Bifurcation of On-site and Off-site Solitary Waves of Discrete Nonlinear Schrödinger Type Equations

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ABSTRACT

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A feature of immeasurable interest in nonlinear systems is that of spatially localized traveling pulses, or solitary waves - states which persist indefinitely in time, focus energy, and facilitate its transfer. Furthermore, in many lattice systems, discreteness effects are important and play a key role in these dynamics.

In this thesis, we construct the multiple families of solitary standing (time-periodic) waves of the discrete, focusing cubically nonlinear Schrödinger equation (DNLS). These states are related to the so-called *Peierls-Nabarro* energy barrier, which refers to the energy difference between these distinct states and is thought to be responsible for the absence of indefinitely traveling, non-deforming solitary (spatially localized) waves of arbitrary velocity in many (non-dissipative) discrete systems. Instead, one observes that traveling waves of these discrete equations radiate energy and deform until they eventually cease to propagate and settle to a stationary time-periodic standing wave centered at a lattice site.

We address two specific cases of DNLS: *(1)* nearest-neighbor coupling on a cubic lattice in dimensions $d = 1, 2, 3$, and *(2)* long-range site coupling in dimension $d = 1$. These states are obtained via a bifurcation analysis with respect to a natural small parameter about the continuum nonlinear Schrödinger equation (NLS) limit. Depending on the spatial dimension, these may be vertex-, bond-, cell-, or face-centered. In the first case of nearest-neighbor coupling, we construct an explicit asymptotic expansion. In the second case of one-dimensional long-range coupling when the decay of the site coupling with respect to distance is sufficiently slow, the continuum limiting NLS equation has Laplacian of fractional power. Finally, we show that the energy difference among distinct states of the same frequency is exponentially small with respect to the small parameter *beyond all polynomial orders*. This provides a rigorous bound for the Peierls-Nabarro barrier.

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To my parents, Kevin and Gill

Part I

Introduction

Chapter 1

Introduction

The (focusing) discrete cubic nonlinear Schrödinger equation (DNLS) is the differential equation

$$i\partial_t u_n(t) = -h^{-2}(\mathcal{L}u)_n(t) - |u_n(t)|^2 u_n(t) \ , \qquad t \in \mathbb{R}, n \in \mathbb{Z}^d, \tag{1.1}$$

governing a complex-valued vector $u(t) = \{u_n(t)\}_{n\in\mathbb{Z}^d}$. Here, $\mathcal{L} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ is a linear, symmetric difference operator defined by

$$(\mathcal{L}u)_n \equiv \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \ (u_m - u_n) \ . \tag{1.2}$$

The range of the nonlocal interaction is characterized by the rate of decay of the coupling sequence, $\{J_m\}$.

DNLS is a Hamiltonian system, expressible in the form

$$i\partial_t u = \frac{\delta\mathcal{H}[u,\overline{u}]}{\delta\overline{u}}, \quad \text{where} \tag{1.3}$$

$$\mathcal{H}_{\mathrm{DNLS}} = \mathcal{H}[u,\overline{u}] = \frac{1}{2h^2}\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} J_{|m-n|}|u_m - u_n|^2 - \frac{1}{2}|u_n|^4 \ . \tag{1.4}$$

We note the following basic result on concerning $\mathcal{L}$ acting on $l^2(\mathbb{Z})$.

Proposition 1.0.1. *Assume $J = \{J_m\}_{m\in\mathbb{Z}}$ is non-negative ($J_m \geqslant 0$), symmetric ($J_m = J_{|m|}$), and in $J \in l^1(\mathbb{Z})$. Then,*

1. *$\mathcal{L}$ is a bounded linear operator on $l^2(\mathbb{Z})$.*

2. *$\mathcal{L}$ is self-adjoint.*

3. *$-\mathcal{L}$ is non-negative.*

4. *The spectrum of $-\mathcal{L}$ is continuous and equal to $[0, M_\star]$, where*

$$M_\star = 4 \max_{q \in [-\pi,\pi]} \sum_{m=1}^{\infty} J_m \ \sin^2\left(\frac{qm}{2}\right). \tag{1.5}$$

Proof: Young's inequality implies boundedness. The other details of the proof are presented in Appendix D.

The initial value problem (1.1) with initial data $u_n(0) = f_n \in l^2(\mathbb{Z}^d)$ is globally well-posed in the sense that for each $f = \{f_n\}_{n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$ there exists a unique global solution $u(t) = \{u_n(t)\}_{n\in\mathbb{Z}^d} \in C^1([0,\infty), l^2(\mathbb{Z}^d))$ to (1.1). This result follows from a standard contraction mapping argument applied to the equivalent integral equation formulation of the initial value problem; see, for example, [Kirkpatrick et al., 2012]. Their proof is formulated in one dimension but applies in arbitrary dimension, since $\|f\|_{l^\infty(\mathbb{Z}^d)} \lesssim \|f\|_{l^2(\mathbb{Z}^d)}$.

Time translation invariance implies that $\mathcal{H}[u,\overline{u}]$ is time-invariant on solutions and the invariance $u \mapsto e^{i\theta}u$ implies the time-invariance of

$$\mathcal{N}[u,\overline{u}] = \sum_{n\in\mathbb{Z}} |u_n|^2. \tag{1.6}$$

See Appendix B for details.

We will address two particular cases of (1.1). The simplest case is that of DNLS with nearest-neighbor coupling for general spatial dimension $d \geqslant 1$. Letting

$$J_{|m|} \equiv \begin{cases} 1 & : |m| = 1 \\ 0 & : |m| \geqslant 1 \end{cases}, \tag{1.7}$$

recovers the standard discrete Laplacian (second order finite-difference operator):

$$(\mathcal{L}u)_n = (\delta^2 u)_n \equiv \sum_{|m-n|=1} u_m - 2du_n, \tag{1.8}$$

such that DNLS becomes

$$i\partial_t u_n(t) = -h^{-2}(\delta^2 u)_n(t) - |u_n(t)|^2 u_n(t)\ , \qquad t \in \mathbb{R}, \quad n \in \mathbb{Z}^d, \tag{1.9}$$

DNLS with nearest-neighbor coupling is addressed in Chapter 3.

The second case of (1.1) which we address is DNLS with general nonlocal coupling, $\{J_m\}_{m\in\mathbb{Z}} \in l^1(\mathbb{Z})$, restricted to spatial dimension $d = 1$. Here, (1.1) becomes

$$i\partial_t u_n(t) = -h^{-2} \sum_{m\in\mathbb{Z}} J_{|m-n|}(u_m(t) - u_n(t)) - |u_n(t)|^2 u_n(t) \ , \qquad t \in \mathbb{R},\ n \in \mathbb{Z}, \tag{1.10}$$

The two most common cases of (1.10) on which we will focus correspond respectively to the polynomial and exponential (Kac-Baker) decay of the coupling sequence:

$$(1) \qquad J_m^s = \frac{1}{m^{1+2s}}, \tag{1.11}$$

$$(2) \qquad J_m^\infty = e^{-\gamma m}, \qquad \gamma > 0. \tag{1.12}$$

Here, we have taken $J_0^s \equiv 0$. Our analysis of the local (nearest-neighbor) case (1.9) will inform and motivate our approach to the nonlocal case (1.10). We remark that when $d = 1$, the former equation (1.9) is a specific case of the latter equation (1.10) when $J_0 = 0, J_{\pm 1} = 1$ and $J_m = 0, |m| \geqslant 1$. DNLS with nonlocal coupling (1.10) is addressed in Chapter 4.

Of particular interest are time-harmonic and spatially localized solutions:

$$\begin{aligned} &u_n(t) = e^{-i\omega t}\ g_n, \qquad \omega < 0, \\ &g = \{g_n\}_{n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d). \end{aligned} \tag{1.13}$$

Substitution of (1.13) into (1.1) yields the nonlinear eigenvalue problem

$$\begin{aligned} &\omega\ g_n = -h^{-2}\ (\mathcal{L}g)_n - |g_n|^2 g_n, \\ &n \in \mathbb{Z}^d, \qquad g = \{g_n\}_{n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d). \end{aligned} \tag{1.14}$$

which is an infinite system of algebraic equations. The main focus of this thesis is the study of solutions to (1.14) via bifurcation theory about the continuum ($\omega \to 0$ or $h \to 0$) limit, along with their respective energies. In particular, solutions are constructed as bifurcations from the zero state at the edge of the continuous spectrum of the operator. A detailed summary of this approach is given in Section 1.2.

Outline of the introduction: In Section 1.1, we motivate our interest in solutions to (1.14) via a summary of the physical applications of DNLS and a heuristic investigation of the dynamics of traveling pulses in lattice systems. In Section 1.2, we outline our approach to the construction of solutions to (1.14) via bifurcation theory and in Section 1.3, we summarize the main results of this thesis. In Section 1.4, we summarize previous results which are related to the main results herein, and Section 1.5 is devoted to the discussion of possible future work and extensions to the presented results. Finally, in Section 1.6, we outline the structure of the thesis.

1.1 Physical and Mathematical Motivation

For the past century, discrete nonlinear dynamical systems have proven to be of rich scientific interest in a diverse range of disciplines. Mathematical interest in such systems may be traced back to the seminal experiments of Fermi, Pasta, and Ulam in the 1950s, in which numerical simulations of a series of masses connected by springs illustrated that nonlinear systems could exhibit wildly counter-intuitive and unpredictable behavior [Fermi et al., 1955]. One feature of immeasurable interest in general nonlinear systems is that of spatially localized traveling pulses, or solitary waves. Here, dispersive behavior is counterbalanced by nonlinear focusing effects to produce states which persist indefinitely in time. Such states are particularly important in that they focus energy and facilitate its transfer. However, in many such systems, discreteness effects are important and play a key role in the dynamics of solutions [Braun and Kivshar, 1998; Friesecke and Pego, 1999; Herrmann, 2010; Arancibia-Bulnes and Ruiz-Suárez, 2002; Ponson et al., 2010; Kevrekidis, 2009a; Kevrekidis, 2009b].

Discrete nonlinear dispersive systems such as DNLS arise in the context of nonlinear optics, *e.g.* [Aceves et al., 1994b; Aceves et al., 1994a; Eisenberg et al., 1998; Eisenberg et al., 1999; Naether and Vicencio, 2011], dynamics of biological molecules, *e.g.* [Davydov and Kislukha, 1973; Eilbeck et al., 1985], and condensed matter physics. For example, they arise in the study of intrinsic localized modes in anharmonic crystal lattices, *e.g.* [A.S. Barker and Sievers, 1975; Takeno et al., 1988]. See also [Smerzi et al., 2002; Hennig et al., 2010]. In these fields, discrete systems arise either as phenomenological models or as *tight-binding* approximations; see also [MacKay et al., 2008; Pelinovsky and Schneider, 2010]. There is also a natural interest in such systems as discrete numerical approximations of equations in continuum models.

Nonlocal discrete models such as the nonlocal form of DNLS addressed in this thesis are of particular interest in several of the fields above where atomic interactions occur at length scales considerably larger than those accounted for by models with nearest-neighbor coupling [Gaididei et al., 1997; Mingaleev et al., 1999; Davydov, 1971; Kevrekidis et al., 2001; Kac and Helfand, 1963; G.A. Baker, 1961].

We remark here that the aforementioned nonlocal discrete models, along with our bifurcation analysis, are intimately associated with continuum differential and pseudo-differential equations such as the nonlinear Schrödinger equation with fractional-power Laplacian (see (4.14)). Continuum models with nonlocal effects arise in several contexts including path-integral formulations of quantum mechanics [Laskin, 2002], deep water internal and small-amplitude surface wave fluid dynamics [Benjamin, 1967; Ono, 1975; Weinstein, 1987], semi-relativistic quantum mechanics (astrophysics) [Lieb and Yau, 1987; Frölich and Lenzmann, 2007], and as the continuum limits of the aforementioned nonlocal discrete models [Kirkpatrick et al., 2012; Tarasov, 2006; Tarasov and Zaslavsky, 2006; Gaididei et al., 1996].

To mathematically motivate our main results, we begin with a study of the dynamics of traveling localized pulses in discrete systems such as DNLS (1.1). For simplicity of presentation and in order to facilitate a clear contrast with continuum (non-discrete) systems which have continuous symmetries or invariances, we focus in this section on the case of nearest-neighbor DNLS. We begin with a brief discussion of the standard continuum nonlinear Schrödinger (NLS) equation with cubic nonlinearity:

$$i\partial_t u(x,t) = -\Delta_x \ u(x,t) - |u(x,t)|^2 u(x,t), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R} \ . \tag{1.15}$$

NLS has, for any frequency $\omega < 0$, a standing wave solution:

$$u_\omega(x,t) = e^{-i\omega t} \ \psi_{|\omega|}(x), \tag{1.16}$$

where $\psi_{|\omega|}(x)$ is the unique *positive* solution to the nonlinear elliptic problem

$$-\Delta_x \psi_{|\omega|} - \left|\psi_{|\omega|}\right|^2 \psi_{|\omega|} \ = \ \omega \ \psi_{|\omega|}, \quad u \in H^1(\mathbb{R}^d), \tag{1.17}$$

which is real-valued, radially symmetric and decreasing to zero at spatial infinity [Bourgain, 1999; Kwong, 1989; Strauss, 1977; Sulem and Sulem, 1999; Tao, 2006; Weinstein, 1983]; see also Proposition 3.1.2. We refer to this solution as the *ground state* of NLS. Note that $\psi_{|\omega|}(x) = \sqrt{|\omega|} \ \psi_1 \left(\sqrt{|\omega|}x\right)$.

Since NLS is Galilean invariant, these solutions can be "boosted" to generate solitary *traveling* waves. For any velocity, v, and frequency ω,

$$e^{iv\cdot(x-vt)}\ u_\omega(x-2vt,t)\ =\ e^{-i\omega t}e^{iv\cdot(x-vt)}\ \psi_{|\omega|}(x-2vt)\ , \tag{1.18}$$

is a solution to NLS.

In contrast, recall nearest-neighbor DNLS (1.9):

$$\begin{aligned} i\partial_t u_n &= -h^{-2}(\delta^2 u)_n - |u_n|^2 u_n\ ,\ n\in\mathbb{Z}^d, \\ \left(\delta^2 u\right)_n &= \sum_{|m-n|=1} u_m - 2d\ u_n. \end{aligned} \tag{1.19}$$

which is a Hamiltonian (non-dissipative) system expressible in the form

$$i\partial_t u = \frac{\delta\mathcal{H}[u,\overline{u}]}{\delta\overline{u}},\quad \text{where} \tag{1.20}$$

$$\mathcal{H}_{\text{DNLS}} = \mathcal{H}[u,\overline{u}] = \frac{1}{2h^2}\sum_{|n-j|=1}\sum_{n\in\mathbb{Z}^d}|u_j-u_n|^2 - \frac{1}{2}|u_n|^4\ , \tag{1.21}$$

with conserved (time-invariant) quantities $\mathcal{H}_{\text{DNLS}} = \mathcal{H}[u,\overline{u}]$ and

$$\mathcal{N}_{\text{DNLS}} = \mathcal{N}[u,\overline{u}] = \sum_{n\in\mathbb{Z}^d}|u_n|^2. \tag{1.22}$$

As stated previously, in analogy with NLS (1.15), DNLS (1.19) is known to have discrete solitary standing waves [Eilbeck et al., 1985; Kevrekidis, 2009a; Weinstein, 1999]

$$u_n(t) = e^{-i\omega t}g_n,\quad n\in\mathbb{Z}^d\quad \text{where } \omega<0\ . \tag{1.23}$$

Here, g satisfies the discrete elliptic problem, an infinite system of algebraic equations:

$$-h^{-2}(\delta^2 g)_n - |g_n|^2 g_n\ =\ \omega\ g_n \qquad \|g\|^2_{l^2(\mathbb{Z}^d)} = \sum_{n\in\mathbb{Z}^d}|g_n|^2<\infty. \tag{1.24}$$

In spatial dimension $d=1$, equation (1.24) has been shown numerically and analytically to have two families of unimodal solutions which are positive, real-valued, symmetric, and respectively centered about a lattice site ("on-site," or vertex-centered) or between two lattice sites ("off-site," or bond-centered). See Figure 1.1. Such states are generalized to higher spatial dimensions $d=2$ and $d=3$, where (1.24) has, up to reflections, $d+1$ families of symmetric solutions which are "on-site" (vertex-centered) or "off-site" (bond-, face-, or cell-centered). See Definitions 2.0.1 and

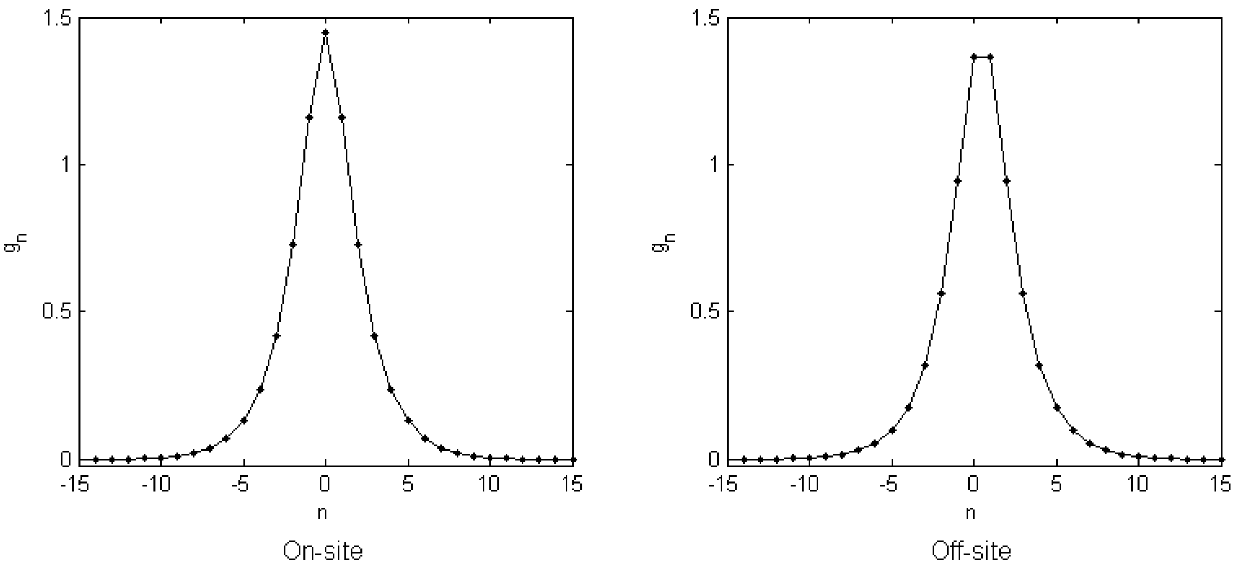

Figure 1.1: On-site symmetric (left) and off-site symmetric or bond-centered (right) standing wave solutions of nearest-neighbor DNLS for $d = 1$. Discrete solitary wave profiles are defined by the values at discrete points, nh (here $h = 0.6$ and $\omega = 1$). Plotted are linearly interpolated profiles.

2.0.2 in Chapter 2 for rigorous definitions of these families of solutions, respectively in dimension $d = 1$ and general spatial dimension $d \geqslant 1$.

Continuing our comparison with continuum NLS, we are led to ask:

Question: Are there discrete solitary traveling waves of DNLS?

This question has been studied at least since the pioneering article of Peyrard and Kruskal [Peyrard and Kruskal, 1984] which considers the propagation of discrete kinks for the discrete sine-Gordon and ϕ^4 models. Numerical evidence strongly supports the claim that there are no discrete traveling waves of DNLS. Figure 1.2 displays several simulations which shed light on the question. For a range of values of h, we solve the initial value problem for DNLS (1.19) in spatial dimension $d = 1$ with initial condition

$$u_n(0) = e^{ivx_n}\ \psi_{|\omega|}(x_n), \quad x_n = nh,\ n \in \mathbb{Z}\ , \tag{1.25}$$

obtained by evaluating $e^{ivx}\psi_{|\omega|}(x)$ at the lattice sites $x_n = nh,\ n \in \mathbb{Z}$. For the continuum limit, the solution is given by (1.18). Plotted in figure 1.2 is the location of the $\mathrm{argmax}_{n\in\mathbb{Z}}|u_n(t)|$ versus t for the parameter choices $v = 0.1$ and $\omega = 1$. The continuum limit ($h = 0$) corresponds to the dashed straight line of slope $2v = 0.2$, the (group) speed of the solitary wave envelope $\psi^{|\omega|}$ in (1.18). For a range of $t \geqslant 0$, the other curves begin linearly but then level off and approach a constant value $x_n = x_{n_\star(h)}$.

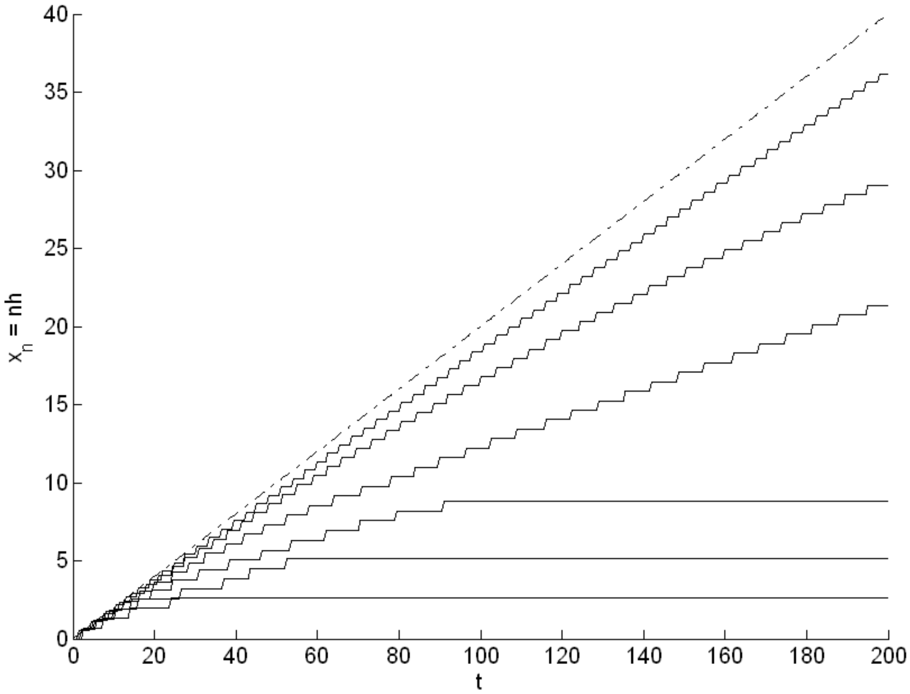

Figure 1.2: Position of peak magnitude for traveling pulses of DNLS initial value problem in spatial dimension $d = 1$, with $h = .54, .58, .61, .6295, .64$ and $.66$ and initial condition (1.25) for $v = 0.1$ and $\omega = 1$.

An examination of the time-evolving solution profiles shows a rightward moving localized structure, which continuously radiates small amplitude dispersive waves ("phonons," or lattice vibrational modes), slows down and eventually relaxes to a discrete solitary wave, which is "pinned" to a lattice site.

The phenomenology developed in the physics literature to explain this phenomenon is as follows. The discrete dynamical system, DNLS, supports solitary standing waves which are centered "on-site" and "off-site". Note, for the continuum (translation invariant NLS, (1.15)) limit, that the values of time-invariant quantities

$$\mathcal{H}_{\rm NLS}[u] = \int_{\mathbb{R}^d} |\nabla u|^2 - \frac{1}{2}|u|^4 \; dx, \quad \mathcal{N}_{\rm NLS}[u] = \int_{\mathbb{R}^d} |u|^2 \; dx$$

for $u = u_\omega(x + x_0, t)$ are independent of x_0. For DNLS, in contrast, the on-site waves are of lower energy and are therefore stable. An amount of energy equal to this energy difference, the

Peierls-Nabarro (PN) barrier, must be expended to move a discrete soliton from one lattice site to an adjacent one. This energy is dissipated through the radiation of small amplitude dispersive waves (phonons) to infinity. Since a quantum of energy is lost at each transition, the translating localized wave structure eventually no longer exceeds the PN barrier and converges asymptotically to a discrete on-site (stable) solitary wave. A mathematically rigorous study of these phenomena is an open problem.

The goal of this thesis is to present a detailed study of *on-site symmetric* (vertex-centered) and *off-site symmetric* (bond-centered, face-centered or cell-centered) discrete solitary standing waves of DNLS.

Remark 1.1.1. *Another class of discrete nonlinear systems which exhibits similar phenomena are lattice-nonlinear Klein-Gordon models, e.g. the discrete sine-Gordon equation and the discrete* ϕ^4 *equation [Peyrard and Kruskal, 1984; Kevrekidis and Weinstein, 2000]. A study of the latter stages of asymptotic relaxation to a stable on-site kink, via radiative (non-dissipative) decay phenomena is pursued in [Kevrekidis and Weinstein, 2000]. The mechanism is resonant coupling of discrete and continuum modes and resulting radiation damping, studied in the setting of continuum nonlinear wave equations in, for example, [Weinstein and Yeary, 1996; Soffer and Weinstein, 1999; Buslaev and Sulem, 2003; Soffer and Weinstein, 2004].*

Remark 1.1.2. *We note that there are discrete systems which have translating, non-deforming solitary traveling waves: (a) The Ablowitz-Ladik (AL) lattice, a discrete integrable system resulting from a special discretization of the one-dimensional NLS [Ablowitz and Ladik, 1976] , (b) FPU lattices in the supersonic regime [Friesecke and Pego, 1999; Friesecke and Wattis, 1994; Pankov and Rothos, 2011; Herrmann, 2010] and (c) the nonlinear Klein-Gordon lattice [Bates and Zhang, 2006] in the supersonic regime.*

Remark 1.1.3. *As previously stated, the continuum nonlinear Schroedinger equation (NLS),* $i\partial_t u(x,t) = (-\Delta)^p u(x,t) - |u(x,t)|^2 u(x,t)$*, with local dispersion corresponding to* $p = 1$ *is Galilean invariant. Hence, any solitary standing wave of NLS can be boosted to give a solitary traveling wave. In [Krieger et al., 2013; Hong and Sire, 2015], non-deforming traveling solitary waves of FNLS* $i\partial_t u(x,t) = (-\Delta)^p u(x,t) - |u(x,t)|^2 u(x,t), \quad 1/2 \leqslant p < 1$ *were been shown to exist. This is in direct contrast with the discrete setting, where the PNB phenomenon occurs due to the breaking* *of continuous translational symmetries. In [Krieger et al., 2013], traveling waves of the form* $u(x,t) = e^{it}U_v(x - vt), \quad |v| < 1$ *are shown to occur for* $p = 1$. *In [Hong and Sire, 2015], waves of the form* $u(x,t) = e^{-it(|v|^{2p} - \omega^{2p})}\, U_{\omega,v}\left(x - 2tp|v|^{2p-2}v\right), \quad \omega > 0, v \in \mathbb{R}$ *for* $1/2 < p < 1$.

1.2 Approach: Bifurcation of Nonlinear Eigenstates from the Edge of the Continuous Spectrum

We seek to understand the on-site symmetric and off-site symmetric discrete solitary standing waves via bifurcation methods about a small-amplitude, long-wavelength "continuum" limit which we describe heuristically in this section. For a detailed description of the strategy of our proofs and our asymptotic analysis, see Sections 3.1.2 and 4.1.2. Recall the nonlinear eigenvalue problem (1.14) for localized time-periodic standing wave solutions of DNLS:

$$\begin{aligned} &\omega g_n = -h^{-2}\ (\mathcal{L}g)_n - |g_n|^2 g_n \\ &n \in \mathbb{Z}^d, \qquad g = \{g_n\}_{n \in \mathbb{Z}^d} \in l^2(\mathbb{Z}^d). \end{aligned} \tag{1.26}$$

with the linear difference operator $\mathcal{L} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$:

$$\begin{aligned} &(\mathcal{L}u)_n \equiv \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq n}} J_{|m-n|}\ (u_m - u_n), \\ &\{J_{|m|}\}_{m \in \mathbb{N}^d} \in l^1(\mathbb{N}^d). \end{aligned} \tag{1.27}$$

We seek spatially localized on-site and off-site solutions of (1.24) in two related distinguished limits, expressed in terms of the frequency ω and lattice spacing h:

(L1) *Continuum limit; frequency* $\omega < 0$ *fixed and lattice (grid) spacing* $h \to 0$. This is a limit of interest in numerical computations. Solutions are expected to approach those of continuum NLS.

(L2) *Homogenized long wave limit; fixed lattice spacing* h *and* $\omega \to 0$. Here, there is large scale separation between the width of the discrete standing wave and the lattice spacing. Solutions are expected to approach a (homogenized or averaged) continuum NLS equation.

We will find, in particular, that the power of the Laplacian in the continuum NLS limit will be determined by the decay of the coupling coefficients $\{J_{|m|}\}_{m\in\mathbb{N}^d}$ in the operator $\mathcal{L}$. For nearest-neighbor coupling or sufficiently fast coupling coefficient decay, the limiting equation will use the standard Laplacian; for slow decay, the limiting equation will involve a Laplacian of fractional power.

The two limits (L1) and (L2) may be studied together through the introduction of a single parameter. Introduce

$$\omega = -\epsilon^2 \neq 0, \qquad g_n = h^{-1} G_n, \qquad \kappa(\alpha) \equiv \epsilon\ h\ . \tag{1.28}$$

where the "effective frequency" $\kappa(\alpha) > 0$ is a continuous function with $\kappa(\alpha) \downarrow 0$ as $\alpha \to 0$. The appropriate choice of $\kappa(\alpha) > 0$ will be determined by the decay of the coupling coefficient sequence $\{J_{|m|}\}_{m\in\mathbb{Z}^d}$ as $|m| \to \infty$. Then by (1.26), $G = \{G_n\}_{n\in\mathbb{Z}^d}$ satisfies the nonlinear eigenvalue problem

$$-\kappa(\alpha)\ G_n = -(\mathcal{L}G)_n - |G_n|^2 G_n\ , \qquad G \in l^2(\mathbb{Z}^d)\ . \tag{1.29}$$

Thus, the limits (L1) and (L2) are reduced to the study of DNLS for lattice spacing $h = 1$ and $\alpha \to 0$.

Nonlinear bound states arise as bifurcations of non-trivial localized states from the zero state at frequency $\kappa(0) = 0$, the endpoint of the continuous spectrum of the operator $\mathcal{L}$. As is the case for the limit (L2) above, we expect that solutions to (1.29) for low effective frequencies $0 < \kappa(\alpha) \ll 1$ will have small amplitude and large wavelength, such that the effects of the discrete lattice on the solutions are mitigated and they behave like the analogous continuum NLS problem.

In order to compare the spatially discrete and spatially continuous problems, it is natural to work, respectively, with the discrete and continuous Fourier transforms. These are both functions of a continuum variable (momentum, respectively, quasi-momentum). Let $\widehat{g}(q) = \mathcal{F}_D[g](q)$ denote the discrete Fourier transform on $\mathbb{Z}$ of the sequence $g = \{g_n\}_{n\in\mathbb{Z}}$ and let $\widetilde{f}(q) = \mathcal{F}_C[f](q)$ denote the continuous Fourier transform on $\mathbb{R}$ of $f : \mathbb{R} \to \mathbb{C}$; see Chapter 2 for definitions and Appendix C for a discussion of key properties.

We now use nearest-neighbor DNLS (1.9) with $d = 1$ as a simple heuristic example, where

$$-(\mathcal{L}G)_n = -(\delta^2 G)_n = -(G_{n+1} + G_{n-1} - 2G_n). \tag{1.30}$$

The continuous spectrum of the operator $-\delta^2 : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is the interval $[0, 4]$. Nearest-neighbor DNLS in $d = 1$ is then given by

$$-\kappa(\alpha)\; G_n = -(G_{n+1} + G_{n-1} - 2G_n) - (G_n)^3, \tag{1.31}$$

where we have dropped the complex modulus in the nonlinearity since we are seeking real-valued solutions (in fact, for nearest-neighbor DNLS when $d = 1$, $l^2(\mathbb{Z})$ solutions to (1.29) must necessarily be real-valued up to a constant phase factor; see [Kevrekidis, 2009a]).

Next, recall the nonlinear eigenvalue problem for the unique positive ground state of continuum NLS (1.17) for $d = 1$:

$$-\alpha^2\; \psi_{\alpha^2}(x) = -\partial_x^2\; \psi_{\alpha^2}(x) - (\psi_{\alpha^2}(x))^3, \tag{1.32}$$

which scales as $\psi_{\alpha^2}(x) = \alpha\; \psi_1(\alpha x)$. This implies that $\psi_{\alpha^2} \to 0$ point-wise as $\alpha \to 0$, and furthermore, that

$$\|\psi_{\alpha^2}\|^2_{L^2(\mathbb{R})} \sim \alpha. \tag{1.33}$$

We see that solutions ψ_{α^2} bifurcate from zero-state at the lower edge of the continuous spectrum of $-\partial_x^2$ given by the half-interval $[0, \infty)$. See Figure 1.3. Note that the continuous Fourier transform of $\psi_1(x)$, denoted by $\widetilde{\psi_1}(Q) = \mathcal{F}_C[\psi_1](Q)$, solves

$$-\;\widetilde{\psi_1}(Q) = |Q|^2\; \widetilde{\psi_1}(Q) - \frac{1}{(2\pi)^2}\; \widetilde{\psi_1} * \widetilde{\psi_1} * \widetilde{\psi_1}(Q). \tag{1.34}$$

Motivated by the scaling of the continuum NLS problem, we take $\kappa(\alpha) = \alpha^2$ such that (1.31) becomes

$$-\alpha^2\; G_n = -(G_{n+1} + G_{n-1} - 2G_n) - (G_n)^3. \tag{1.35}$$

Applying the discrete Fourier transform to (1.35), we obtain

$$-\alpha^2\; \widehat{G}(q) = 4\sin^2(q/2)\; \widehat{G}(q) - \frac{1}{(2\pi)^2}\widehat{G} *_1 \widehat{G} *_1 \widehat{G}(q). \tag{1.36}$$

Define the rescaled quasimomentum: $Q = q/\alpha$ and $\widehat{\Phi}(Q) = \widehat{G}(q/\alpha)$. Substitution into (1.36) and a change of variables gives

$$-\;\widehat{\Phi}(q) = \frac{4}{\alpha^2}\sin^2(Q\alpha/2)\; \widehat{\Phi}(Q) - \frac{1}{(2\pi)^2}\widehat{\Phi} *_\alpha \widehat{\Phi} *_\alpha \widehat{\Phi}(Q). \tag{1.37}$$

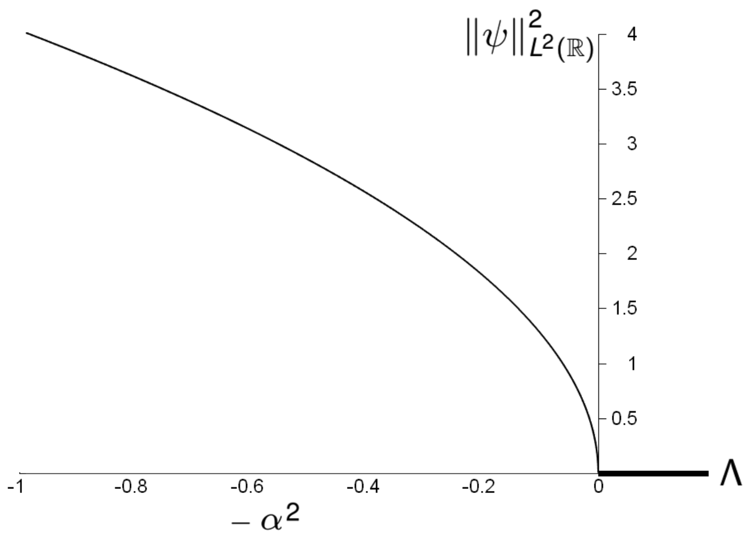

Figure 1.3: Continuum NLS eigenvalue problem: bifurcation of solutions to $\Lambda\psi(x) = -\partial_x^2\ \psi(x) - (\psi(x))^3$ from the continuous spectrum with squared $L^2(\mathbb{R})$ norm as a function of $\Lambda = -\alpha^2$. The thick dark line marks the semi-infinite band of continuous spectrum, the interval $[0, \infty)$, of the second derivative, $-\partial_x^2$.

We find asymptotically for low frequencies $|Q|\alpha \ll 1$ that

$$\frac{4}{\alpha^2}\sin^2(Q\alpha/2) \sim |Q|^2,$$
$$\widehat{\Phi} *_\alpha \widehat{\Phi} *_\alpha \widehat{\Phi}(Q) \sim \widehat{\Phi} * \widehat{\Phi} * \widehat{\Phi}(Q),$$

such that (1.37) becomes

$$-\ \widehat{\Phi}(Q) = |Q|^2\ \widehat{\Phi}(Q) - \frac{1}{(2\pi)^2}\ \widehat{\Phi} * \widehat{\Phi} * \widehat{\Phi}(Q) + \mathcal{O}(\alpha^2), \tag{1.38}$$

Thus, to leading order, we obtain the Fourier transform of the continuum NLS equation, (1.34), along with the leading order behavior

$$\|G\|^2_{l^2(\mathbb{Z})} \sim \|\psi_{\alpha^2}\|^2_{L^2(\mathbb{R})} \sim \alpha. \tag{1.39}$$

See Figure 1.4.

Remark 1.2.1. *Our discussion in this section does not address the technical issue of how we choose which of the two branches, "on-site" or "off-site," to follow in Figure 1.4. Our results also prove that the splitting between the two branches in Figure 1.4 is exponentially small with respect to the small parameter, α.*

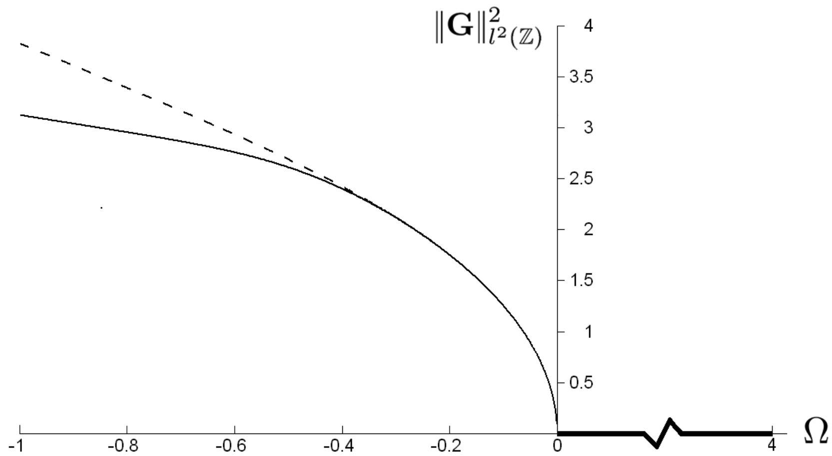

Figure 1.4: Bifurcation of on-site (solid line) and off-site (dashed line) solutions of $\Omega\ G_n = -(G_{n+1} + G_{n-1} - 2G_n) - (G_n)^3$ from the continuous spectrum with squared $l^2(\mathbb{Z})$ norm as a function of $\Omega = -\alpha^2$. The thick dark line marks the finite band of continuous spectrum, the interval $[0, 4]$, of the discrete Laplacian, $-\delta^2$.

We also note here that in order to make rigorous the heuristic argument given in this section (i.e. to utilize the localization of solitary wave solutions at low frequencies), we use a Lyapunov-Schmidt reduction strategy: we first solve for the quasi-momentum components of the solitary wave for $\alpha|Q| \sim 1$ (high-frequency components of the solitary wave), in terms of those for $0 \leqslant \alpha|Q| \ll 1$ (low-frequency components of the solitary wave). The solutions of the low-frequency equation can be studied perturbatively about the continuum NLS limit using the implicit function theorem, stated and proven in Appendix F.

Again, see Sections 3.1.2 and 4.1.2 for a more technical discussion of our approach.

1.3 Statement of Results

As stated in Section 1.2, we are interested in solutions to the nonlinear eigenvalue problem (1.29):

$$-\kappa(\alpha)\ G_n = -(\mathcal{L}G)_n - |G_n|^2 G_n\ , \qquad G \in l^2(\mathbb{Z}^d) \tag{1.40}$$

where $\mathcal{L} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ is given by

$$(\mathcal{L}u)_n \equiv \sum_{m \in \mathbb{Z}^d} J_{|m-n|}\ (u_m - u_n),$$
$$\{J_{|m|}\}_{m \in \mathbb{Z}^d} \in l^1(\mathbb{Z}^d). \tag{1.41}$$

and where $\kappa(\alpha) > 0$ is a continuous function, to be determined, with $\kappa(\alpha) \downarrow 0$ as $\alpha \to 0$.

In Chapter 3, we construct families of symmetric solutions to (1.40) for nearest-neighbor coupling

$$(\mathcal{L}G)_n = (\delta^2 G)_n = \sum_{|m-n|=1} G_m - 2dG_n. \tag{1.42}$$

in dimensions $d = 1, 2, 3$, and provide estimates on their energy differences.

In Chapter 4, we construct families of symmetric solutions to (1.40) for nonlocal coupling

$$(\mathcal{L}G)_n = \sum_{m \in \mathbb{Z}} J_{|m-n|}(G_m - G_n). \tag{1.43}$$

in dimension $d = 1$, and again provide estimates on their energy differences.

1.3.1 Nearest-neighbor Cubic DNLS: Bifurcation of Solitary Waves in Dimensions $d = 1, 2,$ and 3

In Chapter 3, we consider cubic DNLS with nearest-neighbor coupling, (1.9), for which the associated nonlinear eigenvalue problem (1.40) is given by

$$\begin{aligned} -\alpha^2 G_n &= -(\delta^2 G)_n - G_n^3, \qquad n \in \mathbb{Z}^d, \\ (\delta^2 G)_n &= \sum_{|m-n|=1} G_m - 2dG_n, \end{aligned} \tag{1.44}$$

where $d = 1, 2, 3$.

Let $\psi_{\alpha^2}(x) \in H^1(\mathbb{R}^d)$ be the unique positive solution to continuum NLS (see Proposition 3.1.1 for details):

$$-\alpha^2\ \psi_{\alpha^2}(x) = -\Delta_x\ \psi_{\alpha^2}(x) - (\psi_{\alpha^2}(x))^3, \qquad x \in \mathbb{R}^d. \tag{1.45}$$

Our results are as follows:

1. **Theorem 3.2.1: Bifurcation of solutions in dimension** $d = 1$. We show that, for $\alpha > 0$ sufficiently small, there exist two families of symmetric solutions to (1.44): on-site (vertex-centered) and off-site (bond-centered) bound states (see Figure 1.1 and Definition 2.0.1 for further details). Up to integer lattice translations, these families to leading order are respectively given by $\psi_{\alpha^2}(n)$ and $\psi_{\alpha^2}(n - 1/2)$. Furthermore, up to any finite integer $K \geqslant 1$,

we derive an asymptotic expansion in powers of α^2 for the on-site and off-site states, given in terms of K functionals $\mathcal{G}_k, k = 1, \ldots, K$, of ψ_{α^2}:

$$G_n^{\text{on}} = \psi_{\alpha^2}(n) + \sum_{k=1}^{K} \alpha^{2k}\ \mathcal{G}_k\left[\psi_{\alpha^2}\right](n) + \mathcal{O}(\alpha^{2K+2}),$$

$$G_n^{\text{off}} = \psi_{\alpha^2}(n-1/2) + \sum_{k=1}^{K} \alpha^{2k}\ \mathcal{G}_k\left[\psi_{\alpha^2}\right](n-1/2) + \mathcal{O}(\alpha^{2K+2}).$$

In particular, we show that up to any finite polynomial order in the small parameter α, the on-site and off-site states are half-integer shifts of the same continuous function.

2. **Theorem 3.2.2: Bifurcation of solutions in dimension $d = 1, 2, 3$.** We generalize Theorem 3.2.1 to dimensions $d = 1, 2, 3$. We show that, for $\alpha > 0$ sufficiently small, there exist (up to reflection) $d+1$ families of symmetric solutions to (1.44): on-site (vertex-centered) and off-site (bond-, face-, and cell-centered) bound states (see Definition 2.0.2, Table 2, and Figure 2.1). Up to integer lattice translations, these families to leading order are respectively given by $\psi_{\alpha^2}(n - \sigma)$, where $\sigma \in \{0, 1/2\}^d$ determines the centering of the symmetric state on the lattice. Furthermore, up to any finite integer $K \geqslant 1$, we derive an asymptotic expansion in powers of α^2 for the on-site and off-site states, given in terms of K functionals $\mathcal{G}_k, k = 1, \ldots, K$, of ψ_{α^2}:

$$G_n^{\sigma} = \psi_{\alpha^2}(n-\sigma) + \sum_{k=1}^{K} \alpha^{2k}\ \mathcal{G}_k\left[\psi_{\alpha^2}\right](n-\sigma) + \mathcal{O}(\alpha^{2K+2}).$$

Our results are limited to $1 \leqslant d \leqslant 3$. A well-known argument [Sulem and Sulem, 1999] shows that no non-trivial solutions to (1.45) exist for $d \geqslant 4$, such that there can be no bifurcation of a non-trivial DNLS solution. See Remark 3.2.1 for more details.

3. **Theorem 3.2.3: Exponential smallness of the Peierls-Nabarro barrier**. Recall the conserved quantities of nearest-neighbor DNLS (for lattice spacing $h = 1$ under our rescaling):

$$\mathcal{H}[G] = \sum_{j=1}^{d} \sum_{n \in \mathbb{Z}^d} |G_{n+e^{(j)}} - G_n|^2 - \frac{1}{2}|G_n|^4,$$

$$\mathcal{N}[G] = \sum_{n \in \mathbb{Z}^d} |G_n|^2.$$

Here, $e^{(j)}$ is the unit vector in the jth coordinate direction. We show that for $\alpha > 0$ sufficiently small and for any two solutions $G^{\sigma_1} = \{G_n^{\sigma_1}\}_{n \in \mathbb{Z}^d}$ and $G^{\sigma_2} = \{G_n^{\sigma_2}\}_{n \in \mathbb{Z}^d}$ of (1.44) characterized by Theorem 3.2.2, there exist constants $D_1 > 0$, $D_2 > 0$ such that

$$\left| \mathcal{N}\left[G^{\sigma_1}\right] - \mathcal{N}\left[G^{\sigma_2}\right] \right| + \left| \mathcal{H}\left[G^{\sigma_1}\right] - \mathcal{H}\left[G^{\sigma_2}\right] \right| \leqslant D_1 \ \alpha^{2-d} \ e^{-D_2/\alpha}.$$

Thus, the energy differences between the on-site and off-site symmetric solutions to DNLS are exponentially small *beyond all polynomial orders in* α, as suggested by the polynomial expansions in Theorems 3.2.1 and 3.2.2.

1.3.2 Nonlocal Cubic DNLS: Bifurcation of Solitary Waves in Dimension $d = 1$

In Chapter 4, we consider cubic DNLS with nonlocal coupling, (1.10), for which the associated nonlinear eigenvalue problem (1.40) is given by:

$$\begin{aligned} -\kappa(\alpha)\ G_n &= -(\mathcal{L}G)_n - G_n^3, \qquad n \in \mathbb{Z}, \\ (\mathcal{L}G)_n &= \sum_{m\in\mathbb{Z}} J_{|m-n|}(G_m - G_n). \end{aligned} \tag{1.46}$$

Here, the continuous function $\kappa(\alpha) > 0$ is still to be determined, with $\kappa(\alpha) \to 0$ as $\alpha \downarrow 0$.

Let $1/4 < p \leqslant 1$ and let $\psi_{\alpha^{2p}}(x) \in H^p(\mathbb{R}^d)$ be the unique positive solution to continuum NLS with fractional power Laplacian (see Proposition 4.1.1 for details):

$$-\alpha^{2p}\ \psi_{\alpha^{2p}}(x) = (-\Delta_x)^p\ \psi_{\alpha^{2p}}(x) - (\psi_{\alpha^{2p}}(x))^3, \qquad x \in \mathbb{R}^d. \tag{1.47}$$

Here, $(-\Delta_x)^p$ is the fractional Laplacian, which is a pseudo-differential operator defined via the continuum Fourier transform (see Chapter 2 for details) by

$$\begin{aligned} (-\Delta_x)^p f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{iqx}\ |q|^{2p}\ \widetilde{f}(q)dq, \\ \widetilde{f}(q) = \mathcal{F}_C[f](q) &= \int_{\mathbb{R}} e^{-iqx}\ f(x)dx. \end{aligned}$$

Our results are as follows:

1. **Theorem 4.2.1: Bifurcation of solutions for polynomially or exponentially decaying coupling.** Let $1/4 < s \leqslant \infty$, $J_0^s \equiv 0$, and let

$$J_m = J_m^s = \begin{cases} \frac{C_s^{-1}}{m^{1+2s}} & : 1/4 < s < 1 \\ C_\infty^{-1}\ e^{-\gamma m} & : s = \infty \end{cases}, \quad m > 0.$$

where $C_s > 0$ is a constant to be determined and $\gamma > 0$ is arbitrary. Furthermore, let $J_0^s \equiv 0$. Fix $p = \min(s, 1)$, with $\psi_{\alpha^{2p}}$ the solution to (1.47). We again show that, for $\alpha > 0$ sufficiently small, there exist two families of symmetric solutions to (1.44): on-site (vertex-centered) and off-site (bond-centered) bound states (see Figure 1.1 and Definition 2.0.1 for further details). Up to integer lattice translations and for an appropriate choice of the function $\kappa(\alpha) = \kappa(\alpha, s)$, these families to leading order are respectively given by $(\kappa(\alpha)^{1/2}/\alpha^p)\ \psi_{\alpha^{2p}}(n)$ and $(\kappa(\alpha)^{1/2}/\alpha^p)\ \psi_{\alpha^{2p}}(n - 1/2)$. Therefore, when $s \geqslant 1$ (short- to intermediate range coupling), the limiting equation (up to scaling) of nonlocal DNLS for $\alpha \to 0$ is the continuum NLS equation with standard Laplacian; when $1/4 < s < 1$ (long-range coupling), the limiting equation is the continuum NLS equation with fractional (nonlocal) Laplacian of power s.

In analogy with Theorem 3.2.2, our results are limited to $1/4 < s < \infty$. The same well-known argument (given in [Sulem and Sulem, 1999] for the nearest-neighbor case; see Appendix E for details) shows that no non-trivial solutions to (1.47) exist for $p \leqslant 1/4$, such that there can be no bifurcation of a non-trivial DNLS solution. See Remark 4.2.1 for more details.

2. **Theorem 4.2.2: Exponential smallness of the Peierls-Nabarro barrier**. Recall the conserved quantities of nearest-neighbor DNLS (for lattice spacing $h = 1$ under our rescaling):

$$\mathcal{H}[G] = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} J_{|m-n|}\ |G_m - G_n|^2 - \frac{1}{2}|G_n|^4,$$
$$\mathcal{N}[G] = \sum_{n\in\mathbb{Z}} |G_n|^2.$$

Fix $\eta = \min(2s, 1)$. We show that for $\alpha > 0$ sufficiently small and for the on-site and off-site solutions $G^{\text{on}} = \{G_n^{\text{on}}\}_{n\in\mathbb{Z}^d}$ and $G^{\text{off}} = \{G_n^{\text{off}}\}_{n\in\mathbb{Z}^d}$ of (4.1) characterized by Theorem 4.2.1, there exist constants $D_1 > 0$, $D_2 > 0$ such that

$$\left|\ \mathcal{N}\left[G^{\text{on}}\right] - \mathcal{N}\left[G^{\text{off}}\right]\ \right| + \left|\ \mathcal{H}\left[G^{\text{on}}\right] - \mathcal{H}\left[G^{\text{off}}\right]\ \right| \leqslant D_1\ \left(\frac{\kappa(\alpha)}{\alpha}\right)\ e^{-D_2/\alpha^\eta}.$$

Thus, the energy differences between the on-site and off-site symmetric solutions to DNLS are again exponentially small *beyond all polynomial orders in* α.

1.4 Related Previous Results

1. *The Peierls-Nabarro barrier:* Campbell and Kivshar [Campbell and Kivshar, 1993] provided an early formal calculation of the energy difference between off-site and on-site discrete solitary waves (PN barrier) in a small perturbation of the Ablowitz-Ladik (AL-DNLS) system. AL-DNLS, a completely integrable Hamiltonian system, is the particular discretization of 1D NLS in which $|u_n(t)|^2$ is replaced by $\frac{1}{2}(|u_{n-1}(t)|^2 + |u_{n+1}(t)|^2)$ in (1.1). AL-DNLS has localized discrete *breather* solutions which can be centered about *any* point in the continuum, $\mathbb{R}$. The results in [Campbell and Kivshar, 1993] suggest that the off-site state of DNLS has larger Hamiltonian energy than the on-site state for fixed l^2 norm, an *effective PN barrier* which is exponentially small in the perturbative parameter. Also in the perturbed AL-DNLS setting, Kapitula and Kevrekidis [Kapitula and Kevrekidis, 2001] perform linear spectral analysis about on- and off-site states and reach conclusions consistent with those in [Campbell and Kivshar, 1993]. Oxtoby and Barashenkov studied, via asymptotic analysis beyond all polynomial orders of the small parameter, the radiative decay and pinning of DNLS pulses [Oxtoby and Barashenkov, 2007].

2. *Radiation damping:* Here, we recall Remark 1.1.1. Radiative (non-dissipative) decay phenomena such as that exhibited in DNLS systems were studied in similar discrete systems (lattice-nonlinear Klein-Gordon models, *e.g.* the discrete sine-Gordon equation and the discrete ϕ^4 equation) in [Peyrard and Kruskal, 1984; Kevrekidis and Weinstein, 2000]. A study of the latter stages of asymptotic relaxation to a stable on-site kink via this mechanism is pursued in [Kevrekidis and Weinstein, 2000]. The resonant coupling of discrete and continuum modes and resulting radiation damping is also studied in the setting of continuum nonlinear wave equations in, for example, [Weinstein and Yeary, 1996; Soffer and Weinstein, 1999; Buslaev and Sulem, 2003; Soffer and Weinstein, 2004].

3. *Existence of DNLS solitary standing waves:* Weinstein [Weinstein, 1999] studied, by variational methods, the existence of discrete solitary standing waves of DNLS in $\mathbb{Z}^d$, with general homogeneous polynomial nonlinearity; see also [Weinstein, 1983; Malomed and Weinstein, 1996]. Here, such waves are realized as minimizers (nonlinear ground states) of the Hamiltonian, $\mathcal{H}_{\mathrm{DNLS}}$, subject to fixed l^2 norm. Such ground states, when they exist, are nonlinearly

orbitally stable. Particular attention is given to conditions on the nonlinearity and spatial dimension for which there is an *excitation threshold*, a positive l^2 threshold below which there does not exist any bound state [Weinstein, 1999]. This variational method of construction *does not* give information on whether nonlinear ground states are on-site or off-site; for further connections between these results and the results of this thesis, see Section 3.2.1.

4. *Direct construction of on-site and off-site solitary waves:* Variational methods can also be used to construct on-site and off-site solitary standing waves solutions of (1.29) by appropriately constraining the function classes in which critical points are sought. Bambusi and Penati [Bambusi and Penati, 2010] construct on-site and off-site solitary standing wave solutions near the continuum limit ($0 < \alpha^2 \ll 1$) of the one-dimensional DNLS by combining a finite element approach with the above variational formulation. Herrmann [Herrmann, 2011] constructs on-site and off-site solitary standing waves for any $\alpha^2 > 0$ as the infinite period limit of variationally obtained periodic standing waves; see also [Pankov, 2006]. Chong, Pelinovsky, and Schneider [Chong et al., 2012] construct, in an asymptotic limit, on-site and off-site states which lie near formal collective coordinate (variational) approximations.

5. *Analysis of DNLS with nonlocal coupling:* In [Gaididei et al., 1997], Gaididei et. al. study the on-site and off-site solutions to nonlocal DNLS(4.1) for $d = 1$ with polynomial coupling decay (1.11) analytically and numerically. They investigate the stability of the solitary waves at different powers of the coupling decay and predict the critical value below which the continuum limiting behavior of the DNLS solutions is described by the nonlocal (Fractional) NLS equation (see Theorem 4.2.1). In [Mingaleev et al., 1999], Mingaleev et. al. also address one-dimensional nonlocal DNLS and include an investigation of the Kac-Baker exponential coupling decay (1.12).

In [Kirkpatrick et al., 2012], Kirkpatrick et. al. address the continuum limiting behavior, for small lattice spacing $h \to 0$, of the one-dimensional nonlocal DNLS initial value problem (4.1). They study general sequences of coupling coefficients with both polynomial and stronger than polynomial decay, and confirm that the time-dependent initial value problem in the continuum limit is fractional NLS below a sufficient value of polynomial decay. Several aspects of our analysis in Chapter 4 of time-periodic solutions are inspired by their approach.

6. *Band-edge bifurcation results for the Gross-Pitaevskii equation* In [Ilan and Weinstein, 2010], Ilan and Weinstein construct time-periodic solitary wave solutions to the Gross-Pitaevskii (GP) equation $i\partial_t\psi = -\Delta\psi + V(x)\psi - |\psi|^{2\sigma}\psi$, where V is a smooth, real-valued and spatially periodic potential. The periodicity of GP may be thought of as analogous to the discreteness of DNLS. They construct, via multiple scale expansion, nonlinear eigenstates which may be centered at any point of symmetry of the potential V (analogous to the discrete on-site and off-site states) and which bifurcate from the endpoints of the spectral bands of the linear part of GP. The limiting behavior of such states as the eigenvalue approaches the band edge is a homogenized "effective" NLS equation. Their approach is analogous to that used int his thesis to construct nonlinear bound states; in particular, they utilize the Lyapunov-Schmidt reduction strategy (see Remark 1.2.1 and Sections 3.1.2 and 4.1.2).

7. *Analysis of continuum limit NLS equations:* As already stated, our analysis of nearest-neighbor DNLS relies extensively on a characterization of the properties of the unique (positive) ground state solution of continuum NLS:

$$\psi(x) - \Delta_x\psi(x) - (\psi(x))^3 = 0, \qquad x \in \mathbb{R}^d, \tag{1.48}$$

along with the spectral properties of the operator obtained by linearizing this equation about the ground state:

$$L_+ = 1 - \Delta_x - 3\psi(x)^2. \tag{1.49}$$

These results are well-known and are summarized in [Bourgain, 1999; Kwong, 1989; Strauss, 1977; Sulem and Sulem, 1999; Tao, 2006; Weinstein, 1983]. See Propositions 3.1.1 and 3.1.3.

Our analysis of nonlocal DNLS in $d = 1$ also utilizes the aforementioned results when the coupling decay is sufficiently strong that the limiting NLS equation is local. When the continuum limiting equation is fractional (nonlocal) NLS, we require analogous results characterizing unique (positive) ground state solution of continuum NLS:

$$\psi(x) + (-\Delta_x)^p\ \psi(x) - (\psi(x))^3 = 0, \qquad 1/4 < p \leqslant 1, \qquad x \in \mathbb{R}. \tag{1.50}$$

along with the linearized operator

$$L_+ = 1 + (-\Delta_x)^p - 3\psi(x)^2. \tag{1.51}$$

Frank and Lenzmann summarized the properties of the ground state (including more general polynomial power nonlinearity) and proved its uniqueness for $d = 1$ in [Frank and Lenzmann, 2013]; their analysis also provides a spectral characterization of L_+. See Propositions 4.1.1 and 3.1.3. We also note that they extend these results to higher spatial dimension in [Frank et al., to appear], which would be essential in extending our analysis of nonlocal DNLS to higher spatial dimension $d \geqslant 1$.

1.5 Future Directions

1. *Application of our strategy to the existence of solutions to other nonlinear lattice equations:* As illustrated in Section 1.1, localized standing wave solutions to discrete nonlinear systems play a fundamental role in the dynamics of traveling waves. The techniques of this thesis from bifurcation and harmonic analysis may be employed to establish a broad existence theory for such solutions to discrete dynamical systems in the continuum limit.

   Various dispersive lattice equations exhibit centering-dependent families of standing waves analogous to those of DNLS [Kevrekidis and Weinstein, 2000]. Equations such as these may also be generalized to various geometries other than the standard $d-$dimensional rectangular lattice or to lattices with spatially varying coupling strength. In the case of anisotropic coupling coefficients, the continuous spectrum of the associated linear part of such nonlinear systems may have multiple bands [Brillouin, 1953], and I will require the more general Floquet-Bloch theory to address states which bifurcate from the various spectral band edges [Ilan and Weinstein, 2010]. One may also seek to extend the existence theory away from the continuum limit by restricting our analysis to function spaces corresponding to solution families' respective symmetries (lattice centering) and employing a combination of a priori estimates and variational characterizations of solutions (see, for example, [Weinstein, 1999; Frank and Lenzmann, 2013; Frank et al., to appear]).

2. *Local stability analysis: competition between on-site and off-site states:* Conditions for the local stability (and instability) of dispersive solitary waves have been established in a number of seminal papers [Vakhitov and Kolokolov, 1973; Weinstein, 1985; Weinstein, 1986; Grillakis

et al., 1987; Grillakis, 1988; Jones, 1988] (see also the discussion in Section 3.2.1). Following their approach with respect to DNLS, one seeks to address the eigenvalues of the operators

$$L_-^{\text{disc}} = \kappa(\alpha) - \mathcal{L} - (G_n)^2, \qquad L_+^{\text{disc}} = \kappa(\alpha) - \mathcal{L} - 3(G_n)^2, \tag{1.52}$$

where G_n is a solution to DNLS. A large wealth of literature exists characterizing the coupling of discrete to continuum modes of similar operators in various contexts (radiation damping) [Soffer and Weinstein, 1999; Kevrekidis and Weinstein, 2000; Soffer and Weinstein, 2004]. In particular, the emergence of discrete eigenmodes of the L_- and L_+ operators (1.52) and their coupling with the continuous spectrum are responsible for the Peierls-Nabarro settling phenomenon and the local asymptotic stability of stationary states. This analysis would also potentially employ dispersive estimates on the linear discrete Schrödinger operator [Kevrekidis and Stefanov, 2005].

One may extend the analysis of this thesis to illustrate that the eigenvalues of (1.52) bifurcate exponentially from the zero modes of the associated operators of the continuum limit. The spectral analysis of such operators often also relies on oscillation theory for Schrödinger operators and Perron-Frobenius arguments [Weinstein, 1986; Frank and Lenzmann, 2013; Frank et al., to appear], and analogous results from the theory of discrete Jacobi operators [Teschl, 1996; Krüger and Teschl, 2009] may also prove necessary. However, the exponential smallness *beyond all polynomial orders of the small parameter* [Kapitula and Kevrekidis, 2001] of the bifurcating eigenvalues renders standard asymptotic analysis insufficient for exposing the PN pinning mechanism, and tools such as singular perturbation theory and Borel summation of trans-series [Costin, 2009; Oxtoby and Barashenkov, 2007] may be required. Such techniques may also be used to make rigorous the heuristic perturbative approaches to the Peierls-Nabarro trapping behavior found in physics literature [Kivshar and Campbell, 1993].

3. *Global dynamical behavior and capture:* A global picture of the dynamical behavior of solutions to discrete systems such as DNLS, in particular the trapping of a solution in the stable energy basin of a vertex-centered standing wave, requires analysis more novel and sophisticated than the local orbital and asymptotic stability methods employed previously. One recent example of such a technique is the use of nonlinear monotonicity formulae employed by Martel and

Merle [Martel and Merle, 2000] in the context of the critical generalized KdV equation. While DNLS is not dissipative due to conservation of the l^2 norm, more refined energy methods such as this may be essential in following the concentration of mass in traveling pulses of the system.

Other global techniques from dynamical systems may also be essential to the completion of this project. One such example of this is the compactness technique applied by Tao in [Tao, 2007; Tao, 2008] to NLS type equations, wherein he examines the existence of local or global attractors. By establishing the existence of an invariant subset of an appropriate space under the flow of DNLS, I will be able to prove the decomposition of a large class of solutions into a radiative part plus the attractive stationary state.

1.6 Thesis Outline

This thesis is structured in four chapters, the bibliography, and six appendices. Chapter 3 is adapted from the peer-reviewed publication [Jenkinson and Weinstein, 2015]. Chapter 4 is adapted from the publication which is currently in preparation, [Jenkinson and Weinstein, in preparation].

Ch. 1 This chapter introduces the nonlinear eigenvalue problem for DNLS and motivates it with a presentation of past results and the dynamics of discrete systems.

Ch. 2 We present notation, definitions, and conventions which are used consistently throughout the thesis.

Ch. 3 We construct solitary waves of nearest-neighbor DNLS (1.9) in spatial dimension $d = 1, 2, 3$. We also compare their respective energies, related to the Peierls-Nabarro barrier discussed in Chapter 1.

Ch. 4 We construct solitary waves of nonlocal DNLS (1.10) in spatial dimension $d = 1$ and compare their respective energies. Chapters 3 and 4 are structured similarly.

Ap. A We introduce function spaces and inequalities which are used in various places throughout this thesis.

Ap. B We provide that the power and Hamiltonian are conserved (time-invariant) quantities of the DNLS flow.

Ap. C We summarize properties of the discrete Fourier transform, defined in Chapter 2.

Ap. D We summarize spectral properties of the nonlocal discrete difference operator $\mathcal{L}$, defined in (1.2) and characterized by Proposition 1.0.1.

Ap. E we provide several energy identities which are analogous to the well-known Pohozaev identities of the standard (local) NLS equation

Ap. F We provide a statement of the implicit function theorem, a variant of the version given in [Nirenberg, 2001]. We also use it to prove a general lemma for the existence of solutions to equations of a specific form, which is used extensively in the proofs in Chapters 3 and 4.

Ap. G We introduce several special functions and use them to derive asymptotic expansions of the Fourier symbols of the discrete difference operator $\mathcal{L}$ which appears in DNLS. We use these expansions in the distributional sense in our bifurcation analysis.

Ap. H We provide a general lemma which is used to prove the exponential decay of solitary waves in the Fourier variable. These lemmata apply to continuum NLS and DNLS.

Chapter 2

Preliminaries, Notation and Conventions

In this chapter, we introduce several important definitions and notation. **All notation in this chapter remains consistent throughout the thesis. Any definitions and notations which are specific to each chapter will be noted as such.**

Define the discrete Fourier transform (DFT) of the sequence $f = \{f_n\}_{n\in\mathbb{Z}^d} \in l^1(\mathbb{Z}^d) \cap l^2(\mathbb{Z}^d)$ by

$$\widehat{f}(q) = \mathcal{F}_{D}[f](q) \equiv \sum_{n\in\mathbb{Z}^d} f_n e^{-iq\cdot n} \ . \tag{2.1}$$

for $q \in \mathbb{R}^d$. Since $\widehat{f}(q + 2\pi e^{(j)}) = \widehat{f}(q)$, where $e^{(j)}$ denotes the j^{th} standard basis element in $\mathbb{R}^d$, we shall view $\widehat{f}(q)$ as being defined on the torus $\mathbb{T}^d \simeq \mathcal{B}/\pi\mathbb{Z}^d$, where $\mathcal{B}$ is fundamental period cell (*Brillouin zone*),

$$\mathcal{B} \equiv [-\pi, \pi]^d \ . \tag{2.2}$$

Thus, $\widehat{f}$ is completely determined by its values on $\mathcal{B}$. We shall also make use of the scaled *Brillouin zone*,

$$\mathcal{B}_\alpha \ \equiv \Big[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha} \Big]^d \qquad \text{with} \qquad \mathbb{T}^d \simeq \mathcal{B}_\alpha \Big/ \frac{\pi}{\alpha}\mathbb{Z}^d. \tag{2.3}$$

The inverse discrete Fourier transform is defined by

$$f_n = \Big(\mathcal{F}_{D}^{-1}[\widehat{f}] \Big)_n = \frac{1}{(2\pi)^d} \int_{\mathcal{B}} \widehat{f}(q) e^{iq\cdot n} dq \ . \tag{2.4}$$

A summary of key properties of the discrete Fourier transform is included in Appendix C.

A function $F(q),\ q \in \mathbb{R}^d$ is said to be $2\pi-$ *periodic* if for all $j \in \{1,\ldots,d\} : F(q) = F(q+2\pi e^{(j)})$. A function $F(q),\ q \in \mathbb{R}^d$ is said to be $2\pi\sigma-$ *pseudo-periodic* if there exists a $\sigma \in \mathbb{R}$ such that $F(q) = e^{2\pi i\sigma}F(q+2\pi e^{(j)}),\ j = 1,\ldots,d$. Throughout the thesis, we shall follow the convention that $C > 0$ and $C_j,\ j \in \mathbb{N}$ refer to constants, which may not necessarily be the same constant between any two inequalities.

For a and b in $L^1_{\text{loc}}(\mathbb{R}^d)$, we define the convolution on $\mathcal{B}_\alpha$ by

$$a *_\alpha b(q) = \int_{\mathcal{B}_\alpha} a(\xi)b(q-\xi)d\xi, \tag{2.5}$$

Proposition 2.0.1. *For a,b, and c $2\pi\sigma/\alpha-$ pseudo-periodic, with fundamental periodic cell $\mathcal{B}_\alpha$, this convolution satisfies the usual commutative and associative properties:*

$$a *_\alpha b = b *_\alpha a \qquad \text{and} \qquad (a *_\alpha b) *_\alpha c = a *_\alpha (b *_\alpha c). \tag{2.6}$$

Remark 2.0.1. *We shall on occasion be convolving up to three functions which are not $2\pi\sigma/\alpha-$ pseudo-periodic, in which case the grouping of terms with parentheses is essential.*

The standard convolution is given by

$$f * g(q) = \int_{\mathbb{R}^d} f(\xi)g(q-\xi)d\xi, \tag{2.7}$$

when defined. For $f \in L^2(\mathbb{R}^d)$, the continuous Fourier transform and its inverse are given by

$$\widetilde{f}(q) = \mathcal{F}_C[f](q) \equiv \int_{\mathbb{R}^d} f(x)e^{-iq\cdot x}dx, \tag{2.8}$$

$$\text{and} \qquad f(x) = \mathcal{F}_C^{-1}[\widetilde{f}](x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widetilde{f}(q)e^{iq\cdot x}dq\ . \tag{2.9}$$

We now present the rigorous definition of on-site and off-site discrete states in one spatial dimension $d = 1$, followed by the higher dimensional generalization.

