Columbia University

Department of Economics
Discussion Paper Series

Learning From a Piece of Pie

Pierre-André Chiappori
Olivier Donni
Ivana Komunje

Discussion Paper No.: 1011-05

Department of Economics
Columbia University
New York, NY 10027

May 2011

Learning From a Piece of Pie*

Pierre-André Chiappori[†] Olivier Donni[‡] Ivana Komunjer[§]

Revised Version: May 1, 2011

Abstract

We investigate the empirical content of the Nash solution to two-player bargaining games. The bargaining environment is described by a set of variables that may affect agents' preferences over the agreement sharing, the status quo outcome, or both. The outcomes (i.e., whether an agreement is reached, and if so the individual shares) and the environment (including the size of the pie) are known, but neither are the agents' utilities nor their threat points. We consider both a deterministic version of the model in which the econometrician observes the shares as deterministic *functions* of the variables under consideration, and a stochastic one in which because of latent disturbances only the joint distribution of incomes and outcomes is recorded. We show that in the most general framework any outcome can be rationalized as a Nash solution. However, even mild exclusion restrictions generate strong implications that can be used to test the Nash bargaining assumption. Stronger conditions further allow to recover the underlying structure of the bargaining, and in particular, the cardinal representation of individual preferences in the absence of uncertainty. An implication of this finding is that empirical works entailing Nash bargaining could (and should) use much more general and robust versions than they usually do.

Keywords: Testability, Identifiability, Bargaining, Nash Solution, Cardinal Utility

JEL codes: C71, C78, C59

---

*Paper presented at seminars in Chicago and Paris. We thank participants, and in particular Jaap Abbring, Bernard Fortin, Jim Heckman and Federico Perali for useful comments and suggestions. Financial support from the NSF (grant SBR9729559) and the ANR (grant ANR-08-FASHS-018-02) is gratefully acknowledged. Errors are ours.

[†]Department of Economics, Columbia University, 1014 International Affairs Building, 420 West 118th St., New York, NY 10025. Email: pc2167@columbia.edu.

[‡]THEMA and University of Cergy-Pontoise, 33 Boulevard du Port, 95011 Cergy-Pontoise Cedex, France. E-mail: olivier.donni@eco.u-cergy.fr.

[§]Department of Economics, University of California San Diego, 9500 Gilman Drive, La Jolla CA 92093. Email: komunjer@ucsd.edu.

I passed by his garden, and marked, with one eye,
How the Owl and the Panter were sharing a pie:
The Panther took pie-crust, and gravy, and meat,
While the Old had the dish as its share of the treat.
When the pie was all finished, the Owl, as a boon,
Was kindly permitted to pocket the spoon:
While the Panther received knife and fork with a growl,
And concluded the banquet by...
Lewis Caroll (Alice's Adventures in Wonderland, 1866)

1 Introduction

Economic applications of the Nash solution The axiomatic theory of bargaining, originated in a fundamental paper by John F. Nash (1950), has provided a simple and elegant framework to resolve the indeterminateness of the terms of bargaining. As such, it has been widely applied in economics. For instance, several works model the firm's decision process as a bargaining game between the management and the workers, represented by a union. Most of the time, such models assume the original Nash solution (de Menil, 1971, and Hamermesh, 1973); and several papers have tried to test this assumption (Bognanno and Dworkin, 1975; Bowlby and Schriver, 1978; Svejnar 1986, for instance). The modern analysis of employment contracts is another example of application. The analysis is often based on search models in which, once a meeting results in an employment contract, the parties bargain over the distribution of the surplus (see for instance Moscarini 2005 or Postel-Vinay and Robin 2006 for a recent survey). Bargaining theory has also been used in the analysis of international cooperation for fiscal and trade policies (Chari and Kehoe, 1990), negotiations in joint venture operations (Svejnar and Smith, 1984) or the sharing of profit in cartels (Harrington, 1991) and oligopoly (Fershtman and Muller, 1986), just to name a few. Last but not least, bargaining models have played a prominent role in recent analyses of household behavior. During the last two decades, several models accounting for the fact that spouses' goals may differ—and, therefore, that the decision process at stake has crucial consequences on the outcomes—have emerged. While the collective approach pioneered by Chiappori (1988a, 1992) relies on the sole assumption that the intrahousehold decision process is efficient, Manser and Brown (1980), McElroy and Horney (1981), Lundberg and Pollak (1993), who consider couples, and Kotlikoff, Shoven and Spivak (1986), who concentrate on negotiations between parents and children, introduce additional structure by explicitly refering to a Nash-bargaining equilibrium concept.[1] An interesting problem is whether (and under what conditions) the additional structure provided by Nash-bargaining results in either additional testable predictions on behavior, or a more accurate identification of individual preferences and decision processes.[2]

The Nash solution, or its generalization to situations with asymmetric bargaining power, is used in the large majority of the contributions just listed, and can be considered as the main solution concept to the bargaining problem, at least when the latter arises under a general form. From an empirical perspective, a few contributions consider specific examples in which players follow an explicit bargaining protocol, the details of which are moreover known to the econometrician; then non cooperative bargaining theory (and more precisely a non cooperative bargaining model constructed to exactly mimic the rules under consideration) is a natural tool to be taken to data. Most of the time, however, the bargaining environment is not known, or even not properly defined ex ante.[3] Then the Nash solution is regularly employed as the reduced form of a more complicated strategic bargaining process.

Is Nash Bargaining empirically relevant? While Nash bargaining is an elegant and convenient tool for approaching an old and important problem, its empirical relevance has not received the attention it deserves. As a benchmark example, consider a game in which two players, 1 and 2, bargain about a pie of size y. If the players agree on some sharing (ρ_1, ρ_2) with $\rho_1 \geqslant 0$, $\rho_2 \geqslant 0$ and $\rho_1 + \rho_2 = y$, it is implemented. If not, each agent s (with $s = 1, 2$) receives some reservation payment x_s. The setting of the process (i.e., the size y and the reservation payments x_s), as well as its outcome (whether an agreement is reached, and if so the individual shares ρ_s) are typically observable by an outside econometrician; however, individual utilities are not. Let us now assume that agents use a Nash bargaining solution. What is the empirical content of this assumption? Specifically, denoting $\rho_1 = \rho$ and $\rho_2 = y - \rho$, a Nash-bargaining agreement exists if and only if one can find some $\rho \in [0, y]$ such that $U^1(\rho) \geqslant T^1(x_1)$ and $U^2(y - \rho) \geqslant T^2(x_2)$; then the Nash solution ρ solves a program of the type:

$$\max_{\rho} \left(U^1(\rho) - T^1(x_1)\right) \cdot \left(U^2(y - \rho) - T^2(x_2)\right), \tag{1}$$

[1]Note that since Nash-bargaining generates efficient outcomes, the Nash-bargaining approach is at any rate a particular case of the 'collective' framework, see Chiappori and Donni (2009) and Browning, Chiappori and Weiss (forthcoming).

[2]See Chiappori (1988b, 1991) and McElroy and Horney (1990) for an early exchange on this issue.

[3]Intrahousehold bargaining is a prime example of this situation; while the idea that spouses bargain over decisions to be made is natural (and has been used by many contributors), the bargaining game is mostly informal, and cannot be described by fixed rules.

for some functions U^1, U^2, T^1, T^2 where the utility function of s is denoted U^s if an agreement is reached, and T^s in the opposite case.[4] What does this structure imply (if anything) on the relationship between (y, x_1, x_2) and ρ?

The traditional approach arbitrarily assumes a specific (usually linear) form for individual utilities.[5] Under such an assumption, the sharing function solves the program:

$$\max_{\rho} (\rho - x_1) \cdot (y - \rho - x_2),$$

giving the simple, linear form $\rho = \frac{1}{2}(y + x_1 - x_2)$ whenever $y \geqslant x_1 + x_2$. While this prediction is indeed testable, it totally relies on the linearity assumption; since the Nash bargaining outcome depends on the cardinal representation of individual preferences, any deviation from linear utilities will give a different form for the resulting shares.[6] It follows that any test based on the above program is a joint test of two assumptions, one general (Nash bargaining), and another very specific (linear utilities). A rejection is likely to be considered as inconclusive, since the burden of rejection can always be put on the specific and often ad hoc linearity assumption.

From a methodological perspective, assuming linear forms contradicts the generally accepted rule in empirical economics, whereby preferences should be recovered from the data rather than assumed a priori. This remark, in turn, raises two questions. First, is it possible to test the Nash bargaining assumption *without previous knowledge of individual utilities*? And second, can the utility players derive from the consumption of either their share of the pie or their reservation payment be recovered from the sole observation of the bargaining outcomes?

In the present paper, we address these two questions—the testability of Nash bargaining models and the identifiability of the underlying structure from observed behavior—in a general framework. In our setting, the environment is described by a set of variables that may affect agents' preferences over the agreement sharing, the status quo outcome, or both. A key role will be played by the econometrician's *prior information* on the structure of the model at stake. In a non-parametric spirit, this information will be described by some (broad) classes to which the utility or threat functions are known to belong. We are mainly interested in situations in which this prior information is limited. We thus do

[4]Note that we allow in principle these utilities to differ. For instance, in the case of households, individuals may have different preferences, say, different marginal rates of substitution between consumption and leisure, when married than when single.

[5]This approach is explained and discussed by Svejnar (1980). The household literature is an obvious exception.

[6]In the example above, for instance, if the utility of agent 1 is $U(\rho) = \sqrt{\rho}$ instead of $U(\rho) = \rho$, the solution becomes $\rho = \frac{1}{3}y + \frac{2}{9}x_1 - \frac{1}{3}x_2 + \frac{2}{9}\sqrt{x_1(x_1 + 3y - 3x_2)}$.

not assume that the econometrician knows the parametric form of the utility and threat functions, but simply that these functions satisfy some exclusion restrictions.[7] Our basic question can thus be precisely restated in the following way: *is the prior information sufficient to achieve (i) testability of the Nash bargaining theory, and (ii) identifiability of the underlying structural model*?

Regarding the identifiability issue, an interesting aspect is that Nash solutions are not invariant to monotonic transformations of utility functions. It follows that one may in principle retrieve a *cardinal* representation of preferences. While the identification of cardinal preferences is a standard problem in economics, the present situation is original in that it does not involve uncertainty. Whether concavity of utility functions matter in bargaining because of risk aversion—as suggested both by Nash's initial article[8] and by (some of) the non cooperative foundations of Nash bargaining—or for unrelated reasons is an interesting conceptual problem, on which our findings shed a new light. This issue will be further discussed in the concluding Section.

Deterministic versus stochastic models Economic models are, in general, stochastic. As noted by Marschak (1950), however, important distinct properties of the models can be brought out even if we assume all the variables to be measured exactly and to conform exactly to the predictions of economic theory. In particular, important insights into the problem of testability and identifiability of Nash bargaining models can be obtained even if we assume the econometrician has access to 'ideal data' in which she observes individual shares as *deterministic functions* of the variables entering the game (in our introductory example, our observer would know ρ_1 and ρ_2 as functions of (y, x_1, x_2)). Thus stated, the problem is the counterpart, in a bargaining context, of well known results in consumer theory—namely, that a smooth demand function can be derived from utility maximization under linear budget constraint if and only if it satisfies homogeneity, adding up and the Slutsky conditions, and that the underlying utility can then be recovered up to an increasing transform. In other words, the first perspective can be summarized as follows: Find an equivalent, for the Nash bargaining setting, of Slutsky relationships in consumer theory.

The perspective just sketched, however, is largely hypothetical; it relies on the availability of 'ideal data', in which a smooth relationship between the fundamentals of the bargaining process and its outcomes is not affected by any latent disturbances. In prac-

[7]To put it in a Popperian perspective (Popper, 1959), we do not want the falsifiability of Nash bargaining to be entirely driven by ad hoc auxiliary hypotheses such as particular functional forms of individual utility functions.

[8]We thank an anonymous referee for emphasizing this point.

tice, unobserved individual heterogeneity is paramount which affects the output of the game; such feature can only be accommodated in a stochastic version of the bargaining model. In this case, we typically observe a *joint distribution* of incomes and outcomes: in our example, of $(y, x_1, x_2, \rho_1, \rho_2)$. Note that the first (deterministic) perspective is but a statistically degenerate version of the second (stochastic), in the sense that it is relevant only when the distribution is degenerate (i.e., when its support is born by the graph of some deterministic function mapping (y, x_1, x_2) to (ρ_1, ρ_2)). In the second context, we may again ask whether the Nash bargaining framework imposes restrictions on the observed distribution, and whether, conversely, the distribution allows one to identify the underlying structure. The answer, however, now depends not only on the bargaining framework but also on the stochastic structure attached to it. A key remark, here, is that identifiability of the deterministic model is a necessary, although not sufficient, condition for identifiability of its stochastic version; obviously, if two different structural models generate the same 'ideal' demand function, there is no hope whatsoever of empirically distinguishing them.

Main findings In the present paper, we successively address the two problems. We first consider the deterministic version of the model. We show that, in its most general version, Nash bargaining is not testable: any Pareto efficient rule can be rationalized as the outcome of a Nash bargaining process.[9] We then introduce simple exclusion restrictions; namely, we assume that (i) threat point utilities do not depend on the size of the surplus over which agents bargain, and (ii) for each agent s, there exists (at least) one variable, say x_s, that only affects this agent's utility and threat functions (i.e., if $s' \neq s$, neither $U^{s'}$ nor $T^{s'}$ depend on x_s). Then the Nash model generates strong testable restrictions that take the form of a Partial Differential Equation (PDE) in the function ρ; these are reminiscent of Slutsky conditions. In addition, if either one of the pairs of functions (U^s, T^s) is known or if there exists, for each agent, a variable that enters the agent's threat point but not her utility when an agreement is reached, then both individual utility and threat functions can be cardinally identified (i.e., identified up to an affine transform). In particular, in the benchmark example (1) given above, the conditions are satisfied; therefore both testability and identifiability are achieved. Note also that the majority of these results remain valid, *mutatis mutandis*, in the case of the Generalized Nash solution.[10]

We then move to a stochastic version of the model, in which each individual surplus

[9] That Pareto efficiency by itself may generate testable restrictions is a classic finding of the literature on collective decision making in households; see for instance Chiappori and Donni (2011) and Browning, Chiappori and Weiss (forthcoming) for a general presentation.

[10] Some of them are also relevant for other solution concepts of bargaining theory.

is the sum of a deterministic component (equal, as above, to the difference between the utility of the share and the reservation utility) and an individual-specific random term. It follows that the Nash solution now involves two random terms. Coming back to our benchmark example, the program would now become:

$$\max_{\rho} \left(U^1(\rho) - T^1(x_1) + \epsilon_1\right) \cdot \left(U^2(y-\rho) - T^2(x_2) + \epsilon_2\right).$$

Here, the ϵ_s's can be seen as latent disturbances reflecting some unobserved heterogeneity between couples. In particular, we want to allow these random terms to be correlated with each other in an arbitrary way, reflecting the fact that the initial match between the negotiating partners is typically not random, but reflects some degree of (positive or negative) assortativeness. What the econometrician observes is the joint distribution of $(y, x_1, x_2, \rho_1, \rho_2)$ over the population under consideration—and we now assume this distribution is not degenerate.

In this stochastic setting, we can show the following result: under the same exclusion restrictions as before plus a conditional independence assumption, testable restrictions are generated, and individual utility and threat functions are cardinally identified. In other words, the deterministic results do extend to the stochastic framework, even though the stochastic structure under consideration involves an unknown bivariate error distribution.

Related works The empirical content of game theory is undoubtedly a topical issue as illustrated by several recent contributions. For example, Sprumont (2001) considers, from the revealed preferences viewpoint, a non-cooperative game played by a finite number of players, each of whom can choose a strategy from a finite set. Ray and Zhou (2001) adopt a similar set-up but focus on extensive-form games. Other related papers include Bossert and Sprumont (2002, 2003), Carvajal, Ray and Snyder (2004), Xu and Zhou (2007). Nonetheless, our contribution differs in many respects from what is generally done. Firstly, our subject matter—the Nash solution—has never been investigated in spite of the various applications of bargaining models in economics. Secondly, our methodology is not based on revealed preferences. The inspiration of the present paper, in fact, is more closely related to the work of Chiappori (1988, 1992) and its numerous sequels (Chiappori and Ekeland, 2006 and 2009) on the empirical implications of Pareto efficiency. This methodology is probably more appropriate for the empirical implementation of theoretical results. Moreover, our results are extended to the case of a model with unobserved random terms. Thirdly, the emphasis of this paper is largely on the identification problem, which is generally ignored by the authors cited above (with the exception of Chiappori and Ekeland, 2009).

The organization of the paper is as follows. In the next Section, we develop the general model and show that neither testability, nor identification obtain without a priori information on utility and threat functions. In Section 3 we introduce additional structure into the model, and show that testability obtains under mild assumptions on utility and threat functions. In Section 4, we note that identification requires stronger assumptions, of which several examples are given. Section 5 presents the stochastic version of the model and the main results. In the concluding Section, we discuss the potential applications of the results.

2 The deterministic model

2.1 The framework

We consider a game of Nash bargaining (denoted by NB hereafter) in which two players, 1 and 2, share a pie of size y. The bargaining environment is described by a vector x of n variables and we assume that (y, x) vary continuously within some convex, compact subset $\mathcal{S}$ of $\mathbb{R}_+ \times \mathbb{R}^n$ with non empty interior. We let $\mathcal{N}$ denote the subset of $\mathcal{S}$ on which no agreement is reached (so that agents receive their reservation payment), and $\mathcal{M}$ the subset on which an agreement is reached, with $\mathcal{S} = \mathcal{M} \cup \mathcal{N}$. Sharing is observed over $\mathcal{M}$. For notational convenience, we define the *sharing function* ρ as the share of the pie allocated to player 1, i.e., $\rho(y, x) = \rho_1(y, x)$ (then $y - \rho(y, x) = \rho_2(y, x)$). Let $U^s(\rho_s, x)$ denote the utility of player s (with $s = 1, 2$) when an agreement is reached and the sharing ρ is implemented. Similarly, let $T^s(y, x)$ denote the threat function of player s, i.e., utility when no agreement is reached and the reservation payments are made. The functions $U^s(\rho_s, x)$ and $T^s(y, x)$ may in general be different.

The players' behavior is then defined as follows: an agreement is reached if and only if there exists a sharing (ρ_1, ρ_2) with $\rho_1 \geqslant 0$, $\rho_2 \geqslant 0$, and $\rho_1 + \rho_2 = y$, such that

$$T^s(y, x) \leqslant U^s(\rho_s, x), \quad s = 1, 2, \tag{2}$$

i.e., the allocation $(T^1(y, x), T^2(y, x))$ lies within the Pareto frontier; in that case, the observed sharing $(\rho_1 = \rho, \rho_2 = y - \rho)$ solves:

$$\max_{0 \leqslant \rho \leqslant y} \left(U^1(\rho, x) - T^1(y, x)\right) \cdot \left(U^2(y - \rho, x) - T^2(y, x)\right). \tag{3}$$

The set of all functions $U^s(\rho_s, x)$ (resp. $T^s(y, x)$) that are compatible with the a priori restrictions is denoted by $\mathcal{U}^s$ (resp. $\mathcal{T}^s$).

