(q)|$ is small for all s and $q \in L$. We require $H^{\tau_i}_s$ to be monotone-decreasing in s and $H_s\big|_L < 0$. These final conditions ensure that the $\iota^{\tau_i}_{(a,b)}$ respects the action-filtration.

By definition,

$$\left(\iota_{(a,b)} \circ i\right)^* ([X]) = (\iota_{(a,b)} \circ c^{-1})^*([\eta_{-1}]) = (\iota^{\tau_0}_{(a,b)} \circ c^{-1})^*([\eta_{-1}]) = (\iota^{\tau_{-1}}_{(a,b)})^*([\eta_{-1}]), \tag{37}$$

where the last equality follows from the equality

$$(\iota^{\tau_i}_{(a,b)} \circ c^{i-1})^* = (\iota^{\tau_{i-1}}_{(a,b)})^* \tag{38}$$

derived in the proof of Lemma 6.

The first term in the equation $0 = \delta^*(X)$ is

$$0 = \left\{\partial^{fl}(\zeta_i) + c^{i-1}(\eta_{i-1}) - \eta_i\right\}_{i=-1}^{-\infty}$$

Composing with $\iota^{\tau_i}_{(a,b)}$ yields the componentwise equalities

$$\iota^{\tau_i}_{(a,b)}(\eta_i) = \iota^{\tau_i}_{(a,b)} \circ \partial^{fl}(\zeta_i) + \iota^{\tau_i}_{(a,b)} \circ c^{i-1}(\eta_{i-1}).$$

The proof of Lemma 5 shows that $(\iota^{\tau_i}_{(a,b)} \circ \partial^{fl})^* = 0$, and so

$$(\iota^{\tau_i}_{(a,b)})^*([\eta_i]) = (\iota^{\tau_i}_{(a,b)} \circ c^{i-1})^*([\eta_{i-1}]). \tag{39}$$

for any i. If $(\iota^{\tau_0}_{(a,b)} \circ c^{-1})^*([\eta_{-1}]) \neq 0$, then by equations (37), (38), and (39),

$$(\iota^{\tau_i}_{(a,b)})^*([\eta_i]) = (\iota^{\tau_{-1}}_{(a,b)})^*([\eta_{-1}]) \neq 0. \tag{40}$$

By assumption, the action $\mathcal{A}$ of every non-zero summand of $\iota^{\tau_i}_{(a,b)}(\eta_i)$ is in (a,b). As $\lim\limits_{i\to -\infty} \tau_i = -\infty$, we may choose i so that

$$\sup_{q\in L} H^{\tau_i}_0(q) < a - b. \tag{41}$$

If $\eta_i = 0$ for all $i >> 0$ then (40) is a contradiction. Otherwise, choose a large i so (41) holds and $\eta_i \neq 0$. Write $\eta_i = \sum_{\ell=1}^{m_i} \eta_i^\ell$, where $\eta_i^\ell = a_\ell T^{\alpha_\ell} \mathfrak{y}_i^\ell$ for some $\mathfrak{y}_i^\ell \in o_{x_\ell}; x_\ell \in \mathcal{P}(H^{\tau_i})$, $\alpha_\ell \in \mathbb{R}$, and $a_\ell \in \mathbb{K}^*$, and where $\mathcal{A}_{H^{\tau_i}}(\eta_i^\ell) \in (a,b)$. Suppose that $w(s,t)$ is a Floer solution of $X_{H^{\tau_i}_s}$ with positive asymptotic limit some x_ℓ. A contribution of w to $\iota^{\tau_i}_{(a,b)}(\eta_i)$ is of the form $kT^{\alpha_\ell + \omega(w\#(-\widetilde{x_\ell}))+\kappa}p$, where $k \in \mathbb{K}$, $p \in \mathrm{Crit}(f)$ and $\kappa \geq 0$ is the (necessarily non-negative) area of the J-holomorphic discs in a rigid pearly trajectory starting at p and ending at $w(0,1)$.

Recall that the action is decreased by Floer trajectories induced by monotone-decreasing homotopies. By assumption, $\alpha_\ell + \mathcal{A}_{H^{\tau_i}}(x_\ell) \in (a,b)$. Thus,

$$a < \alpha_\ell + \mathcal{A}_{H^{\tau_i}}(x_\ell) \leq \alpha_\ell + \omega(w\#(-\widetilde{x}_\ell)) + \int_0^1 H^{\tau_i}_0(w(0,t))dt,$$

and so by (41) and the non-negativity of κ,

$$b = a + (b-a) < a - \int_0^1 H^{\tau_i}_0(w(0,t))dt < \alpha_\ell + \omega(-\widetilde{x}_\ell\#w) + \kappa.$$

It follows that $0 = [kT^{\alpha_\ell+\omega(w\#(-\widetilde{x_\ell}))+\kappa}p] \in CF^*_{(a,b)}(L;E_\gamma)$. We conclude that

$$(\iota^{\tau_0}_{(a,b)} \circ c^{-1})^*([\eta_{-1}]) = (\iota^{\tau_i}_{(a,b)})^*([\eta_i]) = 0.$$

□

EXAMPLE 1) The total space of the line bundle $\mathcal{O}(-k) \longrightarrow \mathbb{C}P^m$ is monotone whenever $1 \leq k \leq m$, and contains a Lagrangian torus in the radius-$\frac{1}{\sqrt{\pi(1+m-k)}}$ sphere bundle that satisfies the conditions of Theorem 3 [45]. It follows that $\widehat{SH}^*(W;\Lambda) \neq 0$ for any Liouville cobordism W containing the sphere bundle of radius $\frac{1}{\sqrt{\pi(1+m-k)}}$.

Chapter 4

Computations for negative line bundles

4.1 Line bundles over projective space

Let $E = \mathrm{Tot}\left(\mathcal{O}(-k) \xrightarrow{\rho} \mathbb{C}P^m\right)$ be a line bundle over projective space satisfying $1 \leq k \leq m$. As shown in [42], [43], and [45], the constraints on k allow for a monotone symplectic structure on E. For completeness we repeat this construction.

Let ω_{FS} be the Fubini-Studi form on $\mathbb{C}P^m$, scaled so that $\langle [\omega_{FS}], [\mathbb{C}P^1] \rangle = 1$. The scaling of ω_{FS} implies that $c_1^{TM} = (1+m)[\omega_{FS}]$ and $c_1^E = -k[\omega_{FS}]$; the latter condition defines E as a *negative line bundle*. Let J be an ω_{FS}-compatible complex structure on $\mathbb{C}P^m$. Let $|\cdot|$ be a Hermitian metric on E with induced Chern curvature $\mathcal{F}$. Define a radial coordinate r by $r(w) = |w|$ and a fiber-wise angular one-form on the complement of the zero-section by

$$\alpha = \frac{1}{4\pi k} d^c \log(r^2)$$

so that

$$d\alpha = \frac{1}{4\pi k} dd^c \log(r^2) = \frac{i}{2\pi k} \partial\bar{\partial} \log(r^2) = -\frac{i}{2\pi k} \rho^* \mathcal{F} \equiv -\frac{1}{k} \rho^* c_1^E \equiv \rho^* \omega_{FS}.$$

Note that α defines a contact one-form on the corresponding prequantization bundle. Let

$$\Omega = (1 + k\pi r^2)d\alpha + 2k\pi r dr \wedge \alpha = d\alpha + d(k\pi r^2 \alpha)$$

be a symplectic form on the complement of the zero section. Extend Ω smoothly over the zero-section by

$$\Omega\big|_{\text{zero section}} = -\frac{i}{2\pi k}\rho^*\mathcal{F} + \{\text{area form of fiber}\}.$$

Then Ω is a symplectic form on E and $[\Omega] = [\rho^*\omega_{FS}]$, and

$$c_1^{TE} = \rho^* c_1^{TM} + \rho^* c_1^E = (1+m)[\rho^*\omega_{FS}] - k[\rho^*\omega_{FS}] = (1+m-k)[\Omega].$$

The purpose of this chapter is to prove the following theorem computing action-completed symplectic cohomology for monotone negative line bundles over projective space.

THEOREM 2 *Let* $E = \mathrm{Tot}\left(\mathcal{O}(-k) \xrightarrow{\rho} \mathbb{C}P^m\right)$ *be a negative line bundle with* $1 \le k \le m$, *and equip* E *with a Hermitian metric that induces an angular form* α *satisfying* $d\alpha = \rho^*\omega_{FS}$, *a radial coordinate* r, *and a symplectic form* $\Omega = d\alpha + d(k\pi r^2\alpha)$. *Let* $A_{(R_1,R_2)}$ *be the annulus subbundle between radii* R_1 *and* R_2. *Then*

$$\widehat{SH}^*(A_{(R_1,R_2)};\Lambda) \cong \begin{cases} SH^*(E;\Lambda) & R_1 < \frac{1}{\sqrt{\pi(1+m-k)}} \le R_2 \\ 0 & \textit{else} \end{cases}.$$

The line bundle $\mathcal{O}(-k) \longrightarrow \mathbb{C}P^m$ contains a Lagrangian torus in the radius–$\frac{1}{\sqrt{\pi(1+m-k)}}$ sphere bundle that satisfies all of the conditions of Theorem 3 [45]. If W is a Liouville cobordism between two sphere bundles, then Theorem 3 guarantees that $\widehat{SH}^*(W;\Lambda) \neq 0$ if W contains the radius–$\frac{1}{\sqrt{\pi(1+m-k)}}$ sphere bundle. Theorem 2 is thus an illustration of the converse phenomenon.

Cieliebak-Frauenfelder-Oancea showed in [19] that if M is instead a Liouville domain then the uncompleted symplectic cohomology of a trivial cobordism $W \subset \widetilde{M}$ containing ∂M is isomorphic to the uncompleted Rabinowitz Floer homology of ∂M. We expect a relationship in this flavor between $\widehat{RFH}^*(\Sigma;\Lambda)$ and the Rabinowitz Floer homology of a contact hypersurface Σ in a negative line bundle E studied by Albers-Kang [9]. In particular, Albers-Kang showed that the Rabinowitz Floer homology of a sphere bundle in E of radius less than $\frac{1}{\sqrt{\pi(1+m-k)}}$ vanishes. While they claim that this vanishing result extends to the sphere bundle of radius $\frac{1}{\sqrt{\pi(1+m-k)}}$, Theorem 2 implies otherwise.

We first compute action-completed symplectic cohomology and homology of the disc bundles of radii R_1 and R_2, then show how these computations imply Theorem 2. For simplicity of notation we set $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$ in this chapter. Thus, each orientation line o_x appearing as a generator of a Floer complex is replaced by the corresponding critical point x. It will be clear from the method of proof that the results extend to arbitrary coefficient fields.

4.2 Symplectic cohomology of disc bundles

THEOREM 5 *Let* (E, Ω) *be a degree* $-k$ *monotone negative line bundle over* m*-dimensional complex projective space with monotonicity constant* $\tau = 1 + m - k$*. Let* D_R *be the disc subbundle of radius* R*. Then*

$$\widehat{SH^*}(D_R; \Lambda) \cong \begin{cases} 0 & R < \frac{1}{\sqrt{\pi\tau}} \\ SH^*(E; \Lambda) & R \geq \frac{1}{\sqrt{\pi\tau}} \end{cases} .$$

To compute symplectic cohomology we will use a particular family of almost-complex structures and a particular family of Hamiltonians.

4.2.1 The Hamiltonians

Fix a radius $R \in (0, \infty)$. We will construct a chain complex generated by one-periodic orbits that cluster near the boundary of D_R. Fix a constant $C > 0$ and let $\{h_n \colon \mathbb{R} \longrightarrow \mathbb{R}\}_{n \in \mathbb{N}}$ be a family of functions where each h_n is

1. convex and monotone increasing on $\mathbb{R}_{\geq 0}$,
2. bounded in absolute value by C on $[0, k\pi R^2]$, and
3. of slope $\frac{n}{k} + \frac{1}{2k}$ on $(k\pi R_n^2, \infty)$, for some $R_n < R$.

Further assume that the sequence $\{R_n\}$ tends to R as n tends to ∞ (see Figure 4.1). To simplify later proofs, we also require that

1. h_n and h_n' are monotone increasing, and
2. $h_n \leq h_{n+1}$ everywhere.

Choose a perfect Morse function function $f\colon \mathbb{C}P^m \longrightarrow \mathbb{R}$ that is $\mathcal{C}^2$-small. Let r be the radial coordinate on the fibers of E determined by the Hermitian metric. Define a family of Hamiltonians $\{H_n\colon E \longrightarrow \mathbb{R}\}_{n\in\mathbb{N}}$ by

$$H_n = h_n(k\pi r^2) + (1 + k\pi r^2)\rho^* f.$$

We assume that the one-periodic orbits of H_n are transversally nondegenerate. For example, we can take each h_n to be a smoothing of a piecewise linear function, as in Figure 4.1.

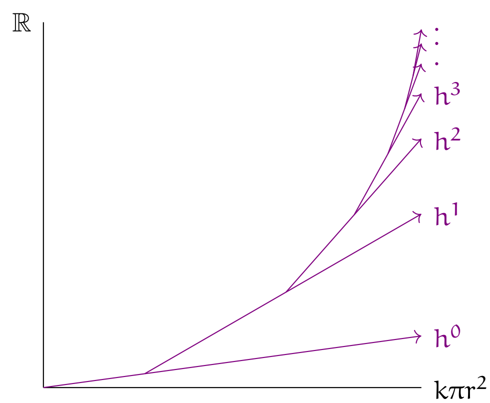


Figure 4.1: A family of Hamiltonians $\{h^n(k\pi r^2)\}$

Let R_α be the unique vector field satisfying

$$\begin{cases} \alpha(R_\alpha) = 1 \\ dr(R_\alpha) = 0 \\ i_{R_\alpha} d\alpha = 0 \end{cases}.$$

Note that $R_\alpha\big|_{r=\mathfrak{r}}$ is the Reeb vector field of the contact form $\alpha\big|_{\{r=\mathfrak{r}\}}$ and the simply-covered orbits of R_α have period $\frac{1}{k}$. As

$$dH_n = 2k\pi r(h_n'(k\pi r^2) + \rho^* f)dr + (1 + k\pi r^2)\rho^* df,$$

the Hamiltonian vector field of H_n is

$$X_{H_n} = (h_n'(k\pi r^2) + \rho^* f)R_\alpha + X_f^h,$$

where X_f^h is the unique vector field in the horizontal distribution H satisfying $\rho_*(X_f^h) = X_f$. Thus, the one periodic orbits of H_n correspond precisely to the orbits of $R_\alpha\big|_{D_R}$ of period between one and n that lie in fibers above the critical points of f, and the critical points of f itself.

For each n choose an $\mathbb{R}$-family of functions $h_n^s \colon \mathbb{R} \to \mathbb{R}$, monotonely decreasing in s, with $h_n^s = h_n$ when $s >> 0$ and $h_n^s = h_{n+1}$ when $s << 0$. Set $H_n^s = h_n^s(k\pi r^2) + (1 + k\pi r^2)\rho^* f$.

4.2.2 The almost-complex structure

Let $\mathcal{J}(\omega_{std})$ be the space of S^1-families of almost-complex structures on $\mathbb{C}$ compatible with the standard symplectic form, and such that each almost-complex structure is cylindrical in a neighborhood of periodic orbits and at infinity. Let $\mathcal{J}(\omega_{FS})$ be the space of S^1-families of almost-complex structures on $\mathbb{CP}^m$ compatible with ω_{FS}. Choose $i_t \in \mathcal{J}(\omega_{std})$ and $j_t \in \mathcal{J}(\omega_{FS})$. The one-form α determines a splitting of TE into a vertical component $V \cong \mathbb{C}$ and horizontal component H. Let $L_t(H, V)$ be the space of S^1-families of linear maps from H to V. Let $\mathcal{U}$ be an open set comprised of small neighborhoods of the circle bundles on which live non-constant periodic orbits of each H_n, as well as small neighborhoods of the constant orbits. Let $\mathfrak{B}(i_t, j_t)$ be the elements $B_t \in L_t(H, V)$ with compact support in the complement of $\mathcal{U}$, and satisfying $i_t B_t + B_t \rho^* j_t = 0$ for all t. We will use this data to define a set of almost-complex structures. These conditions will then ensure that Floer trajectories "flow outward", that Floer trajectories converge to periodic orbits, and that J_t is almost-complex. Define

$$\mathcal{J}(\Omega) = \left\{ J_t = \begin{bmatrix} i_t & B_t \\ 0 & \rho^* j_t \end{bmatrix} \in \mathrm{End}(TE) \;\middle|\; i_t \in \mathcal{J}(\omega_{std}), j_t \in \mathcal{J}(\omega_{FS}), B_t \in \mathfrak{B}(i_t, j_t), J_t \text{ is } \Omega\text{–tame} \right\}.$$

4.2.3 The coefficient fields

Let T be a formal variable of degree -2τ, and let

$$\Gamma = \mathbb{K}[T, T^{-1}]$$

the ring of Laurent polynomials in T. Define Γ_a to be the subgroup of Γ comprised of polynomials of minimum power greater than a. The Novikov field is the completion of Γ:

$$\Lambda = \varinjlim_{a} \varprojlim_{b} \Gamma_a \big/ \Gamma_b \,.$$

4.2.4 The chain complex

To be able to set up Floer theory we need to either perturb the Hamiltonian or use Morse-Bott methods. We have applied a Morse perturbation directly to $\mathbb{C}P^m$ to rid ourselves of the "horizontally-degenerate" critical points. We use Morse-Bott methods on the remaining S^1 families.