Definition 2.0.1 (On-site symmetric and off-site symmetric states in one spatial dimension)**.** *Let $g = \{g_n\}_{n\in\mathbb{Z}}$.*

1. *A solution to equation (1.24) is referred to as on-site symmetric if for all $n \in \mathbb{Z}$, it satisfies*

$$g_n = g_{-n}. \tag{2.10}$$

In this case, g is symmetric about $n = 0$.

2. *A solution to equation (1.24) is referred to as off-site symmetric or bond-centered symmetric if for all $n \in \mathbb{Z}$, it satisfies*

$$g_n = g_{-n+1}. \tag{2.11}$$

In this case, g is symmetric about the point halfway between $n = 0$ and $n = 1$.

The following proposition characterizes on-site symmetric and off-site symmetric states on $\mathbb{Z}$ in terms of the structure of their discrete Fourier transforms. The higher dimensional analogue ($d \geqslant 1$) is stated in Proposition 2.0.3. The proof is given in Appendix C.

Proposition 2.0.2. *(a) Let $G = \{G_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ be real and on-site symmetric in the sense of Definition 2.0.1. Then, $\widehat{G}(q) = \mathcal{F}_D[G]$, the discrete Fourier transform of G, is real-valued and symmetric. Conversely, if $\widehat{G}(q)$ is real and symmetric, then $\mathcal{F}_D^{-1}[\widehat{G}]$, its inverse discrete Fourier transform, is real and on-site symmetric.*

(b) If $G = \{G_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ is real and off-site symmetric in the sense of Definition 2.0.1, then

$$\widehat{G}(q) = e^{-iq/2}\widehat{K}(q), \tag{2.12}$$

where $\widehat{K}(q)$ is real and symmetric. Conversely, if $\widehat{G}(q) = e^{-iq/2}\widehat{K}(q)$, where $\widehat{K}(q)$ is real and symmetric, then $\mathcal{F}_D^{-1}[\ \widehat{G}\ ]$ is real and off-site symmetric.

The following definition generalizes Definition 2.0.1 to general spatial dimension $d \geqslant 1$.

Definition 2.0.2 (σ-centered states in general spatial dimension)**.** *Let $G = \{G_n\}_{n\in\mathbb{Z}^d}$ be a solution to equation (3.33) and let $\sigma \in \{0, 1/2\}^d$. We say that G is σ- centered if it is symmetric about the point σ in space. That is, for each spatial component $k = 1, \ldots, d$, we have $G_n = G_m^{(k)}$ where $m^{(k)} = (n_1, \ldots, n_k + 2\sigma_k, \ldots, n_d)^T \in \mathbb{Z}^d$. Note that this definition is consistent with its one-dimensional analogue given in Definition 2.0.1. See Table 2 and Figure 2.1 for an illustration of possible σ-centerings in spatial dimensions $d = 1, 2, 3$.*

We now generalize Proposition 2.0.2 to $d \geqslant 1$.

Proposition 2.0.3. *If $G = \{G_n\}_{n\in\mathbb{Z}^d}$ is real and σ-centered, then $\widehat{G}(q) = e^{-iq\cdot\sigma}\widehat{K}(q)$, where $\widehat{K}(q)$ is real and symmetric. Conversely, if $\widehat{G}(q) = e^{-iq\cdot\sigma}\widehat{K}(q)$, where $\widehat{K}(q)$ is real-valued and symmetric, then $\mathcal{F}_D^{-1}[\widehat{G}]$ is real and σ-centered. This is consistent with the one-dimensional analogue given in Proposition 2.0.2.*

d	$\sigma \in \mathbb{Z}^d/2$	**Centering**
1	0	Vertex-centered (On-site)
	1/2	Bond-centered (Off-site)
2	(0, 0)	Vertex-centered (On-site)
	(0, 1/2)	Bond-centered (Off-site)
	(1/2, 0)	Bond-centered (Off-site)
	(1/2, 1/2)	Cell-centered (Off-site)
3	(0, 0, 0)	Vertex-centered (On-site)
	(0, 0, 1/2)	Bond-centered (Off-site)
	(0, 1/2, 0)	Bond-centered (Off-site)
	(1/2, 0, 0)	Bond-centered (Off-site)
	(0, 1/2, 1/2)	Face-centered (Off-site)
	(1/2, 0, 1/2)	Face-centered (Off-site)
	(1/2, 1/2, 0)	Face-centered (Off-site)
	(1/2, 1/2, 1/2)	Cell-centered (Off-site)

Table 2.1: Centering of discrete solitions in dimensions $d = 1,\ 2,\ 3$.

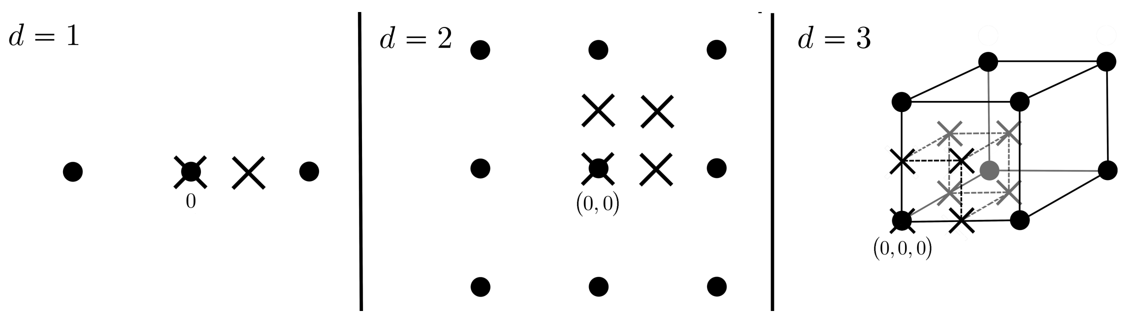

Figure 2.1: Centerings of solitary standing waves, up to integer translations, in dimensions $d = 1, 2, 3$.

The following are notations used throughout:

1. The inner product,

$$\langle f, g \rangle = \langle f, g \rangle_{L^2(\mathcal{B})} = \int_{\mathcal{B}} \overline{f(q)}\ g(q)\ dq, \tag{2.13}$$

   where $\overline{f}$ is the complex conjugate of f.

2. $\|f\|^2_{l^2(\mathbb{Z}^d)} = \sum_{n \in \mathbb{Z}^d} |f_n|^2$.

3. $L^2(A)$, the space of functions satisfying $\|f\|_{L^2(A)} = \left(\int_A |f(x)|^2 dx\right)^{1/2} < \infty$.

4. $H^1(A)$, the space of functions satisfying $\|f\|_{L^2(A)} = \left(\|f\|^2_{L^2(A)} + \|\nabla f\|^2_{L^2(A)}\right)^{1/2} < \infty$.

5. $L^{2,a}(A)$, the space of functions satisfying $\|f\|_{L^{2,a}(A)} = \left(\int_A (1+|q|^2)^a |f(q)|^2 dq\right)^{1/2} < \infty$, where $L^{2,0}(A) = L^2(A)$.

6. $L^{2,a}_{\text{even}}(A)$, the space of functions $f \in L^{2,a}(A)$ which are even. That is, for any $\tau_j = \pm 1$, $j = 1, \ldots d$, $f(\tau_1 x_1, \ldots \tau_d x_d) = f(x_1, \ldots x_d)$. We also refer to $L^2_{\text{even}}(A) = L^{2,0}_{\text{even}}(A)$. We also refer to these as symmetric functions.

7. $H^a(\mathbb{R}^d)$, the space of functions such that $\widetilde{f} = \mathcal{F}_C[f] \in L^{2,a}(\mathbb{R}^d)$, with $\|f\|_{H^a(\mathbb{R}^d)} \equiv \|\widetilde{f}\|_{L^{2,a}(\mathbb{R}^d)}$.

8. The forward finite difference operator,

$$(\delta_j f)_n = f_{n+e^{(j)}} - f_n, \tag{2.14}$$

for $j \in \{1, \ldots, d\}$ where $e^{(j)}$ is the standard unit vector in the jth coordinate direction. When $d = 1$, we also use the short-hand

$$(\delta f)_n = (\delta_1 f)_n = f_{n+1} - f_n. \tag{2.15}$$

9. For $d = 1$ and $f = \{f_n\}_{n \in \mathbb{Z}}$,

$$\left(\delta^2 f\right)_n = f_{n+1} + f_{n-1} - 2f_n, \tag{2.16}$$

the one-dimensional discrete Laplacian, and for general dimension, d:

$$\left(\delta^2 f\right)_n = \sum_{|j-n|=1} f_j - 2d\ f_n\ , \tag{2.17}$$

10. $\chi_A(x) = \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$, the indicator function for a set A, and $\overline{\chi}_A = 1 - \chi_A$.

11. $f(\alpha) = \mathcal{O}(\alpha^\infty)$ if for all $n \geqslant 1$, $f(\alpha) = \mathcal{O}(\alpha^n)$.

We often use the expression $f_1 \lesssim f_2$, which indicates that there exists some constant $C > 0$ such that

$$f_1 \leqslant C\ f_2. \tag{2.18}$$

We will fix $a > d/2$ such that $H^a(\mathbb{R}^d)$ forms an algebra, with

$$\|f_1 f_2\|_{H^a(\mathbb{R}^d)} \lesssim \|f_1\|_{H^a(\mathbb{R}^d)} \; \|f_2\|_{H^a(\mathbb{R}^d)}\,, \tag{2.19}$$

or equivalently by the Plancherel identity,

$$\left\|\widetilde{f}_1 * \widetilde{f}_2\right\|_{L^{2,a}(\mathbb{R}^d)} \lesssim \left\|\widetilde{f}_1\right\|_{L^{2,a}(\mathbb{R}^d)} \; \left\|\widetilde{f}_2\right\|_{L^{2,a}(\mathbb{R}^d)}. \tag{2.20}$$

See also [Dohnal and Uecker, 2009] and Lemma A.0.4.

Part II

Main Results

Chapter 3

Nearest-neighbor Cubic DNLS: Bifurcation of Solitary Waves in Dimensions $d = 1, 2,$ and 3

3.1 Introduction

In this chapter, we address solitary wave solutions to the nearest-neighbor discrete nonlinear Schrödinger equation (nearest-neighbor DNLS) in spatial dimension $d = 1, 2, 3$:

$$i\partial_t u_n = -h^{-2}(\delta^2 u)_n - |u_n|^2 u_n \ , \qquad n \in \mathbb{Z}^d, \qquad u = \{u_n\}_{n \in \mathbb{Z}^d}. \tag{3.1}$$

Here, $h > 0$ is the lattice spacing parameter and δ^2 is the d-dimensional discrete (centered-difference) Laplacian, given by

$$\left(\delta^2 \mathbf{u}\right)_n = \sum_{|j-n|=1} u_j - 2d \ u_n, \tag{3.2}$$

where $j \in \mathbb{Z}^d$ are taken from the set of adjacent nearest-neighbor points to $n \in \mathbb{Z}^d$ (of unit distance from n).

We recall several details about nearest-neighbor DNLS from Chapter 1. Nearest-neighbor DNLS is a Hamiltonian system, expressible in the form

$$i\partial_t u = \frac{\delta \mathcal{H}[u, \overline{u}]}{\delta \overline{u}}, \quad \text{where} \tag{3.3}$$

$$\mathcal{H}_{\text{DNLS}} = \mathcal{H}[u, \overline{u}] = \frac{1}{h^2} \sum_{j=1}^{d} \sum_{n \in \mathbb{Z}^d} |(\delta_j u)_n|^2 - \frac{1}{2}|u_n|^4. \tag{3.4}$$

Here, δ_j is the jth discrete forward difference operator,

$$(\delta_j u)_n = u_{n+e^{(j)}} - u_n \ , \tag{3.5}$$

where $e^{(j)}$ is the unit vector in the jth coordinate direction.

The initial value problem

$$\begin{aligned} &i\partial_t u_n(t) = -h^{-2}(\delta^2 u)_n(t) - |u_n(t)|^2 u_n(t) \ , \ n \in \mathbb{Z}^d, \quad t \geqslant 0 \\ &u_n(0) = f_n \in l^2(\mathbb{Z}^d), \end{aligned} \tag{3.6}$$

is globally well-posed in $l^2(\mathbb{Z}^d)$, in the sense that for each $f = \{f_n\}_{n \in \mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$ there exists a unique global solution $u(t) = \{u_n(t)\}_{n \in \mathbb{Z}^d} \in C^1([0, \infty), l^2(\mathbb{Z}^d))$ to (3.6). This result follows from a standard contraction mapping argument applied to the equivalent integral equation formulation of the initial value problem; see, for example, [Kirkpatrick et al., 2012]; their proof is formulated in one dimension but applies in arbitrary dimension, since $\|f\|_{l^\infty(\mathbb{Z}^d)} \lesssim \|f\|_{l^2(\mathbb{Z}^d)}$.

Furthermore, the functionals $\mathcal{H}[u, \overline{u}]$ and

$$\mathcal{N}_{\text{DNLS}} = \mathcal{N}[u, \overline{u}] = \sum_{n \in \mathbb{Z}^d} |u_n|^2 \tag{3.7}$$

are conserved quantities (time - invariant) for solutions of DNLS. For details, see Appendix B.

Following the approach set forth in Section 1.2, we introduce the rescaling

$$g_n = h^{-1} G_n, \tag{3.8}$$

and set

$$\alpha \ \equiv \ \sqrt{|\omega|} \ \ h \ . \tag{3.9}$$

Then, $G = \{G_n\}_{n \in \mathbb{Z}^d}$ satisfies the nonlinear eigenvalue problem

$$-\alpha^2 G_n = -(\delta^2 G)_n - |G_n|^2 G_n \ , \qquad G \in l^2(\mathbb{Z}^d) \ . \tag{3.10}$$

We will study solutions to (3.10) as bifurcations of non-trivial localized states from the zero state at frequency $\alpha^2 = 0$, the endpoint of the continuous spectrum of $-(\delta^2)$.

The main results of Chapter 3 concern the existence and energetic properties of localized solutions of (3.10) for α^2 small:

1. **Theorems 3.2.1 ($d = 1$) and 3.2.2 (general spatial dimension); bifurcation of onsite and off-site states of DNLS:** In dimensions $d = 1, 2$ or 3, there exist families of on-site (vertex-centered) symmetric and off-site (bond- , face- and cell- centered) symmetric solitary standing waves of (3.10) (in the sense of Definitions 2.0.1 and 2.0.2), which bifurcate from the continuum limit ($\alpha \downarrow 0$) ground state solitary wave of NLS.

2. **Theorem 3.2.3; exponential smallness of the Peierls-Nabarro barrier:** Then, there exist positive constants α_0 and $C > 0$ such that for all $0 < \alpha < \alpha_0$, we have:

$$\begin{aligned} \left| \mathcal{N}\left[G^{\alpha,\text{on}}\right] - \mathcal{N}\left[G^{\alpha,\text{off}}\right] \right| &\lesssim \alpha^{2-d}\ e^{-C/\alpha}, \\ \left| \mathcal{H}\left[G^{\alpha,\text{on}}\right] - \mathcal{H}\left[G^{\alpha,\text{off}}\right] \right| &\lesssim \alpha^{2-d}\ e^{-C/\alpha}\ . \end{aligned} \tag{3.11}$$

Here,

$$\mathcal{N}[G] \equiv \mathcal{N}\left[G, \overline{G}\right] = \sum_{n\in\mathbb{Z}^d} |G_n|^2, \tag{3.12}$$

and

$$\mathcal{H}[G] \equiv \mathcal{H}\left[G, \overline{G}\right] = \sum_{j=1}^{d} \sum_{n\in\mathbb{Z}^d} |G_{n+e^{(j)}} - G_n|^2 - \frac{1}{2}|G_n|^4 \tag{3.13}$$

are the corresponding square $l^2(\mathbb{Z}^d)$ norm (Power) and Hamiltonian of (3.10) (for effective lattice spacing $h = 1$). The quantities

$$\mathcal{N}\left[G^{\alpha,\text{on}}\right] - \mathcal{N}\left[G^{\alpha,\text{off}}\right] \quad \text{and} \quad \mathcal{H}\left[G^{\alpha,\text{on}}\right] - \mathcal{H}\left[G^{\alpha,\text{off}}\right] \tag{3.14}$$

are related to the PN barrier; see the earlier discussion in Section 1.1. Theorem 3.2.3 applies for $d = 1, 2, 3$ where $G^{\alpha,\text{on}}$ corresponds to a vertex-centered state and $G^{\alpha,\text{off}}$ corresponds to a bond-, face-, or cell-centered state.

Throughout this chapter, we use ψ_{α^2} to refer to the unique *positive*, symmetric solution to the local cubic nonlinear Schrödinger equation (NLS) with frequency $\alpha^2 > 0$ in dimensions $d = 1, 2$, or 3,

$$NLS: \qquad 0 = \alpha^2 \ \psi_{\alpha^2}(x) - \Delta_x \psi_{\alpha^2}(x) - \left(\psi_{\alpha^2}(x)\right)^3, \qquad x \in \mathbb{R}^d. \tag{3.15}$$

We also frequently use the convention $\psi(x) \equiv \psi_1(x)$ to mean the solution to

$$0 = \psi(x) - \Delta_x \psi(x) - \psi(x)^3, \qquad x \in \mathbb{R}^d. \tag{3.16}$$

We establish the properties of ψ_{α^2} and its continuous Fourier transform in Section 3.1.3.

3.1.1 Outline of the Chapter

We finish the introduction of this chapter, first by describing our the strategy of our proof in Section 3.1.2. We also establish important properties of the continuum NLS solitary wave in Section 3.1.3 which we in our bifurcation analysis throughout the chapter.

In Section 3.2, we rigorously state our results on solitary waves of nearest-neighbor DNLS in Theorems 3.2.1 through 3.2.3, and also discuss associated numerical results and connections to the stability the solitary waves.

In Section 3.2.1, we supplement our results with numerical simulations and discuss them within the broader context of the temporal stability of solitary waves.

In Section 3.3, we begin the proof of Theorem 3.2.1 and characterize one-dimensional DNLS in the Fourier domain. We focus on the problem for $d = 1$ for the sake of clarity and later address the higher dimensional generalization. In particular, we formally derive an asymptotic expansion of the discrete solitary waves in powers of the small parameter α in Section 3.3.2.

We then proceed to rigorously justify the expansion in the following Section 3.4 by decomposing the error from our asymptotic expansion into its high- and low-frequency components. Here, we solve for the high-frequency components as a (small) functional of the low-frequency components and the small parameter α.

In Section 3.5, we turn our attention to solving the low-frequency equation, which is perturbatively associated with the continuum limit.

In Section 3.6, we use the analysis of the previous three sections to complete the proof of Theorem 3.2.1.

In Section 3.7, we prove Theorem 3.2.3 for $d = 1$.

In Section 3.8, we summarize the changes which must be made to the one-dimensional proof to generalize it to $d = 1, 2, 3$, thereby completing the proofs of Theorems 3.2.2 and 3.2.3.

3.1.2 Strategy of the Proofs of Theorems 3.2.1 - 3.2.3

We outline the strategy for dimension $d = 1$. As noted above in Section 1.2, the limit $\alpha \to 0$ in (3.30) is related to the continuum NLS limit. In order to compare the spatially discrete and spatially continuous problems, it is natural to work, respectively, with the discrete and continuous Fourier transforms. These are both functions of a continuum variable (momentum, respectively, quasi-momentum).

Let $\widehat{g}(q) = \mathcal{F}_D[g](q)$ denote the discrete Fourier transform on $\mathbb{Z}$ of the sequence $g = \{g_n\}_{n\in\mathbb{Z}}$ and let $\widetilde{f}(q) = \mathcal{F}_c[f](q)$ denote the continuous Fourier transform on $\mathbb{R}^d$ of $f : \mathbb{R}^d \to \mathbb{C}$; see Chapter 2 for definitions and Appendix C for a discussion of key properties.

To prove our main results, we first rewrite the equation for a DNLS standing wave profile (onsite or offsite) $g = g^{\sigma,\alpha} \in l^2(\mathbb{Z})$, (3.10), in discrete Fourier space for $G(q) = e^{-i\sigma q}\widehat{K}(q)$, where $K = \widehat{K}^{\sigma,\alpha}$ satisfies $K(q + 2\pi) = \widehat{K}(q)$. Motivated by Proposition 2.0.2, we seek $\widehat{K}(q)$ symmetric and real-valued. Here, $\sigma = 0$ corresponds to the on-site case and $\sigma = 1/2$ to the off-site case. Note that $\widehat{K}(q)$ is determined by its restriction, $\widehat{\phi}(q)$, to the fundamental cell $q \in \mathcal{B} = [-\pi, \pi]$ (Brillouin zone); we occassionally suppress the dependence on α or σ for notational convenience.

Define the rescaled quasimomentum: $Q = q/\alpha$. We obtain the following equation for $\widehat{\Phi}(Q) = \widehat{\phi}(q/\alpha)$, defined for $Q \in \mathcal{B}_\alpha = [-\pi/\alpha, \pi/\alpha]$, the rescaled (stretched) Brillouin zone:

$$\mathcal{D}^{\sigma,\alpha}[\widehat{\Phi}] \equiv [1 + M_\alpha(Q)]\ \widehat{\Phi}(Q) - \frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2}\left(\widehat{\Phi} * \widehat{\Phi} * \widehat{\Phi}\right)(Q) + R_1^\sigma[\widehat{\Phi}](Q) = 0, \tag{3.17}$$

Here, $\chi_{\mathcal{B}_\alpha}(Q)$ is characteristic function of $\mathcal{B}_\alpha$, $M_\alpha(Q)$ is the scaled Fourier symbol of the discrete Laplacian:

$$M_\alpha(Q) \equiv \frac{1}{\alpha^2} M(Q\alpha) = \frac{4}{\alpha^2}\sin^2\left(\frac{Q\alpha}{2}\right) \tag{3.18}$$

and

$$R_1^\sigma[\widehat{\Phi}](Q) = -\frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2}\sum_{m=\pm1} e^{2m\pi i\sigma}\left(\widehat{\Phi} * \widehat{\Phi} * \widehat{\Phi}\right)(Q - 2m\pi/\alpha). \tag{3.19}$$

It is important to note that the nonlinear operator, $\widehat{\Phi} \mapsto \mathcal{D}^{\sigma,\alpha}[\widehat{\Phi}]$, depends on σ, which designates the case of on-site or off-site states, only through the "± 1 side-band" term R_1^σ.

Reasoning formally, we see that (4.19) converges to:

$$(1 + |Q|^2)\widetilde{\psi}_1(Q) \;-\; \frac{1}{4\pi^2}\;\left(\widetilde{\psi_1} * \widetilde{\psi_1} * \widetilde{\psi_1}\right)(Q) = 0, \tag{3.20}$$

the equation for the Fourier transform on $\mathbb{R}$ of the continuum solitary wave, $\psi_1(x)$. In observing this we have used: (a) $M_\alpha(Q) \to |Q|^2$, for bounded Q, as $\alpha \to 0$, (b) $\mathcal{B}_\alpha \to \mathbb{R}$ as $\alpha \to 0$, and (c) $\chi_{\mathcal{B}_\alpha}(Q)R_1^\sigma[\widehat{\Phi}](Q) \to 0$ as $\alpha \to 0$ for $\widehat{\Phi}(Q)$ localized; see Lemma 3.4.1. Therefore, at leading order in α the behavior of $\widehat{\Phi}^{\sigma,\alpha}$ appears to be $\widetilde{\psi}_1(Q)$.

In fact, we show in Theorems 3.2.1 and 3.2.2 that for any non-negative integer J, there solutions of (3.30), $\widehat{\Phi} = \widehat{\Phi^{\alpha,\sigma}}(Q)$ for $\sigma = 0$ (on-site), and $\sigma = 1/2$ (offsite), of the form:

$$\widehat{\Phi^{\alpha,\sigma}}(Q) = e^{-i\alpha Q\sigma}\left[\widetilde{\psi_1}\,(Q) + \alpha^2\; F_1[\widetilde{\psi_1}]\,(Q) + \cdots + \alpha^{2J}\; F_J[\widetilde{\psi_1}]\,(Q) + \widehat{E_J^{\alpha,\sigma}}\,(Q)\right], \tag{3.21}$$

The mappings $\widetilde{\psi_1} \mapsto F_j[\widetilde{\psi_1}], j \geqslant 1$, are nonlocal nonlocal mappings defined below. For any $J \geqslant 1$, the sum: $\widetilde{\psi_1}\,(Q) + \alpha^2\; F_1[\widetilde{\psi_1}]\,(Q) + \cdots + \alpha^{2J}\; F_J[\widetilde{\psi_1}]$ is independent of σ and $\|\widehat{E_J^{\alpha,\sigma}}\|_{L^2(-\pi/\alpha,\pi/\alpha)}$ is of order α^{2J+1} as $\alpha \downarrow 0$. Hence, up to the phase factor related to the centering of the wave relative to the spatial lattice, the difference between on-site and off-site states is *beyond all polynomial orders in* α for $\alpha \downarrow 0$.

From the previous discussion, the on-site and off-site discrete standing solitary standing waves, $G^{\alpha,\text{on}}$ and $G^{\alpha,\text{off}}$ can be constructed from (3.21), by rescaling $Q = q/\alpha$ and inverting the discrete Fourier transform,

Theorem 3.2.3, which bounds the differences $\mathcal{N}[G^{\alpha,\text{on}}] - \mathcal{N}[G^{\alpha,\text{off}}]$ and $\mathcal{H}[G^{\alpha,\text{on}}] - \mathcal{H}[G^{\alpha,\text{off}}]$, related to the Peirls-Nabarro barrier, are exponentially small, is derived using the continuity of $\mathcal{N}[\cdot]$ and $\mathcal{H}[\cdot]$, and the Plancherel identity as follows:

$$\left|\mathcal{N}[G^{\alpha,\text{off}}] - \mathcal{N}[G^{\alpha,\text{on}}]\right|^2 \lesssim \alpha\; \int_{\mathbb{R}}(1 + |Q|^2)\;|\; \widehat{\Phi^{\alpha,1/2}}(Q) - \widehat{\Phi^{\alpha,0}}(Q)\;|^2\; dQ \tag{3.22}$$

Proposition 3.7.2 bounds $\widehat{\Phi^{\text{diff}}}(Q) \equiv \widehat{\Phi^{\alpha,1/2}}(Q) - \widehat{\Phi^{\alpha,0}}(Q)$, whose equation may be derived from the difference of the equations: $\mathcal{D}^{\alpha,\sigma}[\widehat{\Phi^{\alpha,\sigma}}]Q) = 0,\ \sigma = 0, 1/2$; see (4.19). The norm of $\widehat{\Phi^{\text{diff}}}$ is shown to be controlled by that of the ± 1 side-band terms, R_1^σ. The latter is shown to be exponentially small, using the exponential decay, uniformly in α small, of $\widehat{\Phi^{\alpha,\sigma}}(Q),\ \sigma = 0, 1/2$; see Appendix H.

We obtain:

$$\int_{\mathbb{R}} (1 + |Q|^2) \mid \widehat{\Phi^{\alpha,1/2}}(Q) - \widehat{\Phi^{\alpha,0}}(Q) \mid^2 \, dQ \lesssim e^{-C/\alpha}. \tag{3.23}$$

Finally we remark that the relation of (4.19) to the continuum limit NLS equation (4.26) was based on formal convergence argument, as $\alpha \downarrow 0$, for fixed scaled quasimomentum, $Q \in [-\pi/\alpha, \pi/\alpha]$. To make the arguments rigorous we use a Lypapunov-Schmidt reduction strategy. We first solve the quasi-momentum components of $\widehat{E_J^{\alpha,\sigma}}(Q)$ for $\alpha^{r-1} \leqslant |Q| \leqslant \pi/\alpha,\ (0 < r < 1)$ (high frequency components of $\widehat{E_J^{\alpha,\sigma}}$) in terms of those for $0 \leqslant |Q| \leqslant \alpha^{r-1}$ (low frequency components of $\widehat{E_J^{\alpha,\sigma}}$). The solutions of the low-frequency equation can be studied perturbatively about the continuum NLS limit using the implicit function theorem.

3.1.3 Properties of the NLS Ground State Standing Waves

The following results summarize properties of the NLS solitary standing wave ("soliton") and its Fourier transform on $\mathbb{R}^d$. See, for example, references [Bourgain, 1999; Kwong, 1989; Strauss, 1977; Sulem and Sulem, 1999; Tao, 2006; Weinstein, 1983].

Proposition 3.1.1 (NLS Ground State)**.** *There exists a unique positive $H^1(\mathbb{R}^d)$ solution $\psi_{|\omega|}(x)$ to* (1.17) *which is real-valued, symmetric about $x = 0$ and decaying to zero at infinity. Moreover,*

1. *$\psi_{|\omega|}(x)$ is in Schwartz class, $\mathcal{S}(\mathbb{R}^d)$ and is exponentially decaying:*

$$|\psi^{|\omega|}(x)| \lesssim \frac{e^{-\sqrt{|\omega|}\ |x|}}{|x|^{\frac{d-1}{2}}}. \tag{3.24}$$

2. *$\psi_{|\omega|}(x) = \sqrt{|\omega|}\ \psi_1(\sqrt{|\omega|}x)$.*

3. *Uniqueness up to Phase and Spatial-Translation: Any solution of* (1.17) *is of the form $e^{i\theta}\ \psi_{|\omega|}(x - x_0)$ for some $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$.*

We shall also require the detailed properties of the Fourier transform of $\psi_{|\omega|}$:

Proposition 3.1.2 (Fourier transform of NLS Ground State)**.**

The Fourier transform, $\widetilde{\psi_{|\omega|}}(q) = \mathcal{F}_c[\psi_{|\omega|}](q)$, satisfies the equation

$$\omega\ \widetilde{\psi_{|\omega|}}(q) = |q|^2\widetilde{\psi_{|\omega|}}(x)\ -\ \frac{1}{4\pi^2}\ \widetilde{\psi_{|\omega|}} * \widetilde{\psi_{|\omega|}} * \widetilde{\psi_{|\omega|}}(q), \qquad q \in \mathbb{R}^d. \tag{3.25}$$

and has the following properties:

1. *Scaling:*

$$\widetilde{\psi_{|\omega|}}(q) = \left(\frac{1}{|\omega|}\right)^{(d-1)/2} \widetilde{\psi_1}\left(\frac{q}{\sqrt{|\omega|}}\right). \tag{3.26}$$

2. *Exponential decay bound: Let $a > d/2$. Then, there exists a positive constant C_0 such that*

$$\left\| e^{C_0|q|}\ \widetilde{\psi_1}(q)\right\|_{L^{2,a}(\mathbb{R}^d_q)} \lesssim 1. \tag{3.27}$$

The exponentially weighted $L^{2,a}$ bound, (3.27), follows from Lemma H.0.17 in Appendix H.

In our bifurcation analysis, an important role is played by the the operator $f \mapsto L_+ f$, the linearization of the stationary NLS equation with $\omega = 1$, (1.17), about ψ_1. Here,

$$L_+ \equiv 1 - \Delta_x - 3(\psi_1(x))^2\ , \tag{3.28}$$

In Fourier space, $\widetilde{f}(q) \mapsto \widetilde{L_+}\widetilde{f}(q)$, where

$$\widetilde{L_+} = 1 + |q|^2 - \frac{3}{4\pi^2}\ \widetilde{\psi_1} * \widetilde{\psi_1} * \tag{3.29}$$

In particular, we require a characterization of the $L^2(\mathbb{R}^d)-$ kernel of L_+; see [Weinstein, 1985; Kwong, 1989; Bourgain, 1999; Tao, 2006].

Proposition 3.1.3. *Assume $1 \leqslant d \leqslant 3$.*

1. *The continuous spectrum of L_+ is given by the half-line $[1, \infty)$.*

2. *Zero is an isolated eigenvalue with corresponding eigenspace,*

$$\text{kernel}(L_+) = \text{span}\{\partial_{x_j}\psi_1(x),\ j = 1, \ldots, d\ \}.$$

3. *Equivalently, the kernel of $\widetilde{L_+}$ is spanned by the functions $q_j\widetilde{\psi_1}(q),\ j = 1, \ldots, d.$*

4. *$L_+ : H^a_{\text{even}}(\mathbb{R}^d) \to H^{a-2}_{\text{even}}(\mathbb{R}^d)$ is an isomorphism.*

5. *$\widetilde{L_+} : L^{2,a}_{\text{even}}(\mathbb{R}^d) \to L^{2,a-2}_{\text{even}}(\mathbb{R}^d)$ is an isomorphism.*

3.2 Main Results

We begin with precise statements of our results on the existence of discrete solitary standing waves in spatial dimensions $d = 1, 2, 3$. We first give the simpler statement in dimension $d = 1$, referring to the two types of solutions of Definition 2.0.1; see Figure 1.1.

Theorem 3.2.1. *(Discrete DNLS solitary waves on $\mathbb{Z}$) Let $\psi_{|\omega|}(x)$ denote the ground state of NLS; see Proposition 3.1.1 with frequency ω; see Proposition 3.1.1. Consider the nonlinear eigenvalue problem governing real-valued one-dimensional DNLS standing waves:*

$$-\alpha^2 G_n^\alpha = -(\delta^2 G^\alpha)_n - (G_n^\alpha)^3, \qquad G^\alpha \in l^2(\mathbb{Z}). \tag{3.30}$$

1. *Fix an integer $J \geqslant 0$. There exist mappings $\mathcal{G}_j : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, for $j = 0, 1, \ldots, J$ and a positive constant $\alpha_0 = \alpha_0[J] > 0$ such that for all $0 < \alpha < \alpha_0$, the following holds:*
   There exist two families of real-valued symmetric solutions to DNLS (3.30). These are on-site (vertex-centered) and off-site (bond-centered) solutions in the sense of Definition 2.0.1, which to leading order satisfy:

$$\begin{aligned} G_n^{\alpha,\mathrm{on}} &\approx \mathcal{G}_0\left[\psi_1\right](n) \equiv \psi_{\alpha^2}(n) = \alpha\ \psi_1(\alpha\ n), \\ G_n^{\alpha,\mathrm{off}} &\approx \mathcal{G}_0\left[\psi_1\right](n-1/2) = \psi_{\alpha^2}(n-1/2) = \alpha\ \psi_1\left(\alpha\ [n-1/2]\right). \end{aligned} \tag{3.31}$$

   More precisely,

   On-site symmetric (vertex-centered):

$$\begin{aligned} G_n^{\alpha,\mathrm{on}} &= \textstyle\sum_{j=0}^J\ \alpha^{2j}\ \mathcal{G}_j\left[\psi_1\right](n) + \mathcal{E}_n^{\alpha,J,\mathrm{on}}, \qquad n \in \mathbb{Z}, \\ \text{where}\quad \|\mathcal{G}_j[\psi_1](n)\|_{l^2(\mathbb{Z}_n)} &\sim \alpha^{1/2}, \qquad \left\|\mathcal{E}^{\alpha,J,\mathrm{on}}\right\|_{l^2(\mathbb{Z})} \lesssim \alpha^{2J+5/2}, \end{aligned} \tag{3.32}$$

   Off-site symmetric (bond-centered):

$$\begin{aligned} G_n^{\alpha,\mathrm{off}} &= \textstyle\sum_{j=0}^J\ \alpha^{2j}\ \mathcal{G}_j\left[\psi_1\right](n-1/2) + \mathcal{E}_n^{\alpha,\mathrm{off}}, \qquad n \in \mathbb{Z}, \\ \text{where}\quad \left\|\mathcal{G}_j[\psi_1]\left(n-\tfrac{1}{2}\right)\right\|_{l^2(\mathbb{Z}_n)} &\sim \alpha^{1/2}, \qquad \left\|\mathcal{E}^{\alpha,J,\mathrm{off}}\right\|_{l^2(\mathbb{Z})} \lesssim \alpha^{2J+5/2}. \end{aligned}$$

In Definition 3.8.1, we generalize the notion of on-site symmetric (vertex-centered) and off-site symmetric (bond-centered) waves to the notion of $\sigma-$ centered waves.

Theorem 3.2.2. *[Discrete solitary waves in $\mathbb{Z}^d$] Let $\psi_{|\omega|}(x)$ denote the ground state of NLS with frequency ω; see Proposition 3.1.1. Consider the nonlinear eigenvalue problem governing real-valued discrete solitary waves of DNLS:*

$$-\alpha^2 G_n^\alpha = -(\delta^2 G^\alpha)_n - (G_n^\alpha)^3, \qquad n \in \mathbb{Z}^d, \qquad G^\alpha \in l^2(\mathbb{Z}^d). \tag{3.33}$$

Fix $0 \leqslant J < \infty$, $J \in \mathbb{N}$. There exists a constant $\alpha_0 = \alpha_0[J] > 0$ and $J > 0$ functionals $\mathcal{G}_j : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that for all $0 < \alpha < \alpha_0$, the following holds:

1. *For every $d-$ tuple $\sigma \in \{0, 1/2\}^d$ (see Table 3.2), there exists a real-valued $\sigma-$ centered solution of* (3.30)*; see Definition 2.0.2:*

$$G_n^{\alpha,\sigma} = \sum_{j=0}^{J} \alpha^{2j}\ \mathcal{G}_j\left[\psi_1\right](n-\sigma) + \ \mathcal{E}_n^{\alpha,J,\sigma}, \quad \text{where}$$
$$\|\mathcal{G}_j[\psi_1]\,(n-\sigma)\|_{l^2(\mathbb{Z}_n^d)} \sim \alpha^{1-d/2}, \qquad \|\mathcal{E}^{\alpha,J,\sigma}\|_{l^2(\mathbb{Z}^d)} \lesssim \alpha^{2J+3-d/2}. \tag{3.34}$$

   In particular, $\mathcal{G}_0\left[\psi_1\right](n) \equiv \psi_{\alpha^2}(n) = \alpha\ \psi_1(\alpha\ n)$.

2. *There are 2^d solutions, one for each $\sigma \in \{0, 1/2\}^d$. Modulo reflections, there are $d+1$ distinct centerings which are labeled in Table 3.2.*

Conjecture 3: The 2^d solutions in Theorem 3.2.2 are locally unique for each σ up to lattice translations by some $M \in \mathbb{Z}^d$. That is, if $\{J_n^{\alpha,\sigma}\}_{n\in\mathbb{Z}^d}$ is a solution to (3.33) with $\alpha < \alpha_0$, it must be centered about $\sigma + M$ for $\sigma \in \{0, 1/2\}^d$ and $M \in \mathbb{Z}^d$, and we must have $J_n^{\alpha,\sigma} = \pm G_{n+M}^{\alpha,\sigma}$.

Remark 3.2.1. *In Theorem 3.2.2, we assume $1 \leqslant d \leqslant 3$. We do not expect there to be a bifurcation from at zero frequency for dimensions $d \geqslant 4$. Indeed, the $\alpha^2 \to 0$ rescaled-limit of such bifurcating states is the solitary standing wave solution of continuum NLS, satisfying $-\Delta u + u - u^3 = 0$. A well-known argument [Sulem and Sulem, 1999] based on "Pohozaev" / virial identities shows that the equation $-\Delta u + u - u^p = 0$ has $H^1(\mathbb{R}^d)$ solutions only if $p < (d+2)/(d-2)$. For the cubic case, $p = 3$, this implies $d \leqslant 3$; see, for example, [Strauss, 1977; Sulem and Sulem, 1999]. Therefore, there can be no bifurcation for $d \geqslant 4$.*

d	$\sigma \in \mathbb{Z}^d/2$	**Centering**
1	0	Vertex-centered (On-site)
	1/2	Bond-centered (Off-site)
2	(0, 0)	Vertex-centered (On-site)
	(0, 1/2)	Bond-centered (Off-site)
	(1/2, 0)	Bond-centered (Off-site)
	(1/2, 1/2)	Cell-centered (Off-site)
3	(0, 0, 0)	Vertex-centered (On-site)
	(0, 0, 1/2)	Bond-centered (Off-site)
	(0, 1/2, 0)	Bond-centered (Off-site)
	(1/2, 0, 0)	Bond-centered (Off-site)
	(0, 1/2, 1/2)	Face-centered (Off-site)
	(1/2, 0, 1/2)	Face-centered (Off-site)
	(1/2, 1/2, 0)	Face-centered (Off-site)
	(1/2, 1/2, 1/2)	Cell-centered (Off-site)

Table 3.1: Centering of discrete solitions in dimensions $d = 1,\ 2,\ 3$.

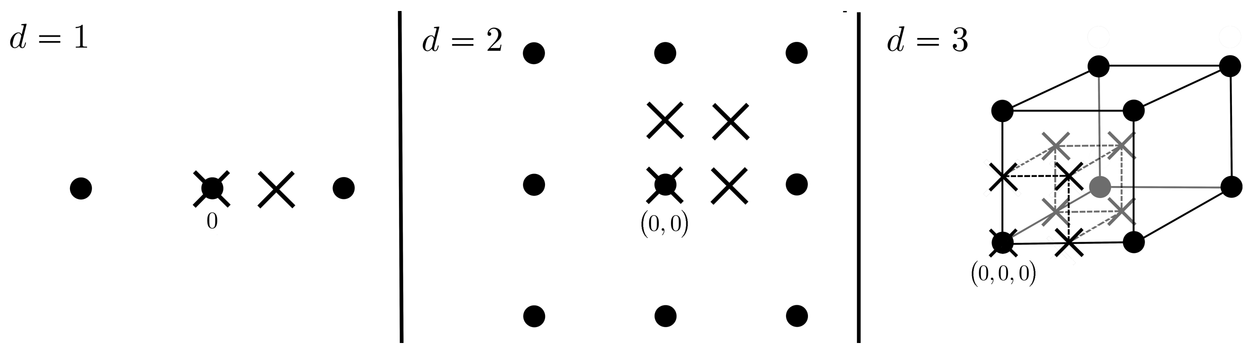

Figure 3.1: Centerings of solitary standing waves in dimensions $d = 1, 2, 3$.

Theorem 3.2.3. *[Exponential smallness of Peierls-Nabarro barrier] Let $\sigma_1, \sigma_2 \in \{0, 1/2\}^d$. There exist constants $\alpha_0 > 0, C$ and $D > 0$ such that for all $0 < \alpha < \alpha_0$,*

$$\left|\mathcal{N}[G^{\alpha,\sigma_1}] - \mathcal{N}[G^{\alpha,\sigma_2}]\right| = \left|\|G^{\alpha,\sigma_1}\|^2_{l^2(\mathbb{Z}^d)} - \|G^{\alpha,\sigma_2}\|^2_{l^2(\mathbb{Z}^d)}\right| \leqslant D\ \alpha^{2-d}\ e^{-C/\alpha},$$

$$\text{and} \qquad \left|\mathcal{H}[G^{\alpha,\sigma_1}] - \mathcal{H}[G^{\alpha,\sigma_2}]\right| \leqslant D\ \alpha^{2-d}\ e^{-C/\alpha}. \tag{3.35}$$

3.2.1 Numerical Results and Connections with Stability

Let $L_+^{\text{disc}} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ be

$$L_+^{\text{disc}} \equiv \alpha^2 - (\delta^2 \cdot)_n - 3\left(G_n^{\alpha,\sigma}\right)^2, \quad n \in \mathbb{Z}^d, \tag{3.36}$$

the linearized DNLS operator (the linearized continuum NLS operator is denoted by L_+; see Proposition 3.1.3), and $\mathcal{N}$ and $\mathcal{H}$ be as given in (3.12) and (3.13) respectively. Known results in stability theory [Grillakis et al., 1987; Weinstein, 1986] provide two sufficient conditions for the stability of the solitary wave solution $G^{\alpha,\sigma}$ are given by

Condition 1: L_+^{disc} has one negative eigenvalue.

Condition 2: The following condition holds:

$$\frac{d}{d\alpha}\mathcal{N}\left[G^{\alpha,\sigma}\right] = \frac{d}{d\alpha}\|G^{\alpha,\sigma}\|_{l^2(\mathbb{Z})} = -\frac{1}{\alpha^2}\frac{d}{d\alpha}\mathcal{H}\left[G^{\alpha,\sigma}\right] > 0. \tag{3.37}$$

Condition 1 in particular follows if $G^{\alpha,\sigma}$ is a so-called nonlinear ground state of DNLS; one such characterization of the ground state is the minimizer of the variational problem:

$$\text{Minimize } \mathcal{H}\left[f\right] \text{ subject to } \mathcal{N}\left[f\right] = \|f\|^2_{l^2(\mathbb{Z}^d)} = \nu > 0 \text{ fixed.} \tag{3.38}$$

The minimizer of (3.38) necessarily solves time-independent DNLS (3.33) with Lagrange multiplier $\alpha(\nu)^2$ [Weinstein, 1999]. Furthermore, one may easily observe that the minimizer is strictly non-negative and real-valued up to a complex phase by replacing it with its complex magnitude; it is therefore natural to expect one of the solitary wave solutions of Theorem 3.2.2 to be the ground state. Numerical results such as those in Figure 3.1 and analogous simulations for $d = 2, 3$ show that traveling pulses of time-dependent DNLS tend to settle to vertex-centered time-periodic solitary waves. Thus, for certain values of $\alpha > 0$, we expect the vertex-centered (on-site) states of DNLS to satisfy Conditions 1 and 2 above (and in particular, to be the ground state solution to (3.38)). This is further supported by Figures 3.2-3.4 below.

Figures 3.2-3.4 display the numerically computed values of $\mathcal{N}[G^{\alpha,\sigma}]$ and $\mathcal{H}[G^{\alpha,\sigma}]$ as functions of $\alpha > 0$ along with $\mathcal{H}[G^{\alpha,\sigma}]$ as a function of $\mathcal{N}[G^{\alpha,\sigma}]$ (with $\alpha > 0$ as a parameter) for dimension $d = 1, 2, 3$. Observe that in the limit $\alpha \to 0$, the squared l^2 norm ($\mathcal{N}$) of the vertex-, bond-, cell-, and face-centered solutions all approach the value $\|\psi_{\alpha^2}\|^2_{L^2(\mathbb{R}^d)} \sim \alpha^{2-d}$, at a rate given by

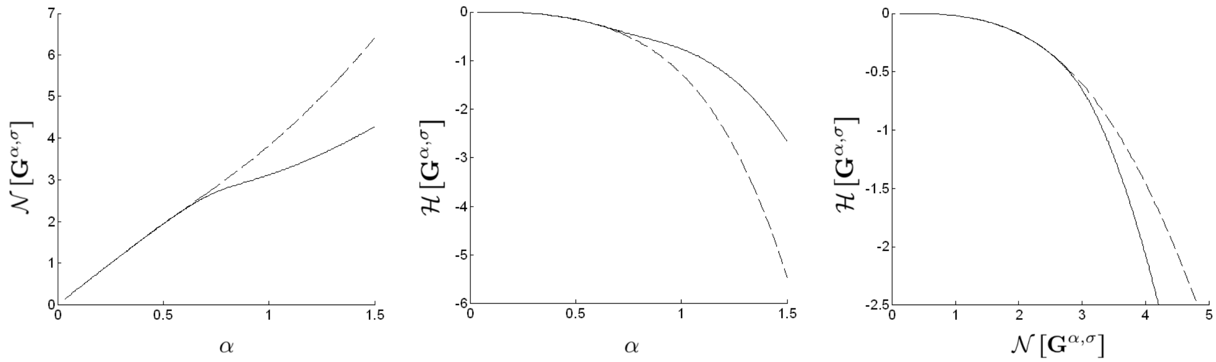

Figure 3.2: "Energy" surfaces for vertex-centered (solid line) and bond-centered (dashed line) solitary waves in $d = 1$.

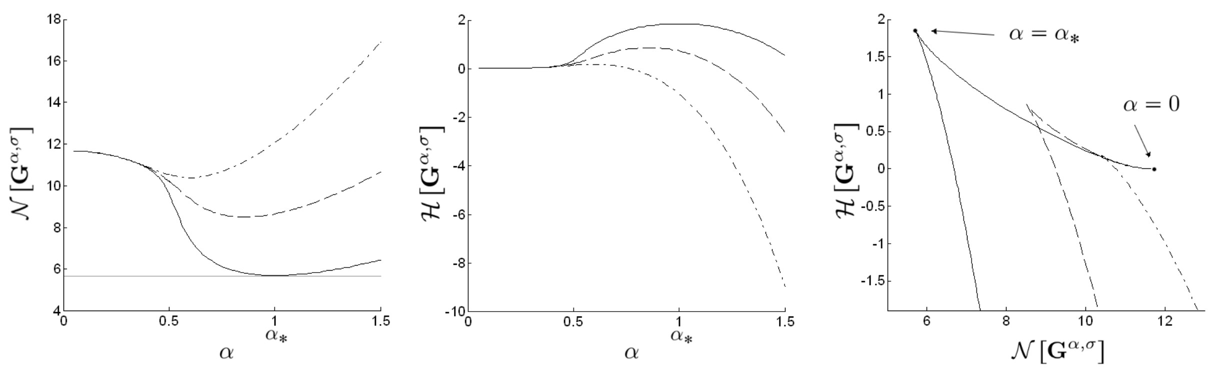

Figure 3.3: "Energy" curves for vertex-centered (solid line), bond-centered (long dashed line), and cell-centered (double dashed line) solitary waves in $d = 2$. In the right-hand plot, the solid line extends beneath the dashed lines indefinitely; for fixed $\mathcal{N}$, there is always a vertex-centered state with lower $\mathcal{H}$ than the other two states. In the first two plots, at $\alpha = \alpha_*$ the vertex-centered curves have critical points. The gray horizontal line indicates the value of $\mathcal{N}$ corresponding to the excitation threshold, which is attained for $\alpha = \alpha_*$ and below which no bound states exist [Weinstein, 1999].

$\|\mathcal{E}^{\alpha,J=1,\sigma}\|^2_{l^2(\mathbb{Z}^d)} \lesssim \alpha^{6-d}$, which is consistent with Theorem 3.2.1. Furthermore, the l^2 norms and Hamiltonians of the vertex-, bond-, cell-, and face-centered solutions are exponentially close to each other, consistent with Theorem 3.2.3.

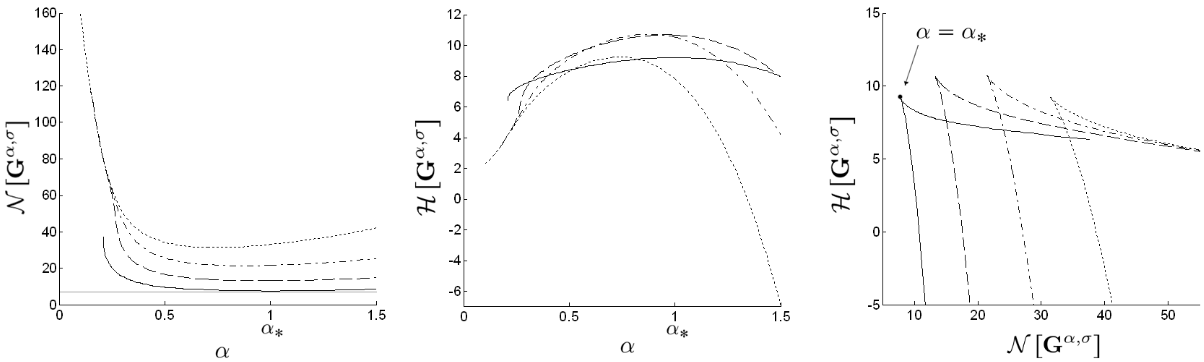

Figure 3.4: "Energy" curves for vertex-centered (solid line), bond-centered (long dashed line), face-centered (double dashed line), and cell-centered (short dashed line) solitary waves in $d = 3$. In the right-hand plot, the solid line extends beneath the dashed lines indefinitely; for fixed $\mathcal{N}$, there is always a vertex-centered state with lower $\mathcal{H}$ than the other two states. At $\alpha = \alpha_*$ the vertex-centered curves have critical points in the first two plots. In the right hand plot, as $\alpha \to 0$, the right hand curves approach the point $(\mathcal{N}, \mathcal{H}) = (\infty, 0)$. The gray horizontal line indicates the value of $\mathcal{N}$ corresponding to the excitation threshold, which is attained for $\alpha = \alpha_*$ and below which no bound states exist [Weinstein, 1999].

For $d = 1$, Figure 3.2 suggests that the vertex-centered (on-site) solitary wave ($\sigma = 0$) satisfies Conditions 1 and 2; of the two positive solitary wave solutions, it is the minimizer of $\mathcal{H}$ subject to fixed $\mathcal{N}$. Furthermore, for all $\alpha > 0$, $\mathcal{N}\left[G^{\alpha,\sigma=0}\right]$ is monotonically increasing while $\mathcal{H}\left[G^{\alpha,\sigma=0}\right]$ is monotonically decreasing;

$$\text{for all} \ \ \alpha > 0, \ \mathcal{N}\left[G^{\alpha,\sigma=0}\right] = -\frac{1}{\alpha^2}\mathcal{H}\left[G^{\alpha,\sigma=0}\right] > 0. \tag{3.39}$$

For $d = 2$ and $d = 3$, there exists a minimum value of $\mathcal{N}\left[G^{\alpha,\sigma}\right] = \|G^{\alpha,\sigma}\|^2_{l^2(\mathbb{Z})^d} > 0$ such that a solution to time-independent DNLS (3.33) exists. This is consistent with the existence of "excitation thresholds," the minimum value of $\nu > 0$ such that the variational problem (3.38) has a ground state, in dimensions $d = 2, 3$ as proven in [Weinstein, 1999]. In particular, in figures 3.3-3.4, the vertex-centered (on-site, $\sigma = 0$) solutions to 3.33 for $\alpha > \alpha_*$ appear to be the ground state solutions to DNLS, where $\alpha_* \simeq 1$ is the value for which $G^{\alpha,\sigma=0}$ satisfies

$$\frac{d}{d\alpha}\mathcal{N}\left[G^{\alpha,\sigma=0}\right]\bigg|_{\alpha=\alpha_*} = -\frac{1}{\alpha_*^2}\frac{d}{d\alpha}\mathcal{H}\left[G^{\alpha,\sigma=0}\right]\bigg|_{\alpha=\alpha_*} = 0, \tag{3.40}$$

above which for $\alpha > \alpha_*$ we have

$$\frac{d}{d\alpha}\mathcal{N}\left[G^{\alpha,\sigma=0}\right] = -\frac{1}{\alpha^2}\frac{d}{d\alpha}\mathcal{H}\left[G^{\alpha,\sigma=0}\right] > 0. \tag{3.41}$$

Thus, for $\alpha > \alpha_*$, we expect $G^{\alpha,\sigma=0}$ to satisfy Conditions 1 and 2 above and thus be stable.

Remark 3.2.2. *The vertex-centered solitary waves ($\sigma = 0$) given in Theorem 3.2.2 are shown to exist only for $0 < \alpha < \alpha_0 < \alpha_*$. According to Figures 3.2-3.3, in dimensions $d = 2, 3$ they will satisfy*

$$\frac{d}{d\alpha}\mathcal{N}\left[G^{\alpha,\sigma=0}\right] = -\frac{1}{\alpha^2}\frac{d}{d\alpha}\mathcal{H}\left[G^{\alpha,\sigma=0}\right] < 0, \tag{3.42}$$

and in turn we expect them to be unstable. This is in contrast with $d = 1$ where, as stated above, we expect the on-site solutions of Theorem 3.2.1 to be stable for any $\alpha > 0$.

3.3 Beginning of the Proof of Theorem 3.2.1; Formulation of DNLS in Fourier Space for $d = 1$

For ease of presentation, we focus initially on the one-dimensional case ($d = 1$). We adapt the current discussion to dimensions $d = 2, 3$ (Theorem 3.2.2) in Section 3.8.

Applying the discrete Fourier transform (2.1) to equation (3.30), governing $G = G^\alpha$, we obtain an equivalent equation for the discrete Fourier transform, $\widehat{G}(q) = \widehat{G^\alpha}(q)$:

$$\begin{aligned} \widehat{DNLS}[\widehat{G}](q) \ &\equiv \left[\alpha^2 + M(q)\right]\widehat{G}(q) - \left(\frac{1}{2\pi}\right)^2 \widehat{G} *_1 \widehat{G} *_1 \widehat{G}(q) = 0, \\ \widehat{G}(q + 2\pi) &= \widehat{G}(q), \end{aligned} \tag{3.43}$$

where we recall the definition of the convolution $f *_1 g$ on $\mathcal{B} = \mathcal{B}_1$ in (2.5) and its properties (Appendix C). Here, $M(q)$ denotes the (discrete) Fourier symbol of the 1-dimensional discrete Laplacian with unit lattice-spacing:

$$M(q)\ \widehat{G}(q) \ = \ \widehat{(\delta^2 G)}(q) = 4\sin^2(q/2)\ \widehat{G}(q)\ . \tag{3.44}$$

By Proposition 2.0.2 we have that

1. if G is onsite symmetric, then $\widehat{G}(q) = \widehat{K}(q)$, where $\widehat{K}(q)$ is real and symmetric, and

2. if G is offsite symmetric, then $\widehat{G}(q) = e^{-iq/2}\widehat{K}(q)$, where $\widehat{K}(q)$ is real and symmetric.

We therefore seek $\widehat{G}(q)$ in the form

$$\widehat{G^\sigma}(q) \; = \; e^{-i\sigma q} \; \widehat{K^\sigma}(q), \quad \sigma = 0, 1/2 \tag{3.45}$$

$$\widehat{K^\sigma}(q) = \widehat{K^\sigma}(-q), \quad \overline{\widehat{K^\sigma}(q)} = \widehat{K^\sigma}(q) \tag{3.46}$$

Substitution of (3.45) into (3.43) yields

$$\left[\alpha^2 + M(q)\right] \widehat{K^\sigma}(q) - \left(\frac{1}{2\pi}\right)^2 \widehat{K^\sigma} *_1 \widehat{K^\sigma} *_1 \widehat{K^\sigma}(q) = 0, \quad q \in \mathbb{R}, \tag{3.47}$$

$$\widehat{K^\sigma}(q + 2\pi) = e^{2\pi i\sigma} \; \widehat{K^\sigma}(q), \qquad \sigma = 0, 1/2 \; . \tag{3.48}$$

The "Bloch" phase factor, $e^{2\pi i\sigma}$ (equal to ± 1) encodes the on-site and off-site cases.

Lemma 3.3.1. *Let $A(q)$, defined on $\mathbb{R}$, be $2\pi\sigma-$ pseudo-periodic, i.e. $A(q + 2\pi) = e^{2\pi i\sigma} \; A(q)$. Then, $A(q)$ is completely determined by its values on $\mathcal{B} = [-\pi, \pi]$ and has the representation:*

$$A(q) = \sum_{m\in\mathbb{Z}} \chi_{\mathcal{B}}(q - 2m\pi)A(q - 2m\pi)e^{2\pi i m\sigma}. \tag{3.49}$$

Proof of Lemma 3.3.1: Since $A(q) = e^{2\pi i\sigma}A(q - 2\pi)$, we have $A(q) = e^{2\pi i m\sigma}A(q - 2\pi m)$ for all $m \in \mathbb{Z}$. Hence,

$$A(q) = A(q) \sum_{m\in\mathbb{Z}} \chi_{\mathcal{B}}(q - 2m\pi) = \sum_{m\in\mathbb{Z}} \chi_{\mathcal{B}}(q - 2m\pi)A(q - 2m\pi)e^{2\pi i m\sigma}. \qquad \square$$

By Lemma 3.3.1 we may express $\widehat{K^\sigma}(q)$, for any $q \in \mathbb{R}$, explicitly in terms of its values on $q \in \mathcal{B}$. In particular, we set

$$\widehat{K^\sigma}(q) = \widehat{\phi^\sigma}(q), \qquad q \in \mathcal{B}. \tag{3.50}$$

Extending (3.50) to $q \in \mathbb{R}$, we have:

$$\widehat{K^\sigma}(q) \equiv \sum_{m\in\mathbb{Z}} \chi_{\mathcal{B}}(q - 2m\pi) \; \widehat{\phi^\sigma}(q - 2m\pi) \; e^{2m\pi i\sigma}, \qquad \text{and} \qquad \widehat{G^\sigma}(q) = e^{-i\sigma q} \; \widehat{K^\sigma}(q). \tag{3.51}$$

Note that $\widehat{\phi^\sigma}(q - 2m\pi) = \chi_{\mathcal{B}}(q - 2m\pi) \; \widehat{\phi^\sigma}(q - 2m\pi)$ is supported on $\{q : q \in 2m\pi + \mathcal{B} = [(2m - 1)\pi, (2m + 1)\pi]\}$. Therefore,

$$\chi_{\mathcal{B}}(q) \; \widehat{K^\sigma}(q) = \widehat{\phi^\sigma}(q), \qquad \text{and} \qquad \chi_{\mathcal{B}}(q) \; \widehat{G^\sigma}(q) = e^{-iq\sigma} \; \widehat{\phi^\sigma}(q). \tag{3.52}$$

Equations (3.51) encode the required $2\pi-$ periodicity of $\widehat{G^\sigma}(q)$ and the $2\pi\sigma-$ pseudo-periodicity of $\widehat{K^\sigma}(q)$ on all $\mathbb{R}$,. Furthermore,

$$\widehat{G^\sigma}(q) \text{ and } \widehat{K^\sigma}(q) \text{ are completely specified by } \widehat{\phi^\sigma}(q) \text{ for } q \in \mathcal{B}. \tag{3.53}$$

With a view toward deriving an equation from (3.47) determining $\widehat{\phi^\sigma}(q)$ for $q \in \mathcal{B}$, we require a lemma which facilitates simplification of the convolution terms in (3.47).

Lemma 3.3.2. *Let $\widehat{A}(q), \widehat{B}(q), \widehat{C}(q)$ be bounded $2\pi\sigma-$ pseudo-periodic functions of $q \in \mathbb{R}$. Then*

$$\chi_{\mathcal{B}}(q)\widehat{A} *_1 \widehat{B} *_1 \widehat{C}(q) = \chi_{\mathcal{B}}(q) \sum_{m=-1}^{1} e^{2\pi i m\sigma} \widehat{A} *_1 \left[\widehat{B} *_1 \left(\chi_{\mathcal{B}}\widehat{C}\right)\right](q - 2m\pi). \tag{3.54}$$

Note that since $\chi_{\mathcal{B}}(q)$ is not pseudo-periodic, the convolution in (3.54) is not associative.

Proof of Lemma 3.3.2: Observe that $q, \xi, \zeta \in \mathcal{B} = [-\pi, \pi]$ implies that $q - \xi - \zeta \in [-3\pi, 3\pi]$. Therefore, $q, \xi, \zeta \in \mathcal{B}$ implies $q - \xi - \zeta - 2m\pi \in \mathcal{B} = [-\pi, \pi]$ if and only if $m \in \{-1, 0, 1\}$, and that $\chi_{\mathcal{B}}(q - \xi - \zeta - 2m\pi) = 0$ for $m \notin \{-1, 0, 1\}$. Applying Lemma 3.3.1 to $\widehat{C}(q)$ we have

$$\begin{aligned}
&\chi_{\mathcal{B}}(q)\widehat{A} *_1 \left[\widehat{B} *_1 \widehat{C}\right](q) \\
&= \chi_{\mathcal{B}}(q) \int_{\mathcal{B}} d\xi \int_{\mathcal{B}} d\zeta\ \widehat{A}(\xi)\widehat{B}(\zeta) \sum_{m\in\mathbb{Z}} \chi_{\mathcal{B}}(q - \xi - \zeta - 2m\pi)\ \widehat{C}(q - \xi - \zeta - 2m\pi)e^{2m\pi i\sigma} \\
&= \chi_{\mathcal{B}}(q) \int_{\mathcal{B}} d\xi \int_{\mathcal{B}} d\zeta\ \widehat{A}(\xi)\widehat{B}(\zeta) \sum_{m=-1}^{1} \chi_{\mathcal{B}}(q - \xi - \zeta - 2m\pi)\ \widehat{C}(q - \xi - \zeta - 2m\pi)e^{2m\pi i\sigma} \\
&= \chi_{\mathcal{B}}(q) \sum_{m=-1}^{1} e^{2m\pi i\sigma} \widehat{A} *_1 \left[\widehat{B} *_1 \left(\chi_{\mathcal{B}}\widehat{C}\right)\right](q - 2m\pi). \qquad \square
\end{aligned} \tag{3.55}$$

Applying Lemma 3.3.2 and (3.51), we have:

Proposition 3.3.1. *Equation (3.47) for $\widehat{K^\sigma}(q)$ on $q \in \mathbb{R}$ is equivalent to the following equation for the compactly supported function $\widehat{\phi^\sigma}(q) = \chi_{\mathcal{B}}(q)\ \widehat{\phi^\sigma}(q)$:*

$$\begin{aligned}
&[\alpha^2 + M(q)]\ \widehat{\phi^\sigma}(q) - \frac{\chi_{\mathcal{B}}(q)}{4\pi^2} \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma} \right)(q) \\
&\qquad - \frac{\chi_{\mathcal{B}}(q)}{4\pi^2} \sum_{m=\pm 1} e^{2m\pi i\sigma} \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma} \right)(q - 2m\pi) = 0
\end{aligned} \tag{3.56}$$

where $M(q)\widehat{G}(q) = 4\sin^2(q/2)\ \widehat{G}(q) = \widehat{\delta^2 G}(q)$; see (3.44).

Proof of Proposition 3.3.1: We project (3.47) onto $\mathcal{B}$, decompose $\widehat{K^\sigma}(q)$ with (3.51), and apply Lemma 3.3.2 to (3.47) get

$$\left[\alpha^2 + M(q)\right] \widehat{\phi^\sigma}(q) - \frac{\chi_{\mathcal{B}}}{4\pi^2} \sum_{m=-1}^{1} e^{2\pi i m \sigma} \widehat{\phi^\sigma} *_1 \left[\widehat{\phi^\sigma} *_1 \left(\chi_{\mathcal{B}} \widehat{\phi^\sigma}\right)\right](q - 2m\pi) = 0. \tag{3.57}$$

We observe that the multiplication of (3.57) by $\overline{\chi}_{\mathcal{B}}(q)$ implies <u>a-priori</u> that

$$\left[\alpha^2 + M(q)\right] \overline{\chi}_{\mathcal{B}}(Q)\, \widehat{\phi^\sigma}(q) = 0 \qquad \Longrightarrow \qquad \overline{\chi}_{\mathcal{B}}(Q)\, \widehat{\phi^\sigma}(q) = 0, \tag{3.58}$$

where we have used that $M(q) \geqslant 0$. Thus, we may write for $m = -1, 0, 1$,

$$\begin{aligned} \widehat{\phi^\sigma} *_1 \left[\widehat{\phi^\sigma} *_1 \left(\chi_{\mathcal{B}}\, \widehat{\phi^\sigma}\right)\right](q - 2m\pi) &= \left(\chi_{\mathcal{B}}\, \widehat{\phi^\sigma}\right) * \left(\chi_{\mathcal{B}}\, \widehat{\phi^\sigma}\right) * \left(\chi_{\mathcal{B}}\, \widehat{\phi^\sigma}\right)(q - 2m\pi) \\ &= \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma}\right)(q - 2m\pi), \end{aligned} \tag{3.59}$$

This completes the proof of Proposition 3.3.1. □

3.3.1 Rescaled Equation for $\widehat{\phi^\sigma}$

As discussed in Section 3.1.2, we expect that for $\alpha \ll 1$:

$$\widehat{\phi^\sigma}(q) \sim \widetilde{\psi_{\alpha^2}}(q) = \widetilde{\psi_1}\left(\frac{q}{\alpha}\right), \tag{3.60}$$

where $\widetilde{\psi_1}$ denotes the Fourier transform of the continuum NLS solitary wave. We therefore study (3.56) using rescaling which makes explicit the relation between DNLS and the continuum (NLS) limit for α small: We introduce:

$$\begin{aligned} &\text{Rescaled momentum:} && Q \equiv q/\alpha, \quad Q \in \mathcal{B}_\alpha = [-\pi/\alpha, \pi/\alpha], \\ &\text{Rescaled projection:} && \chi_{\mathcal{B}_\alpha}(Q) \equiv \chi_{\mathcal{B}}(Q\alpha) = \chi_{[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}]}(Q), \\ &\text{Rescaled wave:} && \widehat{\Phi^\sigma}(Q) \equiv \widehat{\phi^\sigma}(Q\alpha) = \widehat{\phi^\sigma}(q), \\ &\text{Rescaled discrete Fourier symbol:} && M_\alpha(Q) \equiv \frac{1}{\alpha^2} M(Q\alpha) = \frac{4}{\alpha^2} \sin^2\left(\frac{Q\alpha}{2}\right). \end{aligned} \tag{3.61}$$

The following proposition is a formulation of Proposition 3.3.1 in terms of functions of the rescaled quasi-momentum, Q:

Proposition 3.3.2. *Equation (3.47) for $\widehat{K^\sigma}(q)$ on $q \in \mathbb{R}$ is equivalent to the following equation for $\widehat{\Phi^\sigma}(Q) = \chi_{\mathcal{B}_\alpha}(q)\ \widehat{\Phi^\sigma}(Q)$, compactly supported on $\mathcal{B}_\alpha = [-\pi/\alpha, \pi/\alpha]$:*

$$\mathcal{D}^{\sigma,\alpha}[\widehat{\Phi^\sigma}](Q) \ \equiv\ [1 + M_\alpha(Q)]\ \widehat{\Phi^\sigma}(Q) - \frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2}\ \left(\ \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma}\ \right)(Q) \\ +\ R_1^\sigma[\widehat{\Phi^\sigma}](Q) = 0\ , \tag{3.62}$$

where $R_1^\sigma[\widehat{\Phi^\sigma}]$ contains the ± 1-sideband contributions:

$$R_1^\sigma[\widehat{\Phi^\sigma}](Q) \equiv -\frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2} \sum_{m=\pm 1} e^{2m\pi i\sigma}\ \left(\ \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma}\ \right)(Q - 2m\pi/\alpha), \tag{3.63}$$

and $M_\alpha(Q) = \frac{4}{\alpha^2}\sin^2(\frac{\alpha Q}{2})$; see (3.61).

To prove Proposition 3.3.2 we need to re-express the convolutions in (3.56) in terms of $\widehat{\Phi^\sigma}(Q)$. For this we use the following lemma, proved by change of variables.