The utility functions $U^s(\rho_s, x)$ and the threat functions $T^s(y, x)$ are assumed to be unknown to the econometrician. We assume in this section that the econometrician observes the 'ideal data' referred to in the introduction; namely, she observes, for any value of the variables y and x, whether or not an agreement is reached, and if so, the share ρ_s of player s as a function $\rho_s(y,x)$ of the relevant variables (with $\rho_1(y,x)+\rho_2(y,x)=y$). Two questions then arise: are the observables compatible with the model in (2) and (3)? and second, do they allow the econometrician to uncover the functions $U^s(\rho_s,x)$ and $T^s(y,x)$ that generated them? More formally, we shall use the following definitions.

Definition 1 *The observables* $(\{\mathcal{M},\mathcal{N}\},\rho)$ *are compatible with NB if and only if there exist two utility functions* $U^s\in\mathcal{U}^s$ *and two threat functions* $T^s\in\mathcal{T}^s$*, with* $s=1,2$*, such that, for any* $(y,x)\in\mathcal{M}$*,* (2) *and* (3) *are satisfied. If the functions* (U^s,T^s) *are unique up to a common affine transform, then the NB model* (2)-(3) *is said to be identified.*

Two remarks are in order. First, what we can recover is, at best, a *cardinal representation* of the functions under consideration: if we replace (U^s,T^s) with the affine transforms $(\alpha_sU^s+\beta_s,\alpha_sT^s+\beta_s)$ program (3) is not modified. Moreover, the α_s and β_s can themselves be functions of (some of) the variables at stake—an issue that will be clarified below. Second, the present framework cannot be used to test Pareto optimality. Indeed, as the pie is supposed to be entirely 'eaten up' by players, efficiency is automatically imposed.

2.2 A negative result

The answers to the two questions raised above, testability and identifiability, obviously depend on the prior information one is willing to exploit in the framework at stake. Our first result is that a fully general setting is simply too general. Specifically, if the form of threat functions is not restricted, then the answer to both testability and identifiability questions is negative: NB cannot generate testable predictions on observed outcomes, and the observation of the outcome does not allow to recover preferences. Note that, interestingly, this claim is valid even if the utilities U^1 and U^2, relevant when an agreement is reached, are known. This is stated formally in the following proposition.

Proposition 1 *Let* $\rho(y,x)$ *be some function defined over* $\mathcal{M}$*, and whose range is included in* $[0,\,y]$*. Then, for any pair of utility functions* (U^1,U^2) *there exist two threat functions* (T^1,T^2) *such that the agents' behavior* $(\{\mathcal{M},\mathcal{N}\},\rho)$ *is compatible with NB.*

Proof. The proof of Proposition 1 is simple: given any pair of functions (U^1,U^2) one can define (T^1,T^2) by:

$$\begin{aligned} T^s(y,x) &= U^s(\rho_s(y,x),x) \text{ if } (y,x)\in\mathcal{M}, \\ T^s(y,x) &> U^s(y,x) \qquad\qquad \text{if } (y,x)\in\mathcal{N}. \end{aligned}$$

Then for any (y, x) in $\mathcal{N}$, no agreement can be reached, whereas for any (y, x) in $\mathcal{M}$, the sharing $(\rho_1(y, x), \rho_2(y, x))$ is the only one compatible with individual rationality; thus it is obviously the NB allocation. ■

The intuition behind this result is straightforward: it is always possible to chose the status quo utilities (T^1, T^2) equal to the agents' respective utilities at the observed outcome whenever an agreement is reached (so that, in practice, the chosen point is the only feasible point compatible with individual rationality), while making sure that (T^1, T^2) is outside the Pareto frontier when agents are observed to disagree. Simple as it may seem, this argument still conveys two important messages. One is that when threat points are unknown, NB has no empirical content (beyond Pareto efficiency); any efficient outcome can be reconciled with NB. Secondly, the observation of the outcome brings no information on preferences (and in particular the concavity of the utility functions): any utility function can be made compatible with observed outcomes, using ad hoc threat points. Finally, it is important to stress that these negative results are by no means specific to NB. The proof applies to any bargaining concept satisfying individual rationality—a very mild requirement indeed.

2.3 Bargaining structure

The negative result above does not mean that NB (or, for that matter, bargaining theory altogether) cannot be tested, but simply that more structure is needed to achieve that goal. To continue the comparison initiated in the Introduction, consumer theory is not testable (beyond the trivial property of adding-up) in a general setting where utility is a function of prices and income in a general way (Pollak, 1977). It is the exclusion of prices and income from the arguments of the utility function that generates the well-known Slutsky constraints. Therefore, we first restrict the sets $\mathcal{U}^s$ of the players' utility functions.

Assumption U.1 *(a) There exists a partition $x = (x_1, x_2, \bar{x})$, with $x_1 = (x_{11}, \ldots, x_{1n_1})$, $x_2 = (x_{21}, \ldots, x_{2n_2})$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{\bar{n}})$ and $n_1 \geqslant 1, n_2 \geqslant 1$, $n = n_1 + n_2 + \bar{n}$, such that U^s does not depend on $x_{s'}$, where $s, s' = 1, 2$ and $s \neq s'$; i.e., $U^s(\rho_s, x) = U^s(\rho_s, x_s, \bar{x})$. (b) For $s = 1, 2$, the function $U^s(\rho_s, x_s, \bar{x})$ is three times continuously differentiable, strictly increasing and concave in ρ_s, and twice continuously differentiable in x_s.*

Essentially, U.1 says that some variables are excluded from the arguments of utility functions. For instance, the utility function of one player may depend on her own characteristics (such as her age and education) but not on those of the other player.

We further restrict the sets $\mathcal{T}^s$ of the players' threat functions.

Assumption T.1 *(a) For* $s = 1, 2$*, the function* $T^s(y, x)$ *does not depend on either* $x_{s'}, s' \neq s$*, nor the size of the pie* y*; i.e.,* $T^s(y, x) = T^s(x_s, \bar{x})$*. (b) For* $s = 1, 2$*, the function* $T^s(x_s, \bar{x})$ *is twice continuously differentiable in* x_s*.*

The differentiability of T^s is sufficient to obtain some restrictions on the sharing function. However, our interpretation of testability is more demanding and we introduce, in addition, some exclusion restrictions. The additional structure given by this assumptions should a priori increase significantly the empirical content of the bargaining game. The exclusion of y is standard; it is typical, for instance, of situations where the variables x fully capture any information the agent has about outside opportunities (e.g. future y) that determine its threat point. The exclusion of $x_{s'}$ provides the key structure needed for testability.[11] The smoothness of T^s together with that of U^s guarantees that the sharing function is sufficiently differentiable for our purpose.

Finally, for the sake of simplicity, we concentrate on the case for which the solution of the Nash program is interior. This is formalized by the next assumption.

Assumption S.1 *For any* $(y, x) \in \mathcal{M}$*, the sharing* (ρ_1, ρ_2) *is interior; i.e.,* $\rho_s > 0$ *for* $s = 1, 2$*.*

This condition will be automatically satisfied, for instance, if $\lim_{\rho_s \to 0} \partial U^s / \partial \rho_s = \infty$. In the benchmark example of the Introduction, if $x_s > 0$ for $s = 1, 2$, then a sufficient condition for S.1 is that the utilities be strictly increasing and that the normalization $U^s(0) = T^s(0) = 0$ holds for $s = 1, 2$.

3 Testability: the deterministic case

In this section we study the properties of the NB model under U.1 and T.1 in the deterministic case. It is straightforward to check that the negative result of Proposition 1 continues to hold even if the utility functions satisfy U.1. However, under the additional restrictions given in T.1, the answer to the testability question is now positive: there exist strong testable restrictions on ρ generated by the NB approach. For the sake of presentation, we separately consider two cases depending on whether or not disagreements are observed for some $(y, x) \in \mathcal{S}$.

[11] Indeed testable restrictions can be obtained without the exclusion of y provided that x_1 and x_2 are multi-dimensional vectors.

3.1 The general agreement case

We first leave aside the situations in which the players either disagree or are indifferent between agreeing and disagreeing, and make the simplifying assumption (which will be relaxed in the next subsection) that an agreement is always reached. Formally:

Assumption S.2 *For any* $(y, x) \in \mathcal{S}$*, there exists a sharing* (ρ_1, ρ_2) *such that* $U^s(\rho_s, x) - T^s(y, x) > 0$ *for* $s = 1, 2$.

Under U.1, T.1, S.1 and S.2, the sharing function ρ is then defined as a function of (y, x) over the entire space $\mathcal{S}$ and has a range included in $]\,0,\, y\,[$. Assuming NB, this function solves the problem:

$$\max_{0 \leqslant \rho \leqslant y} \ln\left(U^1(\rho, x_1, \bar{x}) - T^1(x_1, \bar{x})\right) + \ln\left(U^2(y - \rho, x_2, \bar{x}) - T^2(x_2, \bar{x})\right).$$

Since U^1 and U^2 are strictly concave in ρ_1 and ρ_2, respectively, a sharing rule ρ is a solution to the above program if and only if it solves the first order condition:

$$R^1(\rho, x_1, \bar{x}) = R^2(y - \rho, x_2, \bar{x}), \tag{4}$$

where we have let

$$R^s(\rho_s, x_s, \bar{x}) \equiv \frac{\partial U^s(\rho_s, x_s, \bar{x})/\partial \rho_s}{U^s(\rho_s, x_s, \bar{x}) - T^s(x_s, \bar{x})}. \tag{5}$$

The first identification result we shall present can easily be stated in the case of the Generalized Nash bargaining (GNB) provided the bargaining weights δ_s do not depend on the other player's characteristics, i.e., $\delta_s = \delta_s(x_s, \bar{x})$. The optimization problem then becomes:

$$\max_{0 \leqslant \rho \leqslant y} \delta_1(x_1, \bar{x}) \ln\left(U^1(\rho, x_1, \bar{x}) - T^1(x_1, \bar{x})\right) + \delta_2(x_2, \bar{x}) \ln\left(U^2(y - \rho, x_2, \bar{x}) - T^2(x_2, \bar{x})\right)$$

If so, the first order conditions remain (4) if the R^s are redefined as:

$$R^s(\rho_s, x_s, \bar{x}) \equiv \delta_s(x_s, \bar{x}) \frac{\partial U^s(\rho_s, x_s, \bar{x})/\partial \rho_s}{U^s(\rho_s, x_s, \bar{x}) - T^s(x_s, \bar{x})}.$$

We now proceed to derive the NB restrictions in this generalized framework. The first result is the following:

Proposition 2 *Suppose Assumptions U.1, T.1, S.1 and S.2 hold, and that the bargaining weights* $\delta_s(x_s, \bar{x})$ *are twice continuously differentiable in* x_s*. If the agents' behavior*

$(\{\mathcal{M},\mathcal{N}\},\rho)$ is compatible with GNB, then the function $\rho\,(y,x)$ is twice continuously differentiable and satisfies:

$$0<\frac{\partial\rho}{\partial y}(y,x)<1. \tag{6}$$

Moreover, for any $(y,x)\in\mathcal{S}$,

$$\left(1-\frac{\partial\rho}{\partial y}(y,x)\right)^{-1}\left(\frac{\partial\rho}{\partial x_{1i}}(y,x)\right) \text{ is a function } \Phi_i^1 \text{ of } (\rho,x_1,\bar{x}) \text{ alone,} \tag{7}$$

$$\left(\frac{\partial\rho}{\partial y}(y,x)\right)^{-1}\left(\frac{\partial\rho}{\partial x_{2j}}(y,x)\right) \text{ is a function } \Phi_j^2 \text{ of } (y-\rho,x_2,\bar{x}) \text{ alone,} \tag{8}$$

where the functions $(\Phi_1^1,\ldots,\Phi_{n_1}^1)$ and $(\Phi_1^2,\ldots,\Phi_{n_2}^2)$ satisfy (Slutsky) conditions:

$$\frac{\partial\Phi_i^1}{\partial x_{1i'}}+\Phi_{i'}^1\frac{\partial\Phi_i^1}{\partial\rho_1} = \frac{\partial\Phi_{i'}^1}{\partial x_{1i}}+\Phi_i^1\frac{\partial\Phi_{i'}^1}{\partial\rho_1}, \tag{9}$$

$$\frac{\partial\Phi_j^2}{\partial x_{2j'}}+\Phi_{j'}^2\frac{\partial\Phi_j^2}{\partial\rho_2} = \frac{\partial\Phi_{j'}^2}{\partial x_{2j}}+\Phi_j^2\frac{\partial\Phi_{j'}^2}{\partial\rho_2}, \tag{10}$$

for every $i,i'=1,\ldots,n_1$ and $j,j'=1,\ldots,n_2$.

Conversely, suppose the observed agents' behavior $(\{\mathcal{M},\mathcal{N}\},\rho)$ satisfies Assumptions S.1 and S.2 with a sharing rule that satisfies conditions (6) through (10). Then, $(\{\mathcal{M},\mathcal{N}\},\rho)$ can be rationalized as the NB solution of a model in which the utilities and threat functions satisfy Assumptions U.1 and T.1, respectively.

A proof of Proposition 2 is in Appendix. Put in words, this proposition establishes three properties:

1. When the information about the structure of the game is described by U.1 and T.1, the GNB solution can be falsified (in Popper's, 1959, terms) by observable behavior. Specifically, condition (6) states that any increase in the size of the pie must benefit both agents; it is a direct consequence of the exclusion of the size of the pie from the arguments of threat functions. Moreover, conditions (7) and (8) translate the particular separable structure of the first order condition (4) into a property of the sharing rule. Indeed, the function R^1 has $(\rho_1,x_1,\bar{x})$ as arguments but not of y and x_2 while the function R^2 has $(\rho_2,x_2,\bar{x})$ as arguments but not of y and x_1. Precisely, the variables (x_2,y) will affect the left-hand side of the first order condition (4) only in so far as the share of player 1 is modified. Therefore, any simultaneous change in (x_2,y) that leaves unchanged the share of player 1 must keep constant $R^1(\rho_1,x_1,\bar{x})$ as well. Since $R^1(\rho_1,x_1,\bar{x})$ coincides with $R^2(\rho_2,x_2,\bar{x})$ for the (Generalized) Nash solution, the change in (x_2,y) must be such that $R^2(\rho_2,x_2,\bar{x})$ is constant. Hence it depends on $(\rho_2,x_2,\bar{x})$ alone. This is the intuition of condition (8). The same argument applies, *mutatis mutandis*, to condition (7).

2. Conversely, these conditions are sufficient. If they are satisfied, then there exists a bargaining model for which the solution coincides with the sharing rule under consideration. Specifically, one can construct utilities $\bar{U}^s$ and threat functions $\bar{T}^s$, $s = 1, 2$, that satisfy Assumptions U.1 and T.1, respectively, and such that the corresponding $\bar{R}^1$ and $\bar{R}^2$ defined in (5) solve the first order condition (4) of the NB program.

3. Interestingly, the previous statement is true for both NB and GNB: the necessary conditions for GNB appear to be sufficient for NB. In particular, in the structure just presented, NB and GNB are empirically indistinguishable: any function that is compatible with GNB is also compatible with NB, possibly for different utility functions.

Additional remarks Two additional remarks can be made at this point. Firstly, conditions (7) and (8) can equivalently be stated in terms of second order partial derivatives of ρ. Indeed, differentiating again the expressions in (7) and (8), respectively, with respect to y and x_{2j}, and y and x_{1i}, gives:

$$\frac{\partial \rho}{\partial x_{1i}} \left(\frac{\partial^2 \rho}{\partial x_{2j} \partial y} \frac{\partial \rho}{\partial y} - \frac{\partial^2 \rho}{\partial y^2} \frac{\partial \rho}{\partial x_{2j}} \right) + \left(1 - \frac{\partial \rho}{\partial y} \right) \left(\frac{\partial^2 \rho}{\partial x_{1i} \partial x_{2j}} \frac{\partial \rho}{\partial y} - \frac{\partial^2 \rho}{\partial x_{1i} \partial y} \frac{\partial \rho}{\partial x_{2j}} \right) = 0,$$

for every $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. This condition is equivalent to both (7) and (8). Secondly, our result can be generalized, in the following sense. Conditions (6)-(10) are not specific to the (Generalized) NB solution: other solutions to the bargaining problem may satisfy these conditions. Indeed, from the proof of Proposition 2, one can see that any sharing function which can be rationalized by the maximization of an additively separable index such as $\sum_s f^s(\rho_s, x_s, \bar{x})$, for some functions f^s that are smooth, increasing and concave in ρ_s, will satisfy conditions (6) to (10). If a solution concept can be described by such a maximization, then it satisfies the Independence of Irrelevant Alternatives (IIA) but the converse is not true (Peters and Wakker, 1991). For this reason, the conditions stated in the proposition above are related to IIA although they cannot be interpreted as a formal test of this axiom.[12] The set of solution concepts that can be described by such a maximization include not only the Generalized Nash solution, but also the Egalitarian

[12] For a precise definitions of the axioms and a taxonomy of the solutions, the reader is referred to Thomson (1994). Lensberg (1987) characterizes the axioms that are necessary and sufficient to describe all the bargaining solutions that can be represented by the maximization of such an additively separable index. This characterization, unfortunately, requests a variable number of players, which limits the applicability of this result here.

solution and, as a limit case, the Utilitarian solution, although it excludes the Kalai-Smorodinsky solution. Technically, a convenient family of functions f^s can summarize the main solution concepts compatible with (6) to (10) and deserves a particular attention. It is defined as follows:

$$f^s = \begin{cases} \delta_s \left((U^s - T^s)^\gamma / \gamma \right) & \text{if } \gamma \neq 0 \\ \delta_s \ln (U^s - T^s) & \text{if } \gamma = 0 \end{cases}, \tag{11}$$

with $\gamma \leqslant 1$ and $\delta_s > 0$. This family of functions generates, if $\gamma = 0$, the Generalized Nash solution with constant weight (and if, in addition, $\delta_1 = \delta_2$, the (symmetric) Nash solution). The (symmetric or asymmetric) Utilitarian solution is obtained if $\gamma = 1$ and the (symmetric or asymmetric) Egalitarian solution if $\gamma \to -\infty$.[13] All these forms are empirically indistinguishable, in the sense that any sharing rule ρ that is compatible with one of them is also compatible with any other, possibly for different utilities.