Let $CF^*(H_n;\Gamma)$ be the Floer cochain complex associated to a Hamiltonian H_n, as above, and an almost-complex structure in $\mathcal{J}(\Omega)$. Each transversely non-degenerate orbit contributes two generators to $CF^*(H_n;\Gamma)$ representing the maximum and minimum of a perfect Morse function on the orbit. We $\mathbb{Z}$-grade a generating orbit x by choosing the fibre disc as a capping disc $\tilde{x}$ of the underlying Reeb orbit. The grading $|x|$ of x is defined to be the Conley-Zehnder index μ_{H_n} of $\tilde{x}$, as defined in [15], with a shift. The following computation appears in the literature in various guises, for example [9][15][38][39][41].

LEMMA 8 *Let x be a periodic orbit of H_n corresponding to a critical point of a perfect Morse function $\mathfrak{f}$ on an underlying Reeb orbit of period ℓ. Let $\mu_f(\rho\circ x)$ be the index of $\rho\circ x$ as a critical point of f. Then*

$$\mu_{H_n}(T^a\tilde{x}) = -2\ell + 2\tau a + \mu_f(\rho\circ x) - m - 1$$

if x corresponds to the minimum of $\mathfrak{f}$ and

$$\mu_{H_n}(T^a\tilde{x}) = -2\ell + 2\tau a + \mu_f(\rho\circ x) - m$$

if x corresponds to the maximum of $\mathfrak{f}$.

SKETCH OF PROOF: The pointwise splitting of TE into horizontal and vertical distributions H and V extends over the pullback bundle $\tilde{x}^*TE$. We write $\tilde{x}^*TE \cong TD^2 \oplus H$. The flow ϕ_t of X_{H_n} respects these distributions: $d\phi_t|_H \subset H$ and $d\phi_t|_{TD^2} \subset TD^2$. By the additivity axiom of Conley-Zehnder indices it suffices to compute $\mu_{CZ}(d\phi_t|_{TD^2})$ and $\mu_{CZ}(d\phi_t|_H)$. If ϕ_t^f is the flow of X_f on $\mathbb{C}P^m$, then $\rho\circ\phi_t = \phi_t^f$ and so $d\rho\circ d\phi_t = d\phi_t^f$. But $d\rho|_H = \mathrm{Id}$, and so

$$\mu_{CZ}(d\phi_t|_H) = \mu_{CZ}(d\phi_t^f) = \mu_f(\rho\circ x) - m,$$

where the final equality holds because f is a $\mathcal{C}^2$-small Morse function on a manifold of real dimension $2m$.

Oancea computes in [41] that the Robbin-Salamon index of $d\phi_t|_{TD^2}$ is $i_{RS} = 2\ell + \frac{1}{2}$. A computation in [15] shows that

$$\mu_{H_n}(x) = \begin{cases} -i_{RS} - \frac{1}{2} & x \text{ corresponds to a minimum} \\ -i_{RS} + \frac{1}{2} & x \text{ corresponds to a maximum} \end{cases}.$$

The result follows.

□

To ease notation we shift all gradings up by $m + \frac{1}{2}$. Thus,

$$|T^a x| = -2\ell + 2\tau a + \mu_f(\rho \circ x) \pm \frac{1}{2}.$$

In the previous chapters we defined Floer theory through non-degenerate Hamiltonians. In this chapter we will use the methods of cascades discussed in subsection 2.2.1. Let $\mathbf{u} = (c_m, u_m, \dots, c_0)$ be a rigid cascade corresponding to H_n with $c_m \in \text{im}(x)$ and $c_0 \in \text{im}(y)$. As hinted in subsection 2.2.1, u contributes a term $T^{\omega((-\tilde{x})\#\mathbf{u}\#\tilde{y})}y$ to the differential ∂^{fl}, where $(-\tilde{x})\#\mathbf{u}\#\tilde{y}$ is the sphere formed by gluing $\tilde{x}$ with reversed orientation and $\tilde{y}$ with the same orientation to the concatenated cylinder $u_m\#\dots\#u_1$. Geometrically, we are replacing the capping $\tilde{y}$ with the capping $\tilde{x}\#(-\mathbf{u})$. The action of a generator $T^a x$ is then defined to be

$$\mathcal{A}_{H_n}(T^a x) = a - \int_{D^2} \tilde{x}^*\Omega + \int_0^1 H_n(x(t))dt.$$

Analogously to the set up of Chapter 2, from this we construct the action-completed symplectic cochain complex

$$\widehat{SC^*}(D_R; \Lambda) = \left(\varprojlim_b \varinjlim_a \operatorname*{hocolim}_{n \in \mathbb{N}} CF^*_{(a,b)}(H_n; \Gamma), \partial \right) \tag{42}$$

where ∂ is constructed from the Floer differential and the continuation maps $\{c_n\}$ that are induced by the Hamiltonians H^s_n.

In subsection 4.2.5 we show that (42) is well-defined.

4.2.5 Moduli spaces of Floer solutions

LEMMA 9 (REGULARITY (ALBERS-KANG [9])) *There exists a comeager subset of $\mathcal{J}(\Omega)$ for which the finite-energy cascades of H_n and H_n^s, for any n, are cut out transversally.*

PROOF: Let $\mathcal{J}^{reg}(\omega_{FS}) \subset \mathcal{J}(\omega_{FS})$ be the subset for which finite-energy Floer solutions of the equation

$$\partial_s u + j_t(\partial_t u - X_f),$$

as well as simple j_t-holomorphic spheres, are regular. Similarly, let $\mathcal{J}^{reg}(\omega_{std}) \subset \mathcal{J}(\omega_{std})$ be the subset for which the finite-energy Floer solutions of each of the equations

$$\left\{\partial_s u + i_t(\partial_t u - X_{h_n(k\pi r^2)+f(p)(1+k\pi r^2)})\right\}_{p\in \mathrm{Crit}(f)}$$

are regular. It follows from the usual arguments that $\mathcal{J}^{reg}(\omega_{FS})$ and $\mathcal{J}^{reg}(\omega_{std})$ are each comeager [38]. Take $j_t \in \mathcal{J}^{reg}(\omega_{FS})$ and $i_t \in \mathcal{J}^{reg}(\omega_{std})$. Let x and $T^a y$ be weighted one-periodic orbits of H_n and let $\mathcal{M}(x, T^a y)$ be the moduli space of Floer solutions from x to y of weight α. There are now two cases:

1. Elements of $\mathcal{M}(x, T^a y)$ pass through the complement of $\mathcal{U}$. This occurs if x and y live in different fibers of E or $\alpha \neq 0$. In these cases, we need to show that, for generic $B_t \in \mathfrak{B}(i_t, j_t)$, the linearized Floer equation associated to H_n and $J_t = \begin{bmatrix} i_t & B_t \\ 0 & \rho^* j_t \end{bmatrix}$ is surjective. Recall the pointwise splitting of the tangent bundle $T_pE \cong V_p \oplus H_p$. As j_t is regular, surjectivity in the horizontal direction is automatically achieved. We need to show surjectivity in the vertical direction. Let $(v, h) \in T_pE$ be a tangent vector with $h \neq 0$, written with respect to the splitting $V_p \oplus H_p$. Fix $B \in \mathfrak{B}(i_t, j_t)$. By the methods in [38] it suffices to find $C \in T_{B_\tau}\mathfrak{B}(i_\tau, j_\tau)$ satisfying

$$\begin{cases} Ch = v \\ i_\tau C + C j_\tau = 0. \end{cases}$$

   Define C by $Ch = v$, $Cj_\tau h = -i_\tau v$, and $C\big|_{span\{h, jh\}^\perp} = 0$. Then

$$(i_\tau C + C j_\tau)h = i_\tau v - i_\tau v = 0$$

and

$$(i_\tau C + Cj_\tau)j_\tau h = i_\tau Cj_\tau h - Ch = i_\tau(-i_\tau \nu) - \nu = \nu - \nu = 0.$$

2. In the other case, x and y live in the same fiber E_p and $\alpha = 0$. Let $u \in \mathcal{M}(x, y)$. It follows from regularity of j_t that $\rho \circ u$ is constant, and so u remains in the fiber. Thus, the moduli space $\mathcal{M}(x, y)$ is canonically identified with the moduli space of Floer trajectories from x to y in $\mathbb{C}$ satisfying the Floer equation of the pair $(h_n(\pi r^2) + (1 + \pi r^2)f(p), i_t)$. This is a manifold of the expected dimension $|x| - |y|$, and so regularity is achieved.

□

LEMMA 10 (GROMOV COMPACTNESS I) *(Gromov compactness I) Sequences of cascades of H_n or H_n^s between two fixed periodic orbits remain in a compact region of E, for any n.*

PROOF: This follows from an integrated maximum principle, as in the proof of Lemma 12. Indeed, by assuming that i_t is cylindrical and $B_t = 0$ outside of D_R, we can assume that all Floer solutions appearing in a cascade remain in D_R.

□

LEMMA 11 (GROMOV COMPACTNESS II) *Bubbling does not occur in the limit of sequences of index-0, 1 and index-2 cascades defining the chain complex.*

PROOF: By monotonicity of E, all J-holomorphic spheres have index at least two. They therefore will not be seen by moduli spaces of cascades of dimension less than two. These moduli spaces include those appearing in the arguments showing that

- the differential is well-defined as a map on vector spaces (dimension 0);
- continuation maps are well-defined as maps on vector spaces (dimension 0);
- continuation maps are chain maps (dimension 1); and
- continuation maps are invariant under choice of underlying Hamiltonian families (dimension 1).

It remains to consider the dimension-2 moduli spaces of cascades that appear in showing $\partial^2 = 0$. Note that only index-2 bubbles will *a priori* interfere, and so it suffices to consider the case $k = m$, that is, $E = \mathrm{Tot}(\mathcal{O}(-m) \longrightarrow \mathbb{CP}^m)$. Let $\mathcal{M}_2(x, y)$ be the two-dimensional component of the moduli space of cascades connecting period orbits x and y, and associated with generic Floer data (H_n, J). Suppose that bubbling occurs within the moduli space $\mathcal{M}_2(x, y)$. By an argument in [46], any bubble must intersect a dimension-0 component of $\mathcal{M}(x, y)$; in particular, $x = y$ and any bubble passes through x. We will show that x lies on the zero-section.

Let $v\colon \Sigma \longrightarrow E$ be a non-constant J-holomorphic sphere of index 2. As the symplectic form on E is exact away from the zero-section, Σ must intersect the zero-section (else, by Stokes' Theorem, Σ would have zero symplectic area, contradicting that v is non-constant). Assume for contradiction that Σ leaves the zero-section. Recall that we chose i_t cylindrical in a neighborhood of each circle bundle containing a non-constant periodic orbit. Choose a generic disc bundle $\mathcal{D}$ containing no non-constant periodic orbits, but such that i_t is cylindrical on $\partial\mathcal{D}$. Suppose that v leaves $\mathcal{D}$. Apply the integrated maximum principle from [3] to $v^{-1}(E \setminus \mathcal{D})$. This computation shows that the symplectic area of $v^{-1}(E \setminus \mathcal{D})$ is negative, contradicting J-holomorphicity.

Thus, the image of v only intersects constant orbits, and we conclude that x is a constant orbit contained in the zero section. Usually one could argue that an S^1-family of J_t-holomorphic spheres have codimension three, and therefore do not generically intersect a zero-dimensional constant orbit. However, the requirement $B_t = 0$ in a neighborhood of the constant orbits means that we cannot necessarily perturb an almost-complex structure in a direction required to "push" it off of a constant orbit. We will instead show that no sequence of maps in $\mathcal{M}_2(x, x)$ converges to a broken trajectory. This will show that, even if $\mathcal{M}_2(x, x)$ "sees" bubbling, it does not affect the computation $\partial^2 = 0$.

Suppose that a sequence of cascades $u_i \in \mathcal{M}_2(x, x)$ converges to a broken cascade $u \in \mathcal{M}_1(x, z) \times \mathcal{M}_1(z, x)$. As f is a perfect Morse function, the space of one-dimensional cascades of (f, j_t) is empty. z must therefore be a Reeb orbit on some hypersurface of some period $k \in \mathbb{Z}_{>0}$. But by Lemma 12 this is impossible.

□

Define $\mathfrak{w}(x)$ to be the winding number of a non-constant periodic orbit x, viewed as a map from the circle into $\mathbb{C}^*$. Define $\mathfrak{w}(x)$ of a constant orbit x to be zero.

LEMMA 12 *Let* u *be a solution of Floer's equation (11) with* $\lim_{s\to\pm\infty} u(s,t) = x_\pm(t)$. *Then* $\mathfrak{w}(x_+) \geq \mathfrak{w}(x_-)$.

PROOF: Assume for contradiction that $\mathfrak{w}(x_-) > \mathfrak{w}(x_+)$. Say that $x_\pm$ lives in the sphere bundle of radius $\sigma_\pm$. If $x_\pm$ are non-constant then the R_α-orbit underlying $x_\pm$ has period $\frac{1}{k}\mathfrak{w}(x_\pm)$. From the earlier computation $X_{H_n} = (h_n'(k\pi r^2) + \rho^* f)R_\alpha + X_f^h$ it follows that $h_n'(k\pi\sigma_\pm^2) + \rho^* f(x_\pm) = \frac{1}{k}\mathfrak{w}(x_\pm)$. The winding numbers are integers, and so

$$h_n'(k\pi\sigma_-^2) + \rho^* f(x_-) \geq h_n'(k\pi\sigma_+^2) + \rho^* f(x_+) + \frac{1}{k}.$$

By the smallness of f we can assume that

$$h_n'(k\pi\sigma_-^2) - h_n'(k\pi\sigma_+^2) \geq \rho^* f(x_+) - \rho^* f(x_-) + \frac{1}{k} > 0.$$

As h_n is convex, we deduce that $\sigma_- > \sigma_+$. If x_+ is constant and x_- is non-constant it follows immediately that $0 = \sigma_+ < \sigma_-$.

Choose a generic circle subbundle $\mathcal{S}$ of radius σ, with $\sigma_+ < \sigma < \sigma_-$, and on which $B = 0$ and i_t is cylindrical. For example, if σ is close to σ_+ or σ_- these conditions will, by construction, be met. Let $\mathcal{D}$ be the region bounded by $\mathcal{S}$, and denote $\Sigma = u^{-1}(E \setminus \mathcal{D})$. Let $v \colon \Sigma \longrightarrow E \setminus \mathcal{D}$ be the restriction of u. We will equate Σ with its image under the inclusion into $\mathbb{R} \times S^1$ and use the coordinates (s,t) induced on $\mathrm{Int}(\Sigma)$.

Let $c_x = h_n(k\pi x^2) - h_n'(k\pi x^2)k\pi x^2$ be the y-intercept of the tangent line to h_n at $k\pi x^2$. Then on $\mathcal{S}$, $H_n = h_n'(k\pi\sigma^2)k\pi\sigma^2 + (1 + k\pi\sigma^2)\rho^* f + c_\sigma$ (and $X_{H_n} = (h_n'(k\pi\sigma^2) + \rho^* f)R_\alpha + X_f^h$). In particular,

$$H_n\big|_{\mathcal{S}} = (1 + k\pi\sigma^2)\alpha(X_{H_n}) - h_n'(k\pi\sigma^2) + c_\sigma.$$

We employ the integrated maximum principle. Let $\partial\Sigma_+$ be the boundary component of Σ mapping into $\mathcal{S}$. Note that we have chosen J to be Ω-tame, so that

$$E_J(v) := \frac{1}{2}\int_\Sigma \left(\Omega(\partial_s v, J\partial_s v) + \Omega\left(\partial_t v - X_{H_n}(v), J(\partial_t v - X_{H_n}(v)\right)\right) ds \wedge dt \geq 0$$

Shuffling terms, we have

$$\begin{aligned}
E_J(v) &= \int_\Sigma v^*\Omega - v^* dH_n \otimes dt \\
&= \int_{\partial\Sigma_+} (1+k\pi r^2)v^*\alpha - H_n(v(s,t))dt - \int_{S^1} (1+k\pi r^2)x_-^*\alpha - H_n(x_-(t))dt \\
&= \int_{\partial\Sigma_+} (1+k\pi\sigma^2)v^*\alpha - (1+k\pi\sigma^2)\alpha(X_{H_n}) \otimes dt + \left(h_n'(k\pi\sigma^2) - c_\sigma\right) dt \\
&\qquad - \int_{S^1} (1+k\pi\sigma_-^2)x_-^*\alpha - H_n(x_-(t))dt
\end{aligned}$$

We consider this final equation in pieces. A solution $v(s,t)$ of Floer's equation satisfies

$$(dv - X_{H_n} \otimes dt)^{(0,1)} = 0.$$

We have that, on $\mathcal{S}$, $dr \circ i_t = -2k\pi r\alpha$ and $B_t = 0$. The former implies that JR_α is proportional to ∂_r and the latter implies that JX_f^h lives in the horiziontal distribution. So altogether, $\alpha(JX_{H_n}) = 0$. Thus, the first term equals

$$\begin{aligned}
\int_{\partial\Sigma_+} &(1+k\pi\sigma^2)\alpha(dv - X_{H_n} \otimes dt) + \left(h_n'(k\pi\sigma^2) - c_\sigma\right) dt \\
&= \int_{\partial\Sigma_+} -(1+k\pi\sigma^2)\alpha \circ J(dv - X_{H_n} \otimes dt) \circ j + \left(h_n'(k\pi\sigma^2) - c_\sigma\right) dt \\
&= \int_{\partial\Sigma_+} -\frac{1+k\pi\sigma^2}{2k\pi\sigma} dr \circ dv \circ j + \left(h_n'(k\pi\sigma^2) - c_\sigma\right) dt \\
&\leq \int_{\partial\Sigma_+} \left(h_n'(k\pi\sigma^2) - c_\sigma\right) dt
\end{aligned}$$