Lemma 3.3.3. *Suppose that $\widehat{a}(q) = \widehat{A}(Q)$, $\widehat{b}(q) = \widehat{B}(Q)$, and $\widehat{c}(q) = \widehat{C}(Q)$, where $Q = q/\alpha$. Then*

$$\left(\widehat{a} * \widehat{b} * \widehat{c}\right)(q) = \alpha^2\ \left(\widehat{A} * \widehat{B} * \widehat{C}\right)(Q). \tag{3.64}$$

Applying the rescalings (3.61) and Lemma 3.3.3 to (3.56) and then dividing by α^2, we obtain (3.62).

Note that for small $|\alpha|$, the scaled symbol, $M_\alpha(Q) = \frac{4}{\alpha^2}\sin^2\left(\frac{Q\alpha}{2}\right)$, has the expansion in powers of α^2 for fixed $Q \in \mathbb{R}$:

$$M_\alpha(Q) = 2\sum_{j=0}^{\infty} \frac{\alpha^{2j}\ (-1)^j\ |Q|^{2j+2}}{(2j+2)!} \ =\ |Q|^2 - \frac{\alpha^2\ |Q|^4}{12} + \frac{\alpha^4\ |Q|^6}{360} + \mathcal{O}\left(\alpha^6\ |Q|^8\right). \tag{3.65}$$

Using truncations of the expansion of $M_\alpha(Q)$ we shall, for any $J = 0, 1, 2, \dots$, construct $\widehat{\Phi}^\sigma(Q)$ in the form of a finite expansion in power of α^{2j}, $j = 0, \dots, J$, with an error term of which is of order α^{2J+2} plus a corrector of higher order. For each J, the polynomial expansion in α^2 is *independent of* σ. The construction is summarized in the following:

Proposition 3.3.3. *Fix $J \geqslant 0$, $a > 1/2$, and $\sigma \in \{0, 1/2\}$. Then there exist a constant $\alpha_0 = \alpha_0[a, J, \sigma] > 0$, and J mappings $F_j : L^{2,a}_{\text{even}}(\mathbb{R}) \to L^{2,a}_{\text{even}}(\mathbb{R})$, $j = 0, \dots, J$, and a unique, real-valued* *function* $\widehat{E_J^{\alpha,\sigma}} \in L^{2,a}_{\rm even}(\mathbb{R})$ *such that for all* $0 < \alpha < \alpha_0$,

$$\widehat{\Phi^\sigma}(Q) \;=\; \chi_{\mathcal{B}_\alpha}(Q)\; \widetilde{\psi_1}(Q) \;+\; \sum_{j=1}^{J} \alpha^{2j}\; \chi_{\mathcal{B}_\alpha}(Q)\; F_j\left[\widetilde{\psi_1}\right](Q) + \widehat{E_J^{\alpha,\sigma}}(Q), \tag{3.66}$$

solves equation (3.62) *with the error bound:*

$$\left\| \widehat{E_J^{\alpha,\sigma}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2J+2} \tag{3.67}$$

$F_j,\ j \geqslant 1$, *defined in Proposition 4.7.3 below, is independent of* σ *and* α, *and*

$$\widehat{E_J^{\alpha,\sigma}}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\; \widehat{E_J^{\alpha,\sigma}}(Q). \tag{3.68}$$

Remark 3.3.1. *By Proposition 3.3.3, since the polynomial expansion in* α^2 *is completely determined by* $\widetilde{\psi_1}(Q)$, $\widehat{\Phi^\sigma}(Q)$ *is completely specified once we have contructed* $\widehat{E_J^{\alpha,\sigma}}(Q)$ *for* $Q \in \mathbb{R}$. *And, in turn by the rescalings* $Q = q/\alpha$ *and* $\widehat{\Phi^\sigma}(Q) = \widehat{\phi^\sigma}(Q\alpha) = \widehat{\phi^\sigma}(q)$, (3.53) *implies that* $\widehat{G^\sigma}(q)$ *and* $\widehat{K^\sigma}(q)$ *are completely specified by* $\widehat{E_J^{\alpha,\sigma}}(q/\alpha)$ *for* $q \in \mathbb{R}$. *Therefore, Proposition 3.3.3 completely characterizes* $\widehat{G^\sigma}(q)$.

The proof of Proposition 3.3.3 extends over sections 3.3.2 through 3.6.1. In Section 3.3.2, we formally derive and construct the functionals $F_j\left[\cdot\right]$, appearing in the expansion (3.66). We then contruct and bound the corrector $\widehat{E_J^{\alpha,\sigma}}(Q)$ in Sections 3.4 and 3.5 by using a Lyapunov-Schmidt reduction strategy.

3.3.2 Formal Asymptotic Expansion for $\widehat{\Phi^\sigma}$, the Solution of (3.62)

A solution to (3.62)-(3.63), $\widehat{\Phi}$, is compactly supported on $\mathcal{B}_\alpha$. Our approach to solving (3.62)-(3.63) is to first construct formal solution of the related equation:

$$\left[1 + M_\alpha(Q)\right]\; F(Q) - \frac{1}{4\pi^2}\left(F * F * F\right)(Q) = 0\ . \tag{3.69}$$

Each term in this power series, $F_j^\alpha(Q)$, will have support on all $\mathbb{R}$ and can be shown to decay exponentially as $|Q| \to \infty$. The deviation of (4.172) from (3.62) are terms of the form:

$$-\Big(1 - \chi_{\mathcal{B}_\alpha}(Q)\Big)\left[1 + M_\alpha(Q)\right] F(Q) + \frac{1}{4\pi^2}\ \Big(1 - \chi_{\mathcal{B}_\alpha}(Q)\Big)\left(F^\alpha * F^\alpha * F^\alpha\right)(Q) + R_1^{\alpha,\sigma}[F^\alpha](Q), \tag{3.70}$$

whose norms can be shown to be beyond all polynomial orders in α as $\alpha \to 0$, *i.e.* $\mathcal{O}(\alpha^m)$, for all $m \geqslant 1$, in $L^{2,a}(\mathbb{R}; dQ)$ with $a > 1/2$. Therefore, we expect that if $\widehat{\Phi^{\alpha,\sigma}}(Q)$ is a solution of (3.62), then the function $\chi_{\mathcal{B}_\alpha}(Q)\ F^\alpha(Q)$, where F^α solves (4.172) formally solves (3.62) with an error which is beyond all polynomial orders in α^2, *i.e.*

$$\left\| \mathcal{D}^{\alpha,\sigma} \left[\chi_{\mathcal{B}_\alpha}\ F^\alpha \right] \right\|_{L^{2,a}(\mathbb{R};dQ)} = \mathcal{O}(\alpha^\infty) \tag{3.71}$$

Using the power series expansion (3.65), we shall construct a formal power series expansion in α^2 for $F^\alpha(Q)$:

$$F^\alpha(Q) = \sum_{j=0}^{\infty} \alpha^{2j}\ F_j(Q), \tag{3.72}$$

The sequence of truncated sums, $S_J^\sigma = \sum_{j=0}^{\infty} \alpha^{2j}\ F_j$, is a sequence of approximate solutions with decreasing residuals: $\|\mathcal{D}^{\alpha,\sigma}\left[S_J^\sigma\right]\|_{L^{2,a}} = \mathcal{O}(\alpha^{2J+2})$. In the coming sections, construct a solution $\Phi^{\sigma,\alpha} = S_J^\sigma + E_J^{\sigma,\alpha}$ by a Lyapunov-Schmidt procedure.

We now turn to the construction of the terms in series (4.173). Substitution of (4.173) into (4.174), we obtain a hierarchy of equations for F_j.

$\mathcal{O}(\alpha^0)$ **equation** :

$$\left[1 + |Q|^2\right]\ F_0(Q) - \left(\frac{1}{2\pi}\right)^2\ F_0 * F_0 * F_0(Q) = 0. \tag{3.73}$$

Equation (4.176) is the Fourier transform of continuum NLS (3.25). Denote by

$$F_0(Q) = \widetilde{\psi_1}(Q) = \mathcal{F}_C\left[\psi_1\right](Q), \tag{3.74}$$

where $\psi_1(x)$ is the unique (up to translation) positive and decaying solution of NLS. $\psi_1(x)$ is real-valued and radially symmetric about some point, which we take to be $x = 0$. By Proposition 3.1.2 there exists $C_0 > 0$ such that

$$e^{C_0|Q|}\ \widetilde{\psi_1}(Q) \in L^{2,a}(\mathbb{R})\ . \tag{3.75}$$

At each order in α^2, we shall derive an equation for $F_j(Q)$ of the general form:

$$\widetilde{L_+}\ F^\sharp(Q) = F^\flat(Q). \tag{3.76}$$

It is important for us to understand how decay properties $F^\flat(Q)$ propagate to the solution $F^\sharp(Q)$.

Proposition 3.3.4. *Fix $a > 1/2$. Suppose that $F^\flat(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ)$ and that there exists a constant $C_0 \geqslant C_\flat > 0$ such that $e^{C_\flat |Q|} F^\flat(Q) \in L^{2,a}(\mathbb{R}; dQ)$, where C_0 is defined in* (4.178). *Then there exists a solution of* (4.179), *$F^\sharp(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ)$. Furthermore, we have $e^{C_\flat |Q|} F^\sharp(Q) \in L^{2,a}(\mathbb{R}; dQ)$.*

Since $F^\flat(Q)$ is even it is $L^2(\mathbb{R}; dQ)$ orthogonal to the kernel of $\widetilde{L_+} = \text{span}\{Q\widetilde{\psi_1}(Q)\}$ (see Proposition 3.1.3). Therefore, $F^\sharp = \left(\widetilde{L_+}\right)^{-1} F^\flat \in L^{2,a+2}_{\text{even}}(\mathbb{R}; dQ)$; see Proposition 3.1.3. A detailed proof that the exponential decay rate is preserved is deferred until the end of this section.

We now turn to the hierarchy of equations at order α^{2j}, beginning with $j = 1$. We find

$$\left[1 + |Q|^2\right] \ F_1(Q) - 3 \ \frac{1}{4\pi^2} \ \widetilde{\psi_1} * \widetilde{\psi_1} * F_1(Q) = \frac{|Q|^4}{12} \ \widetilde{\psi_1}(Q), \tag{3.77}$$

or

$\mathcal{O}(\alpha^2)$ **equation** :
$$\widetilde{L_+} \ F_1(Q) = \frac{|Q|^4}{12} \ \widetilde{\psi_1}(Q). \tag{3.78}$$

Here, $L_+ = -\partial_x^2 + 1 - 3\psi_1^2(x)$, is the linearization of the continuum NLS operator about $\psi_1(x)$. By Proposition 3.1.3, $F_1(Q) = \left(\widetilde{L_+}\right)^{-1} \left(\frac{|Q|^4}{12} \ \widetilde{\psi_1}(Q)\right) \in L^{2,a}_{\text{even}}(\mathbb{R})$. Since (4.181) has real-valued forcing, F_1 is real-valued. Let $C_\flat = 3C_0/4$ and note $\|e^{C_\flat|Q|}|Q|^4\widetilde{\psi_1}\|_{L^{2,a}} \lesssim \|e^{C_0|Q|}\widetilde{\psi_1}\|_{L^{2,a}}$ Therefore, $e^{\frac{3C_0}{4}|Q|}F_1(Q) \in L^{2,a}(\mathbb{R}; dQ)$ for $a > 1/2$.

We now proceed to inductively construct and bound the sequence $F_j(Q), \ j \geqslant 1$ using Proposition 4.7.2 and the following two lemmata.

Lemma 3.3.4. *Fix $a > 1/2$. Suppose that $\widetilde{f_1}, \widetilde{f_2} \in L^{2,a}_{\text{even}}(\mathbb{R})$. Then $\widetilde{f_1} * \widetilde{f_2} \in L^{2,a}_{\text{even}}(\mathbb{R})$. Suppose further that there exist $c_1, c_2 > 0$ such that $e^{c_1|Q|}\widetilde{f_1}(Q) \in L^{2,a}(\mathbb{R})$, $e^{c_2|Q|}\widetilde{f_2}(Q) \in L^{2,a}(\mathbb{R}_Q)$. Then for $c_3 = \min(c_1, c_2)$, we have $e^{c_3|Q|} \ \widetilde{f_1} * \widetilde{f_2}(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}_Q)$ and*

$$\left\| \ e^{c_3|Q|} \ (\widetilde{f_1} * \widetilde{f_2})(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \left\| \ e^{c_3|Q|} \ \widetilde{f_1}(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)} \left\| \ e^{c_3|Q|} \ \widetilde{f_2}(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{3.79}$$

Proof of 4.7.2: Clearly, $\widetilde{f_1} * \widetilde{f_2} \in L^{2,a}(\mathbb{R})$ by (2.20) (i.e. $\|\widetilde{f_1} * \widetilde{f_2}\|_{L^{2,a}(\mathbb{R})} \lesssim \|\widetilde{f_1}\|_{L^{2,a}(\mathbb{R})} \ \|\widetilde{f_2}\|_{L^{2,a}(\mathbb{R})}$). To see that $\widetilde{f_1} * \widetilde{f_2}$ is even for $\widetilde{f_1}$ and $\widetilde{f_2}$ even, we write

$$\begin{aligned} \widetilde{f_1} * \widetilde{f_2}(Q) &= \int_{\mathbb{R}} \ \widetilde{f_1}(\xi) \ \widetilde{f_2}(Q - \xi) \ d\xi = \int_{\mathbb{R}} \ \widetilde{f_1}(-\xi) \ \widetilde{f_2}(-Q + \xi) \ d\xi \\ &= \int_{\mathbb{R}} \ \widetilde{f_1}(\xi) \ \widetilde{f_2}(-Q - \xi) \ d\xi = \widetilde{f_1} * \widetilde{f_2}(-Q), \end{aligned} \tag{3.80}$$

where we changed variables $\xi \mapsto -\xi$ in the last step.

Next, observe that for any $\xi \in \mathbb{R}$, and $c_1, c_2 > 0$, and $c_3 = \min(c_1, c_2)$,

$$c_3|Q| \leqslant c_2|Q-\xi| + c_1|\xi| \qquad \Longrightarrow \qquad e^{c_3|Q|} \leqslant e^{c_2|Q-\xi|}\, e^{c_1|\xi|}, \tag{3.81}$$

and in turn

$$\begin{aligned} \left| e^{c_3|Q|}\ \widetilde{f}_1 * \widetilde{f}_2(Q) \right| &\leqslant \int_{\mathbb{R}} e^{c_3|Q|}\ |\widetilde{f}_1(\xi)|\ |\widetilde{f}_2(Q-\xi)|\ d\xi \\ \leqslant \int_{\mathbb{R}} e^{c_1|\xi|}\ |\widetilde{f}_1(\xi)|\ e^{c_2|Q-\xi|}\ |\widetilde{f}_2(Q-\xi)|\ d\xi &= \left(e^{c_1|\cdot|}|\widetilde{f}_1|\right) * \left(e^{c_2|\cdot|}|\widetilde{f}_2|\right)(Q). \end{aligned} \tag{3.82}$$

Applying (2.20) again implies that $e^{c_3|Q|}\ \widetilde{f}_1 * \widetilde{f}_2(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}_Q)$ This completes the proof of 4.7.2.
□

Lemma 3.3.5. *Fix $a > 1/2$ and $k \in \mathbb{N}$. Suppose that $\widetilde{f} \in L^{2,a}_{\text{even}}(\mathbb{R})$ and that there exists $c_1 > 0$ such that $e^{c_1|Q|}\widetilde{f}(Q) \in L^{2,a}(\mathbb{R})$. Then $|Q|^{2k}\ \widetilde{f} \in L^{2,a}_{\text{even}}(\mathbb{R})$ and for any $0 < c_2 < c_1$, we have*

$$\left\| e^{c_2|Q|}|Q|^{2k}\widetilde{f} \right\|_{L^{2,a}(\mathbb{R})} \leqslant \left(\frac{2k}{c_1 - c_2}\right)^{2k}\ e^{-2k}\ \left\|\ e^{c_1|Q|}\ \widetilde{f}(Q)\ \right\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{3.83}$$

Proof of 4.7.3: Observe that on $|Q| \in [0, \infty)$, the continuously differentiable positive function $e^{-(c_1-c_2)|Q|}|Q|^{2k}$ has a single minimum at $|Q| = 0$ and a single maximum when

$$\begin{aligned} & e^{-(c_1-c_2)|Q|}|Q|^{2k-1}\left((c_2 - c_1)|Q| + 2k\right) = 0, \qquad |Q| > 0 \\ \Longleftrightarrow \quad & |Q_{\max}| = \frac{2k}{c_1 - c_2}, \end{aligned} \tag{3.84}$$

where the function takes the value

$$e^{-(c_1-c_2)|Q_{\max}|}|Q_{\max}|^{2k} = \left(\frac{2k}{c_1 - c_2}\right)^{2k}\ e^{-2k} \tag{3.85}$$

Therefore,

$$\begin{aligned} \left\| e^{c_2|Q|}|Q|^{2k}\widetilde{f} \right\|_{L^{2,a}(\mathbb{R})} &= \left\| e^{c_1|Q|}e^{-(c_1-c_2)|Q|}|Q|^{2k}\widetilde{f} \right\|_{L^{2,a}(\mathbb{R})} \\ &\leqslant \left(\frac{2k}{c_1 - c_2}\right)^{2k}\ e^{-2k}\left\| e^{c_1|Q|}|Q|^{2k}\widetilde{f} \right\|_{L^{2,a}(\mathbb{R})}. \end{aligned} \tag{3.86}$$

This completes the proof of 4.7.3. □

We now proceed to inductively construct and bound the sequence $F_j(Q),\ j \geqslant 1$ of terms in the asymptotic expansion (4.173) using Proposition 4.7.2 and Lemmata 4.7.2 and 4.7.3.

Proposition 3.3.5. *Let $j \geqslant 1$. The equation for F_j at order $\mathcal{O}(\alpha^{2j})$, independent of α and σ, is given by*

$$\mathcal{O}(\alpha^{2j})\ \textbf{equation}: \quad \widetilde{L_+}\ F_j(Q) = 2\sum_{k=0}^{j-1} \frac{(-1)^{k-j+1}\ |Q|^{2j-2k+2}\ F_k(Q)}{(2j-2k+2)!}, \\ + \frac{1}{(2\pi)^2} \sum_{\substack{k+l+z=j \\ 0\leqslant k,l,z<j}} F_k * F_l * F_z(Q) \equiv H_j\left[F_0, \ldots, F_{j-1}\right](Q), \tag{3.87}$$

and has the unique solution

$$F_j = \left(\widetilde{L_+}\right)^{-1} \left(H_j\left[F_0, \ldots, F_{j-1}\right]\right) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ). \tag{3.88}$$

Furthermore, F_j is real-valued and $e^{C_j|Q|}\ F_j(Q) \in L^{2,a}(\mathbb{R}; dQ)$, where $C_j \equiv C_0\left(\frac{1}{2} + \frac{1}{2^{j+1}}\right) \geqslant \frac{C_0}{2}$ and $C_0 > 0$ is as in (4.178).

Proof of Proposition 4.7.3: We induct to solve at each order in α^{2j}. Let $F_0(Q) \equiv \widetilde{\psi_1}(Q)$, which solves (4.176), is real-valued, and satisfies $e^{C_0|Q|}F_0(Q) \in L^{2,a}(\mathbb{R}; dQ)$. Fix $m \geqslant 2$ and assume that for $1 \leqslant j \leqslant m-1$, $F_j(Q) \in L^{2,a}_{\text{even}}(\mathbb{R})$ satisfies (4.184) and is real-valued. Furthermore, assume that

$$e^{C_j|Q|}F_j(Q) \in L^{2,a}(\mathbb{R}; dQ), \qquad C_j \equiv C_0\left(\frac{1}{2} + \frac{1}{2^{j+1}}\right) \geqslant \frac{C_0}{2}. \tag{3.89}$$

We have already proven above that these inductive hypotheses hold for $j = 1$. We expand

$$M_\alpha(Q) = 2\sum_{j=0}^{\infty} \frac{\alpha^{2j}\ (-1)^j\ |Q|^{2j+2}}{(2j+2)!}, \qquad F^\alpha(Q) = \sum_{j=0}^{\infty} \alpha^{2j}\ F_j(Q), \tag{3.90}$$

and substitute into (4.172). Using (4.176) for $F_0(Q) \equiv \widetilde{\psi_1}(Q)$ we obtain

$$\sum_{j=1}^{\infty} \alpha^{2j}\ \widetilde{L_+}\ F_j(Q) = 2\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{2j+2k}(-1)^{j+1}|Q|^{2j+2}}{(2j+2)!}F_k(Q) + \sum_{j=1}^{\infty} \frac{\alpha^{2j}}{(2\pi)^2} \sum_{\substack{k+l+z=j \\ 0\leqslant k,l,z<j}} F_k * F_l * F_m(Q)$$

$$= \sum_{j=1}^{\infty} \alpha^{2j}\left(2\sum_{k=0}^{j-1} \frac{(-1)^{k-j+1}\ |Q|^{2j-2k+2}}{(2j-2k+2)!}\ F_k(Q) + \frac{1}{4\pi^2} \sum_{\substack{k+l+z=j \\ 0\leqslant k,l,z<j}} F_k * F_l * F_z(Q)\right). \tag{3.91}$$

Applying the inductive hypothesis (4.184) for $1 \leqslant j \leqslant m-1$ and dividing by α^{2m}, (4.188) becomes

$$
\begin{aligned}
&\widetilde{L_+}\ F_m(Q) + \alpha^2 \left[\sum_{j=m+1}^{\infty} \alpha^{2j-2(m+1)}\ \widetilde{L_+}\ F_j(Q) \right] \\
&= 2 \sum_{k=0}^{m-1} \frac{(-1)^{k-m+1}\ |Q|^{2m-2k+2}\ F_k(Q)}{(2m-2k+2)!} + \frac{1}{(2\pi)^2} \sum_{\substack{k+l+z=m \\ 0\leqslant k,l,z<m}} F_k * F_l * F_z(Q) \\
&\quad + \alpha^2 \left[\sum_{j=m+1}^{\infty} \alpha^{2j-2(m+1)} \left(2 \sum_{k=0}^{j-1} \frac{(-1)^{k-j+1}\ |Q|^{2j-2k+2}}{(2j-2k+2)!}\ F_k(Q) + \frac{1}{4\pi^2} \sum_{\substack{k+l+z=j \\ 0\leqslant k,l,z<j}} F_k * F_l * F_z(Q) \right) \right].
\end{aligned}
\tag{3.92}
$$

Since $2j - 2(m+1) \geqslant 0$ for $j \geqslant m+1$, the bracketed terms with coefficient α^2 are $\mathcal{O}(\alpha^2)$. Therefore the terms of order precisely α^{2m} are given by (4.184). This establishes the case: $j = m$.

We now prove that (4.184) has a solution, F_m satisfying (4.186) with $j = m$. First, applying Lemmata 4.7.2 and 4.7.3 to the right hand side of (4.184) for $j = m$, H_m, we have that $H_m \in L^{2,a}_{\text{even}}$ with the bound:

$$
\left\| \ e^{C_m|Q|}\ H_m \left[F_0, \ldots, F_{m-1}\right](Q) \ \right\|_{L^{2,a}(\mathbb{R};dQ)} \lesssim \lambda_m, \tag{3.93}
$$

where $C_m = C_0 \left(\frac{1}{2} + \frac{1}{2^{m+1}} \right)$ and

$$
\begin{aligned}
\lambda_m \equiv\ & 2 \sum_{k=0}^{m-1} \frac{e^{-(2m-2k+2)}}{(2m-2k+2)!} \left[\frac{2m-2k+2}{C_k - C_m} \right]^{2m-2k+2} \left\| \ e^{C_k|\cdot|}\ F_k \ \right\|_{L^{2,a}(\mathbb{R})} \\
&+ \frac{1}{(2\pi)^2} \sum_{\substack{k+l+z=m \\ 0\leqslant k,l,z<m}} \left\| \ e^{C_k|\cdot|}\ F_k \ \right\|_{L^{2,a}(\mathbb{R})} \left\| \ e^{C_l|\cdot|}\ F_l \ \right\|_{L^{2,a}(\mathbb{R})} \left\| \ e^{C_z|\cdot|}\ F_z \ \right\|_{L^{2,a}(\mathbb{R})}.
\end{aligned}
\tag{3.94}
$$

Proposition 4.7.2 implies that there exists a unique solution $F_m(Q) \in L^{2,a}_{\text{even}}(\mathbb{R})$ to equation (4.184) with $e^{C_m|Q|} F_m(Q) \in L^{2,a}(\mathbb{R})$. Finally, to see that F_m is real-valued, note that equation (4.184) for F_m is a linear with inhomogeneous forcing on the right-hand-side given by $H_m \left[F_0, \ldots, F_{m-1}\right]$, which is necessarily real-valued for F_j real-valued, $j = 0, \ldots, m-1$. This completes the proof of Proposition 4.7.3. □

Completion of the proof of Proposition 4.7.2: By assumption, we have $F^\flat(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ)$, and there exists a constant $C_0 \geqslant C_\flat > 0$ such that $e^{C_\flat|Q|} F^\flat(Q) \in L^{2,a}(\mathbb{R}; dQ)$. We also know that there exists a solution of (4.179), $F^\sharp(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ)$:

$$\widetilde{L_+}\ F^\sharp(Q) = F^\flat(Q). \tag{3.95}$$

We seek to show that $e^{C_\flat|Q|} F^\sharp(Q) \in L^{2,a}(\mathbb{R}; dQ)$.

We rewrite equation (3.95) as

$$F^\sharp(Q) = \frac{1}{1+|Q|^2}\ \left(F^\flat(Q) + \frac{3}{(2\pi)^2}\ \widetilde{\psi_1} * \widetilde{\psi_1} * F^\sharp(Q) \right). \tag{3.96}$$

Multiply through by $e^{C_\flat|Q|}$ to get

$$e^{C_\flat|Q|} F^\sharp(Q) = \frac{e^{C_\flat|Q|}}{1+|Q|^2}\ \left(F^\flat(Q) + \frac{3}{(2\pi)^2}\ \widetilde{\psi_1} * \widetilde{\psi_1} * F^\sharp(Q) \right). \tag{3.97}$$

We now project $F^\sharp$ onto frequencies near ($|Q| \leqslant \frac{1}{\epsilon}$) and away from zero ($|Q| > \frac{1}{\epsilon}$) for an appropriate choice of $\epsilon > 0$. Define the set A_ϵ and its associated projections:

$$\begin{aligned} A_\epsilon &= \left\{ Q \ :\ |Q| \leqslant \frac{1}{\epsilon} \right\}, \\ 1 &= \chi_{A_\epsilon}(Q) + \overline{\chi}_{A_\epsilon}(Q), \qquad F^\sharp = \chi_{A_\epsilon} F^\sharp + \overline{\chi}_{A_\epsilon} F^\sharp. \end{aligned} \tag{3.98}$$

First, note that since $C_\flat > 0$, we have $\chi_{A_\epsilon}(Q)\ e^{C_\flat|Q|} \leqslant e^{C_\flat/\epsilon}$, and in turn,

$$\left\|\ \chi_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{C_\flat/\epsilon}\ \left\|F^\sharp\right\|_{L^{2,a}(\mathbb{R})}. \tag{3.99}$$

Next, we may project (3.97) onto $|Q| > 1/\epsilon$:

$$\begin{aligned} \overline{\chi}_{A_\epsilon}(Q) e^{C_\sharp|Q|} F^\sharp(Q) = &\frac{\overline{\chi}_{A_\epsilon}(Q)\ e^{C_\sharp|Q|}}{1+|Q|^2}\ \Bigg[F^\flat(Q) + \frac{3}{(2\pi)^2}\ \widetilde{\psi_1} * \widetilde{\psi_1} * \left(\overline{\chi}_{A_\epsilon}\ F^\sharp \right)(Q) \\ &+ \frac{3}{(2\pi)^2}\ \widetilde{\psi_1} * \widetilde{\psi_1} \left(\chi_{A_\epsilon}\ F^\sharp \right)(Q) \Bigg]. \end{aligned} \tag{3.100}$$

Note that since $1 + |Q|^2 \geqslant |Q|^2$, we have

$$\frac{\overline{\chi}_{A_\epsilon}(Q)}{1+|Q|^2} \leqslant \frac{\overline{\chi}_{A_\epsilon}(Q)}{|Q|^2} \leqslant \epsilon^2. \tag{3.101}$$

From (3.99), (3.100), (3.101), and Lemma 4.7.2, we therefore have

$$\begin{aligned} &\left\|\ \overline{\chi}_{A_\epsilon} e^{C_\flat|\cdot|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R})} \lesssim\ \epsilon^2\ \left\| e^{C_\flat|Q|} F^\flat \right\|_{L^{2,a}(\mathbb{R}_Q)} \\ &\quad + \frac{3}{4\pi^2}\ \epsilon^2\ \left\| e^{C_\flat|Q|}\ \widetilde{\psi_1} \right\|^2_{L^{2,a}(\mathbb{R}_Q)} \left(\left\|\ \overline{\chi}_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)} + \left\|\ \chi_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \right) \\ &\leqslant \epsilon^2\ \left\| e^{C_\flat|Q|} F^\flat \right\|_{L^{2,a}(\mathbb{R}_Q)} \\ &\quad + \frac{3}{4\pi^2}\ \epsilon^2\ \left\| e^{C_0|Q|}\ \widetilde{\psi_1} \right\|^2_{L^{2,a}(\mathbb{R}_Q)} \left(\left\|\ \overline{\chi}_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)} + e^{C_\flat/\epsilon}\ \left\|\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R})} \right). \end{aligned} \tag{3.102}$$

or for some constant $C > 0$,

$$\left(1 - C\frac{3}{4\pi^2}\ \epsilon^2\ \left\| e^{C_0|Q|}\ \widetilde{\psi_1} \right\|^2_{L^{2,a}(\mathbb{R}_Q)}\right)\ \left\|\ \overline{\chi}_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)}$$
$$\lesssim \epsilon^2\ \left\| e^{C_\flat|Q|}\ F^\flat\ \right\|_{L^{2,a}(\mathbb{R}_Q)} + \frac{3}{4\pi^2}\ \epsilon^2\ \left\| e^{C_0|Q|}\ \widetilde{\psi_1} \right\|^2_{L^{2,a}(\mathbb{R}_Q)}\ e^{C_\flat/\epsilon}\ \|\ F^\sharp\ \|_{L^{2,a}(\mathbb{R})}. \tag{3.103}$$

It follows that for a sufficiently small choice of ϵ dependent only on $\widetilde{\psi_1}$ and $C_0 > 0$,

$$\left\|\ \overline{\chi}_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \epsilon^2\ \left\| e^{C_\flat|Q|}\ F^\flat\ \right\|_{L^{2,a}(\mathbb{R}_Q)}$$
$$+ \frac{3}{4\pi^2}\ \epsilon^2\ \left\| e^{C_0|Q|}\ \widetilde{\psi_1} \right\|^2_{L^{2,a}(\mathbb{R}_Q)}\ e^{C_\flat/\epsilon}\ \|\ F_j\ \|_{L^{2,a}(\mathbb{R})}. \tag{3.104}$$

and therefore by the triangle inequality and (3.99),

$$\left\|\ e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \leqslant \left\|\ \chi_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)} + \left\|\ \overline{\chi}_{A_\epsilon} e^{C_\flat|Q|}\ F^\sharp\ \right\|_{L^{2,a}(\mathbb{R}_Q)}$$
$$\lesssim \epsilon^2\ \left\| e^{C_\flat|Q|}\ F^\flat\ \right\|_{L^{2,a}(\mathbb{R}_Q)}$$
$$+ \left(1 + \frac{3}{4\pi^2}\ \epsilon^2\ \left\| e^{C_0|Q|}\ \widetilde{\psi_1} \right\|^2_{L^{2,a}(\mathbb{R}_Q)}\right)\ e^{C_\flat/\epsilon}\ \|\ F^\sharp\ \|_{L^{2,a}(\mathbb{R})}. \tag{3.105}$$

This completes the proof of Proposition 4.7.2. □

3.4 Rigorous Justification of Asymptotic Series 3.66 and Proof of Proposition 3.3.3

Fix $J \geqslant 0$ and define the truncated asymptotic expansion

$$S^\alpha_J(Q) \equiv \sum_{j=0}^{J}\ \alpha^{2j}\ \chi_{\mathcal{B}_\alpha}(Q)\ F_j\left[\widetilde{\psi_1}\right](Q). \tag{3.106}$$

where

$$F_0[\widetilde{\psi_1}](Q) \equiv \widetilde{\psi_1}(Q), \qquad \text{and} \qquad F_j[\widetilde{\psi_1}](Q) \equiv F_j(Q), \qquad j \geqslant 1, \tag{3.107}$$

with $F_j \in L^{2,a}(\mathbb{R})$ prescribed in Proposition 4.7.3. Note that (3.106) is the projection of the first $J+1$ terms of the formal asymptotic expansion F^α in (4.173). Note that

$$(I - \chi_{\mathcal{B}_\alpha})|F_j(Q)| = (I - \chi_{\mathcal{B}_\alpha})e^{-C_j|Q|} \cdot e^{C_j|Q|}|F_j(Q)| \leqslant e^{-\pi C_j/\alpha} \cdot e^{C_j|Q|}|F_j(Q)|, \tag{3.108}$$

from which we see that the Fourier tail neglected in (3.106) is exponentially small in α. In the subsequent sections, we use the following consequence of Proposition 4.7.3.

Proposition 3.4.1. *For S_J^α defined in* (3.106)*, we have $S_J^\alpha \in L^{2,a}$ and, for $\alpha > 0$ sufficiently small, the bounds*

$$\|S_J^\alpha\|_{L^{2,a}(\mathbb{R})} \lesssim 1, \qquad \text{and} \qquad \left\| e^{C_S|\cdot|}\ S_J^\alpha \right\|_{L^{2,a}(\mathbb{R})} \lesssim 1, \tag{3.109}$$

where $C_S = \min\{C_0, ..., C_J\}$, and where C_j are the constants prescribed in Proposition 4.7.3.

3.4.1 Equation for the remainder, $\widehat{E_J^{\alpha,\sigma}}$

In order to prove Proposition 3.3.3, we seek an equation for $\widehat{E_J^{\alpha,\sigma}}(Q) = \widehat{\Phi^\sigma}(Q) - S_J^\alpha(Q)$. The equation for $\widehat{\Phi^\sigma}$ is (3.62) given in Proposition 3.3.2. Substituting into (3.62), we obtain the following equation for $\widehat{E_J^{\alpha,\sigma}}$.

Proposition 3.4.2. *Equation* (3.62) *is equivalent to the following closed equation for $\widehat{E_J^{\alpha,\sigma}}(Q)$ on $Q \in \mathbb{R}$:*

$$[1 + M_\alpha(Q)]\ \widehat{E^{\sigma,\alpha}}(Q) - 3\ \chi_{\mathcal{B}_\alpha}(Q)\ \left(\frac{1}{2\pi}\right)^2 \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_J^{\alpha,\sigma}}(Q) = \mathcal{R}_{J,1}^\sigma\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right](Q), \tag{3.110}$$

where $\mathcal{R}_{J,1}^\sigma\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right]$ is defined in (3.111)*.*

Remark 3.4.1. *Note that the operator on the left-hand side has formal limit $\widetilde{L_+}$, where L_+ is the linearized continuum NLS operator displayed in Proposition 3.1.3.*

Proof of Proposition 3.4.2: Substitution of $\widehat{\Phi^\sigma} = S_J^\alpha + \widehat{E_J^{\alpha,\sigma}}$ into (3.62) yields, after some manipulation, (3.110), where

$$\mathcal{R}_{J,1}^\sigma\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right] \equiv \mathcal{D}^{\sigma,\alpha}[S_J^\alpha] + R_{\text{L}}^\sigma\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right] + R_{\text{NL}}^\sigma\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right]. \tag{3.111}$$

R_{L}^σ contains terms which are linear in $\widehat{E_J^{\alpha,\sigma}}$ but which are higher order in α, and R_{NL}^σ contains terms which are nonlinear in $\widehat{E_J^{\alpha,\sigma}}$; they are respectively given by

$$R^\sigma_{\text{L}}\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right](Q) \equiv \chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\left[\ \sum_{m=-1}^{1} e^{2m\pi i\sigma}\ S^\alpha_J * S^\alpha_J * \widehat{E_J^{\alpha,\sigma}}(Q - 2m\pi/\alpha) - \ \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_J^{\alpha,\sigma}}(Q)\right],$$

$$R^\sigma_{\text{NL}}\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right](Q) \equiv \chi_{\mathcal{B}_\alpha}(Q)\ \left(\frac{1}{2\pi}\right)^2 \sum_{m=-1}^{1} e^{2m\pi i\sigma}\left[3\ S^\alpha_J * \widehat{E_J^{\alpha,\sigma}} * \widehat{E_J^{\alpha,\sigma}}(Q - 2m\pi/\alpha) + \widehat{E_J^{\alpha,\sigma}} * \widehat{E_J^{\alpha,\sigma}} * \widehat{E_J^{\alpha,\sigma}}(Q - 2m\pi/\alpha)\right]. \tag{3.112}$$

Note that $\mathcal{R}^\sigma_{J,1}\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right] = \chi_{\mathcal{B}_\alpha}\ \mathcal{R}^\sigma_{J,1}\left[\alpha, \widehat{E_J^{\alpha,\sigma}}\right]$. This completes the proof of Proposition 3.4.2. $\square$

3.4.2 Coupled System for High and Low Frequency Components of $\widehat{E_J^{\alpha,\sigma}}$

We now embark on the construction of a solution $\widehat{E_J^{\alpha,\sigma}} \in L^{2,a}(\mathbb{R})$ to (3.110) for $\alpha > 0$ sufficiently small. Our strategy is to formulate the equation for $\widehat{E_J^{\alpha,\sigma}}$ as an equivalent coupled system for its high and low frequency components. Let r be such that $0 < r < 1$. Define the sharp spectral cutoff functions

$$\chi_{\text{lo}}(Q) = \chi\left(|Q| \leqslant \alpha^{r-1}\right) \quad \text{and} \quad \chi_{\text{hi}}(Q) = \chi\left(|Q| > \alpha^{r-1}\right), \tag{3.113}$$
$$\text{where} \quad 1 = \chi_{\text{lo}}(Q) + \chi_{\text{hi}}(Q),$$

Note that $\chi_{\text{lo}}(Q)\ \chi_{\mathcal{B}_\alpha}(Q) = \chi_{\text{lo}}(Q)$, while $\chi_{\text{hi}}(Q)\ \chi_{\mathcal{B}_\alpha}(Q) = \chi_{\mathcal{B}_\alpha}(Q) - \chi_{\text{lo}}(Q)$. For general $\widehat{A}(Q)$, defined for $Q \in \mathbb{R}$, we introduce its localizations near and away from $Q = 0$:

$$\widehat{A}_{\text{lo}}(Q) = \left(\chi_{\text{lo}}\widehat{A}\right)(Q) \equiv \chi_{\text{lo}}(Q)\widehat{A}(q),$$
$$\text{and} \qquad \widehat{A}_{\text{hi}}(Q) = \left(\chi_{\text{hi}}\widehat{A}\right)(Q) \equiv \chi_{\text{hi}}(Q)\widehat{A}(q). \tag{3.114}$$

In particular, we use χ_{lo} and χ_{hi} to localize $\widehat{E_J^{\alpha,\sigma}}$ on $|Q| \leqslant \alpha^{r-1}$ and $|Q| > \alpha^{r-1}$:

$$\widehat{E_{\text{lo}}^{\alpha,\sigma}}(Q) = \chi_{\text{lo}}(Q)\widehat{E_J^{\alpha,\sigma}}(Q) \quad \text{and} \quad \widehat{E_{\text{hi}}^{\alpha,\sigma}}(Q) = \chi_{\text{hi}}(Q)\widehat{E_J^{\alpha,\sigma}}(Q). \tag{3.115}$$

Thus, we can express $\widehat{E_J^{\alpha,\sigma}}$ as follows:

$$\widehat{E_J^{\alpha,\sigma}}(Q) = \widehat{E_{\text{lo}}^{\alpha,\sigma}}(Q) + \widehat{E_{\text{hi}}^{\alpha,\sigma}}(Q). \tag{3.116}$$

NOTE: *Since the analysis for the cases $\sigma = 0$ (onsite) and $\sigma = 1/2$ (offsite) in sections 3.4.2 - 3.5.5 are very similar, in order to keep the notation less cumbersome we omit the superscripts α and σ, when the context is clear, and shall instead write:*

$$\widehat{E}(Q) = \widehat{E_J^{\alpha,\sigma}}(Q), \qquad \widehat{E_{\rm lo}}(Q) = \widehat{E_{\rm lo}^{\alpha,\sigma}}(Q), \qquad \widehat{E_{\rm hi}}(Q) = \widehat{E_{\rm hi}^{\alpha,\sigma}}(Q). \tag{3.117}$$

The following Proposition is obtained by applying the spectral projections $\chi_{\rm lo}$ and $\chi_{\rm hi}$ to (3.110).

Proposition 3.4.3. *Equation (3.110) is equivalent to the following coupled system of equations for the low and high frequency components, $\widehat{E_{\rm lo}}$ and $\widehat{E_{\rm hi}}$, of $\widehat{E}$ on $Q \in \mathbb{R}$:*

Low Frequency Equation:

$$\begin{aligned}[1 + M_\alpha(Q)]\ \widehat{E_{\rm lo}}(Q) - \chi_{\rm lo}(Q)\frac{3}{4\pi^2}\left(\widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm hi}}(Q)\right)& \\ = \chi_{\rm lo}(Q)\ \mathcal{R}^\sigma_{J,1}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}\right](Q),&\end{aligned} \tag{3.118}$$

High Frequency Equation:

$$\begin{aligned}[1 + M_\alpha(Q)]\ \widehat{E_{\rm hi}}(Q) - \chi_{\rm hi}(Q)\ \chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\left(\widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm hi}}(Q)\right)& \\ = \chi_{\rm hi}(Q)\ \mathcal{R}^\sigma_{J,1}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}\right](Q).&\end{aligned} \tag{3.119}$$

Here, $\mathcal{R}^\sigma_{J,1}$ is defined in (3.111).

3.4.3 Solving for $\widehat{E_{\rm hi}}\left[\alpha, \widehat{E_{\rm lo}}\right]$ and Reduction to a Closed Equation for the Low-frequency Components, $\widehat{E_{\rm lo}}$

We shall solve (3.118) and (3.119) via a Lyapunov-Schmidt reduction strategy [Nirenberg, 2001]; see the discussion of section 3.1.2. We first solve for $\widehat{E_{\rm hi}} = \widehat{E_{\rm hi}}\left[\alpha, \widehat{E_{\rm lo}}\right]$ as a functional of $\widehat{E_{\rm lo}}$ and α, with α sufficiently small. We view the equation of $\widehat{E_{\rm hi}}$ as depending on parameters $\alpha \in \mathbb{R}, |\alpha| \ll 1$ and a function $\Gamma(Q) = \widehat{E_{\rm lo}}(Q) \in L^{2,a}(\mathbb{R})$:

$$\begin{aligned}[1 + M_\alpha(Q)]\ \widehat{E_{\rm hi}}(Q) - \chi_{\rm hi}(Q)\ \chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\left(\widetilde{\psi_1} * \widetilde{\psi_1} * \Gamma(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm hi}}(Q)\right)& \\ -\chi_{\rm hi}(q)\ \mathcal{R}^\sigma_{J,1}[\alpha, \Gamma + \widehat{E_{\rm hi}}](Q) = 0\ .&\end{aligned} \tag{3.120}$$

In the following proposition, we construct $\widehat{E_{\rm lo}} \mapsto \widehat{E_{\rm hi}}\left[\alpha, \widehat{E_{\rm lo}}\right]$. Note that since $0 < r < 1$, $\lim_{\alpha \to 0} \alpha^{2-2r} = 0$.

Proposition 3.4.4. *Set $0 < r < 1$.*

1. *There exist constants $\alpha_0, \beta_0 > 0$, such that for all $\alpha \in (0, \alpha_0)$, equation (3.120) defines a unique mapping*

$$(\alpha, \Gamma) \mapsto \widehat{E_{\rm hi}}[\alpha, \Gamma],$$
$$\widehat{E_{\rm hi}} : [0,1] \times B_{\beta_0}(0) \to L^{2,a}(\mathbb{R})$$

   where $B_{\beta_0}(0) \subset L^{2,a}(\mathbb{R})$ and $\widehat{E_{\rm hi}}[\alpha, \Gamma]$ is the unique solution to the high frequency equation (3.120) *(see also* (3.119)*). In particular, if $\Gamma \in L^{2,a}_{\rm even}$, then $\widehat{E_{\rm hi}}[\alpha, \Gamma] \in L^{2,a}_{\rm even}$.*

2. *Moreover, this mapping is C^1 with respect to Γ and for all $(\alpha, \Gamma) \in [0, \alpha_0) \times B_{\beta_0}(0)$, and there exists some constant $C > 0$ such that*

$$\left\| \widehat{E_{\rm hi}}[\alpha, \Gamma] \right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \; \|\Gamma\|_{L^{2,a}(\mathbb{R})} + e^{-C/\alpha^{1-r}}. \tag{3.121}$$

$$\left\| \; D_\Gamma \widehat{E_{\rm hi}}[\alpha, \Gamma] \; \right\|_{L^{2,a}(\mathbb{R}) \longrightarrow L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r}, \tag{3.122}$$

   where the implicit constants are dependent on α_0 and β_0.

3. *$\widehat{E_{\rm hi}}[\alpha, \Gamma]$ is supported on $Q \in \left[-\frac{\pi}{\alpha}, -\alpha^{r-1}\right) \cap \left(\alpha^{r-1}, \frac{\pi}{\alpha}\right]$ such that $\widehat{E_{\rm hi}}[\alpha, \Gamma] = \chi_{\rm hi} \; \chi_{\mathcal{B}_\alpha} \; \widehat{E_{\rm hi}}[\alpha, \Gamma]$.*

Proof of Proposition 3.4.4: Since $0 < 1 \leqslant 1 + M_\alpha(Q)$ for all α positive and small, we may rewrite (3.120) (see also (3.119)) as

$$\widehat{E_{\rm hi}}(Q) - \chi_{\rm hi}(Q) \; [1 + M_\alpha(Q)]^{-1} \Bigg(\chi_{\mathcal{B}_\alpha}(Q) \; \frac{3}{4\pi^2} \; \left[\widetilde{\psi_1} * \widetilde{\psi_1} * \Gamma(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm hi}}(Q) \right] - \mathcal{R}^\sigma_{J,1}[\alpha, \Gamma + \widehat{E_{\rm hi}}](Q) \Bigg) = 0. \tag{3.123}$$

We apply the implicit function theorem to obtain $\widehat{E_{\rm hi}}$ as a functional of α and Γ. The key steps in the proof are given in Propositions 3.4.5 through 3.4.8. The proofs of these propositions are deferred until the following Section 3.4.4. We first give bounds on the latter two terms in equation (3.123).

Proposition 3.4.5. *There exists a constant $\alpha_0 > 0$, such that for all (Γ, α) with $0 < \alpha < \alpha_0$, and $\Gamma \in L^{2,a}(\mathbb{R})$, we have the bounds*

$$\left\| \chi_{\rm hi}\ \chi_{\mathcal{B}_\alpha}\ [1 + M_\alpha(\cdot)]^{-1}\ \widetilde{\psi_1} * \widetilde{\psi_1} * \Gamma \right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \left\|\Gamma\right\|_{L^{2,a}(\mathbb{R})}, \tag{3.124}$$

$$\left\| \chi_{\rm hi}\ \chi_{\mathcal{B}_\alpha}\ [1 + M_\alpha(\cdot)]^{-1}\ \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm hi}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \left\|\widehat{E_{\rm hi}}\right\|_{L^{2,a}(\mathbb{R})}, \tag{3.125}$$

$$\begin{aligned} \left\| \chi_{\rm hi}\ [1 + M_\alpha(\cdot)]^{-1}\ \mathcal{R}^\sigma_{J,1}[\alpha, \Gamma + \widehat{E_{\rm hi}}] \right\|_{L^{2,a}(\mathbb{R})} &\lesssim e^{-C/\alpha^{1-r}} \\ + \alpha^{2-2r} \Bigg[\left\|\Gamma\right\|_{L^{2,a}(\mathbb{R})} + \left\|\widehat{E_{\rm hi}}\right\|_{L^{2,a}(\mathbb{R})} + \left\|\Gamma\right\|^2_{L^{2,a}(\mathbb{R})} + \left\|\Gamma\right\|_{L^{2,a}(\mathbb{R})} \left\|\widehat{E_{\rm hi}}\right\|_{L^{2,a}(\mathbb{R})} + \left\|\widehat{E_{\rm hi}}\right\|^2_{L^{2,a}(\mathbb{R})} \\ + \left\|\Gamma\right\|^3_{L^{2,a}(\mathbb{R})} + \left\|\Gamma\right\|^2_{L^{2,a}(\mathbb{R})} \left\|\widehat{E_{\rm hi}}\right\|_{L^{2,a}(\mathbb{R})} + \left\|\Gamma\right\|_{L^{2,a}(\mathbb{R})} \left\|\widehat{E_{\rm hi}}\right\|^2_{L^{2,a}(\mathbb{R})} + \left\|\widehat{E_{\rm hi}}\right\|^3_{L^{2,a}(\mathbb{R})} \Bigg]. \end{aligned} \tag{3.126}$$

We may write equation (3.123) as $\mathcal{A}[\alpha, \Gamma, \widehat{E_{\rm hi}}](Q) = 0$, where $\mathcal{A} : (0, \infty) \times L^{2,a}(\mathbb{R}) \times L^{2,a}(\mathbb{R}) \longrightarrow L^{2,a}(\mathbb{R})$ is defined by

$$\begin{aligned} \mathcal{A}\left[\alpha, \Gamma, \widehat{E_{\rm hi}}\right](Q) &\equiv \widehat{E_{\rm hi}}(Q) - \chi_{\rm hi}(Q)\ [1 + M_\alpha(Q)]^{-1} \\ &\cdot \left(\frac{3}{4\pi^2}\ \chi_{\mathcal{B}_\alpha}(Q) \left[\widetilde{\psi_1} * \widetilde{\psi_1} * \Gamma(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm hi}}(Q) \right] + \mathcal{R}^\sigma_{J,1}[\alpha, \Gamma + \widehat{E_{\rm hi}}](Q) \right). \end{aligned} \tag{3.127}$$

By Proposition 3.4.5, we can extend $\mathcal{A}$ as a continuous $L^{2,a}(\mathbb{R})$ operator valued function of $\alpha \in [0, \infty)$ as follows:

$$\mathcal{A}[\alpha, \Gamma, \widehat{E_{\rm hi}}](Q) \equiv \begin{cases} \widehat{E_{\rm hi}}(q) - \chi_{\rm hi}(Q)\ [1 + M_\alpha(Q)]^{-1} \\ \cdot\Big(\frac{3}{4\pi^2}\ \chi_{\mathcal{B}_\alpha}(Q) \left[\widetilde{\psi_1} * \widetilde{\psi_1} * \Gamma(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm hi}}(Q) \right] \\ \qquad + \mathcal{R}^\sigma_{J,1}[\alpha, \Gamma + \widehat{E_{\rm hi}}](Q) \Big), & \text{for } \alpha > 0, \\ \widehat{E_{\rm hi}}(Q) & \text{for } \alpha = 0. \end{cases} \tag{3.128}$$

We now summarize the properties of the mapping $\mathcal{A}$ in the following proposition.

Proposition 3.4.6. *1. The mapping $\mathcal{A} : [0, \infty) \times L^{2,a}(\mathbb{R}) \times L^{2,a}(\mathbb{R}) \longrightarrow L^{2,a}(\mathbb{R})$*

$$(\alpha, \Gamma, \widehat{E_{\rm hi}}) \longmapsto \mathcal{A}[\alpha, \Gamma, \widehat{E_{\rm hi}}],$$

is continuous at $(0, 0, 0)$.

2. $\mathcal{A}[0, 0, 0] = 0$ in $L^{2,a}(\mathbb{R})$.

3. *<u>Differential with respect to $\widehat{E_{\rm hi}}$</u>: For $\widehat{f} \in L^{2,a}(\mathbb{R})$, introduce the operator on $L^{2,a}(\mathbb{R})$*

$$\begin{aligned} D_Z\Big(\mathcal{R}^{\sigma}_{J,1}[\alpha, Z]\Big)\ \widehat{f}(Q) = \ & R^{\sigma}_{\rm L}\left[\alpha, \widehat{f}\right](Q) \\ & + \chi_{\mathcal{B}_\alpha}(Q)\ \left(\frac{1}{2\pi}\right)^2 \sum_{m=-1}^{1} e^{2m\pi i\sigma}\ \Big[6\ S^{\alpha}_J * Z * \widehat{f}(Q - 2m\pi/\alpha) \\ & \qquad + 3\ Z * Z * \widehat{f}(Q - 2m\pi/\alpha)\Big]. \end{aligned} \tag{3.129}$$

For any $(\alpha, \Gamma, \widehat{E_{\rm hi}}) \in [0, \infty) \times L^{2,a}(\mathbb{R}) \times L^{2,a}(\mathbb{R})$, $\mathcal{A}$ is Fréchet differentiable with respect to $\widehat{E_{\rm hi}}$ with $D_{\widehat{E_{\rm hi}}}\mathcal{A}[\alpha, \Gamma, \widehat{E_{\rm hi}}]$, continuous at $(0, 0, 0)$,

$$D_{\widehat{E_{\rm hi}}}\mathcal{A}[\alpha, \Gamma, \widehat{E_{\rm hi}}] = \begin{cases} I - \chi_{\rm hi}\ \ [1 + M_\alpha(Q)]^{-1}\ \Big[\frac{3}{4\pi^2}\ \chi_{\mathcal{B}_\alpha}\ \ \widetilde{\psi_1} * \widetilde{\psi_1} * \\ \qquad\qquad + D_Z\Big(\mathcal{R}^{\sigma}_{J,1}[\alpha, \Gamma + \widehat{E_{\rm hi}}]\Big)\Big], & \text{for } \alpha > 0 \\ I & \text{for } \alpha = 0\ \ . \end{cases} \tag{3.130}$$

4. *$D_{\widehat{E_{\rm hi}}}\mathcal{A}[0, 0, 0] = I$ is an isomorphism of $L^{2,a}(\mathbb{R})$ onto $L^{2,a}(\mathbb{R})$.*

5. *<u>Differential with respect to Γ</u>: For any $(\alpha, \Gamma, \widehat{E_{\rm hi}}) \in [0, \infty) \times L^{2,a}(\mathbb{R}) \times L^{2,a}(\mathbb{R})$, the mapping $\Gamma \mapsto \mathcal{A}[\alpha, \Gamma, \widehat{E_{\rm hi}}]$ is Fréchet differentiable. Here,*

$$D_\Gamma\mathcal{A}[\alpha, \Gamma, \widehat{E_{\rm hi}}] = \begin{cases} -\chi_{\rm hi}\ \ [1 + M_\alpha(Q)]^{-1}\ \Big[\frac{3}{4\pi^2}\ \chi_{\mathcal{B}_\alpha}\ \ \widetilde{\psi_1} * \widetilde{\psi_1} * \\ \qquad\qquad + D_Z\Big(\mathcal{R}^{\sigma}_{J,1}[\alpha, \Gamma + \widehat{E_{\rm hi}}]\Big)\Big], & \text{for } \alpha > 0 \\ 0 & \text{for } \alpha = 0\ \ . \end{cases} \tag{3.131}$$

By Proposition 3.4.6, the mapping $\mathcal{A}$ satisfies the hypotheses of a variation of the implicit function theorem stated in Theorem F.0.2. Therefore, there exist $\alpha_0, \beta_0, \kappa > 0$ such that for all

$$\alpha \in [0, \alpha_0) \ \ \text{and} \ \ \ \|\Gamma\|_{L^{2,a}(\mathbb{R})} < \beta_0\ , \tag{3.132}$$

there exists a unique map, differentiable with respect to Γ,

$$\begin{aligned} & (\alpha, \Gamma) \longmapsto \widehat{E_{\rm hi}}\,[\alpha, \Gamma]\,, \\ & \widehat{E_{\rm hi}} : \Big\{(\alpha, \Gamma) \in [0, \alpha_0) \times L^{2,a}(\mathbb{R}) \Big|\ \ \alpha \in [0, \alpha_0) \ \ \text{and} \ \ \ \|\Gamma\|_{L^{2,a}(\mathbb{R})} < \beta_0\Big\} \longmapsto L^{2,a}(\mathbb{R}), \\ & \Big\|\widehat{E_{\rm hi}}\,[\alpha, \Gamma]\Big\|_{L^{2,a}(\mathbb{R})} \leqslant \kappa, \qquad \text{with} \qquad \lim_{(\alpha,\Gamma)\to(0,0)} \widehat{E_{\rm hi}}\,[\alpha, \Gamma] = 0, \end{aligned} \tag{3.133}$$

which solves

$$\mathcal{A}\left[\alpha, \Gamma, \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]\right](Q) = 0 \quad \text{for} \quad \alpha \in [0, \alpha_0) \quad \text{and} \quad \|\Gamma\|_{L^{2,a}(\mathbb{R})} < \beta_0. \tag{3.134}$$

Furthermore, the mapping $\Gamma \longmapsto \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]$ is Fréchet differentiable with derivative $D_\Gamma \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]$.

Proposition 3.4.7. *The mapping* $(\alpha, \Gamma) \longmapsto \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]$*,* $\widehat{E_{\text{hi}}} : [0, \alpha_0) \times B_{\beta_0}(0) \longmapsto L^{2,a}(\mathbb{R})$ *satisfies the bounds*

$$\left\|\widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \|\Gamma\|_{L^{2,a}(\mathbb{R})} + e^{-C/\alpha^{1-r}} \tag{3.135}$$

$$\left\|D_\Gamma \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]\right\|_{L^{2,a}(\mathbb{R}) \longmapsto L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r}, \tag{3.136}$$

for some constant $C > 0$*.*

The mapping $(\alpha, \widehat{E_{\text{lo}}}) \mapsto \widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]$ is supported on $\left[-\frac{\pi}{\alpha}, -\alpha^{r-1}\right) \cap \left(\alpha^{r-1}, \frac{\pi}{\alpha}\right]$. Finally we note that the above proof can be adapted to show that the mapping $\widehat{E_{\text{lo}}} \mapsto \widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]$ maps the space of even functions into itself.

Proposition 3.4.8. *For* $(\alpha, \Gamma) \in [0, \alpha_0) \times B_{\beta_0}(0)$*, the mapping* $\Gamma \mapsto \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]$ *from Proposition 3.4.4 maps* $\Gamma \in L^{2,a}_{\text{even}}(\mathbb{R})$ *into* $L^{2,a}_{\text{even}}(\mathbb{R})$*.*

Proof of Proposition 3.4.8: We follow the steps given in the proof of Proposition 3.4.4 but restrict the mapping $\mathcal{A} : [0, \infty) \times L^{2,a}_{\text{even}}(\mathbb{R}) \times L^{2,a}_{\text{even}}(\mathbb{R}) \to L^{2,a}(\mathbb{R})$ such that it only acts upon even functions. Since $M_\alpha(Q)$ and S^α_J are even by Proposition 4.7.3 and definition (3.106), and by Lemma 4.7.2, $\mathcal{A}$ as defined in (3.127) maps even functions to even functions. Thus, we have $\mathcal{A} : \mathcal{A} : [0, \infty) \times L^{2,a}_{\text{even}}(\mathbb{R}) \times L^{2,a}_{\text{even}}(\mathbb{R}) \to L^{2,a}_{\text{even}}(\mathbb{R})$. Furthermore, $D_{\widehat{E_{\text{hi}}}} \mathcal{A}[0, \Gamma, 0] = I$ as defined in (3.130) is now an isomorphism from $L^{2,a}_{\text{even}}(\mathbb{R})$ to $L^{2,a}_{\text{even}}(\mathbb{R})$, and all properties given in Proposition 3.4.6 hold for the restriction since $L^{2,a}_{\text{even}}(\mathbb{R}) \subset L^{2,a}(\mathbb{R})$. Therefore, following the proof above and applying the implicit function theorem guarantees that there exists a unique map $(\alpha, \widehat{E_{\text{lo}}}) \longmapsto \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]$, $\widehat{E_{\text{hi}}} : [0, \alpha_0) \times B_{\beta_0}(0) \longmapsto L^{2,a}_{\text{even}}(\mathbb{R})$ with $B_{\beta_0}(0) \in L^{2,a}_{\text{even}}(\mathbb{R})$ and defined for $0 < \alpha < \alpha_0$.

This mapping and the mapping obtained in Proposition 3.4.4 uniquely solve the same equation (3.123) on their respective domains. Since $L^{2,a}_{\text{even}}(\mathbb{R}) \subset L^{2,a}(\mathbb{R})$, it follows that the mappings must coincide on $[0, \alpha_0) \times L^{2,a}_{\text{even}}(\mathbb{R}) \times L^{2,a}_{\text{even}}(\mathbb{R})$. Therefore, $\widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]$ must be even when Γ is even.

Finally, recall from (3.123) that the equation for $\widehat{E_{\text{hi}}}\,[\alpha,\Gamma]$ is given by

$$\widehat{E_{\text{hi}}}(Q) - \chi_{\text{hi}}(Q)\;[1+M_\alpha(Q)]^{-1}\bigg(\chi_{\mathcal{B}_\alpha}(Q)\;\frac{3}{4\pi^2}\;\Big[\widetilde{\psi_1}*\widetilde{\psi_1}*\Gamma(Q)+\widetilde{\psi_1}*\widetilde{\psi_1}*\widehat{E_{\text{hi}}}(Q)\Big] \\ -\mathcal{R}^{\sigma}_{J,1}[\alpha,\Gamma+\widehat{E_{\text{hi}}}](Q)\bigg) = 0. \qquad (3.137)$$

Recall from Proposition 3.4.2 that $\mathcal{R}^{\sigma}_{J,1}[\alpha,\Gamma+\widehat{E_{\text{hi}}}] = \chi_{\mathcal{B}_\alpha}\;\mathcal{R}^{\sigma}_{J,1}[\alpha,\Gamma+\widehat{E_{\text{hi}}}]$. We therefore project (3.137) onto the set $\mathbb{R}\setminus\left[-\frac{\pi}{\alpha},-\alpha^{r-1}\right)\cap\left(\alpha^{r-1},\frac{\pi}{\alpha}\right]$ via multiplication by $1-\chi_{\text{hi}}(Q)\chi_{\mathcal{B}_\alpha}(Q)$ to get

$$\Big[1-\chi_{\text{hi}}(Q)\chi_{\mathcal{B}_\alpha}(Q)\Big]\;\widehat{E_{\text{hi}}}(Q) = 0. \qquad \square \qquad (3.138)$$

Taking $\Gamma = \widehat{E_{\text{lo}}}$ completes the proof of Proposition 3.4.4. □

3.4.4 Technical Details in the Proof of Proposition 3.4.4 from Section 3.4.3

In this section we elaborate on the proofs of technical propositions used in Section 3.4.3. These technical points are used to prove Proposition 3.4.4, which constructs the high-frequency component of the error: $(\alpha,\widehat{E_{\text{lo}}}) \mapsto \widehat{E_{\text{hi}}}\left[\alpha,\widehat{E_{\text{lo}}}\right]$. To construct this map, we solve equation (3.123) using the implicit function theorem. Propositions 3.4.5 and 3.4.6 together assert that the hypotheses of the implicit function theorem hold.

Proof of Proposition 3.4.5: We let $a > 1/2$ and make extensive use of (2.20):

$$\left\|\widetilde{f_1}*\widetilde{f_2}\right\|_{L^{2,a}(\mathbb{R}^d)} \lesssim \left\|\widetilde{f_1}\right\|_{L^{2,a}(\mathbb{R}^d)}\;\left\|\widetilde{f_2}\right\|_{L^{2,a}(\mathbb{R}^d)}. \qquad (3.139)$$

We also require the following lemma, stated generally since we use it in multiple sections of the proofs, which addresses the exponential smallness of "shifted" ($m = \pm 1$ by our convention) convolutions of exponentially decaying functions.

Lemma 3.4.1. *Let $a > 1/2$ and let $m = \pm 1$. Let $\widehat{f} \in L^{2,a}(\mathbb{R})$ and such that for $C > 0$, $e^{\,C|q|}\widehat{f}(q) \in L^{2,a}(\mathbb{R})$. Then*

$$\left\|\;\chi_{\mathcal{B}_\alpha}\;\widehat{f}*\widehat{f}*\widehat{f}(Q-2m\pi/\alpha)\;\right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim\; e^{-C\pi/\alpha}\;\left\|\;e^{C|Q|}\;\widehat{f}\;\right\|^3_{L^{2,a}(\mathbb{R}_Q)}. \qquad (3.140)$$

Proof of Lemma 3.4.1: We will multiply the convolution in (3.140) by the identity

$$1 = e^{-C\;|Q-2m\pi/\alpha|}\;e^{\;C\;|Q-2m\pi/\alpha|}, \qquad (3.141)$$

and show that the exponential growth may be distributed to the terms of the convolution. We then use the exponential decay of $\widehat{f}$ with the fact that $|Q - 2m\pi/\alpha| \geqslant \pi/\alpha$ for $Q \in \mathcal{B}_\alpha$ (or $|Q| \leqslant \pi/\alpha$) to get the pointwise bound

$$\chi_{\mathcal{B}_\alpha}(Q)\ e^{-C\ |Q-2m\pi/\alpha|} \leqslant e^{-C\pi/\alpha}. \tag{3.142}$$

To distribute the exponential growth across the convolution, first observe that for any $\xi, \zeta \in \mathbb{R}$,

$$|Q - 2m\pi/\alpha| \ \leqslant\ |Q - \xi - \zeta - 2m\pi/\alpha| + |\xi| + |\zeta|, \tag{3.143}$$

and therefore

$$e^{\ C\ |Q-2m\pi/\alpha|} \ \leqslant\ e^{\ C\ |Q-\xi-\zeta-2m\pi/\alpha|}\ e^{\ C\ |\xi|}\ e^{\ C\ |\zeta|}. \tag{3.144}$$

In turn,

$$\begin{aligned}
&e^{\ C\ |Q-2m\pi/\alpha|}\ |\ \widehat{f} * \widehat{f} * \widehat{f}(Q - 2m\pi/\alpha)\ | \\
&\quad \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\ C\ |Q-2m\pi/\alpha|}\ |\widehat{f}(\xi)|\ |\widehat{f}(\zeta)|\ |\widehat{f}(Q - \xi - \zeta - 2m\pi/\alpha)|\ d\xi\ d\zeta \\
&\quad \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\ C|\xi|}\ |\widehat{f}(\xi)|\ e^{\ C|\zeta|}\ |\widehat{f}(\zeta)|\ e^{\ C|Q-\xi-\zeta-2m\pi/\alpha|} |\widehat{f}(Q - \xi - \zeta - 2m\pi/\alpha)|\ d\xi\ d\zeta \\
&\quad = \left(e^{\ C|\cdot|}|\widehat{f}|\right) * \left(e^{\ C|\cdot|}|\widehat{f}|\right) * \left(e^{\ C\ |\cdot|}|\widehat{f}|\right)(Q - 2m\pi/\alpha).
\end{aligned} \tag{3.145}$$

We now combine (3.141), (3.142), (3.145), and (3.139) to get

$$\begin{aligned}
&\left\| \ \chi_{\mathcal{B}_\alpha}\ \widehat{f} * \widehat{f} * \widehat{f}(Q - 2m\pi/\alpha)\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \\
&\quad = \left\| \ \chi_{\mathcal{B}_\alpha}\ e^{-C\ |Q-2m\pi/\alpha|}\ e^{\ C\ |Q-2m\pi/\alpha|}\ \widehat{f} * \widehat{f} * \widehat{f}(Q - 2m\pi/\alpha)\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \\
&\quad \leqslant e^{-C\pi/\alpha}\ \left\| \ \left(e^{\ C|\cdot|}|\widehat{f}|\right) * \left(e^{\ C|\cdot|}|\widehat{f}|\right) * \left(e^{\ C|\cdot|}|\widehat{f}|\right)(Q - 2m\pi/\alpha)\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \\
&\quad \leqslant\ e^{-C\pi/\alpha}\ \left\| \ e^{\ C\ |Q|}\widehat{f}\ \right\|^3_{L^{2,a}(\mathbb{R}_Q)}.
\end{aligned} \tag{3.146}$$

This completes the proof of Lemma 3.4.1. □

Proposition 3.4.9. *For any function $\widehat{f} \in L^{2,a}(\mathbb{R})$, we have*

$$\left\| \ \chi_{\text{hi}}\ \chi_{\mathcal{B}_\alpha}\ \ [1 + M_\alpha(Q)]^{-1}\ \ \widehat{f}\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim\ \alpha^{2-2r} \left\|\ \widehat{f}\ \right\|_{L^{2,a}(\mathbb{R})}. \tag{3.147}$$

Proof of Proposition 3.4.9: Observe that on $Q \in [-\pi/2, \pi/2]$,

$$|\sin(Q)| \geqslant \frac{2}{\pi}|Q|. \tag{3.148}$$

We then square this and rescale with $Q \mapsto Q\alpha/2$ to get

$$M_\alpha(Q) = \frac{4}{\alpha^2}\sin^2(Q\alpha/2) \geqslant \frac{4}{\pi^2}|Q|^2. \tag{3.149}$$

Therefore,

$$1 + M_\alpha(Q) \geqslant \frac{4}{\pi^2}|Q|^2, \qquad Q \in \mathcal{B}_\alpha. \tag{3.150}$$

Since the operator $\chi_{\rm hi}$ projects onto the set where $|Q| \geqslant \alpha^{r-1}$ or $|Q|^{-1} \leqslant \alpha^{1-r}$, we have

$$\chi_{\rm hi}\ \chi_{\mathcal{B}_\alpha}\ [1 + M_\alpha(Q)]^{-1} \leqslant \frac{\pi^2}{4}\ \chi_{\rm hi}\ \chi_{\mathcal{B}_\alpha}\ |Q|^{-2} \lesssim\ \alpha^{2-2r}. \tag{3.151}$$

This completes the proof of Proposition 3.4.9. □

Next, we bound the individual terms of $\mathcal{R}^\sigma_{J,1}\left[\alpha, \Gamma + \widehat{E}_{\rm hi}\right]$.