Parametric example 1. The previous result can be exploited to test whether players make use of the Nash solution (or any alternative solution in the family just described). The simplest way is to translate conditions (6)-(10) into constraints on the parameters of a functional form.

As an illustration, consider the following specification for the sharing function (where, for the sake of notational simplicity, we omit $\bar{x}$ and assume that the variables x_s are one dimensional):

$$\rho = y \cdot \mathcal{L} \left(a_{00} + a_{01} x_1 + a_{02} x_2 + a_{11} x_1^2 + a_{22} x_2^2 + a_{12} x_1 x_2 \right),$$

where $\mathcal{L}(x) = 1/(1 + \exp(x))$ is the logistic distribution function. In words, the respective shares ρ/y are taken to be independent of y and logistic transformations of a general second order approximation in (x_1, x_2). Therefore, $\rho(y, x_1, x_2)$ is necessarily between 0 and y. Moreover, condition (6) is globally satisfied. Conditions (7)-(10) require that $a_{12} = 0$; indeed, we have that:

$$\left(1 - \frac{\partial \rho}{\partial y} (y, x_1, x_2) \right)^{-1} \left(\frac{\partial \rho}{\partial x_1} (y, x_1, x_2) \right) = -\rho \left(a_{01} + 2 x_1 a_{11} + x_2 a_{12} \right)$$
$$\left(\frac{\partial \rho}{\partial y} (y, x_1, x_2) \right)^{-1} \left(\frac{\partial \rho}{\partial x_2} (y, x_1, x_2) \right) = -(y - \rho) \left(a_{02} + x_1 a_{12} + 2 x_2 a_{22} \right),$$

and the first (resp. second) right-hand term is a function of ρ and x_1 (resp. $y - \rho$ and x_2) alone if and only if $a_{12} = 0$. Hence an econometric test of the Nash solution, under U.1–T.1, boils down to testing that $a_{12} = 0$.

[13] The first order condition in the Egalitarian case is simply given by $U_1 - T_1 = U_2 - T_2$.

We now show how utility and threat functions can be constructed. If the restriction $a_{12} = 0$ is satisfied, then $\rho g_1(x_1) = (y-\rho)g_2(x_2)$, where $g_1(x_1) \equiv k \exp(a_{00} + a_{01}x_1 + a_{11}x_1^2)$, $g_2(x_2) \equiv k \exp(-(a_2 x_{02} + a_{22}x_2^2))$, and the constant $k > 0$ is chosen so that $g_1(x_1) > 1$ and $g_2(x_2) > 1$. Next, for $s = 1, 2$, let

$$\bar{R}^s(\rho_s, x_s, \bar{x}) \equiv \frac{1}{\rho_s g_s(x_s)} \quad \text{so} \quad \bar{R}^1(\rho, x_1) = \bar{R}^2(y-\rho, x_2),$$

and $\bar{R}^s > 0$ and $\partial \bar{R}^s / \partial \rho_s < 0$. One can then construct the utility functions $\bar{U}^s$ for arbitrary choices of the threat points $\bar{T}^s$ by solving a differential equation; for $\bar{U}^1$, for example:

$$\frac{\partial \bar{U}^1(\rho_1, x_1)/\partial \rho_1}{\bar{U}^1(\rho_1, x_1) - \bar{T}^1(x_1)} = \frac{1}{\rho_1 g_1(x_1)}.$$

Thus:

$$\bar{U}^1(\rho_1, x_1) = K(x_1)\, \rho_1^{1/g_1(x_1)} + \bar{T}^1(x_1),$$

for some function $K(x_1) > 0$. The utility $\bar{U}^1$ is of the CRRA form, with a coefficient of relative risk aversion $1 - 1/g_1(x_1)$ that varies with x_1. Since $g_1(x_1) > 1$, $\bar{U}^1$ satisfies U.1.

Note that this example can be generalized to an approximation of any order.

3.2 Outside and along the agreement frontier

In the previous subsection, it is assumed that cooperation always generates a positive surplus that can be shared between the players. From now on, we consider a more general case: $\mathcal{M} \subseteq \mathcal{S}$ so that the possibility of a disagreement between the players, or an agreement along the boundary of $\mathcal{M}$, can no longer be excluded. To begin with, it is worth noting that, when $(y, x) \in \mathcal{N}$, i.e., the players do not agree about the sharing of the pie, the outside observer can learn next to nothing about the underlying structure of the bargaining. One can only infer from the observation of a disagreement that the status quo point must lie outside the Pareto frontier.

The study of the agreement frontier—the locus where the players are indifferent whether the agreement is reached or not—is much more interesting. Formally, the agreement frontier $\mathcal{F}$ is defined by the points that belong to the intersection of the closure of the agreement set $\mathcal{M}$ and the closure of the non-agreement set $\mathcal{N}$, that is, $\mathcal{F} = \text{cl}(\mathcal{M}) \cap \text{cl}(\mathcal{N})$. In what follows, we assume that $\mathcal{F} \neq \varnothing$. Along this frontier, one observes the sharing of the pie, as a function of the size of the pie and the set of environmental variables, and knows that, by definition, the bargaining surplus is exactly equal to zero. If $(y, x) \in \mathcal{F}$, then each agent is indifferent between her share of the pie and her reservation payment, i.e.,

$$(y, x) \in \mathcal{F} \quad \Longrightarrow \quad U^1(\rho, x) = T^1(x) \text{ and } U^2(y - \rho, x) = T^2(x),$$

for some ρ, with $0 \leqslant \rho \leqslant y$.[14] The agreement frontier $\mathcal{F}$, if non-empty, is observable by construction. It should have some features that can be tested. Indeed, the following proposition presents a set of testable restrictions which are based on the observation of the sole agreement frontier.

Proposition 3 *Suppose Assumptions U.1 and T.1 hold. If $\{\mathcal{M},\mathcal{N}\}$ is compatible with NB, then there exists a subset $\mathcal{B}$ in $\mathbb{R}^n$, and a twice continuously differentiable function $\sigma(x)$ defined over $\mathcal{B}$, such that $y=\sigma(x)$ if and only if $(y,x)\in\mathcal{F}$, and*

(i) *if $(y,x)\in\mathcal{M}$ and $x\in\mathcal{B}$, then $y\geqslant\sigma(x)$,*

(ii) *if $(y,x)\in\mathcal{N}$ and $x\in\mathcal{B}$, then $y\leqslant\sigma(x)$.*

Moreover, the function $\sigma(x)$ is additive in the sense that $\sigma(x)=\sigma_1(x_1,\bar{x})+\sigma_2(x_2,\bar{x})$ for some functions $\sigma_1(x_1,\bar{x})$ and $\sigma_2(x_2,\bar{x})$.

A proof of Proposition 3 is in Appendix. The first part of the proposition states that the equation characterizing the agreement frontier can be written as: $y=\sigma(x)$, whereby players agree about the sharing of a pie if and only if its size exceeds some reservation value $\sigma(x)$. The second part of the proposition yields a strong testable restriction on the form of the agreement frontier.

The conditions at stake here involve only the agreement frontier. This implies that when $\mathcal{F}\neq\varnothing$, NB can be tested even without observing the sharing of the pie. Unsurprisingly, stronger conditions obtain when the sharing of the pie is actually observed. This is formally stated as follows:

Proposition 4 *Suppose Assumptions U.1, T.1 and S.1 hold. If the agents' behavior $(\{\mathcal{M},\mathcal{N}\},\rho)$ is compatible with NB, then for any (y,x) in $\mathcal{F}$,*

$$\frac{\partial\sigma}{\partial x_{1i}}=\frac{\partial\rho/\partial x_{1i}}{1-\partial\rho/\partial y}\quad \text{and}\quad \frac{\partial\sigma}{\partial x_{2j}}=-\frac{\partial\rho/\partial x_{2j}}{\partial\rho/\partial y},$$

for every $i=1,\dots,n_1$ and $j=1,\dots,n_2$.

4 Identifiability: the deterministic case

4.1 A non identifiability result

We now consider the identification problem; i.e., we ask whether the utility and threat functions can be retrieved from the observation of the sharing function. We put the emphasis on what happens inside the agreement frontier (outside this frontier, identification

[14]Technically, the converse is not necessarily true. Indeed, it is possible that, for some (y,x) that do not belong to $\mathcal{F}$, the surplus of the players is exactly equal to zero. This will be the case if these points belong to $\mathcal{N}$ but are not in the neighborhood of $\mathcal{M}$.

cannot be reached). It is clear from the form of the problem that NB is invariant by affine transformation of individual utilities. Moreover, the affine transformation of U^s may actually depend on the common variables $(x_s, \bar{x})$ in an arbitrary way. Therefore, we say that utility functions U^s and $\bar{U}^s$ (resp. threat functions T^s and $\bar{T}^s$) are different if and only if there does not exist functions $\alpha(x_s, \bar{x}) > 0$ and $\beta(x_s, \bar{x})$ such that $U^s = \alpha(x_s, \bar{x}) \bar{U}^s + \beta(x_s, \bar{x})$ (resp. $T^s = \alpha(x_s, \bar{x}) \bar{T}^s + \beta(x_s, \bar{x})$).

The main conclusion then is that the model is not identified. Formally, we have the following result:

Proposition 5 *Suppose the observed agents' behavior $(\{\mathcal{M}, \mathcal{N}\}, \rho)$ satisfies Assumptions S.1 and S.2 with a sharing rule that satisfies conditions (6) through (10). Then there exists a continuum of different utility and threat functions satisfying Assumptions U.1 and T.1, respectively, for which the agents' behavior is compatible with NB.*

Specifically, let $(\bar{U}^1, \bar{U}^2, \bar{T}^1, \bar{T}^2)$ be one such solution (with corresponding $\bar{R}^1$ and $\bar{R}^2$ defined in (5)). Then, (U^1, U^2, T^1, T^2) is a solution if and only if the corresponding R^1 and R^2 in (5) satisfy:

$$R^s(\rho_s, x_s, \bar{x}) = G\left(\bar{R}^s(\rho_s, x_s, \bar{x}), \bar{x}\right), \quad s = 1, 2,$$

with $G(\cdot, \bar{x})$ positive, strictly increasing and differentiable on the range of $\bar{R}^s$, and such that $\left[\partial \bar{R}^s / \partial \rho_s\right] G_r(\bar{R}^s, \bar{x}) + \left[G(\bar{R}^s, \bar{x})\right]^2 < 0$, where G_r denotes the derivative of $G(\cdot, \bar{x})$. For any choice of G, the utilities U^s and threat functions T^s are determined up to an affine increasing transform that may depend on $(x_s, \bar{x})$.

A proof of Proposition 5 is in Appendix. A consequence of this result, incidentally, is that along the agreement frontier, the conditions in Propositions 3 and 4 are *not* sufficient. Indeed the functions $R^s(\rho_s, x_s, \bar{x})$ for $s = 1, 2$ are defined up to some positive increasing function $G(\cdot, \bar{x})$, i.e. $R^s(\rho_s, x_s, \bar{x}) = G\left(\bar{R}^s(\rho_s, x_s, \bar{x}), \bar{x}\right)$ where $\bar{R}^s(\rho_s, x_s, \bar{x})$ is a known particular solution to (4) that satisfies $R^s > 0$ and $\partial \bar{R}^s / \partial \rho_s < 0$. Remember now that

$$R^s(\rho_s, x_s, \bar{x}) = \frac{\partial U^s(\rho_s, x_s, \bar{x}) / \partial \rho_s}{U^s(\rho_s, x_s, \bar{x}) - T^s(x_s, \bar{x})}.$$

Hence, any particular solution $\bar{R}^s(\rho_s, x_s, \bar{x})$ has to satisfy a boundary condition, i.e., $\lim_{y \to \sigma(x)} \bar{R}^s(\rho_s(y, x), x_s, \bar{x}) = \infty$.

The intuition of Proposition 5 is that, at best, the functions R^1 and R^2 in expression (4) are defined up to some (common) mapping G. This is illustrated below.

Parametric example 2. Coming back to our numerical example, with the logistic-quadratic specification for the sharing function:

$$\rho = y \cdot \mathcal{L}\left(a_{00} + a_{01}x_1 + a_{02}x_2 + a_{11}x_1^2 + a_{22}x_2^2\right),$$

one can see that for $s = 1, 2$, the functions R^s are given by:

$$R^s\left(\rho_s, x_s\right) = G\left(\frac{1}{\rho_s g_s(x_s)}\right),$$

where G is an arbitrary function that is positive and strictly increasing on the range of $\bar{R}^s(\rho_s, x_s) = 1/[\rho_s g_s(x_s)]$, and g_s are the functions defined previously. For any choice of G, one can recover the utility functions for arbitrary choices of the threat points by solving a differential equation. Whenever G is such that $(\partial \bar{R}^s/\partial \rho_s)G_r(\bar{R}^s) + [G(\bar{R}^s)]^2 < 0$, the corresponding utilities are strictly concave. For example, consider the following family of transformations:

$$G(u) \equiv \alpha \cdot c^{\alpha-1} \cdot u^{\alpha}, \quad \alpha \geqslant 1,$$

where the constant $c > 0$ is chosen such that $\rho_s \geqslant c$, $s = 1, 2$. For U^1, we have:

$$U^1\left(\rho, x_1\right) = K\left(x_1\right) \exp\left[-\frac{\alpha}{\alpha-1}\left(\frac{1}{g_1(x_1)}\right)^{\alpha}\left(\frac{c}{\rho_1}\right)^{\alpha-1}\right] + T^1(x_1), \quad \text{if } \alpha > 1,$$

and $U^1(\rho, x_1) = \bar{U}^1(\rho, x_1)$ if $\alpha = 1$, with $\bar{U}^1$ as defined previously, and $K(x_1) > 0$. The utility U^1 exhibits decreasing relative risk aversion with a coefficient of relative risk aversion equal to $\alpha\left[1 - (c/\rho_1)^{\alpha-1}/(g_1(x_1))^{\alpha}\right]$.

4.2 Identifying assumptions

We now provide two examples of additional assumptions that enable to recover the underlying structural model from observed behavior. The argument is presented for the NB case; the GNB case is discussed at the end of the subsection.

4.2.1 Case 1: one known utility function

If the utility and threat functions of one player are known from other sources, stronger identifiability results can be derived. Let us assume that the functions (U^2, T^2) are *known*. Keeping in mind that $\partial\rho/\partial y > 0$, the function ρ can be globally inverted in y on $\mathcal{S}$ giving y as some function $\theta\left(\rho, x_1, x_2, \bar{x}\right)$. The first order condition (4) then becomes:

$$R^1\left(\rho, x_1, \bar{x}\right) = R^2\left(\theta\left(\rho, x_1, x_2, \bar{x}\right) - \rho, x_2, \bar{x}\right), \tag{12}$$

where both R^2 and θ are known functions. This can then be integrated as:

$$U^1(\rho_1, x_1, \bar{x}) - T^1(x_1, \bar{x}) = K(x_1, x_2, \bar{x}) \cdot \exp\left(\int_{\rho_1^0}^{\rho_1} R^2(\theta(r, x_1, x_2, \bar{x}) - r, x_2, \bar{x})\, dr\right),$$

for some $\rho_1^0 \in]0, y[$ and some function $K(x_1, x_2, \bar{x}) > 0$. Since the right-hand side of the above equation does not depend on x_2, we necessarily have:

$$\frac{1}{K}\frac{\partial K}{\partial x_{j2}} + \int_{\rho_1^0}^{\rho_1}\left[\frac{\partial \theta}{\partial x_{j2}} \cdot \frac{\partial R^2}{\partial \rho_2} + \frac{\partial R^2}{\partial x_{j2}}\right] dr = 0, \tag{13}$$

for all $j = 1, \ldots, n_2$. Differentiating again with respect to ρ_1 yields additional restrictions that the sharing function must satisfy for all $j = 1, \ldots, n_2$,

$$\frac{\partial \theta(\rho, x_1, x_2, \bar{x})}{\partial x_{j2}} = -\frac{\partial R^2(\theta(\rho, x_1, x_2, \bar{x}) - \rho, x_2, \bar{x}) / \partial x_{j2}}{\partial R^2(\theta(\rho, x_1, x_2, \bar{x}) - \rho, x_2, \bar{x}) / \partial \rho_2}. \tag{14}$$

Combining (13) and (14) then shows that K does not depend on x_2, so

$$U^1(\rho_1, x_1, \bar{x}) - T^1(x_1, \bar{x}) = K(x_1, \bar{x}) \cdot \exp\left(\int_{\rho_1^0}^{\rho_1} R^2(\theta(r, x_1, x_2, \bar{x}) - r, x_2, \bar{x})\, dr\right).$$

We conclude that the difference $U^1 - T^1$ is identified up to a multiplicative function of $(x_1, \bar{x})$; in particular, the corresponding index of risk aversion is identified exactly.

Consider, in particular, the case of an affine ('risk neutral') utility function:

$$U^2(\rho_2, x_2, \bar{x}) = \alpha(x_2, \bar{x}) + \beta(x_2, \bar{x}) \cdot \rho_2.$$

This may be the case, for instance, if agent 2 represents a risk-neutral employer who bargains with a risk averse worker (or trade union). If so, the R^2 function in (12) equals $R^2(\rho_2, x_2, \bar{x}) = 1/(\rho_2 - \gamma(x_2, \bar{x}))$, where $\gamma \equiv (T^2 - \alpha)/\beta$. A first consequence of the linearity assumption is that only the ratio γ is relevant in the maximization program. A second consequence is that γ is exactly identified even if only U^2 is known. Indeed, we know from previous results that the R^2 function is known to be identified up to an increasing transform G. Then using the assumption on U^2 pins down G, which in turn identifies γ. Moreover, under linearity the additional restrictions in (14) become

$$\frac{\partial \theta(\rho, x_1, x_2, \bar{x})}{\partial x_{2j}} = \frac{\partial \gamma(x_2, \bar{x})}{\partial x_{2j}},$$

which in turn implies that

$$\frac{\partial^2 \theta(\rho, x_1, x_2, \bar{x})}{\partial x_{1i} \partial x_{2j}} = \frac{\partial^2 \theta(\rho, x_1, x_2, \bar{x})}{\partial \rho \partial x_{2j}} = 0,$$

for all $i = 1, \ldots, n_1$ and all $j = 1, \ldots, n_2$. Finally, we then also have

$$U^1(\rho_1, x_1, \bar{x}) - T^1(x_1, \bar{x}) = K(x_1, \bar{x}) \cdot \exp\int_{\rho_1^0}^{\rho_1} \frac{dr}{\theta(r, x_1, x_2, \bar{x}) - r - \gamma(x_2, \bar{x})}$$

where θ and γ are known functions, and $\rho_1^0 \in]0, y[$ and $K(x_1, \bar{x}) > 0$ are arbitrary.