We also have

$$H_n(x_-(t)) = h_n'(k\pi\sigma_-^2)k\pi\sigma_-^2 + c_{\sigma_-} + (1+k\pi\sigma_-^2)\rho^* f(x_-(t)) = (1+k\pi\sigma_-^2)\frac{\mathfrak{w}(x_-)}{k} - h_n'(k\pi\sigma^2) + c_{\sigma_-}$$

and so the second term becomes

$$\begin{aligned}
-\int_{S^1} (1+k\pi r^2)x_-^*\alpha + \int_0^1 H_n(x_-(t))dt &= -\frac{\mathfrak{w}(x_-)}{k}(1+k\pi\sigma_-^2) + \frac{\mathfrak{w}(x_-)}{k}(1+k\pi\sigma_-^2) + c_{\sigma_-} - h_n'(k\pi\sigma_-^2) \\
&= \int_{S^1} \left(-h_n'(k\pi\sigma_-^2) + c_{\sigma_-}\right) dt
\end{aligned}$$

Altogether,

$$E_J(v) \le \int_{\partial\Sigma_+} \left(h_n'(k\pi\sigma^2) - c_\sigma\right) dt + \int_{S^1} \left(-h_n'(k\pi\sigma_-^2) + c_{\sigma_-}\right) dt. \tag{43}$$

Σ is a collection of bounded regions in $\mathbb{C}^*$ and one unbounded region. As dt is exact on $\mathbb{C}^*$, the bounded regions contribute nothing to the right-hand side of (43). Let $\tilde{\Sigma}$ be the unbounded component. Near infinity Σ is contained in a neighborhood of x_-, and so all boundary components of $\tilde{\Sigma}$ occur within the intersection of Σ with some annulus $(-\infty, R] \times S^1$. Let $\mathfrak{h}$ be a function on $\tilde{\Sigma}$ that is equal to $h_n'(k\pi\sigma^2) - c_\sigma$ on $\tilde{\Sigma} \cap \left((-\infty, R] \times S^1\right)$ and equal to $h_n'(k\pi\sigma_-^2) - c_{\sigma_-}$ for all large radii $r >> R$. Then

$$\begin{aligned} E_J(v) &\le \int_{\partial\Sigma} \mathfrak{h}\, dt = \int_{\tilde{\Sigma}} d(\mathfrak{h}\, dt) \\ &= \int_{\tilde{\Sigma}\cap((-\infty,R]\times S^1)} d(\mathfrak{h}\, dt) + \int_{\tilde{\Sigma}\cap([R,\infty)\times S^1)} d(\mathfrak{h}\, dt) \\ &= \int_{\tilde{\Sigma}\cap((-\infty,R]\times S^1)} d(\mathfrak{h}\, dt) \\ &= (h_n'(k\pi\sigma^2) - h_n'(k\pi\sigma_-^2)) + (c_{\sigma_-} - c_\sigma). \end{aligned}$$

As h_n is convex by assumption and $\sigma < \sigma_-$, both $\left(h_n'(k\pi\sigma^2) - h_n'(k\pi\sigma_-^2)\right) < 0$ and $(c_{\sigma_-} - c_\sigma) < 0$. It follows that

$$E_J(v) \le (h_n'(k\pi\sigma^2) - h_n'(k\pi\sigma_-^2)) + (c_{\sigma_-} - c_\sigma) < 0.$$

which yields the desired contradiction.

□

4.2.6 A heuristic outline of the proof of Theorem 2

We adapt a technique first used by Albers-Kang in [9] in the context of Rabinowitz Floer homology to show the vanishing of symplectic (co)homology for small enough disc bundles.

The choice of Hamiltonians implies that periodic-orbits $x(t)$ of H_n satisfy

$$\begin{cases} \alpha(\dot{x}) = h_n'(k\pi r(x)^2) + f(\rho(x)) \\ dr(\dot{x}) = 0 \\ \rho_* \dot{x} = X_f(\rho \circ x) \end{cases}$$

Thus, $x(t)$ is a Reeb orbit of α lying in the fiber above a critical point of f. Conversely, if p is a critical point of f and E_p is the fiber above p, the one-periodic orbits of the Hamiltonian $g_n : \mathbb{C} \longrightarrow \mathbb{R}$ defined by $g_n = h_n(k\pi r^2) + (1 + k\pi r^2)f(p)$ correspond to one-periodic orbits of H_n in E_p. The choice of almost-complex structure J furthermore implies that any Floer solution in $\mathbb{C}$ of the data (g_n, i_t) represents a Floer solution of (H_n, J). Thus, $CF^*(H_n; \Gamma)$ contains one copy of $CF^*(g_n; \Gamma)$ for each critical point of f. It is shown in Lemma 16 that $CF^*(g_n; \Gamma)$ corresponds to the chain complex shown in Figure 4.2.

$$x_-^0 \longleftarrow x_+^1 \qquad Tx_-^1 \longleftarrow \cdots \qquad \cdots \longleftarrow T^{n-1}x_+^n \qquad T^n x_-^n$$

Figure 4.2: The chain complex $CF^*(g_n; \Gamma)$

After making a (non-essential) assumption, we will conclude that continuation maps act as canonical inclusions. Figure 4.3 is therefore a schematic of the complex $SC^*(\mathbb{C})$.

$$x_-^0 \longleftarrow x_+^1 \qquad Tx_-^1 \longleftarrow Tx_+^2 \qquad Tx_-^2 \longleftarrow Tx_+^3 \qquad Tx_-^3 \longleftarrow \cdots$$

Figure 4.3: The chain complex $SC^*(\mathbb{C})$

We decompose the Floer differential as $\partial = \partial_0 + \partial_{>0}$, where ∂_0 is the component of the differential depicted as the horizontal arrows in Figure 4.3. Let $(x_\pm^n)_p$ be the "$x_\pm^n$" orbit of Figure 4.3 in the fiber E_p. The method of computing symplectic cohomology is roughly as follows.

1. Suppose $(x_+^{n_1})_{p_1}$ is a summand of a cocycle $\mathbf{X}$. Then

$$\partial((x_+^{n_1})_{p_1}) = \partial_0((x_+^{n_1})_{p_1}) + \partial_{>0}((x_+^{n_1})_{p_1}) = (x_-^{n_1-1})_{p_1} + \partial_{>0}((x_+^{n_1})_{p_1}).$$

There must therefore be some $(x_+^{n_2})_{p_2}$, another summand of $\mathbf{X}$, with $(x_-^{n_1-1})_p$ a summand of

$\partial_{>0}((x_+^{n_2})_{p_2})$. But then

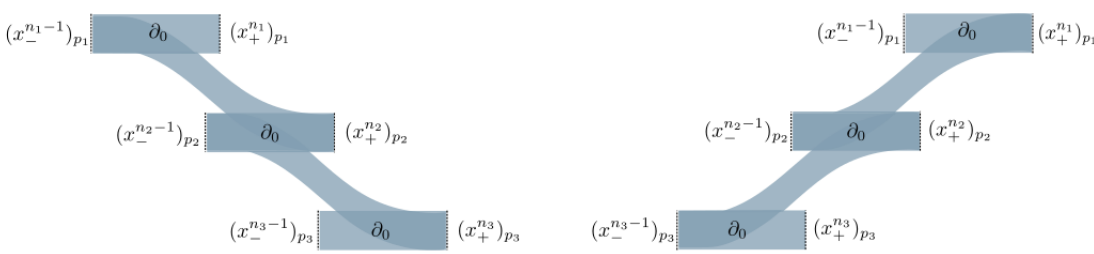
$$\partial((x_+^{n_2})_{p_2}) = (x_-^{n_2-1})_{p_2} + \partial_{>0}((x_+^{n_2})_{p_2})$$

So there is some $(x_+^{n_3})_{p_3}$, a further summand of $\mathbf{X}$, with $(x_-^{n_2-1})_{p_2}$ a summand of $\partial_{>0}$. Iterating, we build a sequence $\{(x_+^{n})_p, (x_+^{n_1})_{p_1}, (x_+^{n_2})_{p_2}, \dots\}$ of summands of $\mathbf{X}$ (see Figure 4.4a). We will show that $\lim_i \mathcal{A}((x_+^{n_i})_{p_i}) \neq \infty$ when $R < \frac{1}{\sqrt{\pi\tau}}$, which will contradict that $\mathbf{X}$ is an element of $SC^*(D_R)$.

2. Similarly, if $(x_-^{n_1})_{p_1}$ is a cocycle then we can build some

$$\mathbf{X} = \sum_{i=0}^{\infty} (x_+^{n_i})_{p_i}$$

with $\partial(\mathbf{X}) = (x_-^{n_1})_{p_1}$ (see Figure 4.4b). We will show that, if $R < \frac{1}{\sqrt{\pi\tau}}$, $\lim_i \mathcal{A}((x_+^{n_i})_{p_i}) = \infty$, and so $(x_-^{n_1})_{p_1}$ is a coboundary.



(a) Trying to form a coycle out of some $(x_+^n)_p$

(b) Trying to form a coboundary via some $(x_+^n)_p$

Figure 4.4: Cocycles and coboundaries

4.2.7 Computing $\widehat{SH}^*(D_R)$ when $R < \frac{1}{\sqrt{\pi\tau}}$

Each periodic orbit x of a Hamiltonian H_n lies in a fiber above a critical point of f, and we choose its lift $\tilde{x}$ to be a capping disk in the fiber. As $\rho \circ \tilde{x}$ is a critical point of f, ρ induces a map from the lifted one-periodic orbits of H_n to the critical points of f with constant capping. Let $\mu_f \colon CF^*(f;\Gamma) \longrightarrow \mathbb{R}$ be the Conley-Zehnder index associated to Novikov-weighted critical points of f. Note that a Floer trajectory of (H_n, J_t) descends to a Floer trajectory of (f, j_t) on the zero section. $CF^*(H_n;\Gamma)$ therefore has a filtration

$$\mathcal{F}_p = \mathbb{K}\left\langle T^a x \in CF^*(H_n;\Gamma) \,\middle|\, \mu_f(T^a \rho \circ x) \geq p \right\rangle, \tag{44}$$

and the differential decomposes as

$$\partial = \partial_0 + \partial_1 + \partial_2 + \dots$$

where $\partial_i(\mathcal{F}_p) \subset \mathcal{F}_{p+i}$. It will be convenient to work with explicit formal sums, as in the following chain complex.

$$\widehat{SC^*_{pr}}(D_R;\Lambda) := \left\{ \sum_{i=0}^{\infty} d_i T^{a_i} x_i \,\middle|\, d_i T^{a_i} x_i \in CF^*(H_{n_i};\Gamma) \text{ for some } n_i; \lim_{i\to\infty} \mathcal{A}_{H_{n_i}}(T^{a_i} x_i) = \infty \right\} \tag{45}$$

We define the differential of (45) as the map induced by the differential on each $CF^*(H_n;\Gamma)$. The "pr" stands for "pre-quotient", as this is a chain complex that computes symplectic cohomology after first identifying generating sums of periodic orbits through continuation maps, as illustrated by the following lemma.

LEMMA 13 *Let* $\mathfrak{c}\colon \widehat{SC^*_{pr}}(D_R;\Lambda) \longrightarrow \widehat{SC^*_{pr}}(D_R;\Lambda)$ *be the linear extension of the continuation maps* $\{\mathfrak{c}_n\}$. *There is an isomorphism*

$$\widehat{SH^*_{pr}}(D_R;\Lambda) \Big/ \mathrm{im}(\mathfrak{c}^* - \mathrm{id}^*) \cong \widehat{SH^*}(D_R;\Lambda). \tag{46}$$

PROOF: First note that there is a chain isomorphism

$$\varprojlim_b \varinjlim_a \bigoplus_{n=0}^{\infty} CF^*_{(a,b)}(H_n;\Gamma) \cong \widehat{SC^*}_{pr}(D_R;\Lambda).$$

Thus, $\widehat{SC^*}(D_R;\Lambda)$ is chain isomorphic to $\widehat{SC^*_{pr}}(D_R;\Lambda) \oplus \widehat{SC^*_{pr}}(D_R;\Lambda)[1]$. If ι_1 is inclusion into the first factor, and π_2 is projection onto the second factor, the exact triangle

$$\begin{array}{ccc} \widehat{SC^*_{pr}}(D_R;\Lambda)[1] & \xrightarrow{\mathfrak{c}-\mathrm{id}} & \widehat{SC^*_{pr}}(D_R;\Lambda) \\ & {\scriptstyle \pi_2}\nwarrow \quad \swarrow{\scriptstyle \iota_1} & \\ & \widehat{SC^*}(D_R;\Lambda) & \end{array} \tag{47}$$

yields a long-exact sequence on homology

$$\cdots \longrightarrow \widehat{SH}^*_{pr}(D_R;\Lambda) \xrightarrow{\iota_1^*} \widehat{SH}^*(D_R;\Lambda) \xrightarrow{\pi_2^*} \widehat{SH}^*_{pr}(D_R;\Lambda) \xrightarrow{\mathfrak{c}^*-\mathrm{id}^*} \widehat{SH}^{*+1}_{pr}(D_R;\Lambda) \longrightarrow \cdots \tag{48}$$

This begets the short exact sequence

$$0 \longrightarrow \widehat{SH}^*_{pr}(D_R;\Lambda) \Big/ \mathrm{im}(\mathfrak{c}^* - \mathrm{id}^*) \longrightarrow \widehat{SH}^*(D_R;\Lambda) \longrightarrow \ker(\mathfrak{c}^* - \mathrm{id}^*) \longrightarrow 0. \tag{49}$$

We want to show that $\ker(\mathfrak{c}^* - \mathrm{id}^*) = 0$. Consider a sum $\mathbf{X} = \sum_{n=0}^{\infty} \eta_n \in \widehat{SH}^*_{pr}(D_R;\Lambda)$, where $\eta_n \in CF^*(H_n;\Gamma)$. By construction, $(\mathfrak{c} - \mathrm{id})(\mathbf{X})$ is a coboundary if and only if each η_n satisfies $\mathfrak{c}_n(\eta_n) - \eta_{n+1} = \partial^{fl}(\zeta_{n+1})$ for some ζ_{n+1}. In particular, η_0 is a coboundary.

We proceed by induction. Suppose that all η_n with $n \leq N$ are coboundaries. Continuation maps preserve coboundaries, and so $\mathfrak{c}_n(\eta_n)$ is a coboundary. It follows that

$$\eta_{n+1} = \mathfrak{c}_n(\eta_n) - \partial^{fl}(\zeta_{n+1})$$

is a coboundary, and so $\mathbf{X}$ is a coboundary as well. Thus, $\ker(\mathfrak{c}^* - \mathrm{id}^*) = 0$, and equation (49) proves the lemma.

□

We now further investigate the behavior of $\mathfrak{c}^*$. Let $\mathfrak{w}(\sum_{i=0}^{j} d_i T^{a_i} x_i)$ be the maximum of the winding numbers of the periodic orbits x_i.

LEMMA 14 *If x is a one-periodic orbit of H_n then $\mathfrak{w}(\mathfrak{c}_n(x)) \leq \mathfrak{w}(x)$*

PROOF: This is the analogous continuation-map statement of Lemma 12, using the fact that the Hamiltonian H_n^s defining c_n is monotone decreasing in s.

□

For simplicity we will make the following assumption:

ASSUMPTION 1 *$h_n = h_{n+1}$ on some interval $[0, k\pi R_n^2]$ such that h_n is linear on $(k\pi R_n^2, k\pi R^2]$. The Hamiltonian H_n^s inducing a continuation map is constant on $[0, k\pi R_n^2]$. In particular, H_n^s is constant on a disc bundle containing all periodic orbits of H_n.*

This assumption is not necessary, but it simplifies later proofs due to the following Corollary of Lemma 14.

COROLLARY 1 *Under Assumption 1,* c_n *coincides with the canonical inclusion, i.e. all Floer solutions contributing to* c_n *are constant.*

PROOF: Lemma 14 implies that any non-constant Floer solution contributing to c_n must leave and re-enter the region of E where $(H_n)_s$ is s-independent. Applying the integrated maximum principle as in Lemma 12 and using the fact that the continuation Hamiltonians h^s_n are monotone decreasing in s now yields the desired result.

□

LEMMA 15 *The action of a sequence of distinct terms*

$$\left\{T^{a_i}z_i \in CF^{\mathfrak{k}}(H_{n_i};\Gamma) \;\middle|\; \mu_f(T^{a_i}z) = p_i\right\}_{i\in\mathbb{N}}$$

grows like $(1-\pi\tau R^2)p_i$.