Proposition 3.4.10 (High frequency residual bound). *Let $0 < r < 1$ and let the operator $\mathcal{D}^{\sigma,\alpha}$ be defined in* (3.62). *We have*

$$\left\|\chi_{\rm hi}\ [1 + M_\alpha(Q)]^{-1}\ \mathcal{D}^{\sigma,\alpha}[S^\alpha_J]\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim\ e^{-C/\alpha^{1-r}}. \tag{3.152}$$

for some constant $C > 0$.

Proof of Proposition 3.4.10: Recall from (3.62) that

$$\begin{aligned}\mathcal{D}^{\sigma,\alpha}[S^\alpha_J](Q) = &-[1 + M_\alpha(Q)]\ S^\alpha_J(Q)\\ &+ \chi_{\mathcal{B}_\alpha}(Q)\frac{1}{4\pi^2}\sum_{m=-1}^{1} e^{2m\pi i\sigma}\ S^\alpha_J * S^\alpha_J * S^\alpha_J(Q - 2m\pi/\alpha),\end{aligned} \tag{3.153}$$

where

$$S^\alpha_J(Q) \equiv \sum_{j=0}^{J} \alpha^{2j}\ \chi_{\mathcal{B}_\alpha}(Q)\ F_j\left[\widetilde{\psi_1}\right](Q). \tag{3.154}$$

such that

$$\chi_{\text{hi}}(Q)\ [1 + M_\alpha(Q)]^{-1}\ \mathcal{D}^{\sigma,\alpha}[S^\alpha_J](Q) = -\chi_{\text{hi}}(Q)\ S^\alpha_J(Q)$$
$$+ \frac{1}{4\pi^2}\ \chi_{\text{hi}}(Q)\ \chi_{\mathcal{B}_\alpha}(Q)\ [1 + M_\alpha(Q)]^{-1} \sum_{m=-1}^{1} e^{2m\pi i\sigma}\ S^\alpha_J * S^\alpha_J * S^\alpha_J(Q - 2m\pi/\alpha). \tag{3.155}$$

For the first term, we observe that since χ_{hi} projects onto the set $|Q| > \alpha^{r-1}$, we have for any $C > 0$,

$$\chi_{\text{hi}}(Q)\ e^{-C|Q|}\ \leqslant e^{-C\alpha^{r-1}}. \tag{3.156}$$

Letting $C = C_S$ be the constant from Proposition 3.4.1, we then have

$$\begin{aligned} \|\ \chi_{\text{hi}}\ S^\sigma_J\ \|_{L^{2,a}(\mathbb{R})} &= \left\|\ \chi_{\text{hi}}\ e^{-C_S|Q|}\ e^{C|Q|}\ S^\sigma_J\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \\ &\lesssim e^{-C_S\alpha^{r-1}}\ \left\|\ e^{C_S|Q|}\ S^\sigma_J\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim e^{-C_S\alpha^{r-1}}. \end{aligned} \tag{3.157}$$

To the $m = \pm 1$ terms in (3.155), we apply Lemma 3.4.1 again with the fact that $e^{C_S|Q|}S^\sigma_J \in L^{2,a}(\mathbb{R})$. This gives for $m = \pm 1$,

$$\|\ \chi_{\mathcal{B}}\ S^\alpha_J * S^\sigma_J * S^\sigma_J(Q - 2m\pi/\alpha)\ \|_{L^{2,a}(\mathbb{R}_Q)} \lesssim\ e^{-C_S\pi/\alpha}. \tag{3.158}$$

To the final $m = 0$ terms in (3.155), we use a similar approach to that used in the proof of Lemma 3.4.1. Specifically, observe that for any $\xi, \zeta \in \mathbb{R}$ and $C > 0$,

$$|Q| \leqslant\ |Q - \xi - \zeta| + |\xi| + |\zeta| \qquad \Longrightarrow \qquad e^{C|Q|}\ \leqslant\ e^{C|Q-\xi-\zeta|}\ e^{C|\xi|}\ e^{C|\zeta|}. \tag{3.159}$$

In turn, for any $\widetilde{f} \in L^{2,a}(\mathbb{R})$ with $e^{C|Q|}\widetilde{f} \in L^{2,a}(\mathbb{R}_Q)$,

$$\begin{aligned} e^{C|Q|}\ |\ \widetilde{f} * \widetilde{f} * \widetilde{f}(Q)\ | &\leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} e^{C|Q|}\ |\widetilde{f}(\xi)|\ |\widetilde{f}(\zeta)|\ |\widetilde{f}(Q - \xi - \zeta)|\ d\xi\ d\zeta \\ &\leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} e^{C|\xi|}\ |\widetilde{f}(\xi)|\ e^{C|\zeta|}\ |\widetilde{f}(\zeta)|\ e^{C|Q-\xi-\zeta|}\ |\widetilde{f}(Q - \xi - \zeta)|\ d\xi\ d\zeta \\ &= \left(e^{C|\cdot|}|\widetilde{f}|\right) * \left(e^{C|\cdot|}|\widetilde{f}|\right) * \left(e^{C|\cdot|}|\widetilde{f}|\right)(Q). \end{aligned} \tag{3.160}$$

Thus, (3.160), (3.156), Proposition 3.4.1 and (3.139) give

$$\begin{aligned} &\|\ \chi_{\text{hi}} S^\sigma_J * S^\sigma_J * S^\sigma_J\ \|_{L^{2,a}(\mathbb{R})} \leqslant e^{-C_S\alpha^{r-1}}\ \left\|\ e^{C_S|\cdot|}\ S^\sigma_J * S^\sigma_J * S^\sigma_J\ \right\|_{L^{2,a}_\alpha(\mathbb{R})} \\ &\leqslant e^{-C_S\alpha^{r-1}}\ \left\|\ \left(e^{C_S|\cdot|}|S^\sigma_J|\right) * \left(e^{C_S|\cdot|}|S^\sigma_J|\right) * \left(e^{C_S|\cdot|}|S^\sigma_J|\right)\right\|_{L^{2,a}(\mathbb{R})} \\ &\lesssim e^{-C_S\alpha^{r-1}}\ \left\|\ e^{C_S|Q|}\ S^\sigma_J\ \right\|^3_{L^{2,a}(\mathbb{R}_Q)} \lesssim\ e^{-C_S\alpha^{r-1}}. \end{aligned} \tag{3.161}$$

To complete the proof of Proposition 3.4.10, we apply Proposition 3.4.9 to the multiplier $\chi_{\text{hi}}\ \chi_{\mathcal{B}_\alpha}\ \ [1 + M_\alpha(Q)]^{-1}$ and note that $\alpha^{2-2r} \leqslant 1$ for sufficiently small α . □

Proposition 3.4.11. *Let* $R_{\text{L}} = \chi_{\mathcal{B}_\alpha} R_{\text{L}}$ *and* $R_{\text{NL}} = \chi_{\mathcal{B}_\alpha} R_{\text{NL}}$ *be as defined in* (3.112). *Then*

$$\begin{aligned}\left\| R_{\text{L}}\left[\alpha, \Gamma + \widehat{E_{\text{hi}}}\right] \right\|_{L^{2,a}(\mathbb{R})} &\lesssim \| \Gamma \|_{L^{2,a}(\mathbb{R})} + \left\| \widehat{E_{\text{hi}}} \right\|_{L^{2,a}(\mathbb{R})}, \\ \left\| R_{\text{NL}}\left[\alpha, \Gamma + \widehat{E_{\text{hi}}}\right] \right\|_{L^{2,a}(\mathbb{R})} &\lesssim \| \Gamma \|^2_{L^{2,a}(\mathbb{R})} + \| \Gamma \|_{L^{2,a}(\mathbb{R})} \left\| \widehat{E_{\text{hi}}} \right\|_{L^{2,a}(\mathbb{R})} \\ &\quad + \left\| \widehat{E_{\text{hi}}} \right\|^2_{L^{2,a}(\mathbb{R})} + \| \Gamma \|^3_{L^{2,a}(\mathbb{R})} + \| \Gamma \|^2_{L^{2,a}(\mathbb{R})} \left\| \widehat{E_{\text{hi}}} \right\|_{L^{2,a}(\mathbb{R})} \\ &\quad + \| \Gamma \|_{L^{2,a}(\mathbb{R})} \left\| \widehat{E_{\text{hi}}} \right\|^2_{L^{2,a}(\mathbb{R})} + \left\| \widehat{E_{\text{hi}}} \right\|^3_{L^{2,a}(\mathbb{R})}. \end{aligned} \tag{3.162}$$

These bounds follow from (3.139). We now apply Propositions 3.4.10, 3.4.9, and 3.4.11 to $\mathcal{R}^{\sigma}_{J,1}\left[\alpha, \Gamma + \widehat{E_{\text{hi}}}\right]$ given in (3.111) to obtain (3.126). This completes the proof of Proposition 3.4.5. □

Proof of Proposition 3.4.6: We apply Propositions 3.4.1 and 3.4.9 along with (3.139) to (3.130) to get

$$\begin{aligned}\left\| D_{\widehat{E_{\text{hi}}}} \left(\mathcal{A}[\alpha, \Gamma, \widehat{E_{\text{hi}}}] \right) - I \right\|_{L^{2,a}(\mathbb{R}) \longrightarrow L^{2,a}(\mathbb{R})} &\lesssim \alpha^{2-2r} \Bigg(1 + \|\Gamma\|_{L^{2,a}(\mathbb{R})} + \|\widehat{E_{\text{hi}}}\|_{L^{2,a}(\mathbb{R})} \\ &\quad + \|\Gamma\|^2_{L^{2,a}(\mathbb{R})} + \|\Gamma\|_{L^{2,a}(\mathbb{R})} \|\widehat{E_{\text{hi}}}\|_{L^{2,a}(\mathbb{R})} + \|\widehat{E_{\text{hi}}}\|^2_{L^{2,a}(\mathbb{R})} \Bigg). \qquad \square \end{aligned} \tag{3.163}$$

Proof of Proposition 3.4.7: By Propositions 3.4.5 and 3.4.6 , we have the map $\widehat{E_{\text{hi}}} : [0, \alpha_0) \times B_{\beta_0}(0) \to L^{2,a}(\mathbb{R}),\ (\alpha, \Gamma) \mapsto \widehat{E_{\text{hi}}}[\alpha, \Gamma]$ which is the unique solution to (3.134),

$$\begin{aligned}\widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right](Q) = \chi_{\text{hi}}(Q)\ [1 + M_\alpha(Q)]^{-1} \Bigg(\chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\ \left[\widetilde{\psi_1} * \widetilde{\psi_1} * \Gamma(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right](Q) \right] \\ - \mathcal{R}^{\sigma}_{J,1}\left[\alpha, \Gamma + \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right]\right](Q) \Bigg) = 0. \end{aligned} \tag{3.164}$$

We now estimate the mapping $\widehat{E_{\text{hi}}}[\alpha, \Gamma]$ in $L^{2,a}(\mathbb{R}^d)$. By (3.132) and (3.133),

$$\alpha \in [0, \alpha_0) \quad \text{and} \quad \| \Gamma \|_{L^{2,a}(\mathbb{R})} \leqslant \beta_0 \qquad \Longrightarrow \qquad \left\| \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right] \right\|_{L^{2,a}(\mathbb{R})} \leqslant \kappa, \tag{3.165}$$

for some constants $\alpha_0, \beta_0, \kappa > 0$. We apply Proposition 3.4.5 and (3.165) to (3.164) to get

$$\left\| \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right] \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C\alpha^{r-1}} + \alpha^{2-2r} \left(\| \Gamma \|_{L^{2,a}(\mathbb{R})} + \left\| \widehat{E_{\text{hi}}}\left[\alpha, \Gamma\right] \right\|_{L^{2,a}(\mathbb{R})} \right). \tag{3.166}$$

Since $\alpha^{2-2r} \to 0$ when $\alpha \to 0$ when $0 < r < 1$, taking α sufficiently small in (3.166) yields (3.135).

Next, the differentiability of $\mathcal{A}$ stated in Proposition 3.4.6 and the implicit function theorem guarantee that $D_\Gamma \widehat{E_{\rm hi}}[\alpha, \Gamma]$ is well-defined and given by

$$\begin{aligned}\left(D_\Gamma \widehat{E_{\rm hi}}[\alpha, \Gamma]\right)\ \widehat{f}(Q) = \chi_{\rm hi}\ \left[1 + M_\alpha(Q)\right]^{-1} \Bigg[\frac{3}{4\pi^2}\ \chi_{\mathcal{B}_\alpha}\ \Bigg(\widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{f}(Q) \\ + \widetilde{\psi_1} * \widetilde{\psi_1} * \left\{D_\Gamma\left(\widehat{E_{\rm hi}}[\alpha, \Gamma]\right)\ \widehat{f}\right\}(Q)\Bigg) \\ + D_\Gamma\left(\mathcal{R}^\sigma_{J,1}\left[\alpha, \Gamma + \widehat{E_{\rm hi}}\left[\alpha, \Gamma\right]\right]\right)\widehat{f}(Q)\Bigg],\end{aligned} \tag{3.167}$$

for any $\widehat{f} \in L^{2,a}(\mathbb{R})$. We apply Propositions 3.4.1 and 3.4.9 along with equations (3.139), (3.165), and (3.165) to (3.167) to get (3.136). This completes the proof of Proposition 3.4.7. □

3.5 Analysis of the Low Frequency Equation, Governing $\widehat{E_{\rm lo}}$

3.5.1 Equation for $\widehat{E_{\rm lo}}$ as a Perturbation of the Continuum NLS Limit via the Operator $\widetilde{L_+}$

Having constructed $\widehat{E_{\rm lo}} \mapsto \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]$, we insert this map into equation (3.118) and obtain the following closed equation for $\widehat{E_{\rm lo}}(Q)$ on $Q \in \mathbb{R}$:

$$\begin{aligned}&\left[1 + M_\alpha(Q)\right]\ \widehat{E_{\rm lo}}(Q) - \chi_{\rm lo}(Q)\frac{3}{4\pi^2}\widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q) \\ &\quad = \chi_{\rm lo}(Q)\ \mathcal{R}^\sigma_{J,1}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right](Q) + \chi_{\rm lo}(Q)\frac{3}{4\pi^2}\widetilde{\psi_1} * \widetilde{\psi_1} * \left(\widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right)(Q).\end{aligned} \tag{3.168}$$

Here, $\mathcal{R}^\sigma_{J,1}$ is given in (3.111).

Noting that the operator on the left-hand-side of (3.168) has a formal $\alpha \downarrow 0$ limit equal to the linearized continuum NLS operator, $\widetilde{L_+}$, we now rewrite (3.168) as a small α perturbation of this limit:

Proposition 3.5.1. *There exists* $0 \leqslant \alpha_1 \leqslant \alpha_0$ *and* $0 < r < 1$ *such that the following holds. For* $\alpha \leqslant \alpha_1$*, equation* (3.168) *may be written as*

$$\widetilde{L_+}\widehat{E_{\rm lo}}(Q) = \mathcal{R}^\sigma_{J,2}[\alpha, \widehat{E_{\rm lo}}](Q). \tag{3.169}$$

$\mathcal{R}^\sigma_{J,2}\left[\alpha, \widehat{E_{\rm lo}}\right](Q)$*, displayed in* (3.177)*, is continuous at* $(0,0) \in [0, \alpha_1) \times L^{2,a}(\mathbb{R})$*. Furthermore, the mapping* $\widehat{E_{\rm lo}} \mapsto \mathcal{R}^\sigma_{J,2}\left[\alpha, \widehat{E_{\rm lo}}\right]$ *is Fréchet differentiable with respect to* $\widehat{E_{\rm lo}}$*, with* $D_{\widehat{E_{\rm lo}}}\mathcal{R}^\sigma_{J,2}[\alpha, \widehat{E_{\rm lo}}]$

displayed in (3.193)*. Finally, we have the bounds*

$$\left\| \mathcal{R}^{\sigma}_{J,2}[\alpha, \widehat{E_{\rm lo}}] \right\|_{L^{2,a-2}(\mathbb{R})} \lesssim \alpha^{2J+2} + \left\| \widehat{E_{\rm lo}} \right\|_{L^{2,a}(\mathbb{R})} + \left\| \widehat{E_{\rm lo}} \right\|^2_{L^{2,a}(\mathbb{R})} + \left\| \widehat{E_{\rm lo}} \right\|^3_{L^{2,a}(\mathbb{R})}, \tag{3.170}$$

$$\left\| D_{\widehat{E_{\rm lo}}} \mathcal{R}^{\sigma}_{J,2}\left[\alpha, \widehat{E_{\rm lo}}\right] \right\|_{L^{2,a}(\mathbb{R}) \to L^{2,a-2}(\mathbb{R})} \lesssim \alpha^{2r} + \alpha^{2-2r} + \left\| \widehat{E_{\rm lo}} \right\|_{L^{2,a}(\mathbb{R})} + \left\| \widehat{E_{\rm lo}} \right\|^2_{L^{2,a}(\mathbb{R})}. \tag{3.171}$$

3.5.2 Part 1 of the Proof of Proposition 3.5.1; Detailed Derivation of Equation (3.169) for $\widehat{E_{\rm lo}}$

We proceed to rewrite equation (3.168) in the form (3.169). First, we use $\chi_{\rm hi} = 1 - \chi_{\rm lo}$ to get

$$\chi_{\rm lo}(Q)\, \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q) = \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q) - \chi_{\rm hi}(Q)\, \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q). \tag{3.172}$$

Next note as a consequence of (3.168) we have $\overline{\chi}_{\rm lo} \widehat{E_{\rm lo}} = 0$. Therefore, we may write:

$$M_\alpha(Q)\, \widehat{E_{\rm lo}}(Q) = |Q|^2\, \widehat{E_{\rm lo}}(Q) + \chi_{\rm lo}\left[M_\alpha(Q) - |Q|^2\right]\, \widehat{E_{\rm lo}}(Q). \tag{3.173}$$

We may now express the left-hand side of (3.168) via (3.172) and (3.173) as

$$[1 + M_\alpha(Q)]\, \widehat{E_{\rm lo}}(Q) - \chi_{\rm lo}(Q) \frac{3}{4\pi^2} \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q) = \widetilde{L_+} \widehat{E_{\rm lo}}(Q) - R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right](Q). \tag{3.174}$$

Here, we have defined the (linear in $\widehat{E_{\rm lo}}$) operator

$$\begin{aligned} R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right](Q) &\equiv \left[\, |Q|^2 - M_\alpha(Q)\, \right]\, \widehat{E_{\rm lo}}(Q) - \chi_{\rm hi}(Q)\, \frac{3}{4\pi^2}\, \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q) \\ &= \chi_{\rm lo}(Q)\, \left[\, |Q|^2 - M_\alpha(Q)\, \right]\, \widehat{E_{\rm lo}}(Q) - \chi_{\rm hi}(Q)\, \frac{3}{4\pi^2}\, \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\rm lo}}(Q). \end{aligned} \tag{3.175}$$

We may now rewrite (3.168) as

$$\widehat{L_+} \widehat{E_{\rm lo}}(Q) = \chi_{\rm lo}(Q)\, \mathcal{R}^{\sigma}_{J,1}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right](Q) + R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right](Q). \tag{3.176}$$

Now define

$$\mathcal{R}^{\sigma}_{J,2}\left[\alpha, \widehat{E_{\rm lo}}\right](Q) \equiv \chi_{\rm lo}(Q)\, \mathcal{R}^{\sigma}_{J,1}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right](Q) + R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right](Q), \tag{3.177}$$

where $\mathcal{R}^{\sigma}_{J,1}$ is defined in (3.111), to get equation (3.169). Finally, the expression for $D_{\widehat{E_{\rm lo}}} \mathcal{R}^{\sigma}_{J,2}[\alpha, \widehat{E_{\rm lo}}]$ is displayed and bounded, together with $\mathcal{R}^{\sigma}_{J,2}[\alpha, \widehat{E_{\rm lo}}]$ in Section 3.5.4.

3.5.3 Some Tools for Estimates on the Low Frequency Equation, (3.169)

To establish the properties of $\mathcal{R}^{\sigma}_{J,2}$ given in Proposition 3.5.1, we require a number of general tools. Recall that $\chi_{\rm hi}(Q) = 1 - \chi_{\rm lo}(Q)$, where $\chi_{\rm lo}(Q) \equiv \chi_{[-\alpha^{r-1},\alpha^{r-1}]}(Q)$ and $0 < r < 1$. The next four lemmata will be used to bound $\mathcal{R}^{\sigma}_{J,2}$.

Lemma 3.5.1. *For any function $\widehat{f} \in L^{2,a-2}(\mathbb{R})$ such that $|Q|^{-2}\widehat{f} \in L^{2,a}(\mathbb{R}_Q)$, we have*

$$\left\|\widehat{f}\right\|_{L^{2,a-2}(\mathbb{R})} \lesssim \left\||Q|^{-2}\widehat{f}\ \right\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{3.178}$$

Proof of Lemma 3.5.1: Observe that

$$\begin{aligned}\left\|\widehat{f}\right\|^2_{L^{2,a-2}(\mathbb{R})} &= \int_{\mathbb{R}} (1+|Q|^2)^{a-2}\ |\widehat{f}(Q)|^2\ dQ = \int_{\mathbb{R}} \frac{|Q|^4}{(1+|Q|^2)^2}\ \ (1+|Q|^2)^a\ |Q|^{-4}\ |\widehat{f}(Q)|^2\ dQ \\ &\leqslant \int_{\mathbb{R}} \frac{|Q|^4}{1+|Q|^4}\ \ (1+|Q|^2)^a\ |Q|^{-4}\ |\widehat{f}(Q)|^2\ dQ \\ &\leqslant \int_{\mathbb{R}} (1+|Q|^2)^a\ |Q|^{-4}\ |\widehat{f}(Q)|^2\ dQ = \left\||Q|^{-2}\widehat{f}\ \right\|^2_{L^{2,a}(\mathbb{R}_Q)}. \qquad \Box\end{aligned} \tag{3.179}$$

Lemma 3.5.2. *For any function $\widehat{f} \in L^{2,a}(\mathbb{R})$, we have*

$$\left\|\ \chi_{\rm hi}\widehat{f}\ \right\|_{L^{2,a-2}(\mathbb{R})} \lesssim \alpha^{2-2r}\ \|\widehat{f}\|_{L^{2,a}(\mathbb{R})}. \tag{3.180}$$

This follows from Lemma 3.5.1 and $\chi_{\rm hi}|Q|^{-2} \leqslant \alpha^{2-2r}$.

Lemma 3.5.3. *Let $0 < r < 1$. For any function $\widehat{f} \in L^{2,a}(\mathbb{R})$, we have*

$$\left\|\ \chi_{\rm lo}\ \left[\ |Q|^2 - M_\alpha(Q)\ \right]\ \widehat{f}\ \right\|_{L^{2,a-2}(\mathbb{R}_Q)} \lesssim \alpha^{2r}\ \left\|\ \widehat{f}\ \right\|_{L^{2,a}(\mathbb{R})}. \tag{3.181}$$

Proof of Lemma 3.5.3: We apply Lemma 3.5.1 to get

$$\left\|\ \chi_{\rm lo}\ \left[\ |Q|^2 - M_\alpha(Q)\ \right]\ \widehat{f}\ \right\|_{L^{2,a-2}(\mathbb{R}_Q)} \lesssim \left\|\ \chi_{\rm lo}\ |Q|^{-2}\ \left[\ |Q|^2 - M_\alpha(Q)\ \right]\ \widehat{f}\ \right\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{3.182}$$

We Taylor expand

$$\begin{aligned}\chi_{\rm lo}(Q)\ |Q|^{-2}\ M_\alpha(Q) &= \chi_{\rm lo}(Q)\ \frac{4}{\alpha^2|Q|^2}\sin^2\left(\frac{Q\alpha}{2}\right) = 2\ \chi_{\rm lo}(Q)\ \sum_{k=0}^{\infty} \frac{(-1)^k\ \alpha^{2k}\ |Q|^{2k}}{(2k+2)!} \\ &= \chi_{\rm lo}(Q) + 2\ \chi_{\rm lo}(Q)\ \sum_{k=1}^{\infty} \frac{(-1)^k\ \alpha^{2k}\ |Q|^{2k}}{(2k+2)!} \\ &= \chi_{\rm lo}(Q) + 2\ \chi_{\rm lo}(Q)\ \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\ \alpha^{2k+2}\ |Q|^{2k+2}}{(2k+4)!}.\end{aligned} \tag{3.183}$$

Thus,

$$\chi_{\text{lo}}(Q)\ |Q|^{-2}\ \left[\ |Q|^2 - M_\alpha(Q)\ \right] = 2\ \chi_{\text{lo}}(Q)\ \sum_{k=0}^{\infty} \frac{(-1)^k\ \alpha^{2k+2}\ |Q|^{2k+2}}{(2k+4)!} \tag{3.184}$$

Since $\chi_{\text{lo}}(Q)$ projects onto $|Q| \leqslant \alpha^{r-1}$, we have

$$\begin{aligned} &\left| \chi_{\text{lo}}(Q)\ \sum_{k=0}^{\infty} \frac{(-1)^k\ \alpha^{2k+2}\ |Q|^{2k+2}}{(2k+4)!} \right| \\ &\leqslant \chi_{\text{lo}}(Q)\ \sum_{k=0}^{\infty} \frac{\alpha^{2k+2}\ |Q|^{2k+2}}{(2k+4)!} \leqslant \alpha^{2r}\ \sum_{k=0}^{\infty} \frac{\alpha^{2kr}}{(2k+4)!} \lesssim \alpha^{2r}, \end{aligned} \tag{3.185}$$

such that (3.182) gives

$$\begin{aligned} &\left\| \chi_{\text{lo}}\ \left[\ |Q|^2 - M_\alpha(Q)\ \right]\ \widehat{f}\ \right\|_{L^{2,a-2}(\mathbb{R}_Q)} \\ &\lesssim \left\| \chi_{\text{lo}}\ |Q|^{-2}\ \left[\ |Q|^2 - M_\alpha(Q)\ \right]\ \widehat{f}\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \alpha^{2r}\ \left\|\widehat{f}\right\|_{L^{2,a}(\mathbb{R})}. \end{aligned} \tag{3.186}$$

□

Lemma 3.5.4. *Suppose that $\widehat{f}_1 \in L^{2,a}(\mathbb{R})$ and there exists $C > 0$ such that $e^{C|Q|}\widehat{f}_1(Q) \in L^{2,a}(\mathbb{R}_Q)$. Then for all $\widehat{f}_2, \widehat{f}_3 \in L^{2,a}(\mathbb{R})$, we have*

$$\left\| \overline{\chi}_{\mathcal{B}_\alpha}\ \widehat{f}_1\ \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C\pi/\alpha}\ \left\| e^{C|Q|}\ \widehat{f}_1(Q)\right\|_{L^{2,a}(\mathbb{R}_Q)}, \tag{3.187}$$

$$\left\| \left(\overline{\chi}_{\mathcal{B}_\alpha} \widehat{f}_1\right) * \widehat{f}_2 * \widehat{f}_3\right\|_{L^{2,a-2}(\mathbb{R})} \lesssim\ e^{-C\pi/\alpha}\ \left\| e^{C|Q|}\ \widehat{f}_1(Q)\right\|_{L^{2,a}(\mathbb{R}_Q)} \left\|\widehat{f}_2\right\|_{L^{2,a}(\mathbb{R})} \left\|\widehat{f}_3\right\|_{L^{2,a}(\mathbb{R})}. \tag{3.188}$$

where $\overline{\chi}_{\mathcal{B}_\alpha}(Q) = \chi_{\{|Q|>\pi/\alpha\}}$.

Proof of Lemma 3.5.4: Observe that

$$\begin{aligned} \left\| \overline{\chi}_{\mathcal{B}_\alpha}\ \widehat{f}_1\ \right\|_{L^{2,a}(\mathbb{R})} &= \left\| \overline{\chi}_{\mathcal{B}_\alpha}\ e^{-C|Q|}\ e^{C|Q|}\ \widehat{f}_1\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \leqslant\ e^{-C\pi/\alpha}\ \left\| \overline{\chi}_{\mathcal{B}_\alpha}\ e^{C|Q|}\ \widehat{f}_1\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \\ &\leqslant\ e^{-C\pi/\alpha}\ \left\| e^{C|Q|}\ \widehat{f}_1\ \right\|_{L^{2,a}(\mathbb{R}_Q)}. \end{aligned} \tag{3.189}$$

Next, by (3.139),

$$\begin{aligned} \left\|\left(\overline{\chi}_{\mathcal{B}_\alpha} \widehat{f}_1\right) * \widehat{f}_2 * \widehat{f}_3\right\|_{L^{2,a}(\mathbb{R})} &\leqslant \left\|\left(\overline{\chi}_{\mathcal{B}_\alpha} \widehat{f}_1\right) * \widehat{f}_2 * \widehat{f}_3\right\|_{L^{2,a}(\mathbb{R})} \\ &\leqslant\ \left\| \overline{\chi}_{\mathcal{B}_\alpha}\ \widehat{f}_1\right\|_{L^{2,a}(\mathbb{R})}\ \left\|\widehat{f}_2\right\|_{L^{2,a}(\mathbb{R})}\ \left\|\widehat{f}_3\right\|_{L^{2,a}(\mathbb{R})} \\ &\leqslant\ e^{-C\pi/\alpha}\ \left\| e^{C|Q|}\ \widehat{f}_1\ \right\|_{L^{2,a}(\mathbb{R}_Q)}\ \left\|\widehat{f}_2\right\|_{L^{2,a}(\mathbb{R})}\ \left\|\widehat{f}_3\right\|_{L^{2,a}(\mathbb{R})} \end{aligned} \tag{3.190}$$

Here, we also use that $\|\widehat{f}\|_{L^{2,a-2}(\mathbb{R})} \leqslant \|\widehat{f}\|_{L^{2,a}(\mathbb{R})}$. □

3.5.4 Part 2 of the Proof of Proposition 3.5.1; Analysis of $\mathcal{R}^{\sigma}_{J,2}$ and Derivation of Estimates (3.170) and (3.171)

To bound $\mathcal{R}^{\sigma}_{J,2} : \mathbb{R} \times L^{2,a}(\mathbb{R}) \mapsto L^{2,a-2}(\mathbb{R})$, we use the following estimates from Proposition 3.4.4, valid for $\alpha < \alpha_0$:

$$\left\| \widehat{E_{\rm hi}} \left[\alpha, \widehat{E_{\rm lo}} \right] \right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \, \| \widehat{E_{\rm lo}} \|_{L^{2,a}(\mathbb{R})} + e^{-C/\alpha^{1-r}}, \tag{3.191}$$

$$\left\| \, D_{\widehat{E_{\rm lo}}} \, \widehat{E_{\rm hi}} \left[\alpha, \widehat{E_{\rm lo}} \right] \, \right\|_{L^{2,a}(\mathbb{R}) \to L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \, . \tag{3.192}$$

For any $\widehat{f} \in L^{2,a}(\mathbb{R})$, a direct computation using (3.177) and the linearity of $R^{\sigma}_{\rm L}$ and $R_{\rm pert}$ in their second argument gives

$$\begin{aligned} D_{\widehat{E_{\rm lo}}} \mathcal{R}^{\sigma}_{J,2}[\alpha, \widehat{E_{\rm lo}}] \; \widehat{f}(Q) &= \chi_{\rm lo}(Q) \; R^{\sigma}_{\rm L} \left[\alpha, \; \chi_{\rm lo} \widehat{f} \right] (Q) \\ &+ \chi_{\rm lo}(Q) \; R^{\sigma}_{\rm L} \left[\alpha, \left(D_{\widehat{E_{\rm lo}}} \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}] \right) \widehat{f} \right] (Q) + R_{\rm pert} \left[\alpha, \widehat{f} \right] (Q) \\ &+ D_{\widehat{E_{\rm lo}}} \left(\chi_{\rm lo} \; R^{\sigma}_{NL} \left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}] \right] \right) \widehat{f}(Q). \end{aligned} \tag{3.193}$$

Here,

$$\begin{aligned} & D_{\widehat{E_{\rm lo}}} \left(\chi_{\rm lo} \; R^{\sigma}_{NL} \left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}] \right] \right) \widehat{f}(Q) \\ & = \chi_{\rm lo}(Q) \; \left(\frac{1}{2\pi} \right)^2 \sum_{m=-1}^{1} e^{2m\pi i \sigma} \Big[6 \; S^{\alpha}_J * \widehat{E^{\alpha,\sigma}_J} * \widehat{f}(Q - 2m\pi/\alpha) \\ & + 3 \; \widehat{E_{\rm lo}} * \widehat{E_{\rm lo}} * \widehat{f}(Q - 2m\pi/\alpha) \\ & + 6 \; S^{\alpha}_J * \widehat{E_{\rm lo}} * \left(D_{\widehat{E_{\rm lo}}} \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}] \cdot \widehat{f} \right) (Q - 2m\pi/\alpha) \\ & + 3 \; \widehat{E_{\rm lo}} * \widehat{E_{\rm lo}} * \left(D_{\widehat{E_{\rm lo}}} \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}] \cdot \widehat{f} \right) (Q - 2m\pi/\alpha) \Big]. \end{aligned} \tag{3.194}$$

We proceed to bound the terms in (3.177) and (3.193). We will often use $\|\widehat{f}\|_{L^{2,a-2}(\mathbb{R})} \leqslant \|\widehat{f}\|_{L^{2,a}(\mathbb{R})}$ without explicitly stating it.

Proposition 3.5.2. *Let $R_{\rm pert}[\alpha, \widehat{f}]$ be as defined in* (3.175)*, $0 < r < 1$, and $0 < \alpha < \alpha_0$. Then for $\widehat{f} \in L^{2,a}(\mathbb{R})$,*

$$\left\| \; R_{\rm pert} \left[\alpha, \widehat{f} \right] \; \right\|_{L^{2,a-2}(\mathbb{R})} \lesssim \left(\alpha^{2r} + \alpha^{2-2r} \right) \; \left\| \; \widehat{f} \; \right\|_{L^{2,a}(\mathbb{R})} . \tag{3.195}$$

Proof of Proposition 3.5.2: From the definition in (3.175), recall that R_{pert} is linear in $\widehat{E_{\text{lo}}}$. We may therefore apply Lemmata 3.5.2 and 3.5.3 to get both estimates. □

Proposition 3.5.3. *Let $0 < r < 1$. Let R^{σ}_{L} be defined in (3.112) and let $\widehat{f} \in L^{2,a}(\mathbb{R})$. Then there exists some α_0 such that for $0 < \alpha < \alpha_0$, we have for some $C > 0$*

$$\left\| \chi_{\text{lo}} \ R^{\sigma}_{\text{L}} \left[\alpha, \ \chi_{\text{lo}} \widehat{f}\right] \right\|_{L^{2,a-2}(\mathbb{R})} \lesssim \alpha^2 \ \left\| \widehat{f} \right\|_{L^{2,a}(\mathbb{R})}, \tag{3.196}$$

$$\left\| \chi_{\text{lo}} \ R^{\sigma}_{\text{L}} \left[\alpha, \widehat{f}\right] \right\|_{L^{2,a-2}(\mathbb{R})} \lesssim \left\| \widehat{f} \right\|_{L^{2,a}(\mathbb{R})}. \tag{3.197}$$

Proof of Proposition 3.5.3: From (3.112), we have for any $\widehat{f} \in L^{2,a}(\mathbb{R})$.

$$\begin{aligned} R^{\sigma}_{\text{L}} \left[\alpha, \widehat{f}\right](Q) \equiv & \chi_{\mathcal{B}_\alpha}(Q) \ \frac{3}{4\pi^2} \left[\ \sum_{m=-1}^{1} e^{2m\pi i \sigma} \ S^{\alpha}_{J} * S^{\alpha}_{J} * \widehat{f}(Q - 2m\pi/\alpha) \right. \\ & \left. - \ \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{f}(Q) \right]. \end{aligned} \tag{3.198}$$

An application of (3.139) gives (4.121).

To obtain (4.120), we first address the $m = 0$ term. We have by definition (3.106),

$$\begin{aligned} S^{\alpha}_{J}(Q) &= \sum_{j=0}^{J} \alpha^{2j} \ \chi_{\mathcal{B}_\alpha}(Q) \ F_j \left[\widetilde{\psi_1}\right](Q) = \chi_{\mathcal{B}_\alpha}(Q) \ \widetilde{\psi_1}(Q) + \sum_{j=1}^{J} \alpha^{2j} \ \chi_{\mathcal{B}_\alpha}(Q) \ F_j \left[\widetilde{\psi_1}\right](Q) \\ &= \left[1 - \overline{\chi}_{\mathcal{B}_\alpha}(Q)\right] \ \widetilde{\psi_1}(Q) + \sum_{j=1}^{J} \alpha^{2j} \ \chi_{\mathcal{B}_\alpha}(Q) \ F_j \left[\widetilde{\psi_1}\right](Q) \\ &= \left[1 - \overline{\chi}_{\mathcal{B}_\alpha}(Q)\right] \ \widetilde{\psi_1}(Q) + R^{\alpha}_{\text{S}}(Q), \end{aligned} \tag{3.199}$$

for residual sum

$$R^{\alpha}_{\text{S}}(Q) \equiv \sum_{j=1}^{J} \alpha^{2j} \ \chi_{\mathcal{B}_\alpha}(Q) \ F_j \left[\widetilde{\psi_1}\right](Q). \tag{3.200}$$

Proposition 4.7.3 (that is, the construction of $F_j \left[\widetilde{\psi_1}\right] \in L^{2,a}(\mathbb{R})$) implies that

$$\left\| R^{\alpha}_{\text{S}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^2. \tag{3.201}$$

Therefore, from (3.198),

$$\begin{aligned} & S^{\alpha}_{J} * S^{\alpha}_{J} * \widehat{f}(Q) - \ \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{f}(Q) \\ &= 2 \left(\overline{\chi}_{\mathcal{B}_\alpha} \widetilde{\psi_1}\right) * \widetilde{\psi_1} * \widehat{f}(Q) + 2 \left(\overline{\chi}_{\mathcal{B}_\alpha} \widetilde{\psi_1}\right) * R^{\alpha}_{\text{S}} * \widehat{f}(Q) - \left(\overline{\chi}_{\mathcal{B}_\alpha} \widetilde{\psi_1}\right) * \left(\overline{\chi}_{\mathcal{B}_\alpha} \widetilde{\psi_1}\right) * \widehat{f}(Q) \\ & \quad - 2 \ \widetilde{\psi_1} * R^{\alpha}_{\text{S}} * \widehat{f}(Q) - R^{\alpha}_{\text{S}} * R^{\alpha}_{\text{S}} * \widehat{f}(Q), \end{aligned} \tag{3.202}$$

such that (3.139), (3.201), and Lemma 3.5.4 give

$$\left\| S_J^\alpha * S_J^\alpha * \widehat{f} - \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{f} \right\|_{L^{2,a}(\mathbb{R})} \lesssim \left(\alpha^2 + e^{-C/\alpha} \right) \left\| \widehat{f} \right\|_{L^{2,a}(\mathbb{R})}. \tag{3.203}$$

The essential difficulty in estimate (4.120) are the $m = \pm 1$ terms from (3.198), of the form $\chi_{\rm lo}(Q)\ S_J^\alpha * S_J^\alpha * \left(\chi_{\rm lo}\widehat{f}\right)(Q + 2\pi/\alpha)$. The difficulty lies in the fact that we do not have a uniform decay estimate on $\widehat{f}$, but require the bound (4.120) to be small in α in order to apply the implicit function theorem to equation (3.169) (see the hypotheses of Lemma 3.5.5). We therefore use the localization of Q and $\widehat{E_{\rm lo}}$ along with the decay of S_J^α.

First observe from definition (3.106) that

$$S_J^\sigma(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ S_J^\sigma(Q). \tag{3.204}$$

Now, take $m = -1$ and observe that for any $Q, \xi, \zeta \in \mathbb{R}$ satisfying

$$|\xi| \leqslant \frac{\pi}{\alpha}, \quad |\zeta| \leqslant \frac{\pi}{\alpha}, \quad |Q| \leqslant \alpha^{r-1}, \quad |\xi + \zeta - Q - 2\pi/\alpha| \leqslant \alpha^{r-1}, \tag{3.205}$$

we have

$$\xi + \zeta - Q - \frac{2\pi}{\alpha} \geqslant -\alpha^{r-1} \quad \Longrightarrow \quad \xi + \zeta \geqslant \frac{1}{\alpha}(2\pi - \alpha^r) + Q \geqslant \frac{2}{\alpha}(\pi - \alpha^r). \tag{3.206}$$

Similarly when $m = 1$ and for any $Q, \xi, \zeta \in \mathbb{R}$ satisfying

$$|\xi| \leqslant \frac{\pi}{\alpha}, \quad |\zeta| \leqslant \frac{\pi}{\alpha}, \quad |Q| \leqslant \alpha^{r-1}, \quad |\xi + \zeta - Q + 2\pi/\alpha| \leqslant \frac{\pi}{\alpha}, \tag{3.207}$$

we have

$$-\xi - \zeta + Q - \frac{2\pi}{\alpha} \geqslant -\alpha^{r-1} \quad \Longrightarrow \quad -\xi - \zeta \geqslant \frac{1}{\alpha}(2\pi - \alpha^r) - Q \geqslant \frac{2}{\alpha}(\pi - \alpha^r). \tag{3.208}$$

In either case $m = \pm 1$, taking α small enough that $\alpha^r \leqslant \pi/2$ gives for any $C > 0$,

$$\begin{aligned} &|\xi| \leqslant \frac{\pi}{\alpha}, \quad |\zeta| \leqslant \frac{\pi}{\alpha}, \quad |Q| \leqslant \alpha^{r-1}, \quad |\xi + \zeta - Q + 2m\pi/\alpha| \leqslant \alpha^{r-1} \\ &\Longrightarrow |\xi| + |\zeta| \geqslant |\xi + \zeta| \geqslant \frac{2}{\alpha}(\pi - \alpha^r) \geqslant \frac{\pi}{\alpha} \\ &\Longrightarrow 1 = e^{C|\xi|}\ e^{C|\zeta|}\ e^{-C(|\xi| + |\zeta|)} \leqslant e^{-C\pi/\alpha}. \end{aligned} \tag{3.209}$$

Thus, by (3.204) and (3.209) we may write

$$
\begin{aligned}
&\left| \chi_{\rm lo}(Q)\ S_J^\alpha * S_J^\alpha * \left(\chi_{\rm lo}\widehat{f}\right)(Q - 2m\pi/\alpha)\ \right| \\
&\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{\rm lo}(Q)\ \chi_{\mathcal{B}_\alpha}(\xi)\ \chi_{\mathcal{B}_\alpha}(\zeta)\ \chi_{\rm lo}(Q - \xi - \zeta - 2m\pi/\alpha) \\
&\qquad\qquad \cdot |S_J^\alpha(\xi)|\ |S_J^\alpha(\zeta)|\ |\widehat{f}(Q - \xi - \zeta - 2m\pi/\alpha)|\ d\xi\ d\zeta \\
&\leqslant e^{-C\pi/\alpha} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{C|\xi|}\ |S_J^\alpha(\xi)|\ e^{C|\zeta|}\ |S_J^\alpha(\zeta)|\ |\widehat{f}(Q - \xi - \zeta - 2m\pi/\alpha)|\ d\xi\ d\zeta, \qquad (3.210)
\end{aligned}
$$

which using Proposition 3.4.1 and (3.139) gives for some $C > 0$,

$$
\begin{aligned}
&\left\| \ \chi_{\rm lo}\ S_J^\alpha * S_J^\alpha * \left(\chi_{\rm lo}\widehat{f}\right)(Q - 2m\pi/\alpha) \ \right\|_{L^{2,a-2}(\mathbb{R}_Q)} \\
&\lesssim e^{-C/\alpha}\ \left\| e^{C|Q|}\ S_J^\alpha \right\|^2_{L^{2,a}(\mathbb{R}_Q)} \left\| \ \widehat{E_{\rm lo}} \ \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha}\ \left\| \ \widehat{E_{\rm lo}} \ \right\|_{L^{2,a}(\mathbb{R})}. \qquad (3.211)
\end{aligned}
$$

This completes the proof of estimate (4.120) and Proposition 3.5.3. □

Proposition 3.5.4. *Let* $0 < r < 1$. *Let* $R^\sigma_{\rm NL}$ *be defined in* (3.112) *and recall its derivative given in* (3.194). *Then there exists* $\alpha_0, C > 0$ *such that for* $0 < \alpha < \alpha_0$,

$$
\begin{aligned}
\left\| \ \chi_{\rm lo}\ R^\sigma_{\rm NL}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right] \ \right\|_{L^{2,a-2}(\mathbb{R})} &\lesssim e^{-C/\alpha^{1-r}} + e^{-C/\alpha^{1-r}}\ \left\| \ \widehat{E_{\rm lo}} \ \right\|_{L^{2,a}(\mathbb{R})} \\
&\quad + \left\| \ \widehat{E_{\rm lo}} \ \right\|^2_{L^{2,a}(\mathbb{R})} + \left\| \ \widehat{E_{\rm lo}} \ \right\|^3_{L^{2,a}(\mathbb{R})}, \qquad (3.212) \\
\left\| \ D_{\widehat{E_{\rm lo}}}\left(\chi_{\rm lo}\ R^\sigma_{NL}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right]\right) \ \right\|_{L^{2,a}(\mathbb{R})\to L^{2,a-2}(\mathbb{R})} &\lesssim e^{-C/\alpha^{1-r}} \\
&\quad + \left\| \ \widehat{E_{\rm lo}} \ \right\|_{L^{2,a}(\mathbb{R})} + \left\| \ \widehat{E_{\rm lo}} \ \right\|^2_{L^{2,a}(\mathbb{R})}. \qquad (3.213)
\end{aligned}
$$

This follows from (3.139) and estimates (3.191) and (3.192).

Proposition 3.5.5. *Let* $\mathcal{D}^{\sigma,\alpha}$ *be defined in* (3.62). *Then there exists some* α_0 *such that for* $0 < r < 1$ *and* $0 < \alpha < \alpha_0$, *we have*

$$
\|\ \chi_{\rm lo}\ \mathcal{D}^{\sigma,\alpha}[S_J^\alpha]\ \|_{L^{2,a-2}(\mathbb{R})} \lesssim \alpha^{2J+2}. \qquad (3.214)
$$

Proof of Proposition 3.5.5: We work out all estimates in $L^{2,a}(\mathbb{R})$ and then use $L^{2,a}(\mathbb{R}) \hookrightarrow L^{2,a-2}(\mathbb{R})$. Recall from (3.62) that

$$
\begin{aligned}
\mathcal{D}^{\sigma,\alpha}[S_J^\alpha](Q) &\equiv -\left[1 + M_\alpha(Q)\right]\ S_J^\alpha(Q) + \chi_{\mathcal{B}_\alpha}(Q)\ \left(\frac{1}{2\pi}\right)^2 \sum_{m=-1}^{1} e^{2m\pi i\sigma}\ S_J^\alpha * S_J^\alpha * S_J^\alpha(Q - 2m\pi/\alpha) \\
&= R_{\rm F,1}[\alpha] + R^\sigma_{\rm F,2}[\alpha] \qquad (3.215)
\end{aligned}
$$

where we have defined

$$R_{\mathrm{F},1}\left[\alpha\right] \equiv \chi_{\mathcal{B}_\alpha}(Q)\ \left(-\left[1 + M_\alpha(Q)\right]\ S_J^\alpha(Q) + \left(\frac{1}{2\pi}\right)^2\ S_J^\alpha * S_J^\alpha * S_J^\alpha(Q)\right),$$
$$R_{\mathrm{F},2}^\sigma\left[\alpha\right] \equiv \chi_{\mathcal{B}_\alpha}(Q)\ \left(\frac{1}{2\pi}\right)^2 \sum_{m=\pm 1} e^{2m\pi i\sigma}\ S_J^\alpha * S_J^\alpha * S_J^\alpha(Q - 2m\pi/\alpha). \tag{3.216}$$

By Lemma 3.4.1, we have for some $C > 0$,

$$\left\|\ \chi_{\mathrm{lo}}\ R_{\mathrm{F},2}^\sigma\ \right\|_{L^{2,a}(\mathbb{R})} \leqslant \left\|\ R_{\mathrm{F},2}^\sigma\ \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha}. \tag{3.217}$$

We will use the derivation of the partial asymptotic expansion S_J^α from Section 3.3.2 to justify bound (3.214). Recall from 3.106 that

$$S_J^\alpha(Q) = \sum_{j=0}^{J} \alpha^{2j}\ \chi_{\mathcal{B}_\alpha}(Q)\ F_j\left[\widetilde{\psi_1}\right](Q) = \sum_{j=0}^{J} \alpha^{2j}\ \chi_{\mathcal{B}_\alpha}(Q)\ F_j(Q), \tag{3.218}$$

where $F_0[\widetilde{\psi_1}] = \widetilde{\psi_1}$ and $F_j[\widetilde{\psi_1}] = F_j \in L^{2,a}(\mathbb{R})$ is characterized by the construction given in Proposition 4.7.3. Since S_J^α is band-limited (unlike F_j), we decompose it such that we may effectively utilize this construction:

$$S_J^\alpha(Q) = \sum_{j=0}^{J} \alpha^{2j}\ \chi_{\mathcal{B}_\alpha}(Q)\ F_j(Q) = \sum_{j=0}^{J} \alpha^{2j}\ \left(1 - \overline{\chi}_{\mathcal{B}_\alpha}(Q)\right)\ F_j(Q)$$
$$= S_{\mathrm{full}}^\alpha(Q) - S_{\mathrm{tail}}^\alpha(Q), \tag{3.219}$$

where

$$S_{J,\mathrm{full}}^\alpha(Q) \equiv \sum_{j=0}^{J} \alpha^{2j}\ F_j(Q), \qquad \text{and} \qquad S_{J,\mathrm{tail}}^\alpha(Q) \equiv \sum_{j=0}^{J} \alpha^{2j}\ \overline{\chi}_{\mathcal{B}_\alpha}(Q)\ F_j(Q). \tag{3.220}$$

Note that $S_{J,\mathrm{full}}^\alpha$ is the truncation of the formal expansion (4.173), with

$$S_{J,\mathrm{full}}^\alpha(Q) = S_J^\alpha(Q) + S_{J,\mathrm{tail}}^\alpha(Q),$$
$$S_J^\alpha(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ S_J^\alpha(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ S_{J,\mathrm{full}}^\alpha(Q),$$
$$S_{J,\mathrm{tail}}^\alpha(Q) = \overline{\chi}_{\mathcal{B}_\alpha}(Q)\ S_{J,\mathrm{tail}}^\alpha(Q) = \overline{\chi}_{\mathcal{B}_\alpha}(Q)\ S_{J,\mathrm{full}}^\alpha(Q). \tag{3.221}$$

By construction in Proposition 4.7.3, $S_{J,\mathrm{full}}^\alpha, S_{J,\mathrm{tail}}^\alpha \in L^{2,a}(\mathbb{R})$ and there exists a constant $C > 0$ such that $e^{C|Q|}S^{\alpha}_{J,\text{full}},\ e^{C|Q|}S^{\alpha}_{J,\text{tail}} \in L^{2,a}(\mathbb{R}_Q)$. Substituting, we have from (3.216) that

$$\begin{aligned}\chi_{\text{lo}}(Q)\ R_{\text{F},1}[\alpha] = \chi_{\text{lo}}(Q)\ &\left(-[1+M_\alpha(Q)]\ S^{\alpha}_{J,\text{full}}(Q) + \left(\frac{1}{2\pi}\right)^2\ S^{\alpha}_{J,\text{full}} * S^{\alpha}_{J,\text{full}} * S^{\alpha}_{J,\text{full}}(Q)\right),\\ &- \chi_{\text{lo}}(Q)\ \left(\frac{1}{2\pi}\right)^2\ \Big[3\ S^{\alpha}_{J,\text{tail}} * S^{\alpha}_{J,\text{full}} * S^{\alpha}_{J,\text{full}}(Q)\\ &\qquad + 3\ S^{\alpha}_{J,\text{tail}} * S^{\alpha}_{J,\text{tail}} * S^{\alpha}_{J,\text{full}}(Q) + S^{\alpha}_{J,\text{tail}} * S^{\alpha}_{J,\text{tail}} * S^{\alpha}_{J,\text{tail}}(Q)\Big]. \end{aligned} \tag{3.222}$$

To all convolutions involving $S_{J,\text{tail}} = \overline{\chi}_{\mathcal{B}_\alpha}\, S_{J,\text{tail}}$, we apply Lemma 3.5.4 to get another exponentially small bound $\lesssim e^{-C\pi/\alpha}$ in $L^{2,a}(\mathbb{R})$.

We finally turn our attention to the terms involving $S^{\alpha}_{J,\text{full}}$, with which we utilize our formal expansion from Section 3.3.2:

$$R_{\text{full}}[\alpha](Q) \equiv \chi_{\text{lo}}(Q)\ \left(-[1+M_\alpha(Q)]\ S^{\alpha}_{J,\text{full}}(Q) + \left(\frac{1}{2\pi}\right)^2\ S^{\alpha}_{J,\text{full}} * S^{\alpha}_{J,\text{full}} * S^{\alpha}_{J,\text{full}}(Q)\right). \tag{3.223}$$

By the construction in Section 3.3.2 and Proposition 4.7.3, we expand

$$\begin{aligned} M_\alpha(Q) &= \frac{4}{\alpha^2}\sin^2\left(\frac{Q\alpha}{2}\right) = 2\sum_{j=0}^{\infty}\frac{\alpha^{2j}\ (-1)^j\ |Q|^{2j+2}}{(2j+2)!},\\ \text{and}\qquad S^{\alpha}_{J,\text{full}}(Q) &\equiv \sum_{j=0}^{J}\alpha^{2j}\ F_j(Q), \end{aligned} \tag{3.224}$$

such that (3.223) becomes

$$\begin{aligned} R_{\text{full}}[\alpha](Q) &= \chi_{\text{lo}}(Q)\Bigg(2\sum_{j=J+1}^{\infty}\frac{\alpha^{2j}\ (-1)^{j+1}\ |Q|^{2j+2}}{(2j+2)!}\ S^{\alpha}_{J,\text{full}}(Q)\\ &\qquad + \frac{1}{(2\pi)^2}\ \sum_{j=J+1}^{3J}\ \alpha^{2j}\ \sum_{\substack{k+l+m=j\\ 0\leqslant k,l,m<j}}\ \widehat{f}_k * \widehat{f}_l * \widehat{f_m}(Q)\Bigg)\\ &= \chi_{\text{lo}}(Q)\ \alpha^{2J+2}\ \Bigg(2\sum_{j=J+1}^{\infty}\frac{\alpha^{2j-2J-2}\ (-1)^{j+1}\ |Q|^{2j+2}}{(2j+2)!}\ S^{\alpha}_{J,\text{full}}(Q)\\ &\qquad + \frac{1}{(2\pi)^2}\ \sum_{j=J+1}^{3J}\ \alpha^{2j-2J-2}\ \sum_{\substack{k+l+m=j\\ 0\leqslant k,l,m\leqslant J}}\ \widehat{f}_k * \widehat{f}_l * \widehat{f_m}(Q)\Bigg). \end{aligned} \tag{3.225}$$

We may clearly use (3.139) with Proposition 4.7.3 to estimate the convolution terms to be $\lesssim 1$ in

$L^{2,a}(\mathbb{R})$ for $\alpha \lesssim 1$. Similarly, we write the first term as

$$\begin{aligned} &2\ \chi_{\rm lo}(Q)\ \sum_{j=J+1}^{\infty} \frac{\alpha^{2j-2J-2}\ (-1)^{j+1}\ |Q|^{2j+2}}{(2j+2)!}\ S^{\alpha}_{J,{\rm full}}(Q) \\ &= 2\ \chi_{\rm lo}(Q)\ \sum_{j=J+1}^{\infty} \frac{\alpha^{2j-2J-2}\ (-1)^{j+1}\ |Q|^{2j-2J-2}}{(2j+2)!}\ |Q|^{2J+4}\ S^{\alpha}_{J,{\rm full}}(Q). \end{aligned} \tag{3.226}$$

which is we see is also $\lesssim 1$ in $L^{2,a}(\mathbb{R})$ since

$$\begin{aligned} &\|\ |Q|^{2J+4}\ S_{J,{\rm full}}\ \|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \left\|\ e^{C_S|Q|}\ S_{J,{\rm full}}\ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim 1, \\ &\chi_{\rm lo}(Q) \left| \sum_{j=J+1}^{\infty} \frac{\alpha^{2j-2J-2}\ (-1)^{j+1}\ |Q|^{2j-2J-2}}{(2j+2)!} \right| \leqslant \chi_{\rm lo}(Q)\ \sum_{j=0}^{\infty} \frac{\alpha^{2j}\ |Q|^{2j}}{(2J+2j+4)!} \\ &\lesssim \sum_{j=0}^{\infty} \frac{\alpha^{2jr}}{(2J+2j+4)!} \lesssim 1, \end{aligned} \tag{3.227}$$

where we have used that $|Q| \leqslant \alpha^{r-1}$ by the projection $\chi_{\rm lo}$. Thus,

$$\|R_{\rm full}[\alpha]\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2J+2}. \tag{3.228}$$

This completes the proof of Proposition 3.5.5. □

We now apply Propositions 3.5.2, 3.5.3, and 3.5.4 and estimates (3.191) and (3.192) to (3.177) and (3.193). This implies estimates (3.170) and (3.171) in Proposition 3.5.1.

3.5.5 Solution of the Low Frequency Equation

In order to solve equation (3.169), we make use of the following general lemma, a consequence of the implicit function theorem F.0.2 and a specific case of Lemma F.0.13; see Appendix F.

Lemma 3.5.5. *Consider the equation*

$$\mathcal{M}[\alpha, f] \equiv \mathfrak{L}f - \mathcal{R}[\alpha, f] = 0, \quad f \in L^{2,a}_{\rm even}(\mathbb{R}). \tag{3.229}$$

1. $\mathfrak{L} : L^{2,a}_{\rm even}(\mathbb{R}) \longmapsto L^{2,a-2}_{\rm even}(\mathbb{R})$ *be an isomorphism.*

2. $\mathcal{R} : [0, \alpha_1)_\alpha \times L^{2,a}_{\rm even}(\mathbb{R}) \longrightarrow L^{2,a-2}_{\rm even}(\mathbb{R})$ *is continuous at* $(0, 0)$*, Fréchet differentiable on* $L^{2,a}_{\rm even}(\mathbb{R})$

3. $\mathcal{R}[0,0] = 0$ *and satisfies the bounds:*

$$\|\mathcal{R}[\alpha, f]\|_{L^{2,a-2}(\mathbb{R})} \lesssim \mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R})} + \|f\|^2_{L^{2,a}(\mathbb{R})} + \|f\|^3_{L^{2,a}(\mathbb{R})}, \tag{3.230}$$

$$\|D_f\mathcal{R}[\alpha, f]\|_{L^{2,a}(\mathbb{R})\to L^{2,a-2}(\mathbb{R})} \lesssim \mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R})} + \|f\|^2_{L^{2,a}(\mathbb{R})}, \tag{3.231}$$

for some continuous function $\mathfrak{K}(\alpha) \geqslant 0$*, satisfying* $\mathfrak{K}(0) = 0$*,* $\mathfrak{K}(\alpha) \to 0$ *as* $\alpha \to 0$*.*

Then there exists a constant $\alpha_2 \leqslant \alpha_1$ *such that for all* $0 < \alpha < \alpha_2$*, the equation* (F.33)*,* $\mathcal{M}[\alpha, f] = 0$*, has a unique solution* $f = f[\alpha] \in L^{2,a}_{\text{even}}(\mathbb{R})$ *satisfying*

$$\|f[\alpha]\|_{L^{2,a}(\mathbb{R})} \lesssim \mathfrak{K}(\alpha). \tag{3.232}$$

We may now apply Lemma 3.5.5 to the rescaled low frequency equation.

Proposition 3.5.6. *Let* $a > 1/2$ *and* $0 < r < 1$*. Then there exists* $0 < \alpha_2 \leqslant \alpha_1$ *such that for all* $\alpha \in (0, \alpha_2)$*, there exists an even (symmetric) solution* $\widehat{E_{\text{lo}}}$ *to (3.169) which satisfies*

$$\left\|\widehat{E_{\text{lo}}}\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2J+2}. \tag{3.233}$$

Furthermore, we have that $\widehat{E_{\text{lo}}} = \chi_{\text{lo}}\widehat{E_{\text{lo}}}$*; that is,* $\widehat{E_{\text{lo}}}(Q)$ *is supported on* $Q \in [-\alpha^{r-1}, \alpha^{r-1}]$*.*

Proof of Proposition 3.5.6: It suffices to show that the hypotheses to Lemma 3.5.5 are satisfied by equation (3.169) for $a > 1/2$. By Proposition 3.1.3 $\widetilde{L_+} : L^{2,a}_{\text{even}}(\mathbb{R}) \to L^{2,a-2}_{\text{even}}(\mathbb{R})$ is an isomorphism. Moreover, by Proposition 3.5.1 the mapping $(\alpha, \hat{E}_{\text{lo}}) \mapsto \mathcal{R}^{\sigma}_{J,2}[\alpha, \hat{E}_{\text{lo}}]$, maps $L^{2,a}_{\text{even}}(\mathbb{R})$ to $L^{2,a-2}_{\text{even}}(\mathbb{R})$, and is continuous at $(\alpha, \widehat{E_{\text{lo}}}) = (0, 0)$. Furthermore, by choosing $\alpha < \alpha_1$ the estimates (3.170) and (3.171) on $\mathcal{R}^{\sigma}_{J,2}[\alpha, \widehat{E_{\text{lo}}}]$ hold. Hence, hypotheses (F.34) and (F.35) of Lemma 3.5.5 are satisfied. Lemma 3.5.5 implies, for $0 < \alpha < \alpha_2 \leqslant \alpha_2$, the existence of $\widehat{E_{\text{lo}}}$ satisfying the bound (3.233). □

3.6 Completion of the Proofs of Proposition 3.3.3 and Theorem 3.2.1

3.6.1 Completion of the Proof of Proposition 3.3.3

First, we summarize the results of sections 3.4.3 through 3.5.5. Proposition 3.4.4 guarantees that a unique solution $\widehat{E_{\text{hi}}}[\alpha, \Gamma] \in L^{2,a}_{\text{even}}(\mathbb{R})$ exists to the high frequency equation (3.119) for any $\Gamma \in$

$L^{2,a}_{\text{even}}(\mathbb{R})$ and $\alpha > 0$ sufficiently small:

$$[1 + M_\alpha(Q)]\ \widehat{E_{\text{hi}}}[\alpha, \Gamma](Q) - \chi_{\text{hi}}(Q)\ \chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\left(\widetilde{\psi_1} * \widetilde{\psi_1} * \Gamma(Q) + \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\text{hi}}}[\alpha, \Gamma](Q)\right)$$
$$= \chi_{\text{hi}}(Q)\ \mathcal{R}^{\sigma}_{J,1}\left[\alpha, \Gamma + \widehat{E_{\text{hi}}}[\alpha, \Gamma]\right](Q). \tag{3.234}$$

with the bound

$$\left\|\widehat{E_{\text{hi}}}[\alpha, \Gamma]\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r}\ \|\Gamma\|_{L^{2,a}(\mathbb{R})} + e^{-C\alpha^{r-1}}, \qquad 0 < r < 1. \tag{3.235}$$

We apply this result with $\Gamma = \widehat{E_{\text{lo}}} \in L^{2,a}_{\text{even}}(\mathbb{R})$ to obtain $\widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}] \in L^{2,a}_{\text{even}}(\mathbb{R})$. Thus, (3.118) may be rewritten as a closed equation (3.168) for $\widehat{E_{\text{lo}}}$:

$$[1 + M_\alpha(Q)]\ \widehat{E_{\text{lo}}}(Q) - \chi_{\text{lo}}(Q)\frac{3}{4\pi^2}\widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\text{lo}}}(Q)$$
$$= \chi_{\text{lo}}(Q)\ \mathcal{R}^{\sigma}_{J,1}\left[\alpha, \widehat{E_{\text{lo}}} + \widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]\right](Q) + \chi_{\text{lo}}(Q)\frac{3}{4\pi^2}\widetilde{\psi_1} * \widetilde{\psi_1} * \left(\widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]\right)(Q). \tag{3.236}$$

By Proposition 3.5.6 there exists for all $0 < \alpha < \alpha_2$, with α_2 sufficiently small, and any $r \in (0, 1)$ a unique solution $\widehat{E_{\text{lo}}} \in L^{2,a}_{\text{even}}(\mathbb{R})$ of (3.169):

$$\widetilde{L_+}\widehat{E_{\text{lo}}}(Q) = \mathcal{R}^{\sigma}_{J,2}[\alpha, \widehat{E_{\text{lo}}}](Q), \tag{3.237}$$

with the bound

$$\left\|\widehat{E_{\text{lo}}}\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2J+2}. \tag{3.238}$$

By Proposition 3.5.1, equation (3.237) is equivalent to (3.236). By Proposition 3.4.3, adding together equations (3.236) and (3.234) (with $\Gamma(q) = \widehat{E_{\text{lo}}}(q)$) gives equation (3.110) for

$$\hat{E}(Q) \ = \ \widehat{E^{\sigma,\alpha}_J}(Q) \ = \ \widehat{E_{\text{lo}}}(Q) + \widehat{E_{\text{hi}}}\left[\alpha, \widehat{E_{\text{lo}}}\right](Q), \qquad \hat{E}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \hat{E}(Q),$$
$$[1 + M_\alpha(Q)]\ \hat{E}(Q) - \chi_{\mathcal{B}_\alpha}(Q)\ \left(\frac{1}{2\pi}\right)^2 \widetilde{\psi_1} * \widetilde{\psi_1} * \hat{E}(Q) = \mathcal{R}^{\sigma}_{J,1}\left[\alpha, \hat{E}\right](Q)\ . \tag{3.239}$$

By Proposition 3.4.4, $\widehat{E_{\text{lo}}} \in L^{2,a}_{\text{even}}(\mathbb{R})$ implies that $\widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}] \in L^{2,a}_{\text{even}}(\mathbb{R})$ and therefore $\hat{E} \in L^{2,a}_{\text{even}}(\mathbb{R})$. Furthermore, by (3.235) and (3.238), we have the bound

$$\left\|\ \hat{E}\ \right\|_{L^{2,a}(\mathbb{R})} \lesssim \left\|\widehat{E_{\text{lo}}}\right\|_{L^{2,a}(\mathbb{R})} + \left\|\widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]\right\|_{L^{2,a}(\mathbb{R})}$$
$$\lesssim \left(1 + \alpha^{2-2r}\right)\left\|\widehat{E_{\text{lo}}}\right\|_{L^{2,a}(\mathbb{R})} + \mathcal{O}(\alpha^\infty) \lesssim \alpha^{2J+2}, \tag{3.240}$$

for $0 < \alpha < \alpha_2$ with α_2 sufficiently small and $0 < r < 1$. Finally, from (3.66) we have

$$\widehat{\Phi^\sigma}(Q) = \sum_{j=0}^{J} \alpha^{2j} \chi_{\mathcal{B}_\alpha}(Q) F_j\left[\widetilde{\psi_1}\right](Q) + \widehat{E_J^{\alpha,\sigma}}(Q) \qquad \in L^{2,a}_{\text{even}}(\mathbb{R}), \tag{3.241}$$

where $F_j[\widetilde{\psi_1}] = F_j \in L^{2,a}_{\text{even}}(\mathbb{R})$ are given in Proposition 4.7.3.

To complete the proof of Proposition 3.3.3, we show that $\widehat{\Phi^\sigma}$ is real-valued. By Proposition 4.7.3, $F_j[\widetilde{\psi_1}]$, $j = 0, \ldots, J$ are real-valued. It therefore suffices to show that $\widehat{E_J^{\alpha,\sigma}}(Q)$ is real-valued.