4.2.2 Case 2: x_s–independent utility functions

An alternative identifying assumption is that x_1 and x_2 are only relevant for the threat points; they have no direct impact on utilities. Typically, it will be the case if x_1 and x_2 represent the payment made to players in the case of disagreement. We now proceed to show that in this context, not only additional restrictions are generated on the shape of the sharing function, but both individual utilities and threat points are uniquely recovered (up to the same affine transform). For the sake of notational simplicity, we hereafter take x_1 and x_2 to be one dimensional; the extension to the general case is straightforward (although it requires tedious notations) and is left to the reader.

Formally, we introduce the following assumptions:

Assumption U.2 *For $s = 1, 2$, the individual utilities U^s are independent of x_s; i.e., $U^s(\rho_s, x_s, \bar{x}) = U^s(\rho_s, \bar{x})$.*

Assumption T.2 *For $s = 1, 2$, the individual threat functions are not constant functions of x_s; i.e., there exists at least one value of $(x_s, \bar{x})$ such that $\partial T^s(x_s, \bar{x})/\partial x_s \neq 0$.*

Assumptions U.2 and T.2 are akin to an exclusion restriction: while the variable x_s is excluded from U^s, it still affects the threat function T^s. In this sense, the requirement is similar to the standard exclusion restrictions used in simultaneous equations systems where identification is driven by the fact that certain variables enter some of the equations in the system while being excluded from the others.[15]

Under U.1, U.2 and T.1, the sharing function $\rho(y, x)$ solves the problem:

$$\max_{0 \leqslant \rho \leqslant y} \left(U^1(\rho, \bar{x}) - T^1(x_1, \bar{x}) \right) \cdot \left(U^2(y - \rho, \bar{x}) - T^2(x_2, \bar{x}) \right),$$

and, as previously, the first order condition is of the form (4), where the functions R^s now exhibit an additional separability property, namely

$$R^s(\rho_s, x_s, \bar{x}) = \frac{\partial U^s(\rho_s, \bar{x}) / \partial \rho_s}{U^s(\rho_s, \bar{x}) - T^s(x_s, \bar{x})}. \tag{15}$$

Our main result is then as follows:

Proposition 6 *Suppose Assumptions U.1, U.2, T.1, T.2, S.1, and S.2 hold, and that x_s is one-dimensional, $s = 1, 2$. Then, the utilities U^s and threat functions T^s are identified up to an affine, increasing transform that may depend on $\bar{x}$.*

[15] Note that in the case of Assumption T.1 where y was excluded from T^s, the requirement that U^s was not a constant function of y was automatically satisfied under the strict concavity assumption U.1.

Moreover, the sharing function must satisfy additional testable restrictions. Assume ρ is four times continuously differentiable in (y, x_1, x_2), and let $\Phi^1(\rho, x_1, \bar{x}) = -(\partial\rho/\partial x_1)/(1-\partial\rho/\partial y)$, and $\Phi^2(y-\rho, x_2, \bar{x}) = (\partial\rho/\partial x_2)/(\partial\rho/\partial y)$ be the functions in Proposition 2. For any x such that $\partial\rho(y,x)/\partial x_s \neq 0$, two cases are possible:

(a) either $\frac{\partial^2 \ln|\Phi^s|}{\partial\rho_s\partial x_s} \neq 0$ for some ρ_s. Then, necessarily

$$\left(\frac{\partial^3 \ln|\Phi^s|/\partial\rho_s\partial x_s^2}{\partial^2 \ln|\Phi^s|/\partial\rho_s\partial x_s}\right) \text{ is a function of } (x_s, \bar{x}) \text{ alone;} \tag{16}$$

(b) or $\frac{\partial^2 \ln|\Phi^s|}{\partial\rho_s\partial x_s} = 0$ for all ρ_s. Then, necessarily

$$\Phi^s \text{ is a function of } (x_s, \bar{x}) \text{ alone.} \tag{17}$$

The complete proof of Proposition 6 is in Appendix. The proposition shows two results. First is an identification result. To give its intuition, let us recall that from Proposition 5, the functions R^s are known to be identified up to an unknown transform $G(\cdot, \bar{x})$. Now, Assumption U.2 implies that the functions R^s must be of the separable form (15). This generates additional restrictions on admissible transformations $G(\cdot, \bar{x})$; specifically, only transformations of the form $G(r, \bar{x}) = r$ are possible. This in turn leads to the identification of U^s and T^s (up to an affine, increasing transform that may depend on $\bar{x}$). The result continues to hold for the GNB model with weights $\delta_s(\bar{x})$ that do not depend on x_s.

The second result of Proposition 6 is that further testable restrictions are generated under U.2 and T.2. The restrictions take different forms depending on whether or not the functions $(\partial\rho/\partial x_1)/(1-\partial\rho/\partial y)$ and $(\partial\rho/\partial x_2)/(\partial\rho/\partial y)$ are of the form $\varphi(\rho, \bar{x}) \cdot \theta(x_s, \bar{x})$, i.e., their logarithms are additively separable in ρ and x_s. In both cases, their intuition is relatively simple. Conditions (16) and (17) are a direct consequence of the exclusion restrictions in U.2. The latter imply that any particular solution for the utility function $\bar{U}^s$ must be independent of x_s for ρ fixed, that is, $\partial\bar{U}^s/\partial x_s = 0$, which implies the restrictions on the sharing function under the form given above.

Parametric example 3. These conditions significantly limit the form of the sharing functions that derive from x_s-independent utilities and are compatible with GNB. Namely, any function of the form

$$\rho = y \cdot g(x) \quad \text{with} \quad 0 < g(x) < 1 \quad \text{and} \quad \frac{\partial}{\partial x_{s'}}\left(\frac{\partial g(x)/\partial x_s}{g(x)\,(1-g(x))}\right) = 0,$$

is incompatible with assumption U.2 unless g is a constant function of x_1 and x_2. In particular, the logistic-quadratic form used in the parametric examples 1 and 2 is not compatible with this setting. This follows directly from (17), since under the specification above $\partial^2 \ln|\Phi^s|/\partial\rho_s\partial x_s = 0$ and yet

$$\frac{\partial\Phi^1(\rho_1, x_1)}{\partial\rho_1} = -\frac{\partial g(x)/\partial x_1}{g(x)\,(1-g(x))} \quad \text{and} \quad \frac{\partial\Phi^2(\rho_2, x_2)}{\partial\rho_2} = \frac{\partial g(x)/\partial x_1}{g(x)\,(1-g(x))}.$$

In other words, an empirical model of bargaining that is using either the logistic-quadratic specification (or actually any version in which each member's fraction of the surplus does not depend on the surplus's size) *must* assume (at least implicitly) that individual utilities in case of an agreement depend on the threat point payments—a strong assumption indeed. This remark illustrates the relevance of a preliminary, theoretical investigation. An empirical specification based on the logistic-quadratic form may be quite appealing (and fit the data); but it is internally inconsistent with the model at stake, at least if one assumes (as it seems natural) that agents care about their threat point utility only insofar as it affects the bargaining outcome.

5 Identifiability: the stochastic case

Finally, let us consider the stochastic version of the model. We now assume that not all of the variables entering the model are observed by the econometrician. Specifically, the model also depends on variables ϵ that are unobservable (to the econometrician) in addition to the observed preference and payoff variables x. In the presence of latent disturbances ϵ (such as unobserved individual heterogeneity) the knowledge of the size of the pie to be shared y and of the observed preference and status quo payoff variables x no longer fully determines the agreement event m and the agreed sharing rule ρ. Instead, the unobservables induce a nondegenerate distribution of (m, ρ) given (y, x). The question of identification then is whether upon observing this distribution, it is possible for the econometrician to uniquely recover the agents' utility and threat functions.

Specifically, we now consider the situation in which the players always reach an agreement and the agreed sharing function solves:

$$\max_{0\leqslant\rho\leqslant y} \left(U^1(\rho,\bar{x}) - T^1(x_1,\bar{x}) + \epsilon_1\right) \cdot \left(U^2(y-\rho,\bar{x}) - T^2(x_2,\bar{x}) + \epsilon_2\right). \tag{18}$$

Here, the econometrician observes the joint distribution of the variables $y \in \mathbb{R}_+$, $x = (x_1, x_2, \bar{x}) \in \mathbb{R}^n$, and the resulting share $\rho \in]\,0, y\,[$, while $(\epsilon_1, \epsilon_2) \in \mathbb{R}_+^2$ remain latent.

We shall maintain the following assumptions on the distribution of (ϵ_1, ϵ_2):

Assumption D.1 $(\epsilon_1, \epsilon_2) \perp (x_1, x_2) \,|\, (y, \bar{x})$.

Assumption D.2 *The conditional distribution* $F_{\epsilon_1,\epsilon_2|\,y,\bar{x}}$ *of* (ϵ_1, ϵ_2) *given* $(y, \bar{x})$ *is absolutely continuous and has full support on* $\mathbb{R}^2_+$.

We shall also reinforce the structure on the threat functions:

Assumption T.3 *The threat functions* T^s *are strictly monotonic in* x_{s1}*, i.e., for every* $(x_s, \bar{x})$*,* $\partial T^s(x_s, \bar{x})/\partial x_{s1} \neq 0$*. Moreover,* T^s *is proper in* x_{s1}*, i.e.,* $\lim_{|x_{s1}|\to\infty} |T^s(x_1, \bar{x})| = \infty$*.*

In words, D.1 states that the model unobservables (ϵ_1, ϵ_2) are conditionally independent of the agent specific variables (x_1, x_2) given the size of the pie y and the common variables $\bar{x}$. This conditional independence property takes the form of exclusion restrictions which will be shown to drive our identification results. D.2 is a support restriction; we use it to guarantee that the conditional distribution of the share ρ given the observables (y, x) is nondegenerate. Since under T.1, the threat functions T^s are continuously differentiable, the requirement in T.3 says that the partial derivatives $\partial T^s/\partial x_{s1}$ are either everywhere (strictly) positive or everywhere (strictly) negative.[16] However, no assumptions are placed on the signs of these derivatives, which as we shall proceed to show, are identifiable from the distribution of the observables. Finally, if one is interested in identifying the agents' utilities U^s and threat functions T^s alone then T.3 can be omitted; it only plays a role in the identification of the conditional distribution $F_{\epsilon_1,\epsilon_2|\,y,\bar{x}}$ of the unobservables.

Before proceeding, we recall several useful definitions. Following the related literature (Koopmans and Reiersøl, 1950; Brown, 1983; Roehrig, 1988; Matzkin, 2003), we call *structure* a particular value of the quintuplet $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$. Note that the model (18) simply corresponds to the set of all structures $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$ that satisfy the a priori restrictions given by U.1, T.1, T.3, D.1 and D.2. Each structure in the model induces a conditional distribution $F_{\rho|\,y,x}$ of the observables, and two structures $(\tilde{U}^1, \tilde{U}^2, \tilde{T}^1, \tilde{T}^2, \tilde{F}_{\tilde{\epsilon}_1,\tilde{\epsilon}_2|\,y,\bar{x}})$ and $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$ are *observationally equivalent* if they generate the same $F_{\rho|\,y,x}$. The model (18) said to be *identified*, if the set of structures that are observationally equivalent to $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$ reduces to a singleton. More formally, the structure $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$ is globally identified if any observationally equivalent structure $(\tilde{U}^1, \tilde{U}^2, \tilde{T}^1, \tilde{T}^2, \tilde{F}_{\tilde{\epsilon}_1,\tilde{\epsilon}_2|\,y,\bar{x}})$ satisfies:

$$\tilde{U}^s(\rho_s, \bar{x}) = U^s(\rho_s, \bar{x}), \quad \tilde{T}^s(x_s, \bar{x}) = T^s(x_s, \bar{x}), \text{ and}$$
$$\tilde{F}_{\tilde{\epsilon}_1,\tilde{\epsilon}_2|\,y,\bar{x}}(t|y, \bar{x}) = F_{\epsilon_1,\epsilon_2|\,y,\bar{x}}(t|y, \bar{x}),$$

for every $(\rho_1, \rho_2, x_1, x_2, t)$, a.e. $(y, \bar{x})$, and $s = 1, 2$.

[16] Note that this is a strengthening of T.2, which was used to achieve identification in Proposition 6.

Note that equality in conditional distribution given (y, x) is denoted by $\underset{|(y,x)}{\sim}$. Then our main result is as follows:

Proposition 7 *Suppose Assumptions U.1, U.2, T.1, T.3, S.1, S.2, D.1 and D.2 hold. Then the structures* $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|y,\bar{x}})$ *and* $(\tilde{U}^1, \tilde{U}^2, \tilde{T}^1, \tilde{T}^2, \tilde{F}_{\tilde{\epsilon}_1,\tilde{\epsilon}_2|y,\bar{x}})$ *are observationally equivalent if and only if there exist functions* $A^1(\bar{x}) > 0$, $A^2(\bar{x}) > 0$, $B^1(\bar{x}) > 0$ *and* $\alpha^1(\bar{x})$, $\alpha^2(\bar{x})$, $\beta^1(\bar{x})$, $\beta^2(\bar{x})$, *such that for every* (r, y, x) *the agents' utilities and threat functions satisfy:*

$$\begin{aligned}
\tilde{U}^1(\rho_1, \bar{x}) &= A^1(\bar{x}) \cdot U^1(\rho_1, \bar{x}) + \alpha^1(\bar{x}), \\
\tilde{U}^2(\rho_2, \bar{x}) &= A^2(\bar{x}) \cdot U^2(\rho_2, \bar{x}) + \alpha^2(\bar{x}), \\
\tilde{T}^1(x_1, \bar{x}) &= B^1(\bar{x}) \cdot T^1(x_1, \bar{x}) + \beta^1(\bar{x}), \\
\tilde{T}^2(x_2, \bar{x}) &= A^2(\bar{x})(A^1(\bar{x}))^{-1} B^1(\bar{x}) \cdot T^2(x_2, \bar{x}) + \beta^2(\bar{x}),
\end{aligned} \tag{19}$$

and the conditional distributions of the unobservables $F_{\epsilon_1,\epsilon_2|y,\bar{x}}$ *and* $\tilde{F}_{\tilde{\epsilon}_1,\tilde{\epsilon}_2|y,\bar{x}}$ *are such that for every* $t \in \mathbb{R}$, *every* $r \in]0, y[$, *and almost every* (y, x),

$$\begin{aligned}
&\frac{\partial \tilde{U}^2(y-r, \bar{x})}{\partial \rho_2} \left[\tilde{U}^1(r, \bar{x}) + \tilde{\epsilon}_1\right] - \frac{\partial \tilde{U}^1(r, \bar{x})}{\partial \rho_1} \left[\tilde{U}^2(y-r, \bar{x}) + \tilde{\epsilon}_2\right] \underset{|(y,x)}{\sim} \\
&\quad A^2(\bar{x}) B^1(\bar{x}) \left\{ \frac{\partial U^2(y-r, \bar{x})}{\partial \rho_2} \left[U^1(r, \bar{x}) + \epsilon_1 + \frac{\beta^1(\bar{x})}{B^1(\bar{x})}\right] \right. \\
&\quad\quad \left. - \frac{\partial U^1(r, \bar{x})}{\partial \rho_1} \left[U^2(y-r, \bar{x}) + \epsilon_2 + \frac{A^1(\bar{x})}{A^2(\bar{x})} \frac{\beta^2(\bar{x})}{B^1(\bar{x})}\right] \right\}.
\end{aligned} \tag{20}$$

A proof of Proposition 7 is in Appendix. This proposition is essentially an identification result. Its main implication is that the joint distribution of the observable variables determines individual preferences and threat points (the U^s and T^s) up to affine transforms (which, obviously, can depend on $\bar{x}$). More precisely, in any two observationally equivalent structures, agents' utilities and treat functions need to be related by simple strictly increasing affine transformations involving unknown functions of $\bar{x}$ (those transformations are then "undone" at the level of the conditional distributions of the disturbances). Moreover, there exists a relationship between the affine transforms (see the form of (19)). The fact that even the unknown threat functions are also related through strictly increasing transformations is surprising at first view given that no restrictions were placed on the direction of monotonicity in T.3; it turns out, however, that the observables identify the signs of $\partial T^s / \partial x_{1s}$ (see Equation (50) in the Appendix) which in turn implies that $B^1(\bar{x}) > 0$. Lastly, the conditional distribution of the random terms is not identified, but if two models are observationally equivalent their respective distributions are related by condition (20).

The previous result can be clarified if we first discuss the normalizations that are needed in our framework. Clearly, if we let $\tilde{U}^1(\rho_1,\bar{x}) = U^1(\rho_1,\bar{x}) + \mu^1(\bar{x})$, $\tilde{T}^1(x_1,\bar{x}) = T^1(x_1,\bar{x}) + \nu^1(\bar{x})$ and $\tilde{\epsilon}_1 = \epsilon_1 - \mu_1(\bar{x}) - \nu_1(\bar{x})$, then the structure $(\tilde{U}^1, U^2, \tilde{T}^1, T^2, F_{\tilde{\epsilon}_1,\epsilon_2|\,y,\bar{x}})$ is observationally equivalent to $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$ for any choice of functions $\mu^1(\bar{x})$ and $\nu_1(\bar{x})$. Analogous result obtains if instead of modifying the utility and threat function of player 1 we do so with player 2. Similarly, if for any $\lambda^1(\bar{x}) > 0$ we let $\bar{U}^1(\rho_1,\bar{x}) = U^1(\rho_1,\bar{x})\lambda^1(\bar{x})$, $\bar{T}^1(x_1,\bar{x}) = T^1(x_1,\bar{x})^1\lambda(\bar{x})$, $\bar{\epsilon}_1 = \epsilon_1\lambda^1(\bar{x})$, then $(\bar{U}^1, U^2, \bar{T}^1, T^2, \bar{F}_{\bar{\epsilon}_1,\epsilon_2|\,y,\bar{x}})$ is again observationally equivalent to $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$; the same holds for player 2. We therefore impose that any U^s, T^s, and ϵ_s $(s = 1,2)$ in (18) satisfy the following normalization conditions:

$$\begin{aligned}
\text{(i) for known } \rho_s^0 \text{ and } k^s(\bar{x}),\ U^s(\rho_s^0,\bar{x}) &= k^s(\bar{x}), \\
\text{(ii) for known } x_s^0 \text{ and } c^s(\bar{x}),\ T^s(x_s^0,\bar{x}) &= c^s(\bar{x}), \\
\text{(iii) for known } \rho_s^* \text{ and } K^s(\bar{x}) > 0,\ \partial U^s(\rho_s^*,\bar{x})/\partial\rho_s &= K^s(\bar{x}).
\end{aligned} \tag{21}$$

Note that, here, the values ρ_s^0, x_s^0 and the functions $k^s(\bar{x}), c^s(\bar{x})$ and $K^s(\bar{x})$ can be arbitrarily chosen. Under the six normalization conditions in (21), we obtain the following Corollary to Proposition 7.