PROOF: Following the methods of Albers-Kang in [9], we can rephrase the action in terms of symplectic area [9]. The first observation is that, for a periodic orbit $T^a z$ with $\mathfrak{w}(z) = \ell$, the index formula can be manipulated:

$$\mathfrak{k} = -2\ell + 2\tau a + \mu_f(\rho(z)) \pm \frac{1}{2} \iff \ell = -\frac{1}{2}\left(\mathfrak{k} - 2\tau a - \mu_f(\rho(z)) \mp \frac{1}{2}\right).$$

Let $r(z)$ be the radial coordinate of z. The action of $T^a z$ can be reformulated as

$$\begin{aligned}
\mathcal{A}_{H_n}(T^a z) &= a - \int_D \tilde{z}^*\Omega + \int_0^1 H_n(z(t)) + (1 + k\pi r(z)^2)\rho^* F(z(t))dt && (50)\\
&\approx a - \int_D \tilde{z}^* d(k\pi r(z)^2\alpha) + \int_0^1 H_n(z(t)) + (1 + k\pi r(z)^2)\rho^* F(z(t))dt && (51)\\
&= a - \ell\pi r(z)^2 + \int_0^1 H_n(z(t)) + (1 + k\pi r(z)^2)\rho^* F(z(t))dt && (52)\\
&= a + \frac{\pi r(z)^2}{2}\left(k - 2\tau a - \mu_f(\rho(z)) \mp \frac{1}{2}\right) + \int_0^1 H_n(z(t)) + (1 + k\pi r(z)^2)\rho^* F(z(t))dt && (53)\\
&= (1 - \pi\tau r(z)^2)a + C(z), && (54)
\end{aligned}$$

where

$$C(z) = \frac{\pi r(z)^2}{2}\left(\mathfrak{k} - \mu_f(\rho(z)) \mp \frac{1}{2}\right) + \int_0^1 H_n(z(t)) + (1 + k\pi r(z)^2)\rho^* F(z(t))dt$$

is uniformly bounded. A sequence of distinct orbits z_i satisfy $r(z_i) \to R$. Thus, $\mathcal{A}_{H_n}(T^{a_i} z_i)$ grows like $(1 - \pi\tau R^2)a_i$.

On the other hand

$$p_i = 2(\tau + k)a_i + \mu_f(\rho(z)),$$

grows linearly in a_i. If $R \neq \frac{1}{\sqrt{\pi\tau}}$ the Lemma immediately follows.

So suppose $R = \frac{1}{\sqrt{\pi\tau}}$. By the assumptions of boundedness and convexity on each h^n, as well as the assumption that $(h^n)'(k\pi R_n^2) = \frac{1}{k} + \frac{1}{2k}$, it follows that

$$\sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right)\left(\pi R_{n+1}^2 - \pi R_n^2\right) < \infty.$$

Therefore, $O(\pi R_{n+1}^2 - \pi R_n^2) < \frac{1}{n^2}$ as $n \to \infty$. Because $\lim_{n\to\infty} R_n = \frac{1}{\sqrt{\pi\tau}}$, this implies that $1 - \pi\tau R_n^2 < \frac{1}{n}$ for large enough n. If $T^a z \in CF^*(H_n; \Gamma)$ for sufficiently large n, then

$$0 < (1 - \pi\tau R_n^2)\mathfrak{w}(z) \leq (1 - \pi\tau R_n^2)n < 1.$$

The Novikov weight a and the winding number $\mathfrak{w}(z)$ are linearly dependent. A sequence of distinct orbits z_i in degree $\mathfrak{k}$ satisfy $\mathfrak{w}(z_i) \to \infty$ as $i \to \infty$, implying that $n_i \to \infty$. The Lemma follows.

□

We can now begin to compute the differential. Each S^1-family of periodic orbits of H_n appears twice in the set of generators of $CF^*(H_n; \Gamma)$, corresponding to the maximum and minimum of a perfect Morse function on S^1.

LEMMA 16 *Let x_+^ℓ, respectively $x_-^{\ell+1}$, be the maximum, respectively minimum, of a perfect Morse function on an S^1-family of Reeb orbits of period ℓ, respectively $\ell + 1$, in the same fiber. Then $\partial_0(x_-^{\ell+1}) = x_+^\ell$.*

PROOF: Let $u(s,t)$ be a Floer solution contributing to $\partial_0(x_+^\ell)$. Then $\rho \circ u = p$ is constant, and so u remains in a single fiber E_p. u therefore corresponds to a Floer solution v on $\mathbb{C}$ satisfying

$$\partial_s v + i_t(\partial_t v - X_{h_n(k\pi r^2) + f(p)(1+k\pi r^2)}) = 0. \tag{55}$$

Conversely, any rigid solution of (55) contributes to ∂_0. It therefore suffices to understand the Floer theory of the data $(h_n(k\pi r^2) + f(p)(1 + k\pi r^2), i_t)$ on $\mathbb{C}$. This is a complex generated by a single one-periodic orbit in every degree between $-n$ and 0 (see Figure 4.2). The degree-zero orbit is the constant orbit at the origin. The even-degree orbits are the maxima of perfect Morse functions on Reeb trajectories, and the odd-degree orbits are the minima. By Assumption 1 the continuation maps act as canonical inclusions, and so the full symplectic cochain complex is rank one in every non-positive degree. As shown in [47], $SH^*(\mathbb{C}) = 0$. The constant orbit therefore needs to lie in the image of the differential; it must be precisely the image of the degree-one orbit. The differential squares to zero, and so the degree-two orbit needs to lie in the image of the differential; it must be precisely the image of the degree-three orbit. Proceeding inductively, the result follows. A schematic is shown in Figure 4.3.

□

LEMMA 17 *If* $\mathbf{X} = \sum_{i=0}^{\infty} c_i T^{a_i} x_i$ *is a cocycle in* $\widehat{SC^{\mathfrak{k}}_{pr}}(D_R;\Lambda)$ *then every* x_i *is either constant or corresponds to the maximum of the perfect Morse function on the underlying trajectory.*

PROOF: Suppose for contradiction that there exists n_0 such that x_{n_0} is the minimum of the perfect Morse function on an underlying trajectory. Recall the filtration (44), and choose p_0 so that $T^{a_{n_0}} x_{n_0} \in \mathcal{F}_{p_0} \setminus \mathcal{F}_{p_0+1}$. By Lemma 17, $\partial_0(T^{a_{n_0}} x_{n_0}) \neq 0$. Thus, there exists $\tau_1 > 0$ and x_{n_1} such that $\partial_0(T^{a_{n_0}} x_{n_0})$ is a summand of $\partial_{\tau_1}(T^{a_{n_1}} x_{n_1})$. Setting $p_1 = p_0 - \tau_1$, we have $T^{a_{n_1}} x_{n_1} \in \mathcal{F}_{p_1} \setminus \mathcal{F}_{p_1+1}$. By degree considerations, using the fact that $f\colon \mathbb{C}P^m \longrightarrow \mathbb{R}$ is perfect, x_{n_1} is also the minimum of a perfect Morse function on an underlying trajectory. Thus, $\partial_0(x_{n_1}) \neq 0$.

Iterating this construction, we find an infinite sequence of terms

$$\{T^{a_{n_0}} x_{n_0} \in \mathcal{F}_{p_0} \setminus \mathcal{F}_{p_0+1}, T^{a_{n_1}} x_{n_1} \in \mathcal{F}_{p_1} \setminus \mathcal{F}_{p_1+1}, T^{a_{n_2}} x_{n_2} \in \mathcal{F}_{p_2} \setminus \mathcal{F}_{p_2+1}, \dots\}$$

with $p_0 > p_1 > p_2 > \dots$ a decreasing sequence of integers. By Lemma 15, $\lim\limits_{i\to\infty} \mathcal{A}(T^{a_{n_i}} x_{n_i}) = -\infty$. Thus, $\lim_{i\to\infty} \mathcal{A}(T^{a_i} x_i) \neq \infty$, contradicting the assumption that $\mathbf{X} \in \widehat{SC^{\mathfrak{k}}_{pr}}(D_R;\Lambda)$. □

PROPOSITION 1

$$\widehat{SH^*_{pr}}(D_R;\Lambda) \Big/ \mathrm{im}(\mathfrak{c}^* - \mathrm{id}^*) = 0.$$

PROOF: Let $\mathbf{X}$ be a cocycle in $\widehat{SC^{\mathfrak{k}}_{pr}}(D_R;\Lambda)$. We will show that there exists $\mathbf{Y} \in \widehat{SC^{\mathfrak{k}-1}_{pr}}(D_R;\Lambda)$ such that $\partial(\mathbf{Y}) \equiv \mathbf{X}$. To ease notation, note that $\partial_0 \colon CF^*(H_n;\Gamma) \longrightarrow CF^*(H_n;\Gamma)$ restricts to a bijection from {minima of Morse functions on S^1-families of Reeb orbits} to {constant trajectories and maxima of Morse functions on S^1-families of Reeb orbits of period less than n}. If x lives in the latter set, we will define $\partial_0^{-1}(x)$ accordingly, and extend Γ-linearly.

By Lemma 15, there is some $p \in \mathbb{Z}$ such that all summands of $\mathbf{X}$ live in $\mathcal{F}_p$. Write $\mathfrak{c}(\mathbf{X}) = \sum_{i=p}^{\infty} \eta_i$, where $\eta_i \in \mathcal{F}_i$. Corollary 1 and Lemma 17 imply that each η_i is in the image of ∂_0. We will build $\mathbf{Y}$ inductively:

Set $\zeta_0 = \partial_0^{-1}(\eta_0)$. Assume there exists $\zeta_0, \zeta_1, \ldots, \zeta_j$ such that

$$\sum_{i=0}^{j} (\partial_0 + \cdots + \partial_{j-i})\zeta_i = \sum_{i=1}^{j} \eta_i$$

in $\widehat{SC^{\mathfrak{k}}_{pr}}(D_R;\Lambda)$. By index considerations, all summands of $\partial_{j-i+1}\zeta_i$ are Novikov-weighted constant orbits or maxima of Morse functions on S^1-families of Reeb orbits. Again by Lemma 17, $\mathfrak{c}\left(\partial_{j-i+1}\zeta_i\right)$ is in the image of ∂_0. Define

$$\zeta_{j+1} = \partial_0^{-1}(\eta_{j+1}) + \sum_{i=0}^{j} \partial_0^{-1}\left(\mathfrak{c}\left(\partial_{j-i+1}\zeta_i\right)\right).$$

Then $\partial_0(\zeta_{j+1}) = \eta_{j+1} + \sum_{i=0}^{j} \mathfrak{c}\left(\partial_{j-i+1}\zeta_i\right)$. It follows that

$$\begin{aligned}\sum_{i=0}^{j+1} (\partial_0 + \cdots + \partial_{j+1-i})\zeta_i &= \sum_{i=0}^{j} \eta_i + \sum_{i=0}^{j} \partial_{j-i+1}\zeta_i + \sum_{i=0}^{j} \mathfrak{c}\left(\partial_{j-i+1}\zeta_i\right) + \eta_{j+1} \\ &\equiv \sum_{i=0}^{j+1} \eta_i.\end{aligned}$$

Now set $\mathbf{Y} = \sum_{i=0}^{\infty} \zeta_0$. By construction $\partial(\mathbf{Y}) \equiv \mathbf{X}$. We need to check that the minimum action of terms in ζ_i goes to ∞ with i. But continuation maps increase μ_f, and so $\zeta_i \in \mathcal{F}_i$. Through Lemma 15, we conclude that $\mathbf{Y} \in \widehat{SC^{\mathfrak{k}-1}_{pr}}(D_R;\Lambda)$. □

4.2.8 Computing $\widehat{SH}^*(D_R;\Lambda)$ when $R \geq \frac{1}{\sqrt{\pi\tau}}$

Let $SH^*(E;\Lambda)$ be the uncompleted symplectic cohomology of E with coefficients in Λ. $SH^*(E;\Lambda)$ is independent of the family of non-degenerate Hamiltonians used to define it, given that each Hamiltonian is of linear slope near infinity and the slopes increase to infinity as one runs through the family. In the context of E, "linear slope near infinity" means that each Hamiltonian is linear in the coordinate $\frac{1+k\pi r^2}{1+k\pi}$. Let $\widetilde{H}_n$ be a Hamiltonian equal to H_n in D_R and linear of slope $(1+k\pi)\frac{n}{k}\frac{1+k\pi r^2}{1+k\pi}$ for $r >> 0$. By the smallness of f, the one-periodic orbits of H_n and $\widetilde{H}_n^s$ are canonically identified, and the integrated maximum principle applies to canonically identify the Floer differentials as well. Let $\widetilde{H}_n^s$ be a monotone decreasing homotopy with $\widetilde{H}_n^s = H_n$ for $s << 0$ and $\widetilde{H}_n^s = \widetilde{H}_n$ for $s >> 0$. Applying the integrated maximum principle as in the proof of Lemma 12 shows that the continuation map induced by $\widetilde{H}_n^s$ acts as the canonical identification, and so $CF^*(H_n;\Gamma) \cong CF^*(\widetilde{H}_n;\Gamma)$. Continuation maps, as canonical inclusions, commute with this identification as well. It follows that a chain complex computing $SH^*(E;\Lambda)$ is

$$SC^*(E;\Lambda) = \operatorname*{hocolim}_{n\in\mathbb{N}} \varprojlim_{b} \varinjlim_{a} CF^*_{(a,b)}(H_n;\Gamma) \tag{56}$$

We assume for the rest of this subsection that $R \geq \frac{1}{\sqrt{\pi\tau}}$. To compute $\widehat{SH}^*(D_R;\Lambda)$ we will need the following fact.

LEMMA 18 *For every* $\mathfrak{k} \in \mathbb{N}$ *there exists a* $B \in \mathbb{R}$ *such that if* $T^a x \in CF^{\mathfrak{k}}(H_n;\Gamma)$, *for any* $n \in \mathbb{N}$, *then* $\mathcal{A}_{H_n}(T^a x) < B$.

PROOF: Let $\mathfrak{w}(x) = \ell$. Manipulating the index formula $\mathfrak{k} = -2\ell + 2\tau a + \mu_f(\rho(z)) \pm \frac{1}{2}$ yields

$$a = \frac{1}{2\tau}\left(\mathfrak{k} + 2\ell - \mu_f(\rho(z)) \mp \frac{1}{2}\right),$$

which is positive whenever

$$2\ell > -\mathfrak{k} + 2m + \frac{1}{2}, \tag{57}$$

where m is the complex dimension of the base. Note that there are finitely many elements $T^s x$ with $|T^s x| = \mathfrak{k}$ that do not satisfy (57). Recall that each Hamiltonian H_n is bounded between 0 and some fixed constant $C > 0$. If $R > \frac{1}{\sqrt{\pi\tau}}$, equation (54) now shows that

$$\mathcal{A}_{H_n}(T^a x) \leq C(x) < \frac{\pi R^2}{2}\left(\mathfrak{k} + 2m + \frac{1}{2}\right) + C + (1 + k\pi R^2) =: C'$$

Define

$$B = \max\left\{\mathcal{A}_{H_n}(T^a x) + 1 \;\middle|\; 2\mathfrak{w}(x) \le -\mathfrak{k} + 2m + \frac{1}{2}\right\} \cup \{C'\}.$$

If $R = \frac{1}{\sqrt{\pi\tau}}$ then the proof of Lemma 15 shows that the action is similarly uniformly bounded, and we define B accordingly.

□

PROPOSITION 2 *For every* $\mathfrak{k} \in \mathbb{Z}$ *there is an isomorphism*

$$\kappa\colon SC^{\mathfrak{k}}(E;\Lambda) \xrightarrow{\simeq} \widehat{SC}^{\mathfrak{k}}(D_R;\Lambda).$$

PROOF: $\widehat{SC^*}(D_R;\Lambda)$ is defined by equation (42). Denote $\varinjlim_a CF^{\mathfrak{k}}_{(a,b)}(H_n;\Gamma)$ by $CF^{\mathfrak{k}}_{(-\infty,b)}(H_n;\Gamma)$. We need to show that the limit over n commutes with the limit over b, that is, that

$$\operatorname*{hocolim}_{n\in\mathbb{N}} \varprojlim_b CF^{\mathfrak{k}}_{(-\infty,b)}(H_n;\Gamma) = \varprojlim_b \operatorname*{hocolim}_{n\in\mathbb{N}} CF^{\mathfrak{k}}_{(-\infty,b)}(H_n;\Gamma).$$

Let B be as in Lemma 18. Then for every $b' \ge B$ the map

$$\operatorname*{hocolim}_{n\in\mathbb{N}} CF^{\mathfrak{k}}_{(-\infty,b')}(H_n;\Gamma) \longrightarrow \operatorname*{hocolim}_{n\in\mathbb{N}} CF^{\mathfrak{k}}_{(-\infty,B)}(H_n;\Gamma)$$

is an isomorphism. Therefore,

$$\begin{aligned}\varprojlim_b \operatorname*{hocolim}_{n\in\mathbb{N}} CF^{\mathfrak{k}}_{(-\infty,b)}(H_n;\Gamma) &\cong \operatorname*{hocolim}_{n\in\mathbb{N}} CF^{\mathfrak{k}}_{(-\infty,B)}(H_n;\Gamma)\\ &\cong \operatorname*{hocolim}_{n\in\mathbb{N}} \varprojlim_b CF^{\mathfrak{k}}_{(-\infty,b)}(H_n;\Gamma).\end{aligned}$$

As the colimit over a commutes with the homotopy colimit over n, the result follows.