Recall equation (3.239) for $\widehat{E} = \widehat{E_J^{\alpha,\sigma}}$, which has inhomogeneous forcing $\mathcal{D}^{\sigma,\alpha}[S_J^\alpha]$ contained in $\mathcal{R}^\sigma_{J,1}[\alpha, \widehat{E}]$ (here, the operator $\mathcal{D}^{\sigma,\alpha}$ is defined in (3.62)) . Note that since $S_J^\sigma = \sum_{j=0}^J \alpha^{2j} \chi_{\mathcal{B}_\alpha} F_j[\widetilde{\psi_1}]$ is real-valued and since $e^{2m\pi i\sigma} = \pm 1$ for $\sigma \in \{0, 1/2\}$, the forcing $\mathcal{D}^{\sigma,\alpha}[S_J^\alpha]$ is also real-valued. Now define

$$\widehat{E^\sigma_{\text{im}}}(Q) \equiv \widehat{E}(Q) - \overline{\widehat{E}}(Q) = 2\ i\ \text{Im}\ \widehat{E^\sigma}(Q). \tag{3.242}$$

Subtracting the complex conjugate of equation (3.239) for $\widehat{E}$ from itself then gives the linear equation for $\widehat{E_{\text{im}}}$:

$$[1 + M_\alpha(Q)]\ \widehat{E_{\text{im}}}(Q) - \chi_{\mathcal{B}_\alpha}(Q) \left(\frac{1}{2\pi}\right)^2 \widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E_{\text{im}}}(Q) = \mathcal{R}^\sigma_{J,\text{diff}}\left[\alpha, \widehat{E_{\text{im}}}\right](Q), \tag{3.243}$$

Since $\mathcal{D}^{\sigma,\alpha}[S_J^\alpha]$ is real-valued, $\mathcal{R}^\sigma_{J,\text{diff}}$ contains no inhomogeneous forcing term. As such, a Lyapunov-Schmidt strategy applied to (3.243) yields, for some $0 < r < 1$,

$$\left\|\widehat{E_{\text{im,hi}}}\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \left\|\widehat{E_{\text{im,lo}}}\right\|_{L^{2,a}(\mathbb{R})}, \tag{3.244}$$

$$\left\|\widehat{E_{\text{im,lo}}}\right\|_{L^{2,a}(\mathbb{R})} \lesssim (\alpha^2 + \alpha^{2-2r}) \left(\left\|\widehat{E_{\text{im,lo}}}\right\|_{L^{2,a}(\mathbb{R})} + \left\|\widehat{E_{\text{im,hi}}}\right\|_{L^{2,a}(\mathbb{R})}\right),$$

$$\iff \qquad (1 - C\ [\alpha^2 + \alpha^{2-2r}]) \left\|\widehat{E_{\text{im,lo}}}\right\|_{L^{2,a}(\mathbb{R})} \leqslant 0. \tag{3.245}$$

Taking α small enough that $1 - C\ [\alpha^2 + \alpha^{2-2r}] > 1/2$ implies that $\widehat{E_{\text{im,lo}}} = 0$. Therefore, $\widehat{E}$ is real-valued, which implies that $\widehat{\Phi^\sigma}$ is real-valued. This completes the proof of Proposition 3.3.3. □

3.6.2 Completion of the proof of Theorem 3.2.1

Above we solved for, $\alpha \mapsto \widehat{\Phi^{\sigma,\alpha}}(Q)$, the discrete Fourier transform of the on- and off-site standing waves as a function of the scaled variable Q, restricted to the scaled Brillouin zone, $\mathcal{B}_\alpha =$

$[-\pi/\alpha, \pi/\alpha]$. To complete the proof we use this to construct $\widehat{\phi^{\sigma,\alpha}}(q)$, defined on $\mathcal{B} = [-\pi, \pi]$. From (3.241) we have

$$\widehat{\phi^{\sigma}}(q) = \chi_{\mathcal{B}}(q)\widehat{\phi^{\sigma}}(q) = \sum_{j=0}^{J} \alpha^{2j}\ \chi_{\mathcal{B}}(q)\ F_j\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right) + \widehat{E_J^{\alpha,\sigma}}\left(\frac{q}{\alpha}\right),$$
$$\left\|F_j\left[\widetilde{\psi_1}\right]\left(\frac{\cdot}{\alpha}\right)\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{1/2}, \qquad \left\|\widehat{E_J^{\alpha,\sigma}}\left(\frac{\cdot}{\alpha}\right)\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2J+5/2}. \tag{3.246}$$

The bounds (3.246) follow since $F_j\left[\widetilde{\psi_1}\right](Q)$ and $\widehat{E_J^{\alpha,\sigma}}(Q)$ have order one $L^{2,a}(\mathbb{R}_Q)$ norm, and using the general bound on $q \mapsto f(q/\alpha)$ in $L^{2,a}(\mathbb{R}_q)$:

$$\left\|f\left(\frac{q}{\alpha}\right)\right\|^2_{L^{2,a}(\mathbb{R}_q)} = \int_{\mathbb{R}} \left(1+|q|^2\right)^a \left|f\left(\frac{q}{\alpha}\right)\right|^2 dq \leqslant \int_{\mathbb{R}} \left(1+\frac{|q|^2}{\alpha^2}\right)^a \left|f\left(\frac{q}{\alpha}\right)\right|^2 dq$$
$$= \alpha \int_{\mathbb{R}} \left(1+|Q|^2\right)^a \left|f\left(Q\right)\right|^2 dQ = \alpha\, \|f\|^2_{L^{2,a}(\mathbb{R}_Q)}. \tag{3.247}$$

Next, the ($2\pi-$ periodic in q) discrete Fourier transform of $\alpha \mapsto \{G_n^{\sigma,\alpha}\}_{n\in\mathbb{Z}}$ is $\widehat{G^{\sigma,\alpha}}(q) = e^{-i\sigma q}\widehat{K^{\sigma,\alpha}}(q)$, where

$$\widehat{K^{\sigma,\alpha}}(q) = \sum_{m\in\mathbb{Z}} \chi_{\mathcal{B}}(q-2m\pi)\ \widehat{\phi^{\sigma,\alpha}}(q-2m\pi)\ e^{2m\pi i\sigma}, \tag{3.248}$$

This implies the expansion on the Brillouin zone $q \in \mathcal{B} = [-\pi, \pi]$:

$$\widehat{G^{\alpha,\sigma}}(q) = e^{-iq\sigma}\ \widehat{\phi^{\sigma}}(q) = e^{-iq\sigma}\left(\sum_{j=0}^{J} \alpha^{2j}\ \chi_{\mathcal{B}}(q)\ F_j\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right) + \widehat{E_J^{\alpha,\sigma}}\left(\frac{q}{\alpha}\right)\right), \qquad q \in \mathcal{B}. \tag{3.249}$$

Define for $1 \leqslant j \leqslant J$,

$$\mathcal{G}_j[\psi_1](n) = \mathcal{F}_D^{-1}\left[\ F_j\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right)\right]_n. \tag{3.250}$$

Therefore, for any σ, in particular $\sigma = 0, 1/2$,

$$\mathcal{G}_j[\widetilde{\psi_1}](n-\sigma) = \mathcal{F}_D^{-1}\left[e^{-iq\sigma}\ F_j\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right)\right]_n. \tag{3.251}$$

Next, note that the leading order term in (3.249) may be written in terms of ψ_{α^2}:

$$\mathcal{F}_D^{-1}\left[\ F_0\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right)\right]_n = \mathcal{F}_D^{-1}\left[\widetilde{\psi_1}\left(\frac{q}{\alpha}\right)\right]_n = \frac{1}{2\pi}\int_{\mathcal{B}} e^{iqn}\ \widetilde{\psi_1}\left(\frac{q}{\alpha}\right) dq$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{iqn}\ \widetilde{\psi_1}\left(\frac{q}{\alpha}\right) dq + \frac{1}{2\pi}\int_{|q|\geqslant\pi} e^{iqn}\ \widetilde{\psi_1}\left(\frac{q}{\alpha}\right) dq$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{iqn}\ \widetilde{\psi_{\alpha^2}}(q) dq + \frac{1}{2\pi}\int_{|q|\geqslant\pi} e^{iqn}\ \widetilde{\psi_1}\left(\frac{q}{\alpha}\right) dq$$
$$= \psi_{\alpha^2}(n) + \frac{1}{2\pi}\int_{|q|\geqslant\pi} e^{iqn}\ \widetilde{\psi_1}\left(\frac{q}{\alpha}\right) dq. \tag{3.252}$$

Let

$$G_{\text{resid}}(n) \equiv \frac{1}{2\pi} \int_{|q| \geqslant \pi} e^{iqn} \; \widetilde{\psi_1}\left(\frac{q}{\alpha}\right) dq = \mathcal{F}_C^{-1}[\chi_{\mathcal{B}} \widetilde{\psi_\alpha^2}](n). \tag{3.253}$$

Note that $G_{\text{resid}}(n)$ is real-valued by Proposition 2.0.2 (i.e. since $\widetilde{\psi_1}$ is symmetric). Then the Poisson summation formula gives

$$\begin{aligned} \|G_{\text{resid}}(n)\|^2_{l^2(\mathbb{Z}_n)} &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\mathcal{F}_C^{-1}[\overline{\chi}_{\mathcal{B}} \widetilde{\psi_\alpha^2}](n) \right)^2 \\ &= \sum_{k \in \mathbb{Z}} \left(\overline{\chi}_{\mathcal{B}} \widetilde{\psi_\alpha^2} \right) * \left(\overline{\chi}_{\mathcal{B}} \widetilde{\psi_\alpha^2} \right)(k) \lesssim e^{-C/\alpha}. \end{aligned} \tag{3.254}$$

We may now define

$$\begin{aligned} &\mathcal{G}_0[\psi_1](n) = \psi_{\alpha^2}(n) = \alpha \; \psi_1(\alpha \; n), \\ &\mathcal{E}_n^{\alpha,J,\sigma} \equiv \mathcal{F}_D^{-1}\left[e^{-iq\sigma} \widehat{E_J^{\alpha,\sigma}} \left(\frac{q}{\alpha}\right)\right]_n + G_{\text{resid}}(n). \end{aligned} \tag{3.255}$$

Applying the inverse discrete Fourier transform (2.4) to $\widehat{G^{\alpha,\sigma}}(q)$ in (3.249) gives the branches of on-site ($\sigma = 0$) and off-site ($\sigma = 1/2$) discrete solitary waves (3.32). We use the Plancherel identity with the bounds (3.246) to get

$$\left\| \mathcal{G}_0[\widetilde{\psi_1}](n-\sigma) \right\|_{l^2(\mathbb{Z}_n)} \lesssim \frac{1}{(2\pi)^2} \; \left\| e^{-iq\sigma} F_0 \left[\widetilde{\psi_1}\right] \left(\frac{q}{\alpha}\right) \right\|_{L^2(\mathcal{B};dq)} + e^{-C/\alpha} \lesssim \alpha^{1/2}, \tag{3.256}$$

and for $1 \leqslant j \leqslant J$,

$$\begin{aligned} &\left\| \mathcal{G}_j[\widetilde{\psi_1}](n-\sigma) \right\|_{l^2(\mathbb{Z}_n)} = \frac{1}{(2\pi)^2} \; \left\| e^{-iq\sigma} F_j \left[\widetilde{\psi_1}\right] \left(\frac{q}{\alpha}\right) \right\|_{L^2(\mathcal{B};dq)} \lesssim \left\| F_j \left[\widetilde{\psi_1}\right] \left(\frac{q}{\alpha}\right) \right\|_{L^{2,a}(\mathbb{R}_q)} \lesssim \alpha^{1/2}, \\ &\left\| \mathcal{E}^{\alpha,J,\sigma} \right\|_{l^2(\mathbb{Z})} \lesssim \left\| e^{-iq\sigma} E_J^{\alpha,\sigma} \left(\frac{q}{\alpha}\right) \right\|_{L^2(\mathcal{B};dq)} + e^{-C/\alpha} \; \lesssim \left\| e^{-iq\sigma} E_J^{\alpha,\sigma} \left(\frac{q}{\alpha}\right) \right\|_{L^{2,a}(\mathbb{R}_q)} \lesssim \alpha^{2J+5/2}. \end{aligned} \tag{3.257}$$

To complete the proof of Theorem 3.2.1, we show that the solitary wave $G_n^{\sigma,\alpha} = \mathcal{F}_D^{-1}[\widehat{G^{\sigma,\alpha}}]_n$ corresponds for $\sigma = 0$ to a real-valued, on-site symmetric solitary wave and corresponds for $\sigma = 1/2$ to a real-valued, off-site symmetric solitary wave. By Proposition 2.0.2, it suffices to show that $\widehat{K^\sigma}(q)$ is symmetric (even) and real-valued. Proposition 3.3.3 gives that $\widehat{\Phi^\sigma} \in L^{2,a}_{\text{even}}(\mathbb{R}_Q)$ is real-valued, which implies by 3.247 that its rescaling $\widehat{\phi^\sigma} \in L^{2,a}_{\text{even}}(\mathbb{R}_q)$ is real-valued. Since $e^{2m\pi i\sigma} = \pm 1$ for $\sigma \in \{0, 1/2\}$, $\widehat{K^\sigma}$ as given in (3.248) will be also be even and real-valued. This completes the proof of Theorem 3.2.1.

3.7 Exponential Smallness of the Peierls-Nabarro Barrier; Proof of Theorem 3.2.3 for $d = 1$

We now prove Theorem 3.2.3 for dimension $d = 1$ in sections 3.7 through 3.7.1. Recall the definitions

$$\mathcal{N}[G] = \|G\|^2_{l^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |G_n|^2,$$
$$\mathcal{H}[G] = \sum_{n\in\mathbb{Z}} |G_{n+1} - G_n|^2 - \frac{1}{2}|G_n|^4 = \|\delta G\|^2_{l^2(\mathbb{Z})} - \frac{1}{2}|G_n|^4. \tag{3.258}$$

By Theorem 3.2.1, for α sufficiently small, there exist solutions $G^{\alpha,\text{on}} = \{G^{\alpha,\text{on}}_n\}_{n\in\mathbb{Z}}$ and $G^{\alpha,\text{off}} = \{G^{\alpha,\text{off}}_n\}_{n\in\mathbb{Z}}$ to DNLS: $-\alpha^2 G^{\alpha,\sigma}_n = -(\delta^2 G)_n - (G_n)^3$. We shall prove that there exists a constant $C > 0$ such that for α sufficiently small,

$$\textbf{PN Barrier for } d = 1: \qquad \left|\mathcal{N}[G^{\text{off}}] - \mathcal{N}[G^{\text{on}}]\right| \lesssim \alpha\ e^{-C/\alpha} \tag{3.259}$$
$$\left|\mathcal{H}[G^{\text{off}}] - \mathcal{H}[G^{\text{on}}]\right| \lesssim \alpha\ e^{-C/\alpha}. \tag{3.260}$$

The following identity allows us to equate the nonlinear term in $\mathcal{H}$ with terms involving $\|G\|^2_{l^2(\mathbb{Z})}$ and $\|\delta G\|^2_{l^2(\mathbb{Z})}$.

Proposition 3.7.1. *Suppose that $G = \{G_n\}_{n\in\mathbb{Z}}$ solves DNLS: $-\alpha^2 G_n = -(\delta^2 G)_n - (G_n)^3$. Then G satisfies*

$$\alpha^2\|G\|^2_{l^2(\mathbb{Z})} + \|\delta G\|^2_{l^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |G_n|^4. \tag{3.261}$$

Proof of Proposition 3.7.1: Multiplying DNLS by G_n and summing over $n \in \mathbb{Z}$ gives the result.

□

Now recall from (3.51) that

$$\widehat{G^\sigma}(q) = \mathcal{F}_D\left[G^\sigma\right](q) = e^{-iq\sigma}\ \widehat{K^\sigma}(q), \qquad q \in \mathbb{R},\ \sigma \in \{0, 1/2\}, \tag{3.262}$$

where $\widehat{K^\sigma}(q)$ is even, real-valued, and $2\pi\sigma$- pseudoperiodic, and where $\chi_{\mathcal{B}}(q)\ \widehat{K^\sigma}(q) = \widehat{\phi^\sigma}(q)$.

Next, observe that by the Plancherel identity and since $\widehat{\phi^\sigma}$ is real-valued,

$$\begin{aligned}
2\pi \left|\mathcal{N}[G^{\rm off}] - \mathcal{N}[G^{\rm on}]\right| &= 2\pi \left|\|G^{\rm off}\|^2_{l^2(\mathbb{Z})} - \|G^{\rm on}\|^2_{l^2(\mathbb{Z})}\right| = \left|\|\widehat{G^{\rm off}}\|^2_{L^2(\mathcal{B})} - \|\widehat{G^{\rm on}}\|^2_{L^2(\mathcal{B})}\right| \\
&= \left|\|\widehat{K^{\rm off}}\ \|^2_{L^2(\mathcal{B})} - \|\widehat{K^{\rm on}}\|^2_{L^2(\mathcal{B})}\right| = \left|\|\widehat{\phi^{\rm off}}\ \|^2_{L^2(\mathcal{B})} - \|\widehat{\phi^{\rm on}}\|^2_{L^2(\mathcal{B})}\right| \\
&\leqslant \left\|\widehat{\phi^{\rm off}} + \widehat{\phi^{\rm on}}\right\|_{L^2(\mathcal{B})} \left\|\widehat{\phi^{\rm off}} - \widehat{\phi^{\rm on}}\right\|_{L^2(\mathcal{B})} .
\end{aligned} \tag{3.263}$$

Similarly, by Lemma C.0.6 and the Plancherel identity,

$$\begin{aligned}
2\pi \left| \left(\sum_{n\in\mathbb{Z}} |G^{\rm on}_{m+n} - G^{\rm on}_n|^2 - |G^{\rm on}_{m+n} - G^{\rm on}_n|^2\right) \right| &= 2\pi \left|\|\delta G^{\rm on}\|^2_{l^2(\mathbb{Z})} - \|\delta G^{\rm off}\|^2_{l^2(\mathbb{Z})}\right| \\
&= \left| \left\| \sin(m\cdot/2)\ \widehat{G^{\rm on}}\right\|^2_{L^2(\mathcal{B})} - \left\| \sin(m\cdot/2)\ \widehat{G^{\rm on}}\right\|^2_{L^2(\mathcal{B})} \right| \\
&= \left| \left\| \sin(m\cdot/2)\ \widehat{\phi^{\rm on}}\right\|^2_{L^2(\mathcal{B})} - \left\| \sin(m\cdot/2)\ \widehat{\phi^{\rm on}}\right\|^2_{L^2(\mathcal{B})} \right| \\
&\leqslant \left\|\widehat{\phi^{\rm off}} + \widehat{\phi^{\rm on}}\right\|_{L^2(\mathcal{B})} \left\|\widehat{\phi^{\rm off}} - \widehat{\phi^{\rm on}}\right\|_{L^2(\mathcal{B})} .
\end{aligned} \tag{3.264}$$

Note also since $\widehat{\phi^\sigma}(q) = \chi_{\mathcal{B}}(q)\widehat{\phi^\sigma}(q) = \widehat{\Phi^\sigma}(q/\alpha) = \widehat{\Phi^\sigma}(Q)$,

$$\left\|\widehat{\phi^{\rm off}} - \widehat{\phi^{\rm on}}\right\|_{L^2(\mathcal{B})} = \left\|\widehat{\phi^{\rm off}} - \widehat{\phi^{\rm on}}\right\|_{L^2(\mathbb{R})} = \alpha^{1/2} \left\|\widehat{\Phi^{\rm off}} - \widehat{\Phi^{\rm on}}\right\|_{L^2(\mathbb{R})} \lesssim \alpha^{1/2} \left\|\widehat{\Phi^{\rm off}} - \widehat{\Phi^{\rm on}}\right\|_{L^{2,a}(\mathbb{R})} . \tag{3.265}$$

Proposition 3.7.1, the bound $\|\widehat{\phi^{\rm off}} + \widehat{\phi^{\rm on}}\|_{L^2(\mathcal{B})} \lesssim \alpha^{1/2}$, and equations (3.263), (3.264), and (3.265) give

$$\left|\mathcal{N}[G^{\rm off}] - \mathcal{N}[G^{\rm on}]\right| + \left|\mathcal{H}[G^{\rm off}] - \mathcal{H}[G^{\rm on}]\right| \lesssim \alpha \left\|\widehat{\Phi^{\rm off}} - \widehat{\Phi^{\rm on}}\right\|_{L^{2,a}(\mathbb{R})} , \tag{3.266}$$

We shall prove that $\widehat{\Phi^{\rm off}}(Q) - \widehat{\Phi^{\rm on}}(Q)$ is exponentially small in $L^{2,a}(\mathbb{R})$.

Proposition 3.7.2. *Let $\alpha_0 > 0$ be that prescribed in Proposition 3.3.3. Then for $0 < \alpha < \alpha_0$, there exists a constant $C > 0$ such that*

$$\left\|\widehat{\Phi^{\rm off}} - \widehat{\Phi^{\rm on}}\right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha}. \tag{3.267}$$

We prove Proposition 3.7.2 in the subsequent section. Theorem 3.2.3 follows directly from Proposition 3.7.2 and (3.266). □

3.7.1 Estimation of the Difference $\widehat{\Phi^{\rm off}} - \widehat{\Phi^{\rm on}}$

We embark on the proof of Proposition 3.7.2. Recall from Proposition 3.3.3 that $\widehat{\Phi^{\sigma,\alpha}} \in L^{2,a}(\mathbb{R})$ is well-defined, $\|\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim 1$, for α sufficiently small and satisfies equation (3.62). Note that $\widehat{\Phi^{\sigma,\alpha}}$ is supported on $\mathcal{B}_\alpha = [-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}]$, an interval which grows as $\alpha \downarrow 0$. We begin by proving a uniform decay bound for $\widehat{\Phi^{\sigma,\alpha}}$.

Proposition 3.7.3. *Let $0 < \alpha < \alpha_2$. Then there exists a constant $\mu = \mu\left[\|\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)}\right] > 0$ such that $\widehat{\Phi^{\sigma,\alpha}}$ satisfies $\|e^{\mu|Q|}\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \|\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)}$.*

Proof of Proposition 3.7.3: We apply Lemma H.0.17 from the appendix to equation (3.62) for $\widehat{\Phi^{\sigma,\alpha}}$. To see that the hypotheses of the lemma are satisfied, it suffices to observe that

$$M_\alpha(Q) = \frac{4}{\alpha^2}\sin^2\left(\frac{Q\alpha}{2}\right) \geqslant \frac{4}{\pi^2}\,|Q|^2, \qquad Q \in \mathcal{B}_\alpha. \tag{3.268}$$

and for $m \in \{-1, 0, 1\}$ and $Q \in \mathcal{B}_\alpha$, $|Q| \leqslant |Q - 2m\pi/\alpha|$. □

Next, we derive the equation for $\widehat{\Phi^{\rm diff}} = \widehat{\Phi^{\rm off}} - \widehat{\Phi^{\rm on}}$.

Proposition 3.7.4. *Let $0 < \alpha < \alpha_0$. Then $\widehat{\Phi^{\rm diff}}(q) = \widehat{\Phi^{\rm off}}(q) - \widehat{\Phi^{\rm on}}(q)$ solves the following linear equation:*

$$\begin{aligned}
&[1 + M_\alpha(Q)]\ \widehat{\Phi^{\rm diff}}(q) - \chi_{\mathcal{B}_\alpha}(Q)\frac{1}{4\pi^2}\Big(\widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm diff}}(Q)\\
&+ \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm diff}}(Q) + \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm diff}}(Q) = R_{\rm diff}\left[\widehat{\Phi^{\rm off}}, \widehat{\Phi^{\rm on}}\right](Q),
\end{aligned} \tag{3.269}$$

where the inhomogeneous right-hand side is given by

$$\begin{aligned}
R_{\rm diff}\left[\widehat{\Phi^{\rm off}}, \widehat{\Phi^{\rm on}}\right](Q) = -\chi_{\mathcal{B}_\alpha}(Q)\frac{1}{4\pi^2}\sum_{m=\pm1}\Big(&\ \widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm off}}(Q - 2m\pi/\alpha)\\
&+\ \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm on}}(Q - 2m\pi/\alpha)\Big),
\end{aligned} \tag{3.270}$$

with

$$\left\|R_{\rm diff}\left[\widehat{\Phi^{\rm off}}, \widehat{\Phi^{\rm on}}\right]\right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha}, \quad C > 0. \tag{3.271}$$

Proof of Proposition 4.6.3 : We subtract equation (3.62) for $\sigma = 0$ from the same equation for $\sigma = 1/2$. To estimate the $m = \pm 1$ terms in (3.270), we apply Lemma 3.4.1 along with Proposition 3.7.3. This gives for $m = \pm 1$,

$$\left\| \chi_{\mathcal{B}_\alpha} \ \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma}(\cdot - 2m\pi/\alpha) \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha}, \quad C > 0, \tag{3.272}$$

which gives (3.271). □

We now use Proposition 4.6.3 to prove the exponential bound (3.267) on $\widehat{\Phi^{\rm diff}}$. We use an argument analogous to that used in the proof of Proposition 3.3.3. Here, we only summarize the argument since the details are quite familiar. Introduce

$$\widehat{\Phi^{\rm diff}_{\rm lo}}(q) \equiv \chi_{\rm lo}(q) \ \widehat{\Phi^{\rm diff}}(q), \qquad \text{and} \qquad \widehat{\Phi^{\rm diff}_{\rm hi}}(q) \equiv \chi_{\rm hi}(q) \ \widehat{\Phi^{\rm diff}}(q). \tag{3.273}$$

Estimation of $\widehat{\Phi^{\rm diff}_{\rm hi}}$ gives

$$\left\| \widehat{\Phi^{\rm diff}_{\rm hi}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2-2r} \ e^{-C/\alpha} + \ \alpha^{2-2r} \ \left\| \widehat{\Phi^{\rm diff}_{\rm lo}} \right\|_{L^{2,a}(\mathbb{R})}, \tag{3.274}$$

$\widehat{\Phi^{\rm diff}_{\rm lo}}$ satisfies an inhomogeneous equation forced by $\chi_{\rm lo} \ R_{\rm diff}\left[\widehat{\Phi^{\rm off}}, \widehat{\Phi^{\rm on}}\right]$ which satisfies the exponential bound (3.271). A simple bootstrap argument using (3.274) and the proved bounds on $\widehat{\Phi^{\rm off}}$ and $\widehat{\Phi^{\rm on}}$ gives $\left\| \widehat{\Phi^{\rm diff}_{\rm lo}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha}, \quad C > 0$, for α sufficiently small, which dominates $\widehat{\Phi^{\rm diff}}$. This completes the proof of Proposition 3.7.2. □

3.8 Extension of Our Analysis to Dimensions $d = 2, 3$

In Sections 3.3 through 3.7 we proved Theorems 3.2.1 and 3.2.3, concerning the bifurcation of discrete solitary waves and a bound on the PN-barrier in dimension $d = 1$. In this section, we show how to adapt the the proof to dimensions $d = 2$ and $d = 3$. This will then prove Theorem 3.2.2. For many of the details, we refer to the proofs in one dimension and emphasize those aspects in which the spatial dimension, d, appears explicitly. In particular, the proof of Theorem 3.2.3 for $d = 2, 3$ concerning the PN-barrier bound follows Section 3.7 using the d-dimensional scalings of the discrete solitary waves in this section.

We begin with a generalization of Definition 2.0.1, concerning the different centerings of discrete solitary standing waves.

Definition 3.8.1. *Let $G = \{G_n\}_{n\in\mathbb{Z}^d}$ be a solution to equation (3.33) and let $\sigma \in \{0, 1/2\}^d$. We say that G is σ- centered if it is symmetric about the point σ in space. That is, for each spatial component $k = 1, \ldots, d$, we have $G_n = G_m^{(k)}$ where $m^{(k)} = (n_1, \ldots, n_k + 2\sigma_k, \ldots, n_d)^T \in \mathbb{Z}^d$. Note that this definition is consistent with its one-dimensional analogue given in Definition 2.0.1.*

Applying the d-dimensional discrete Fourier transform to DNLS, we obtain

$$\widehat{DNLS}[\widehat{G^\alpha}](q) \equiv \left[\alpha^2 + M(q)\right]\widehat{G^\alpha}(q) - \left(\frac{1}{2\pi}\right)^{2d} \widehat{G^\alpha} *_1 \widehat{G^\alpha} *_1 \widehat{G^\alpha}(q) = 0, \tag{3.275}$$

$$\widehat{G^\alpha}(q + 2\pi e^{(k)}) = \widehat{G^\alpha}(q). \tag{3.276}$$

Here, $e^{(k)}$ is the unit vector in the kth coordinate direction, $q_k = e^{(k)} \cdot q$ and

$$M(q) = 4\sum_{k=1}^{d} \sin^2(q_k/2). \tag{3.277}$$

Note that $\widehat{G^\alpha}$ is periodically tiled over d-cubes of volume $(2\pi)^d$ in $\mathbb{R}^d$.

Proposition 3.8.1. *If $G = \{G_n\}_{n\in\mathbb{Z}^d}$ is real and σ-centered, then $\widehat{G}(q) = e^{-iq\cdot\sigma}\widehat{K}(q)$, where $\widehat{K}(q)$ is real and symmetric. Conversely, if $\widehat{G}(q) = e^{-iq\cdot\sigma}\widehat{K}(q)$, where $\widehat{K}(q)$ is real-valued and symmetric, then $\mathcal{F}_D^{-1}[\widehat{G}]$ is real and σ-centered. This is consistent with the one-dimensional analogue given in Proposition 2.0.2.*

We take $\sigma = \{\sigma_k\}_{k=1,\ldots,d}$ where $\sigma_k \in \{0, 1/2\}$ and seek $\widehat{G}(q)$ in the form

$$\widehat{G^\sigma}(q) \ = \ e^{-i\sigma\cdot q}\ \widehat{K^\sigma}(q), \qquad \widehat{K^\sigma}(q) = \widehat{K^\sigma}(-q), \qquad \overline{\widehat{K^\sigma}(q)} = \widehat{K^\sigma}(q), \tag{3.278}$$

By Lemma C.0.7, the periodicity of $\widehat{G}(q)$ and (3.278) we have

$$\left[\alpha^2 + M(q)\right]\ \widehat{K^\sigma}(q) - \left(\frac{1}{2\pi}\right)^{2d}\ \widehat{K^\sigma} *_1 \widehat{K^\sigma} *_1 \widehat{K^\sigma}(q) = 0, \quad q \in \mathbb{R}^d, \tag{3.279}$$

$$\widehat{K^\sigma}\left(q + 2\pi e^{(k)}\right) = e^{2\pi i\sigma\cdot e^{(k)}}\ \widehat{K^\sigma}(q), \quad k = 1, \ldots, d. \tag{3.280}$$

The "Bloch" phase factor, $e^{2\pi i\sigma\cdot e^{(k)}}$, in (3.280) is equal to ± 1.

Lemma 3.8.1. *Let $A(q)$, defined on $\mathbb{R}^d$, be $2\pi\sigma-$ pseudo-periodic (condition (3.280)). Then, $A(q)$ is completely determined by its values on $\mathcal{B} = [-\pi, \pi]^d$ and has the representation:*

$$A(q) = \sum_{m\in\mathbb{Z}^d} \chi_{\mathcal{B}}(q - 2m\pi)A(q - 2m\pi)e^{2\pi i m\cdot\sigma}. \quad \square \tag{3.281}$$

By Lemma 3.8.1 we may express $\widehat{K^\sigma}(q)$, for any $q \in \mathbb{R}^d$, explicitly in terms of its values on $q \in \mathcal{B}$. In particular, we set $\widehat{K^\sigma}(q) = \widehat{\phi^\sigma}(q)$ for $q \in \mathcal{B}$, and extend $\widehat{\phi^\sigma}(q)$ to $\mathbb{R}^d$ to get:

$$\widehat{K^\sigma}(q) \equiv \sum_{m\in\mathbb{Z}^d} \chi_{\mathcal{B}}(q - 2m\pi)\ \widehat{\phi^\sigma}(q - 2m\pi)\ e^{2\pi i m\cdot\sigma}, \qquad \text{and} \qquad \widehat{G^\sigma}(q) = e^{-i\sigma\cdot q}\ \widehat{K^\sigma}(q). \tag{3.282}$$

Note that $\widehat{\phi^\sigma}(q - 2m\pi) = \chi_{\mathcal{B}}(q - 2m\pi)\ \widehat{\phi^\sigma}(q - 2m\pi)$ is supported on $\{q : q \in 2m\pi + \mathcal{B}\}$. Therefore,

$$\chi_{\mathcal{B}}(q)\ \widehat{K^\sigma}(q) = \widehat{\phi^\sigma}(q), \qquad \text{and} \qquad \chi_{\mathcal{B}}(q)\ \widehat{G^\sigma}(q) = e^{-iq\cdot\sigma}\ \widehat{\phi^\sigma}(q). \tag{3.283}$$

Equation (3.282) encodes the required $2\pi-$ periodicity of $\widehat{G^\sigma}(q)$ on all $\mathbb{R}^d$, the $2\pi\sigma-$ pseudo-periodicity of $\widehat{K^\sigma}(q)$. Furthermore, $\widehat{G^\sigma}(q)$ and $\widehat{K^\sigma}(q)$ are completely specified by $\widehat{\phi^\sigma}(q)$ for $q \in \mathcal{B}$.

With a view toward obtaining an equation determining $\widehat{\phi^\sigma}(q)$ for $q \in \mathcal{B}$, we require a lemma to simplify the convolution terms in (3.279).

Lemma 3.8.2. *Let $\widehat{A}, \widehat{B}, \widehat{C}$ be bounded $2\pi\sigma-$ pseudo-periodic. Furthermore, let $\widehat{C}$ have the form:*

$$\widehat{C}(q) \equiv \sum_{m\in\mathbb{Z}^d} e^{2\pi i m\cdot\sigma}\chi_{\mathcal{B}}(q - 2m\pi)\ \widehat{C}(q - 2m\pi). \tag{3.284}$$

Then

$$\chi_{\mathcal{B}}(q)\widehat{A} *_1 \widehat{B} *_1 \widehat{C}(q) = \chi_{\mathcal{B}}(q) \sum_{m\in\{-1,0,1\}^d} e^{2\pi i m\cdot\sigma}\ \widehat{A} *_1 \left[\ \widehat{B} *_1 \left(\chi_{\mathcal{B}}\widehat{C}\right)\right](q - 2m\pi). \tag{3.285}$$

This lemma is a simple generalization of Lemma 3.3.2; the proof is nearly identical. Applying Lemma 3.8.2 and (3.282), we have:

Proposition 3.8.2. *Equation (3.279) for $\widehat{K^\sigma}$ on $q \in \mathbb{R}^d$ is equivalent to the following equation for $\widehat{\phi^\sigma}$ on $q \in \mathcal{B}$:*

$$\begin{aligned}[\alpha^2 + M(q)]\ \widehat{\phi^\sigma}(q) - \chi_{\mathcal{B}}(q)\ \left(\frac{1}{2\pi}\right)^{2d} \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma}\right)(q)& \\ - \chi_{\mathcal{B}}(q)\ \left(\frac{1}{2\pi}\right)^{2d} \sum_{\substack{m\in\{-1,0,1\}^d \\ m\neq 0}} e^{2\pi i m\cdot\sigma} \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma}\right)(q - 2m\pi) &= 0,\end{aligned} \tag{3.286}$$

where $M(q)\widehat{G}(q) = 4\sum_{j=1}^d 4\sin^2(q_j/2)\widehat{G}(q) = \widehat{\delta^2 G}(q)$; see (3.277).

Note now that the convolutions are taken over $\mathbb{R}^d$.

3.8.1 Rescaled Equation for $\widehat{\phi^\sigma}$

In analogy with the one-dimensional case discussed in Section 3.3.1, we expect

$$\widehat{\phi^\sigma}(q) \sim \widetilde{\psi_{\alpha^2}}(q) = \alpha^{1-d}\ \widetilde{\psi_1}\left(\frac{q}{\alpha}\right), \qquad \alpha \ll 1, \tag{3.287}$$

(see Proposition 3.1.2). To anticipate this leading order behavior, we study (3.286) by a rescaling which will make explicit the connection, for $\alpha \downarrow 0$, between DNLS and the continuum (NLS) limit. With the goal of obtaining an asymptotic expansion for $\widehat{\phi^\sigma}$ as a functional of $\widetilde{\psi_1}$ in powers of α, we introduce:

Rescaled momentum: $Q \equiv q/\alpha, \qquad Q_k = q_k/\alpha$ and $Q \in \mathcal{B}_\alpha = [-\pi/\alpha, \pi/\alpha]^d$,

Rescaled projection: $\chi_{\mathcal{B}_\alpha}(Q) \equiv \chi_{\mathcal{B}}(Q\alpha) = \chi_{[-\frac{\pi}{\alpha},\frac{\pi}{\alpha}]^d}(Q)$,

Rescaled wave: $\widehat{\Phi^\sigma}(Q) \equiv \alpha^{d-1}\ \widehat{\phi^\sigma}(Q\alpha) = \alpha^{d-1}\ \widehat{\phi^\sigma}(q)$,

Rescaled symbol:
$$M_\alpha(Q) \equiv \frac{1}{\alpha^2} M(Q\alpha) = \frac{4}{\alpha^2}\sum_{k=1}^{d} \sin^2\left(\frac{Q_k\alpha}{2}\right). \tag{3.288}$$

The following proposition is a formulation of Proposition 3.3.1 in terms of functions of the rescaled quasi-momentum, Q:

Proposition 3.8.3. *Equation* (3.47) *for* $\widehat{K^\sigma}(q)$ *on* $q \in \mathbb{R}$ *is equivalent to the following equation for* $\widehat{\Phi^\sigma}(Q) = \chi_{\mathcal{B}_\alpha}(q)\ \widehat{\Phi^\sigma}(Q)$*, compactly supported on* $\mathcal{B}_\alpha = [-\pi/\alpha, \pi/\alpha]$*:*

$$\begin{aligned}\mathcal{D}^{\sigma,\alpha}[\widehat{\Phi^\sigma}](Q) \ \equiv\ & [1 + M_\alpha(Q)]\ \widehat{\Phi^\sigma}(Q) - \frac{\chi_{\mathcal{B}_\alpha}(Q)}{(2\pi)^d}\ \left(\widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma}\right)(Q) \\ & + \ R_1^\sigma[\widehat{\Phi^\sigma}](Q) = 0\ ,\end{aligned} \tag{3.289}$$

where $R_1^\sigma[\widehat{\Phi^\sigma}]$ *contains the* ± 1*-sideband contributions:*

$$R_1^\sigma[\widehat{\Phi^\sigma}](Q) \equiv -\frac{\chi_{\mathcal{B}_\alpha}(Q)}{(2\pi)^d} \sum_{\substack{m\in\{-1,0,1\}^d \\ m\neq 0}} e^{2m\pi i\sigma}\ \left(\widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma}\right)(Q - 2m\pi/\alpha), \tag{3.290}$$

and $M_\alpha(Q) = \frac{4}{\alpha^2}\sum_{j=1}^d \sin^2(\frac{\alpha Q_j}{2})$*; see* (3.288).

To prove Proposition 3.8.3 we need to re-express the convolutions in (3.286) in terms of $\widehat{\Phi^\sigma}(Q)$. For this we use the following lemma, which is proved by change of variables and generalizes Lemma 3.3.3.

Lemma 3.8.3. *Suppose that $\widehat{a}(q) = \widehat{A}(Q)$, $\widehat{b}(q) = \widehat{B}(Q)$, and $\widehat{c}(q) = \widehat{C}(Q)$, where $Q = q/\alpha$. Then*

$$\left(\widehat{a} * \widehat{b} * \widehat{c}\right)(q) = \alpha^{2d} \ \left(\widehat{A} * \widehat{B} * \widehat{C}\right)(Q). \tag{3.291}$$

Applying the rescalings (3.288) and Lemma 3.8.3 to (3.286) and then dividing by α^{3-d}, we obtain (3.289). □

We will again formally Taylor expand, for $\alpha \ll 1$ and $Q \in \mathbb{R}^d$ fixed,

$$\begin{aligned} M_\alpha(Q) = \frac{4}{\alpha^2}\sum_{k=1}^{d} \sin^2\left(\frac{Q_k\alpha}{2}\right) &= 2\sum_{k=1}^{d}\sum_{j=0}^{\infty} \frac{\alpha^{2j}\ (-1)^j\ |Q_k|^{2j+2}}{(2j+2)!} \\ &= \sum_{k=1}^{d}\ |Q_k|^2 - \frac{\alpha^2\ |Q_k|^4}{12} + \frac{\alpha^4\ |Q_k|^6}{360} + \mathcal{O}\left(\alpha^6\ |Q_k|^8\right) \\ &= |Q|^2 - \sum_{k=1}^{d} \frac{\alpha^2\ |Q_k|^4}{12} - \frac{\alpha^4\ |Q_k|^6}{360} + \mathcal{O}\left(\alpha^6\ |Q_k|^8\right). \end{aligned} \tag{3.292}$$

Using truncations of the expansion of $M_\alpha(Q)$ we shall, for any $J = 0, 1, 2, \ldots$, construct $\widehat{\Phi}^\sigma(Q)$ in the form of a finite expansion in power of $\alpha^{2j},\ j = 0, \ldots, J$, with an error term of which is of order α^{2J+2} plus a corrector of higher order. For each J, the polynomial expansion in α^2 is *independent of* σ. The construction is summarized in the following:

Proposition 3.8.4. *Fix $J \geqslant 0$, $a > 1/2$, and $\sigma \in \{0, 1/2\}^d$. Then there exist a constant $\alpha_0 = \alpha_0[a, J, \sigma] > 0$, and J mappings $F_j : L^{2,a}_{\text{even}}(\mathbb{R}^d) \to L^{2,a}_{\text{even}}(\mathbb{R}^d)$, $j = 0, \ldots, J$, and a unique, real-valued function $\widehat{E^{\alpha,\sigma}_J} \in L^{2,a}_{\text{even}}(\mathbb{R}^d)$ such that for all $0 < \alpha < \alpha_0$,*

$$\widehat{\Phi^\sigma}(Q) \ = \ \chi_{\mathcal{B}_\alpha}(Q)\ \widetilde{\psi_1}(Q) \ + \ \sum_{j=1}^{J} \ \alpha^{2j}\ \chi_{\mathcal{B}_\alpha}(Q)\ F_j\left[\widetilde{\psi_1}\right](Q) + \widehat{E^{\alpha,\sigma}_J}(Q), \tag{3.293}$$

solves equation (3.289) *with the error bound:*

$$\left\|\widehat{E^{\alpha,\sigma}_J}\right\|_{L^{2,a}(\mathbb{R}^d)} \lesssim \alpha^{2J+2} \tag{3.294}$$

$F_j,\ j \geqslant 1$, defined in Proposition 3.8.5 below, is independent of σ and α, and

$$\widehat{E^{\alpha,\sigma}_J}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widehat{E^{\alpha,\sigma}_J}(Q). \tag{3.295}$$

Remark 3.8.1. *By Proposition 3.8.4, since the polynomial expansion in α^2 is completely determined by $\widetilde{\psi_1}(Q)$, $\widehat{\Phi^\sigma}(Q)$ is completely specified once we have constructed $\widehat{E_J^{\alpha,\sigma}}(Q)$ for $Q \in \mathbb{R}^d$. And, in turn by the rescalings $Q = q/\alpha$ and $\widehat{\Phi^\sigma}(Q) = \widehat{\phi^\sigma}(Q\alpha) = \widehat{\phi^\sigma}(q)$, $\widehat{G^\sigma}(q)$ and $\widehat{K^\sigma}(q)$ are completely specified by $\widehat{E_J^{\alpha,\sigma}}(q/\alpha)$ for $q \in \mathbb{R}^d$. Therefore, Proposition 3.8.4 completely characterizes $\widehat{G^\sigma}(q)$.*

The formal asymptotic analysis in Section 3.3.2 is easily generalized to dimension $d \geqslant 1$ and $L^{2,a}(\mathbb{R}^d)$ for $a > d/2$ using Propositions 3.1.2 and 3.1.3, and (2.20). We therefore construct and characterize $F_j\left[\widetilde{\psi_1}\right]$ via the following proposition, which itself is a generalization of Proposition 4.7.3.

Proposition 3.8.5. *Let $j \geqslant 1$. The equation for F_j at order $\mathcal{O}(\alpha^{2j})$, independent of α and σ, is given by*

$$\mathcal{O}(\alpha^{2j})\ \textbf{equation}: \qquad \widetilde{L_+}\ F_j(Q) = 2\sum_{l=1}^{d}\sum_{k=0}^{j-1}\frac{(-1)^{k-j+1}\ |Q_l|^{2j-2k+2}\ F_k(Q)}{(2j-2k+2)!}, \\ + \frac{1}{(2\pi)^{2d}}\sum_{\substack{k+l+z=j\\ 0\leqslant k,l,z<j}} F_k * F_l * F_z(Q) \\ \equiv H_j\left[F_0,\ldots,F_{j-1}\right](Q), \tag{3.296}$$

and has the unique solution

$$F_j = \left(\widetilde{L_+}\right)^{-1}\left(H_j\left[F_0,\ldots,F_{j-1}\right]\right) \in L^{2,a}_{\text{even}}(\mathbb{R}^d; dQ). \tag{3.297}$$

Furthermore, F_j is real-valued and $e^{C_j|Q|}\ F_j(Q) \in L^{2,a}(\mathbb{R}^d; dQ)$, where $C_j \equiv C_0\left(\frac{1}{2} + \frac{1}{2^{j+1}}\right) \geqslant \frac{C_0}{2}$ and $C_0 > 0$ is as in (4.178).

The proof closely follows that of Proposition 4.7.3. The remainder of the proof of Proposition 3.8.4 follows that of Proposition 3.3.3 for the 1-d case given in sections 3.4 and 3.5.

3.8.2 Completion of the Proofs of Theorems 3.2.2 and 3.2.3 for $d = 1, 2, 3$

Above we solved for, $\alpha \mapsto \widehat{\Phi^{\sigma,\alpha}}(Q)$, the discrete Fourier transform of the on- and off-site standing waves as a function of the scaled variable Q, restricted to the scaled Brillouin zone, $\mathcal{B}_\alpha =$

$[-\pi/\alpha, \pi/\alpha]^d$. To complete the proof we use this to construct $\widehat{\phi^{\sigma,\alpha}}(q)$, defined on $\mathcal{B} = [-\pi, \pi]^d$. From (3.293) and the scaling $\widehat{\phi^{\sigma}}(q) = \alpha^{1-d}\ \widehat{\Phi}(q/\alpha)$, we have

$$\widehat{\phi^{\sigma}}(q) = \chi_{\mathcal{B}}(q)\widehat{\phi^{\sigma}}(q) = \sum_{j=0}^{J}\ \alpha^{2j+1-d}\ \chi_{\mathcal{B}}(q)\ F_j\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right) + \alpha^{1-d}\ \widehat{E_J^{\alpha,\sigma}}\left(\frac{q}{\alpha}\right),$$
$$\left\| F_j\left[\widetilde{\psi_1}\right]\left(\frac{\cdot}{\alpha}\right)\right\|_{L^{2,a}(\mathbb{R}^d)} \lesssim \alpha^{d/2}, \qquad \left\|\widehat{E_J^{\alpha,\sigma}}\left(\frac{\cdot}{\alpha}\right)\right\|_{L^{2,a}(\mathbb{R}^d)} \lesssim \alpha^{2J+2+d/2}\ . \tag{3.298}$$

The bounds (3.298) follow since $F_j\left[\widetilde{\psi_1}\right](Q)$ and $\widehat{E_J^{\alpha,\sigma}}(Q)$ have order one $L^{2,a}(\mathbb{R}^d_Q)$ norm, and using the general bound on $q \mapsto f(q/\alpha)$ in $L^{2,a}(\mathbb{R}^d_q)$:

$$\begin{aligned}\left\| f\left(\frac{q}{\alpha}\right)\right\|^2_{L^{2,a}(\mathbb{R}^d_q)} &= \int_{\mathbb{R}^d} \left(1+|q|^2\right)^a \left| f\left(\frac{q}{\alpha}\right)\right|^2 dq \leqslant \int_{\mathbb{R}^d} \left(1+\frac{|q|^2}{\alpha^2}\right)^a \left| f\left(\frac{q}{\alpha}\right)\right|^2 dq \\ &= \alpha^d \int_{\mathbb{R}^d} \left(1+|Q|^2\right)^a \left| f\left(Q\right)\right|^2 dQ = \alpha^d \left\| f\right\|^2_{L^{2,a}(\mathbb{R}^d_Q)}\,. \end{aligned} \tag{3.299}$$

Next, the ($2\pi-$ periodic in q) discrete Fourier transform of $\alpha \mapsto \{G_n^{\sigma,\alpha}\}_{n\in\mathbb{Z}^d}$ is $\widehat{G^{\sigma,\alpha}}(q) = e^{-i\sigma\cdot q}\widehat{K^{\sigma,\alpha}}(q)$, where

$$\widehat{K^{\sigma,\alpha}}(q) = \sum_{m\in\mathbb{Z}^d} \chi_{\mathcal{B}}(q-2m\pi)\ \widehat{\phi^{\sigma,\alpha}}(q-2m\pi)\ e^{2\pi i m\cdot\sigma}, \tag{3.300}$$

This implies the expansion on the Brillouin zone $q \in \mathcal{B} = [-\pi, \pi]^d$:

$$\begin{aligned}\widehat{G^{\alpha,\sigma}}(q) &= e^{-iq\cdot\sigma}\ \widehat{\phi^{\sigma}}(q) \\ &= e^{-iq\cdot\sigma}\left(\sum_{j=0}^{J}\ \alpha^{2j+1-d}\ \chi_{\mathcal{B}}(q)\ F_j\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right) + \alpha^{1-d}\ \widehat{E_J^{\alpha,\sigma}}\left(\frac{q}{\alpha}\right)\right), \qquad q\in\mathcal{B}. \end{aligned}\tag{3.301}$$

Define for $1 \leqslant j \leqslant J$,

$$\mathcal{G}_j[\psi_1](n) = \alpha^{1-d}\ \mathcal{F}_D^{-1}\left[\ F_j\left[\widetilde{\psi_1}\right]\left(\frac{q}{\alpha}\right)\right]_n. \tag{3.302}$$

In analogy with the one-dimensional case, we may write

$$\mathcal{G}_0[\psi_1](n) = \psi_{\alpha^2}(n) + G_{\mathrm{resid}}(n), \tag{3.303}$$

where $\|G_{\mathrm{resid}}(n)\|_{l^2(\mathbb{Z}^d_n)} \lesssim e^{-C/\alpha}$, and

$$\begin{aligned}\mathcal{G}_0[\widetilde{\psi_1}](n) &\equiv \psi_{\alpha^2}(n) = \alpha\psi_1(\alpha n), \\ \mathcal{E}_n^{\alpha,J,\sigma} &\equiv \alpha^{1-d}\ \mathcal{F}_D^{-1}\left[e^{-iq\cdot\sigma}\widehat{E_J^{\alpha,\sigma}}\left(\frac{q}{\alpha}\right)\right]_n + G_{\mathrm{resid}}(n), \end{aligned}\tag{3.304}$$

Therefore, for any σ, in particular $\sigma = \{0, 1/2\}^d$,

$$\left\| \mathcal{G}_0[\widetilde{\psi_1}](n-\sigma) \right\|_{l^2(\mathbb{Z}_n)} \lesssim \frac{1}{(2\pi)^2} \left\| e^{-iq\sigma} F_0 \left[\widetilde{\psi_1}\right] \left(\frac{q}{\alpha}\right) \right\|_{L^2(\mathcal{B};dq)} + e^{-C/\alpha} \lesssim \alpha^{1/2}, \tag{3.305}$$

and for $1 \leqslant j \leqslant J$,

$$\mathcal{G}_j[\psi_1](n-\sigma) = \mathcal{F}_D^{-1} \left[e^{-iq\cdot\sigma} \; F_j \left[\widetilde{\psi_1}\right] \left(\frac{q}{\alpha}\right) \right]_n. \tag{3.306}$$

Applying the inverse discrete Fourier transform (2.4) to $\widehat{G^{\alpha,\sigma}}(q)$ in (3.301) gives the branches of vertex-, bond-, face-, and cell-centered discrete solitary waves (3.34). We use the Plancherel identity with the bounds (3.298) to get

$$\begin{aligned}
&\|\mathcal{G}_j[\psi_1](n-\sigma)\|_{l^2(\mathbb{Z}_n^d)} = \frac{\alpha^{1-d}}{(2\pi)^2} \left\| e^{-iq\sigma} F_j \left[\widetilde{\psi_1}\right] \left(\frac{q}{\alpha}\right) \right\|_{L^2(\mathcal{B};dq)} \\
&\lesssim \alpha^{1-d} \left\| F_j \left[\widetilde{\psi_1}\right] \left(\frac{q}{\alpha}\right) \right\|_{L^{2,a}(\mathbb{R}_q^d)} \lesssim \alpha^{1-d/2}, \\
&\left\| \mathcal{E}^{\alpha,J,\sigma} \right\|_{l^2(\mathbb{Z}^d)} \lesssim \frac{\alpha^{1-d}}{(2\pi)^2} \left\| e^{-iq\cdot\sigma} E_J^{\alpha,\sigma} \left(\frac{q}{\alpha}\right) \right\|_{L^2(\mathcal{B};dq)} + e^{-C/\alpha} \\
&\lesssim \alpha^{1-d} \left\| e^{-iq\cdot\sigma} E_J^{\alpha,\sigma} \left(\frac{q}{\alpha}\right) \right\|_{L^{2,a}(\mathbb{R}_q^d)} \lesssim \alpha^{2J+3-d/2}.
\end{aligned} \tag{3.307}$$

The proof of Theorem 3.2.3 for $d = 2, 3$ is almost identical to that from Section 3.7 for $d = 1$.

Chapter 4

Nonlocal Cubic DNLS: Bifurcation of Solitary Waves in Dimension $d = 1$

4.1 Introduction

In this chapter, we address solitary wave solutions to the nonlocal discrete nonlinear Schrödinger equation (nonlocal DNLS) (1.10) in spatial dimension $d = 1$:

$$\begin{aligned} i\partial_t u_n(t) &= -h^{-2}(\mathcal{L}u)_n(t) - |u_n(t)|^2 u_n(t)\ , \qquad t \in \mathbb{R}, n \in \mathbb{Z}, \\ (\mathcal{L}u)_n(t) &= \sum_{m\in\mathbb{Z}} J_{|m-n|}(u_m(t) - u_n(t)). \end{aligned} \tag{4.1}$$

Here, $\{J_{|m|}\}_{m\in\mathbb{N}} \in l^1(\mathbb{N})$ is the sequence of non-negative coupling coefficients. The two most common cases of (1.10) on which we will focus correspond respectively to the polynomial and exponential (Kac-Baker) decay of the coupling sequence:

$$J_m^s \simeq \begin{cases} \frac{1}{m^{1+2s}} & : \quad s > 0 \\ e^{-\gamma m}, \quad \gamma > 0 & : \quad s = \infty \end{cases}, \qquad m > 0. \tag{4.2}$$

Here, $J_0^s \equiv 0$. We shall refer to the case with $s > 1$ as the case of short-range interaction, and the case $0 < s < 1$ as long-range interaction. The case $s = 1$ is the critical or marginal interaction. See also (4.18).

We recall several details about nonlocal DNLS from Chapter 1. Nonlocal DNLS is a Hamiltonian system, expressible in the form

$$i\partial_t \mathbf{u} = \frac{\delta \mathcal{H}[\mathbf{u}, \overline{\mathbf{u}}]}{\delta \overline{\mathbf{u}}}, \quad \text{where} \tag{4.3}$$

$$\mathcal{H}_{\text{DNLS}} = \mathcal{H}[\mathbf{u}, \overline{\mathbf{u}}] = \frac{1}{2h^2} \sum_{n \in \mathbb{Z}^d} \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq n}} J_{|m-n|} |u_m - u_n|^2 - \frac{1}{2} |u_n|^4. \tag{4.4}$$

Let $\{J_m\}_{m \in \mathbb{Z}}$ be an admissible coupling sequence (long- or short-range). Then the initial value problem

$$i\partial_t u_n(t) = -h^{-2} \sum_{m \in \mathbb{Z}} J_{|m-n|} (u_m(t) - u_n(t)) - |u_n(t)|^2 u_n(t) \ , \ n \in \mathbb{Z}^d, \quad t \geqslant 0$$

$$u_n(0) = f_n \in l^2(\mathbb{Z}), \tag{4.5}$$

is globally well-posed in $l^2(\mathbb{Z})$. That is, for each $f = \{f_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$ there is a unique global solution $u(t) = \{u_n(t)\}_{n \in \mathbb{Z}} \in C^1([0, \infty), l^2(\mathbb{Z}))$ to (4.5). This result follows from a standard contraction mapping argument applied to the equivalent integral equation formulation of the initial value problem; again see, for example, [Kirkpatrick et al., 2012].

Furthermore, the functionals $\mathcal{H}[u, \overline{u}]$ and

$$\mathcal{N}_{\text{DNLS}} = \mathcal{N}[u, \overline{u}] = \sum_{n \in \mathbb{Z}} |u_n|^2 \tag{4.6}$$

are conserved quantities (time - invariant) for solutions of DNLS. For details, see Appendix B.

Remark 4.1.1. *Although the dynamics* (4.5) *are defined for any choice of* $\{J_m\}_{m \in \mathbb{N}}$ *in* $l^1(\mathbb{N})$, *we shall henceforth restrict our attention to the case* $\{J_m\}_{m \in \mathbb{N}}$ *in* (4.2) *for* $s > 1/4$. *The constraint on the range of the nonlocal interaction is the range for which fractional NLS equation, related to the continuum limit, has localized solutions. See Proposition 4.1.1, Theorem 4.2.1, Remark 4.2.1, and Appendix E.*

Following the approach set forth in Section 1.2, we introduce the rescaling

$$g_n = h^{-1} G_n, \tag{4.7}$$

and set

$$\kappa_s(\alpha) \equiv \sqrt{|\omega|} \ \ h \ . \tag{4.8}$$

where $\kappa_s(\alpha) > 0$ is a continuous function with $\kappa_s(\alpha) \to 0$ as $\alpha \downarrow 0$, to be determined. Then, $G = \{G_n\}_{n\in\mathbb{Z}^d}$ satisfies the nonlinear eigenvalue problem

$$-\kappa_s(\alpha)\ G_n = -\sum_{m\in\mathbb{Z}} J_{|m-n|}(G_m - G_n) - |G_n|^2 G_n\ , \qquad G \in l^2(\mathbb{Z}^d)\ . \tag{4.9}$$

We will study solutions to (4.9) as bifurcations of non-trivial localized states from the zero state at frequency $\kappa(0) = 0$, the endpoint of the continuous spectrum of $-\mathcal{L}$.

The main results of Chapter 4 concern the existence and energetic properties of localized solutions of (4.9) for α, and therefore $\kappa_s(\alpha)$, small:

1. **Theorem 4.2.1; bifurcation of onsite and off-site states of nonlocal DNLS with polynomially decaying coupling:** Let the sequence of coupling coefficients of nonlocal DNLS be (1) the polynomially decaying sequence (4.2) with coupling decay rate $1/4 < s < \infty$, or (2) the exponentially decaying sequence (4.2) for which we set $s = \infty$. Then there exist families of on-site (vertex-centered) symmetric and off-site (bond-centered) symmetric solitary standing waves of (1.29) (in the sense of Definition 2.0.1), which bifurcate from the continuum limit ($\alpha \downarrow 0$) ground state solitary wave of fractional NLS, displayed in (4.14), with fractional power $p = \min(1, s)$.

2. **Theorem 4.2.2; exponential smallness of the Peierls-Nabarro barrier:** Let $\eta = \min(1, 2s)$. Then, there exist positive constants α_0 and $C > 0$ such that for all $0 < \alpha < \alpha_0$, we have:

$$\begin{aligned} \left|\ \mathcal{N}\left[G^{\alpha,\mathrm{on}}\right] - \mathcal{N}\left[G^{\alpha,\mathrm{off}}\right]\ \right| &\lesssim \frac{\kappa_s(\alpha)}{\alpha}\ e^{-C/\alpha^\eta}, \\ \left|\ \mathcal{H}\left[G^{\alpha,\mathrm{on}}\right] - \mathcal{H}\left[G^{\alpha,\mathrm{off}}\right]\ \right| &\lesssim \frac{\kappa_s(\alpha)}{\alpha}\ e^{-C/\alpha^\eta}\ . \end{aligned} \tag{4.10}$$

   Here,

$$\mathcal{N}[G] \equiv \mathcal{N}\left[G, \overline{G}\right] = \sum_{n\in\mathbb{Z}} |G_n|^2, \tag{4.11}$$

   and

$$\mathcal{H}[G] \equiv \mathcal{H}\left[G, \overline{G}\right] = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\ J^s_{|m-n|}|G_m - G_n|^2 - \frac{1}{2}|G_n|^4 \tag{4.12}$$

are the corresponding square $l^2(\mathbb{Z})$ norm (Mass/Power/Charge) and Hamiltonian of (4.9) (for effective lattice spacing $h = 1$). The quantities

$$\mathcal{N}\left[G^{\alpha,\text{on}}\right] - \mathcal{N}\left[G^{\alpha,\text{off}}\right] \quad \text{and} \quad \mathcal{H}\left[G^{\alpha,\text{on}}\right] - \mathcal{H}\left[G^{\alpha,\text{off}}\right] \tag{4.13}$$

are related to the PN barrier; see the earlier discussion in Section 1.1.

Throughout this chapter, we use $\psi_{\alpha^{2p},p}$ for $1/4 < p \leqslant 1$ to refer to the unique *positive*, symmetric solution to the one-dimensional ($d = 1$) fractional nonlinear Schrödinger equation with cubic nonlinearity (FNLS) and frequency $\alpha^{2p} > 0$:

$$FNLS: \qquad 0 = \alpha^{2p}\ \psi_{\alpha^{2p},p}(x) + (-\Delta_x)^p\ \psi_{\alpha^{2p},p}(x) - \left(\psi_{\alpha^{2p},p}(x)\right)^3, \qquad x \in \mathbb{R}^d. \tag{4.14}$$

We also frequently use the convention, for the case where $\alpha = 1$, $\psi_p(x) \equiv \psi_{1,p}(x)$ to mean the solution to

$$0 = \psi_p(x) + (-\Delta_x)^p\ \psi_p(x) - \psi_p(x)^3, \qquad x \in \mathbb{R}^d. \tag{4.15}$$

We establish the properties of $\psi_{\alpha^{2p},p}$ and its continuous Fourier transform in Section 4.1.3

4.1.1 Outline of the Chapter

We finish the introduction of this chapter, first by describing our the strategy of our proof in Section 4.1.2. We also establish important properties of the continuum fractional NLS (FNLS) solitary wave in Section 4.1.3 which are central to the bifurcation analysis.

In Section 4.2, we state our rigorous results on solitary waves of nonlocal DNLS in Theorems 4.2.1 and 4.2.2.

In Section 4.3, we begin the proof of Theorem 4.2.1 and characterize nonlocal DNLS in the Fourier domain. We expand the Fourier symbol of the nonlocal difference operator in α to determine the fractional power of the continuum limiting Laplacian and take the FNLS solitary wave as a leading order ansatz.

In the following Section 4.4, we rigorously justify our ansatz by decomposing the error from our asymptotic expansion into its high- and low-frequency components. Here, we use the implicit function theorem solve for the high-frequency components as a functional of the low-frequency components and the small parameter α. We also obtain a bound on the high-frequency components which is small in terms of the low-frequency components and the small parameter.

In Section 4.5, we turn our attention to solving the low-frequency equation, by a perturbation argument about the continuum FNLS limit.

In Section 4.5.3, we use the analysis of the previous three sections to complete the proof of Theorem 4.2.1, the existence of on-site and off-site symmetric states of nonlocal DNLS.

In Section 4.6, we prove Theorem 4.2.2 on the PN-barrier.

In Section 4.7, we provide a strategy for generalizing the higher order asymptotic expansion from Section 3.3.2 for the nearest-neighbor DNLS case to the case of DNLS with nonlocal coupling, for specific rational cases of the coupling parameter s.

4.1.2 Strategy of the Proofs

As noted in Section 1.2, the limit $\kappa_s(\alpha) \downarrow 0$ for $\alpha \to 0$ in (4.35) is related to the continuum FNLS limit. In order to compare the spatially discrete and spatially continuous problems, it is natural to work, respectively, with the discrete and continuous Fourier transforms. These are both functions of a continuum variable (momentum, respectively, quasi-momentum).

Let $\widehat{g}(q) = \mathcal{F}_D[g](q)$ denote the discrete Fourier transform on $\mathbb{Z}$ of the sequence $g = \{g_n\}_{n\in\mathbb{Z}}$ and let $\widetilde{f}(q) = \mathcal{F}_C[f](q)$ denote the continuous Fourier transform on $\mathbb{R}$ of $f : \mathbb{R} \to \mathbb{C}$; see Chapter 2 for definitions and Appendix C for a discussion of key properties.

Motivated by Proposition 2.0.2, we first rewrite the equation for a DNLS standing wave profile (on-site or off-site) $G = G^{\sigma,\alpha,s} \in l^2(\mathbb{Z})$, (1.29), in discrete Fourier space for $\widehat{G}(q) = e^{-i\sigma q}\widehat{K}(q)$, where $K = \widehat{K}^{\sigma,\alpha}$ satisfies $K(q + 2\pi) = \widehat{K}(q)$. Here, $\sigma = 0$ corresponds to the on-site case and $\sigma = 1/2$ to the off-site case. Note that $\widehat{K}(q)$ is determined by its restriction, $\widehat{\phi}(q)$, to the fundamental cell $q \in \mathcal{B} = [-\pi, \pi]$ (Brillouin zone); we often suppress the dependence on α, s, and σ for notational convenience.

We obtain the following equation for $\widehat{\phi}(q)$, defined for $q \in \mathbb{R}$:

$$\begin{aligned}[\kappa_s(\alpha) + M^s(q)]\, \widehat{\phi^\sigma}(q) - \frac{\chi_{\mathcal{B}}(q)}{4\pi^2} \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma} \right)(q) \\ - \frac{\chi_{\mathcal{B}}(q)}{4\pi^2} \sum_{m=\pm 1} e^{2m\pi i\sigma} \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma} \right)(q - 2m\pi) = 0. \end{aligned} \tag{4.16}$$

where $\chi_{\mathcal{B}}(q)$ is the characteristic function on $\mathcal{B} = [-\pi, \pi]$. Here, $M^s(q)$ is the discrete Fourier symbol of the operator $-\mathcal{L}$:

$$M^s(q) = 4 \sum_{m=1}^{\infty} J_m^s \, \sin^2(qm/2) \tag{4.17}$$

See Lemma D.0.12 in Appendix C for details. In particular, we take

$$J_m^s \equiv \begin{cases} C_s^{-1} \, m^{-1-2s} & : \quad 1/4 < s < \infty \\ \\ C_\infty^{-1} \, e^{-\gamma \, m} & : \quad s = \infty; \quad \gamma > 0, \quad \text{constant.} \end{cases} \tag{4.18}$$

The constant $C_s > 0$ is chosen so that the expansion of $M_s(q)$, for small q is independent of s; see (4.33). This then leads to an effective limiting ($\alpha \downarrow 0$) equation in which $(-\Delta)^p$ has a coefficient equal to one.

Define the rescaled quasimomentum: $Q = q/\alpha$ and $\widehat{\Phi}(Q) = \rho(\alpha) \, \widehat{\phi}(q/\alpha)$, defined for $Q \in \mathbb{R}$ and where $\rho(\alpha)$ is to be determined. We again use Lemma 3.3.3 to rescale the convolutions in (4.52). Substituting these rescalings into (4.52) and dividing by $\kappa_s(\alpha) \, \rho(\alpha)$, we obtain the following equation for $\widehat{\Phi}(Q)$:

$$\begin{aligned} \mathcal{D}^{\sigma,\alpha}[\widehat{\Phi}] \equiv [1 + M_\alpha^s(q)] \, \widehat{\Phi}(Q) - \frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2} \left(\frac{\alpha^2 \, \rho(\alpha)^2}{\kappa_s(\alpha)} \right) \left(\, \widehat{\Phi} * \widehat{\Phi} * \widehat{\phi^\sigma} \, \right)(Q) \\ + \left(\frac{\alpha^2 \, \rho(\alpha)^2}{\kappa_s(\alpha)} \right) R_1^\sigma[\widehat{\Phi}](Q) = 0. \end{aligned} \tag{4.19}$$

Here, $\chi_{\mathcal{B}_\alpha}(Q)$ is characteristic function of $\mathcal{B}_\alpha$, and $M_\alpha^s(Q)$ is the scaled Fourier symbol of the discrete Laplacian:

$$M_\alpha^s(Q) \equiv \frac{1}{\kappa_s(\alpha)} M^s(Q\alpha) = \frac{4}{\kappa_s(\alpha)} \sum_{m=1}^{\infty} J_m \, \sin^2(Qm\alpha/2), \tag{4.20}$$

and

$$R_1^\sigma[\widehat{\Phi}](Q) = -\frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2} \sum_{m=\pm 1} e^{2m\pi i \sigma} \left(\, \widehat{\Phi} * \widehat{\Phi} * \widehat{\Phi} \, \right)(Q - 2m\pi/\alpha). \tag{4.21}$$

The nonlinear operator, $\widehat{\Phi} \mapsto \mathcal{D}^{\sigma,\alpha}[\widehat{\Phi}]$, depends on σ, which designates the case of on-site or off-site states, only through the "± 1 side-band" term R_1^σ. Note: We will find for $\alpha|Q| \ll 1$,

$$M^s(Q\alpha) \sim \begin{cases} \alpha^{2s} \, |Q|^{2s} & : 1/4 < s < 1 \\ (-\log(\alpha))\alpha^2 \, |Q|^2 & : s = 1 \\ \alpha^2 \, |Q|^2 & : 1 < s \leqslant \infty \end{cases} . \tag{4.22}$$

Therefore, take

$$\kappa_s(\alpha) = \begin{cases} \alpha^{2s} & : 1/4 < s < 1 \qquad (\text{long}-\text{range}) \\ (-\log(\alpha))\alpha^2 & : s = 1 \qquad (\text{critical/marginal}) \\ \alpha^2 & : 1 < s \leqslant \infty \qquad (\text{short}-\text{range}) \end{cases} . \tag{4.23}$$

such that for $\alpha \downarrow 0$,

$$M_s^\alpha(Q) = \frac{1}{\kappa_s(\alpha)} M_s(Q\alpha) \quad \to \quad \begin{cases} |Q|^{2s} & : 1/4 < s < 1 \\ |Q|^2 & : 1 < s \leqslant \infty \end{cases} . \tag{4.24}$$

Returning to equation (4.19), we obtain a nontrivial limiting equation as $\alpha \downarrow 0$ if $\rho(\alpha)$ is chosen such that

$$\frac{\alpha^2\ \rho(\alpha)}{\kappa_s(\alpha)} = 1 \qquad \Longleftrightarrow \qquad \rho(\alpha) = \sqrt{\frac{\kappa_s(\alpha)}{\alpha^2}}. \tag{4.25}$$

Reasoning formally and letting $p \equiv \min(1, s)$, we see that (4.19) converges to:

$$(1 + |Q|^{2p})\ \widetilde{\psi_p}(Q) \ - \ \frac{1}{4\pi^2}\ \left(\widetilde{\psi_p} * \widetilde{\psi_p} * \widetilde{\psi_p}\right)(Q) = 0, \tag{4.26}$$

the equation for $\widetilde{\psi_p}(Q)$, the Fourier transform on $\mathbb{R}$ of the continuum FNLS solitary wave, $\psi_p(x)$. In observing this we have used: (a) $M_\alpha^s(Q) \to |Q|^{2p}$, for bounded Q, as $\alpha \to 0$, (b) $\mathcal{B}_\alpha \to \mathbb{R}$ as $\alpha \to 0$, and (c) $\chi_{\mathcal{B}_\alpha}(Q) R_1^\sigma[\widehat{\Phi}](Q) \to 0$ as $\alpha \to 0$ for $\widehat{\Phi}(Q)$ localized; see Lemma 4.4.2. Therefore, at leading order in α the behavior of $\widehat{\Phi}^{\sigma,\alpha}$ appears to be $\widetilde{\psi_p}(Q)$.