Corollary 1 *Let the assumptions of Proposition 7 hold and assume in addition that the normalization conditions* (21) *are satisfied. Then the structures* $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,\bar{x}})$ *and* $(\tilde{U}^1, \tilde{U}^2, \tilde{T}^1, \tilde{T}^2, \tilde{F}_{\tilde{\epsilon}_1,\tilde{\epsilon}_2|\,y,\bar{x}})$ *are observationally equivalent if and only if for every* $(\rho_s,\bar{x})$*:*

$$\tilde{U}^s(\rho_s,\bar{x}) = U^s(\rho_s,\bar{x}),$$

and there exists a function $B(\bar{x}) > 0$*, such that for every* $(x_s,\bar{x})$*,*

$$\tilde{T}^s(x_s,\bar{x}) - c^s(\bar{x}) = B(\bar{x}) \cdot \left[T^s(x_s,\bar{x}) - c^s(\bar{x})\right],$$

and for every $t \in \mathbb{R}$*, every* $r \in \,]0\,,y\,[$ *and almost every* $(r,y,x_s,\bar{x})$*,*

$$\begin{aligned}
&\frac{\partial U^2(y-r,\bar{x})}{\partial\rho_2}\left[U^1(r,\bar{x}) - c^1(\bar{x}) + \tilde{\epsilon}_1\right] - \frac{\partial U^1(r,\bar{x})}{\partial\rho_1}\left[U^2(y-r,\bar{x}) - c^2(\bar{x}) + \tilde{\epsilon}_2\right] \underset{|(y,x)}{\sim} \\
&B(\bar{x})\left\{\frac{\partial U^2(y-r,\bar{x})}{\partial\rho_2}\left[U^1(r,\bar{x}) - c^1(\bar{x}) + \epsilon_1\right] - \frac{\partial U^1(r,\bar{x})}{\partial\rho_1}\left[U^2(y-r,\bar{x}) - c^2(\bar{x}) + \epsilon_2\right]\right\}.
\end{aligned}$$

It is clear from Corollary 1 that under normalization (21) alone, it is not possible to nonparametrically identify the NB model (18). To achieve identification we still need one more restriction to pin down the function $B(\bar{x})$. For instance, we can impose an additional restriction on the disturbances ϵ_1 and ϵ_2 which would require that for $s = 1,2$:

$$E[\epsilon_s|\,y,\bar{x}] = 0. \tag{22}$$

Letting

$$\epsilon(r,y,\bar{x}) \equiv \partial U^2(y-r,\bar{x})/\partial\rho_2 \cdot \left[U^1(r,\bar{x}) - c^1(\bar{x}) + \epsilon_1\right] \\ -\partial U^1(r,\bar{x})/\partial\rho_1 \cdot \left[U^2(y-r,\bar{x}) - c^2(\bar{x}) + \epsilon_2\right],$$

it then follows that

$$E[\epsilon(r,y,\bar{x})|\, y,\bar{x}] = \frac{\partial U^2(y-r,\bar{x})}{\partial\rho_2}\left[U^1(r,\bar{x}) - c^1(\bar{x})\right] - \frac{\partial U^1(r,\bar{x})}{\partial\rho_1}\left[U^2(y-r,\bar{x}) - c^2(\bar{x})\right],$$

which is a known quantity. Now consider $\tilde{\epsilon}(r,y,\bar{x}) = B(\bar{x})\cdot\epsilon(r,y,\bar{x})$. Provided there exists a value of (r,y) for which the right hand side in the above equation is non-zero for all $\bar{x}$, it then follows that $E[\epsilon(r,y,\bar{x})|\, y,\bar{x}] = E[\tilde{\epsilon}(r,y,\bar{x})|\, y,\bar{x}]$, only if $B(\bar{x}) = 1$. Thus we have another Corollary to Proposition 7:

Corollary 2 *Let the assumptions of Corollary 1 hold and assume in addition that the moment condition* (22) *is satisfied. Then,* (U^1, U^2, T^1, T^2) *and the conditional distribution of* $\epsilon(r,y,\bar{x})$ *given* (y,x) *are identified.*

It remains to be shown when the knowledge of the conditional distribution of $\epsilon(r,y,\bar{x})$ given (y,x) is sufficient to uniquely determine the joint distribution of ϵ_1 and ϵ_2. Note that when (r,y,x) is fixed, then $\epsilon(r,y,\bar{x})$ is simply a linear combination of ϵ_1 and ϵ_2, in which all the coefficients are known. When ϵ_1 and ϵ_2 are conditionally independent, $\epsilon_1 \perp \epsilon_2 \,|\, (y,\bar{x})$, the problem of identifying $F_{\epsilon_1|\,y,\bar{x}}$ and $F_{\epsilon_2|\,y,\bar{x}}$ from the conditional distribution of $\epsilon(r,y,\bar{x})$ becomes the well-known deconvolution problem. If in addition there exists a component z of $(y,\bar{x})$ with at least 2 points in the support such that ϵ_1 and ϵ_2 are conditionally independent of z, $(\epsilon_1,\epsilon_2) \perp z \,|\, (y,\bar{x})$, then observations on multiple linear combinations of ϵ_1 and ϵ_2 are available which permit their distributions to be identified (see, e.g., Bonhomme and Robin, 2010). We thus obtain a final Corollary to Proposition 7:

Corollary 3 *Let the assumptions of Corollary 2 hold and assume in addition that* $\epsilon_1 \perp \epsilon_2 \,|\, (y,\bar{x})$ *and* $(\epsilon_1,\epsilon_2) \perp z \,|\, (y,\bar{x})$ *where* z *is a component of* $(y,\bar{x})$ *with at least 2 points in the support. Then,* $(U^1, U^2, T^1, T^2, F_{\epsilon_1,\epsilon_2|\,y,x})$ *is identified.*

We conclude this section with a brief discussion of identification in an alternative stochastic specification of the NB model. Say that instead of (18) we consider the following specification:

$$\max_{0\leqslant\rho\leqslant y}\left(U^1(\rho,\bar{x}) - T^1(x_1,\bar{x},\epsilon)\right)\cdot\left(U^2(y-\rho,\bar{x}) - T^2(x_2,\bar{x},-\epsilon)\right)$$

in which $(y, x_1, x_2, \bar{x})$ are observed by the econometrician, while ϵ remains unobservable. The idea here is that only one of the variables entering the problem nonseparably is latent. This variable ϵ has opposite effects on the threat functions of the two agents. Assume in addition that the threat functions are strictly increasing in the unobservable component, and that $\epsilon \perp (y, x_1, x_2, \bar{x})$. Then, using arguments similar to those in Matzkin (2003), it is straightforward to show that the normalization condition $\epsilon \sim U(0, 1)$ suffices to nonparametrically identify ϵ as a function of $(\rho, y, x_1, x_2, \bar{x})$. Hence, we are back in the case where all the variables are observed and the results of Section 4 apply.

6 Conclusion

Our main results leads to a significant qualification of the widely accepted views that "*bargaining theory contains very few interesting propositions that can be tested empirically*," to quote Hamermesh (1973, p.1146). Admittedly, testability and identifiability do not obtain in the most general model. If the econometrician knows nothing about the form of utility and threat functions, any (efficient) sharing of the pie is compatible with NB. Nevertheless, whenever utility and threat functions satisfy specific exclusion properties, NB generates strong restrictions on observed behavior. Clearly, the relevance of the exclusion conditions cannot be assessed a priori but depends on the bargaining context.

Our results have potentially important consequences for basically all applications listed in the Introduction. Considering for instance the negotiations between a firm and its employees (or a trade union representing them), the case in which one utility function is known to be linear (as analyzed in Subsection 4.2) seems quite relevant, because the linearity assumption makes often sense on the firm's side. Profit maximization is a standard theoretical axiom, and the firm's risk neutrality can be derived from specific assumptions on, say, complete financial markets. On the contrary, workers' risk aversion is often viewed as a driving force in the design of employment contracts, so a risk neutrality assumption on the worker's side would be quite debatable. Our results suggest that such an assumption is by no means needed. Not only can NB be tested without this assumption, but the worker's preferences (and in particular her risk aversion) can in general be identified from the outcome of the negotiation.

Similarly, regarding household decision making, our results imply that the NB assumption, per se, implies very little beyond efficiency—a conclusion already conjectured by Chiappori (1991). More surprisingly, however, Proposition 2 suggests that mild exclusion restrictions may be sufficient to reverse this conclusion. For instance, in a model with purely private consumption, the form of the intrahousehold sharing rule may indeed be constrained by the NB context, even when the threat points are not explicitly specified, provided that some exclusion restrictions can be assumed to hold.[17] The notion of *distribution factor*, as defined in the collective literature, corresponds exactly to such restrictions. It has been known for some time that distribution factors significantly increase the scope of non parametric identification in a collective framework; our results suggests that they are also crucial in deriving additional predictions and tests stemming from a NB framework. This will be the topic of future research.

Perhaps one of the most promising directions of research opened by these results regards experimental economics. The investigation of bargaining theory in experimental economics dates back to the seminal works by Siegel and Fouraker (1960). A standard problem with experiments of this type is that the observer does not know the players' preferences. As we said in the Introduction, assuming linear preferences may unduly restrict the scope of the test: a joint test of NB and linear preferences is likely to be rejected just because preferences fail to be linear—and then the rejection tells very little about the status of the NB hypothesis. A possible solution, introduced by Roth and Malouf (1979), is to consider players who bargain about probabilities of a lottery. The idea, here, is that linearity immediately follows from the expected utility hypothesis. Note, however, that once again one jointly tests NB and expected utility. Given that expected utility tends to be rejected in experiments, the status of the test (as a test of NB) remains ambiguous at best.

From this point of view, the methodology developed in this paper opens new and interesting directions for future research in this area. Consider again the simple experiment discussed in the Introduction. Our main conclusion is that a cardinal representation of each agent's utility function can be identified from it. This identification does not require any form of uncertainty; in particular, it does not rely on the assumption that preferences under uncertainty are of von Neumann-Morgenstern type. Moreover, the NB structure generates strong testable properties for the sharing function. In principle, this can lead to experimental tests. Indeed, one could first face individuals of a given sample with menus of lotteries, in order to assess their level of risk aversion from their choices; then match the agents by pairs and let them play a two-sided bargaining problem identical to the one discussed in the Introduction, which, from our result, allow to recover their bargaining-relevant utility functions. A comparison—and even a formal test—is then possible. Experiments of this kind will be the topic of future work.

[17]As an example of natural exclusion restrictions, one may consider the assumption that a spouse's threat point does not depend on the spouse's wage (or does so only through some specific function - say, the form of the divorce settlement).

A Appendix

Proof of Proposition 2. The proof of Proposition 2 is in two steps. The first establishes the necessity and the second shows the sufficiency of conditions (6)-(10).

Step 1. Necessity: If the first order condition (4) is differentiated with respect to y and x_{1i}, $i = 1, \dots, n_1$, after rearrangement, one gets:

$$\left(\frac{\partial R^1}{\partial \rho_1} + \frac{\partial R^2}{\partial \rho_2}\right)\left(1 - \frac{\partial \rho}{\partial y}\right) = \frac{\partial R^1}{\partial \rho_1}, \tag{23}$$

$$\left(\frac{\partial R^1}{\partial \rho_1} + \frac{\partial R^2}{\partial \rho_2}\right)\frac{\partial \rho}{\partial x_{1i}} = -\frac{\partial R^1}{\partial x_{1i}}. \tag{24}$$

Firstly, it is easily shown that $\partial R^s / \partial \rho_s < 0$ for $s = 1, 2$ since the functions U^s are strictly increasing and concave in ρ_s. Hence, from (23),

$$\frac{\partial \rho}{\partial y} = \frac{\partial R^2 / \partial \rho_2}{\partial R^1 / \partial \rho_1 + \partial R^2 / \partial \rho_2} \in]0, 1[,$$

which demonstrates the necessity of the condition (6). Secondly, from (23) and (24) together, one also gets:

$$\frac{\partial \rho / \partial x_{1i}}{1 - \partial \rho / \partial y} = -\frac{\partial R^1 / \partial x_{1i}}{\partial R^1 / \partial \rho_1},$$

where the right-hand side is a function Φ_i^1 of $(\rho, x_1, \bar{x})$ alone. This shows that the condition (7) is necessary as well. Moreover, differentiating $\Phi_i^1(\rho, x_1, \bar{x})$ with respect to x_{1i} and $x_{1i'}$, with $i \neq i'$, using cross derivative restrictions, i.e., $\partial^2 \rho / \partial x_{1i} \partial x_{1i'} = \partial^2 \rho / \partial x_{1i'} \partial x_{1i}$, and re-arranging gives condition (9).

Similarly, differentiating the first order condition (4) with respect to y and x_{2j}, $j = 1, \dots, n_2$, then taking ratios, one gets:

$$\frac{\partial \rho / \partial x_{2j}}{\partial \rho / \partial y} = \frac{\partial R^2 / \partial x_{2j}}{\partial R^2 / \partial \rho_2},$$

where the right-hand side is now a function Φ_j^2 of $(y - \rho, x_2, \bar{x})$ alone. This demonstrates that condition (8) is also necessary. Again, working out cross derivative restrictions gives condition (10).

Step 2. Sufficiency: Consider first the case of agent 1. If condition (7) is fulfilled, each ratio $(\partial \rho / \partial x_{1i}) / (1 - \partial \rho / \partial y)$, for every $i = 1, \dots, n_1$, can be written as some function Φ_i^1 of $(\rho, x_1, \bar{x})$. Then define a function $\bar{R}^1(\rho_1, x_1, \bar{x})$ as a particular solution to the system of PDEs:

$$\frac{\partial R^1 / \partial x_{1i}}{\partial R^1 / \partial \rho_1} = -\Phi_i^1(\rho, x_1, \bar{x}), \tag{25}$$

for $i = 1, \ldots, n_1$, with $\partial R^1/\partial\rho_1 < 0$. If condition (9) is satisfied, then any solution to the system of equations (25) is an increasing transform $G(\cdot, \bar{x})$ of $\bar{R}^1(\rho_1, x_1, \bar{x})$. One can always choose $\bar{R}^1$ to be positive over the domain at stake (it suffices to consider $\exp \bar{R}^1$) so

$$\bar{R}^1(\rho_1, x_1, \bar{x}) > 0. \tag{26}$$

Now, by construction, $\bar{R}^1$ (and therefore any transform of $\bar{R}^1$) can be written as a function of $(\rho_2, x_2, \bar{x})$—say, $\bar{R}^2(\rho_2, x_2, \bar{x})$—that is, $\bar{R}^1(\rho_1, x_1, \bar{x}) = \bar{R}^2(\rho_2, x_2, \bar{x})$. Hence $\bar{R}^1$ and $\bar{R}^2$ are solutions to (4). Now, consider the equation:

$$\bar{R}^s(\rho_s, x_s, \bar{x}) \equiv \delta_s(x_s, \bar{x}) \frac{\partial\bar{U}^s(\rho_s, x_s, \bar{x})/\partial\rho_s}{\bar{U}^s(\rho_s, x_s, \bar{x}) - \bar{T}^s(x_s, \bar{x})},$$

(where $\delta_s = 1$ in the NB case) as a PDE in $U^s - T^s$. It can readily be integrated into:

$$\bar{U}^s(\rho_s, x_s, \bar{x}) - \bar{T}^s(x_s, \bar{x}) = K_s(x_s, \bar{x}) \exp\left(\int_{\rho_s^0}^{\rho_s} \bar{R}^s(r, x_s, \bar{x})dr\right),$$

where $\rho_s^0 \in]\,0, y\,[$ and $K_s(x_s, \bar{x}) > 0$ are arbitrary. One can pick up an arbitrary $\bar{T}^s(x_s, \bar{x})$ satisfying Assumption T.1, then $\bar{U}^s(\rho_s, x_s, \bar{x})$ is defined by the previous equation. The function $K^s(x_s, \bar{x})$ does not affect the concavity of utility with respect to income, i.e., the condition above identifies a cardinal representation of the utility function (as a function of the share). Now, note that

$$\frac{\partial\bar{U}^s(\rho_s, x_s, \bar{x})}{\partial\rho_s} = K^s(x_s, \bar{x})\bar{R}^s(\rho_s, x_s, \bar{x}) \exp\left(\int_{\rho_s^0}^{\rho_s} \bar{R}^s(r, x_s, \bar{x})dr\right),$$

so by (26) the expression above is positive. Finally,

$$\frac{\partial^2\bar{U}^s(\rho_s, x_s, \bar{x})}{\partial\rho_s^2} = K^s(x_s, \bar{x}) \left[\frac{\partial\bar{R}^s}{\partial\rho_s} + \left(\bar{R}^s\right)^2\right] \exp\left(\int_{\rho_s^0}^{\rho_s} \bar{R}^s(r, x_s, \bar{x})dr\right),$$

so the particular solution must be such that the term into brackets is negative. Since $\mathcal{S}$ is bounded, that can be obtained by the transform $k \exp \bar{R}^s$ where $k > 0$ is an arbitrary small constant. Hence, $\bar{U}^s$ is a strictly increasing and concave function of ρ_s. Provided K^s is chosen to be twice continuously differentiable in x_s, the required smoothness of $\bar{U}^s$ follows from the assumed smoothness of ρ_s and δ_s, and $\bar{U}^s$ satisfies Assumption U.1. ■

Proof of Proposition 3. Consider the system of equations, which is satisfied for any $(y, x) \in \mathcal{F}$,

$$U^1(\rho, x_1, \bar{x}) - T^1(x_1, \bar{x}) = 0, \tag{27}$$
$$U^2(y - \rho, x_2, \bar{x}) - T^2(x_2, \bar{x}) = 0, \tag{28}$$

which implicitly defines ρ and y as a function of x. Inverting (27) with respect to ρ yields:

$$\rho = \sigma^1 (x_1, \bar{x}) \,. \tag{29}$$

Hence, the sharing function is independent of x_2 and y along the agreement frontier. Similarly, inverting (28) with respect to $y - \rho$ yields:

$$y - \rho = \sigma^2 (x_2, \bar{x}) \,. \tag{30}$$

Then, substituting (29) into equation (30) proves that $\sigma(x)$ is additive in the sense of the proposition. ■

Proof of Proposition 4. The proof of Proposition 4 is a direct consequence of equations (29) and (30). We have $\sigma_1(x_1, x) = \rho(\sigma_1(x_1, \bar{x}) + \sigma_2(x_2, \bar{x}), x_1, x_2, \bar{x})$ so differentiating this expression with respect to x_{1i} and x_{2j} immediately gives the desired conditions. ■