□

4.3 Symplectic homology of disc bundles

Define the action-completed chain complex of D_R, $\widehat{SC_*}(D_R;\Lambda)$, to be the dual complex of $SC^*\widehat{(D_R;\Lambda)}$. Explicitly,

$$\widehat{SC_*}(D_R;\Lambda) = \left(\varprojlim_{b} \varinjlim_{a} \operatorname{holim}_{n\in\mathbb{N}} CF^*_{(a,b)}(-H_n;\Gamma), \partial \right). \tag{58}$$

THEOREM 6 *Let* (E,Ω) *be a degree* $-k$ *monotone negative line bundle over* $\mathbb{C}P^m$ *with monotonicity constant* $\tau = 1+m-n$. *Let* $D_R \longrightarrow \mathbb{C}P^m$ *be the disc subbundle of radius* R. *Then*

$$\widehat{SH_*}(D_R;\Lambda) \cong \begin{cases} 0 & R < \frac{1}{\sqrt{\pi\tau}} \\ SH^*(E) & R \geq \frac{1}{\sqrt{\pi\tau}} \end{cases}.$$

LEMMA 19 *There is a vector-space isomorphism*

$$SH^*(E) \cong SH_*(E).$$

PROOF: Let $V = \Lambda^m$. Ritter defines a Λ-linear map $c\colon V \longrightarrow V$ and computes $SH^*(E)$ as the homology of the chain complex

$$V \xrightarrow{c} V \xrightarrow{c} V \xrightarrow{c} V \xrightarrow{c} \cdots$$

with the zero differential. In particular, $SH^*(E) \cong V\big/\ker(c^m)$. Dualizing, symplectic homology is the homology of

$$\cdots \xrightarrow{c^*} V^* \xrightarrow{c^*} V^* \xrightarrow{c^*} V^* \xrightarrow{c^*} V^*$$

with zero differential, and so $SH_*(E) \cong \operatorname{im}((c^*)^m)$. Finally,

$$\operatorname{im}((c^*)^m) \cong (\ker(c^m))^o \cong \left(V\big/\ker(c^m)\right)^* \cong V\big/\ker(c^m).$$

□

As in the above computation of symplectic cohomology, it will be useful to work with the smaller chain complex

$$\widehat{SC^{pr}_*}(D_R;\Lambda) := \lim_{\overleftarrow{b}} \lim_{\overrightarrow{a}} \prod_{n=0}^{\infty} CF^*_{(a,b)}(-H_n;\Gamma)$$

equipped with the componentwise Floer differential. As in Lemma 13, there is a short exact sequence

$$0 \longrightarrow \widehat{SH^{pr}_*}(D_R;\Lambda) \Big/ \mathrm{im}(\mathfrak{c}^* - \mathrm{id}^*) \longrightarrow \widehat{SH}_*(D_R;\Lambda) \longrightarrow \ker(\mathfrak{c}^* - \mathrm{id}^*) \longrightarrow 0. \tag{59}$$

We will show that, if $R < \frac{1}{\sqrt{\pi\tau}}$, both $\widehat{SH^{pr}_*}(D_R;\Lambda) \Big/ \mathrm{im}(\mathfrak{c}^* - \mathrm{id}^*)$ and $\ker(\mathfrak{c}^* - \mathrm{id}^*)$ are zero. We omit proofs that proceed entirely analogously to the dual proofs for symplectic cohomology.

LEMMA 20 *Let x_+^ℓ, respectively $x_-^{\ell+1}$, be the maximum, respectively minimum, of a perfect Morse function on an S^1-family of Reeb orbits of period ℓ, respectively $\ell+1$. Then $\partial_0(x_+^\ell) = x_-^{\ell+1}$. Furthermore, $\partial_0(x^0) = x_-^1$.*

LEMMA 21 *There exists a constant C_k for each $k \in \mathbb{Z}$ such that if $x \in \bigcup_n \mathcal{P}(H_n)$ has $\mathfrak{w}(x) = \ell$ and $T^a x$ has degree k then $\mathcal{A}_H(T^a x)$ is bounded between $\left(1 - \pi\tau r(z)^2\right)\ell \pm C_k$ and $\mu_f(T^a x)$ is bounded between $\frac{\ell}{\tau} \pm C_k$.*

LEMMA 22 *If $\mathbf{X} = \sum_{i=0}^{\infty} c_i T^{a_i} x_i$ is a cycle in $\widehat{SC^{pr}_k}(D_R;\Lambda)$ such that every x_i corresponds to the minimum of the perfect Morse function on the underlying trajectory then $[\mathbf{X}]$ is a boundary.*

PROOF: Let $x_i \in \mathcal{P}(H_{n_i})$. We proceed analogously to the proof of Proposition 1. By Lemma 20 there exists ζ_0^i such that $\partial_0(\zeta_0^i) = c_i T^{a_i} x_i$. In the notation of the proof of Proposition 1, we set $\zeta_0^i = \partial_0^{-1}(c_i T^{a_i} x_i)$. Inductively apply Lemma 20 to define

$$\zeta_j^i = \sum_{i=0}^{j} \partial_0^{-1} \circ \partial_i(\zeta_{j-i}^i).$$

Note that the Conley-Zehnder index μ_f of each summand of ζ_j^i is $\mu_f(c_i T^{a_i} x_i) + j$. The winding number of elements of $\mathcal{P}(H_{n_i})$ is bounded above by n_i, and so, by Lemma 21, there exists j_i such that $\zeta_j^i = 0$ for all $j > j_i$. Let $\zeta_i = \sum_{j=0}^{j_i} \zeta_j^i$. By construction, $\partial(\zeta_i) = c_i T^{a_i} x_i$.

Define $\mathbf{Y} = \sum_{i=0}^{\infty} \zeta_i$. By construction, $\partial(\mathbf{Y}) = \mathbf{X}$. We need to check that $\mathcal{A}_{H_{n_i}}(\zeta_i) \longrightarrow \infty$ as $i \longrightarrow \infty$. By Lemma 15, if $T^{a_i} x_i \in \mathcal{F}_{p_i} \setminus \mathcal{F}_{p_i - 1}$, then $\lim_{i\to\infty} p_i = \infty$. By construction, $\zeta_i \in SC^{\mathfrak{k}}(D_R;\Lambda) \setminus \mathcal{F}_{p_i-1}$. Again by Lemma 15, $\mathcal{A}_{H_{n_i}}(\zeta_i) \longrightarrow \infty$.

□

LEMMA 23 *If* $\mathbf{X} = \sum_{i=0}^{\infty} c_i T^{a_i} x_i$ *is a cycle in* $\widehat{SC}^{pr}_{\mathfrak{k}}(D_R;\Lambda)$ *such that every* x_i *corresponds to either a constant orbit or the maximum of the perfect Morse function on the underlying trajectory then* $[\mathbf{X}] \in \mathrm{im}(\mathfrak{c}^* - \mathrm{id}^*)$.

PROOF: We first show that $\mathbf{X}$ is a finite sum. Let $\mathbf{X_n} = \sum_{i=i_n}^{I_n} c_i T^{\alpha_i} x_i$ be the projection of $\mathbf{X}$ onto the '$CF^{\mathfrak{k}}(-H_n;\Lambda)$' summand. Note that $\mathbf{X_n}$ is a finite sum. By assumption, $\mathbf{X_n}$ is a cycle. Let $\min_{i_n \le i \le I_n} \mu_f(T^{a_i} x_i)$ be realized by $T^{a_j} x_j$. If $\partial_0(c_j T^{a_j} x_j) \neq 0$ then $\mathbf{X_n}$ is not a cycle: any canceling contributions of $\partial_0(T^{a_j} x_j)$ to $\partial(\mathbf{X_n})$ would come from elements $T^{a_i} x_i$ of strictly smaller Conley-Zehnder f-index than $T^{a_j} x_j$, contradicting the choice of $T^{a_j} x_j$. Lemma 20 now implies that $\mathfrak{w}(x_j) = n$. Thus, if the projection of $\mathbf{X}$ onto the '$CF^{\mathfrak{k}}(-H_n;\Lambda)$' summand is non-zero, $\mathbf{X}$ includes a summand with periodic-orbit of winding-number $-n$.

By Lemma 21 and the assumption that $\mathcal{A}_H(T^{a_i} x_i) \longrightarrow \infty$, there exists $N > 0$ such that $\mathfrak{w}(x_i) \geq -N$ for all i. Thus, the projection of $\mathbf{X}$ onto the '$CF^{\mathfrak{k}}(-H_n;\Lambda)$' summand is zero for all $n > N$. Write $\mathbf{X} = \sum_{n=0}^{N} \eta_n$, for $\eta_n \in CF^{\mathfrak{k}}(H_n;\Lambda)$. Define

$$\mathbf{Y} = \sum_{n=0}^{N} \sum_{j=i}^{N} \mathfrak{c}^{j-n}(\eta_j),$$

where we set $\mathfrak{c}^0 = \mathrm{id}$. Then

$$\begin{aligned}
(\mathfrak{c} - \mathrm{id})(\mathbf{Y}) &= \sum_{n=0}^{N} \sum_{j=n}^{N} \mathfrak{c}^{j-n+1}(\eta_j) - \sum_{n=0}^{N} \sum_{j=n}^{N} \mathfrak{c}^{j-n}(\eta_j) \\
&= \sum_{n=0}^{N} \sum_{j=n+1}^{N} \left(\mathfrak{c}^{j-n}(\eta_j) - \mathfrak{c}^{j-n}(\eta_j)\right) + \sum_{n=0}^{N} \eta_n \\
&= \mathbf{X}.
\end{aligned}$$

Thus, $(\mathfrak{c}^* - \mathrm{id}^*)([\mathbf{Y}]) = [\mathbf{X}]$.

□

LEMMA 24 $\ker(\mathfrak{c}^* - \mathrm{id}) = 0$.

PROOF: Lemma 22 and the proof of Lemma 23 show that any degree-$\mathfrak{k}$ cycle $[\mathbf{X}] \neq 0$ in $\ker(\mathfrak{c}^* - \mathrm{id}^*)$ is a finite sum. Write $\mathbf{X} = \sum_{n=0}^{N} \eta_n$, for $\eta_n \in CF^{\mathfrak{k}}(H_n;\Lambda) \setminus \{0\}$. Without loss of generality, assume η_N is non-zero in $HF^*(H_N;\Lambda)$. Let π_N be projection onto the Nth factor, and note that π_N is a chain map. Then

$$\pi_N((\mathfrak{c} - \mathrm{id})(\mathbf{X})) = \pi_N(\eta_N) + \pi_N(\sum_{i=0}^{N-1} \mathfrak{c}(\eta_{i+1}) - \eta_i)$$
$$= \eta_N.$$

By assumption, $[\eta_N] \neq 0$, and so $[(\mathfrak{c} - \mathrm{id})(\mathbf{X}) \neq 0$. □

PROPOSITION 3 *If* $R < \frac{1}{\sqrt{\pi\tau}}$ *then* $\widehat{SH_k}(D_R;\Lambda) = 0$.

PROOF: By degree considerations either the conditions of Lemma 22 or of Lemma 23 hold. The result now follows from the exact sequence (59).

□

LEMMA 25 *For every* $\mathfrak{k}$ *there exists* A *such that if* $T^a x \in CF^{\mathfrak{k}}(-H_n;\Gamma)$, *for any* $n \in \mathbb{Z}_{\geq 0}$, *then* $\mathcal{A}_{H_n}(T^a x) > A$.

PROPOSITION 4 *There is an isomorphism* $\kappa\colon \widehat{SC_*}(D_R;\Lambda) \longrightarrow SC_*(E)$ *if* $R \geq \frac{1}{\sqrt{\pi\tau}}$.

Theorem 6 now follows from Lemma 19, Proposition 3, and Proposition 4.

4.4 Symplectic cohomology of annulus bundles

Recall the chain map

$$\widehat{\mathfrak{c}}\colon \widehat{SC_*}(D_{R_1};\Lambda) \longrightarrow \widehat{SC^*}(D_{R_2};\Lambda) \tag{60}$$

from Chapter 2. Let $A_{(R_1,R_2)}$ be the annulus subbundle between radii R_1 and R_2. The action-completed symplectic cohomology of $A_{(R_1,R_2)}$, denoted by $\widehat{SH^*}(A_{(R_1,R_2)};\Lambda)$, is defined to be the homology of the cone of $\widehat{\mathfrak{c}}$. We now wish to prove Theorem 2:

THEOREM 2 *Let* $E = \mathrm{Tot}\left(\mathcal{O}(-k) \xrightarrow{\rho} \mathbb{C}P^m\right)$ *be a negative line bundle with* $1 \leq k \leq m$, *and equip* E *with a Hermitian metric that induces an angular form* α *satisfying* $d\alpha = \rho^*\omega_{FS}$, *a radial coordinate* r, *and a symplectic form* $\Omega = d\alpha + d(k\pi r^2\alpha)$. *Let* $A_{(R_1,R_2)}$ *be the annulus subbundle between radii* R_1 *and* R_2.

Then

$$\widehat{SH^*}(A_{(R_1,R_2)};\Lambda) \cong \begin{cases} SH^*(E;\Lambda) & R_1 < \frac{1}{\sqrt{\pi(1+m-k)}} \leq R_2 \\ 0 & \text{else} \end{cases}.$$

To prove Theorem 2 we recall the construction in [44] of symplectic cohomology. Define a Hamiltonian $\mathcal{H}_n = (1+\pi r^2)(n+\frac{1}{2}+\rho^* f)$ for each $n \in \mathbb{Z}$. These Hamiltonians have precisely m one-periodic orbits: the constant orbits corresponding to the the critical points of f on the zero-section. Let $g_t = e^{2\pi i t}$ be the Hamiltonian action rotating the fibers; this action corresponds to the Hamiltonian $K = 1+\pi r^2$. It follows that $g^*(\mathcal{H}_n) = \mathcal{H}_{n-1}$. Let $\tilde{g}_t$ be a lift of g_t to $\mathcal{L}M$ preserving the constant discs mapping to critical points of f.

Define an isomorphism $\mathcal{S}\colon CF^*(H;J) \longrightarrow CF^{*+2}(g^*H;g^*J)$ by $u \mapsto \tilde{g}^{-1}\cdot u$.

THEOREM 7 (RITTER, [44])

$$\begin{array}{ccc} CF^*(\mathcal{H}_n;J) & \xrightarrow{c_n} & CF^*(\mathcal{H}_{n+1};J) \\ \downarrow \mathcal{S} & & \downarrow \mathcal{S} \\ CF^{*+2}(\mathcal{H}_{n-1};g^*J) & \xrightarrow{c_{n-1}} & CF^{*+2}(\mathcal{H}_n;g^*J) \end{array}$$

It follows from Theorem 7 that the complex

$$CF^*(\mathcal{H}_0;J) \xrightarrow{c_0} CF^*(\mathcal{H}_1;J) \xrightarrow{c_1} CF^*(\mathcal{H}_2;J) \xrightarrow{c_2} \cdots$$

is isomorphic to the complex

$$\begin{array}{ccccc} CF^*(\mathcal{H}_0;J) & & & CF^*(\mathcal{H}_1;J) & \\ & \searrow^{\mathcal{S}} & & \nearrow^{\mathcal{S}^{-1}} \quad \searrow^{\mathcal{S}^2} & \\ & CF^{*+2}(\mathcal{H}_{-1};g^*J) \xrightarrow{c_{-1}} CF^{*+2}(\mathcal{H}_0;g^*J) & & & CF^{*+4}(\mathcal{H}_{-1};g^*J) \xrightarrow{c_{-1}} \cdots \end{array}$$

which is isomorphic to

$$\begin{array}{ccccccc} & CF^{*+2}(\mathcal{H}_{-1}) & & CF^{*+4}(\mathcal{H}_{-1}) & & \cdots \\ {}^{\mathcal{S}}\nearrow & & \searrow^{c_{-1}} \quad {}^{\mathcal{S}}\nearrow & & \searrow^{c_{-1}} \quad {}^{\mathcal{S}}\nearrow & \\ CF^*(\mathcal{H}_0) & & CF^{*+2}(\mathcal{H}_0) & & CF^{*+4}(\mathcal{H}_0) & \end{array} \tag{61}$$

We have thus far defined the chain complexes through models for homotopy limits and colimits. To fit with the above description of symplectic cohomology and to be compatible with the results in [16] we now switch to definitions of symplectic cohomology and homology as direct and inverse limits of Floer complexes. To be precise, we abuse notation and define

$$\widehat{SC^*}(D_R;\Lambda) := \varprojlim_b \varinjlim_a \varinjlim_n CF^*_{(a,b)}(H_n;\Gamma)$$

and

$$\widehat{SC_*}(D_R;\Lambda) := \varprojlim_b \varinjlim_a \varprojlim_n CF^*_{(a,b)}(H_n;\Gamma).$$

This is justified by Lemma 26.

LEMMA 26 *There are quasi-isomorphisms*

$$\varinjlim_n CF^*_{(a,b)}(H_n;\Gamma) \cong \operatorname*{hocolim}_{n\in\mathbb{N}} CF^*_{(a,b)}(H_n;\Gamma)$$

and

$$\varprojlim_n CF^*_{(a,b)}(H_n;\Gamma) \cong \operatorname*{holim}_{n\in\mathbb{N}} CF^*_{(a,b)}(H_{-n};\Gamma).$$

PROOF: The colimit case is shown in [5] in the proof of Lemma 3.12. The inverse limit case follows similarly.

□

REMARK 14) The quasi-isomorphisms of Proposition 2 and Proposition 4 induce quasi-isomorphisms

$$\kappa\colon SC^*(E;\Lambda) \longrightarrow \widehat{SC^*}(D_R;\Lambda) \qquad \text{and} \qquad \kappa\colon \widehat{SC_*}(D_R;\Lambda) \longrightarrow SC_*(E;\Lambda)$$

when $R > \frac{1}{\sqrt{\pi\tau}}$.

The final ingredient needed to prove Theorem 2 is the computation of the map

$$\mathfrak{c}^*\colon \widehat{SH_*}(D_{R_1};\Lambda) \longrightarrow \widehat{SH^*}(D_{R_2};\Lambda)$$

when both groups are non-zero.