The on-site and off-site discrete standing solitary waves, $G^{\alpha,\text{on}}$ and $G^{\alpha,\text{off}}$ can be constructed from by rescaling $Q = q/\alpha$ with

$$\widehat{G}(q) = e^{-iq\sigma}\widehat{\phi}(q) = e^{-iq\sigma}\ \sqrt{\frac{\kappa_s(\alpha)}{\alpha^2}}\ \widehat{\Phi}\left(\frac{Q}{\alpha}\right) \sim e^{-iq\sigma}\ \sqrt{\frac{\kappa_s(\alpha)}{\alpha^2}}\ \widehat{\Phi}\left(\frac{q}{\alpha}\right)\ \widetilde{\psi_p}\left(\frac{q}{\alpha}\right), \qquad q \in \mathcal{B}, \tag{4.27}$$

and inverting the discrete Fourier transform.

The proof of Theorem 4.2.2 is parallel to that for the nearest-neighbor case; see Section 3.1.2. Here too we require the exponential decay, uniformly in α small, of $\widehat{\Phi^{\alpha,\sigma}}(Q)$, $\sigma = 0, 1/2$. Using the exponential decay lemmata from Appendix H, we obtain the standard decay $\widehat{\Phi^{\alpha,\sigma}}(Q) \sim e^{-C|Q|}$ for $1/2 \leqslant s \leqslant \infty$ and the weaker decay $\widehat{\Phi^{\alpha,\sigma}}(Q) \sim e^{-C|Q|^{2s}}$ for $1/4 < s < 1/2$. These decay rates govern the size of the estimates in Theorem 4.2.2.

As in the nearest-neighbor case in Chapter 3, we note that the relation of (4.19) to the continuum limit FNLS equation (4.26) was based on formal convergence argument, as $\alpha \downarrow 0$, for fixed scaled quasimomentum, $Q \in [-\pi/\alpha, \pi/\alpha]$. To make the arguments rigorous we again use a Lypapunov-Schmidt reduction strategy. We first solve the quasi-momentum components of $\widehat{E_J^{\alpha,\sigma}}(Q)$ for $\lambda(\alpha)^{-1} \leqslant |Q| \leqslant \pi/\alpha$, (where $\lambda(\alpha) > \alpha$ and $\lambda(\alpha) \downarrow 0$ as $\alpha \to 0$) (high frequency components of $\widehat{E_J^{\alpha,\sigma}}$) in terms of those for $0 \leqslant |Q| \leqslant \lambda(\alpha)^{-1}$ (low frequency components of $\widehat{E_J^{\alpha,\sigma}}$). The solutions of the low-frequency equation can be studied perturbatively about the continuum FNLS limit using the implicit function theorem.

4.1.3 Properties of the Continuum Solitary Wave, $\psi_{|\omega|,p}$, on $\mathbb{R}$

The following results summarize properties of the FNLS solitary standing wave ("soliton") and its Fourier transform on $\mathbb{R}$. See, for example, references [Frank and Lenzmann, 2013; Frank et al., to appear].

Proposition 4.1.1 (FNLS Ground State)**.** *For $1/4 < p \leqslant 1$ consider the equation governing p-FNLS solitary standing waves with arbitrary frequency $\omega < 0$. There exists a unique positive $H^p(\mathbb{R})$ solution $\psi_{|\omega|}(x)$ to* (1.17) *which is real-valued, symmetric about $x = 0$ and decaying to zero at infinity. Moreover,*

1. *$u \in H^{2p+1}(\mathbb{R})$.*
2. *For all $k \geqslant 0$, $\psi_{|\omega|}(x),\ k \in H^k(\mathbb{R})$ and satisfies the decay estimate: $|\psi_1(x)| \lesssim (1 + |x|^{1+2p})^{-1}$.*
3. *$\psi_{|\omega|}(x) = \sqrt{|\omega|}\ \psi_1(|\omega|^{1/2p}x)$.*
4. *Uniqueness up to Phase and Spatial-Translation: Any positive solution of* (1.17) *is of the form $e^{i\theta}\ \psi_{|\omega|}(x - x_0)$ for some $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}$.*

Remark 4.1.2. *A well-known argument based on "Pohozaev" / virial identities shows that the equation $(-\Delta)^p u + u - u^m = 0$ does not admit non-trivial $H^{2p+1}(\mathbb{R}^d) \cap L^{m+1}(\mathbb{R}^d)$ solutions when $m \geqslant (d + 2p)/(d - 2p)$ (where if $d - 2p \leqslant 0$, all $m > 0$ are admissible). For the cubic case $m = 3$ in dimension $d = 1$, this implies $p > 1/4$; see [Strauss, 1977; Sulem and Sulem, 1999; Ros-Oton and Serra, 2014].*

The following result concerning $\widetilde{\psi_{|\omega|}} = \mathcal{F}_C[\psi_{|\omega|}]$, the Fourier transform of $\psi_{|\omega|}$, is a consequence of Proposition 4.1.1 and Lemma H.0.17.

Proposition 4.1.2 (Fourier transform of NLS Ground State)**.**
Fix $1/4 < p \leqslant 1$. The Fourier transform, $\widetilde{\psi_{|\omega|}}(Q) = \mathcal{F}_C[\psi_{|\omega|}](Q)$, satisfies the equation

$$\left(\ |\omega| \ + \ |Q|^{2p} \ \right) \ \widetilde{\psi_{|\omega|}}(Q) \ - \ \frac{1}{4\pi^2} \ \widetilde{\psi_{|\omega|}} * \widetilde{\psi_{|\omega|}} * \widetilde{\psi_{|\omega|}}(Q) \ = \ 0, \qquad \widetilde{\psi_{|\omega|}} \in L^{2,a}(\mathbb{R}). \tag{4.28}$$

Moreover, $\widetilde{\psi_{|\omega|}}$ satisfies the following properties:

1. *Scaling:*

$$\widetilde{\psi_{|\omega|}}(Q) = |\omega|^{(p-1)/2p} \ \widetilde{\psi_1}\left(\frac{Q}{|\omega|^{1/2p}}\right) = |\omega|^{(p-1)/2p} \ \widetilde{\psi}\left(\frac{Q}{|\omega|^{1/2p}}\right). \tag{4.29}$$

2. *Exponential decay: Let $a > 1/2$ and set $\eta = \eta(s) = \min(1, 2s)$. Then, there exists a positive constant $\mu = \mu[\|\widetilde{\psi}\|_{L^{2,a}(\mathbb{R})}]$ such that*

$$\left\| e^{\mu|Q|^\eta} \ \widetilde{\psi}(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \left\| \widetilde{\psi}(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{4.30}$$

In our bifurcation analysis, a central role is played by the mapping

$$f \mapsto L_+^p f \equiv \left(\ 1 + (-\Delta)^p \ - 3(\psi(x))^2 \ \right) f(x). \tag{4.31}$$

L_+^p is linearization of the stationary p-FNLS equation with $\omega = -1$, (1.17), about the ground state ψ. Correspondingly, in Fourier space, we have

$$\widetilde{f}(Q) \mapsto \widetilde{L_+^p} \widetilde{f}(Q) \equiv \left(\ 1 + |Q|^{2p} - \frac{3}{4\pi^2} \ \ \widetilde{\psi} * \widetilde{\psi} * \right) \widetilde{f}(Q) \tag{4.32}$$

We require, in particular, a characterization of the $L^2(\mathbb{R})-$ kernel of L_+^p. This was obtained in the important recent work [Frank and Lenzmann, 2013; Frank et al., to appear].

Proposition 4.1.3. 1. *The continuous spectrum of L_+^p is given by the half-line $[1, \infty)$.*

2. *Zero is an isolated eigenvalue of L_+^p with corresponding eigenspace,* $\text{kernel}(L_+^p) = \text{span}\Big\{\partial_x \psi(x)\Big\}$.

3. *Zero is an isolated eigenvalue of* $\widetilde{L_+^p}$ *with corresponding eigenspace,*
   $\mathrm{kernel}(\widetilde{L_+^p}) = \mathrm{span}\left\{Q\ \widetilde{\psi}(Q)\right\}.$

4. $L_+^p : H^a_{\rm even}(\mathbb{R}) \to H^{a-2p}_{\rm even}(\mathbb{R})$ *is an isomorphism.*

5. $\widetilde{L_+^p} : L^{2,a}_{\rm even}(\mathbb{R}) \to L^{2,a-2p}_{\rm even}(\mathbb{R})$ *is an isomorphism.*

4.2 Main Results

We begin with precise statements of our results on the existence of discrete solitary standing waves for non-nearest neighbor interactions in spatial dimension $d = 1$. For $1/4 < s < \infty$, set $J_0^s = 0$ and $J_m^s = C_s^{-1} m^{-1-2s},\ \ m > 0$, giving

$$(\mathcal{L}^s G)_n = \frac{1}{C_s} \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{(G_m - G_n)}{|m-n|^{1+2s}},$$

$$\text{where} \qquad C_s = \begin{cases} -2\Gamma(-2s)\cos(\pi s) & : 1/4 < s < 1 \\ 1 & : s = 1 \\ \zeta(2s-1) & : s > 1 \end{cases} . \tag{4.33}$$

For $s = \infty$, set $J_0^s = 0$ and $J_m^s = C_\infty^{-1} e^{-\gamma m},\ \gamma > 0,\ m > 0$, giving

$$(\mathcal{L}^s G)_n = \frac{1}{C_\infty} \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} e^{-\gamma\ |m-n|}(G_m - G_n), \qquad \text{where} \qquad C_\infty = \frac{\exp(\gamma)(\exp(\gamma)+1)}{(\exp(\gamma)-1)^3}. \tag{4.34}$$

Theorem 4.2.1. *(Nonlocal DNLS solitary waves on $\mathbb{Z}$) Consider the nonlinear eigenvalue problem governing real-valued standing waves of the discrete nonlocal DNLS* (4.9)*:*

$$-\kappa_s(\alpha)\ G_n^\alpha = -(\mathcal{L}_s G^\alpha)_n - (G_n^\alpha)^3, \qquad G^\alpha \in l^2(\mathbb{Z}). \tag{4.35}$$

where $1/4 < s \leqslant \infty$ and $\kappa_s(\alpha)$ is the $s-$ dependent scaling displayed in (4.23)*. Let $p = p(s) =$* $\min(s,1)$ *and denote by $\psi(x) \equiv \psi_1(x)$ denote the ground state of the $p(s)$-FNLS equation,* (1.17)*; see Proposition 4.1.1.*

Then, there exists $\alpha_0 = \alpha_0[s] > 0$ such that for all $0 < \alpha < \alpha_0$, nonlocal DNLS has two real-valued solitary wave solutions to (4.35)*. These are on-site (lattice point- centered) and off-site (bond-centered) symmetric states. To leading order in α, these are given by $\sigma = 0, 1/2$ translates of $\psi(x)$ sampled on $\mathbb{Z}$. The leading order expansions with correctors are given by:*

On-site symmetric (vertex-centered):

$$G_n^{\alpha,\text{on}} = \kappa_s(\alpha)^{1/2}\ \psi(\alpha\ n) + \mathcal{E}_n^{\alpha,\text{on}}, \qquad n \in \mathbb{Z}$$

Off-site symmetric (bond-centered):

$$G_n^{\alpha,\text{off}} = \kappa_s(\alpha)^{1/2}\ \psi(\alpha\ [n - 1/2]) + \mathcal{E}_n^{\alpha,\text{off}}, \qquad n \in \mathbb{Z}\ .$$

Here, $\|\mathcal{E}^{\alpha,\text{on}}\|_{l^2(\mathbb{Z})} + \|\mathcal{E}^{\alpha,\text{off}}\|_{l^2(\mathbb{Z})} \lesssim \mathfrak{e}_1(\alpha) \to 0$ as $\alpha \to 0$, where the $s-$ dependent rate $\mathfrak{e}_1(\alpha)$ is is given by

$$\mathfrak{e}_1(\alpha) \equiv \begin{cases} \alpha^{3/2-s} & : 1/4 < s < 1 \\ \left(\frac{\alpha}{-\log(\alpha)}\right)^{1/2} & : s = 1 \\ \alpha^{2s-3/2} & : 1 < s < 2 \\ (-\log(\alpha))\alpha^{5/2} & : s = 2 \\ \alpha^{5/2} & : 2 < s \leqslant \infty. \end{cases} \qquad (4.36)$$

Note: the relative $l^2(\mathbb{Z})$ norm of the corrector to the leading term, $\mathfrak{e}_1(\alpha) \div (\kappa_s(\alpha)/\alpha)^{\frac{1}{2}} \to 0$ as $\alpha \to 0$.

Remark 4.2.1. *In Theorem 4.2.1, we assume $p = \min(1, s) > 1/4$. We do not expect there to be a bifurcation from at zero frequency for coupling decay $p \leqslant 1/4$. Indeed, the $\kappa_s(\alpha) \to 0$ rescaled-limit of such bifurcating states is the solitary standing wave solution of continuum fractional NLS, satisfying $(-\Delta_x)^p u + u - u^3 = 0$. A well-known argument ([Sulem and Sulem, 1999] for local case $p = 1$) based on "Pohozaev" / virial identities shows that the equation $(-\Delta u)^p + u - u^x = 0$ has $H^{2p+1}(\mathbb{R}^d)$ solutions only if $x < (d + 2s)/(d - 2p)$. For the cubic case, $x = 3$, this implies $p > 1/4$; see Appendix E. Therefore, there can be no bifurcation for $p \leqslant 1/4$.*

Theorem 4.2.2. *[Exponential smallness of Peierls-Nabarro barrier] Let $\eta = \min(2s, 1)$. There exist constants $\alpha_0 > 0, C$ and $D > 0$ such that for all $0 < \alpha < \alpha_0$,*

$$\left|\mathcal{N}[G^{\alpha,\text{off}}] - \mathcal{N}[G^{\alpha,\text{on}}]\right| = \left|\|G^{\alpha,\text{off}}\|_{l^2(\mathbb{Z})}^2 - \|G^{\alpha,\text{on}}\|_{l^2(\mathbb{Z})}^2\right| \leqslant D\left(\frac{\kappa_s(\alpha)}{\alpha}\right)\ e^{-C/\alpha^\eta}, \quad \text{and}$$
$$\left|\mathcal{H}[G^{\alpha,\text{off}}] - \mathcal{H}[G^{\alpha,\text{on}}]\right| \leqslant D\left(\frac{\kappa_s(\alpha)}{\alpha}\right)\ e^{-C/\alpha^\eta}. \qquad (4.37)$$

Remark 4.2.2. *(Extensions) We note that our results for non-local coupling sequences* $\{J^s_{|m|}\}_{m\in\mathbb{Z}}$ *can be extended to general non-local coupling sequences* $\{\mathcal{J}^s_{|m|}\}_{m\in\mathbb{Z}}$*, with asymptotic decay rate* $\sim |m|^{-1-2s}$*. If* $1/4 < s \leqslant 1$*, one may only extend under appropriate assumptions on the rate of decay of* $\mathcal{J}^s_{|m|}/J^s_{|m|} - 1$ *as* $|m| \to \infty$*.*

The results may also be extended to coupling sequences $\{\mathcal{J}^\infty_{|m|}\}_{m\in\mathbb{Z}}$ *with asymptotic decay faster than any algebraic power in* m*. Note that this case includes any coupling sequence with compact support, including that of the nearest-neighbor (centered-difference) Laplacian.*

In Chapter 3, our results apply to cubic DNLS with nearest-neighbor coupling in dimensions $d = 1, 2, 3$*. The dimensionality of the bifurcation result is restricted by nonexistence of non-trivial standing wave solutions to the continuum NLS equation. We note here that we expect to be able to extend our results on the cubic nonlocal DNLS to dimension* $d \geqslant 2$ *under the similar restriction* $2 \leqslant d < 4p$*. See Remark 4.1.2.*

4.3 Beginning of the proof of Theorem 4.2.1; Nonlocal in Fourier space

We follow the proof of Theorem 3.2.1 closely and refer to it when convenient. Recall that

$$(\mathcal{L}^s G)_n = \sum_{m\in\mathbb{Z}} J^s_{|m-n|}\ (G_m - G_n), \tag{4.38}$$

where we set $J^s_0 \equiv 0$ and

$$J^s_m = \begin{cases} \frac{1}{C_s\ m^{1+2s}} & : 1/4 < s < \infty \\ \frac{1}{C_\infty}\ e^{-\gamma\ m} & : s = \infty \end{cases}, \qquad m > 0, \tag{4.39}$$

for $\gamma > 0$ arbitrary and

$$C_s = \begin{cases} -2\Gamma(-2s)\cos(\pi s) & : 1/4 < s < 1 \\ 1 & : s = 1 \\ \zeta(2s-1) & : 1 < s \leqslant \infty \\ \frac{\exp(\gamma)(\exp(\gamma)+1)}{(\exp(\gamma)-1)^3} & : s = \infty \end{cases}. \tag{4.40}$$

We also set

$$\kappa_s(\alpha) = \begin{cases} \alpha^{2s} & : 1/4 < s < 1 \\ (-\log(\alpha))\alpha^2 & : s = 1 \\ \alpha^2 & : 1 < s \leqslant \infty \end{cases} \tag{4.41}$$

Applying the discrete Fourier transform (2.1) to equation (4.35), governing $G = G^{\alpha,s}$, we obtain an equivalent equation for the discrete Fourier transform, $\widehat{G}(q) = \widehat{G^{\alpha}}(q)$:

$$\widehat{DNLS}[\widehat{G}](q) \equiv \left[\kappa_s(\alpha) + M^s(q)\right]\widehat{G}(q) - \left(\frac{1}{2\pi}\right)^2 \widehat{G} *_1 \widehat{G} *_1 \widehat{G}(q) = 0,$$
$$\widehat{G}(q + 2\pi) = \widehat{G}(q), \tag{4.42}$$

where we recall the definition of the convolution $f *_1 g$ on $\mathcal{B} = \mathcal{B}_1$ in (2.5) and its properties (Appendix C). Here, $M^s(q)$ denotes the (discrete) Fourier symbol of the 1-dimensional discrete difference operator

$$M^s(q)\ \widehat{G}(q) = -\widehat{(\mathcal{L}^s G)}(q) = 4 \sum_{m=1}^{\infty} J_m \sin^2(qm/2)\ \widehat{G}(q)\ . \tag{4.43}$$

See Lemma D.0.12. By Proposition 2.0.2 we have that

1. if G is onsite symmetric, then $\widehat{G}(q) = \widehat{K}(q)$, where $\widehat{K}(q)$ is real and symmetric, and
2. if G is offsite symmetric, then $\widehat{G}(q) = e^{-iq/2}\widehat{K}(q)$, where $\widehat{K}(q)$ is real and symmetric,

and seek $\widehat{G}(q)$ in the form

$$\widehat{G^{\sigma}}(q) = e^{-i\sigma q}\ \widehat{K^{\sigma}}(q), \quad \sigma = 0, 1/2 \tag{4.44}$$
$$\widehat{K^{\sigma}}(q) = \widehat{K^{\sigma}}(-q), \quad \overline{\widehat{K^{\sigma}}(q)} = \widehat{K^{\sigma}}(q) \tag{4.45}$$

Substitution of (4.44) into (4.42) yields

$$\left[\kappa_s(\alpha) + M^s(q)\right]\widehat{K^{\sigma}}(q) - \left(\frac{1}{2\pi}\right)^2 \widehat{K^{\sigma}} *_1 \widehat{K^{\sigma}} *_1 \widehat{K^{\sigma}}(q) = 0, \quad q \in \mathbb{R}, \tag{4.46}$$
$$\widehat{K^{\sigma}}(q + 2\pi) = e^{2\pi i\sigma}\ \widehat{K^{\sigma}}(q), \qquad \sigma = 0, 1/2\ . \tag{4.47}$$

The “Bloch” phase factor, $e^{2\pi i\sigma}$ (equal to ± 1) encodes the on-site and off-site cases.

By Lemma 3.3.1 we may express $\widehat{K^\sigma}(q)$, for any $q \in \mathbb{R}$, explicitly in terms of its values on $q \in \mathcal{B}$. In particular, we set

$$\widehat{K^\sigma}(q) = \widehat{\phi^\sigma}(q), \qquad q \in \mathcal{B}. \tag{4.48}$$

Extending (4.48) to $q \in \mathbb{R}$, we have:

$$\widehat{K^\sigma}(q) \equiv \sum_{m\in\mathbb{Z}} \chi_{\mathcal{B}}(q - 2m\pi)\ \widehat{\phi^\sigma}(q - 2m\pi)\ e^{2m\pi i\sigma}, \qquad \text{and} \qquad \widehat{G^\sigma}(q) = e^{-i\sigma q}\ \widehat{K^\sigma}(q). \tag{4.49}$$

Note that $\widehat{\phi^\sigma}(q - 2m\pi) = \chi_{\mathcal{B}}(q - 2m\pi)\ \widehat{\phi^\sigma}(q - 2m\pi)$ is supported on $\{q : q \in 2m\pi + \mathcal{B} = [(2m - 1)\pi, (2m + 1)\pi]\}$. Therefore,

$$\chi_{\mathcal{B}}(q)\ \widehat{K^\sigma}(q) = \widehat{\phi^\sigma}(q), \qquad \text{and} \qquad \chi_{\mathcal{B}}(q)\ \widehat{G^\sigma}(q) = e^{-iq\sigma}\ \widehat{\phi^\sigma}(q). \tag{4.50}$$

Equations (4.49) encode the required $2\pi-$ periodicity of $\widehat{G^\sigma}(q)$ and the $2\pi\sigma-$ pseudo-periodicity of $\widehat{K^\sigma}(q)$ on all $\mathbb{R}$,. Furthermore,

$$\widehat{G^\sigma}(q) \text{ and } \widehat{K^\sigma}(q) \text{ are completely specified by } \widehat{\phi^\sigma}(q) \text{ for } q \in \mathcal{B}. \tag{4.51}$$

With a view toward deriving an equation from (4.46) determining $\widehat{\phi^\sigma}(q)$ for $q \in \mathcal{B}$, we require a lemma which facilitates simplification of the convolution terms in (4.46).

Applying Lemma 3.3.2 and (4.49), we have:

Proposition 4.3.1. *Equation* (4.46) *for* $\widehat{K^\sigma}(q)$ *on* $q \in \mathbb{R}$ *is equivalent to the following equation for the compactly supported function* $\widehat{\phi^\sigma}(q) = \chi_{\mathcal{B}}(q)\ \widehat{\phi^\sigma}(q)$*:*

$$\begin{aligned}[\alpha^2 + M^s(q)]\ \widehat{\phi^\sigma}(q) - \frac{\chi_{\mathcal{B}}(q)}{4\pi^2}\ \left(\ \widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma}\ \right)(q) \\ -\ \frac{\chi_{\mathcal{B}}(q)}{4\pi^2} \sum_{m=\pm 1} e^{2m\pi i\sigma}\ \left(\ \widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma}\ \right)(q - 2m\pi) = 0\end{aligned} \tag{4.52}$$

where $M^s(q)\widehat{G}(q) = 4\sum_{m=1}^{\infty} J^s_m\ \sin^2(qm/2)\ \widehat{G}(q) = -(\widehat{\mathcal{L}^s G})(q)$*; see* (4.43)

Proof of Proposition 4.3.1: We project (4.46) onto $\mathcal{B}$, decompose $\widehat{K^\sigma}(q)$ with (4.49), and apply Lemma 3.3.2 to (4.46) get

$$[\kappa_s(\alpha) + M^s(q)]\ \widehat{\phi^\sigma}(q) - \frac{\chi_{\mathcal{B}}}{4\pi^2} \sum_{m=-1}^{1} e^{2\pi i m\sigma} \widehat{\phi^\sigma} *_1 \left[\widehat{\phi^\sigma} *_1 \left(\chi_{\mathcal{B}} \widehat{\phi^\sigma}\right)\right](q - 2m\pi) = 0. \tag{4.53}$$

We observe that the multiplication of (4.53) by $\overline{\chi}_{\mathcal{B}}(q)$ implies a-priori that

$$[\kappa_s(\alpha) + M^s(q)]\ \overline{\chi}_{\mathcal{B}}(Q)\ \widehat{\phi^\sigma}(q) = 0 \qquad \Longrightarrow \qquad \overline{\chi}_{\mathcal{B}}(Q)\ \widehat{\phi^\sigma}(q) = 0, \tag{4.54}$$

where we have used that $\kappa_s(\alpha), M(q) \geqslant 0$ (for α sufficiently small when $s = 1$). Thus, we may write for $m = -1, 0, 1$,

$$\begin{aligned}\widehat{\phi^\sigma} *_1 \left[\widehat{\phi^\sigma} *_1 \left(\chi_{\mathcal{B}}\ \widehat{\phi^\sigma}\right)\right](q - 2m\pi) &= \left(\chi_{\mathcal{B}}\ \widehat{\phi^\sigma}\right) * \left(\chi_{\mathcal{B}}\ \widehat{\phi^\sigma}\right) * \left(\chi_{\mathcal{B}}\ \widehat{\phi^\sigma}\right)(q - 2m\pi)\\ &= \left(\widehat{\phi^\sigma} * \widehat{\phi^\sigma} * \widehat{\phi^\sigma}\right)(q - 2m\pi),\end{aligned} \tag{4.55}$$

This completes the proof of Proposition 4.3.1. □

4.3.1 Rescaled Equation for $\widehat{\phi^\sigma}$

As discussed in Section 4.1.2, we expect that for $\alpha \ll 1$:

$$\widehat{\phi^\sigma}(q) \sim \left(\frac{\kappa_s(\alpha)^{1/2}}{\alpha}\right)\ \widetilde{\psi_{\alpha^{2p},p}}(q) = \left(\frac{\kappa_s(\alpha)^{1/2}}{\alpha}\right)\ \widetilde{\psi_{1,p}}\left(\frac{q}{\alpha}\right), \tag{4.56}$$

where $\widetilde{\psi_{1,p}} = \widetilde{\psi_p}$ denotes the Fourier transform of the continuum NLS solitary wave. We therefore study (4.52) using rescaling which makes explicit the relation between DNLS and the continuum (NLS) limit for α small: We introduce:

Rescaled momentum: $Q \equiv q/\alpha, \quad Q \in \mathcal{B}_\alpha = [-\pi/\alpha, \pi/\alpha],$

Rescaled projection: $\chi_{\mathcal{B}_\alpha}(Q) \equiv \chi_{\mathcal{B}}(Q\alpha) = \chi_{[-\frac{\pi}{\alpha},\frac{\pi}{\alpha}]}(Q),$

Rescaled wave: $\widehat{\Phi^\sigma}(Q) \equiv \left(\frac{\alpha}{\kappa_s(\alpha)^{1/2}}\right)\widehat{\phi^\sigma}(Q\alpha) = \left(\frac{\alpha}{\kappa_s(\alpha)^{1/2}}\right)\widehat{\phi^\sigma}(q),$

Rescaled Fourier symbol: $M^s_\alpha(Q) \equiv \frac{1}{\kappa_s(\alpha)} M^s(Q\alpha) = \frac{4}{\kappa_s(\alpha)}\sum_{m=1}^{\infty} J^s_m\ 4\sin^2(Qm\alpha/2).$ (4.57)

The following proposition is a formulation of Proposition 4.3.1 in terms of functions of the rescaled quasi-momentum, Q:

Proposition 4.3.2. *Equation* (4.46) *for* $\widehat{K^\sigma}(q)$ *on* $q \in \mathbb{R}$ *is equivalent to the following equation for* $\widehat{\Phi^\sigma}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widehat{\Phi^\sigma}(Q)$*, compactly supported on* $\mathcal{B}_\alpha = [-\pi/\alpha, \pi/\alpha]$*:*

$$\begin{aligned}\mathcal{D}^{\sigma,\alpha}[\widehat{\Phi^\sigma}](Q)\ \equiv\ [1 + M^s_\alpha(Q)]\ \widehat{\Phi^\sigma}(Q) - \frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2}\ \left(\ \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma}\ \right)(Q)\\ +\ R^\sigma_1[\widehat{\Phi^\sigma}](Q) = 0\ ,\end{aligned} \tag{4.58}$$

where $R_1^\sigma[\widehat{\Phi^\sigma}]$ contains the ± 1-sideband contributions:

$$R_1^\sigma[\widehat{\Phi^\sigma}](Q) \equiv -\frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2} \sum_{m=\pm 1} e^{2m\pi i\sigma} \left(\widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} \right)(Q - 2m\pi/\alpha), \tag{4.59}$$

and $M_\alpha^s(Q) = \frac{4}{\kappa_s(\alpha)} \sum_{m=1}^\infty J_m^s \sin^2(Qm\alpha/2)$; *see* (4.57).

Applying the rescalings (4.57) and Lemma 3.3.3 to (4.52) and then dividing by $\kappa_s(\alpha)^{3/2}/\alpha$, we obtain (4.58).

The next proposition states the sense in which $M_\alpha^s(Q) \to |Q|^{2p}$ ($p = \min(1, s)$) as $\alpha \to 0$. This facilates the solution of $\mathcal{D}^{\sigma,\alpha}\left[\widehat{\Phi}\right](Q) = 0$ for $\widehat{\Phi}^{\sigma,\alpha}(Q)$, perturbatively about the solution of the limiting $p(s)-$ FNLS limit equation (4.29).

Proposition 4.3.3. *Let $0 < \eta \leqslant 1$ and fix $p = \min(1, s)$. Suppose $e^{C|Q|^\eta}\widehat{g}(Q) \in L^{2,a}(\mathbb{R}_Q)$ for some $C > 0$. Then, as $\alpha \to 0$*

$$\left\| \chi_{\mathcal{B}_\alpha}(Q) \left[M_\alpha^s(Q) - |Q|^{2p} \right] \widehat{g}(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \sqrt{\alpha}\ \mathfrak{e}_1(\alpha). \tag{4.60}$$

Proof of Proposition 4.3.3: The heart of the matter is Proposition G.0.9 which states that for $|Q| \leqslant \pi/\alpha$ ($Q \in \mathcal{B}_\alpha$), there exists a function $f_s(Q;\alpha)$ such that $\max_{Q \in \mathcal{B}_\alpha} |f_s(Q;\alpha)| \lesssim 1$ and

$$M_\alpha^s(Q) = \frac{1}{C_s\ \kappa_s(\alpha)} M^s(Q\alpha) = |Q|^{2p} + \begin{cases} \alpha^{2-2s}\ f_s(Q;\alpha)\ |Q|^2 & : \quad s < 1 \\ \frac{f_s(Q;\alpha)}{-\log(\alpha)} \left(\frac{3}{2} - \log(|Q|)\right)\ |Q|^2 & : \quad s = 1 \\ \alpha^{2s-2}\ f_s(Q;\alpha)\ |Q|^{2s} & : \quad 1 < s < 2 \\ (-\log(\alpha))\ \alpha^2\ f_s(Q;\alpha)\ |Q|^4 & : \quad s = 2 \\ \alpha^2\ f_s(Q;\alpha)\ |Q|^4 & : \quad 2 < s \leqslant \infty \end{cases} . \tag{4.61}$$

Using (4.61), $\chi_{\mathcal{B}_\alpha}(Q)\ \left[M^s_\alpha(Q) - |Q|^{2p}\right]\widehat{g}(Q)$ can be bounded by $\sqrt{\alpha}\ \mathfrak{e}_1(\alpha)$ (see (4.36)) times a constant in terms of $\|\ |Q|^{2j}\ \widehat{g}(Q)\|_{L^{2,a}(\mathbb{R}_Q)}$, $\ \|\ (\log(|Q|))\ |Q|^{2j}\ \widehat{g}(Q)\|_{L^{2,a}(\mathbb{R}_Q)}$ $(j = 1, 2)$, which is finite by hypothesis on $\widehat{g}$. This completes the proof of Proposition 4.3.3. □

4.4 Bifurcation Analysis and the Proof of Theorem 4.4.1

Our main results on bifurcation are derived from the following result:

Theorem 4.4.1. *Fix $a > 1/2$. Consider the nonlinear eigenvalue problem (4.35) for onsite ($\sigma = 0$) and offsite ($\sigma = 1/2$) bound states of nonlocal DNLS with nonlocal interaction parameter $s \in (1/2, \infty]$. Let ψ denote the ground state of the p-FNLS equation, (4.28), $\psi + (-\Delta)^p\psi - \psi^3 = 0$, where $p = p(s) = \min(1, s)$, and let $\widetilde{\psi}$ be its continuous Fourier transform. Then there exist a constant $\alpha_0 = \alpha_0[a, \sigma, s] > 0$, and a unique, real-valued function $\widehat{E^{\alpha,\sigma}} \in L^{2,a}_{\text{even}}(\mathbb{R})$, with $\widehat{E^{\alpha,\sigma}} = \chi_{\mathcal{B}_\alpha}\widehat{E^{\alpha,\sigma}}$, defined for all $0 < \alpha < \alpha_0$, such that*

$$\widehat{\Phi^{\alpha,\sigma}}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widetilde{\psi}(Q) + \widehat{E^{\alpha,\sigma}}(Q), \tag{4.62}$$

solves (4.58), with corrector bound: $\left\|\widehat{E^{\alpha,\sigma}}\right\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{\frac{1}{2}}\mathfrak{e}_1(\alpha) \to 0\ \alpha \to 0$. $\mathfrak{e}_1(\alpha)$ is displayed in (4.36).

4.4.1 Equation for the Remainder, $\widehat{E^{\alpha,\sigma}}$

For $\widehat{\Phi} = \widehat{\Phi^{\alpha,\sigma}}$ we take the ansatz

$$\widehat{\Phi}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widetilde{\psi}(Q) + \widehat{E^{\alpha,\sigma}}(Q) \equiv S(Q) + \widehat{E}(Q). \tag{4.63}$$

Here, $S(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widetilde{\psi}(Q)$. To prove Theorem 4.4.1, we derive an equation for $\widehat{E}(Q)$ and then construct and bound its solution. Recall, by Proposition 4.3.2, that $\widehat{\Phi}$ must satisfy $\mathcal{D}^{\sigma,\alpha}[\widehat{\Phi}](Q) = 0$; see (4.58). Substitution of the ansatz (4.63) into this equation yields the required equation for $\widehat{E}(Q)$.

Proposition 4.4.1. *$\mathcal{D}^{\sigma,\alpha}[\widehat{\Phi}] = 0$ is equivalent to the following equation for $\widehat{E}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widehat{E}(Q)$:*

$$[1 + M^s_\alpha(Q)]\ \ \widehat{E}(Q) - 3\ \chi_{\mathcal{B}_\alpha}(Q)\ \frac{1}{(2\pi)^2}\ \widetilde{\psi} * \widetilde{\psi} * \widehat{E}(Q) = \mathcal{R}^\sigma_1\left[\alpha, \widehat{E}\right](Q), \quad \widehat{E} \in L^{2,a}(\mathbb{R}). \tag{4.64}$$

where

$$\mathcal{R}_1^\sigma\left[\alpha, \widehat{E}\right] \equiv \mathcal{D}^{\sigma,\alpha}[S] + R_{\text{L}}^\sigma\left[\alpha, \widehat{E}\right] + R_{\text{NL}}^\sigma\left[\alpha, \widehat{E}\right]. \tag{4.65}$$

R_{L}^σ is linear in $\widehat{E}$ and of order $\mathcal{O}(1)$ in α on $Q \in \mathcal{B}_\alpha$. R_{NL}^σ contains terms which are nonlinear in $\widehat{E}$:

$$R_{\text{L}}^\sigma\left[\alpha, \widehat{E}\right](Q) \equiv \chi_{\mathcal{B}_\alpha}(Q)\, \frac{3}{4\pi^2}\left[\sum_{m=-1}^{1} e^{2m\pi i\sigma}\left(S * S * \widehat{E}\right)(Q - 2m\pi/\alpha) - \left(\widetilde{\psi_1} * \widetilde{\psi_1} * \widehat{E}\right)(Q)\right],$$

$$R_{\text{NL}}^\sigma\left[\alpha, \widehat{E}\right](Q) \equiv \chi_{\mathcal{B}_\alpha}(Q)\left(\frac{1}{2\pi}\right)^2 \sum_{m=-1}^{1} e^{2m\pi i\sigma}\left[3\left(S * \widehat{E} * \widehat{E}\right)(Q - 2m\pi/\alpha) + \left(\widehat{E} * \widehat{E} * \widehat{E}\right)(Q - 2m\pi/\alpha)\right]. \tag{4.66}$$

Remark 4.4.1. *Note that the operator on the left-hand side has formal limit $\widetilde{L_+^p}$, where L_+^p is the linearized continuum FNLS operator displayed in Proposition 4.1.3.*

4.4.2 Coupled System for High and Low Frequency Components of $\widehat{E} = \widehat{E^{\alpha,\sigma}}$

We now embark on the construction $\widehat{E} = \widehat{E^{\alpha,\sigma}} \in L^{2,a}(\mathbb{R})$ for $\alpha > 0$ sufficiently small. Our strategy is to reformulate the equation for $\widehat{E}$ as an equivalent coupled system for its high and low frequency components.

Let r be such that $0 < r < 1$. Define the spectral cutoff parameter:

$$\lambda(\alpha) = \lambda_s(\alpha) \equiv \begin{cases} \alpha^{1-r} & : s \neq 1 \\ \frac{1}{(-\log(\alpha))} & : s = 1 \end{cases}. \tag{4.67}$$

Note that $\lambda_s(\alpha) \to 0$ as $\alpha \to 0$.

Next, define the sharp spectral cutoffs onto low and high frequency regimes:

$$\chi_{\text{lo}}(Q) = \chi\left(|Q| \leqslant \frac{1}{\lambda_s(\alpha)}\right) \quad \text{and} \quad \chi_{\text{hi}}(Q) = \chi\left(|Q| > \frac{1}{\lambda_s(\alpha)}\right), \tag{4.68}$$

where $1 = \chi_{\text{lo}}(Q) + \chi_{\text{hi}}(Q)$. Note that $\chi_{\text{lo}}\ \chi_{\mathcal{B}_\alpha} = \chi_{\text{lo}}$, while $\chi_{\text{hi}}\ \chi_{\mathcal{B}_\alpha} = \chi_{\mathcal{B}_\alpha} - \chi_{\text{lo}}$. For general $\widehat{A}$, defined on $\mathbb{R}$, introduce its localizations near and away from $Q = 0$:

$$\widehat{A}_{\text{lo}}(Q) = \left(\chi_{\text{lo}}\widehat{A}\right)(Q) \equiv \chi_{\text{lo}}(Q)\widehat{A}(q), \tag{4.69}$$

$$\widehat{A}_{\text{hi}}(Q) = \left(\chi_{\text{hi}}\widehat{A}\right)(Q) \equiv \chi_{\text{hi}}(Q)\widehat{A}(q), \tag{4.70}$$

In particular, we use $\chi_{\rm lo}$ and $\chi_{\rm hi}$ to localize $\widehat{E} = \widehat{E^{\alpha,\sigma}}$, where $|Q| \leqslant \lambda_s(\alpha)$ and where $|Q| > \lambda_s(\alpha)$:

$$\widehat{E_{\rm lo}}(Q) = \chi_{\rm lo}(Q)\widehat{E}(Q) \quad \text{and} \quad \widehat{E_{\rm hi}}(Q) = \chi_{\rm hi}(Q)\widehat{E}(Q), \tag{4.71}$$

and therefore $\widehat{E}(Q) = \widehat{E_{\rm lo}}(Q) + \widehat{E_{\rm hi}}(Q)$.

NOTE: *Since the analysis for the cases $\sigma = 0$ (onsite) and $\sigma = 1/2$ (offsite) in sections 4.4.2 - 4.5.3 are very similar, in order to keep the notation less cumbersome we omit the superscripts α and σ, when the context is clear, and shall instead write:*

$$\widehat{E}(Q) = \widehat{E}(Q), \qquad \widehat{E_{\rm lo}}(Q) = \widehat{E_{\rm lo}}(Q), \qquad \widehat{E_{\rm hi}}(Q) = \widehat{E_{\rm hi}}(Q). \tag{4.72}$$

The following Proposition is obtained by applying the spectral projections $\chi_{\rm lo}$ and $\chi_{\rm hi}$ to (4.64).

Proposition 4.4.2. *If $\widehat{E}$ is a solution of* (4.64) *then its low and high frequency components, $\widehat{E_{\rm lo}}$ and $\widehat{E_{\rm hi}}$, solve the coupled system:*

Low Frequency Equation:

$$\begin{aligned}[1 + M^s_\alpha(Q)]\ \widehat{E_{\rm lo}}(Q) - \chi_{\rm lo}(Q)\frac{3}{4\pi^2}\left(\widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm lo}}(Q) + \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm hi}}(Q)\right)&\\ = \chi_{\rm lo}(Q)\ \mathcal{R}^\sigma_1\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}\right](Q),&\end{aligned} \tag{4.73}$$

High Frequency Equation:

$$\begin{aligned}[1 + M^s_\alpha(Q)]\ \widehat{E_{\rm hi}}(Q) - \chi_{\rm hi}(Q)\ \chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\left(\widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm lo}}(Q) + \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm hi}}(Q)\right)&\\ = \chi_{\rm hi}(Q)\ \mathcal{R}^\sigma_1\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}\right](Q).&\end{aligned} \tag{4.74}$$

Here, $\mathcal{R}^\sigma_1$ is defined in (4.65)*. Conversely, if $(\widehat{E_{\rm lo}}, \widehat{E_{\rm hi}})$ solves* (4.73)-(4.74)*, then $\widehat{E} \equiv \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}$ solves* (4.64)*.*

We solve (4.73) and (4.74) via the Lyapunov-Schmidt reduction strategy used in Chapter 3 in the proofs of Theorems 3.2.1 and 3.2.3; see Section 4.1.2.

4.4.3 Reduction to a Closed Equation for the Low Frequency Projection, $\widehat{E_{\text{lo}}}$

We first solve for $\widehat{E_{\text{hi}}} = \widehat{E_{\text{hi}}}\left[\alpha, \widehat{E_{\text{lo}}}\right]$ as a functional of $\widehat{E_{\text{lo}}}$, by viewing the equation of $\widehat{E_{\text{hi}}}$ as depending on parameters $\alpha \in \mathbb{R}, |\alpha| \ll 1$ and a function $\Gamma(Q) = \widehat{E_{\text{lo}}}(Q) \in L^{2,a}(\mathbb{R})$.

$$[1 + M^s_\alpha(Q)]\ \widehat{E_{\text{hi}}}(Q) - \chi_{\text{hi}}(Q)\ \chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\left(\widetilde{\psi} * \widetilde{\psi} * \Gamma(Q) + \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\text{hi}}}(Q)\right)$$
$$-\chi_{\text{hi}}(Q)\ \mathcal{R}^\sigma_1[\alpha, \Gamma + \widehat{E_{\text{hi}}}](Q) = 0\ . \tag{4.75}$$

In the following proposition, we construct $\widehat{E_{\text{lo}}} \mapsto \widehat{E_{\text{hi}}}\left[\alpha, \widehat{E_{\text{lo}}}\right]$.

Proposition 4.4.3. *Set* $0 < r < 1$, $p = \min(s, 1)$, *and* $\eta = \min(2s, 1)$.

1. *There exist constants* $\alpha_0, \beta_0 > 0$, *such that for all* $\alpha \in (0, \alpha_0)$, *equation* (4.75) *defines a unique mapping* $(\alpha, \Gamma) \mapsto \widehat{E_{\text{hi}}}[\alpha, \Gamma]$, $\widehat{E_{\text{hi}}} : [0, 1] \times B_{\beta_0}(0) \to L^{2,a}(\mathbb{R})$ *where* $B_{\beta_0}(0) \subset L^{2,a}(\mathbb{R})$ *such that* $\widehat{E_{\text{hi}}}[\alpha, \Gamma]$ *is the unique solution to* (4.75) *(see also* (4.74)*). Also, if* $\Gamma \in L^{2,a}_{\text{even}}$, *then* $\widehat{E_{\text{hi}}}[\alpha, \Gamma] \in L^{2,a}_{\text{even}}$.

2. *The mapping is* C^1 *with respect to* Γ, *and there exists* $C > 0$, *such that for all* $(\alpha, \Gamma) \in [0, \alpha_0) \times B_{\beta_0}(0)$

$$\left\|\widehat{E_{\text{hi}}}[\alpha, \Gamma]\right\|_{L^{2,a}(\mathbb{R})} \lesssim \begin{cases} \alpha^{2p(1-r)}\ \|\Gamma\|_{L^{2,a}(\mathbb{R})} + e^{-C/\lambda(\alpha)^\eta} & : s \neq 1 \\ \frac{1}{(-\log(\alpha)}\ \|\Gamma\|_{L^{2,a}(\mathbb{R})} + e^{-C/\lambda_s(\alpha)^\eta} & : s = 1 \end{cases} \tag{4.76}$$

$$\left\|\ D_\Gamma \widehat{E_{\text{hi}}}[\alpha, \Gamma]\ \right\|_{L^{2,a}(\mathbb{R}) \longrightarrow L^{2,a}(\mathbb{R})} \lesssim \begin{cases} \alpha^{2p(1-r)} & : s \neq 1 \\ \frac{1}{(-\log(\alpha))} & : s = 1 \end{cases}. \tag{4.77}$$

The implicit constants depend only on α_0 *and* β_0.

3. $\widehat{E_{\text{hi}}}[\alpha, \Gamma]$ *is supported on* $Q \in \left[-\frac{\pi}{\alpha}, -\frac{1}{\lambda_s(\alpha)}\right) \cap \left(\frac{1}{\lambda_s(\alpha)}, \frac{\pi}{\alpha}\right]$; $\widehat{E_{\text{hi}}}[\alpha, \Gamma](Q) = \chi_{\text{hi}}\ \chi_{\mathcal{B}_\alpha}\ \widehat{E_{\text{hi}}}[\alpha, \Gamma]$.

Proof of Proposition 4.4.3: The proof follows that of Proposition 3.4.4, which is the analogous proposition for nearest-neighbor DNLS. We summarize the proof and elaborate where the proof differs. Since $0 < 1 \leqslant 1 + M^s_\alpha(Q)$ for all α positive and small, we may rewrite (4.75) as

$$\widehat{E_{\text{hi}}}(Q) - \chi_{\text{hi}}(Q)\ [1 + M^s_\alpha(Q)]^{-1}\left(\chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\ \left[\widetilde{\psi} * \widetilde{\psi} * \Gamma(Q) + \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\text{hi}}}(Q)\right]\right.$$
$$\left. -\mathcal{R}^\sigma_1[\alpha, \Gamma + \widehat{E_{\text{hi}}}](Q)\right) = 0. \tag{4.78}$$

The following results (Propositions 4.4.4, 4.4.5 and Lemma 4.4.2) are the essential ingredients required to extend the arguments from the proof of Proposition 3.4.4 (for nearest-neighbor DNLS) to DNLS with nonlocal interactions.

Proposition 4.4.4. *For any function $\widehat{f} \in L^{2,a}(\mathbb{R})$, we have*

$$\left\| \chi_{\text{hi}} \; \chi_{\mathcal{B}_\alpha} \;\; [1 + M_\alpha^s(Q)]^{-1} \;\; \widehat{f} \;\right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \begin{cases} \alpha^{2p(1-r)} \; \left\| \; \widehat{f} \; \right\|_{L^{2,a}(\mathbb{R})} & : s \neq 1 \\ \frac{1}{(-\log(\alpha))} \left\| \; \widehat{f} \; \right\|_{L^{2,a}(\mathbb{R})} & : s = 1 \end{cases} . \tag{4.79}$$

Proof of Proposition 4.4.4: We require the following lemma.

Lemma 4.4.1. *Let $p = \min(1, s)$. There exists a constant, $C = C_s$, such that for $s \neq 1$ and $Q \in \mathcal{B}_\alpha$, $M_\alpha^s(Q) \geqslant C\; |Q|^{2p}$. If $s = 1$, then $M_\alpha^s(Q) \geqslant \frac{C}{(-\log(\alpha))}\; |Q|^2$.*

We defer the proof of Lemma 4.4.1 until the end of this section. Recall that $\chi_{\text{hi}}(Q)$ projects onto the set $|Q| \geqslant \frac{1}{\lambda_s(\alpha)}$, where $\lambda_s(\alpha) \to 0$, is given by (4.67). Now let $s \neq 1$. From Lemma 4.4.1, we have $\chi_{\text{hi}}(Q)\; \chi_{\mathcal{B}_\alpha}(Q)\;\; [1 + M_\alpha^s(Q)] \geqslant C\; \chi_{\text{hi}}(Q)\; \chi_{\mathcal{B}_\alpha}(Q)\; |Q|^{2p} \geqslant C\lambda_s(\alpha)^{-2p} = C\; \alpha^{2p(r-1)}$. Next, let $s = 1$. Then Lemma 4.4.1 gives $\chi_{\text{hi}}(Q)\; \chi_{\mathcal{B}_\alpha}(Q)\;\; [1 + M_\alpha^s(Q)] \geqslant \chi_{\text{hi}}(Q)\; \chi_{\mathcal{B}_\alpha}(Q)\; \frac{C}{(-\log(\alpha))}\; |Q|^2 \geqslant \frac{C}{(-\log(\alpha))} \frac{1}{\lambda_s(\alpha)^2} = C\; (-\log(\alpha))$. This completes the proof of Proposition 4.4.4. □

We also require the following lemma, stated generally due to its broader use, which establishes exponential smallness of "shifted" ($m = \pm 1$ by our convention) convolutions of exponentially decaying functions.

Lemma 4.4.2. *Fix $0 < \eta \leqslant 1$. Let $a > 1/2$. Let $\widehat{f} \in L^{2,a}(\mathbb{R})$ be such that for $C > 0$, $e^{\,C|Q|^\eta} \widehat{f}(Q) \in L^{2,a}(\mathbb{R}_Q)$. Then, (a) for $m = \pm 1$,*

$$\left\| \chi_{\mathcal{B}_\alpha}(Q) \;\; \left(\widehat{f} * \widehat{f} * \widehat{f}\right)(Q - 2m\pi/\alpha) \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim e^{-C\pi^\eta/\alpha^\eta} \left\| e^{C|Q|^\eta} \;\; \widehat{f}(Q) \right\|^3_{L^{2,a}(\mathbb{R}_Q)} \tag{4.80}$$

and (b)

$$\left\| \chi_{\text{hi}}(Q) \;\; \left(\widehat{f} * \widehat{f} * \widehat{f}\right)(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim e^{-C/(\lambda_s(\alpha))^\eta)} \left\| e^{C|Q|^\eta} \;\; \widehat{f}(Q) \right\|^3_{L^{2,a}(\mathbb{R}_Q)} \tag{4.81}$$

Proof of Lemma 4.4.2: We first prove (a). For any $\xi, \zeta \in \mathbb{R}$ we have $|Q - 2m\pi/\alpha|^\eta \leqslant |Q - \xi - \zeta - 2m\pi/\alpha|^\eta + |\xi|^\eta + |\zeta|^\eta$, since $0 < \eta \leqslant 1$ (Lemma A.0.5). Therefore,

$$1 \leqslant e^{-C|Q-2m\pi/\alpha|^\eta} \; e^{C|Q-\xi-\zeta-2m\pi/\alpha|^\eta} \; e^{C|\xi|^\eta} \; e^{C|\zeta|^\eta} \; . \tag{4.82}$$

Moreover, if $|Q| \leqslant \pi/\alpha$, it follows that $|Q - 2m\pi/\alpha| \geqslant \pi/\alpha$ and therefore $\chi_{\mathcal{B}_\alpha}(Q) e^{-C|Q-2m\pi/\alpha|^\eta} \leqslant e^{-C\pi^\eta/\alpha^\eta}$. Now distributing the exponentials we have

$$\begin{aligned} \chi_{\mathcal{B}_\alpha}(Q) \left(\widehat{f} * \widehat{f} * \widehat{f}\right)(Q - 2m\pi/\alpha) &= \chi_{\mathcal{B}_\alpha}(Q) \int\int \widehat{f}(Q - 2m\pi/\alpha - \xi - \zeta) \widehat{f}(\xi) \widehat{f}(\zeta) \; d\xi d\zeta \\ \leqslant \chi_{\mathcal{B}_\alpha}(Q) \int\int & e^{-C|Q-2m\pi/\alpha|^\eta} e^{C|Q-2m\pi/\alpha-\xi-\zeta|^\eta} \\ & \cdot \widehat{f}(Q - 2m\pi/\alpha - \xi - \zeta) \; e^{C|\xi|^\eta} \widehat{f}(\xi) \;\; e^{C|\zeta|^\eta} \widehat{f}(\zeta) \; d\xi d\zeta. \end{aligned}$$

Therefore,

$$\begin{aligned} & \Big| \; \chi_{\mathcal{B}_\alpha}(Q) \left(\widehat{f} * \widehat{f} * \widehat{f}\right)(Q - 2m\pi/\alpha) \; \Big| \\ & \quad \leqslant e^{-C\pi^\eta/\alpha^\eta} \int e^{C|\zeta|^\eta} |\widehat{f}(\zeta)| \; d\zeta \int e^{C|Q-2m\pi/\alpha-\xi-\zeta|^\eta} |\widehat{f}(Q - 2m\pi/\alpha - \xi - \zeta)| \; e^{C|\xi|^\eta} |\widehat{f}(\xi)| \; d\xi \\ & \quad e^{-C\pi^\eta/\alpha^\eta} \; \left(\widehat{f}_\sharp * \widehat{f}_\sharp * \widehat{f}_\sharp\right)(Q), \qquad \text{where} \;\; \widehat{f}_\sharp(Q) \equiv e^{C|Q|^\eta} |\widehat{f}(Q)|. \end{aligned}$$

To complete the proof of (a) we integrate over all $Q \in \mathbb{R}$ and apply the convolution estimate (2.20) twice. The proof of (b) is very similar. Here, $m = 0$, but χ_{hi} localizes the integrand on the set $|Q| \geqslant 1/\lambda_s(\alpha)$. On this set we then use $1 \leqslant e^{-C|Q|^\eta} \; e^{C|Q-\xi-\zeta|^\eta} \; e^{C|\xi|^\eta} \; e^{C|\zeta|^\eta} \leqslant e^{-C/(\lambda_s(\alpha))^\eta} e^{C|Q-\xi-\zeta|^\eta} \; e^{C|\xi|^\eta} \; e^{C|\zeta|^\eta}$ to complete the proof. □

Finally, we bound the forcing terms in $\chi_{\text{hi}} \mathcal{R}_1^\sigma[\alpha, \Gamma + \widehat{E_{\text{hi}}}]$ which drive $\widehat{E}_{\text{hi}}$.

Proposition 4.4.5. *Assume $s > 1/4$. Let $p = \min(1, s)$, $\eta = \min(1, 2s)$ and let the spectral cutoff parameter, $\lambda_s(\alpha)$, be given as in* (4.67). *Let the operator $\mathcal{D}^{\sigma,\alpha}$ be defined in* (4.58) *and recall that $S(Q) = \chi_{\mathcal{B}_\alpha}(Q) \widetilde{\psi}(Q)$. Then, there exists $C > 0$, such that*

$$\Big\| \chi_{\text{hi}}(Q) \;\; [1 + M_\alpha^s(Q)]^{-1} \;\; \mathcal{D}^{\sigma,\alpha}[S](Q) \; \Big\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \; e^{-C/\lambda(\alpha)^\eta}. \tag{4.83}$$

Proof of Proposition 4.4.5: Recall from Proposition 4.1.2 that for some $\mu > 0$, $\|e^{\mu|Q|^\eta} \widetilde{\psi}(Q)\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim 1$, where $\eta = \min(2s, 1)$. By the definition of $\mathcal{D}^{\sigma,\alpha}[S]$ displayed in (4.58) we have

$$[1 + M^s_\alpha(Q)]^{-1} \ \mathcal{D}^{\sigma,\alpha}[S]$$
$$= S(Q) - \frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2} \ [1 + M^s_\alpha(Q)]^{-1} \sum_{m=-1,0,1} e^{2m\pi i\sigma} \ (S * S * S)(Q - 2m\pi/\alpha) \ . \tag{4.84}$$

The $m = \pm 1$ terms and the $m = 0$ term are bounded, respectively, by parts (a) and (b) of Lemma 4.4.2.

We conclude this section with the proof of Lemma 4.4.1. Recall the expression for $M^s_\alpha(Q)$ in (4.57). For $s > 1$, $\kappa_s(\alpha) = \alpha^2$ and therefore $4\sin^2(q/2) \geqslant \frac{4}{\pi^2}|q|^2$, $q \in [-\pi, \pi]$ gives $M^s_\alpha(Q) \geqslant \frac{4}{\alpha^2} J_1 \sin^2(Q\alpha/2) \geqslant \frac{4J_1}{\pi^2} \ |Q|^2$. Next assume $s = 1$. Then $\kappa_s(\alpha) = (-\log(\alpha)) \ \alpha^2$, $C_s = 1$, and similarly, $M^s_\alpha(Q) \geqslant \frac{4}{\alpha^2(-\log(\alpha))} \sin^2(Q\alpha/2) \geqslant \frac{4}{\pi^2(-\log(\alpha))} \ |Q|^2$.

Finally, suppose that $s < 1$ (thus $p(s) = \min(1, s) = s$). We work with $M^s(q)$ in the original variable $q \in \mathcal{B} = [-\pi, \pi]$, First consider $|q| \geqslant 1$, where $|q|^2 \geqslant |q|^{2s}$.

$$M^s(q) = \frac{1}{C_s} \sum_{m=1}^{\infty} \frac{4\sin^2(qm/2)}{m^{1+2s}} \geqslant \frac{4}{C_s} \sin^2(q/2) \geqslant \frac{4}{C_s \ \pi^2} \ |q|^2 \geqslant \frac{4}{C_s \ \pi^2}|q|^{2s},$$
$$1 \leqslant |q| \leqslant \pi. \tag{4.85}$$

For $0 \leqslant |q| < 1$, Proposition G.0.9 (and Remark G.0.3 for $s = 1/2$) give

$$\begin{aligned} M^s(q) &= |q|^{2s} + \frac{2}{C_s} \sum_{j=1}^{\infty} \frac{\zeta(1 + 2s - 2j)}{(2j)!} \ (-1)^{j+1} \ |q|^{2j} \\ &= |q|^{2s} \left(1 + \frac{2}{C_s} \sum_{j=1}^{\infty} \frac{\zeta(1 + 2s - 2j)}{(2j)!} \ (-1)^{j+1} \ |q|^{2j-2s}\right). \end{aligned} \tag{4.86}$$

If we let $|q| \leqslant q_* \equiv \min\left\{1, \left(\frac{4}{C_s} \ \sum_{j=1}^{\infty} \frac{|\zeta(1+2s-2j)|}{(2j)!}\right)^{-\frac{1}{2-2s}}\right\}$,

$$\begin{aligned} \left|\frac{2}{C_s} \sum_{j=1}^{\infty} \frac{\zeta(1 + 2s - 2j)}{(2j)!} \ (-1)^{j+1} \ |q|^{2j-2s}\right| &\leqslant \frac{2}{C_s} \sum_{j=1}^{\infty} \frac{|\zeta(1 + 2s - 2j)|}{(2j)!} \ |q|^{2j-2s} \\ &\leqslant \frac{2}{C_s} \sum_{j=1}^{\infty} \frac{|\zeta(1 + 2s - 2j)|}{(2j)!} \ |q|^{2-2s} \leqslant \frac{1}{2}. \end{aligned} \tag{4.87}$$

such that from (4.86),

$$M^s(q) = |q|^{2s} \left(1 + \frac{2}{C_s} \sum_{j=1}^{\infty} \frac{\zeta(1 + 2s - 2j)}{(2j)!} \ (-1)^{j+1} \ |q|^{2j-2s}\right) \geqslant \frac{|q|^{2s}}{2}, \qquad |q| \leqslant q_*. \tag{4.88}$$

It remains to address $q_* \leqslant |q| < 1$ (if $q_* = 1$, we are done for $s < 1$). On this interval, we need to find C_* such that

$$M^s(q) = \frac{1}{C_s} \sum_{m=1}^{\infty} \frac{4\sin^2(qm/2)}{m^{1+2s}} \geqslant \frac{4}{C_s}\sin^2(q/2) \geqslant \frac{4}{C_s\ \pi^2}\ |q|^2 \geqslant C_*\ |q|^{2s}, \qquad q_* \leqslant |q| < 1. \quad (4.89)$$

where we have again used $\sin^2(q/2) \geqslant \frac{1}{\pi^2}|q|^2$. Take $C_* \equiv \max_{q_* \leqslant |q| < 1} \left(\frac{4\ |q|^{2-2s}}{C_s\ \pi^2}\right)$ and let $C \equiv \min\left\{\frac{4}{C_s\pi^2},\ \frac{C_*}{2}\right\}$. Therefore, from (4.85), (4.88), and (4.89), and with the rescaling $Q = q/\alpha$, there exists some $C > 0$ such that we have for $s < 1$,

$$\begin{aligned} & M^s(q) = \frac{1}{C_s} \sum_{m=1}^{\infty} \frac{4\sin^2(qm/2)}{m^{1+2s}} \geqslant C\ |q|^{2s}, \qquad q \in \mathcal{B} = [-\pi, \pi] \\ \Longrightarrow \qquad & M^s_\alpha(Q) = \frac{1}{C_s\ \alpha^{2s}} \sum_{m=1}^{\infty} \frac{4\sin^2(Qm\alpha/2)}{m^{1+2s}} \geqslant C\ |Q|^{2s}, \qquad Q \in \mathcal{B}_\alpha = \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]. \end{aligned} \quad (4.90)$$

This completes the Proof of Lemma 4.4.1. □

4.5 Solution of the Low Frequency Equation for $\widehat{E_{\text{lo}}}$

4.5.1 Equation for $\widehat{E_{\text{lo}}}$ as a Perturbation of the Continuum FNLS Limit

Insertion of the map $\widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]$, obtain in Proposition 4.4.3, into equation (4.73) yields a closed equation for $\widehat{E_{\text{lo}}}$:

$$\begin{aligned} & [1 + M^s_\alpha(Q)]\ \widehat{E_{\text{lo}}}(Q) - \chi_{\text{lo}}(Q)\frac{3}{4\pi^2}\widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\text{lo}}}(Q) \\ & \quad = \chi_{\text{lo}}(Q)\ \mathcal{R}^\sigma_1\left[\alpha, \widehat{E_{\text{lo}}} + \widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]\right](Q) + \chi_{\text{lo}}(Q)\frac{3}{4\pi^2}\widetilde{\psi} * \widetilde{\psi} * \left(\widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}]\right)(Q). \end{aligned} \quad (4.91)$$

Here, $\mathcal{R}^\sigma_1[\alpha, \widehat{E}]$ is displayed in (4.65).

Now note that the operator on the left-hand-side of (4.91) has a formal $\alpha \downarrow 0$ limit equal to the linearized continuum FNLS operator, $\widetilde{L^p_+}$. Hence, we now reexpress (4.91) as a small α perturbation of this limit:

Proposition 4.5.1. *Let $p = p(s) = \min(s, 1)$. There exists $0 \leqslant \alpha_1 \leqslant \alpha_0$ and $0 < r < 1$ such that:*

1. *Equation* (4.91) *for* $\widehat{E_{\rm lo}}$ *may be rewritten as*

$$\widetilde{L_+^p}\,\widehat{E_{\rm lo}}(Q) = \mathcal{R}_2^\sigma[\alpha, \widehat{E_{\rm lo}}](Q), \quad \text{where} \tag{4.92}$$

$$\mathcal{R}_2^\sigma\left[\alpha, \widehat{E_{\rm lo}}\right](Q) \equiv \chi_{\rm lo}(Q)\ \mathcal{R}_1^\sigma\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right](Q) + R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right](Q), \quad \text{and} \tag{4.93}$$

$$R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right](Q) \equiv \chi_{\rm lo}(Q)\ \left[\ |Q|^{2p} - M_\alpha^s(Q)\ \right]\ \widehat{E_{\rm lo}}(Q) - \chi_{\rm hi}(Q)\ \frac{3}{4\pi^2}\ \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm lo}}(Q). \tag{4.94}$$

2. $\mathcal{R}_2^\sigma\left[\alpha, \widehat{E_{\rm lo}}\right](Q)$ *is continuous at* $(0,0) \in [0,\alpha_1) \times L^{2,a}(\mathbb{R})$. *Furthermore, the mapping* $\widehat{E_{\rm lo}} \mapsto \mathcal{R}_2^\sigma\left[\alpha, \widehat{E_{\rm lo}}\right]$ *is Fréchet differentiable with respect to* $\widehat{E_{\rm lo}}$, *with* $D_{\widehat{E_{\rm lo}}};\mathcal{R}_2^\sigma[\alpha, \widehat{E_{\rm lo}}]$ *is displayed below in* (4.117). *Finally, we have the bounds*

$$\left\|\mathcal{R}_2^\sigma[\alpha, \widehat{E_{\rm lo}}]\right\|_{L^{2,a-2p}(\mathbb{R})} \lesssim \alpha^{\frac{1}{2}}\mathfrak{e}_1(\alpha) + \left\|\widehat{E_{\rm lo}}\right\|_{L^{2,a}(\mathbb{R})} + \left\|\widehat{E_{\rm lo}}\right\|_{L^{2,a}(\mathbb{R})}^2 + \left\|\widehat{E_{\rm lo}}\right\|_{L^{2,a}(\mathbb{R})}^3, \tag{4.95}$$

$$\left\|D_{\widehat{E_{\rm lo}}}\mathcal{R}_2^\sigma\left[\alpha, \widehat{E_{\rm lo}}\right]\right\|_{L^{2,a}(\mathbb{R})\to L^{2,a-2p}(\mathbb{R})} \lesssim \mathfrak{e}_2(\alpha) + \left\|\widehat{E_{\rm lo}}\right\|_{L^{2,a}(\mathbb{R})} + \left\|\widehat{E_{\rm lo}}\right\|_{L^{2,a}(\mathbb{R})}^2, \tag{4.96}$$

where $\mathfrak{e}_1(\alpha)$ *is displayed in* (4.36) *and* $\mathfrak{e}_2(\alpha)$ *is given by*

$$\mathfrak{e}_2(\alpha) = \begin{cases} \alpha^{2r(1-s)} + \alpha^{2s(1-r)} & : s < 1 \\ \frac{\log(-\log(\alpha))}{(-\log(\alpha))} & : s = 1 \\ \alpha^{2r(s-1)} + \alpha^{2(1-r)} & : 1 < s < 2 \\ (-\log(\alpha))\alpha^{2r} + \alpha^{2(1-r)} & : s = 2 \\ \alpha^{2r} + \alpha^{2(1-r)} & : s > 2 \end{cases}. \tag{4.97}$$

Proposition 4.5.1 is proven in Sections 4.5.1.1 through 4.5.2. In Section 4.5.1, we derive equation (4.92). In Section 4.5.1.3, we provide tools which we use in Section 4.5.2 to prove estimates (4.95) and 4.96.

We then proceed to solve (4.92) in Proposition 4.5.6 of Section 4.5.3 using the strategy detailed in Section 4.5.1.2.

4.5.1.1 Derivation of Equation (4.92) for $\widehat{E_{\rm lo}}$

We follow the derivation of the low-frequency equation found in Section 3.5.2 for the case of nearest-neighbor DNLS, and rewrite equation (4.91) in the form (4.92). First, we use $\chi_{\rm hi} = 1 - \chi_{\rm lo}$ to get

$$\chi_{\rm lo}(Q)\ \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm lo}}(Q) = \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm lo}}(Q) - \chi_{\rm hi}(Q)\ \widetilde{\psi} * \widetilde{\psi} * \widehat{E_{\rm lo}}(Q). \tag{4.98}$$

Since $\overline{\chi}_{\rm lo}\widehat{E_{\rm lo}} = 0$, we may write:

$$M_\alpha^s(Q)\ \widehat{E_{\rm lo}}(Q) = |Q|^{2p}\ \widehat{E_{\rm lo}}(Q) + \chi_{\rm lo}\left[M_\alpha^s(Q) - |Q|^{2p}\right]\ \widehat{E_{\rm lo}}(Q). \tag{4.99}$$

We now express the left-hand side of (4.91) via (4.98) and (4.99) as $\widetilde{L_+^p}\widehat{E_{\rm lo}}(Q) - R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right](Q)$, where the linear operator $\widehat{E_{\rm lo}} \mapsto R_{\rm pert}\left[\alpha, \widehat{E_{\rm lo}}\right]$ is given in (4.94). Finally (4.91) may be written in the desired form (4.92), where $\mathcal{R}_2^\sigma\left[\alpha, \widehat{E_{\rm lo}}\right]$ is given by (4.93). and we obtain (4.92). Finally, the expression for $D_{\widehat{E_{\rm lo}}}\mathcal{R}_2^\sigma[\alpha, \widehat{E_{\rm lo}}]$ is displayed in (4.117) and is bounded, together with $\mathcal{R}_2^\sigma[\alpha, \widehat{E_{\rm lo}}]$ in Section 4.5.2.

4.5.1.2 Strategy for Solving the Low Frequency Equation

We study the solvability of the low frequency equation: $\mathcal{M}[\alpha, \widehat{E_{\rm lo}}] \equiv \widetilde{L_+^p}\widehat{E_{\rm lo}} - \mathcal{R}_2^\sigma[\alpha, \widehat{E_{\rm lo}}] = 0$ by applying a following variant of the implicit function theorem (Theorem F.0.2 of Appendix F).