Proof of Proposition 5. The proof directly follows from that of Proposition 2. Indeed, we have seen that any solution R^s, $s = 1, 2$, to the system of PDEs (25) can be written as:

$$R^s(\rho_s, x_s, \bar{x}) = G\left(\bar{R}^s(\rho_s, x_s, \bar{x}), \bar{x}\right),$$

where $\bar{R}^s$ is the particular solution constructed in the proof of Proposition 2 and $G(\cdot, \bar{x})$ is an arbitrary strictly increasing function. The utilities U^s and threat functions T^s are then obtained as solutions to:

$$\frac{\partial U^s(\rho_s, x_s, \bar{x})/\partial \rho_s}{U^s(\rho_s, x_s, \bar{x}) - T^s(x_s, \bar{x})} = G\left(\bar{R}^s(\rho_s, x_s, \bar{x}), \bar{x}\right),$$

i.e.

$$U^s(\rho_s, x_s, \bar{x}) - T^s(x_s, \bar{x}) = K_s(x_s, \bar{x}) \exp\left(\int_{\rho_s^0}^{\rho_s} G\left(\bar{R}^s(r, x_s, \bar{x}), \bar{x}\right) dr\right), \tag{31}$$

where $\rho_s^0 \in]\,0, y\,[$ and $K_s(x_s, \bar{x}) > 0$ are arbitrary. One can pick up an arbitrary $\bar{T}^s(x_s, \bar{x})$ satisfying Assumption T.1, then $\bar{U}^s(\rho_s, x_s, \bar{x})$ is defined by (31). Differentiating (31) with respect to ρ_s gives:

$$\frac{\partial U^s(\rho_s, x_s, \bar{x})}{\partial \rho_s} = K^s(x_s, \bar{x}) G\left(\bar{R}^s(\rho_s, x_s, \bar{x}), \bar{x}\right) \exp\left(\int_{\rho_s^0}^{\rho_s} G\left(\bar{R}^s(r, x_s, \bar{x}), \bar{x}\right) dr\right),$$

so to preserve the positivity of R^s (and hence the positivity of $\partial U^s/\partial \rho_s$), the function $G(\cdot, \bar{x})$ should be positive over the range of $\bar{R}^s$. In addition

$$\frac{\partial^2 \bar{U}^s(\rho_s, x_s, \bar{x})}{\partial \rho_s^2} = K^s(x_s, \bar{x}) \left[\frac{\partial \bar{R}^s}{\partial \rho_s} G_r(\bar{R}^s) + \left(G(\bar{R}^s)\right)^2\right] \exp\left(\int_{\rho_s^0}^{\rho_s} G\left(\bar{R}^s(r, x_s, \bar{x}), \bar{x}\right) dr\right),$$

where G_r denotes the partial derivative of G with respect to its first argument, i.e. $G_r(\bar{R}^s) = \partial G(\bar{R}^s(\rho_s, x_1, \bar{x}), \bar{x})/\partial r$. Hence $G(\cdot, \bar{x})$ should be differentiable and such that

$$\frac{\partial \bar{R}^s}{\partial \rho_s} G_r(\bar{R}^s) + \left(G(\bar{R}^s)\right)^2 < 0,$$

in order to ensure the strict concavity of U^s. ■

Proof of Proposition 6. The proof is in two steps. The first shows identifiability and the second derives the testable implications.

Step 1: Identification. Consider first agent 1. From the proof of Proposition 5, we know that $R^1(\rho_1, x_1, \bar{x})$ is of the form:

$$R^1(\rho_1, x_1, \bar{x}) = G\big(\bar{R}^1(\rho_1, x_1, \bar{x}), \bar{x}\big), \tag{32}$$

where G is strictly positive and strictly increasing in its first argument on the range of $\bar{R}^1$, and $\bar{R}^1$ is a known function that solves the system of PDEs in (25). We now show that under the additional exclusion restriction U.2-T.2, $G(\cdot, \bar{x})$ is necessarily of the form $G(u, \bar{x}) = u$. Let $\bar{U}^1(\rho_1, \bar{x})$ and $\bar{T}(x_1, \bar{x})$ be the utility and threat functions that are obtained from $\bar{R}^1$ by solving:

$$\bar{R}^1(\rho_1, x_1, \bar{x}) = \frac{\partial \bar{U}^1(\rho_1, \bar{x})/\partial \rho_1}{\bar{U}^1(\rho_1, \bar{x}) - \bar{T}^1(x_1, \bar{x})} = \frac{\partial}{\partial \rho_1} \ln\left(\bar{U}^1(\rho_1, \bar{x}) - \bar{T}^1(x_1, \bar{x})\right), \tag{33}$$

by using the same integration steps as in the proof of Proposition 2, i.e.

$$\bar{U}^1(\rho_1, \bar{x}) - \bar{T}^1(x_1, \bar{x}) = K(\bar{x}) \exp\left(\int_{\rho_1^0}^{\rho_1} \bar{R}^1(r, x_1, \bar{x}) dr\right), \tag{34}$$

where $\rho_1^0 \in]0, y[$ and $K(\bar{x}) > 0$ are arbitrary. Since the left-hand side of (34) is additively separable in ρ_1 and x_1, the same must hold for the right-hand side of (34). Using the fact that for any function $f(a, b)$ we have:

$$\exp\left(f(a, b)\right) = g(a) + h(b) \quad \text{if and only if} \quad \frac{\partial^2 f}{\partial a \partial b} + \frac{\partial f}{\partial a} \cdot \frac{\partial f}{\partial b} = 0,$$

we get the following restrictions on $\bar{R}^1$:

$$\frac{\partial \bar{R}^1(\rho_1, x_1, \bar{x})}{\partial x_1} + \bar{R}(\rho_1, x_1, \bar{x}) \cdot \int^{\rho_1} \frac{\partial \bar{R}^1(r, x_1, \bar{x})}{\partial x_1} dr = 0. \tag{35}$$

We now show that R^1 in (32) satisfies the same restriction (35) only if $G(r, \bar{x}) = r$. Letting $G_r(r, \bar{x}) \equiv \partial G(r, \bar{x})/\partial r$ and writing (35) for $R^1(\rho_1, x_1, \bar{x}) = G\big(\bar{R}^1(\rho_1, x_1, \bar{x}), \bar{x}\big)$ gives

$$\frac{\partial \bar{R}^1(\rho_1, x_1, \bar{x})}{\partial x_1} \cdot G_r\left(\bar{R}^1(\rho_1, x_1, \bar{x}), \bar{x}\right) + \\ G\left(\bar{R}^1(\rho_1, x_1, \bar{x}), \bar{x}\right) \cdot \int^{\rho_1} \frac{\partial \bar{R}^1(r, x_1, \bar{x})}{\partial x_1} \cdot G_r\left(\bar{R}^1(r, x_1, \bar{x}), \bar{x}\right) dr = 0. \tag{36}$$

Combining (35) and (36) and using the fact that $G(\cdot, \bar{x}) > 0$ over the range of $\bar{R}^1$, gives:

$$\int^{\rho_1} \frac{\partial \bar{R}^1(r, x_1, \bar{x})}{\partial x_1} \cdot G_r\left(\bar{R}^1(r, x_1, \bar{x}), \bar{x}\right) dr = \frac{\bar{R}^1(\rho_1, x_1, \bar{x}) \cdot G_r\left(\bar{R}^1(\rho_1, x_1, \bar{x}), \bar{x}\right)}{G\left(\bar{R}^1(\rho_1, x_1, \bar{x}), \bar{x}\right)} \int^{\rho_1} \frac{\partial \bar{R}^1(r, x_1, \bar{x})}{\partial x_1} dr. \tag{37}$$

Differentiating (37) with respect to ρ_1 and using again (35) then gives:

$$\frac{\partial \bar{R}^1}{\partial x_1}\left[G_r(\bar{R}^1) + \frac{1}{\bar{R}^1} \cdot \frac{\partial}{\partial \rho_1}\Big(\frac{\bar{R}^1 \cdot G_r(\bar{R}^1)}{G(\bar{R}^1)}\Big) - \frac{\bar{R}^1 \cdot G_r(\bar{R}^1)}{G(\bar{R}^1)}\right] = 0. \tag{38}$$

Two cases are then possible. Either it is the case that $\partial \bar{R}^1(\rho_1, x_1, \bar{x})/\partial x_1 = 0$ everywhere. Using (33) this can only hold if $\partial \bar{T}^1(x_1, \bar{x})/\partial x_1 = 0$ everywhere which we assumed away in T.2. This means that necessarily:

$$\bar{R}^1\left[G_r\left(\bar{R}^1\right) - \frac{\bar{R}^1 \cdot G_r\left(\bar{R}^1\right)}{G\left(\bar{R}^1\right)}\right] = -\frac{\partial}{\partial \rho_1}\Big(\frac{\bar{R}^1 \cdot G_r(\bar{R}^1)}{G(\bar{R}^1)}\Big). \tag{39}$$

Now, holding $\bar{x}$ fixed, note that the left-hand side of (39) is a function of $\bar{R}^1$ only, so the same must hold for the right-hand side. Two cases are possible. Either $\partial \bar{R}^1/\partial \rho_1$ is a function of $\bar{R}^1$ only; since from (33),

$$\frac{\partial \bar{R}^1}{\partial \rho_1} = \frac{\partial^2 \bar{U}^1/\partial \rho_1^2}{\bar{U}^1 - \bar{T}^1} - \frac{\left(\partial \bar{U}^1/\partial \rho_1\right)^2}{(\bar{U}^1 - \bar{T}^1)^2} = \left(\frac{\partial \bar{U}^1/\partial \rho_1}{\partial^2 \bar{U}^1/\partial \rho_1^2}\right) \cdot \bar{R}^1 - (\bar{R}^1)^2,$$

this would imply that $\left(\partial \bar{U}^1/\partial \rho_1\right) / \left(\partial^2 \bar{U}^1/\partial \rho_1^2\right)$ is a function of $\bar{R}^1$ only; under U.2, the latter can hold only if $\partial \bar{R}^1/\partial x_1 = 0$ everywhere, i.e. $\partial \bar{T}^1/\partial x_1 = 0$ everywhere, which we excluded in T.2. Thus, it is necessarily the case that the right-hand side of (39) is zero, and since $\partial \bar{R}^1/\partial \rho_1 < 0$, this is only possible if

$$\frac{\partial}{\partial \bar{R}^1}\Big(\frac{\bar{R}^1 \cdot G_r(\bar{R}^1)}{G(\bar{R}^1)}\Big) = 0 \quad \text{i.e.} \quad \bar{R}^1 \cdot G_r(\bar{R}^1) = a_0 \cdot G(\bar{R}^1), \tag{40}$$

where a_0 is some positive constant (that is a function of $\bar{x}$). The only solution to (40) is $G(r, \bar{x}) = C(\bar{x}) \cdot r^{a_0(\bar{x})}$ with $a_0(\bar{x}) > 0$ and $C(\bar{x}) > 0$. Plugging back into (39) and using $a_0(\bar{x}) > 0$ then implies $C(\bar{x}) \cdot r^{a_0(\bar{x})-1} = 1$. Thus $a(\bar{x}) = 1$, $C(\bar{x}) = 1$ and the only functions G that satisfy (39) are of the form: $G(r, \bar{x}) = r$, and U^1 and T^1 are identified (up to an increasing affine transform whose coefficients depend on $\bar{x}$). In the GNB case with weights $\delta_s(\bar{x})$ that do not depend on x_s we have:

$$U^1(\rho_1, \bar{x}) - T^1(x_1, \bar{x}) = K(\bar{x}) \exp\left(\int_{\rho_1^0}^{\rho_1} \frac{G\left(\bar{R}^1(r, x_1, \bar{x}), \bar{x}\right)}{\delta_1(\bar{x})} dr\right),$$

where as before $\rho_0 \in]0, y[$ and $K(\bar{x}) > 0$ are arbitrary. Using the same reasoning as before shows that (39) now holds with $G(\cdot, \bar{x})/\delta_1(\bar{x})$ instead of $G(\cdot, \bar{x})$. Then necessarily $G(r, \bar{x}) = \delta_1(\bar{x}) \cdot r$, which shows that U^1 and T^1 are identified as before.

Step 2: Testable Implications. We now derive the testable implications on ρ. Consider agent 1 and let

$$\Phi^1(\rho, x_1, \bar{x}) \equiv -\frac{\partial \rho(y, x)/\partial x_1}{1 - \partial \rho(y, x)/\partial y},$$

as in the proof of Proposition 2; then $\Phi^1 = (\partial \bar{R}^1/\partial x_1)/(\partial \bar{R}^1/\partial \rho_1)$ so

$$\Phi^1(\rho_1, x_1, \bar{x}) = \frac{\left(\partial \bar{T}^1(x_1, \bar{x})/\partial x_1\right)\left(\partial \bar{U}^1(\rho_1, \bar{x})/\partial \rho_1\right)}{\left(\partial \bar{U}^1(\rho_1, \bar{x})/\partial \rho_1\right)^2 - \left(\partial^2 \bar{U}^1(\rho_1, \bar{x})/\partial \rho_1^2\right)\left(\bar{U}^1(\rho_1, \bar{x}) - \bar{T}^1(x_1, \bar{x})\right)}. \tag{41}$$

Under T.2, we know there exists at least one $(x_1, \bar{x})$ such that for every ρ_1 we have $\Phi^1(\rho_1, x_1, \bar{x}) \neq 0$. We can then take logarithms of the above expression and differentiate with respect to x_1 to get, for all ρ_1:

$$\frac{\Phi^1_x}{\Phi^1} = \frac{\partial}{\partial x_1} \ln \left| \frac{\partial \bar{T}^1}{\partial x_1} \right| - \left[\frac{\partial}{\partial \rho_1} \ln \left(\frac{\partial \bar{U}^1}{\partial \rho_1} \right) \right] \cdot \Phi^1, \tag{42}$$

where $\Phi^1_x \equiv \partial \Phi^1(\rho_1, x_1, \bar{x})/\partial x_1$. First, we express the term $\partial \ln(\partial \bar{U}^1/\partial \rho_1)/\partial \rho_1$ as a function of Φ^1 and its derivatives. For this, we differentiate (42) with respect to ρ_1 and then again with respect to x_1 to obtain:

$$\begin{aligned}
-\frac{\partial}{\partial \rho_1} \frac{\Phi^1_x}{\Phi^1} &= \left[\frac{\partial^2}{\partial \rho_1^2} \ln \left(\frac{\partial \bar{U}^1}{\partial \rho_1} \right) \right] \cdot \Phi^1 + \left[\frac{\partial}{\partial \rho_1} \ln \left(\frac{\partial \bar{U}^1}{\partial \rho_1} \right) \right] \cdot \Phi^1_\rho \\
-\frac{\partial^2}{\partial \rho_1 \partial x_1} \frac{\Phi^1_x}{\Phi^1} &= \left[\frac{\partial^2}{\partial \rho_1^2} \ln \left(\frac{\partial \bar{U}^1}{\partial \rho_1} \right) \right] \cdot \Phi^1_x + \left[\frac{\partial}{\partial \rho_1} \ln \left(\frac{\partial \bar{U}^1}{\partial \rho_1} \right) \right] \cdot \Phi^1_{\rho x}.
\end{aligned}$$

Two cases are then possible. Either it is the case that for some ρ_1, $\partial^2 \ln |\Phi^1|/\partial \rho_1 \partial x_1 \neq 0$, so a unique solution to this system exists and we have:

$$\frac{\partial}{\partial \rho_1} \ln \left(\frac{\partial \bar{U}^1}{\partial \rho_1} \right) = -\frac{\Phi^1_x \cdot \frac{\partial}{\partial \rho_1}\left(\frac{\Phi^1_x}{\Phi^1}\right) - \Phi^1 \cdot \frac{\partial^2}{\partial \rho_1 \partial x_1}\left(\frac{\Phi^1_x}{\Phi^1}\right)}{\Phi^1_x \cdot \Phi^1_\rho - \Phi^1 \cdot \Phi^1_{\rho x}} = -\frac{\Phi^1 \cdot \frac{\partial^2}{\partial \rho_1 \partial x_1}\left(\frac{\Phi^1_x}{\Phi^1}\right) - \Phi^1_x \cdot \frac{\partial}{\partial \rho_1}\left(\frac{\Phi^1_x}{\Phi^1}\right)}{\Phi^{1\,2} \cdot \frac{\partial}{\partial \rho_1}\left(\frac{\Phi^1_x}{\Phi^1}\right)}.$$

Combining the above with (42) then gives:

$$\frac{\partial}{\partial x_1} \ln \left| \frac{\partial \bar{T}^1}{\partial x_1} \right| = \frac{\Phi^1_x}{\Phi^1} + \Phi^1 \cdot \frac{\Phi^1 \cdot \frac{\partial^2}{\partial \rho_1 \partial x_1}\left(\frac{\Phi^1_x}{\Phi^1}\right) - \Phi^1_x \cdot \frac{\partial}{\partial \rho_1}\left(\frac{\Phi^1_x}{\Phi^1}\right)}{\Phi^{1\,2} \cdot \frac{\partial}{\partial \rho_1}\left(\frac{\Phi^1_x}{\Phi^1}\right)} = \frac{\frac{\partial^3 \ln |\Phi^1|}{\partial \rho_1 \partial x_1^2}}{\frac{\partial^2 \ln |\Phi^1|}{\partial \rho_1 \partial x_1}},$$

which is a function of $(x_1, \bar{x})$ alone. Analogous result follows by considering agent 2, so (16) holds. Or it is the case that for all ρ_1, $\partial^2 \ln |\Phi^1|/\partial \rho_1 \partial x_1 = 0$. By (42), this implies that $\partial[\big(\partial\ln(\partial\bar{U}^1/\partial\rho_1)/\partial\rho_1\big)\cdot\Phi^1]/\partial\rho_1 = 0$, so $\partial[\big(\big(\partial\ln(\partial\bar{U}^1/\partial\rho_1)/\partial\rho_1\big)\cdot\Phi^1\big)^{-1}]/\partial\rho_1 = 0$, which by (41) implies

$$\frac{\partial}{\partial\rho_1}\left[\frac{(\partial\bar{U}^1/\partial\rho_1)^2}{(\partial^2\bar{U}^1/\partial\rho_1^2)} - \bar{U}^1\right] = 0, \tag{43}$$

for all ρ_1. Integrating (43) three times and using U.2 and U.1 then implies that there exist $\alpha(\bar{x}) > 0$, $\beta(\bar{x})$, and $\mu(\bar{x}) > 0$ such that:

$$\bar{U}^1(\rho_1,\bar{x}) = -\alpha(\bar{x})\exp\left(-\mu(\bar{x})\cdot\rho_1\right) + \beta(\bar{x}). \tag{44}$$