PROPOSITION 5 *Let* $R_1 > \frac{1}{\sqrt{\pi\tau}}$. *There is a quasi-isomorphism* $\mathfrak{c}\colon SC_*(E) \longrightarrow SC^*(E)$ *such that the* *following diagram commutes.*

$$\begin{array}{ccc} SH_*(E;\Lambda) & \xrightarrow{\mathfrak{c}^*} & SH^*(E;\Lambda) \\ \simeq\big\uparrow \kappa^* & & \kappa^* \big\downarrow \simeq \\ \widehat{SH}_*(D_{R_1};\Lambda) & \xrightarrow{\widehat{\mathfrak{c}}^*} & \widehat{SH}^*(D_{R_2};\Lambda) \end{array} \tag{62}$$

The proof of Proposition 5 uses a proposition due to Cieliebak-Frauenfelder [16]. We recall this result. Let $(\mathfrak{A}, \leq)$ and $(\mathfrak{B}, \leq)$ be two ordered sets and let $G = \{G_{\mathfrak{a}}^{\mathfrak{b}}\}_{\mathfrak{a}\in A, \mathfrak{b}\in B}$ be a family of abelian groups double-indexed by A and B. Assume that for all $\mathfrak{b} \in B$ and $\mathfrak{a}_1 \leq \mathfrak{a}_2 \in A$ there exists a homomorphism $\pi^{\mathfrak{b}}_{\mathfrak{a}_2,\mathfrak{a}_1} \colon G^{\mathfrak{b}}_{\mathfrak{a}_1} \longrightarrow G^{\mathfrak{b}}_{\mathfrak{a}_2}$ and for all $\mathfrak{a} \in A$ and $\mathfrak{b}_1 \leq \mathfrak{b}_2 \in B$ there exists a homomorphism $\iota^{\mathfrak{a}}_{\mathfrak{b}_1,\mathfrak{b}_2} \colon G^{\mathfrak{b}_1}_{\mathfrak{a}} \longrightarrow G^{\mathfrak{b}_2}_{\mathfrak{a}}$. Call (G, π, ι) a bidirect system. Fixing $\mathfrak{a}$, the direct limit of the maps $\iota^{\mathfrak{a}}_{\mathfrak{b}_1,\mathfrak{b}_2}$ over B induce a map $\iota^{\mathfrak{a}}$. Fixing $\mathfrak{b}$, the inverse limit of the maps $\pi^{\mathfrak{b}}_{\mathfrak{a}_1,\mathfrak{a}_2}$ over A induce a map $\pi_{\mathfrak{b}}$.

LEMMA 27 (PROPOSITION 1.3 IN [16]) *There exists unique homomorphisms $\kappa^{\mathfrak{b}}$ and κ such that the following diagram commutes.*

$$\begin{array}{ccccc} G^{\mathfrak{b}}_{\mathfrak{a}} & \xleftarrow{\pi^{\mathfrak{b}}_{\mathfrak{a}}} & \varprojlim G^{\mathfrak{b}} & \xrightarrow{\iota^{\mathfrak{b}}} & \varinjlim \varprojlim G \\ \big\downarrow \iota^{\mathfrak{b}}_{\mathfrak{a}} & & \big\downarrow \kappa^{\mathfrak{b}} & \swarrow \kappa & \\ \varinjlim G_{\mathfrak{a}} & \xleftarrow{\pi_{\mathfrak{a}}} & \varprojlim \varinjlim G & & \end{array} \tag{63}$$

PROOF OF PROPOSITION 5: Let $H_{-1} = -H_0$. The quasi-isomorphism $\mathfrak{c}$ is defined by the composition

$$SC_*(E) \longrightarrow CF^*(H_{-1}) \longrightarrow CF^*(H_0) \longrightarrow SC^*(E).$$

We first check that the induced map $\mathfrak{c}$ is a quasi-isomorphism. As symplectic homology and cohomology and the continuation map c_{-1} only depend on the behavior of the Hamiltonians at infinity, we can use Ritter's model to understand $\mathfrak{c}$.

We omit grading shifts from the notation. Dualizing (61), $SC_*(E)$ is isomorphic to the inverse limit

$$\begin{array}{ccccccccc} CF_*(\mathcal{H}_0) & & & & CF_*(\mathcal{H}_0) & & & & CF_*(\mathcal{H}_0) & \\ & \nwarrow^{\mathcal{S}^*} & & \swarrow_{c^*_{-1}} & & \nwarrow^{\mathcal{S}^*} & & \swarrow_{c^*_{-1}} & & \nwarrow^{\mathcal{S}^*} \\ & & CF_*(\mathcal{H}_{-1}) & & & & CF_*(\mathcal{H}_{-1}) & & & & \cdots \end{array}$$

Poincarè duality in Floer theory implies that there are commutative diagrams

$$\begin{CD} CF_*(\mathcal{H}_0) @>{c_{-1}^*}>> CF_*(\mathcal{H}_{-1}) \\ @V{\simeq}VV @V{\simeq}VV \\ CF^*(\mathcal{H}_{-1}) @>{c_{-1}}>> CF^*(\mathcal{H}_0) \end{CD} \qquad\qquad \begin{CD} CF_*(\mathcal{H}_{-1}) @>{\mathcal{S}^*}>> CF_*(\mathcal{H}_0) \\ @V{\simeq}VV @V{\simeq}VV \\ CF^*(\mathcal{H}_0) @>{\mathcal{S}}>> CF^*(\mathcal{H}_{-1}) \end{CD}$$

We therefore have the diagram

$$\begin{array}{ccccccccc}
 & & CF^*(\mathcal{H}_{-1}) & & & & CF^*(\mathcal{H}_{-1}) & & & \\
 & \overset{\mathcal{S}}{\nearrow} & & \overset{c_{-1}}{\searrow} & & \overset{\mathcal{S}}{\nearrow} & & \overset{c_{-1}}{\searrow} & & \\
CF^*(\mathcal{H}_0) & & & & CF^*(\mathcal{H}_0) & & & & \cdots & = SC^*(E) \\
{\scriptstyle c_{-1}}\uparrow & & & & & & & & & \uparrow{\scriptstyle \mathfrak{c}} \\
CF^*(\mathcal{H}_{-1}) & & & & CF^*(\mathcal{H}_{-1}) & & & & \cdots & = SC_*(E) \\
 & \underset{\mathcal{S}}{\nwarrow} & & \underset{c_{-1}}{\swarrow} & & \underset{\mathcal{S}}{\nwarrow} & & \underset{c_{-1}}{\swarrow} & & \\
 & & CF^*(\mathcal{H}_0) & & & & CF^*(\mathcal{H}_0) & & &
\end{array}$$

As $CF^*(\mathcal{H}_0)$ has rank m, the image of the map $(c_{-1} \circ \mathcal{S})^i$ stabilizes when $i = m$. Thus, the bottom complex is isomorphic to $\operatorname{im}((c_{-1} \circ \mathcal{S})^m) \subset CF^*(\mathcal{H}_0)$ and the top complex is isomorphic to $CF^*(\mathcal{H}_0) \big/ \ker((c_{-1} \circ \mathcal{S})^m)$. Under these isomorphims the map $\mathfrak{c}$ becomes

$$\operatorname{im}((c_{-1} \circ \mathcal{S})^m) \xrightarrow{c_{-1} \circ \mathcal{S}} CF^*(\mathcal{H}_0) \longrightarrow CF^*(\mathcal{H}_0) \big/ \ker((c_{-1} \circ \mathcal{S})^m).$$

By the stabilization of $c_{-1} \circ \mathcal{S}$, we have $(c_{-1} \circ \mathcal{S})\big(\operatorname{im}((c_{-1} \circ \mathcal{S})^m)\big) = \operatorname{im}((c_{-1} \circ \mathcal{S})^m)$. In particular, $(c_{-1} \circ \mathcal{S})\big|_{\operatorname{im}((c_{-1} \circ \mathcal{S})^m)}$ is injective, and so $\operatorname{im}((c_{-1} \circ \mathcal{S})^m)$ injects into $CF^*(\mathcal{H}_0) \big/ \ker((c_{-1} \circ \mathcal{S})^m)$. As these vector spaces have the same dimension, $\mathfrak{c}$ is an isomorphism. But by construction the differentials of both $SC^*(E)$ and $SC_*(E)$ are zero. Thus, $\mathfrak{c}^* = \mathfrak{c}$ is an isomorphism.

To check that the diagram commutes it suffices to check that

$$\begin{array}{ccc}
\varinjlim_a \varprojlim_n CF^*_{(a,\infty)}(H_n) & \xrightarrow{\widetilde{\pi}} & \varinjlim_a CF^*_{(a,\infty)}(H_{-1}) \\
{\scriptstyle \kappa}\downarrow & \overset{\pi_{-1}}{\nearrow} & \\
\varprojlim_n \varinjlim_a CF^*_{(a,\infty)}(H_n) & &
\end{array}$$

and

$$\begin{array}{ccc}
\varprojlim_b CF^*_{(-\infty,b)}(H_0) & \longrightarrow & \varprojlim_b \varinjlim_n CF^*_{(-\infty,b)}(H_n) \\
\downarrow & \overset{\kappa}{\nearrow} & \\
\varinjlim_n \varprojlim_b CF^*_{(-\infty,b)}(H_n) & &
\end{array}$$

commute. Noting that the κ appearing here corresponds to the κ appearing in (63), this follows from Lemma 27. For example, let $\mathfrak{A}$ be the indexing set $n \in \mathbb{N}$ and let $\mathfrak{B}$ be the indexing set $a \in \mathbb{R}$. In the notation of the proposition, it then suffices to show that

$$\widetilde{\pi} \circ \iota^a = \iota^a_{-1} \circ \pi^a_{-1}, \tag{64}$$

since every element of $\varinjlim_a \varprojlim_n CF^*_{(a,\infty)}(H_n)$ lies in the image of some ι^a. By construction, (64) holds.

□

REMARK 15) The maps κ can always be defined, not just when the completed and uncompleted Floer theories coincide. Thus, the commutativity of the diagram in Proposition 5, eschewing the horizontal isomorphism symbols, holds for all radii.

COROLLARY 2 *The uncompleted symplectic cohomology of an annulus bundle, defined in Chapter 2 as the cone of $\mathfrak{c}^*$, vanishes for all radii:*

$$SH^*(A_{(R_1,R_2)};\Lambda) = 0.$$

Theorem 1 says that there is a long-exact sequence

$$\cdots \longrightarrow \widehat{SH}_*(D_{R_1};\Lambda) \xrightarrow{\widehat{\mathfrak{c}}^*} \widehat{SH}^*(D_{R_2};\Lambda) \longrightarrow \widehat{SH}^*(A_{(R_1,R_2)};\Lambda) \longrightarrow \widehat{SH}_{*+1}(D_{R_1};\Lambda) \longrightarrow \cdots \tag{65}$$

PROOF OF THEOREM 2: If $R_2 < \frac{1}{\sqrt{\pi\tau}}$ then by Theorem 5 and Theorem 6 both $\widehat{SH}^*(D_{R_2};\Lambda) = 0$ and $\widehat{SH}_*(D_{R_1};\Lambda) = 0$. It follows from the long-exact sequence (65) that $\widehat{SH}_*(A_{(R_1,R_2)};\Lambda) = 0$. Similarly, if $R_1 > \frac{1}{\sqrt{\pi\tau}}$, Proposition 5 says that the map $\widehat{SH}_*(D_{R_1};\Lambda) \longrightarrow \widehat{SH}^*(D_{R_2};\Lambda)$ in the long-exact sequence is an isomorphism. This implies that $\widehat{SH}_*(A_{(R_1,R_2)};\Lambda) = 0$ as well. However, if $R_1 < \frac{1}{\sqrt{\pi\tau}} < R_2$, then Theorem 6 says that $\widehat{SH}_*(D_{R_1};\Lambda) = 0$ and Theorem 5 says that $\widehat{SH}^*(D_{R_2};\Lambda) \cong SH^*(E)$, implying that

$$\widehat{SH}^*(A_{(R_1,R_2)};\Lambda) \cong \widehat{SH}^*(D_{R_2};\Lambda) \cong SH^*(E).$$

□

The following Theorem, due to Ritter, computes uncompleted symplectic cohomology for these bundles [43].

THEOREM 8 *Let* E *be the total space of the negative line bundle* $\mathcal{O}(-k) \longrightarrow \mathbb{C}P^m$. *Then*

$$SH^*(E) = \begin{cases} \Lambda[x] \Big/ (x^{1+m-k} - (-k)^k T) & 1 \le k < 1 + \frac{m}{2} \\ \text{has rank a multiple of } 1+m-k & 1 + \frac{m}{2} \le k \le m \end{cases}.$$

Combing Theorem 2 and Theorem 8 yields Corollary 3.

COROLLARY 3 *If* $R_1 < \frac{1}{\sqrt{\pi\tau}} < R_2$ *then the completed symplectic cohomology of* $A_{(R_1,R_2)}$ *is*

$$\widehat{SH^*}(A_{(R_1,R_2)};\Lambda) \cong \Lambda^{1+m-k}.$$

4.5 Closed-string mirror symmetry

Let E be the complex line bundle $\mathcal{O}(-k) \xrightarrow{\rho} \mathbb{C}P^m$. E is a toric variety whose image under the moment map is

$$\Delta := \left\{ (v_1, \dots, v_{m+1}) \in \mathbb{R}^{m+1} \;\middle|\; v_i \ge 0 \,\forall i \in \{1, \dots, m+1\};\ -v_1 - \dots - v_m + k v_{m+1} \ge -1 \right\}$$

(see, for example, Subsection 7.6 in [42] or Subsection 12.5 in [45]).

Let $\mathbb{K} = \mathbb{C}$ and set $\Lambda^* = \Lambda \setminus \{0\}$. Recall the valuation val, defined in (27). The mirror of E is the subset of $(\Lambda^*)^{m+1}$ given by

$$E^\vee := \left\{ (z_1, \dots, z_{m+1}) \in (\Lambda^*)^{m+1} \;\middle|\; (val(z_1), \dots, val(z_{m+1})) \in \Delta^o \right\},$$

equipped with superpotential

$$\mathcal{W} \colon E^\vee \longrightarrow \Lambda \tag{66}$$

$$(z_1, z_2, \dots, z_{m+1}) \mapsto z_1 + z_2 + \dots + z_m + z_{m+1} + T z_1^{-1} z_2^{-1} \dots z_m^{-1} z_{m+1}^k. \tag{67}$$

(See Example 7.12 in [42] or Proposition 4.2 in [11].) Mirror symmetry predicts an isomorphism between the symplectic cohomology of a toric variety and the Jacobian of $\mathcal{W}$. For example, computations in [45] confirm that

$$SH^*(E;\Lambda) \cong \Lambda[z_1^\pm, z_2^\pm, \dots, z_{m+1}^\pm] \Big/ (\partial_{z_1} \mathcal{W}, \partial_{z_2} \mathcal{W}, \dots, \partial_{z_{m+1}} \mathcal{W}) =: Jac(\mathcal{W}). \tag{68}$$

This story generalizes to domains of restricted size. Let $A_{(R_1,R_2)}E$ be the annulus bundle between radii R_1 and R_2 in E. The mirror of $A_{(R_1,R_2)}E$ is

$$A_{(R_1,R_2)}E^\vee := \left\{(z_1,\dots,z_{m+1}) \in E^\vee \;\middle|\; \pi R_1^2 \le val(z_{m+1}) \le \pi R_2^2\right\},$$

equipped with $\mathcal{W}|_{A_{(R_1,R_2)}E^\vee}$.

For $I = (i_1,\dots,i_{m+1}) \in \mathbb{R}^{m+1}$ and $\boldsymbol{z} = (z_1,\dots,z_{m+1})$, denote $(z_1^{i_1},\dots,z_{m+1}^{i_{m+1}})$ by $\boldsymbol{z}^I$. We denote the ring of functions on $A_{(R_1,R_2)}E^\vee$ in the variable $\boldsymbol{z}$ by $\mathcal{O}(A_{(R_1,R_2)}E^\vee)_{\boldsymbol{z}}$, where

$$\mathcal{O}(A_{(R_1,R_2)}E^\vee)_{\boldsymbol{z}} = \left\{\sum_{i=0}^{\infty} c_i \boldsymbol{z}^{I_i} \;\middle|\; c_i \in \Lambda;\ I_i \in \mathbb{R}^{m+1};\ \lim_{i\to\infty} val(c_i\boldsymbol{z}^{I_i}) = \infty \ \forall\, \boldsymbol{z} \in A_{(R_1,R_2)}E^\vee\right\}.$$

A straight-forward computation shows that

$$Jac(\mathcal{W}|_{A_{(R_1,R_2)}E^\vee}) \cong \mathcal{O}(A_{(R_1,R_2)})_{z_{m+1}} \Big/ (1-(-k)^k T z_{m+1}^{-1-m+k}).$$

If $\pi R_2^2 < \frac{1}{1+m-k}$, then $val(Tz_{m+1}^{-1-m+k}) > 0$ for all $z_{m+1} \in A_{(R_1,R_2)}$. It follows that $1-(-k)^k T z_{m+1}^{-1-m+k}$ is a unit in $\mathcal{O}(A_{(R_1,R_2)})_{z_{m+1}}$, and so

$$Jac(\mathcal{W}|_{A_{(R_1,R_2)}E^\vee}) = 0.$$

Similarly, if $\pi R_1^2 > \frac{1}{1+m-k}$, then $val(T^{-1}z_{m+1}^{1+m-k}) > 0$ for all $z_{m+1} \in A_{(R_1,R_2)}$, and so again

$$Jac(\mathcal{W}|_{A_{(R_1,R_2)}E^\vee}) = 0.$$

If $\pi R_1^2 \le \frac{1}{1+m-k} \le \pi R_2^2$ then

$$Jac(\mathcal{W}|_{A_{(R_1,R_2)}E^\vee}) \cong \Lambda[z^\pm] \Big/ (1-(-k)^k T z^{-1-m+k}).$$

We have therefore proved the following expression of closed-string mirror symmetry

$$\widehat{SH^*}(A_{(R_1,R_2)}E;\Lambda) \cong Jac\left(\mathcal{W}|_{A_{(R_1,R_2)}E^\vee}\right).$$

Chapter 5

The tautological bundle over $\mathbb{C}P^1$

In this chapter we illustrate Theorem 2 by specializing to $E = \mathrm{Tot}(\mathcal{O}(-1) \longrightarrow \mathbb{C}P^1$ and constructing an explicit symplectic cochain complex. Theorem 2 yields the following example, which we will verify by explicit computation.