Lemma 4.5.1. *Consider the equation*

$$\mathcal{M}[\alpha, f] \equiv \mathfrak{L}f - \mathcal{R}[\alpha, f] = 0, \quad f \in L^{2,a}_{\rm even}(\mathbb{R}). \tag{4.100}$$

1. $\mathfrak{L} : L^{2,a}_{\rm even}(\mathbb{R}) \longmapsto L^{2,a-2p}_{\rm even}(\mathbb{R})$ *be an isomorphism for* $0 < p \leqslant 1$

2. $\mathcal{R} : [0, \alpha_1)_\alpha \times L^{2,a}_{\rm even}(\mathbb{R}) \longrightarrow L^{2,a-2p}_{\rm even}(\mathbb{R})$ *is continuous at* $(0,0)$*, Fréchet differentiable on* $L^{2,a}_{\rm even}(\mathbb{R})$

3. $\mathcal{R}[0,0] = 0$ *and satisfies the bounds:*

$$\|\mathcal{R}[\alpha, f]\|_{L^{2,a-2p}(\mathbb{R})} \lesssim \mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R})} + \|f\|^2_{L^{2,a}(\mathbb{R})} + \|f\|^3_{L^{2,a}(\mathbb{R})}, \tag{4.101}$$

$$\|D_f\mathcal{R}[\alpha, f]\|_{L^{2,a}(\mathbb{R}) \to L^{2,a-2p}(\mathbb{R})} \lesssim \mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R})} + \|f\|^2_{L^{2,a}(\mathbb{R})}, \tag{4.102}$$

for some continuous function $\mathfrak{K}(\alpha) \geqslant 0$*, satisfying* $\mathfrak{K}(0) = 0$*,* $\mathfrak{K}(\alpha) \to 0$ *as* $\alpha \to 0$*.*

Then there exists a constant $\alpha_2 \leqslant \alpha_1$ *such that for all* $0 < \alpha < \alpha_2$*, the equation* (F.33)*,* $\mathcal{M}[\alpha, f] = 0$*, has a unique solution* $f = f[\alpha] \in L^{2,a}_{\rm even}(\mathbb{R})$ *satisfying*

$$\|f[\alpha]\|_{L^{2,a}(\mathbb{R})} \lesssim \mathfrak{K}(\alpha). \tag{4.103}$$

In order to construct $\widehat{E_{\text{lo}}}$ by application of Lemma 4.5.1, we must verify the hypotheses for $\mathcal{M}[\alpha, f] \equiv \widetilde{L^p_+} f - \mathcal{R}_2^g[\alpha, f]$ with $\mathfrak{L} = \widetilde{L^p_+}$ and $\mathcal{R} = \mathcal{R}_2^g[\alpha, f]$. Note, by Proposition 4.1.3, that the mapping $\widetilde{L^p_+} : L^{2,a}_{\text{even}}(\mathbb{R}) \to L^{2,a-2p}_{\text{even}}(\mathbb{R})$ is an isomorphism. Hence we must verify that $(\alpha, f) \mapsto \mathcal{R}_2^g[\alpha, f]$ satisfies the necessary hypotheses. This is done in the remainder of Section 4.5, culminating in Proposition 4.5.6.

4.5.1.3 Tools for Estimation of the Low Frequency Equation

To establish the properties of $\mathcal{R}_2^g$ given in Proposition 4.5.1, we require a number of general tools. Recall that $\chi_{\text{hi}} = 1 - \chi_{\text{lo}}$, where $\chi_{\text{lo}} \equiv \chi_{[-\lambda(\alpha)^{-1}, \lambda(\alpha)^{-1}]}$, where $\lambda_s(\alpha)$ is defined in (4.67). Note first that since $(1 + |Q|^2)^{-2p} \leqslant |Q|^{-2p}$, we have

Lemma 4.5.2. *Let $0 < p \leqslant 1$. For any function $\widehat{g} \in L^{2,a-2p}(\mathbb{R})$ such that $|Q|^{-2p}\ \widehat{g} \in L^{2,a}(\mathbb{R}_Q)$, we have*

$$\|\widehat{g}\|_{L^{2,a-2p}} \lesssim \left\| |Q|^{-2p} \widehat{g}(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} . \tag{4.104}$$

The bound $\chi_{\text{hi}}(Q)|Q|^{-2p} \leqslant \lambda(\alpha)^{2p}$ and Lemma 4.5.2 imply:

Lemma 4.5.3. *Let $0 < p \leqslant 1$. For any function $\widehat{g} \in L^{2,a}(\mathbb{R})$, we have*

$$\| \chi_{\text{hi}}(Q)\ \widehat{g}(Q) \|_{L^{2,a-2p}(\mathbb{R}_Q)} \lesssim \lambda(\alpha)^{2p} \|\widehat{g}\|_{L^{2,a}(\mathbb{R})}. \tag{4.105}$$

Lemma 4.5.4. *Let $0 < r < 1$ and $p = \min(1, s)$. For any function $\widehat{g} \in L^{2,a}(\mathbb{R})$, we have*

$$\left\| \chi_{\text{lo}}(Q)\ \left[\ |Q|^{2p} - M^s_\alpha(Q)\ \right]\ \widehat{g}(Q) \right\|_{L^{2,a-2p}(\mathbb{R}_Q)} \lesssim \mathfrak{d}_s(\alpha)\ \|\ \widehat{g}\ \|_{L^{2,a}(\mathbb{R})} , \tag{4.106}$$

where $\mathfrak{d}_s(\alpha)$ is given by

$$\mathfrak{d}_s(\alpha) = \begin{cases} \alpha^{2r(1-s)} & : s < 1 \\ \frac{\log(-\log(\alpha))}{(-\log(\alpha))} & : s = 1 \\ \alpha^{2r(s-1)} & : 1 < s < 2 \\ (-\log(\alpha))\alpha^{2r} & : s = 2 \\ \alpha^{2r} & : 2 < s \leqslant \infty \end{cases} . \tag{4.107}$$

Proof of Lemma 4.5.4: Recall $p = p(s) = \min(s, 1)$. Proposition G.0.9 implies there exists $f_s(Q; \alpha)$ such that $\sup_{Q \in \mathcal{B}_\alpha} |f_s(Q; \alpha)| \lesssim 1$ and $M^s_\alpha(Q) - |Q|^{2p(s)}$ is equal to $f_s(Q; \alpha)$ times a function

of α and Q, displayed in (4.61). Recall that $p = p(s) = \min(s,1)$ and $\lambda_s(\alpha) = \begin{cases} \alpha^{1-r} & : s \neq 1 \\ \frac{1}{(-\log(\alpha))} & : s = 1 \end{cases}$.
Furthermore, recall that $\chi_{\text{lo}}(Q) = \chi\left(|Q| \leqslant \lambda(\alpha)^{-1}\right)$, and that $\lambda(\alpha)^{-1} \leqslant \pi/\alpha$ for α sufficiently small, implying $\sup_Q \chi_{\text{lo}}(Q)\ |f_s(Q;\alpha)| \lesssim 1$.

First, suppose that $s \neq 1$. We apply Lemma 4.5.2 to get

$$\begin{aligned} &\left\| \chi_{\text{lo}}(Q)\ \left[\ |Q|^{2p} - M_\alpha^s(Q)\ \right]\ \widehat{g}(Q) \right\|_{L^{2,a-2p}(\mathbb{R}_Q)} \\ &\lesssim \left\| \chi_{\text{lo}}(Q)\ |Q|^{-2p}\ \left[\ |Q|^{2p} - M_\alpha^s(Q)\ \right]\ \widehat{g}(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)}. \end{aligned} \tag{4.108}$$

Hence, by (4.61) to prove the desired bound it suffices to bound the supremum of

$$\begin{aligned} &\chi_{\text{lo}}(Q)\ |Q|^{-2p}\ \left[\ |Q|^{2p} - M_\alpha^s(Q)\ \right] \\ &= \chi_{\text{lo}}(Q) \begin{cases} f_s(Q;\alpha)\ |\alpha Q|^{2(1-s)} & : \quad s < 1 \\ f_s(Q;\alpha)\ |\alpha Q|^{2(s-1)} & : \quad 1 < s < 2 \\ (-\log(\alpha))\ \ f_s(Q;\alpha)\ |\alpha Q|^2 & : \quad s = 2 \\ f_s(Q;\alpha)\ |\alpha Q|^2 & : \quad 2 < s \leqslant \infty \end{cases}. \end{aligned} \tag{4.109}$$

For $s \neq 1$, $\chi_{\text{lo}}(Q)$ projects onto the set where $|\alpha Q| \leqslant \alpha\lambda_s(\alpha)^{-1} = \alpha^r$ and (4.106) follows in this case.

Finally, consider the case where $s = 1$. In this case, $p = \min(s,1) = 1$. Here we take $\lambda_s(\alpha) = (-\log(\alpha))^{-1} > 1$. To manage the logarithmic term in (4.108), note that

$$\begin{aligned} &|\log(|Q|)|\ |Q|^2 \leqslant \frac{1}{2e} \qquad \text{when} \qquad |Q| \leqslant 1, \\ \text{and} \qquad &|\log(|Q|)| \leqslant \log(-\log(\alpha)) \qquad \text{when} \qquad 1 < |Q| \leqslant \lambda(\alpha)^{-1} = -\log(\alpha). \end{aligned} \tag{4.110}$$

We therefore address the intervals $|Q| \leqslant 1$ and $|Q| \geqslant 1$ separately via projections. We use Lemma

4.5.2 and the fact that $(1+|Q|^2)^{-2} \leqslant 1$ to get

$$\begin{aligned} &\left\| \chi_{\rm lo}(Q) \ \left[\ |Q|^2 - M_\alpha^s(Q) \ \right] \ \widehat{g}(Q) \ \right\|_{L^{2,a-2}(\mathbb{R}_Q)} \\ &\leqslant \left\| \chi(|Q| \leqslant 1) \ \chi_{\rm lo}(Q) \ \left[\ |Q|^2 - M_\alpha^s(Q) \ \right] \ \widehat{g}(Q) \ \right\|_{L^{2,a-2}(\mathbb{R}_Q)} \\ &\quad + \left\| \chi(|Q| > 1) \ \chi_{\rm lo}(Q) \ \left[\ |Q|^2 - M_\alpha^s(Q) \ \right] \ \widehat{g}(Q) \ \right\|_{L^{2,a-2}(\mathbb{R}_Q)} \\ &\leqslant \left\| \chi(|Q| \leqslant 1) \ \left[\ |Q|^2 - M_\alpha^s(Q) \ \right] \ \widehat{g}(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)} \\ &\quad + \left\| \chi(|Q| > 1) \ \chi_{\rm lo}(Q) \ |Q|^{-2} \ \left[\ |Q|^2 - M_\alpha^s(Q) \ \right] \ \widehat{g}(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)} \end{aligned} \tag{4.111}$$

The term localized on $|Q| \leqslant 1$ is bounded, using the first estimate in (4.110) and (4.108), by $\chi(|Q| \leqslant 1) \ \left| \ |Q|^2 - M_\alpha^s(Q) \ \right| \lesssim \frac{1}{-\log(\alpha)}$. The term localized to $|Q| \geqslant 1$ is bounded using the second estimate in (4.110) and (4.108), by $\chi(|Q| \geqslant 1)\chi_{\rm lo}(Q) \ |Q|^{-2} \ \left| \ |Q|^2 - M_\alpha^s(Q) \ \right| \lesssim \frac{\log(-\log(\alpha))}{(-\log(\alpha))}$. Lemma 4.5.4 is proved. $\square$

Lemma 4.5.5. *Suppose that $\widehat{f}_1 \in L^{2,a}(\mathbb{R})$ and $e^{C|Q|^\eta}\widehat{f}_1(Q) \in L^{2,a}(\mathbb{R}_Q)$ for some $\eta, C > 0$. Further, let $\widehat{f}_2, \widehat{f}_3 \in L^{2,a}(\mathbb{R})$. Then, (with $\overline{\chi}_{\mathcal{B}_\alpha}(Q) = \chi(|Q| > \pi/\alpha)$),*

$$\left\| \ \overline{\chi}_{\mathcal{B}_\alpha} \ \widehat{f}_1 \ \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C\pi^\eta/\alpha^\eta} \ \left\| \ e^{C|Q|^\eta} \ \widehat{f}_1(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)}, \tag{4.112}$$

$$\left\| \left(\overline{\chi}_{\mathcal{B}_\alpha} \widehat{f}_1 \right) * \widehat{f}_2 * \widehat{f}_3 \right\|_{L^{2,a-2p}(\mathbb{R})} \lesssim \ e^{-C\pi^\eta/\alpha^\eta} \ \left\| \ e^{C|Q|^\eta} \ \widehat{f}_1(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} \left\| \widehat{f}_2 \right\|_{L^{2,a}(\mathbb{R})} \left\| \widehat{f}_3 \right\|_{L^{2,a}(\mathbb{R})}. \tag{4.113}$$

The first follows from $\overline{\chi}_{\mathcal{B}_\alpha}(Q) \ e^{-C|Q|^\eta} e^{C|Q|^\eta} \leqslant e^{-C\pi^\eta/\alpha^\eta} e^{C|Q|^\eta}$. The second follows from the first and the convolution estimate (2.20).

4.5.2 Part 2 of the Proof of Proposition 4.5.1; Analysis of $\mathcal{R}_2^\sigma$ and Derivation of Estimates (4.95) and (4.96)

To bound $\mathcal{R}_2^\sigma : \mathbb{R} \times L^{2,a}(\mathbb{R}) \mapsto L^{2,a-2p}(\mathbb{R})$, we use the following estimates from Proposition 4.4.3, valid for $0 < \alpha < \alpha_0$:

$$\left\| \widehat{E_{\rm hi}}[\alpha, \Gamma] \right\|_{L^{2,a}(\mathbb{R})} \lesssim \begin{cases} \alpha^{2p(1-r)} \ \|\Gamma\|_{L^{2,a}(\mathbb{R})} + e^{-C/\lambda(\alpha)^\eta} & : s \neq 1 \\ \frac{1}{(-\log(\alpha)} \ \|\Gamma\|_{L^{2,a}(\mathbb{R})} + e^{-C/\lambda(\alpha)} & : s = 1 \end{cases} \tag{4.114}$$

$$\left\| \ D_\Gamma \widehat{E_{\rm hi}}[\alpha, \Gamma] \ \right\|_{L^{2,a}(\mathbb{R}) \longrightarrow L^{2,a}(\mathbb{R})} \lesssim \begin{cases} \alpha^{2p(1-r)} & : s \neq 1 \\ \frac{1}{(-\log(\alpha)} & : s = 1 \end{cases} . \tag{4.115}$$

Here, $\eta = \min(2s, 1)$. Note also from (4.92) that

$$\widehat{E_{\rm lo}}(Q) = \chi_{\rm lo}(Q) \widehat{E_{\rm lo}}(Q). \tag{4.116}$$

For any $\hat{f} \in L^{2,a}(\mathbb{R})$, a direct computation using (4.93), (4.116), and the linearity of $R^\sigma_{\rm L}$ and $R_{\rm pert}$ in their second argument gives

$$\begin{aligned} D_{\widehat{E_{\rm lo}}} \mathcal{R}^\sigma_2[\alpha, \widehat{E_{\rm lo}}] \ \hat{f}(Q) &= \chi_{\rm lo}(Q) \ R^\sigma_{\rm L} \left[\alpha, \ \chi_{\rm lo} \hat{f}\right](Q) \\ &+ \chi_{\rm lo}(Q) \ R^\sigma_{\rm L} \left[\alpha, \left(D_{\widehat{E_{\rm lo}}} \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right) \hat{f}\right](Q) + R_{\rm pert} \left[\alpha, \hat{f}\right](Q) \\ &+ D_{\widehat{E_{\rm lo}}} \left(\chi_{\rm lo} \ R^\sigma_{NL} \left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right]\right) \hat{f}(Q). \end{aligned} \tag{4.117}$$

Here,

$$\begin{aligned} & D_{\widehat{E_{\rm lo}}} \left(\chi_{\rm lo} \ R^\sigma_{NL} \left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right]\right) \hat{f}(Q) \\ & = \chi_{\rm lo}(Q) \ \left(\frac{1}{2\pi}\right)^2 \sum_{m=-1}^{1} e^{2m\pi i \sigma} \left[6 \ S^\alpha_J * \widehat{E^{\alpha,\sigma}_J} * \hat{f}(Q - 2m\pi/\alpha)\right. \\ & + 3 \ \widehat{E_{\rm lo}} * \widehat{E_{\rm lo}} * \hat{f}(Q - 2m\pi/\alpha) \\ & + 6 \ S^\alpha_J * \widehat{E_{\rm lo}} * \left(D_{\widehat{E_{\rm lo}}} \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}] \cdot \hat{f}\right)(Q - 2m\pi/\alpha) \\ & \left. + 3 \ \widehat{E_{\rm lo}} * \widehat{E_{\rm lo}} * \left(D_{\widehat{E_{\rm lo}}} \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}] \cdot \hat{f}\right)(Q - 2m\pi/\alpha)\right]. \end{aligned} \tag{4.118}$$

We now proceed to bound $\mathcal{R}^\sigma_2$, given in (4.93) and $D_{\widehat{E_{\rm lo}}} \mathcal{R}^\sigma_2$, given in (4.117), as maps from $L^{2,a}$ to $L^{2,a-2p}$.

Proposition 4.5.2. *Let $0 < r < 1$, and $0 < \alpha < \alpha_0$. $R_{\rm pert}[\alpha, \hat{f}]$, defined in (4.94), satisfies*

$$\left\| \ R_{\rm pert} \left[\alpha, \hat{f}\right] \ \right\|_{L^{2,a-2p}(\mathbb{R})} \lesssim \ \mathfrak{e}_2(\alpha) \ \left\| \ \hat{f} \ \right\|_{L^{2,a}(\mathbb{R})}. \tag{4.119}$$

Here, $\mathfrak{e}_2(\alpha) \to 0$ as $\alpha \to 0$ is displayed in (4.97).

This proposition follows from Lemmata 4.5.3 and 4.5.4, and the convolution estimate (2.20). Note that $\mathfrak{e}_2(\alpha) = \mathfrak{d}_s(\alpha) + \lambda(\alpha)^{2p}$, where $\mathfrak{d}_s(\alpha)$ is given in (4.107).

Proposition 4.5.3. *Let $0 < r < 1$ and $\eta = \min(2s, 1)$. Let $R^\sigma_{\rm L}$ be defined in (4.66) and let $\widehat{f} \in L^{2,a}(\mathbb{R})$. Then there exist $\alpha_0, C > 0$ such that for $0 < \alpha < \alpha_0$,*

$$\left\| \chi_{\rm lo}\ R^\sigma_{\rm L}\left[\alpha,\ \chi_{\rm lo}\widehat{f}\right] \right\|_{L^{2,a-2p}(\mathbb{R})} \lesssim e^{-C/\alpha^\eta} \left\| \widehat{f} \right\|_{L^{2,a}(\mathbb{R})}, \tag{4.120}$$

$$\left\| \chi_{\rm lo}\ R^\sigma_{\rm L}\left[\alpha, \widehat{f}\right]\right\|_{L^{2,a-2p}(\mathbb{R})} \lesssim \left\| \widehat{f} \right\|_{L^{2,a}(\mathbb{R})}. \tag{4.121}$$

Proof of Proposition 4.5.3: We use the convolution estimate (2.20) extensively. From (4.66), we have for any $\widehat{f} \in L^{2,a}(\mathbb{R})$.

$$\chi_{\rm lo}(Q) R^\sigma_{\rm L}\left[\alpha, \widehat{f}\right](Q) \equiv \chi_{\rm lo}(Q)\chi_{\mathcal{B}_\alpha}(Q)\ \frac{3}{4\pi^2}\left[\sum_{m=-1,0,1} e^{2m\pi i\sigma}\ S * S * \widehat{f}(Q - 2m\pi/\alpha) - \ \widetilde{\psi} * \widetilde{\psi} * \widehat{f}(Q)\right], \tag{4.122}$$

which is linear in its second argument. First, we address the $m = 0$ term. Recall that $S(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widetilde{\psi}(Q)$. Since $\chi_{\mathcal{B}_\alpha} + \overline{\chi}_{\mathcal{B}_\alpha} = 1$, we have

$$S * S * \widehat{f} - \ \widetilde{\psi} * \widetilde{\psi} * \widehat{f} = -2\left(\overline{\chi}_{\mathcal{B}_\alpha}\widetilde{\psi}\right) * \widetilde{\psi} * \widehat{f} + \left(\overline{\chi}_{\mathcal{B}_\alpha}\widetilde{\psi}\right) * \left(\overline{\chi}_{\mathcal{B}_\alpha}\widetilde{\psi}\right) * \widehat{f}. \tag{4.123}$$

Therefore, by Lemma 4.5.5

$$\left\| S * S * \widehat{f} - \ \widetilde{\psi} * \widetilde{\psi} * \widehat{f}\right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha^\eta} \left\|\widehat{f}\right\|_{L^{2,a}(\mathbb{R})}. \tag{4.124}$$

For the $m = \pm 1$ terms, first note that by the convolution estimate (2.20),

$$\left\|S * S * \widehat{f}(Q - 2m\pi/\alpha)\right\|_{L^{2,a}(\mathbb{R}_Q)} \leqslant \left\|\widehat{f}\right\|_{L^{2,a}(\mathbb{R})}. \tag{4.125}$$

Estimates (4.124) and (4.125) together imply (4.121).

To prove (4.120) we consider the $m = \pm 1$ terms. The integral to be bounded has variables $Q, \xi, \zeta \in \mathbb{R}$ constrained by $|\xi| \leqslant \frac{\pi}{\alpha}$, $|\zeta| \leqslant \frac{\pi}{\alpha}$, $|Q| \leqslant \frac{1}{\lambda(\alpha)}$, and $|\xi + \zeta - Q - 2\pi m/\alpha| \leqslant \frac{1}{\lambda(\alpha)}$. First assume $m = 1$ we have $\xi + \zeta - Q - \frac{2\pi}{\alpha} \geqslant -\frac{1}{\lambda(\alpha)}$ and therefore $\xi + \zeta \geqslant \frac{2\pi}{\alpha} - \frac{1}{\lambda(\alpha)} + Q \geqslant \frac{2\pi}{\alpha}\left(1 - \frac{\alpha}{\lambda(\alpha)}\right)$. Similarly when $m = -1$ we have $-\xi - \zeta + Q - \frac{2\pi}{\alpha} \geqslant -\frac{1}{\lambda(\alpha)}$ and therefore $-\xi - \zeta \geqslant \frac{\pi}{\alpha} - \frac{1}{\lambda(\alpha)} - Q \geqslant \frac{2\pi}{\alpha}\left(1 - \frac{\alpha}{\lambda(\alpha)}\right)$. Note also, by the expression for $\lambda_s(\alpha)$ in (4.67), that $\alpha/\lambda_s(\alpha) \to 0$ as $\alpha \to 0$. Therefore, in either case $m = \pm 1$, taking α small enough that $\alpha/\lambda(\alpha) \leqslant 1/2$ gives (using Lemma

A.0.5) $|\xi|^\eta + |\zeta|^\eta \geqslant |\xi+\zeta|^\eta \geqslant \left(\frac{2\pi}{\alpha}\right)^\eta \left(1 - \frac{\alpha}{\lambda(\alpha)}\right)^\eta \geqslant \frac{\pi^\eta}{\alpha^\eta}$. Hence, under these constraints on Q, ξ, ζ, we have $1 = e^{C|\xi|^\eta}\ e^{C|\zeta|^\eta}\ e^{-C(|\xi|^\eta + |\zeta|^\eta)} \leqslant e^{-C\pi^\eta/\alpha^\eta}$. Therefore,

$$\begin{aligned}
&\left|\chi_{\rm lo}(Q)\ S * S * \left(\chi_{\rm lo}\widehat{f}\right)(Q - 2m\pi/\alpha)\ \right| \\
&\leqslant\ \chi_{\rm lo}(Q) \int_{\mathbb{R}}\int_{\mathbb{R}} |(\chi_{\mathcal{B}_\alpha} S)(\xi)|\ |(\chi_{\mathcal{B}_\alpha} S)(\zeta)|\ |(\chi_{\rm lo}\widehat{f})(Q - \xi - \zeta - 2m\pi/\alpha)|\ d\xi\ d\zeta \\
&\leqslant e^{-C\pi^\eta/\alpha^\eta} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{C|\xi|^\eta}\ |S(\xi)|\ e^{C|\zeta|^\eta}\ |S(\zeta)|\ |\widehat{f}(Q - \xi - \zeta - 2m\pi/\alpha)|\ d\xi\ d\zeta,
\end{aligned} \tag{4.126}$$

which using the convolution estimate (2.20) gives, for some $C > 0$,

$$\begin{aligned}
&\left\|\ \chi_{\rm lo}(Q)\ S * S * \left(\chi_{\rm lo}\widehat{f}\right)(Q - 2m\pi/\alpha)\ \right\|_{L^{2,a-2p}(\mathbb{R}_Q)} \\
&\lesssim e^{-C/\alpha^\eta}\ \left\| e^{C|Q|^\eta}\ S(Q)\ \right\|^2_{L^{2,a}(\mathbb{R}_Q)}\ \left\|\ \widehat{f}\ \right\|_{L^{2,a}(\mathbb{R})}
\end{aligned} \tag{4.127}$$

Estimates (4.124) and (4.127) complete the proof of Proposition 4.5.3. □

Proposition 4.5.4. *Let* $0 < r < 1$*. Let* $R^\sigma_{\rm NL}$ *be defined in* (4.66) *and recall its derivative given in* (4.118)*. Then there exists* $\alpha_0, C > 0$ *such that for* $0 < \alpha < \alpha_0$*,*

$$\begin{aligned}
\left\|\ \chi_{\rm lo}\ R^\sigma_{\rm NL}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right]\ \right\|_{L^{2,a-2}(\mathbb{R})} &\lesssim e^{-C/\lambda(\alpha)^\eta} + e^{-C/\lambda(\alpha)^\eta}\ \left\|\ \widehat{E_{\rm lo}}\ \right\|_{L^{2,a}(\mathbb{R})} \\
&\quad + \left\|\ \widehat{E_{\rm lo}}\ \right\|^2_{L^{2,a}(\mathbb{R})} + \left\|\ \widehat{E_{\rm lo}}\ \right\|^3_{L^{2,a}(\mathbb{R})},
\end{aligned} \tag{4.128}$$

$$\begin{aligned}
\left\|\ D_{\widehat{E_{\rm lo}}}\left(\chi_{\rm lo}\ R^\sigma_{NL}\left[\alpha, \widehat{E_{\rm lo}} + \widehat{E_{\rm hi}}[\alpha, \widehat{E_{\rm lo}}]\right]\right)\ \right\|_{L^{2,a}(\mathbb{R})\to L^{2,a-2}(\mathbb{R})} &\lesssim e^{-C/\lambda(\alpha)^\eta} \\
&\quad + \left\|\ \widehat{E_{\rm lo}}\ \right\|_{L^{2,a}(\mathbb{R})} + \left\|\ \widehat{E_{\rm lo}}\ \right\|^2_{L^{2,a}(\mathbb{R})}.
\end{aligned} \tag{4.129}$$

This follows from estimates (4.114) and (4.115), using the convolution estimate (2.20).

Proposition 4.5.5. *Let* $p = \min(1, s)$ *and* $0 < r < 1$*. Recall* $\mathfrak{e}_1(\alpha)$*, given in* (4.36) *. Let* $\mathcal{D}^{\sigma,\alpha}$ *be defined in* (4.58)*. There exists* $\alpha_0 > 0$ *such that for* $0 < \alpha < \alpha_0$*,*

$$\|\ \chi_{\rm lo}\ \mathcal{D}^{\sigma,\alpha}[S]\ \|_{L^{2,a-2p}(\mathbb{R})}\ \lesssim \sqrt{\alpha}\ \mathfrak{e}_1(\alpha). \tag{4.130}$$

Proof of Proposition 4.5.5: We follow the proof of Proposition 3.5.5 from the nearest-neighbor DNLS case and use $\|\widehat{f}\|_{L^{2,a-2p}(\mathbb{R})} \leqslant \|\widehat{f}\|_{L^{2,a}(\mathbb{R})}$ extensively. Recall from (4.58) that

$$\mathcal{D}^{\sigma,\alpha}[S](Q) = R_{\rm F,1}[\alpha] + R^\sigma_{\rm F,2}[\alpha] \tag{4.131}$$

where we have defined

$$R_{\text{F},1}[\alpha] \equiv \chi_{\mathcal{B}_\alpha}(Q) \left(-\left[1 + M^s_\alpha(Q)\right] \ S(Q) + \left(\frac{1}{2\pi}\right)^2 \ (S * S * S)(Q) \right),$$

$$R^\sigma_{\text{F},2}[\alpha] \equiv \chi_{\mathcal{B}_\alpha}(Q) \left(\frac{1}{2\pi}\right)^2 \sum_{m=\pm 1} e^{2m\pi i\sigma} \ (S * S * S)(Q - 2m\pi/\alpha). \tag{4.132}$$

By Lemma 4.4.2, we have for some $C > 0$,

$$\left\| \chi_{\text{lo}} \ R^\sigma_{\text{F},2} \right\|_{L^{2,a-2p}(\mathbb{R})} \leqslant \left\| R^\sigma_{\text{F},2} \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha^\eta}. \tag{4.133}$$

Next, we write $S(Q) = \widetilde{\psi}(Q) - \overline{\chi}_{\mathcal{B}_\alpha}(Q) \ \widetilde{\psi}(Q)$. Substituting into (4.132), we have

$$\begin{aligned} \chi_{\text{lo}}(Q) \ R_{\text{F},1}[\alpha] = \chi_{\text{lo}}(Q) & \left(-\left[1 + M^s_\alpha(Q)\right] \ \widetilde{\psi}(Q) + \left(\frac{1}{2\pi}\right)^2 \ \widetilde{\psi} * \widetilde{\psi} * \widetilde{\psi}(Q) \right), \\ & - \chi_{\text{lo}}(Q) \left(\frac{1}{2\pi}\right)^2 \left[3 \ \left(\overline{\chi}_{\mathcal{B}_\alpha} \ \widetilde{\psi}\right) * \widetilde{\psi} * \widetilde{\psi}(Q) \right. \\ & \qquad + 3 \ \left(\overline{\chi}_{\mathcal{B}_\alpha} \ \widetilde{\psi}\right) * \left(\overline{\chi}_{\mathcal{B}_\alpha} \ \widetilde{\psi}\right) * \widetilde{\psi}(Q) \\ & \qquad \left. + \left(\overline{\chi}_{\mathcal{B}_\alpha} \ \widetilde{\psi}\right) * \left(\overline{\chi}_{\mathcal{B}_\alpha} \ \widetilde{\psi}\right) * \left(\overline{\chi}_{\mathcal{B}_\alpha} \ \widetilde{\psi}\right)(Q) \right]. \end{aligned} \tag{4.134}$$

To all convolutions involving $\overline{\chi}_{\mathcal{B}_\alpha} \ \widetilde{\psi}$, we apply Lemma 4.5.5 to get another exponentially small bound $\lesssim e^{-C\pi^\eta/\alpha^\eta}$ in $L^{2,a}(\mathbb{R})$.

We finally turn our attention to the remaining terms in (4.134):

$$\begin{aligned} & \chi_{\text{lo}}(Q) \left(-\left[1 + |Q|^{2p}\right] \ \widetilde{\psi}(Q) + \left(\frac{1}{2\pi}\right)^2 \ \widetilde{\psi} * \widetilde{\psi} * \widetilde{\psi}(Q) + \left[|Q|^{2p} - M^s_\alpha(Q)\right] \widetilde{\psi}(Q) \right) \\ & = \chi_{\text{lo}}(Q) \ \left[|Q|^{2p} - M^s_\alpha(Q)\right] \widetilde{\psi}(Q), \end{aligned} \tag{4.135}$$

We apply Proposition 4.3.3 to obtain $\left\| \chi_{\text{lo}}(Q) \ \left[|Q|^{2p} - M^s_\alpha(Q)\right] \widetilde{\psi}(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \sqrt{\alpha} \ \mathfrak{e}_1(\alpha)$. This completes the proof of Proposition 4.5.5. □

We now apply Propositions 4.5.2 through 4.5.5, and estimates (4.114) and (4.115) to (4.93) and (4.117). This implies estimates (4.95) and (4.96) in Proposition 4.5.1.

4.5.3 Solution of the Low Frequency Equation

We may now apply Lemma 4.5.1 to the rescaled low frequency equation (4.92).

Proposition 4.5.6. *Let $a > 1/2$, $p = \min(s, 1)$, and $0 < r < 1$. Then there exists $0 < \alpha_2 \leqslant \alpha_1$ such that for all $\alpha \in (0, \alpha_2)$, there exists an even (symmetric) solution $\widehat{E_{\text{lo}}}$ to (4.92) which satisfies*

$$\left\| \widehat{E_{\text{lo}}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim \sqrt{\alpha}\ \mathfrak{e}_1(\alpha), \tag{4.136}$$

where $\mathfrak{e}_1(\alpha)$ is given in (4.36). Furthermore, we have that $\widehat{E_{\text{lo}}} = \chi_{\text{lo}} \widehat{E_{\text{lo}}}$; that is, $\widehat{E_{\text{lo}}}(Q)$ is supported on $Q \in \left[-\frac{1}{\lambda(\alpha)}, \frac{1}{\lambda(\alpha)}\right]$, where $\lambda(\alpha) \to 0$ as $\alpha \to 0$ is defined by $\lambda(\alpha) = \begin{cases} \alpha^{r-1}, 0 < r < 1 & : s \neq 1 \\ \frac{-1}{\log(\alpha)} & : s = 1 \end{cases}$.

Proof of Proposition 4.5.6: By Proposition 4.1.3 $\widetilde{L_+^p} : L^{2,a}_{\text{even}}(\mathbb{R}) \to L^{2,a-2p}_{\text{even}}(\mathbb{R})$ is an isomorphism. Moreover, by Proposition 4.5.1 the mapping $(\alpha, \hat{E}_{\text{lo}}) \mapsto \mathcal{R}_2^\sigma[\alpha, \hat{E}_{\text{lo}}]$, maps $L^{2,a}_{\text{even}}(\mathbb{R})$ to $L^{2,a-2p}_{\text{even}}(\mathbb{R})$, and is continuous at $(\alpha, \widehat{E_{\text{lo}}}) = (0, 0)$. Furthermore, by choosing $\alpha < \alpha_1$ the estimates (4.95) and (4.96) on $\mathcal{R}_2^\sigma[\alpha, \widehat{E_{\text{lo}}}]$ hold. Hence, hypotheses (F.34) and (F.35) of Lemma 4.5.1 are satisfied. The Implicit Function Theorem (Lemma 4.5.1) implies, for $0 < \alpha < \alpha_2 \leqslant \alpha_1$, the existence of $\widehat{E_{\text{lo}}}$ satisfying the bound (4.136). □

To complete the proofs of Theorems 4.4.1 and 4.2.1 , we proceed as follows. From Proposition 4.4.3 we obtain $\widehat{E_{\text{hi}}}[\alpha, \widehat{E_{\text{lo}}}] \in L^{2,a}_{\text{even}}(\mathbb{R})$. Then, $\widehat{E^{\alpha,\sigma}}(Q) = \widehat{E_{\text{lo}}}(Q) + \widehat{E_{\text{hi}}}\left[\alpha, \widehat{E_{\text{lo}}}\right](Q)$ solves the corrector equation (4.64), and

$$\widehat{\Phi^{\alpha,\sigma}}(Q) = \chi_{\mathcal{B}_\alpha}(Q)\ \widetilde{\psi}(Q) + \widehat{E^{\alpha,\sigma}}(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}), \tag{4.137}$$

is a solution to (4.58). As discussed in Section 4.3, $G^{\alpha,\sigma}(q), \sigma = 0, 1/2$, is constructed from $\widehat{\phi^{\sigma,\alpha}}(q)$ via

$$\widehat{\phi^\sigma}(q) = \chi_{\mathcal{B}}(q)\widehat{\phi^\sigma}(q) = \left(\frac{\kappa_s(\alpha)}{\alpha^2}\right)^{1/2} \left[\chi_{\mathcal{B}}(q)\ \widetilde{\psi}\left(\frac{q}{\alpha}\right) + \widehat{E^{\alpha,\sigma}}\left(\frac{q}{\alpha}\right)\right], \tag{4.138}$$

These details are similar to those in the local discrete case and we refer to Section 3.8.2.

4.6 Bound on the Peierls-Nabarro Barrier for Nonlocal DNLS - Proof of Theorem 4.2.2;

We now prove the differences $\mathcal{N}[G^{\alpha,\text{on}}] - \mathcal{N}[G^{\alpha,\text{off}}]$ and $\mathcal{H}[G^{\alpha,\text{on}}] - \mathcal{H}[G^{\alpha,\text{off}}]$ are bounded in magnitude by a quantity of order: $(\kappa_s(\alpha)/\alpha) \cdot\ e^{-C/\alpha^\eta}$, for α small. Here, $G^{\alpha,\text{on}} = \{G_n^{\alpha,\text{on}}\}_{n\in\mathbb{Z}}$ and

$G^{\alpha,\text{off}} = \{G_n^{\alpha,\text{off}}\}_{n\in\mathbb{Z}}$ to the nonlocal DNLS equation, and $\eta = \eta(s) = \min(2s, 1)$, governs the exponential decay rate of the continuum limit ground state solitary wave $p(s)-$ FNLS; see Proposition 4.1.2.

Proposition 4.6.1. *Suppose $G = \{G_n\}_{n\in\mathbb{Z}}$ is real-valued and solves nonlocal DNLS equation with interaction parameter s: $0 = \kappa_s(\alpha)\ G_n - (\mathcal{L}^s G)_n - (G_n)^3, \quad n \in \mathbb{Z}$. Then,*

$$\sum_{n\in\mathbb{Z}} |G_n|^4 = \kappa_s(\alpha) \sum_n |G_n|^2 + \frac{1}{2} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \sum_{n\in\mathbb{Z}} J_{|m|} |G_{m+n} - G_n|^2 \tag{4.139}$$

Proof of Proposition 4.6.1: Multiply the system by G_n, sum over all $n \in \mathbb{Z}$ and then sum by parts. □

Using Proposition 4.6.1 and following the arguments leading to equation (3.266) for the case of DNLS with nearest-neighbor interactions we obtain:

$$\Big|\mathcal{N}[G^{\text{off}}] - \mathcal{N}[G^{\text{on}}]\Big| + \Big|\mathcal{H}[G^{\text{off}}] - \mathcal{H}[G^{\text{on}}]\Big| \lesssim \left(\frac{\kappa_s(\alpha)}{\alpha}\right) \left\|\widehat{\Phi^{\text{off}}} - \widehat{\Phi^{\text{on}}}\right\|_{L^{2,a}(\mathbb{R})}, \tag{4.140}$$

Therefore, bounds on the PN-barrier are reduced to bounds on $\widehat{\Phi^{\text{off}}} - \widehat{\Phi^{\text{on}}}$ in $L^{2,a}(\mathbb{R})$. Here, $\widehat{\Phi^{\text{on}}} \equiv \widehat{\Phi^{\alpha,\sigma=0}}$ and $\widehat{\Phi^{\text{off}}} \equiv \widehat{\Phi^{\alpha,\sigma=1/2}}$. Theorem 4.2.2 now follows directly from the following proposition, proved in the next section.

Proposition 4.6.2. *Let $\alpha_0 > 0$ be that prescribed in Theorem 4.2.1 and fix $\eta = \min(2s, 1)$. Then for $0 < \alpha < \alpha_0$, there exists a constant $C > 0$ such that*

$$\left\|\widehat{\Phi^{\text{off}}} - \widehat{\Phi^{\text{on}}}\right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C/\alpha^\eta}. \tag{4.141}$$

4.6.1 Estimation of the difference $\widehat{\Phi^{\text{off}}} - \widehat{\Phi^{\text{on}}}$; proof of Proposition 4.6.2

The idea is to derive a non-homogeneous equation for $\widehat{\Phi^{\text{diff}}} \equiv \widehat{\Phi^{\text{off}}} - \widehat{\Phi^{\text{on}}}$. We shall prove that this equation is driven by terms which are exponential small in α due to uniform decay bounds on $e^{\mu|Q|^\eta}\widehat{\Phi^{\alpha,\sigma}}(Q)$, $\eta = \min(s, 1)$, for some $\mu > 0$.

Proposition 4.6.3. *Let $0 < \alpha < \alpha_0$. Then $\widehat{\Phi^{\text{diff}}}(q) = \widehat{\Phi^{\text{off}}}(q) - \widehat{\Phi^{\text{on}}}(q)$ solves the following* <u>*linear*</u>

equation:

$$[1 + M_\alpha^s(Q)] \ \widehat{\Phi^{\rm diff}}(q) - \chi_{\mathcal{B}_\alpha}(Q)\frac{1}{4\pi^2}\Big(\widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm diff}}(Q)$$
$$+ \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm diff}}(Q) + \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm diff}}(Q) = R_{\rm diff}\left[\widehat{\Phi^{\rm off}}, \widehat{\Phi^{\rm on}}\right](Q), \tag{4.142}$$

where the inhomogeneous right-hand side, involving only $m = \pm 1$ side-band terms, is given by

$$R_{\rm diff}\left[\widehat{\Phi^{\rm off}}, \widehat{\Phi^{\rm on}}\right](Q) = -\chi_{\mathcal{B}_\alpha}(Q)\frac{1}{4\pi^2}\sum_{m=\pm 1}\Big(\widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm off}} * \widehat{\Phi^{\rm off}}(Q - 2m\pi/\alpha)$$
$$+ \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm on}} * \widehat{\Phi^{\rm on}}(Q - 2m\pi/\alpha)\Big), \tag{4.143}$$

Recall from Theorem 4.4.1 that $\widehat{\Phi^{\sigma,\alpha}} \in L^{2,a}(\mathbb{R})$ is well-defined, $\|\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim 1$, for α sufficiently small and satisfies equation (4.58). Note that $\widehat{\Phi^{\sigma,\alpha}}$ is supported on $\mathcal{B}_\alpha = [-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}]$, an interval which grows as $\alpha \downarrow 0$. We begin by proving a uniform decay bound for $\widehat{\Phi^{\sigma,\alpha}}$.

Proposition 4.6.4. *For $0 < \alpha < \alpha_2$, let $\widehat{\Phi^{\sigma,\alpha}} \in L^{2,a}$, $a > 1/2$ denote the onsite ($\sigma = 0$) and offsite ($\sigma = 1/2$) nonlocal DNLS solitary waves obtained in Theorem 4.2.1. Let $\eta = \min(2s, 1)$. Then there exist constants $\mu = \mu\left(\|\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}}\right)$ and $C_1 > 0$, independent of $\widehat{\Phi^{\sigma,\alpha}}$ and α, such that*

$$\|e^{\mu|Q|^\eta}\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)} \leqslant C_1 \ \|\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{4.144}$$

Proof of Proposition 4.6.4: $\hat{\Phi}$ solves

$$(1 + M_\alpha^s(Q)) \ \hat{\Phi}(Q) - \frac{1}{(2\pi)^2}\sum_{m=-1}^{1}\chi_{\mathcal{B}_\alpha}(Q)\hat{\Phi} * \hat{\Phi} * \hat{\Phi}(Q - 2m\pi/\alpha). \tag{4.145}$$

We seek to apply Lemma H.0.17 from Appendix H with the identifications $M(Q) \equiv M_\alpha^s(Q)$, $A = \mathcal{B}_\alpha$, $\tau_m = -2\pi m/\alpha$, and $m = -1, 0, 1$. We need to check that there exists $D_M > 0$ such that

$$(a) \quad \frac{\chi_{\mathcal{B}_\alpha}(Q)\ |Q|^\eta}{1 + M_\alpha^s(Q)} \leqslant D_M, \text{and} \quad (b) \quad \chi_{\mathcal{B}_\alpha}(Q)\ |Q| \leqslant |Q - 2m\pi/\alpha|, \qquad m = -1, 0, 1. \tag{4.146}$$

We need only worry about (a) for all $Q \in \mathcal{B}_\alpha$. Consider the cases (i) $1/2 \leqslant s \leqslant \infty, s \neq 1$, (ii) $1/4 < s < 1/2, s \neq 1$, (iii) $s = 1$.

First, (b) holds for $Q \in \mathcal{B}_\alpha = \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$ easily.

Next, we prove (a) : $\|e^{\mu|Q|}\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)} \leqslant C_1 \ \|\widehat{\Phi^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R}_Q)}$ for $s \geqslant 1/2$ and $s \neq 1$. Here, $\eta = \min(1, 2s) = 1$, and we apply Lemma H.0.17 to equation (4.58) for $\widehat{\Phi^{\sigma,\alpha}}$, which gives decay

$\sim\ e^{-C|Q|}$ for $\widehat{\Phi^{\sigma,\alpha}}$. To see that the hypotheses of the lemma are satisfied, first note that for $m \in \{-1, 0, 1\}$ and $Q \in \mathcal{B}_\alpha$, $|Q| \leqslant |Q - 2m\pi/\alpha|$. For $s \neq 1$ and $p = \min(1, s)$, recall from Lemma 4.4.1 that there exists $C > 0$ such that $M_\alpha^s(Q) \geqslant C|Q|^{2p}, \quad Q \in \mathcal{B}_\alpha$. This implies that, by maximization over Q,

$$\frac{\chi_{\mathcal{B}_\alpha}(Q)\ |Q|}{1 + M_\alpha^s(Q)} \leqslant \frac{\chi_{\mathcal{B}_\alpha}(Q)\ |Q|}{1 + C\ |Q|^{2p}} \leqslant \frac{1}{2p\ C^{1/2p}}\,(2p-1)^{(2p-1)/2p}\,, \tag{4.147}$$

where we understand the right-hand side to mean C^{-1} for $s = p = 1/2$.

Next, we prove (a) for $1/4 < s < 1/2$ such that $\eta = \min(1, 2s) = 2s$. We again apply Lemma H.0.17, which gives weaker decay $\sim\ e^{-C|Q|^{2s}}$ for $\widehat{\Phi^{\sigma,\alpha}}$. We again recall from Lemma 4.4.1 (since for $s < 1$, $p = \min(1, s) = s$) that there exists $C > 0$ such that $M_\alpha^s(Q) \geqslant C|Q|^{2s}, \quad Q \in \mathcal{B}_\alpha$. This implies that

$$\frac{\chi_{\mathcal{B}_\alpha}(Q)\ |Q|^{2s}}{1 + M_\alpha^s(Q)} \leqslant \frac{\chi_{\mathcal{B}_\alpha}(Q)\ |Q|^{2s}}{1 + C\ |Q|^{2s}} \leqslant \frac{1}{C}. \tag{4.148}$$

Finally, to satisfy (a) from (4.146) when $s = 1$, we require the following proposition.

Proposition 4.6.5. *Let $s = 1$ and let the spectral cutoff projections $\chi_{\rm hi}$ and $\chi_{\rm lo}$ be as defined in* (4.68)*. Then for α sufficiently small, there exists some $C > 0$ such that*

$$(a)\quad \chi_{\mathcal{B}_\alpha}(Q)\ \chi_{\rm hi}(Q)\ M_\alpha^s(Q) \geqslant C|Q| \qquad \textit{and} \qquad (b)\quad \chi(|Q| > 1)\ \chi_{\rm lo}(Q)\ M_\alpha^s(Q) \geqslant C|Q|^2.$$

We defer the proof until later in this section. Proposition 4.6.5 implies that

$$\begin{aligned}
\frac{\chi_{\mathcal{B}_\alpha}(Q)\ |Q|}{1 + M_\alpha^s(Q)} &= \chi_{\mathcal{B}_\alpha}(Q)\left(\frac{\chi(|Q| \leqslant 1)\ |Q|}{1 + M_\alpha^s(Q)} + \frac{\chi(|Q| > 1)\ |Q|}{1 + M_\alpha^s(Q)}\right) \leqslant 1 + \chi_{\mathcal{B}_\alpha}(Q)\left(\frac{\chi(|Q| > 1)\ |Q|}{1 + M_\alpha^s(Q)}\right) \\
&\leqslant 1 + \frac{\chi(|Q| > 1)\ \chi_{\rm lo}(Q)\ |Q|}{1 + M_\alpha^s(Q)} + \frac{\chi_{\mathcal{B}_\alpha}(Q)\ \chi_{\rm hi}(Q)\ |Q|}{1 + M_\alpha^s(Q)} \\
&\leqslant 1 + \frac{\chi(|Q| > 1)\ \chi_{\rm lo}(Q)\ |Q|}{1 + C|Q|^2} + \frac{\chi_{\mathcal{B}_\alpha}(Q)\ \chi_{\rm hi}(Q)\ |Q|}{1 + C|Q|} \\
&\leqslant 1 + \frac{1}{C^{1/2}} + \frac{1}{C} = D_{s=1},
\end{aligned} \tag{4.149}$$

such that the hypotheses of Lemma H.0.17 are again satisfied when $s = 1$.

This completes the proof of Proposition 4.6.4. □

Proof of Proposition 4.6.5: Recall that for $s = 1$, the the spectral projections give

$$\chi_{\rm lo}(Q)\ |Q| \leqslant (-\log(\alpha)), \qquad\qquad \chi_{\rm hi}(Q)\ |Q| \geqslant (-\log(\alpha)). \tag{4.150}$$

Furthermore, recall from Lemma 4.4.1 that for $s = 1$, $\chi_{\mathcal{B}_\alpha}(Q)\ M_s^\alpha(Q) \geqslant \frac{C}{(-\log(\alpha))}|Q|^2$, such that

$$\chi_{\mathcal{B}_\alpha}(Q)\ \chi_{\rm hi}(Q)\ M_s^\alpha(Q) \geqslant \frac{C}{(-\log(\alpha))}\ \chi_{\rm hi}(Q)\ |Q|^2 \geqslant C|Q|. \tag{4.151}$$

This proves (a). Next, we use the expansion from Proposition G.0.9 for $s = 1$ and $Q \in \mathcal{B}_\alpha = \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$:

$$M_s^\alpha(Q) = |Q|^2 + \frac{1}{-\log(\alpha)}\ f_s(Q;\alpha)\ \left(\frac{3}{2} - \log(|Q|)\right)\ |Q|^2 \tag{4.152}$$

$$= |Q|^2\ \left[1 + \frac{1}{-\log(\alpha)}\ f_s(Q;\alpha)\ \left(\frac{3}{2} - \log(|Q|)\right)\right], \qquad |f_s(Q;\alpha)| \lesssim 1. \tag{4.153}$$

Recall that $\lambda(\alpha) = \frac{1}{(-\log(\alpha))}$ for $s = 1$. Furthermore, recall that $\chi_{\rm lo}(Q) = \chi\left(|Q| \leqslant \lambda(\alpha)^{-1}\right)$, where $\lambda(\alpha)^{-1} \leqslant \pi/\alpha$ for α sufficiently small, which implies that $\chi_{\rm lo}(Q)\ |f_s(Q;\alpha)| \lesssim 1$. Therefore, we have

$$\chi(|Q| > 1)\ \chi_{\rm lo}(Q)\left|\frac{1}{-\log(\alpha)}\ f_s(Q;\alpha)\ \left(\frac{3}{2} - \log(|Q|)\right)\right| \lesssim \frac{1 + \log(-\log(\alpha))}{(-\log(\alpha))}. \tag{4.154}$$

such that for α sufficiently small, there exists $C > 0$ such that

$$\chi(|Q| > 1)\ \chi_{\rm lo}(Q)\ \left(1 + \frac{1}{-\log(\alpha)}\ f_s(Q;\alpha)\ \left(\frac{3}{2} - \log(|Q|)\right)\right) \geqslant C. \tag{4.155}$$

This completes the proof of Proposition 4.6.5. □

Next, we derive the equation for $\widehat{\Phi^{\rm diff}} = \widehat{\Phi^{\rm off}} - \widehat{\Phi^{\rm on}}$.

Proposition 4.6.6. *Let* $0 < \alpha < \alpha_0$ *and* $\eta = \min(2s, 1)$*. Then,, the non-homogeneous source term in* (4.142) *for* $\widehat{\Phi^{\rm diff}}(q)$ *and satisfies the bound:* $\left\|R_{\rm diff}\left[\widehat{\Phi^{\rm off}}, \widehat{\Phi^{\rm on}}\right]\right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C\pi^\eta/\alpha^\eta}$.

Proof of Proposition 4.6.6 : We apply Lemma 4.4.2 and Proposition 4.6.4. This gives

$$\left\|\chi_{\mathcal{B}_\alpha}(Q)\ \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma} * \widehat{\Phi^\sigma}(Q - 2m\pi/\alpha)\right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim e^{-C\pi^\eta/\alpha^\eta}, \qquad m = \pm 1. \qquad \square$$

We now use Proposition 4.6.6 to prove the exponential bound (4.141) on $\widehat{\Phi^{\rm diff}}$. We use a Lyapunov-Schmidt reduction argument, which is analogous to that used in the proof of Theorem 4.4.1. We summarize the argument since the details are now quite familiar. Introduce

$$\widehat{\Phi^{\rm diff}_{\rm lo}}(q) \equiv \chi_{\rm lo}(q)\ \widehat{\Phi^{\rm diff}}(q), \qquad \text{and} \qquad \widehat{\Phi^{\rm diff}_{\rm hi}}(q) \equiv \chi_{\rm hi}(q)\ \widehat{\Phi^{\rm diff}}(q). \tag{4.156}$$

Solving for $\widehat{\Phi^{\mathrm{diff}}_{\mathrm{hi}}}$ as a functional of $\widehat{\Phi^{\mathrm{diff}}_{\mathrm{lo}}}$ and estimation of the mapping yields:

$$\left\| \widehat{\Phi^{\mathrm{diff}}_{\mathrm{hi}}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim \begin{cases} \alpha^{2p(1-r)} \left(\left\| \widehat{\Phi^{\mathrm{diff}}_{\mathrm{lo}}} \right\|_{L^{2,a}(\mathbb{R})} + e^{-C\pi^\eta/\alpha^\eta} \right) & : s \neq 1 \\ \frac{1}{(-\log(\alpha))} \left(\left\| \widehat{\Phi^{\mathrm{diff}}_{\mathrm{lo}}} \right\|_{L^{2,a}(\mathbb{R})} + e^{-C\pi/\alpha} \right) & : s = 1 \end{cases}, \tag{4.157}$$

$\widehat{\Phi^{\mathrm{diff}}_{\mathrm{lo}}}$ satisfies an inhomogeneous equation forced by $\chi_{\mathrm{lo}}\ R_{\mathrm{diff}}\left[\widehat{\Phi^{\mathrm{off}}}, \widehat{\Phi^{\mathrm{on}}}\right]$ which satisfies the exponential bound of Proposition 4.6.6. A simple bootstrap argument using (4.157) and the bounds on $\widehat{\Phi^{\mathrm{off}}}$ and $\widehat{\Phi^{\mathrm{on}}}$ give $\left\| \widehat{\Phi^{\mathrm{diff}}_{\mathrm{lo}}} \right\|_{L^{2,a}(\mathbb{R})} \lesssim e^{-C\pi^\eta/\alpha^\eta}$, for α sufficiently small. Since $\widehat{\Phi^{\mathrm{diff}}} = \widehat{\Phi^{\mathrm{diff}}_{\mathrm{lo}}} + \widehat{\Phi^{\mathrm{diff}}_{\mathrm{hi}}}$, the bound (4.157) implies the assertion of Proposition 4.6.2. $\square$

4.7 Higher Order Expansions in α of $G^{\alpha,\mathrm{on}}$ and $G^{\alpha,\mathrm{off}}$

In this section, we provide a general outline for how to extend the expansion of $G^{\alpha,\sigma}$ to higher order in α, for α small. The precise expansion parameter depends on the analytic properties of $M_s^\alpha(Q)$, which is expressible in terms of the polylogarithms (see Appendix G). We have

Proposition 4.7.1. *Let $p = \min(1, s)$ and let $M_s^\alpha(Q)$ be given in (4.43) with J_m^s given in (4.18). Then for $|Q| \leqslant \pi/\alpha$,*

$$M_\alpha^s(Q) = |Q|^{2p} + \begin{cases} \frac{2}{C_s} \sum_{j=1}^\infty \frac{\zeta(1+2s-2j)}{(2j)!} (-1)^{j+1} \alpha^{2j-2s} |Q|^{2j} & : s < 1 \\ \hline \frac{1}{C_s\ (-\log(\alpha))} \left[\frac{3}{2} - \log(|Q|)\right] |Q|^2 \\ + \frac{2}{C_s\ (-\log(\alpha))} \sum_{j=2}^\infty \frac{\zeta(3-2j)}{(2j)!} (-1)^{j+1} \alpha^{2j-2} |Q|^{2j} & : s = 1 \\ \hline \frac{-2\Gamma(-2s)\ \cos(\pi s)}{C_s} \alpha^{2s-2} |Q|^{2s} \\ + \frac{2}{C_s} \sum_{j=2}^\infty \frac{\zeta(1+2s-2j)}{(2j)!} (-1)^{j+1} \alpha^{2j-2} |Q|^{2j} & : 1 < s < \infty, \\ & s \notin \mathbb{N} \\ \hline \frac{2(-1)^s}{C_s\ (2s)!} \left[-\left(\sum_{j=1}^{2s} \frac{1}{j}\right) \right. \\ \qquad \left. + \log(\alpha) + \log(|Q|) \right] \alpha^{2s-2} |Q|^{2s} \\ + \frac{2}{C_s} \sum_{\substack{j=2 \\ j \neq s}}^\infty \frac{\zeta(1+2s-2j)}{(2j)!} (-1)^{j+1} \alpha^{2j-2} |Q|^{2j} & : 1 < s < \infty, \\ & s \in \mathbb{N} \\ \hline \frac{2}{C_s} \sum_{k=1}^\infty \left(\sum_{m=1}^\infty m^{2k+2} e^{-\gamma m}\right) \frac{(-1)^k\ \alpha^{2k}\ |Q|^{2k+2}}{(2k+2)!} & : s = \infty \end{cases}. \tag{4.158}$$

Proof of Proposition 4.7.1: We follow the approach in Appendix G and Proposition G.0.9.

Let $q = Q\alpha \in \mathcal{B} = [-\pi, \pi]$. Recall that $M^s(q) = \frac{1}{C_s} \sum_{m=1}^\infty \frac{4\sin^2(qm/2)}{m^{1+2s}}$ for $0 < s < \infty$ and $M^s(q) = \frac{1}{C_\infty} \sum_{m=1}^\infty 4e^{-\gamma m} \sin^2(qm/2)$, $\gamma > 0$, for $s = \infty$. Here, $C_s > 0$ is defined in (4.33). Also recall that $\kappa_s(\alpha)$ is defined in (4.23).

Case 1: Let $1 + 2s \notin \mathbb{N},\ s \neq \infty$. Expand $4\sin^2(qm/2) = 2 - e^{iqm} - e^{-iqm}$, and note that since $|q| \leqslant \pi < 2\pi$, we may apply Lemma G.0.14 to obtain (G.6):

$$C_s\ M^s(q) = -2\Gamma(-2s)\ \cos(\pi s)\ |q|^{2s} + 2\sum_{j=1}^{\infty} \frac{\zeta(1+2s-2j)}{(2j)!}\ (-1)^{j+1}\ |q|^{2j}. \tag{4.159}$$

Applying the rescaling $q = Q\alpha$ gives

$$\begin{aligned} M^s_\alpha(Q) =& \frac{1}{\kappa_s(\alpha)} M^s(Q\alpha) = -\frac{2\Gamma(-2s)\ \cos(\pi s)}{C_s\ \kappa_s(\alpha)}\ \alpha^{2s}\ |Q|^{2s} \\ &+ \sum_{j=1}^{\infty} \frac{\zeta(1+2s-2j)}{C_s\ \kappa_s(\alpha)\ (2j)!}\ (-1)^{j+1}\alpha^{2j}\ |Q|^{2j}. \end{aligned} \tag{4.160}$$

The series is absolutely convergent on $Q = q/\alpha \in \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$ due to Proposition G.0.8. To obtain (4.158) for $1 + 2s \notin \mathbb{N}, s \neq \infty$, recall that $C_s = -2\Gamma(-2s)\ \cos(\pi s)$, $\kappa_s(\alpha) = \alpha^{2s}$ for $s < 1$ and $C_s = \zeta(2s-1), \quad \kappa_s(\alpha) = \alpha^2$ for $s > 1$.

Case 2: Let $1 + 2s \in \mathbb{N},\ s \neq \infty$. Again expand $4\sin^2(qm/2) = 2 - e^{iqm} - e^{-iqm}$, and note that since $|q| \leqslant \pi < 2\pi$, we may apply Lemma G.0.15 to obtain (G.8):

$$\begin{aligned} M^s(q) = \frac{2\cos(\pi s)}{(2s)!}\left[-\left(\sum_{j=1}^{2s}\frac{1}{j}\right) + \log(|q|)\right]\ |q|^{2s} + \frac{\pi\ \cos\left(\pi(s-1/2)\right)}{(2s)!}\ |q|^{2s} \\ + 2\sum_{\substack{j=1 \\ j\neq s}}^{\infty} \frac{\zeta(1+2s-2j)}{(2j)!}\ (-1)^{j+1}\ |q|^{2j}. \end{aligned} \tag{4.161}$$

Applying the rescaling $q = Q\alpha$ gives

$$\begin{aligned} M^s_\alpha(Q) =& \frac{1}{\kappa_s(\alpha)} M^s(Q\alpha) \\ =& \frac{2\cos(\pi s)}{C_s\ \kappa_s(\alpha)\ (2s)!}\left[-\left(\sum_{j=1}^{2s}\frac{1}{j}\right) + \log(\alpha) + \log(|Q|)\right]\ \alpha^{2s}\ |Q|^{2s} \\ &+ \frac{\pi\ \cos\left(\pi(s-1/2)\right)}{C_s\ \kappa_s(\alpha)\ (2s)!}\alpha^{2s}\ |Q|^{2s} + 2\sum_{\substack{j=1 \\ j\neq s}}^{\infty} \frac{\zeta(1+2s-2j)}{C_s\ \kappa_s(\alpha)\ (2j)!}\ (-1)^{j+1}\alpha^{2j}\ |Q|^{2j}. \end{aligned} \tag{4.162}$$

The infinite series is absolutely convergent on $Q = q/\alpha \in \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$ due to Proposition G.0.8. To obtain (4.158) for for $1+2s \in \mathbb{N}, s \neq \infty$, recall that $C_s = -2\Gamma(-2s)\cos(\pi s)$, $\kappa_s(\alpha) = \alpha^{2s}$ for $s = 1/2$ and $C_s = \zeta(2s-1), \quad \kappa_s(\alpha) = \alpha^2$ for $s > 1$. For $s = 1$, recall that $C_s = 1$ and $\kappa_s(\alpha) = (-\log(\alpha))\ \alpha^2$. We now repeat Remark G.0.3 here.

Remark 4.7.1. *Note that one of the first two terms in* (G.9) *will be zero, depending on whether* $s \in \mathbb{N}$ *or* $s = (2k+1)/2$ *for* $k \in \mathbb{N}$.

Furthermore, in the case where $s = (2k+1)/2$ *for* $k \in \mathbb{N}$, *the series in* (G.9) *is finite. To see this, observe from Proposition G.0.8 that the "trivial zeros" of the zeta function on the real line lie occur at negative even integers. In the series above, this occurs when* $1+2s-2j \leqslant -2 \implies j \geqslant s+3/2$.

We also note that for $s = (2k+1)/2$ *for* $k \in \mathbb{N}$, *the expansion* (G.9) *is compatible with the expansion* (G.7) *where we assumed that* $1 + 2s \notin \mathbb{N}$. *To see this we use Euler's reflection formula from Proposition G.0.7 to obtain, for* $k \in \mathbb{N}$, *equation* (G.10)*:*

$$\lim_{s \to \frac{2k-1}{2}} \; -2\ \Gamma(-2s)\ \cos(\pi s) = \left.\frac{\pi(-1)^{s-1/2}}{(2s)!}\right|_{s=\frac{2k-1}{2}}. \tag{4.163}$$

Case 3: Finally, let $s = \infty$. We have $M^s_\alpha(q) = \frac{1}{C_\infty\ \alpha^2}\sum_{m=1}^{\infty} 4e^{-\gamma m}\sin^2(Qm\alpha/2)$, $\gamma > 0$, $\kappa_s(\alpha) = \alpha^2$, and $Q \in [-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}]$. Taylor expansion about $Q\alpha = 0$ gives

$$\frac{4\sin^2(Qm\alpha/2)}{\alpha^2} = 2\sum_{k=0}^{\infty}\frac{(-1)^k\ \alpha^{2k}\ m^{2k+2}\ |Q|^{2k+2}}{(2k+2)!}. \tag{4.164}$$

In turn,

$$M^s_\alpha(Q) = \frac{1}{C_s}\left(\sum_{m=1}^{\infty} m^2\ e^{-\gamma m}\right)\ |Q|^2 + \frac{2}{C_s}\sum_{k=1}^{\infty}\left(\sum_{m=1}^{\infty} m^{2k+2}\ e^{-\gamma m}\right)\frac{(-1)^k\ \alpha^{2k}\ |Q|^{2k+2}}{(2k+2)!}. \tag{4.165}$$

Observe that

$$\left(\sum_{m=1}^{\infty} m^2\ e^{-\gamma m}\right) = \frac{\exp(\gamma)(\exp(\gamma)+1)}{(\exp(\gamma)-1)^3} = C_\infty. \tag{4.166}$$

This gives (4.158) for $s = \infty$. This completes the proof of Proposition 4.7.1. □

Consider the equation (4.58), $\mathcal{D}^{\sigma,\alpha}\left[\widehat{\Phi}\right] = 0$, for the rescaled s-nonlocal DNLS solitary wave $\widehat{\Phi}(Q)$ supported on $\mathcal{B}_\alpha$. Consider the case where $s \notin \mathcal{N}$, $s = b_1/b_2$ where $b_1, b_2 \in \mathbb{N}$. By Lemma 4.4.2 and Proposition 4.6.4, we have

$$\left\| R^\sigma_1[\widehat{\Phi^\sigma}]\right\|_{L^{2,a}(\mathbb{R})} \lesssim \mathcal{O}(\alpha^\infty). \tag{4.167}$$

Furthermore, we have

Lemma 4.7.1. *Let* $s \notin \mathbb{N}$ *and* $s = b_1/b_2$ *reduced rational for* $b_1, b_2 \in \mathbb{N}$. *Then for* $\tau = 2/b_2$ *and* $Q \in \mathcal{B}_\alpha$, *there exists a sequence* $M_k \in \mathbb{R}$, $k = 1, 2, \ldots$, *such that* $\{M_k\}_{k\in\mathbb{N}} \in l^1(\mathbb{N})$ *and*

$$M^\alpha_s(Q) = |Q|^{2p} + \sum_{k=1}^{\infty} M_k\ \alpha^{j\tau}\ |Q|^{j\tau+2p}. \tag{4.168}$$

Proof of Lemma 4.7.1: First consider the case $1/4 < s < 1$. Then $p = \min(1, s) = s$ and by (4.158) with $\tau = 2/b_2$,

$$\begin{aligned} M_s^\alpha(Q) &= |Q|^{2s} + \frac{2}{C_s} \sum_{j=1}^{\infty} \frac{\zeta(1+2s-2j)}{(2j)!} \; (-1)^{j+1} \; \alpha^{2j-2s} \; |Q|^{2j-2s} \; |Q|^{2s} \\ &= |Q|^{2s} + \frac{2}{C_s} \sum_{j=1}^{\infty} \frac{\zeta(1+2s-2j)}{(2j)!} \; (-1)^{j+1} \; \alpha^{\tau(b_2 j - b_1)} \; |Q|^{\tau(b_2 j - b_1)} \; |Q|^{2s}. \end{aligned} \tag{4.169}$$

Taking $M_k \equiv \frac{2 \; \zeta(1+2s-2j)}{C_s \; (2j)!} \; (-1)^{j+1}$ if $k = b_2 j - b_1$ and $M_k \equiv 0$ otherwise completes the proof for $1/4 < s < 1$.

Next, consider $1 < s < \infty$, $s \notin \mathbb{N}$. Then $p = \min(1, s) = 1$ and by (4.158) with $\tau = 2/b_2$,

$$\begin{aligned} M_s^\alpha(Q) &= |Q|^2 + \frac{2}{C_s} \Bigg(- \Gamma(-2s) \; \cos(\pi s) \; \alpha^{2s-2} \; |Q|^{2s-2} \\ &\qquad + \sum_{j=2}^{\infty} \frac{\zeta(1+2s-2j)}{(2j)!} \; (-1)^{j+1} \; \alpha^{2j-2} \; |Q|^{2j-2} \Bigg) \; |Q|^2 \\ &= |Q|^2 + \frac{2}{C_s} \Bigg(- \Gamma(-2s) \; \cos(\pi s) \; \alpha^{\tau(b_1 - b_2)} \; |Q|^{\tau(b_1 - b_2)} \\ &\qquad + \sum_{j=2}^{\infty} \frac{\zeta(1+2s-2j)}{(2j)!} \; (-1)^{j+1} \; \alpha^{\tau b_2 (j-1)} \; |Q|^{\tau b_2 (j-1)} \Bigg) \; |Q|^2. \end{aligned} \tag{4.170}$$

Taking $M_k = \frac{-2\Gamma(-2s) \; \cos(\pi s)}{C_s}$ if $k = b_1 - b_2$, $M_k \equiv \frac{2 \; \zeta(1+2s-2j)}{C_s \; (2j)!} \; (-1)^{j+1}$ if $k = b_2(j-1)$ and $M_k \equiv 0$ otherwise completes the proof of Lemma 4.7.1 for $1 < s < \infty$, $s \notin \mathbb{N}$. □

Using the above expansion of $M_s^\alpha(Q)$, we construct sequence of solutions to (4.58), which to arbitrary order $(\alpha^\tau)^J$, can be approximated by a sums of the form:

$$S_J^\sigma = \chi_{\mathcal{B}_\alpha}(Q) \; F_0(Q) + \sum_{j=1}^{J} \chi_{\mathcal{B}_\alpha}(Q) \; \alpha^{j\tau} \; F_j(Q), \qquad J \in \mathbb{N}, \tag{4.171}$$

with decreasing residuals: $\|\mathcal{D}^{\alpha,\sigma} \left[S_J^\sigma\right] \|_{L^{2,a}} = \mathcal{O}\left(\alpha^{\tau(J+1)}\right)$ and higher order corrector:

Theorem 4.7.1. *Let $s \notin \mathbb{N}$ and $s = b_1/b_2$ reduced rational for $b_1, b_2 \in \mathbb{N}$. Then there exists, for any $J \geqslant 1$, S_J^σ given by (4.171), such that $\|\mathcal{D}^{\alpha,\sigma} \left[S_J^\sigma\right] \|_{L^{2,a}} = \mathcal{O}\left(\alpha^{\tau(J+1)}\right)$. Furthermore, $\widehat{\Phi^{\sigma,\alpha}}$, the solution of (4.58), is given by $\widehat{\Phi^{\sigma,\alpha}} = S_J^\sigma + \widehat{E_J^{\sigma,\alpha}}$, where $\|\widehat{E_J^{\sigma,\alpha}}\|_{L^{2,a}(\mathbb{R})} \lesssim \alpha^{2J+2}$.*

We next explain the construction of the sequence $F_j(Q)$, $j \geqslant 0$. The proof of Theorem 4.7.1 uses a Lyapunov-Schmidt strategy analogous to that used in the proof of Theorem 4.4.1. The steps in the proof are analogous to those of the nearest neighbor case; see Section 3.3.2.