Finally, plugging (44) back into (41) shows that

$$\Phi^1(\rho_1,x_1,\bar{x}) = \frac{\partial\bar{T}^1(x_1,\bar{x})/\partial x_1}{\mu(\bar{x})\cdot\big[\beta(\bar{x}) - \bar{T}^1(x_1,\bar{x})\big]}, \quad \text{so} \quad \frac{\partial\Phi^1(\rho_1,x_1,\bar{x})}{\partial\rho_1} = 0.$$

Analogous result follows by considering agent 2, which shows (17). ■

Proof of Proposition 7. A sharing rule $\rho(y,x)$ is a solution to (18) if and only if it satisfies the first order condition:

$$R^1(\rho,x_1,\epsilon_1,\bar{x}) = R^2(y-\rho,x_2,\epsilon_2,\bar{x}), \tag{45}$$

where

$$R^s(\rho_s,x_s,\epsilon_s,\bar{x}) \equiv \frac{\partial U^s(\rho_s,\bar{x})/\partial\rho_s}{U^s(\rho_s,\bar{x}) - T^s(x_s,\bar{x}) + \epsilon_s}.$$

Since $\partial R^s/\partial\rho_s < 0$, (45) implicitly defines a unique solution $\rho = \rho(y,x_1,x_2,\epsilon_1,\epsilon_2,\bar{x})$. Consider any $r \in]\,0,y\,[$ and note that we have $\rho(y,x_1,x_2,\epsilon_1,\epsilon_2,\bar{x}) \leqslant r$ if and only if $R^1(r,x_1,\epsilon_1,\bar{x}) - R^2(y-r,x_2,\epsilon_2,\bar{x}) \geqslant 0$, that is

$$\begin{aligned}\frac{\partial U^2(y-r,\bar{x})}{\partial\rho_2}\left[U^1(r,\bar{x})+\epsilon_1\right] - \frac{\partial U^1(r,\bar{x})}{\partial\rho_1}\left[U^2(y-r,\bar{x})+\epsilon_2\right] \leqslant \\ -\frac{\partial U^1(r,\bar{x})}{\partial\rho_1}T^2(x_2,\bar{x}) + \frac{\partial U^2(y-r,\bar{x})}{\partial\rho_2}T^1(x_1,\bar{x}).\end{aligned} \tag{46}$$

Now let $\Theta(r,y,x_1,x_2,\bar{x}) \equiv \Pr\left\{\rho \leqslant r \,\middle|\, y,x_1,x_2,\bar{x}\right\}$ be the conditional distribution of the shares ρ, observed for given $(y,x_1,x_2,\bar{x})$. Moreover, let

$$G(r,y,x_1,x_2,\bar{x}) \equiv -\frac{\partial U^1(r,\bar{x})}{\partial\rho_1}T^2(x_2,\bar{x}) + \frac{\partial U^2(y-r,\bar{x})}{\partial\rho_2}T^1(x_1,\bar{x}) \tag{47}$$

$$\begin{aligned}F(t,r,y,\bar{x}) \equiv \Pr\Bigg\{&\frac{\partial U^2(y-r,\bar{x})}{\partial\rho_2}\left[U^1(r,\bar{x})+\epsilon_1\right] \\ &- \frac{\partial U^1(r,\bar{x})}{\partial\rho_1}\left[U^2(y-r,\bar{x})+\epsilon_2\right] \leqslant t \,\Bigg|\, y,x_1,x_2,\bar{x}\Bigg\}.\end{aligned} \tag{48}$$

Note that the probability $F(, r, y, \bar{x})$ does not depend on (x_1, x_2) because (i) (x_1, x_2) do not enter U^1 nor U^2, and (ii) (ϵ_1, ϵ_2) is conditionally independent of (x_1, x_2) (D.1). Then, we have that:

$$\Theta(r, y, x_1, x_2, \bar{x}) = F\big(G(r, y, x_1, x_2, \bar{x}), r, y, \bar{x}\big). \tag{49}$$

In particular, if $x_1 = (x_{11}, \ldots, x_{1n_1})$, then for every $1 \leqslant s \leqslant n_1$, we have:

$$\frac{\partial \Theta(r, y, x_1, x_2, \bar{x})}{\partial x_{1i}} = \frac{\partial T^1(x_1, \bar{x})}{\partial x_{1i}} \frac{\partial U^2(y - r, \bar{x})}{\partial \rho_2} \frac{\partial F\big(G(r, y, x_1, x_2, \bar{x}), r, y, \bar{x}\big)}{\partial t}. \tag{50}$$

Similarly, if $x_2 = (x_{21}, \ldots, x_{2n_2})$, then for every $1 \leqslant j \leqslant n_2$, we have:

$$\frac{\partial \Theta(r, y, x_1, x_2, \bar{x})}{\partial x_{2j}} = -\frac{\partial T^2(x_2, \bar{x})}{\partial x_{2j}} \frac{\partial U^1(r, \bar{x})}{\partial \rho_1} \frac{\partial F\big(G(r, y, x_1, x_2, \bar{x}), r, y, \bar{x}\big)}{\partial t}. \tag{51}$$

In particular, we focus on (50) when $i = 1$ and on (51) when $j = 1$. Note that under T.3 we have

$$\frac{\partial T^1(x_1, \bar{x})}{\partial x_{11}} \neq 0 \text{ and } \frac{\partial T^2(x_2, \bar{x})}{\partial x_{21}} \neq 0 \quad \text{for all } x = (x_1, x_2, \bar{x}),$$

while D.2 ensures that $\partial F(t, r, y, \bar{x})/\partial t > 0$ for every $t \in \mathbb{R}$. Taking ratios of (50) and (51) obtained for $i = j = 1$ we then have:

$$\frac{\partial \Theta(r, y, x_1, x_2, \bar{x})/\partial x_{11}}{\partial \Theta(r, y, x_1, x_2, \bar{x})/\partial x_{21}} = -\frac{\partial T^1(x_1, \bar{x})/\partial x_{11}}{\partial T^2(x_2, \bar{x})/\partial x_{21}} \frac{\partial U^2(y - r, \bar{x})/\partial \rho_2}{\partial U^1(r, \bar{x})/\partial \rho_1}$$

so

$$\ln \left| \frac{\partial \Theta(r, y, x_1, x_2, \bar{x})/\partial x_{11}}{\partial \Theta(r, y, x_1, x_2, \bar{x})/\partial x_{21}} \right| = \ln \left| \frac{\partial T^1(x_1, \bar{x})}{\partial x_{11}} \right| - \ln \left| \frac{\partial T^2(x_2, \bar{x})}{\partial x_{21}} \right| + \ln \frac{\partial U^2(y - r, \bar{x})}{\partial \rho_2} - \ln \frac{\partial U^1(r, \bar{x})}{\partial \rho_1}$$

for every (y, r) and every $x = (x_1, x_2, \bar{x})$. Now consider the change of variables $\rho_1 \equiv r$ and $\rho_2 \equiv y - r$. We then obtain that for every (ρ_1, ρ_2) and every $x = (x_1, x_2, \bar{x})$,

$$h(\rho_1, \rho_2, x_1, x_2, \bar{x}) = \ln \left| \frac{\partial T^1(x_1, \bar{x})}{\partial x_{11}} \right| - \ln \left| \frac{\partial T^2(x_2, \bar{x})}{\partial x_{21}} \right| + \ln \frac{\partial U^2(\rho_2, \bar{x})}{\partial \rho_2} - \ln \frac{\partial U^1(\rho_1, \bar{x})}{\partial \rho_1} \tag{52}$$

where we have let

$$h(\rho_1, \rho_2, x_1, x_2, \bar{x}) \equiv \ln \left| \frac{\partial \Theta(\rho_1, \rho_1 + \rho_2, x_1, x_2, \bar{x})/\partial x_{11}}{\partial \Theta(\rho_1, \rho_1 + \rho_2, x_1, x_2, \bar{x})/\partial x_{21}} \right|.$$

Step 1: Identification of U^1. Differentiating (52) with respect to ρ_1 gives:

$$\frac{\partial h(\rho_1, \rho_2, x_1, x_2, \bar{x})}{\partial \rho_1} = -\frac{\partial}{\partial \rho_1} \ln \frac{\partial U^1(\rho_1, \bar{x})}{\partial \rho_1}.$$

Integrating from some $\rho_1^* \in]0, y[$ we then obtain:

$$\frac{\partial U^1(\rho_1, \bar{x})}{\partial \rho_1} = K^1(\bar{x}) \exp\left[- \int_{\rho_1^*}^{\rho_1} \frac{\partial h(u, \rho_2, x_1, x_2, \bar{x})}{\partial \rho_1} du \right],$$

where $K^1(\bar{x}) \equiv \partial U^1(\rho_1^*, \bar{x})/\partial \rho_1 > 0$ is an unknown function. We can again integrate from some $\rho_1^0 \in]0, y[$ which gives:

$$U^1(\rho_1, \bar{x}) = K^1(\bar{x}) \int_{\rho_1^0}^{\rho_1} \exp\left[- \int_{\rho_1^*}^{v} \frac{\partial h(u, \rho_2, x_1, x_2, \bar{x})}{\partial \rho_1} du \right] dv + k^1(\bar{x})$$

where $k^1(\bar{x}) \equiv U^1(\rho_1^0, \bar{x})$ is unknown. Hence, the utility of agent 1 is determined up to a strictly increasing affine transformation (in $\bar{x}$) of a known utility function $\bar{U}^1(\rho_1, \bar{x})$:

$$U^1(\rho_1, \bar{x}) = K^1(\bar{x}) \cdot \bar{U}^1(\rho_1, \bar{x}) + k^1(\bar{x}), \quad K^1(\bar{x}) > 0, \tag{53}$$

where we have let

$$\bar{U}^1(\rho_1, \bar{x}) \equiv \int_{\rho_1^0}^{\rho_1} \exp\left[- \int_{\rho_1^*}^{v} \frac{\partial h(u, \rho_2, x_1, x_2, \bar{x})}{\partial \rho_1} du \right] dv.$$

Step 2: Identification of U^2. Differentiating (52) with respect to ρ_2 gives:

$$\frac{\partial h(\rho_1, \rho_2, x_1, x_2, \bar{x})}{\partial \rho_2} = \frac{\partial}{\partial \rho_2} \ln \frac{\partial U^2(\rho_2, \bar{x})}{\partial \rho_2}.$$

Following the same reasoning as above and integrating twice from some $(\rho_2^*, \rho_2^0) \in]0\,, y\,[^2$, we have:

$$U^2(\rho_2, \bar{x}) = K^2(\bar{x}) \int_{\rho_2^0}^{\rho_1} \exp\left[\int_{\rho_2^*}^{v} \frac{\partial h(\rho_1, u, x_1, x_2, \bar{x})}{\partial \rho_2} du \right] dv + k^2(\bar{x})$$

where the functions $k^2(\bar{x}) \equiv U^2(\rho_2^0, \bar{x})$ and $K^2(\bar{x}) \equiv \partial U^2(\rho_2^*, \bar{x})/\partial \rho_2 > 0$ are unknown. Hence, the utility of agent 2 is also determined up to a strictly increasing affine transformation (in $\bar{x}$) of a known utility function $\bar{U}^2$,

$$U^2(\rho_2, \bar{x}) = K^2(\bar{x}) \cdot \bar{U}^2(\rho_2, \bar{x}) + k^2(\bar{x}), \quad K^2(\bar{x}) > 0, \tag{54}$$

where we have let

$$\bar{U}^2(\rho_2, \bar{x}) \equiv \int_{\rho_2^0}^{\rho_1} \exp\left[\int_{\rho_2^*}^{v} \frac{\partial h(\rho_1, u, x_1, x_2, \bar{x})}{\partial \rho_2} du \right] dv.$$

Step 3: Identification of T^1. Differentiating (52) with respect to x_{1k} ($1 \leqslant k \leqslant n_1$), we get that:

$$\frac{\partial h(\rho_1, \rho_2, x_1, x_2, \bar{x})}{\partial x_{1k}} = \frac{\partial}{\partial x_{1k}} \ln \left| \frac{\partial T^1(x_1, \bar{x})}{\partial x_{11}} \right|.$$

In particular, consider the partial derivative with respect to x_{11}, i.e. $k = 1$. For some c_{11} define:

$$t_1(\bar{x}, x_1) \equiv \int_{c_{11}}^{x_{11}} \frac{\partial h(\rho_1, \rho_2, u, x_{12}, \ldots, x_{1n_1}, x_2, \bar{x})}{\partial x_{11}} du.$$

Then, we have that:

$$\ln \left| \frac{\partial T^1(x_1, \bar{x})}{\partial x_{11}} \right| = t_1(x_1, \bar{x}) + g_1(x_{12}, \ldots, x_{1n_1}, \bar{x}), \tag{55}$$

where $g_1(x_{12}, \ldots, x_{1n_1}, \bar{x})$ is an unknown function. Differentiating with respect to x_{12} gives:

$$\frac{\partial}{\partial x_{12}} \ln \left| \frac{\partial T^1(\bar{x}, x_1)}{\partial x_{11}} \right| - \frac{\partial t_1(\bar{x}, x_1)}{\partial x_{12}} = \frac{\partial}{\partial x_{12}} g_1(x_{12}, \ldots, x_{1n_1}, \bar{x}),$$

that is

$$\begin{aligned} \frac{\partial}{\partial x_{12}} g_1(x_{12}, \ldots, x_{1n_1}, \bar{x}) &= \frac{\partial h(\rho_1, \rho_2, x_1, x_2, \bar{x})}{\partial x_{12}} - \int_{c_{11}}^{x_{11}} \frac{\partial^2 h(\rho_1, \rho_2, u, x_{12}, \ldots, x_{1n_1}, x_2, \bar{x})}{\partial x_{11} \partial x_{12}} du \\ &\equiv \sigma_2(x_{12}, \ldots, x_{1n_1}, \bar{x}). \end{aligned}$$

Note that the function $\sigma_2(x_{12}, \ldots, x_{1n_1}, \bar{x})$ on the right hand side of the above equality is known; hence, we can integrate with respect to x_{12} to get:

$$\begin{aligned} g_1(x_{12}, \ldots, x_{1n_1}, \bar{x}) &= \int_{c12}^{x_{12}} \sigma_2(u, x_{13}, \ldots, x_{1n_1}, \bar{x}) du + g_2(x_{13}, \ldots, x_{1n_1}, \bar{x}) \\ &\equiv t_2(x_{12}, \ldots, x_{1n_1}, \bar{x}) + g_2(x_{13}, \ldots, x_{1n_1}, \bar{x}). \end{aligned}$$

Plugging back into (55) we get that:

$$\ln \left| \frac{\partial T^1(x_1, \bar{x})}{\partial x_{11}} \right| = t_1(x_{11}, \ldots, x_{1n_1}, \bar{x}) + t_2(x_{12}, \ldots, x_{1n_1}, \bar{x}) + g_2(x_{13}, \ldots, x_{1n_1}, \bar{x}).$$

Repeating the same reasoning as above for x_{13} etc all the way to x_{1n_1} we get that:

$$\ln \left| \frac{\partial T^1(x_1, \bar{x})}{\partial x_{11}} \right| = t^1(x_1, \bar{x}) + g^1(\bar{x}),$$

where the known function $t^1(x_1, \bar{x})$ is defined as the sum of the recursively computed functions $t_k(x_{1k}, \ldots, x_{1n_1}, \bar{x})$, and $g^1(\bar{x})$ is the unknown residual function. Since from (50) we know that the sign of $\partial T^1(x_1,\bar{x})/\partial x_{11}$ is the same as that of $\partial\Theta(\rho_1,\rho_1+\rho_2,x_1,x_2,\bar{x})/\partial x_{11}$, the above implies:

$$\frac{\partial T^1(x_1,\bar{x})}{\partial x_{11}} = C^1(\bar{x})\cdot \operatorname{sgn}\left(\frac{\Theta(\rho_1,\rho_1+\rho_2,x_1,x_2,\bar{x})}{\partial x_{11}}\right)\cdot \exp\left[t^1(x_1,\bar{x})\right],$$

where $C^1(\bar{x}) = \exp[g^1(\bar{x})] > 0$ is an unknown function. This gives that:

$$\frac{\partial T^1(x_1,\bar{x})}{\partial x_{11}} = C^1(\bar{x})\cdot \tau(x_1,\bar{x}) \tag{56}$$

where

$$\tau(x_1,\bar{x}) \equiv \operatorname{sgn}\left(\frac{\Theta(\rho_1,\rho_1+\rho_2,x_1,x_2,\bar{x})}{\partial x_{11}}\right)\cdot \exp\left[t^1(x_1,\bar{x})\right].$$

Integrating (56) with respect to x_{11} from some constant d_{11} then gives:

$$T^1(x_1,\bar{x}) = C^1(\bar{x})\cdot \int_{d_{11}}^{x_{11}} \tau(u,x_{12},\ldots,x_{1n_1},\bar{x})du + D_1(x_{12},\ldots,x_{1n_1},\bar{x}),$$

for some unknown function $D_1(x_{12},\ldots,x_{1n_1},\bar{x})$. Differentiating the above with respect to x_{12} then gives that:

$$\begin{aligned}\frac{\partial}{\partial x_{12}} D_1(x_{12},\ldots,x_{1n_1},\bar{x}) = \\ \frac{\partial T^1(x_1,\bar{x})}{\partial x_{12}} - C^1(\bar{x})\cdot \int_{d_{11}}^{x_{11}} \frac{\partial \tau(u,x_{12},\ldots,x_{1n_1},\bar{x})}{\partial x_{12}} du.\end{aligned} \tag{57}$$

Now consider again (50); taking the ratio of the expression obtained for $i=2$ and $i=1$ gives:

$$\frac{\partial T^1(x_1,\bar{x})}{\partial x_{12}} = \frac{\partial T^1(x_1,\bar{x})}{\partial x_{11}}\cdot \frac{\partial\Theta(\rho_1,\rho_1+\rho_2,x_1,x_2,\bar{x})/\partial x_{12}}{\partial\Theta(\rho_1,\rho_1+\rho_2,x_1,x_2,\bar{x})/\partial x_{11}}. \tag{58}$$

Combining (58) with (56) then shows that $\partial T^1(x_1,\bar{x})/\partial x_{12}$ is known up to a multiplication by the same function $C^1(\bar{x})$; this means that the right hand side of (57) is known up to a multiplication by the unknown function $C^1(\bar{x}) > 0$. We can then integrate with respect to x_{12}. Following the same recursive reasoning as before, it follows that:

$$T^1(x_1,\bar{x}) = C^1(\bar{x})\cdot \bar{T}^1(x_1,\bar{x}) + c^1(\bar{x}), \quad C^1(\bar{x}) > 0, \tag{59}$$

where the function $\bar{T}^1(x_1,\bar{x})$ is known (and defined recursively), while $C^1(\bar{x}) > 0$ and $c^1(\bar{x})$ are unknown. This means that agent 1's threat function T^1 is determined up to an increasing affine transformation in $\bar{x}$.