EXAMPLE 2) Let E be the total space of the line bundle $\mathcal{O}(-1) \longrightarrow \mathbb{C}P^1$ with area one exceptional divisor. Let W be a cobordism in E between a sphere bundle of radius R_1 (possibly empty) and a sphere bundle of radius R_2, with $R_1 \leq R_2$. Then

$$\widehat{SH}^*(W; \Lambda) \cong \begin{cases} \Lambda & R_1 \leq \frac{1}{\sqrt{\pi}} \leq R_2 \\ 0 & \text{otherwise} \end{cases} . \tag{69}$$

E contains a monotone Lagrangian with non-vanishing Floer homology in the sphere bundle of radius $\frac{1}{\sqrt{\pi}}$. It is shown in Figure 5.1 as the purple dot, and the sphere bundle it lies in is the dashed purple line. The three dotted black lines represent families of Maslov disks with boundary on L.

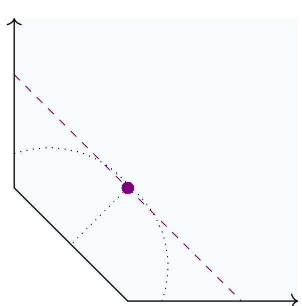


Figure 5.1: The moment polytope of $\mathrm{Tot}\left(\mathcal{O}(-1) \to \mathbb{C}P^1\right)$

5.1 Setting up Floer theory

As in Chapter 4, we set $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$. We adopt the notation from Chapter 4, so that E is the complex line bundle

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & E \\ & & \downarrow \rho \\ & & \mathbb{CP}^1 \end{array}$$

with Chern class $c_1(E) = -[\omega_{FS}]$, where ω_{FS} is the Fubini-Studi form rescaled to give $\mathbb{CP}^1$ an area of one. The unit sphere bundle SE has a contact form α satisfying $d\alpha = \rho^*\omega_{FS}$. This defines a symplectic form Ω on E so that on the complement of the zero section E is symplectomorphic to

$$\left(\left(\frac{1}{1+\pi}, \infty\right) \times S^3, d((1+\pi r^2)\alpha)\right),$$

where r is the coordinate on the interval $(\frac{1}{1+\pi}, \infty)$. The Reeb orbits of the contact form $(1 + \pi r^2)\alpha\big|_{r=1}$ traverse the fibers of the subbundle $SE \xrightarrow{\rho} \mathbb{CP}^1$, and the simple orbits have period $1 + \pi$ [43].

Let $f\colon \mathbb{CP}^1 \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-small Morse function with two critical points: a maximum value at N (north) and a minimum value at S (south). The function f has a Hamiltonian vector field X_f defined through ω_{FS}. Fixing a radius R, let $\{h_n\}_{n\in\mathbb{N}}$ be a family of Hamiltonians as in Chapter 4, and assume for computational simplicity that $h(0) = 0$ (see Figure 4.1). Consider solutions to the equations

$$\begin{cases} \alpha\left(\frac{d}{dt}x(t)\right) = (h_n'(\pi r^2) + \rho^* f) \\ dr\left(\frac{d}{dt}x(t)\right) = 0 \\ \frac{d}{dt}(\rho \circ x)(t) = X_f(x(t)) \end{cases} \tag{70}$$

Solutions of (70) are in bijection with the set of Reeb orbits on SE with projection to $\mathbb{CP}^1$ equal to N or to S.

Each non-constant solution of (70) occurs in an S^1-family. Apply Morse-Bott methods to fix a maximum M and a minimum m of each S^1 family. Define $CF^*(H_n;\Gamma)$ to be the chain complex over Γ generated by the maximum M and minimum m of each family of solutions of (70).

The generators of $CF^*(H_n;\Gamma)$ occur in six flavors: there are 4 Reeb orbits of period $k(1+\pi)$ for each $k \in \{1, \dots, n\}$ that correspond to a unique choice of S or N and m or M. There are also the minimum and maximum constant orbits N and S themselves. Let us fix notation. x_-^0 is the constant orbit mapping to N and y^0_- is the constant orbit mapping to S. x^k_+ is the maximum M of the family of period-$k(1+\pi)$ orbits lying above N and x^k_- is the minimum m. y^k_+ is the maximum M of the family of period-$k(1+\pi)$ orbits lying above S and y^k_- is the minimum m.

An element

$$T^s z; z \in \{x^k_\pm, y^k_\pm\}, s \in \mathbb{R}$$

of $CF^*(H_n;\Gamma)$ is $\mathbb{R}$-graded by

$$\mu(T^s z) = -2k + 2s + \mu_F(\rho \circ z) \pm \frac{1}{2}, \tag{71}$$

where $\mu_F(\rho \circ z)$ is the Morse index of the critical point of F corresponding to $\rho \circ z$. Note that these are cohomological gradings.

Lift a period-k Reeb orbit z to the k-fold fiber disc $\tilde{z}$. Choose a generic, cylindrical almost-complex structure J. The differential ∂^{fl} counts rigid cascades whose components satisfy

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - (h'(\pi r^2) + \rho^* f)R_\alpha - \rho^* X_f\right) = 0, \tag{72}$$

As in Chapter 4, we restrict to the space $\mathcal{J}$ of Ω-tame almost-complex structures of the form

$$J = (J_t)_{t\in S^1} = \left(\begin{bmatrix} i_t & B_t \\ 0 & j_t \end{bmatrix}\right)_{t\in S^1}$$

with respect to the splitting $TE \cong V \oplus H$ induced by α.

5.2 Computing the differential

We use Lemmas 28 – 31 to determine the differential of $\widehat{SC^*}(E;\Lambda)$. A cartoon of the Floer complex is given in Figure 5.2. The "horizontal" differentials correspond to Floer trajectories in the fiber above a critical point. The "diagonal" differentials correspond to Floer trajectories whose projection onto $\mathbb{C}P^1$ either covers all of $\mathbb{C}P^1 \setminus \{N, S\}$ (in the case of an arrow from x to y) or is a Morse flow-line of F (in the case of an arrow from y to x).

LEMMA 28 (ALBERS-KANG [9]) *Any solution of (72) with vanishing symplectic area and with both asymptotic limits contained in the same fiber remains wholly in that fiber. Thus, any such trajectory is identified*

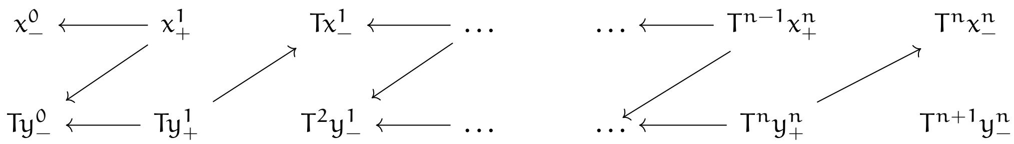


Figure 5.2: The trajectories contributing to $CF^*(H_n;\Gamma)$

with a trajectory of $X_{h^n(\pi r^2)+(1+\pi r^2)f(p)}$ *in* $\mathbb{C}$*, for some* $p \in \{N, S\}$*.*

Utilizing grading considerations and the fact that $SH^*(\mathbb{C};\Lambda) = 0$ (see [47]), the differential restricted to generators in the complex lines above N and S are the horizontal differentials shown in Figure 5.2.

LEMMA 29 *The winding number is decreased by the differential. The only Floer trajectories with both asymptotes contained in the same Reeb orbit have image identically equal to this Reeb orbit.*

PROOF: The first claim is precisely Lemma 12. To see the second claim, suppose $u(s,t)$ is a Floer solution with both positive and negative asymptote some non-constant periodic orbit x. Then $u(s,t)$ either intersects the zero section or has zero energy. In the latter case, u is constant. In the former case, consider the complement of a small neighborhood $\mathcal{U}$ of the zero section. If $v = u|_{u^{-1}(E\setminus\mathcal{U})}$ has only one component then the integrated maximum principal implies $E(v) \leq 0$, and so v must be constant, a contradiction. If v has multiple components, let v' be the component with $\lim_{s\to\infty} v'(s,t) = x$. Then the integrated maximum principal again implies $E(v') \leq 0$.

□

LEMMA 30 (ALBERS-KANG [9]) *If* u *is a Floer solution of (72) then* $\rho \circ u$ *is a Floer solution of* X_f*; in particular, the Conley-Zehnder index of critical points of* X_f *increases from the positive to the negative asymptote of the trajectory* $\rho \circ u$*.*

LEMMA 31 (RITTER [43]) $HF^*(H_n;\Lambda) \cong \Lambda[x] \big/ (x^2 + T)$. *The induced continuation map on cohomology is*

$$HF^*(H_n;\Lambda) \longrightarrow HF^*(H_{n+1};\Lambda)$$

$$1 \mapsto 1$$

$$x \mapsto T.$$

Lemma 29 also holds for continuation maps induced by $\mathbb{R}$-families of Hamiltonians that are monotone-decreasing in $\mathbb{R}$. We may thus choose continuation maps that act as the canonical inclusions. Allowing all rigid cascades that satisfy Lemmas 28, 29, and 30, that define a differential that squares to zero, and that yield Lemma 31, produces the complex shown in Figure 5.2.

Let $D_R \subset E$ be the disc bundle of radius R. The completed symplectic cochain complex of D_R in degree k is

$$\widehat{SC^k}(E_R;\Lambda) = \Lambda\left\langle \left\{ \sum_{i=0}^{\infty} a_i T^{s_i} z_i \;\middle|\; a_i \in \mathbb{K}; \exists\, n \text{ s.t. } T^{s_i} z_i \in CF^k(H_n;\Gamma); \lim_{i\to\infty} \mathcal{A}(T^{s_i} z_i) = \infty \right\} \right\rangle.$$

As in Chapter 4, the action of an element $T^s z \in CF^*(H_n;\Gamma)$ can be reformulated as

$$\mathcal{A}_{H_n}(T^s z) = (1 - \pi R_n^2)s + C(z),$$

where $C(z) = \frac{\pi R_n^2}{2}\left(k - \mu_F(\rho(z)) \mp \frac{1}{2}\right) + \int_0^1 H_n(z(t)) + \rho^* F(z(t))dt$ is uniformly bounded.

Recall from Theorem 5 that the completed symplectic cohomology of a disc bundle of radius R is

$$\widehat{SH^*}(E_R;\Lambda) \cong \begin{cases} 0 & R < \frac{1}{\sqrt{\pi}} \\ \Lambda & R \geq \frac{1}{\sqrt{\pi}} \end{cases}.$$

This can be seen from Figure 5.2 as follows.

1. Suppose $\pi R^2 < 1$. Then $\lim_{i\to\infty} \mathcal{A}(T^{s_i} z_i) = \lim_{i\to\infty} s_i$. Thus, $\widehat{SC^{\frac{1}{2}}}(E_R;\Lambda)$ includes the element

$$Z = \sum_{i=1}^{\infty} T^{i-1} x_+^i + T^i y_+^i. \tag{73}$$

The differential applied to Z yields

$$\begin{aligned} \partial(Z) &= \sum_{i=1}^{\infty} T^{i-1} x_-^{i-1} + T^i y_-^{i-1} + T^i x_-^i + T^i y_-^{i-1} \\ &= x_-^0. \end{aligned}$$

By T-linearity of the differential, this computation extends to produce an annihilator for any element of the form $T^k x_-^0$.

$T^k x_-^0$, $T^{k+i} x_-^i$, and $T^{k+i+1} y_-^i$ are equivalent in cohomology, and so every cocycle generating

$\widehat{SC^*}(E;\Lambda)$ is killed by a completed coboundary. Similarly, any completed cocycle is killed by formally adding together the annihilators of the individual summands (by construction this formal sum will be an element of $\widehat{SC^*}(E_R;\Lambda)$). Thus, $\widehat{SH^*}(E_R;\Lambda) = 0$.

2. If $\pi R^2 \geq 1$ the infinite sum (73) is no longer an element of $\widehat{SC^*}(E_R;\Lambda)$. The cohomology theory reduces to the uncompleted version and is therefore of rank one.

Similarly, writing the dual of a periodic orbit z as $z^\vee$, the completed symplectic homology of the disc bundle of radius R is generated by the infinite sum

$$\sum_{i=0}^{\infty} T^i (x_+^i)^\vee + T^{i+1}(y_+^i)^\vee$$

precisely when $R \geq \frac{1}{\sqrt{\pi}}$.

Equation (69) when $R_1 < \frac{1}{\sqrt{\pi}}$ now follows from the long-exact sequence (24). The case $R_1 \geq \frac{1}{\sqrt{\pi}}$ follows from the following lemma.

LEMMA 32 *The map* $\mathfrak{c}\colon \widehat{SH_*}(E_{R_1};\Lambda) \longrightarrow \widehat{SH^*}(E_{R_2};\Lambda)$ *defined in (23) is an isomorphism whenever* $R_1 > \frac{1}{\sqrt{\pi}}$.

PROOF: By Poincaré duality, $CF_*(H_0;\Gamma)$ is isomorphic (up to grading) to the cochain complex defined by the Hamiltonian $-H_0 - F \circ \rho$, which we denote by $CF^*(-H_0;\Gamma)$. We will abuse notation and continue to denote the generators of $CF^*(-H_0;\Gamma)$ by x_-^0 and y_-^0. Let $c^{-1}\colon CF^*(-H_0;\Gamma) \longrightarrow CF^*(H_0;\Gamma)$ be a continuation map. If $R_1 > \frac{1}{\sqrt{\pi}}$ then $\widehat{SC_*}(E_{R_1};\Lambda)$ and $\widehat{SC^*}(E_{R_2};\Lambda)$ are canonically isomorphic to the uncompleted theories, and the map $\widehat{SC_*}(E_{R_1};\Lambda) \xrightarrow{\mathfrak{c}} \widehat{SC^*}(E_{R_2};\Lambda)$ is determined by the image of $x_-^0 + T^{-1}y_-^0 \in CF_*(H_0;\Gamma)$ under the composition

$$CF_{-*}(H_0;\Gamma) \cong CF^*(-H_0;\Gamma) \xrightarrow{c^{-1}} CF^*(H_0;\Gamma) \hookrightarrow SC^*(E;\Gamma).$$

Recall the maps $\iota^{-1}_{(a,b)}$ and $\iota^{0}_{(a,b)}$ from Section 3, defined through a Morse-Smale pair (f, g_L) on L, and suppose that f has a unique minimum p. Analogous maps ι^{-1}, respectively ι^0, are defined from the (untruncated) Floer complexes $CF^*(-H_0;\Gamma)$, respectively $CF^*(H_0;\Gamma)$ to $CF^*(L;\Gamma)$. The proof of Lemma 6 extends to the equality

$$(\iota^0 \circ c^{-1})^* = (\iota^{-1})^*. \tag{74}$$

We will use this identity to understand the map c^{-1}.

Let $\mathcal{M}^2(L, p)$ be the Maslov index-2 discs $u\colon (D^2, \partial D^2) \longrightarrow (M, L)$ with $u(1) = p$. We have

$$\sum_{u \in \mathcal{M}^2(L,p)} [\partial u] = 0$$

(see [11], [45]). An index calculation now shows that the quantum differential ∂ on $HF^*(L; \Lambda)$ is the ordinary differential on $H^*(L; \Lambda)$ (Proposition 6.1.4 (a) in [13]). Therefore, p is the only representative of the unit of $HF^*(L; \Gamma)$ in $CF^*(L; \Gamma)$. To analyze the contributions to the unit of ι^0 and ι^{-1}, it therefore suffices to analyze the pearly/Floer trajectory amalgamates that negatively asymptote to p.

Let $g_{\mathbb{C}}$ be the standard metric on $\mathbb{C}$ and let $g_{\mathbb{CP}^1}$ be the standard metric on $\mathbb{CP}^1$. Let

$$g = \begin{bmatrix} g_{\mathbb{C}} & 0 \\ 0 & g_{\mathbb{CP}^1} \end{bmatrix}$$

be a metric on E with respect to the splitting $TE \cong V \oplus H$, as in section 5.1. Choose a generic almost-complex structure J. Denote the quantum cochain complex associated to a Morse-Smale pair (F, g) on M by $QC^*(F)$. Consider a map $\phi^{-1}\colon QC^*(-H - F \circ \rho) \longrightarrow QC^*(L)$, respectively $\phi^0\colon QC^*(H + F \circ \rho) \longrightarrow QC^*(L)$, that counts rigid configurations of the type shown in Figure 5.3. Explicitly, $\phi^{-1}(x)$, respectively $\phi^0(x)$, is the count of rigid configurations $(u_1, \ldots, u_\ell)$ such that

1. $u_i\colon (D^2, \partial D) \longrightarrow (M, L)$ is a J-holomorphic disc that is non-constant if $i < \ell$,
2. $u_1(1) = p$,
3. there exists a unique $t \in (-\infty, 0)$ such that $\Phi_t(u_{i+1}(1)) = u_i(-1)$ for all $i < \ell$, where Φ_t is the time-t flow of f, and
4. there exists a flow line $\beta(t)$ of $-H^{\tau_0} - \rho^* F$, respectively $H^{\tau_0} + \rho^* F$, with $\lim_{t \to \infty} \beta(t) = x$ and $\beta(0) = u_\ell(0, 0)$.