Step 4. Identification of T^2. Now, consider again (52) and combine it with the expressions for T^1, U^1 and U^2 obtained in (59), (54), (53), respectively. It follows that for some known function $t^2(x_2,\bar{x})$, we have:

$$\left|\frac{\partial T^2(x_2,\bar{x})}{\partial x_{21}}\right| = \frac{C^1(\bar{x})K^2(\bar{x})}{K^1(\bar{x})}\cdot \exp\left[t^2(x_2,\bar{x})\right],$$

so using again (51) we get,

$$\frac{\partial T^2(x_2,\bar{x})}{\partial x_{21}} = \frac{C^1(\bar{x})K^2(\bar{x})}{K^1(\bar{x})} \cdot \operatorname{sgn}\left(-\frac{\Theta(\rho_1,\rho_1+\rho_2,x_1,x_2,\bar{x})}{\partial x_{21}}\right) \cdot \exp\left[t^2(x_2,\bar{x})\right]$$

Following the same steps as in Step 3 it then follows that:

$$T^2(x_2,\bar{x}) = \frac{C^1(\bar{x})K^2(\bar{x})}{K^1(\bar{x})}\bar{T}^2(x_2,\bar{x}) + c^2(\bar{x}), \tag{60}$$

in which the function $\bar{T}^2(x_2,\bar{x})$ is known, the function $c^2(\bar{x})$ is unknown, and $C^1(\bar{x}) > 0$, $K^1(\bar{x}) > 0$ and $K^2(\bar{x}) > 0$ are the same unknown functions obtained in (59), (54), (53), respectively. In particular, this means that agent 2's threat function is determined up to an unknown increasing affine transformation (in $\bar{x}$) whose slope depends on those obtained for T^1, U^1, and U^2.

Step 5. Identification of F. Combining the inequality in (46) with the expressions for T^2, T^1, U^2, U^1 obtained in (60), (59), (53) and (54), respectively, we get that $\rho(y,x_1,x_2,\epsilon_1,\epsilon_2,\bar{x}) \leqslant r$ if and only if:

$$\begin{aligned}\frac{K^1(\bar{x})}{C^1(\bar{x})}\Bigg\{&\frac{\partial\bar{U}^2(y-r,\bar{x})}{\partial\rho_2}\left[\bar{U}^1(r,\bar{x}) + \frac{k^1(\bar{x})-c^1(\bar{x})+\epsilon_1}{K^1(\bar{x})}\right] \\ &-\frac{\partial\bar{U}^1(r,\bar{x})}{\partial\rho_1}\left[\bar{U}^2(y-r,\bar{x}) + \frac{k^2(\bar{x})-c^2(\bar{x})+\epsilon_2}{K^2(\bar{x})}\right]\Bigg\} \leqslant \\ &\qquad -\frac{\partial\bar{U}^1(r,\bar{x})}{\partial\rho_1}\bar{T}^2(x_2,\bar{x}) + \frac{\partial\bar{U}^2(y-r,\bar{x})}{\partial\rho_2}\bar{T}^1(x_1,\bar{x}).\end{aligned}$$

Let then

$$\bar{G}(y,r,x_1,x_2,\bar{x}) \equiv -\frac{\partial\bar{U}^1(r,\bar{x})}{\partial\rho_1}\bar{T}^2(x_2,\bar{x}) + \frac{\partial\bar{U}^2(y-r,\bar{x})}{\partial\rho_2}\bar{T}^1(x_1,\bar{x})$$

be the quantity on the right hand side of the above inequality; note that the function $\bar{G}(y,r,x_1,x_2,\bar{x})$ is known. Similar to previously, let also

$$\begin{aligned}\bar{F}(t,r,y,\bar{x}) \equiv \Pr\Bigg\{&\frac{K^1(\bar{x})}{C^1(\bar{x})}\Bigg(\frac{\partial\bar{U}^2(y-r,\bar{x})}{\partial\rho_2}\left[\bar{U}^1(r,\bar{x}) + \frac{k^1(\bar{x})-c^1(\bar{x})+\epsilon_1}{K^1(\bar{x})}\right] \\ &-\frac{\partial\bar{U}^1(r,\bar{x})}{\partial\rho_1}\left[\bar{U}^2(y-r,\bar{x}) + \frac{k^2(\bar{x})-c^2(\bar{x})+\epsilon_2}{K^2(\bar{x})}\right]\Bigg) \leqslant t \,\Bigg|\, y,x_1,x_2,\bar{x}\Bigg\}.\end{aligned} \tag{61}$$

Then, we have that for every $r \in]0\,,y\,[$ and every (y,x):

$$\Theta(r,y,x_1,x_2,\bar{x}) = \bar{F}\big(\bar{G}(y,r,x_1,x_2,\bar{x}),r,y,\bar{x}\big), \tag{62}$$

where $\Theta(r,y,x_1,x_2,\bar{x}) = \Pr\{\rho \leqslant r|\,y,x_1,x_2,\bar{x}\}$ as before. We now show that the above equality determines $\bar{F}(t,r,y,\bar{x})$ for all $t \in \mathbb{R}$. For this, fix $(r,y,x_{12},\ldots,x_{1n_1},x_2,\bar{x})$ and note that under T.3 $\bar{G}$ is strictly increasing in x_{11}. Moreover, $\lim_{|x_{11}|\to\infty} |\bar{G}(y,r,x_1,x_2,\bar{x})| = \infty$. This means that for any $t \in \mathbb{R}$, we have $\bar{G}(y,r,x_1,x_2,\bar{x}) = t$ if and only if

$$x_{11} = \left(\bar{T}^1\right)^{-1}\left(\left[\frac{\partial \bar{U}^2(y-r,\bar{x})}{\partial \rho_2}\right]^{-1}\left[t + \frac{\partial \bar{U}^1(r,\bar{x})}{\partial \rho_1}\bar{T}^2(x_2,\bar{x})\right], x_{12},\ldots,x_{1n_1},\bar{x}\right) \equiv x_{11}(t).$$

Now, we can invert (62) to show that for any $t \in \mathbb{R}$,

$$\bar{F}(t,r,y,\bar{x}) = \Theta(r,y,x_{11}(t),x_{12},\ldots,x_{1n_1},x_2,\bar{x}), \tag{63}$$

which is a known function.

Step 6. Observational Equivalence. We now use the results obtained in (54), (53), (59), (60) and (63) to characterize any two observationally equivalent structures. We start with agent 1's utilities: from (54) two observationally equivalent utilities U^1 and $\tilde{U}^1$ must satisfy

$$U^1(\rho_1,\bar{x}) = K^1(\bar{x})\cdot\bar{U}^1(\rho_1,\bar{x}) + k^1(\bar{x}),\quad K^1(\bar{x}) > 0,$$
$$\tilde{U}^1(\rho_1,\bar{x}) = \tilde{K}^1(\bar{x})\cdot\bar{U}^1(\rho_1,\bar{x}) + \tilde{k}^1(\bar{x}),\quad \tilde{K}^1(\bar{x}) > 0.$$

Let then

$$A^1(\bar{x}) \equiv \frac{\tilde{K}^1(\bar{x})}{K^1(\bar{x})} > 0,\quad \alpha^1(\bar{x}) \equiv \tilde{k}^1(\bar{x}) - A^1(\bar{x})\cdot k^1(\bar{x}).$$

It follows that

$$\tilde{U}^1(\rho_1,\bar{x}) = A^1(\bar{x})\cdot U^1(\rho_1,\bar{x}) + \alpha^1(\bar{x}),\quad A^1(\bar{x}) > 0,$$

where the functions $A^1(\bar{x}) > 0$ and $\alpha^1(\bar{x})$ are unknown. Analogously, using (53) (resp. (59)), any two observationally equivalent utilities for agent 2 (resp. threat functions for agent 1) must satisfy:

$$\tilde{U}^2(\rho_2,\bar{x}) = A^2(\bar{x})\cdot U^2(\rho_2,\bar{x}) + \alpha^2(\bar{x}),\quad A^2(\bar{x}) > 0, \tag{64}$$
$$\tilde{T}^1(x_1,\bar{x}) = B^1(\bar{x})\cdot T^1(x_1,\bar{x}) + \beta^1(\bar{x}),\quad B^1(\bar{x}) > 0, \tag{65}$$

where

$$A^2(\bar{x}) \equiv \frac{\tilde{K}^2(\bar{x})}{K^2(\bar{x})} > 0,\quad \alpha^2(\bar{x}) \equiv \tilde{k}^2(\bar{x}) - A^2(\bar{x})\cdot k^2(\bar{x}),$$
$$B^1(\bar{x}) \equiv \frac{\tilde{C}^1(\bar{x})}{C^1(\bar{x})} > 0,\quad \beta^1(\bar{x}) \equiv \tilde{c}^1(\bar{x}) - B^1(\bar{x})\cdot c^1(\bar{x}).$$

Now, for agent 2's threat functions, using (60) we have

$$\tilde{T}^2(x_2,\bar{x}) = B^1(\bar{x})\frac{A^2(\bar{x})}{A^1(\bar{x})}\cdot T^2(x_2,\bar{x}) + \beta^2(\bar{x}),$$

where

$$\beta^2(\bar{x}) \equiv \tilde{c}^2(\bar{x}) - B^1(\bar{x})\frac{A^2(\bar{x})}{A^1(\bar{x})} \cdot c^2(\bar{x}).$$

Finally, combining all of the above with (63), we have that conditional on (y, x) the unobservables (ϵ_1, ϵ_2) and $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$ must satisfy:

$$\frac{\partial \tilde{U}^2(y-r,\bar{x})}{\partial \rho_2}\left[\tilde{U}^1(r,\bar{x}) + \tilde{\epsilon}_1\right] - \frac{\partial \tilde{U}^1(r,\bar{x})}{\partial \rho_1}\left[\tilde{U}^2(y-r,\bar{x}) + \tilde{\epsilon}_2\right] \underset{|(y,x)}{\sim}$$
$$A^2(\bar{x})B^1(\bar{x})\left\{\frac{\partial U^2(y-r,\bar{x})}{\partial \rho_2}\left[U^1(r,\bar{x}) + \epsilon_1 + \frac{\beta^1(\bar{x})}{B^1(\bar{x})}\right]\right.$$
$$\left. - \frac{\partial U^1(r,\bar{x})}{\partial \rho_1}\left[U^2(y-r,\bar{x}) + \epsilon_2 + \frac{A^1(\bar{x})}{A^2(\bar{x})}\frac{\beta^2(\bar{x})}{B^1(\bar{x})}\right]\right\},$$

where $\underset{|(y,x)}{\sim}$ denotes equality in conditional distribution given (y, x). ■

References

[1] Bognanno, Mario F. and James B. Dworkin, 1975, "Comment: Who 'Wins' in Wage Bargaining," *Industrial and Labor Relations Review*, vol. 28, pp. 570-572.

[2] Bonhomme, Stephane and Jean-Marc Robin, "Generalized Non-Parametric Deconvolution with an Application to Earnings Dynamics," *The Review of Economic Studies*, 2010, 77(2), pp. 491-533.

[3] Bossert, Walter and Yves Sprumont, 2002, "Core Rationalizability in Two-Agent Exchange Economics," *Economic Theory*, vol. 200, pp. 777–791.

[4] Bossert, Walter and Yves Sprumont, 2003, "Efficient and Non-Deteriorating Choice," *Mathematical Social Sciences*, vol. 45, pp. 131-142.

[5] Bowlby, Roger L. and William R. Schriver, 1978, "Bluffing and the 'Split-the-Difference' Theory of Wage Bargaining," *Industrial and Labor Relations Review*, vol. 21, pp. 161-171.

[6] Brown, Bryan W., 1983, "The Identification Problem in Systems Nonlinear in the Variables," *Econometrica*, 51, pp 175-196.

[7] Browning, Martin, Pierre-André Chiappori, and Yoram Weiss, 2010, *Household Economics*. Cambridge University Press, forthcoming.

[8] Carvajal, Andrés, Indrajit Ray and Susan Snyder, 2004, "Equilibrium Behavior in Market and Games: Testable Restrictions and Identification," *Journal of Mathematical Economics*, vol. 40, pp. 1–40.

[9] Chari Varadarajan V. and Patrick J. Kehoe, 1990, "International cooperation of fiscal policy in limiting economies," *Journal of Political Economy*, vol. 98, pp. 617-636.

[10] Chiappori, Pierre-André, 1988a, "Nash Bargained Household Decisions," *International Economic Review*, vol. 32, 791-796.

[11] Chiappori, Pierre-André, 1988b, "Rational Household Labor Supply," *Econometrica,* vol. 56, pp.63-89.

[12] Chiappori, Pierre-André, 1991, "Nash Bargained Household Decisions : a Rejoinder," *International Economic Review*, 32 3, 1991, 761-62.

[13] Chiappori, Pierre-André, 1992, "Collective Labor Supply and Welfare," *Journal of Political Economy,* vol. 100, pp. 437–467.

[14] Chiappori, Pierre-André and Olivier Donni, 2011, "Non-unitary Models of Household Behavior: A Survey of the Literature," in: Jose Alberto Molina (ed.), *Household Economic Behaviors*, Springer-Verlag New York Inc..

[15] Chiappori, Pierre-André and Ivar Ekeland, 2006, "The Micro Economics of Group Behavior: General Characterization," *Journal of Economic Theory*, 130, 1-26

[16] Chiappori, Pierre-André and Ivar Ekeland, 2009, "The Micro Economics of Efficient Group Behavior: Identification," *Econometrica*, Vol. 77, No. 3, 763–799

[17] De Menil, Georges, 1971, *Bargaining: Monopoly Power versus Union Power*, MIT Press, Cambridge Mass.

[18] Fershtman Chaim and Eithan Muller, 1986, "Capital investment and price agreements in semi-collusive markets," *The Rand Journal of Economics*, vol. 17, pp. 214-226.

[19] Harrington Joseph E. (1991), "The determination of price and output quotas in a heterogeneous cartel," *International Economic Review*, vol. 32, pp. 767-792.

[20] Hamermesh, Daniel S., 1973, "Who 'Wins' in Wage Bargaining," *Industrial and Labor Relations Review*, vol. 26, pp. 1146–1149.

[21] Koopmans, Tjalling C., and Olaf Reiersøl, 1950, "The Identification of Structural Characteristics, " *The Annals of Mathematical Statistics*, 21, pp. 165-181.

[22] Kotlikoff Lawrence J., John B. Shoven and Avia Spivak, 1986, "The effect of annuity insurance on savings and inequality," *Journal of Labor Economics*, vol. 4, pp. S183-S207.

[23] Lensberg, Terje, 1987, "Stability and Collective Rationality," *Econometrica*, vol. 55, pp. 935-961.

[24] Lensberg, Terje, 1988, "Stability and the Nash Solution," *Journal of Economic Theory*, vol. 45, pp. 330-341.

[25] Lundberg, Shelly and Robert A. Pollak, 1993, "Separate Spheres Bargaining and the Marriage Market," *Journal of Political Economy*, vol. 101, pp. 988-1010.

[26] Manser, Marilyn and Murray Brown, 1980, "Marriage and Household Decision-making," *International Economic Review,* vol. 22, pp. 31–44.

[27] Marschak, Jacob, (1950), "Statistical Inference in Economics: An Introduction," *Cowles Commission Monograph*, vol.10, pp. 1-50.

[28] Matzkin, Rosa L., 2003, "Nonparametric Estimation of Nonadditive Random Functions, " *Econometrica*, 71, pp. 1339-1375.

[29] McElroy, Marjorie and Mary–Jane Horney, 1981, "Nash-Bargained Household Decisions: Toward a Generalization of the Theory of Demand," *International Economic Review,* vol. 22, pp. 333–349.

[30] McElroy, Marjorie and Mary–Jane Horney, 1990, "Nash-Bargained Household Decisions: Reply," *International Economic Review,* Vol. 31, No. 1, pp. 237-242.

[31] Moscarini, Giuseppe, 2005, "Job Matching and the Wage Distribution," *Econometrica*, vol. 73, pp. 481-516.

[32] Nash, John, 1950, "The Bargaining Problem," *Econometrica*, 28, 155-62.

[33] Peters, Hans and Peter Wakker, 1991, "Independence of Irrelevant Alternatives and Revealed Group Preferences," *Econometrica*, vol. 59, pp. 1787-1801.

[34] Pollak, Robert A., 1977, "Price Dependent Preferences," *American Economic Review*, vol. 67, pp. 64-75.

[35] Popper, Karl R., 1959, *The Logic of Scientific Discovery*, London: Hutchinson and Co. (Publishers) Ltd.

[36] Postel-Vinay, F. and J.-M. Robin, 2006, "Microeconometric Search-Matching Models and Matched Employer-Employee Data," in R. Blundell, W. Newey and T. Persson, editors, Advances in Economics and Econometrics, Theory and Applications, Ninth World Congress, Volume 2, Cambridge: Cambridge University Press.

[37] Ray, Indrajit and Lin Zhou, 2001, "Game Theory via Revealed Preferences," *Games and Economic Behavior*, vol. 37, pp. 415–424.

[38] Roehrig, Charles S., 1988, "Conditions for Identification in Nonparametric and Parametric Models," *Econometrica*, 56, pp. 433-447.

[39] Roth, Alvin E. and Michael W.K. Malouf (1979), "Game-theoretic models and the role of information in bargaining ," *Psychological Review*, vol. 86, pp. 574–594.

[40] Siegel, Sidney and Lawrence E. Fouraker (1960), *Bargaining and Group-Decision Making: Experiments in Bilateral Monopoly*, MacMillan: New-York.

[41] Sprumont, Yves, 2000, "On the Testable Implications of Collective Choice Theories," *Journal of Economic Theory*, vol. 101, pp. 437–456.

[42] Svejnar, Jan, 1980, "On Empirical Testing of the Nash-Zeuthen Bargaining Solution," *Industrial and Labor Relations Review*, vol. 33, pp. 536–542.

[43] Svejnar, Jan, 1986, "Bargaining Power, Fear of Disagreement, and Wage Settlements: Theory and Evidence from U.S. Industry," *Econometrica,* vol. 54, pp. 1055–1078.

[44] Svejnar, Jand and Stephen C. Smith, 1984, "The Economics of Joint Ventures in Less Developed Countries," *Quarterly Journal of Economics*, vol. 99, pp. 149-168.

[45] Thomson, William, 1994, "Cooperative Models of Bargaining," In: R.J. Aumann and S. Hart (eds), *Handbook of Game Theory*, vol. 2, pp. 1237–1277.

[46] Xu, Yongsheng and Lin Zhou, 2007, "Rationalizability of Choice Functions by Game Trees," *Journal of Economic theory*, vol. 134, pp. 548-556.