As in section 3.1, we only consider such configurations up to action by $\mathrm{Aut}(D^2, \pm 1)^{\ell - 1}$, where $\mathrm{Aut}(D^2, \pm 1)$ is the set of automorphisms of D^2 fixing ± 1. The maps ϕ^{-1} and ϕ^0 are the unital component of the dual of the quantum inclusion map studied in Section 5.4 of [13].

If $\pi_p\colon CF^*(L; \Lambda) \to \Lambda \cdot p$ is the projection onto the Λ-span of p, then under the PSS isomorphism,

$$(\pi_p \circ \iota^{-1})^* = (\phi^{-1})^* \qquad \text{and} \qquad (\pi_p \circ \iota^0)^* = (\phi^0)^*. \tag{75}$$

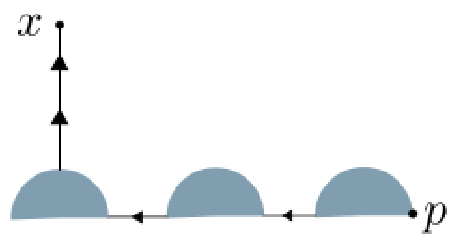


Figure 5.3: Configurations defining $\phi^{-1}(x)$ and $\phi^{0}(x)$

,

The dimension-zero configurations $(u_1, \ldots, u_\ell)$ have $|x| = \mu(u_1) + \cdots + \mu(u_\ell)$, where $|x|$ is the Morse grading and $\mu(u_i)$ is the Maslov index of u_i, [13]. Thus, $\phi^0(y^0_-)$ is a multiple of $T^0 = 1$ and $\phi^0(x^0_-)$ is a multiple of T. In fact, there is precisely one gradient trajectory $\beta(t)$ with $\lim_{t\to\infty} \beta(t) = y$ and $\beta(0) = p$, and so $\phi^0(y^0_-) = 1$. This is the yellow curve in Figure 5.4a.

The configurations contributing to $\phi^0(x^0_-)$ look like a single Maslov index-2 disc u with $u(1) = p$ and $u(0,0)$ intersecting a gradient flow line that converges at positive infinity to x^0_-. There is one such configuration, represented by the green curve in Figure 5.4a. Thus, $\phi^0(x^0_-) = T$.

Similarly, $\phi^{-1}(y^0_-) = 0$ and $\phi^{-1}(x^0_-) = T$, where the only contributing configuration is represented by the blue curve in Figure 5.4b.

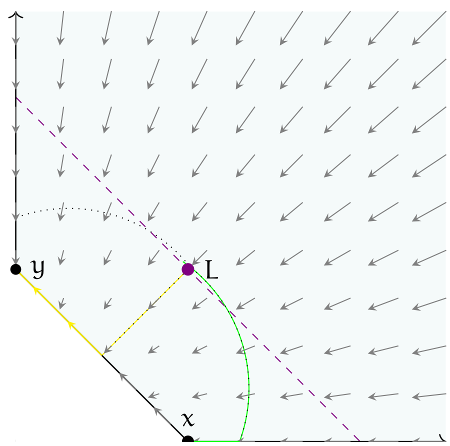
Using equation (74) and the fact that c^{-1} preserves the Conley-Zehnder index (71), we deduce that

$$(c^{-1})^*(y^0_-) \in \{0, x^0_- + Ty^0_-\} \qquad \text{and} \qquad (c^{-1})^*(x^0_-) \in \{T^{-1}x^0_-, y^0_-\}.$$

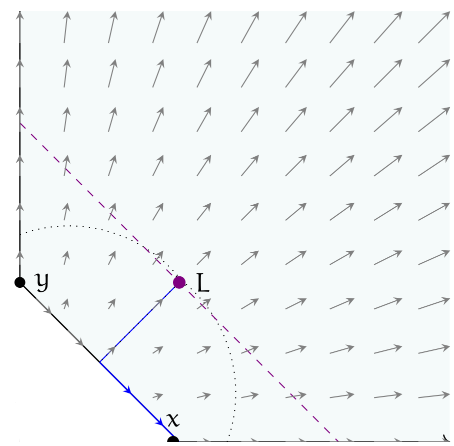
As $[x^0_- + Ty^0_-] = 0$ in $SH^*(E;\Lambda)$, $(c^{-1})^*(y^0_-) = 0$. And $[T^{-1}x^0_-] = [y^0_-]$ generates $SH^*(E;\Lambda)$, so $(c^{-1})^*(x^0_-)$ generates $SH^*(E;\Lambda)$. Thus, $(c^{-1})^*(x^0_- + T^{-1}y^0_-)$ generates $SH^*(E;\Lambda)$. We conclude that the map $\mathfrak{c}\colon SH_*(E_{R_1};\Lambda) \longrightarrow SH^*(E_{R_2};\Lambda)$ is an isomorphism.

□

By the long exact sequence (24), $\widehat{SH}^*(W) = 0$ when $R_1 > \frac{1}{\sqrt{\pi}}$.



(a) Spiked trajectories of $H^{\tau_0} + F \circ \rho$



(b) Spiked trajectories of $-H^{\tau_0} - F \circ \rho$

Figure 5.4: The configurations contributing to ϕ^0 and ϕ^{-1}.

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Appendix

We have assumed in this thesis that M is a monotone symplectic manifold; however, the definitions of action-completed Floer theories translate easily to the exact setting. Suppose M is a Liouville domain, that is, an exact, convex open symplectic manifold, decomposing into a smaller Liouville domain V and a Liouville cobordism W. Define $\widehat{SH^*}(M)$, $\widehat{SH_*}(V)$, and $\widehat{SH^*}(W)$ according to the recipe of Chapter 2.

It transpires that these objects are redundant, as shown by the following Proposition.

PROPOSITION 6 *The action-completed symplectic (co)homology of a Liouville manifold or cobordism* M *is isomorphic to the uncompleted symplectic (co)homology with Novikov coefficients:*

$$\widehat{SH^*}(M;\Lambda) \cong SH^*(M;\Lambda).$$

PROOF: The proof of this Proposition is very similar to those of Proposition 2 and Lemma 26, so we omit some details. Uncompleted symplectic cohomology of a Liouville manifold M with coefficients in Λ is computed via the complex

$$SC^*(M;\Lambda) := \operatorname*{hocolim}_{n\in\mathbb{N}} \varprojlim_{b} \varinjlim_{a} CF^*_{(a,b)}(H_n;\Gamma). \tag{76}$$

Similarly, uncompleted symplectic homology is computed via the complex

$$SC^*(M;\Lambda) := \operatorname*{holim}_{n\in\mathbb{N}} \varprojlim_{b} \varinjlim_{a} CF^*_{(a,b)}(H_{-n};\Gamma).$$

It therefore suffices to check that the limits over the action window commute with both the direct sum and the direct product over the Hamiltonian Floer groups. We will achieve this by showing that the elements in a fixed degree k have action bounded above or below.

First consider symplectic cohomology. Let θ be the choice of primitive for the symplectic form ω on M. Recall that the family of Hamiltonians $\{H_n\}_{n\in\mathbb{N}}$ are universally bounded on M by some constant C. Let $\tilde{x}$ be any capping of a periodic orbit x of some H_n, and suppose x lives in some $\{R\} \times \partial M$. Note that if x is non-constant then

$$-\int_{S^1} x^*\theta = -h_n'(e^R)e^R < 0.$$

whereas a constant orbit x trivially satisfies $\int_{S^1} x^*\theta = 0$. Thus,

$$\mathcal{A}_{H_n}(x) = -\int_{D^2} \tilde{x}^*\omega + \int_0^1 H_n(x)dt = -\int_{S^1} x^*\theta + \int_0^1 H_n(x)dt < C.$$

It follows that

$$\varinjlim_{a} \operatorname*{hocolim}_{n\in\mathbb{N}} CF^*_{(a,b)}(H_n;\Gamma) \cong \varinjlim_{a} \operatorname*{hocolim}_{n\in\mathbb{N}} CF^*_{(a,b')}(H_n;\Gamma)$$

for all $b, b' \geq C$, and so

$$\begin{aligned}
\varprojlim_{b} \varinjlim_{a} \operatorname*{hocolim}_{n\in\mathbb{N}} CF^*_{(a,b)}(H_n;\Gamma) &\cong \varprojlim_{b} \operatorname*{hocolim}_{n\in\mathbb{N}} \varinjlim_{a} CF^*_{(a,b)}(H_n;\Gamma) \\
&\cong \operatorname*{hocolim}_{n\in\mathbb{N}} \varinjlim_{a} CF^*_{(a,C)}(H_n;\Gamma) \\
&\cong \operatorname*{hocolim}_{n\in\mathbb{N}} \varprojlim_{b} \varinjlim_{a} CF^*_{(a,b)}(H_n;\Gamma) \\
&\cong SC^*(M;\Lambda).
\end{aligned}$$

Similarly, if x is a one-periodic orbit of some H_{-n} then

$$\mathcal{A}_{H_{-n}}(x) \geq -C,$$

and the analogous result for symplectic homology follows.

If W is a Liouville cobordism with Liouville filling V then, using the above results, it suffices to prove that there is a commutative diagram

$$\begin{array}{ccc} SC_*(V;\Lambda) & \xrightarrow{\mathfrak{c}} & SC^*(M;\Lambda) \\ \simeq\uparrow & & \downarrow\simeq \\ \widehat{SC_*}(V;\Lambda) & \xrightarrow{\hat{\mathfrak{c}}} & \widehat{SC^*}(M;\Lambda) \end{array} \tag{77}$$

This proof proceeds as in the proof of Lemma 26.

□

Comparisons to the literature

In [19] and [20] Cieliebak-Frauenfelder-Oancea and Cieliebak-Oancea defined the symplectic cohomology of a Liouville cobordism with exact filling. These works became the inspiration and guiding influence for our corresponding definitions in the monotone case. In this section we detail the precise correspondence between our definitions and their work, especially in the context of Proposition 6. These papers defined the symplectic cohomology of a cobordism by studying a directed family $\{\check{H}_\tau\}$ of "v-shaped" Hamiltonians. A "v-shaped" Hamiltonian is any that is $\mathcal{C}^2$ small on the Liouville cobordism W, positive on the complement of V, and linear at infinity. Fix a coefficient field $\mathbb{K}$. Symplectic cohomology was defined as

$$SH^*_{cfo}(M) := \varinjlim_{a} \varinjlim_{\tau} HF^*_{(a,\infty)}(\check{H}_\tau),$$

symplectic homology was defined as

$$SH^{cfo}_*(V) := \varprojlim_{b} \varinjlim_{\tau} HF^*_{(-\infty,b)}(\check{H}_\tau),$$

and the symplectic cohomology of W was defined as

$$SH^*_{cfo}(W) := \varinjlim_{a} \varprojlim_{b} \varinjlim_{\tau} HF^*_{(a,b)}(\check{H}_\tau).$$

The "cfo" tag, standing, of course, for Cieliebak-Frauenfelder-Oancea, is simply to denote the *a priori* difference from our constructions, and it is unnecessary. Indeed, they show that

$$SH^*_{cfo}(M) \cong SH^*(M)$$

and

$$SH_*^{cfo}(V) \cong SH_*(V).$$

REMARK 16) The only discrepancy is that we take homology at the final step, whereas in all of their theories homology is taken first. This does not matter for symplectic cohomology, as taking homology commutes with taking direct limits. It likewise does not matter for symplectic homology because the Mittag-Leffler condition is satisfied (each Floer group $CF_*(H_n)$ is a finite-dimensional vector space).

THEOREM 9 (CIELIEBAK-OANCEA) *There is a long-exact sequence*

$$\cdots \longrightarrow SH_*(V) \xrightarrow{c^*} SH^*(M) \longrightarrow SH^*_{cfo}(W) \longrightarrow SH_{*+1}(V) \longrightarrow \cdots \tag{78}$$

As we are working over a field, the comparable long-exact sequences from Theorem 9 and equation (25), as well as Proposition 6, show the following Proposition.

PROPOSITION 7 *There are isomorphisms*

$$\widehat{SH^*}(W) \cong SH^*(W) \cong SH^*_{cfo}(W).$$

REMARK 17) In the papers [19] and [20] the coefficients are always chosen to lie in a field when working with an exact triangle. These papers apply the inverse limit over b to a long-exact sequence on homology, and the field coefficients ensure that this operation is exact. However, one can, in our specialized circumstance, circumvent this constraint by only taking homology at the final step, thereby applying $\varprojlim_b$ to chain complexes on which the connecting maps are projections. With coefficients in an arbitrary ring, Theorem 9 no longer immediately shows Proposition 7. Thus modifying the definition of $SH^*_{cfo}(M)$ and counterparts, we sketch a proof of Proposition 7 with ring coefficients.

PROOF OF PROPOSITION 7 (WITH RING COEFFICIENTS): The first isomorphism is just restating Proposition 6. We sketch the proof of the second isomorphism.

Recall the family of Hamiltonians $\mathrm{Ad}(V)$ used to define the symplectic homology of V. This was defined with the help of a $\mathcal{C}^2$-small Hamiltonian $\mathcal{H}$. Let $\{\overline{H}_{n,m}\}_{n,m\in\mathbb{N}}$ be a bidirected family of Hamiltonians which is

1. after shifting by a constant, equal to $H^{\tau_{-n}}$ on $V \cup [0, R) \times \partial_- W$,

2. and equal to H^{τ_m} on W_+.

(See Figure A.5.) These are, for all of our purposes, equivalent to the Hamiltonians $\check{H}_\tau$ used by Cieliebak-Oancea. As in equation (22), the orbits in V, denoted $CF^*_V(H_{n,m})$, form a subcomplex of $CF^*(H_{n,m})$. Call

$$\overline{SC}_*(V) := \operatorname*{hocolim}_{m\in\mathbb{N}} \operatorname*{holim}_{n\in\mathbb{N}} CF^*_V(\overline{H}_{n,m})$$

and

$$\overline{SC}^*(M) := \operatorname*{hocolim}_{m\in\mathbb{N}} \operatorname*{holim}_{n\in\mathbb{N}} CF^*(\overline{H}_{n,m}). \tag{79}$$

The inclusions $CF^*_V(H_{n,m}) \hookrightarrow CF^*(\overline{H}_{n,m})$ induce an inclusion

$$\chi\colon \overline{SC}_*(V) \hookrightarrow \overline{SC}^*(M)$$

There is a canonical identification $SC_*(V) = \overline{SC}_*(V)$ and a quasi-isomorphism $\overline{SC}^*(M) \xrightarrow{\sim} SC^*(M)$ such that the following diagram commutes

$$\begin{array}{ccc} \overline{SC}_*(V) & \xrightarrow{\chi} & \overline{SC}^*(M) \\ \| & & \downarrow \\ SC_*(V) & \xrightarrow{\mathfrak{c}} & SC^*(M) \end{array}$$

The Five Lemma now yields a quasi-isomorphism

$$\mathrm{Cone}(\mathfrak{c}) \cong \mathrm{Cone}(\chi).$$

There is a further quasi-isomorphism [51]

$$\mathrm{Cone}(\chi) \cong \overline{SC}^*(M) \big/ \overline{SC}_*(V).$$

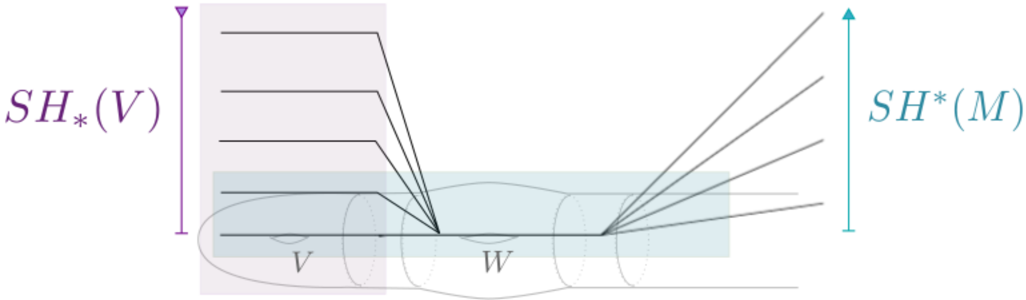

Figure A.5: An alternative definition of $SH^*(W)$

One checks that

$$\overline{SC}^*(M) \Big/ \overline{SC}_*(V) \cong \operatorname*{hocolim}_{m\in\mathbb{N}} \operatorname*{holim}_{n\in\mathbb{N}} CF^*(\overline{H}_{n,m}) \Big/ CF^*_V(\overline{H}_{n,m}).$$

Cieliebak-Oancea show that for every b there exists an n_b such that

$$\varinjlim_{n,m} CF^*_{(a,b)}(\overline{H}_{n,m}) \cong \varinjlim_{m} CF^*_{(a,b)}(\overline{H}_{n_b,m}).$$

Using Lemma 26 to conflate limits and homotopy limits, we deduce that it suffices to show

$$\varinjlim_{m} \varprojlim_{n} CF^*(\overline{H}_{n,m}) \Big/ CF^*_V(\overline{H}_{n,m}) \cong \varinjlim_{m} \varprojlim_{b} CF^*_{(-\infty,b)}(\overline{H}_{n_b,m}),$$

and so it suffices to show a quasi-isomorphism.

$$\varprojlim_{n} CF^*(\overline{H}_{n,m}) \Big/ CF^*_V(\overline{H}_{n,m}) \cong \varprojlim_{b} CF^*_{(-\infty,b)}(\overline{H}_{n_b,m}).$$

for each m. This quasi-isomorphism is induced precisely by the inclusions

$$CF^*_V(\overline{H}_{n_b,m}) \hookrightarrow CF^*_{(b,\infty)}(\overline{H}_{n_b,m}).$$

□