To construct the sequence F_j, we use Lemma 4.7.1 and consider the related equation

$$[1 + M^s_\alpha(Q)]\ F(Q) - \frac{1}{4\pi^2}\left(F * F * F\right)(Q) = 0\ . \tag{4.172}$$

Using the power series expansion from Lemma 4.7.1, we may construct a formal power series expansion in α^τ for $F^\alpha(Q)$:

$$F^\alpha(Q) = F_0(Q) + \sum_{j=1}^{\infty} \alpha^{j\tau}\ F_j(Q), \tag{4.173}$$

Substituting (4.173) into (4.172), we obtain a hierarchy of equations for $F_j(Q)$ at each order of $\alpha^{\tau j}, \quad j \geqslant 0$. Each term $F_j(Q)$ will have support on all $\mathbb{R}$ and can be shown to decay exponentially as $|Q| \to \infty$. The deviation of (4.172) from the equation (4.58), $\mathcal{D}^{\sigma,\alpha}\left[\widehat{\Phi}\right] = 0$, for the rescaled s-nonlocal DNLS solitary wave $\widehat{\Phi}(Q)$ are terms of the form:

$$-\left(1 - \chi_{\mathcal{B}_\alpha}(Q)\right)[1 + M^s_\alpha(Q)]\,F(Q) + \frac{1}{4\pi^2}\ \left(1 - \chi_{\mathcal{B}_\alpha}(Q)\right)(F^\alpha * F^\alpha * F^\alpha)(Q) + R^{\alpha,\sigma}_1[F^\alpha](Q), \tag{4.174}$$

whose norms at each order in $\alpha^{\tau j}$ can be shown to be beyond all polynomial orders in α as $\alpha \to 0$, *i.e.* $\mathcal{O}(\alpha^m)$, for all $m \geqslant 1$, in $L^{2,a}(\mathbb{R}; dQ)$ with $a > 1/2$. Therefore, we expect that if $\widehat{\Phi^{\alpha,\sigma}}(Q)$ is a solution of (4.58), then the function $\chi_{\mathcal{B}_\alpha}(Q)\ F^\alpha(Q)$, where F^α solves (4.172), formally solves (4.58) with an error which is beyond all polynomial orders in α^2, *i.e.*

$$\left\| \mathcal{D}^{\alpha,\sigma}\left[\ \chi_{\mathcal{B}_\alpha}\ F^\alpha\ \right] \right\|_{L^{2,a}(\mathbb{R};dQ)} \ = \ \mathcal{O}(\alpha^\infty) \tag{4.175}$$

We proceed by outlining the derivation of the formal asymptotic expansion. Substituting of (4.173) into (4.174), we obtain a hierarchy of equations for F_j.

$\mathcal{O}(\alpha^0)$ **equation** :

$$\left[1 + |Q|^{2p}\right]\ F_0(Q) - \left(\frac{1}{2\pi}\right)^2\ F_0 * F_0 * F_0(Q) = 0. \tag{4.176}$$

Equation (4.176) is the Fourier transform of continuum FNLS (1.17). Denote by

$$F_0(Q) = \widetilde{\psi}(Q) = \mathcal{F}_C\left[\psi\right](Q), \tag{4.177}$$

where $\psi(x)$ is the unique (up to translation) <u>positive</u> and decaying solution of NLS. $\psi(x)$ is real-valued and radially symmetric about some point, which we take to be $x = 0$. By Proposition 4.1.2 there exists $C_0 > 0$ such that for $\eta = \min(1, 2s)$,

$$e^{C_0|Q|^\eta}\ \widetilde{\psi}(Q) \in L^{2,a}(\mathbb{R}). \tag{4.178}$$

At each order in α^2, we shall derive an equation for $F_j(Q)$ of the general form:

$$\widetilde{L^p_+}\ F^\sharp(Q) = F^\flat(Q). \tag{4.179}$$

It is important for us to understand how decay properties $F^\flat(Q)$ propagate to the solution $F^\sharp(Q)$.

Proposition 4.7.2. *Fix $a > 1/2$ and $\eta = \min(1, 2s)$. Suppose that $F^\flat(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ)$ and that there exists a constant $C_\flat > 0$ such that $e^{C_\flat|Q|^\eta} F^\flat(Q) \in L^{2,a}(\mathbb{R}; dQ)$. Then there exists a solution of* (4.179), *$F^\sharp(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ)$. Furthermore, we have $e^{C_\flat|Q|^\eta} F^\sharp(Q) \in L^{2,a}(\mathbb{R}; dQ)$.*

Since $F^\flat(Q)$ is even it is $L^2(\mathbb{R}; dQ)$ orthogonal to the kernel of $\widetilde{L^p_+} = \text{span}\{Q\widetilde{\psi}(Q)\}$. Therefore, $F^\sharp = \left(\widetilde{L^p_+}\right)^{-1} F^\flat \in L^{2,a+2p}_{\text{even}}(\mathbb{R}; dQ)$; see Proposition 4.1.3. A detailed proof that the exponential decay rate is preserved is given for the case of nearest-neighbor DNLS in Section 3.3.2. The idea is to break $F^\sharp$ into its low ($|Q| \leqslant \epsilon^{-1}$) and high ($|Q| \geqslant \epsilon^{-1}$) frequency components, $F_{\text{lo},\epsilon}$ and $F_{\text{hi},\epsilon}$. The norm $\|e^{C_\flat|Q|^\eta} F_{\text{lo},\epsilon}\|_{L^{2,a}}$ is bounded in terms of $\|F^\sharp\|_{L^{2,a}}$. While the norm $\|e^{C_\flat|Q|^\eta} F_{\text{hi},\epsilon}\|_{L^{2,a}}$ is controlled by a boot-strap argument using that $\chi(|Q| \geqslant \epsilon^{-1})(1 + |Q|^{2p})^{-1}\ F_0 * F_0 * F_{\text{hi},\epsilon}$ has $L^{2,a}-$norm which is bounded by $\sim \epsilon^{2p}\ \|e^{C_\flat|Q|^\eta} F_{\text{hi},\epsilon}\|_{L^{2,a}}$.

We now turn to the hierarchy of equations at order $\alpha^{\tau j}$, beginning with $j = 1$. We find

$$\left[1 + |Q|^{2p}\right]\ F_1(Q) - \frac{3}{4\pi^2}\ \widetilde{\psi} * \widetilde{\psi} * F_1(Q) = -M_1\ |Q|^{2p+\tau}\ \widetilde{\psi}(Q), \tag{4.180}$$

or

$$\mathcal{O}(\alpha^\tau)\ \textbf{equation}: \qquad \widetilde{L^p_+}\ F_1(Q) = -M_1\ |Q|^{2p+\tau}\ \widetilde{\psi}(Q). \tag{4.181}$$

Here, $L_+ = (-\Delta_x)^p + 1 - 3\psi^2(x)$, is the linearization of the continuum NLS operator about $\psi(x)$. By Proposition 4.7.2, $F_1(Q) = -\left(\widetilde{L^p_+}\right)^{-1} \left(M_1\ |Q|^{2p+\tau}\ \widetilde{\psi}(Q)\right) \in L^{2,a}_{\text{even}}(\mathbb{R})$. Since (4.181) has real-valued forcing, F_1 is real-valued. Let $C_\flat = 3C_0/4$ and note $\|e^{C_\flat|Q|^\eta}|Q|^{2p+\tau}\ \widetilde{\psi}\|_{L^{2,a}} \lesssim \|e^{C_0|Q|^\eta}\widetilde{\psi}\|_{L^{2,a}}$ Therefore, $e^{\frac{3C_0}{4}|Q|^\eta} F_1(Q) \in L^{2,a}(\mathbb{R}; dQ)$ for $a > 1/2$.

We now proceed to inductively construct and bound the sequence $F_j(Q),\ j \geqslant 1$ using Proposition 4.7.2 and the following two lemmata, which are proved in detail in Section 3.3.2 for the case $\eta = 1$. The generalization follows the same proof.

Lemma 4.7.2. *Fix $a > 1/2$ and $\eta = \min(1, 2s)$. Suppose that $\widetilde{f}_1, \widetilde{f}_2 \in L^{2,a}_{\text{even}}(\mathbb{R})$. Then $\widetilde{f}_1 * \widetilde{f}_2 \in L^{2,a}_{\text{even}}(\mathbb{R})$. Suppose further that there exist $c_1, c_2 > 0$ such that $e^{c_1|Q|^\eta}\widetilde{f}_1(Q) \in L^{2,a}(\mathbb{R})$, $e^{c_2|Q|^\eta}\widetilde{f}_2(Q) \in L^{2,a}(\mathbb{R}_Q)$. Then for $c_3 = \min(c_1, c_2)$, we have $e^{c_3|Q|^\eta}\ \widetilde{f}_1 * \widetilde{f}_2(Q) \in L^{2,a}_{\text{even}}(\mathbb{R}_Q)$ and*

$$\left\| \ e^{c_3|Q|^\eta}\ (\widetilde{f}_1 * \widetilde{f}_2)(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \left\| \ e^{c_3|Q|^\eta}\ \widetilde{f}_1(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)} \left\| \ e^{c_3|Q|^\eta}\ \widetilde{f}_2(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{4.182}$$

Lemma 4.7.2 is a direct consequence of the convolution estimate (2.20) and appropriately distributing the exponential weights, and $c_3|Q|^\eta - c_1|Q-\xi|^\eta - c_2|\xi|^\eta \leqslant 0$, which follows from Lemma A.0.5.

Lemma 4.7.3. *Fix $a > 1/2$, $\eta = \min(1, 2s)$, and $k \in \mathbb{N}$. Suppose that $\widetilde{f} \in L^{2,a}_{\text{even}}(\mathbb{R})$ and that there exists $c_1 > 0$ such that $e^{c_1|Q|^\eta}\widetilde{f}(Q) \in L^{2,a}(\mathbb{R})$. Then $|Q|^{2k}\ \widetilde{f} \in L^{2,a}_{\text{even}}(\mathbb{R})$ and for any $0 < c_2 < c_1$, we have*

$$\left\| e^{c_2|Q|^\eta}|Q|^{2k}\widetilde{f} \right\|_{L^{2,a}(\mathbb{R})} \leqslant e^{-\frac{2k}{\eta}} \left[\frac{2k}{\eta(c_1-c_2)} \right]^{2k/\eta} \left\| \ e^{c_1|Q|^\eta}\ \widetilde{f}(Q) \ \right\|_{L^{2,a}(\mathbb{R}_Q)}. \tag{4.183}$$

Lemma 4.7.3 follows from $e^{c_2|Q|^\eta}|Q|^{2k}|\widetilde{f}(Q)| \leqslant e^{-\frac{2k}{\eta}} \left[\frac{2k}{\eta(c_1-c_2)}\right]^{2k/\eta} \cdot e^{c_1|Q|^\eta}|\widetilde{f}(Q)|$ and then taking the $L^{2,a}$ norm.

Proposition 4.7.3. *Let $j \geqslant 1$. The equation for F_j at order $\mathcal{O}(\alpha^{\tau j})$, independent of α and σ, is given by*

$\mathcal{O}(\alpha^{\tau j})$ **equation** :

$$\begin{aligned} \widetilde{L^p_+}\ F_j(Q) &= -\sum_{k=0}^{j-1} M_{j-k}\ |Q|^{(j-k)\tau+2p}\ F_k(Q), \\ &+ \frac{1}{(2\pi)^2} \sum_{\substack{k+l+z=j \\ 0\leqslant k,l,z<j}} F_k * F_l * F_z(Q) \\ &\equiv H_j\left[F_0, \ldots, F_{j-1}\right](Q), \end{aligned} \tag{4.184}$$

and has the unique solution

$$F_j = \left(\widetilde{L^p_+}\right)^{-1} \left(H_j\left[F_0, \ldots, F_{j-1}\right] \right) \in L^{2,a}_{\text{even}}(\mathbb{R}; dQ). \tag{4.185}$$

Furthermore, F_j is real-valued and $e^{C_j|Q|^\eta}\ F_j(Q) \in L^{2,a}(\mathbb{R}; dQ)$, where $C_j \equiv C_0\left(\frac{1}{2} + \frac{1}{2^{j+1}}\right) \geqslant \frac{C_0}{2}$ and $C_0 > 0$ is as in (4.178).

Proof of Proposition 4.7.3: We induct to solve at each order in $\alpha^{\tau j}$. Let $F_0(Q) \equiv \widetilde{\psi}(Q)$, which solves (4.176), is real-valued, and satisfies $e^{C_0|Q|^\eta} F_0(Q) \in L^{2,a}(\mathbb{R}; dQ)$. Fix $m \geqslant 2$ and assume that for $1 \leqslant j \leqslant m-1$, $F_j(Q) \in L^{2,a}_{\text{even}}(\mathbb{R})$ satisfies (4.184) and is real-valued. Furthermore, assume that

$$e^{C_j|Q|^\eta} F_j(Q) \in L^{2,a}(\mathbb{R}; dQ), \qquad C_j \equiv C_0 \left(\frac{1}{2} + \frac{1}{2^{j+1}}\right) \geqslant \frac{C_0}{2}. \tag{4.186}$$

We have already proven above that these inductive hypotheses hold for $j = 1$. We expand

$$M_\alpha(Q) = |Q|^{2p} + \sum_{j=1}^{\infty} \alpha^{\tau j}\ M_j\ |Q|^{\tau j + 2p}, \qquad\qquad F^\alpha(Q) = \sum_{j=0}^{\infty} \alpha^{\tau j}\ F_j(Q), \tag{4.187}$$

and substitute into (4.172). Using (4.176) for $F_0(Q) \equiv \widetilde{\psi}(Q)$ we obtain

$$\begin{aligned} \sum_{j=1}^{\infty}\ \alpha^{\tau j}\ \widetilde{L^p_+}\ F_j(Q) &= -\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} \alpha^{\tau j+\tau k}\ M_j\ |Q|^{\tau j+2p}\ F_k(Q) + \sum_{j=1}^{\infty} \frac{\alpha^{2j}}{(2\pi)^2} \sum_{\substack{k+l+z=j\\ 0\leqslant k,l,z<j}} F_k * F_l * F_m(Q) \\ &= \sum_{j=1}^{\infty} \alpha^{2j}\Bigg(-\sum_{k=0}^{j-1} M_{j-k}\ |Q|^{\tau j-\tau k+2p}\ F_k(Q) + \frac{1}{4\pi^2} \sum_{\substack{k+l+z=j\\ 0\leqslant k,l,z<j}} F_k * F_l * F_z(Q)\Bigg). \end{aligned} \tag{4.188}$$

Applying the inductive hypothesis (4.184) for $1 \leqslant j \leqslant m-1$ and dividing by $\alpha^{\tau m}$, (4.188) becomes

$$\begin{aligned} &\widetilde{L^p_+}\ F_m(Q) + \alpha^\tau \left[\sum_{j=m+1}^{\infty} \alpha^{\tau j - \tau(m+1)}\ \widetilde{L^p_+}\ F_j(Q)\right] \\ &= -\sum_{k=0}^{m-1} M_{m-k}\ |Q|^{\tau(m-k)+2p}\ F_k(Q) + \frac{1}{(2\pi)^2} \sum_{\substack{k+l+z=m\\ 0\leqslant k,l,z<m}} F_k * F_l * F_z(Q) \\ &\quad + \alpha^\tau \Bigg[\sum_{j=m+1}^{\infty} \alpha^{\tau j-\tau(m+1)}\Bigg(-\sum_{k=0}^{j-1}\ M_{j-k}\ |Q|^{\tau(j-k)+2p}\ F_k(Q) \\ &\qquad\qquad + \frac{1}{4\pi^2} \sum_{\substack{k+l+z=j\\ 0\leqslant k,l,z<j}} F_k * F_l * F_z(Q)\Bigg)\Bigg]. \end{aligned} \tag{4.189}$$

Since $\tau j - \tau(m+1) \geqslant 0$ for $j \geqslant m+1$, the bracketed terms with coefficient α^τ are $\mathcal{O}(\alpha^\tau)$. Therefore the terms of order precisely $\alpha^{\tau m}$ are given by(4.184). This establishes the case: $j = m$.

We now prove that (4.184) has a solution, F_m satisfying (4.186) with $j = m$. First, applying Lemmata 4.7.2 and 4.7.3 to the right hand side of (4.184) for $j = m$, H_m, we have that $H_m \in L^{2,a}_{\text{even}}$ with the bound:

$$\left\| \ e^{C_m|Q|^\eta}\ H_m\left[F_0, \ldots, F_{m-1}\right](Q)\ \right\|_{L^{2,a}(\mathbb{R};dQ)} \lesssim \lambda_m, \tag{4.190}$$

where $C_m = C_0 \left(\frac{1}{2} + \frac{1}{2^{m+1}}\right)$ and

$$\lambda_m \equiv 2 \sum_{k=0}^{m-1} e^{-\frac{2k}{\eta}} \left[\frac{2k}{\eta(c_1 - c_2)}\right]^{2k/\eta} \left\| e^{C_k|Q|^\eta} F_k(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} + \frac{1}{(2\pi)^2} \sum_{\substack{k+l+z=m \\ 0\leqslant k,l,z<m}} \cdot \left\| e^{C_k|Q|^\eta} F_k(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} \left\| e^{C_l|Q|^\eta} F_l(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} \left\| e^{C_z|Q|^\eta} F_z(Q) \right\|_{L^{2,a}(\mathbb{R}_Q)} . \qquad (4.191)$$

Proposition 4.7.2 implies that there exists a unique solution $F_m(Q) \in L^{2,a}_{\text{even}}(\mathbb{R})$ to equation (4.184) with $e^{C_m|Q|^\eta} F_m(Q) \in L^{2,a}(\mathbb{R})$. Finally, to see that F_m is real-valued, note that equation (4.184) for F_m is a linear with inhomogeneous forcing on the right-hand-side given by $H_m\left[F_0, \ldots, F_{m-1}\right]$, which is necessarily real-valued for F_j real-valued, $j = 0, \ldots, m-1$. This completes the proof of Proposition 4.7.3. □

Part III

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Part IV

Appendices

Appendix A

Spaces and Inequalities

We work extensively in $L^{2,a}(\mathbb{R}^d)$ with $a > d/2$. Here,

$$L^{2,a}(A) = \left\{ f \quad : \quad \left\| (1 + |\cdot|^2)^{a/2} f \right\|_{L^2(A)} = \left(\int_A (1 + |q|^2)^a |f(q)|^2 dq \right)^{1/2} < \infty \right\}. \tag{A.1}$$

Lemma A.0.4. *[Dohnal and Uecker, 2009] Let $a > d/2$ and $f_1, f_2 \in L^{2,a}(\mathbb{R}^d)$. Then there exists a constant $D_a > 0$ such that*

$$\|f_1 * f_2\|_{L^{2,a}(\mathbb{R}^d)} \leqslant D_a \ \|f_1\|_{L^{2,a}(\mathbb{R}^d)} \ \|f_2\|_{L^{2,a}(\mathbb{R}^d)} . \tag{A.2}$$

We require the following subadditivity inequality on several occasions.

Lemma A.0.5. *For $x, y \in \mathbb{R}$ and $0 < p \leqslant 1$, we have*

$$|x + y|^p \leqslant |x|^p + |y|^p. \tag{A.3}$$

Proof of Lemma A.0.5: The result is trivial for $\eta = 1$. The result also holds trivially for $x = 0$ or $y = 0$, so we assume $x \neq 0$ and $y \neq 0$.

First let $x > 0$ and $y < 0$ (or $x < 0$ and $y > 0$). Then

$$|x + y|^\eta \leqslant \max\{|x|^\eta, |y|^\eta\} \leqslant |x|^\eta + |y|^\eta. \tag{A.4}$$

Now let $x > 0$ and $y > 0$. Then

$$\begin{aligned} |x + y|^\eta &= \frac{|x + y|}{|x + y|^{1-\eta}} \leqslant \frac{|x| + |y|}{|x + y|^{1-\eta}} \\ &= \left(\frac{|x|}{|x + y|} \right)^{1-\eta} |x|^\eta + \left(\frac{|y|}{|x + y|} \right)^{1-\eta} |y|^\eta \leqslant |x|^\eta + |y|^\eta, \end{aligned} \tag{A.5}$$

where we have used that

$$\frac{|x|}{|x+y|} \leqslant 1, \qquad \frac{|y|}{|x+y|} \leqslant 1, \tag{A.6}$$

for $x > 0$, $y > 0$. $\square$

Appendix B

Conserved Quantities of DNLS

We consider the general discrete cubic nonlinear Schrödinger equation (DNLS) (1.1):

$$i\partial_t u_n(t) = -h^{-2}(\mathcal{L}u)_n(t) - |u_n(t)|^2 u_n(t) \ , \qquad t \in \mathbb{R}, n \in \mathbb{Z}^d, \tag{B.1}$$

governing a complex-valued vector $u(t) = \{u_n(t)\}_{n\in\mathbb{Z}^d}$. Here, $\mathcal{L} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ is a linear, symmetric difference operator defined by

$$(\mathcal{L}u)_n \equiv \sum_{\substack{m\in\mathbb{Z}^d \\ m\neq n}} J_{|m-n|} \ (u_m - u_n),$$

$$\{J_{|m|}\}_{m\in\mathbb{N}^d} \in l^1(\mathbb{N}^d), \tag{B.2}$$

where the sequence of coupling coefficients $\{J_{|m|}\}_{m\in\mathbb{N}^d}$ is real-valued. Define the following two functionals of $u \in l^2(\mathbb{Z}^d)$, known respectively as the "power" and the Hamiltonian:

$$\mathcal{N}[u] \equiv \|u\|^2_{l^2(\mathbb{Z}^d)} = \sum_{n\in\mathbb{Z}^d} |u_n|^2, \tag{B.3}$$

$$\mathcal{H}[u] \equiv \frac{1}{2h^2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|}|u_m - u_n|^2 - \frac{1}{2}|u_n|^4. \tag{B.4}$$

In the following proposition, we show that the power and Hamiltonian are conserved (time-invariant) quantities of the DNLS flow.

Proposition B.0.4. *Suppose that $u(t) = \{u_n(t)\}_{n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$ solves DNLS (1.1). Then*

$$\frac{d}{dt}\mathcal{N}[u] = 0, \tag{B.5}$$

$$\frac{d}{dt}\mathcal{H}[u] = 0. \tag{B.6}$$

Proof of Proposition B.0.4: First, we prove (B.5). Multiply (1.1) by the complex conjugate $\overline{u}_n(t)$ and sum over $n \in \mathbb{Z}^d$ to get

$$i \sum_{n\in\mathbb{Z}^d} (\partial_t u_n)\ \overline{u}_n = -h^{-2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|}\ (u_m - u_n)\overline{u}_n - \sum_{n\in\mathbb{Z}^d} |u_n|^4. \tag{B.7}$$

Taking the complex conjugate of (B.7) gives

$$-i \sum_{n\in\mathbb{Z}^d} (\partial_t \overline{u}_n)\ u_n = -h^{-2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|}\ (\overline{u}_m - \overline{u}_n)u_n - \sum_{n\in\mathbb{Z}^d} |u_n|^4. \tag{B.8}$$

Subtracting (B.8) from (B.7) gives

$$i \sum_{n\in\mathbb{Z}^d} \left[(\partial_t u_n)\ \overline{u}_n + (\partial_t \overline{u}_n)\ u_n\right] = -h^{-2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left[(u_m - u_n)\overline{u}_n - (\overline{u}_m - \overline{u}_n)u_n\right]. \tag{B.9}$$

The left-hand side of (B.9) gives

$$i \sum_{n\in\mathbb{Z}^d} \left[(\partial_t u_n)\ \overline{u}_n + (\partial_t \overline{u}_n)\ u_n\right] = i \sum_{n\in\mathbb{Z}^d} \partial_t |u_n|^2 = i \frac{d}{dt} \sum_{n\in\mathbb{Z}^d} |u_n|^2, \tag{B.10}$$

while the right-hand side gives

$$\sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left[(u_m - u_n)\overline{u}_n - (\overline{u}_m - \overline{u}_n)u_n\right] = \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left(u_m\ \overline{u}_n - \overline{u}_m\ u_n\right) = 0, \tag{B.11}$$

Here, we changed the order of integration on the second term and used the symmetry of $J_{|m-n|}$ in n and m. Thus, (B.9) gives

$$i \frac{d}{dt} \sum_{n\in\mathbb{Z}^d} |u_n|^2 = 0, \tag{B.12}$$

which is (B.5).

Next, we prove (B.6). Multiply (1.1) by the complex conjugate $\partial_t \overline{u}_n(t)$ and sum over $n \in \mathbb{Z}^d$ to get

$$i \sum_{n\in\mathbb{Z}^d} |\partial_t u_n|^2 = -h^{-2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|}(u_m - u_n)\partial_t \overline{u}_n - |u_n|^2 u_n \partial_t \overline{u}_n. \tag{B.13}$$

Taking the complex conjugate of (B.13) gives

$$-i \sum_{n\in\mathbb{Z}^d} |\partial_t u_n|^2 = -h^{-2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|}(\overline{u}_m - \overline{u}_n)\partial_t u_n - |u_n|^2 \overline{u}_n \partial_t u_n. \tag{B.14}$$

Adding (B.14) to (B.13) gives

$$0 = -h^{-2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left[(u_m - u_n)\partial_t \overline{u}_n + (\overline{u}_m - \overline{u}_n)\partial_t u_n \right] - |u_n|^2 \left[u_n \partial_t \overline{u}_n + \overline{u}_n \partial_t u_n \right]. \quad \text{(B.15)}$$

First, note that

$$|u_n|^2 \left[u_n \partial_t \overline{u}_n + \overline{u}_n \partial_t u_n \right] = \frac{1}{2} \partial_t \left(|u_n|^4 \right). \quad \text{(B.16)}$$

Next,

$$\begin{aligned}
&2 \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left[(u_m - u_n)\partial_t \overline{u}_n + (\overline{u}_m - \overline{u}_n)\partial_t u_n \right] \\
&= \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left[(u_m - u_n)\partial_t \overline{u}_n + (\overline{u}_m - \overline{u}_n)\partial_t u_n \right] \\
&\quad + \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left[(u_n - u_m)\partial_t \overline{u}_m + (\overline{u}_n - \overline{u}_m)\partial_t u_m \right] \\
&= - \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \left[(u_m - u_n)\partial_t (\overline{u}_m - \overline{u}_n) + (\overline{u}_m - \overline{u}_n)\partial_t (u_m - u_n) \right] \\
&= - \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} \partial_t |u_m - u_n|^2 = -\frac{d}{dt} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} |u_m - u_n|^2. \qquad \text{(B.17)}
\end{aligned}$$

Here, we again changed the order of integration and used the symmetry of $J_{|m-n|}$ in n and m. Therefore, (B.15) is equivalent to

$$0 = \frac{d}{dt} \left(\frac{1}{2h^2} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} J_{|m-n|} |u_m - u_n|^2 - \frac{1}{2} |u_n|^4 \right), \quad \text{(B.18)}$$

which is (B.6). $\square$

Appendix C

Properties of the Discrete Fourier Transform

In this section, we prove some general properties of the discrete Fourier transform as defined in (2.1).

Lemma C.0.6. *For any function $u \in l^1(\mathbb{Z}^d) \cap l^2(\mathbb{Z}^d)$,*

$$\widehat{(\delta_j u)}(q) = \left(e^{iq} - 1\right) \hat{u}(q), \tag{C.1}$$

and

$$\widehat{(\delta^2 u)}(q) = \mathcal{F}_D[(\delta^2 u)](q) = -4 \sum_{j=1}^{d} \sin^2(q_j/2)\ \hat{u}(q), \tag{C.2}$$

where q_j is the jth component of $q \in \mathbb{R}^d$.

Lemma C.0.7. *For any two functions $u, v \in l^1(\mathbb{Z}^d) \cap l^2(\mathbb{Z}^d)$, and with their product given by $u \cdot v = \{u_n v_n\}_{n \in \mathbb{Z}^d}$,*

$$\mathcal{F}_D[u \cdot v](q) = \widehat{uv}(q) = (2\pi)^{-1} \hat{u} *_1 \hat{v}(q). \tag{C.3}$$

where the periodic convolution $*_1$ is defined in (2.5).

Lemma C.0.8. *Let $\widehat{u}$ and $\widehat{v}$ be $L^1_{\text{loc}}(\mathbb{R}^d)$ functions. Then if C is any constant, we have*

$$\left(e^{iC(\cdot)}\widehat{u}\right) *_1 \left(e^{iC(\cdot)}\widehat{v}\right)(q) = e^{iCq}\left(\widehat{u} *_1 \widehat{v}\right)(q). \tag{C.4}$$

Lemma C.0.9. *(Commutativity) For 2π-periodic functions $\widehat{u}$, $\widehat{v} \in L^1_{\text{loc}}(\mathbb{R}^d)$, we have*

$$\widehat{u} *_1 \widehat{v}(q) = \widehat{v} *_1 \widehat{u}(q). \tag{C.5}$$

Lemma C.0.10. *For three functions $\widehat{u}, \widehat{v}, \widehat{w} \in L^1_{\text{loc}}(\mathbb{R}^d)$, we have*

$$\widehat{u} *_1 \left[\widehat{v} *_1 \widehat{w}\right](q) = \widehat{v} *_1 \left[\widehat{u} *_1 \widehat{w}\right](q). \tag{C.6}$$

Note that $\widehat{u}, \widehat{v}$, and $\widehat{w}$ need not be periodic.

Proof of Lemma C.0.10: By a simple application of Fubini's theorem, we have

$$\widehat{u} *_1 \left[\widehat{v} *_1 \widehat{w}\right](q) = \int_{\mathcal{B}}\int_{\mathcal{B}} \widehat{u}(\xi)\widehat{v}(\zeta)\widehat{w}(q-\xi-\zeta)d\xi d\zeta$$

$$= \int_{\mathcal{B}}\int_{\mathcal{B}} \widehat{u}(\xi)\widehat{v}(\zeta)\widehat{w}(q-\xi-\zeta)d\zeta d\xi = \widehat{v} *_1 \left[\widehat{u} *_1 \widehat{w}\right](q). \;\square \tag{C.7}$$

Lemma C.0.11. *Suppose that $\widehat{u}$ and $\widehat{v}$ are even functions. Then $\widehat{u} *_\alpha \widehat{v}$ is also even. That is, for any $\tau_j = \pm 1$, $j = 1, \ldots d$, we have $\widehat{u} *_\alpha \widehat{v}(\tau_1 q_1, \ldots \tau_d q_d) = \widehat{u} *_\alpha \widehat{v}(q)$.*

The proof of Lemma C.0.11 also provides us with similar result for standard convolutions on the line.

Corollary C.0.1. *Suppose that $\widehat{u}$ and $\widehat{v}$ are even functions. Then $\widehat{u} * \widehat{v}$ is also even. That is, for any $\tau_j = \pm 1$, $j = 1, \ldots d$, we have $\widehat{u} * \widehat{v}(\tau_1 \xi, \ldots \tau_d q_d) = \widehat{u} * \widehat{v}(q)$.*

Appendix D

Properties of the Operator $\mathcal{L}$

In this section, we prove Proposition 1.0.1 on properties of the non-local linear difference operator $\mathcal{L}$ given by

$$(\mathcal{L}u)_n \equiv \sum_{m\in\mathbb{Z}} J_{|m-n|}\ (u_m - u_n), \qquad \{J_{|m|}\}_{m\in\mathbb{Z}} \in l^1(\mathbb{Z}), \qquad J_0 = 0,\ J_m \geqslant 0. \tag{D.1}$$

Boundness , $\|\mathcal{L}u\|_{l^2(\mathbb{Z})} \leqslant \|J\|_{l^1(\mathbb{Z})}\ \|u\|_{l^2(\mathbb{Z})}$, follows by Young's inequality.

Self-adjointness of $\mathcal{L}$ on $l^2(\mathbb{Z})$ follows since $(\mathcal{L}u)_n = \sum_m \mathcal{L}_{mn}u_m$, where $\mathcal{L}_{mn} = J_{|m-n|} - \|J\|_{l^1(\mathbb{Z})}\delta_{mn}$.

Non-negativity of $-\mathcal{L}$*:* By Young's inequality, $\langle \mathcal{L}u, u\rangle = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} J_{|m-n|}\ u_m\ \overline{u_n} - \|J\|_{l^1(\mathbb{Z})}\ \|u\|^2_{l^2(\mathbb{Z})} \leqslant 0$.

Dispersion relation of $-\mathcal{L}$*:* This is a consequence of the following

Lemma D.0.12. *Assume $J = \{J_m\}_{m\in\mathbb{Z}}$ is non-non-negative, real-valued, symmetric (such that $J_m = J_{-m}$, and in $J \in l^1(\mathbb{Z})$. Then for any function $u \in l^1(\mathbb{Z}) \cap l^2(\mathbb{Z})$,*

$$\widehat{(\mathcal{L}u)}(q) = -4\sum_{m=1}^{\infty} J_m \sin^2(qm/2)\ \widehat{u}(q)\ . \tag{D.2}$$

Proof of Lemma D.0.12: First, observe that

$$\begin{aligned}\mathcal{F}_D\left[\sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} J_{|m-n|}\ u_n\right](q) &= \sum_{n\in\mathbb{Z}} e^{-iqn}\sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} J_{|m-n|}\ u_n = \sum_{n\in\mathbb{Z}} e^{-iqn}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} J_{|m|}\ u_n\\ &= 2\left(\sum_{m=1}^{\infty} J_m\right)\left(\sum_{n\in\mathbb{Z}} e^{-iqn}\ u_n\right) = 2\left(\sum_{m=1}^{\infty} J_m\right)\ \widehat{u}(q).\end{aligned} \tag{D.3}$$

Next

$$\begin{aligned}
\mathcal{F}_{D}\left[\sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} J_{|m-n|}\ u_m\right](q) &= \sum_{n\in\mathbb{Z}} e^{-iqn} \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} J_{|m-n|}\ u_m = \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq m}} e^{-iqn} J_{|m-n|}\ u_m \\
&= \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-iq(n+m)} J_{|n|}\ u_m = \sum_{m\in\mathbb{Z}} e^{-iqm} \left(\sum_{\substack{n\in\mathbb{Z}\\ m\neq 0}} J_{|n|}\ e^{-iqn} \right) u_m \\
&= \left(\sum_{n=1}^{\infty} J_n \left[e^{-iqn} + e^{iqn}\right] \right) \left(\sum_{m\in\mathbb{Z}} e^{-iqm} u_m \right) \\
&= \left(\sum_{m=1}^{\infty} J_m \left[e^{-iqm} + e^{iqm}\right] \right) \widehat{u}(q). \qquad \text{(D.4)}
\end{aligned}$$

Therefore, since $4\sin^2(qm/2) = 2 - e^{-iqm} - e^{iqm}$, we have

$$\mathcal{F}_{D}\left[-\mathcal{L}u\right](q) = \left(\sum_{m=1}^{\infty} J_m \left[e^{-iqm} + e^{iqm} - 2\right] \right) \widehat{u}(q) = -4 \sum_{m=1}^{\infty} J_m \sin^2(qm/2)\ \widehat{u}(q). \ \square \qquad \text{(D.5)}$$

Appendix E

Pohozaev Identities for Fractional NLS

In this appendix, we provide several energy identities which are analogous to the well-known Pohozaev identities of the standard NLS equation [Strauss, 1977; Sulem and Sulem, 1999].

Proposition E.0.5 (Pohozaev identities)**.** *Let* $u \in H^{2p+1}(\mathbb{R}^d) \cap L^{m+1}(\mathbb{R}^d)$ *be a real-valued, positive solution to* $(-\Delta)^p\ u + u - u^m = 0$*. Then*

$$(i) \qquad \int_{\mathbb{R}^d} |(-\Delta)^{p/2} u|^2 dx + \|u\|^2_{L^2(\mathbb{R}^d)} - \|u\|^{m+1}_{L^{m+1}(\mathbb{R}^d)} = 0, \tag{E.1}$$

$$(ii) \qquad \frac{d-2p}{2} \int_{\mathbb{R}^d} |(-\Delta)^{p/2} u|^2 dx + \frac{d}{2}\ \|u\|^2_{L^2(\mathbb{R}^d)} - \frac{d}{m+1}\ \|u\|^{m+1}_{L^{m+1}(\mathbb{R}^d)} = 0, \tag{E.2}$$

In particular, if $m \geqslant (d+2p)/(d-2p)$ *and* $d - 2p > 0$ *, then* $u \equiv 0$*.*

Proof of Proposition E.0.5: *(i)* First, multiply $(-\Delta)^p\ u + u - u^m = 0$ by u and integrate over $\mathbb{R}^d$ to obtain

$$\int_{\mathbb{R}^d} u\ (-\Delta)^p u\ dx + \|u\|^2_{L^2(\mathbb{R}^d)} - \|u\|^{m+1}_{L^{m+1}(\mathbb{R}^d)} = 0. \tag{E.3}$$

Equation (E.1) follows from the Plancherel identity since for $u, v \in H^{2p+1}(\mathbb{R}^d)$ real-valued,

$$\begin{aligned} \int_{\mathbb{R}^d} u\ (-\Delta)^p v\ dx &= \int_{\mathbb{R}^d} \overline{u}\ (-\Delta)^p v\ dx = \int_{\mathbb{R}^d} \overline{\widetilde{u}(q)}\ |q|^{2p} \widetilde{v}(q)\ dq = \int_{\mathbb{R}^d} \overline{|q|^p \widetilde{u}(q)}\ |q|^p \widetilde{v}(q)\ dq \\ &= \int_{\mathbb{R}^d} (-\Delta)^{p/2} \overline{u}\ (-\Delta)^{p/2} v\ dx = \int_{\mathbb{R}^d} \overline{(-\Delta)^{p/2} u}\ (-\Delta)^{p/2} v\ dx \\ &= \int_{\mathbb{R}^d} (-\Delta)^{p/2} u\ (-\Delta)^{p/2} v\ dx. \end{aligned} \tag{E.4}$$

(ii) By assumption, $\nabla u \in L^2(\mathbb{R}^d)$. Next, multiply $(-\Delta)^p\ u + u - u^m = 0$ by $x \cdot \nabla u$ and integrate over $\mathbb{R}^d$ to obtain

$$\int_{\mathbb{R}^d} (x \cdot \nabla u)\ (-\Delta)^p u\ dx + \int_{\mathbb{R}^d} (x \cdot \nabla u)\ u\ x - \int_{\mathbb{R}^d} (x \cdot \nabla u)\ u^m\ dx = 0. \tag{E.5}$$

For the first integral in (E.5), we proceed as in [Ros-Oton and Serra, 2014]. Define $u_\lambda(x) \equiv u(\lambda x)$ and observe that $(-\Delta_x)^{p/2}\ u_\lambda(x) = \lambda^p\ (-\Delta_z)^{p/2}\ u(z)|_{z=\lambda x}$. Next, define $w(x) \equiv (-\Delta_x)^{p/2}\ u(x) = (-\Delta_z)^{p/2}\ u(z)|_{z=x}$. Therefore, by (E.4),

$$\begin{aligned}
\int_{\mathbb{R}^d} (x \cdot \nabla u)\ (-\Delta)^p u\ dx &= \frac{d}{d\lambda}\bigg|_{\lambda=1} \int_{\mathbb{R}^d} u_\lambda(x)\ (-\Delta)^p\ u(x)\ dx \\
&= \frac{d}{d\lambda}\bigg|_{\lambda=1} \int_{\mathbb{R}^d} (-\Delta)^{p/2}\ u_\lambda(x)\ (-\Delta)^{p/2}\ u(x)\ dx \\
&= \frac{d}{d\lambda}\bigg|_{\lambda=1} \left(\lambda^p \int_{\mathbb{R}^d} (-\Delta_z)^{p/2}\ u(z)|_{z=\lambda x}\ (-\Delta_x)^{p/2}\ u(x)\ dx\right) \\
&= \frac{d}{d\lambda}\bigg|_{\lambda=1} \left[\lambda^{\frac{2p-d}{2}} \int_{\mathbb{R}^d} (-\Delta_z)^{p/2}\ u(z)|_{z=\lambda^{1/2}x}\ (-\Delta_z)^{p/2}\ u(z)|_{z=\lambda^{-1/2}x}\ dx\right] \\
&= \frac{2p-d}{2} \int_{\mathbb{R}^d} (-\Delta)^{p/2}\ u(x)\ (-\Delta)^{p/2}\ u(x)\ dx \\
&\qquad + \frac{d}{d\lambda}\bigg|_{\lambda=1} \int_{\mathbb{R}^d} w\left(\lambda^{1/2}x\right)\ w\left(\lambda^{-1/2}x\right)\ dx \\
&= \frac{2p-d}{2} \int_{\mathbb{R}^d} (-\Delta)^{p/2}\ u(x)\ (-\Delta)^{p/2}\ u(x)\ dx
\end{aligned} \tag{E.6}$$

For the second two integrals in (E.5), integrating by parts gives

$$\begin{aligned}
2\int_{\mathbb{R}^d} (x \cdot \nabla u)\ u\ dx &= \int_{\mathbb{R}^d} \nabla \cdot (u^2 x) dx - \int_{\mathbb{R}^d} (\nabla \cdot x) u^2 dx = -d \int_{\mathbb{R}^d} u^2 dx, \\
(m+1)\int_{\mathbb{R}^d} (x \cdot \nabla u)\ u^m\ dx &= \int_{\mathbb{R}^d} \nabla \cdot (u^{m+1} x) dx - \int_{\mathbb{R}^d} (\nabla \cdot x) u^{m+1} dx = -d \int_{\mathbb{R}^d} u^{m+1} dx
\end{aligned} \tag{E.7}$$

This proves (E.2).

Now assume $m \geqslant (d+2p)/(d-2p)$ and $d - 2p > 0$. Multiplying (E.2) by $2/(d-2p)$ and subtracting it from (E.1) gives

$$\left(\frac{-2p}{d-2p}\right)\ \|u\|^2_{L^2(\mathbb{R}^d)} = \left(1 - \frac{2d}{(d-2p)(m+1)}\right)\ \|u\|^{m+1}_{L^{m+1}(\mathbb{R}^d)} \tag{E.8}$$

Since $u \geqslant 0$, both norms are non-negative. Since $d - 2p > 0$, then $p > 0$ then implies that the left-hand side is non-positive. But

$$m \geqslant \frac{d+2p}{d-2p} \qquad \Longrightarrow \qquad (m+1)(d-2p) \geqslant 2d \qquad \Longrightarrow \qquad 1 \geqslant \frac{2d}{(d-2p)(m+1)}, \tag{E.9}$$

implying that the right-hand side is non-negative. Equality can therefore only hold if $u \equiv 0$. □

Appendix F

The Implicit Function Theorem

Given Banach spaces X, Y, and Z and a function $f : X \times Y \to Z$, $(x, y) \mapsto f(x, y)$ satisfying

$$f(0,0) = 0, \tag{F.1}$$

the implicit function theorem in [Nirenberg, 2001] provides a unique solution to $(x, y[x]) \in X \times Y$ to the equation

$$f(x,y) = 0, \tag{F.2}$$

in an open neighborhood of the point $(x, y[x]) = (0, 0)$. In Theorem F.0.2, we restate and prove a version of the theorem which includes explicitly the parameter $\alpha \in \mathbb{R}$, $\alpha > 0$ and relaxes the hypotheses to require only that f and its derivative be continuous at the origin.

In Lemma F.0.13, we then use this version of the theorem to prove a general lemma which provides a unique small solution to nonlinear equations of the form

$$\mathcal{L}\ f(q) = \mathcal{R}[\alpha, f](q), \qquad q \in \mathbb{R}^d, \tag{F.3}$$

where $\mathcal{L} : L^{2,a}_{\text{even}}(\mathbb{R}^d) \longmapsto L^{2,a-2p}_{\text{even}}(\mathbb{R}^d)$, with $0 < p \leqslant 1$, is a bounded and invertible linear operator, and where $\mathcal{R}[0, 0] = 0$. In Chapters 3 and 4s, $\mathcal{L}$ is the linearization of the continuum limiting nonlinear Schrödinger equation (NLS) (L_+ or L^p_+ respectively; see Propositions 3.1.3 and 4.1.3) to which the discrete nonlinear Schrödinger equation (DNLS) is treated as a perturbation, with p corresponding to the degree of nonlocality (the fractional power of the Laplacian of NLS). Solving equations of this form provides a solution to DNLS as a perturbation of the solution to NLS.

Theorem F.0.2. *(Implicit Function Theorem) Assume the following hypotheses.*

1. *X, Y, and Z are Banach spaces.*

2. *The mapping*

$$\begin{aligned} f : [0,1] \times X \times Y &\to Z \\ (\alpha, x, y) &\mapsto f(\alpha, x, y). \end{aligned} \tag{F.4}$$

   satisfies $f(0,0,0) = 0$ and is continuous at $(0,0,0)$.

3. *For all $(\alpha, x) \in [0,1] \times X$, the mapping*

$$y \mapsto f(\alpha, x, y) \tag{F.5}$$

   is Fréchet differentiable which we denote $D_y f(\alpha, x, y) : Y \to Z$. Furthermore, the mapping

$$(\alpha, x, y) \mapsto D_y f(\alpha, x, y) \tag{F.6}$$

   is continuous at $(0,0,0)$.

4. *$D_y f(0,0,0)$ is an isomorphism of Y onto Z.*

Then there exist $\alpha_0, \delta, \kappa > 0$ such that for $(\alpha, x) \in [0, \alpha_0) \times B_\delta(0)$, there exists a unique map $y_* : [0, \alpha_0) \times B_\delta(0) \mapsto Y$ such that

$$y_*[0,0] = 0, \tag{F.7}$$

$$y_*[\alpha, x] \quad \text{is well-defined on} \quad [0, \alpha_0) \times B_\delta(0), \tag{F.8}$$

$$\|y_*[\alpha, x]\|_Y \leqslant \kappa, \tag{F.9}$$

$$\lim_{(\alpha,x)\to(0,0)} y_*[\alpha, x] = y_*[0,0] = 0, \tag{F.10}$$

$$f(\alpha, x, y_*[\alpha, x]) = 0. \tag{F.11}$$

Suppose also that

5. *For all $(\alpha, y) \in [0,1] \times Y$, the mapping*

$$x \mapsto f(\alpha, x, y) \tag{F.12}$$

Fréchet differentiable which we denote $D_x f(\alpha, x, y) : Y \to Z$. Furthermore, the mapping

$$(\alpha, x, y) \mapsto D_x f(\alpha, x, y) \tag{F.13}$$

is continuous at $(0, 0, 0)$.

Then $D_x y_*[\alpha, x] : X \to Y$ exists and is continuous.

Remark F.0.2. *The proof below follows that found in [Nirenberg, 2001]. However, we include the parameter $\alpha > 0$ explicitly and relax the hypotheses and require only that f and its derivative be continuous at the origin.*

Proof of Theorem F.0.2: Let $\mathcal{K} \equiv D_y f(0, 0, 0)$. Observe that

$$f(\alpha, x, y) = 0, \tag{F.14}$$

is equivalent to the equation

$$\mathcal{K}y = \mathcal{K}y - f(\alpha, x, y) \equiv \mathcal{G}_1(\alpha, x, y), \tag{F.15}$$

$$\mathcal{G}_1 : [0, 1] \times X \times Y \to Z. \tag{F.16}$$

Since $\mathcal{K} : Y \to Z$ is an isomorphism, it must have a bounded inverse $\mathcal{K}^{-1} : Z \to Y$ by the bounded inverse theorem, such that we may define

$$y = y - \mathcal{K}^{-1} f(\alpha, x, y) = \mathcal{K}^{-1}\mathcal{K}_1(\alpha, x, y) \equiv \mathcal{K}_2(\alpha, x, y), \tag{F.17}$$

$$\mathcal{K}_2 : [0, 1] \times X \times Y \to Y. \tag{F.18}$$

Note in particular that

$$f(0, 0, 0) = 0 \quad \iff \quad \mathcal{K}_2(0, 0, 0) = 0. \tag{F.19}$$

We seek a fixed point $y_*[\alpha, x]$ of $\mathcal{K}_2(\alpha, x, y)$. In particular, we want to show that

$$y \mapsto \mathcal{K}_2(\alpha, x, y),$$

$$B_\kappa(0) \to B_\kappa(0), \tag{F.20}$$

is a contraction which maps the ball $B_\kappa(0)$ to itself for some $\kappa > 0$.

First we show that on a subset of $[0,1] \times X \times Y$ which is restricted sufficiently close to the origin, G_2 is contracting. That is, we may choose $(\alpha, x, y_1), (\alpha, x, y_2) \in [0,1] \times X \times Y$ small enough such that we have

$$\|\mathcal{K}_2(\alpha, x, y_1) - \mathcal{K}_2(\alpha, x, y_2)\|_Y \leqslant C\|y_1 - y_2\|_Y, \qquad C < 1. \tag{F.21}$$

Observe that

$$\begin{aligned} \mathcal{K}_1(\alpha, x, y_1) - \mathcal{K}_1(\alpha, x, y_2) &= \mathcal{K}y_1 - \mathcal{K}y_2 - f(\alpha, x, y_1) + f(\alpha, x, y_2) \\ &= \mathcal{K}(y_1 - y_2) - \int_0^1 \left[D_y f\big(\alpha, x, ty_1 + (1-t)y_2\big)dt\right](y_1 - y_2) \\ &= \int_0^1 \left[\mathcal{K} - D_y f\big(\alpha, x, ty_1 + (1-t)y_2\big)dt\right](y_1 - y_2). \end{aligned} \tag{F.22}$$

Since $D_y f(\alpha, x, y)$ is continuous at $(0,0,0)$ and since $\mathcal{K} = D_y f(0,0,0)$, for any $\epsilon > 0$ we may choose some $\alpha_0, \delta, \kappa > 0$ such that for $x \in B_\delta(0)$ and $y_1, y_2 \in B_\kappa(0)$, we have

$$\|\mathcal{K} - D_y f\big(\alpha, x, ty_1 + (1-t)y_2\big)\|_{Y\to Z} \leqslant \epsilon, \tag{F.23}$$

$$\Longrightarrow$$

$$\begin{aligned} \|\mathcal{K}_2(\alpha, x, y_1) - \mathcal{K}_2(\alpha, x, y_2)\|_Y &\leqslant \|\mathcal{K}^{-1}\|_{Z\to Y}\ \|\mathcal{K}_1(\alpha, x, y_1) - \mathcal{K}_1(\alpha, x, y_2)\|_Z \\ &\leqslant\ \epsilon\ \|\mathcal{K}^{-1}\|_{Z\to Y}\|y_1 - y_2\|_Y. \end{aligned} \tag{F.24}$$

Thus, we choose

$$\epsilon \leqslant \frac{1}{2}\left(\|\mathcal{K}^{-1}\|_{Z\to Y}\right)^{-1}, \tag{F.25}$$

such that

$$\|\mathcal{K}_2(\alpha, x, y_1) - \mathcal{K}_2(\alpha, x, y_2)\|_Y \leqslant \frac{1}{2}\|y_1 - y_2\|_Y. \tag{F.26}$$

Next, fix $\alpha \in [0, \alpha_0)$, $x \in B_\delta(0)$. We seek to show that for a small enough choice of α_0, δ, and κ, the map $y \mapsto G_2(\alpha, x, y)$ sends the ball $B_\kappa(0)$ into itself, such that

$$\|y\|_Y \leqslant \kappa \quad \Longrightarrow \quad \|G_2(\alpha, x, y)\|_Y \leqslant \kappa. \tag{F.27}$$

By the continuity of f and thus $\mathcal{K}_2$ at $(0,0,0)$, we may choose α_0, δ small enough that

$$\|\mathcal{K}_2(\alpha,x,0) - \mathcal{K}_2(0,0,0)\|_Y \leqslant \frac{1}{2}\kappa. \tag{F.28}$$

Thus, by (F.26) and (F.29),

$$\begin{aligned}\|\mathcal{K}_2(\alpha,x,y)\|_Y &\leqslant \|\mathcal{K}_2(\alpha,x,y) - \mathcal{K}_2(\alpha,x,0)\|_Y + \|\mathcal{K}_2(\alpha,x,0)\|_Y \\ &= \|\mathcal{K}_2(\alpha,x,y) - \mathcal{K}_2(\alpha,x,0)\|_Y + \|\mathcal{K}_2(\alpha,x,0) - \mathcal{K}_2(0,0,0)\|_Y \\ &\leqslant \frac{1}{2}\|y\|_Y + \frac{1}{2}\kappa \leqslant \kappa. \end{aligned} \tag{F.29}$$

Thus, by (F.26) and (F.29), $\mathcal{K}_2 : B_\kappa(0) \to B_\kappa(0)$ is a contraction for $\alpha \in [0,\alpha_0)$, $x \in B_\delta(0)$, and the Banach fixed-point theorem admits a unique fixed point $y = y_*[\alpha,x]$ such that $y_*[0,0] = 0$ and

$$\mathcal{K}_2(\alpha,x,y_*[\alpha,x]) = y_*[\alpha,x] \quad \Longleftrightarrow \quad f(\alpha,x,u[\alpha,x]) = 0. \tag{F.30}$$

Furthermore, (F.26) gives that

$$\begin{aligned} &\|y_*[\alpha,x]\|_Y = \|\mathcal{K}_2(\alpha,x,y_*[\alpha,x])\|_Y \\ &\leqslant \|\mathcal{K}_2(\alpha,x,y_*[\alpha,x]) - \mathcal{K}_2(\alpha,x,0)\|_Y + \|\mathcal{K}_2(\alpha,x,0)\|_Y \\ &\leqslant \frac{1}{2}\|y_*[\alpha,x]\|_Y + \|\mathcal{K}_2(\alpha,x,0)\|_Y, \\ \Longleftrightarrow \quad &\|y_*[\alpha,x]\|_Y \leqslant 2\|\mathcal{K}_2(\alpha,x,0)\|_Y, \end{aligned} \tag{F.31}$$

such that the continuity of $\mathcal{K}_2$ at the origin implies that

$$\lim_{(\alpha,x)\to(0,0)} y_*[\alpha,x] = y_*[0,0] = \lim_{(\alpha,x)\to(0,0)} G(\alpha,x,0) = 0. \tag{F.32}$$

□

We now use Theorem F.0.2 to prove a lemma which characterizes the solution to equation (F.3) and which we use in Chapters 3 and 4.

Lemma F.0.13. *Let $d \geqslant 1$ and consider the equation*

$$\mathcal{J}[\alpha,f] \equiv \mathfrak{L}f - \mathcal{R}[\alpha,f] = 0, \quad f \in L^{2,a}_{\text{even}}(\mathbb{R}^d). \tag{F.33}$$

1. $\mathfrak{L} : L^{2,a}_{\text{even}}(\mathbb{R}^d) \longmapsto L^{2,a-2p}_{\text{even}}(\mathbb{R}^d)$ *be an isomorphism for* $0 < p \leqslant 1$

2. $\mathcal{R} : [0, \alpha_1)_\alpha \times L^{2,a}_{\text{even}}(\mathbb{R}^d) \longrightarrow L^{2,a-2p}_{\text{even}}(\mathbb{R}^d)$ *is continuous at* $(0,0)$*, Fréchet differentiable on* $L^{2,a}_{\text{even}}(\mathbb{R}^d)$

3. $\mathcal{R}[0,0] = 0$ *and satisfies the bounds:*

$$\|\mathcal{R}[\alpha, f]\|_{L^{2,a-2p}(\mathbb{R}^d)} \lesssim \mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R}^d)} + \|f\|^2_{L^{2,a}(\mathbb{R}^d)} + \|f\|^3_{L^{2,a}(\mathbb{R}^d)}, \tag{F.34}$$

$$\|D_f\mathcal{R}[\alpha, f]\|_{L^{2,a}(\mathbb{R}^d) \to L^{2,a-2p}(\mathbb{R}^d)} \lesssim \mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R}^d)} + \|f\|^2_{L^{2,a}(\mathbb{R}^d)}, \tag{F.35}$$

for some continuous function $\mathfrak{K}(\alpha) \geqslant 0$*, satisfying* $\mathfrak{K}(0) = 0$*,* $\mathfrak{K}(\alpha) \to 0$ *as* $\alpha \to 0$*.*

Then there exists a constant $\alpha_2 \leqslant \alpha_1$ *such that for all* $0 < \alpha < \alpha_2$*, the equation* (F.33)*,* $\mathcal{M}[\alpha, f] = 0$*, has a unique solution* $f = f[\alpha] \in L^{2,a}_{\text{even}}(\mathbb{R}^d)$ *satisfying*

$$\|f[\alpha]\|_{L^{2,a}(\mathbb{R}^d)} \lesssim \mathfrak{K}(\alpha). \tag{F.36}$$

Proof of Lemma F.0.13: To prove the lemma, we apply the implicit function theorem. Define the map $\mathcal{J} : \mathbb{R}_+ \times L^{2,a}_{\text{even}}(\mathbb{R}^d) \longrightarrow L^{2,a-2p}_{\text{even}}(\mathbb{R})$:

$$\mathcal{J}[\alpha, f] \equiv \mathfrak{L}f - \mathcal{R}[\alpha, f]. \tag{F.37}$$

Proposition F.0.6. *(Properties of* $\mathcal{J}$*)*

1. *The mapping* $(\alpha, f) \longmapsto \mathcal{J}[\alpha, f]$*,* $\mathcal{J} : \mathbb{R}_+ \times L^{2,a}_{\text{even}}(\mathbb{R}^d) \longmapsto L^{2,a-2p}_{\text{even}}(\mathbb{R}^d)$ *is continuous at* $(0,0)$*.*

2. *For* $\alpha \neq 0$*,* $\mathcal{J}$ *is Fréchet differentiable with respect to* f *with*

$$D_f\mathcal{J}[\alpha, f] = \mathfrak{L} - D_f\mathcal{R}[\alpha, f], \tag{F.38}$$

which is continuous at $(0,0)$ *in* $\mathbb{R}_+ \times L^{2,a}_{\text{even}}(\mathbb{R}^d)$*.*

3. $\mathcal{J}[0,0] = 0$ *in* $L^{2,a-2p}_{\text{even}}(\mathbb{R}^d)$*.*

4. $D_f\mathcal{A}[0,0] = \mathfrak{L}$ *is an isomorphism of* $L^{2,a}_{\text{even}}(\mathbb{R}^d)$ *onto* $L^{2,a-2p}_{\text{even}}(\mathbb{R})$*.*

Proof of Proposition F.0.6: Since $\mathfrak{L}$ and $\mathcal{R}$ are continuous at $(0,0)$, so must be $\mathcal{J}$. Similarly, it must also be Fréchet differentiable on $L^{2,a}_{\text{even}}(\mathbb{R}^d)$ such that

$$D_f\mathcal{J}[\alpha, f] \equiv \mathfrak{L} - D_f\mathcal{R}[\alpha, f]. \tag{F.39}$$

The boundedness of $\mathfrak{L}$ and estimate (F.34) give

$$\begin{aligned}
&\|\mathcal{J}[\alpha, f]\|_{L^{2,a-2p}(\mathbb{R}^d)} \leqslant \|\mathfrak{L}f\|_{L^{2,a-2p}(\mathbb{R}^d)} + \|\mathcal{R}[\alpha, f]\|_{L^{2,a-2p}(\mathbb{R}^d)} \\
&\leqslant \|\mathfrak{L}\|_{L^{2,a}(\mathbb{R}^d)\to L^{2,a-2p}(\mathbb{R}^d)}\|f\|_{L^{2,a}(\mathbb{R}^d)} + \|\mathcal{R}[\alpha, f]\|_{L^{2,a-2p}(\mathbb{R}^d)} \\
&\leqslant C\left(\mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R}^d)} + \|f\|^2_{L^{2,a}(\mathbb{R}^d)} + \|f\|^3_{L^{2,a}(\mathbb{R}^d)}\right), \qquad \text{(F.40)}
\end{aligned}$$

which in turn implies that $\mathcal{J}[0, 0] = 0$. Similarly, estimate $F.35$ implies that

$$\begin{aligned}
&\|D_f\mathcal{J}[\alpha, f] - \mathfrak{L}\|_{L^{2,a}(\mathbb{R}^d)\to L^{2,a-2p}(\mathbb{R}^d)} = \|D_f\mathcal{R}[\alpha, f]\|_{L^{2,a}(\mathbb{R}^d)\to L^{2,a-2p}(\mathbb{R}^d)} \\
&\lesssim\ \mathfrak{K}(\alpha) + \|f\|_{L^{2,a}(\mathbb{R}^d)} + \|f\|^2_{L^{2,a}(\mathbb{R}^d)}. \qquad \text{(F.41)}
\end{aligned}$$

Thus, $D_f\mathcal{J}[0, 0] = \mathfrak{L}$. Since $\mathfrak{L}$ has a kernel spanned by odd functions, it must have a bounded inverse from $L^{2,a-2p}_{\text{even}}(\mathbb{R}^d)$ to $L^{2,a}_{\text{even}}(\mathbb{R}^d)$. □

Proposition F.0.6 implies that $\mathcal{J}$ satisfies the hypotheses of the implicit function theorem as given in Theorem F.0.2. Thus, there exists some $\alpha_1 < \alpha_0$ such that for all $0 < \alpha < \alpha_1$, there exists a mapping $\alpha \longmapsto f[\alpha]$, $f : \mathbb{R}_+ \longmapsto L^{2,a}_{\text{even}}(\mathbb{R}^d)$ which solves

$$\begin{aligned}
&\mathcal{J}\big[\alpha, f[\alpha]\big] = 0 \\
&\Longleftrightarrow \mathfrak{L}f[\alpha] = \mathcal{R}\big[\alpha, f[\alpha]\big]. \qquad \text{(F.42)}
\end{aligned}$$

We now proceed to prove estimate (F.36). We define the operator

$$\mathcal{K}[\alpha, f] \equiv f - \mathfrak{L}^{-1}\mathcal{J}[\alpha, f] = \mathfrak{L}^{-1}\left(\mathfrak{L}f - \mathcal{J}[\alpha, f]\right). \qquad \text{(F.43)}$$

First, observe that $\mathcal{K}[\alpha, f]$ is a contraction if we choose

$$\|f\|_{L^{2,a}(\mathbb{R}^d)} \leqslant 2C\mathfrak{K}(\alpha), \qquad \text{(F.44)}$$

since for two functions $f_1, f_2 \in L^{2,a}_{\text{even}}(\mathbb{R}^d)$, with

$$\|f_1\| \leqslant 2C\mathfrak{K}(\alpha), \quad \|f_2\| \leqslant 2C\mathfrak{K}(\alpha), \qquad \text{(F.45)}$$

we have

$$\mathcal{K}[f_1,\alpha]-\mathcal{K}[f_2,\alpha]=\mathfrak{L}^{-1}\Big[\mathfrak{L}\Big(f_1-f_2\Big)-\Big(\mathcal{J}[f_1,\alpha]+\mathcal{J}[f_2,\alpha]\Big)\Big]$$
$$=\mathfrak{L}^{-1}\left[\mathfrak{L}\Big(f_1-f_2\Big)-\left(\int_0^1 D_f\mathcal{J}\left[tf_1+(1-t)f_2,\alpha\right]dt\right)\Big(f_1-f_2\Big)\right]$$
$$=\mathfrak{L}^{-1}\left[\mathfrak{L}-\int_0^1 D_f\mathcal{J}\left[tf_1+(1-t)f_2,\alpha\right]dt\right]\Big(f_1-f_2\Big)$$
$$=\mathfrak{L}^{-1}\left[\int_0^1 \mathfrak{L}-D_f\mathcal{J}\left[tf_1+(1-t)f_2,\alpha\right]dt\right]\Big(f_1-f_2\Big)$$
$$=\mathfrak{L}^{-1}\left[\int_0^1 D_f\mathcal{R}\left[tf_1+(1-t)f_2,\alpha\right]dt\right]\Big(f_1-f_2\Big), \tag{F.46}$$

and in turn by the hypothesis (F.35),

$$\|\mathcal{K}[f_1,\alpha]-\mathcal{K}[f_2,\alpha]\|_{L^{2,a}(\mathbb{R}^d)}$$
$$\leqslant \|\mathfrak{L}^{-1}\|_{L^{2,a-2p}(\mathbb{R}^d)\mapsto L^{2,a}(\mathbb{R}^d)}\|D_f\mathcal{R}[f_1+f_2,\alpha]\|_{L^{2,a}(\mathbb{R}^d)\mapsto L^{2,a-2p}(\mathbb{R}^d)}\|f_1-f_2\|_{L^{2,a}(\mathbb{R}^d)}$$
$$\leqslant C\Big(\mathfrak{K}(\alpha)+\|f_1\|_{L^{2,a}(\mathbb{R}^d)}+\|f_2\|_{L^{2,a}(\mathbb{R}^d)}+\|f_1\|^2_{L^{2,a}(\mathbb{R}^d)}+\|f_2\|^2_{L^{2,a}(\mathbb{R}^d)}\Big)\|f_1-f_2\|_{L^{2,a}(\mathbb{R}^d)}$$
$$\leqslant C\Big(\mathfrak{K}(\alpha)+4\mathfrak{K}(\alpha)+4\mathfrak{K}(\alpha)^2\Big)\|f_1-f_2\|_{L^{2,a}(\mathbb{R}^d)}. \tag{F.47}$$

Thus, $\mathcal{K}$ is a contraction for α sufficiently small on the ball of radius $2C\mathfrak{K}(\alpha)$ in $L^{2,a}_{\text{even}}(\mathbb{R}^d)$.

Next, observe that by (F.47) and (F.34),

$$\|\mathcal{K}[\alpha,f]\|_{L^{2,a}(\mathbb{R}^d)}=\|\mathcal{K}[\alpha,f]-\mathcal{K}[0,\alpha]+\mathcal{K}[0,\alpha]\|_{L^{2,a}(\mathbb{R}^d)}$$
$$\leqslant \|\mathcal{K}[\alpha,f]-\mathcal{K}[0,\alpha]\|_{L^{2,a}(\mathbb{R}^d)}\ +\ \|\mathcal{K}[0,\alpha]\|_{L^{2,a}(\mathbb{R}^d)}$$
$$=\|\mathcal{K}[\alpha,f]-\mathcal{K}[0,\alpha]\|_{L^{2,a}(\mathbb{R}^d)}\ +\ \|\mathcal{R}[0,\alpha]\|_{L^{2,a}(\mathbb{R}^d)}$$
$$\leqslant C\mathfrak{K}(\alpha)+C\Big(5\mathfrak{K}(\alpha)^2\Big)\|f\|_{L^{2,a}(\mathbb{R}^d)}$$
$$\leqslant C\Big(\mathfrak{K}(\alpha)+10\mathfrak{e}_1(\alpha)^2+8\mathfrak{K}(\alpha)^3\Big)\leqslant 2C\mathfrak{K}(\alpha). \tag{F.48}$$

for α sufficiently small, so $\mathcal{K}$ is a contraction which maps the ball of radius $2C\mathfrak{K}(\alpha)$ to itself.

Thus, there exists a unique solution $f\in L^{2,a}(\mathbb{R}^d)$ to equation F.33:

$$\mathcal{J}[\alpha,f]=\mathfrak{L}f-\mathcal{R}\left[\alpha,f\right]=0, \tag{F.49}$$

such that

$$\|f\|_{L^{2,a}(\mathbb{R}^d)} \lesssim \mathfrak{e}_1(\alpha). \tag{F.50}$$

This completes the proof of Lemma F.0.13. □

Appendix G

Special Functions $\mathrm{Li}_s(z)$, $\zeta(z)$, $\Gamma(z)$, and the Dispersion Relation $M^s(q)$ of $-\mathcal{L}^s$

This section contains key calculations for the proofs of Proposition 4.3.3 and Lemma 4.5.4 on the asymptotics of the scaled dispersion relation $M_\alpha^s(Q)\ :\ M_\alpha^s(Q) \simeq |Q|^{2p}$ as $\alpha \to 0$ for $p = \min(1, s)$ and $0 < s \leqslant \infty$. The key result is given in Proposition G.0.9.

Recall that $M^s(q)$ is the dispersion relation for the operator $-\mathcal{L}^s$. Recall from Lemma D.0.12 that $M^s(q) = \frac{1}{C_s}\sum_{m=1}^{\infty} \frac{4\sin^2(qm/2)}{m^{1+2s}}$ for $0 < s < \infty$ and $M^s(q) = \frac{1}{C_\infty}\sum_{m=1}^{\infty} 4e^{-\gamma m}\sin^2(qm/2),\ \gamma > 0$, for $s = \infty$. As noted in Section 4.1.2, the factor C_s and the α-dependent rescalings are introduced so that the asymptotic coefficient of $|Q|^{2p}$ is one.

We introduce the polylogarithm:

$$\mathrm{Li}_s(z) \equiv \sum_{m=1}^{\infty} \frac{z^m}{m^s}. \tag{G.1}$$

for $z \in \mathbb{C}$, $s \in \mathbb{R}$. Observe in particular that for cases in which $|z| = 1$ and $s > 1$, the series $\mathrm{Li}_s(z)$ clearly converges. We also make use of the ζ and Γ functions, defined to be the analytic continuation on $z \in \mathbb{C}$ of the functions defined by

$$\zeta(z) = \sum_{j=1}^{\infty} \frac{1}{m^z}, \quad \text{for } \operatorname{Re} z > 1, \qquad \Gamma(z) = \int_0^\infty x^{z-1}\, e^{-x} dx \quad \text{for } \operatorname{Re} z > 0. \tag{G.2}$$

The following two propositions ensure that the ζ and Γ functions are well-behaved in all of their appearances in this thesis, in the sense that they are not singular or do not grow asymptotically when they appear in infinite series.

Proposition G.0.7 (Properties of the Γ function on $\mathbb{R}$, [Abramowitz and Stegun, 1972])**.**

1. *For* $z \in \mathbb{R}$ *fixed with* $z > 0$, $\Gamma(z)$ *is continuous and finite.*

2. *(Euler's reflection formula) For any* $z \notin \mathbb{Z}$*, we have* $\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$.

Proposition G.0.8 (Properties of the ζ function on $\mathbb{R}$, [Abramowitz and Stegun, 1972; Edwards, 2001])**.**

1. *For* $z \in \mathbb{R}$ *fixed with* $z \neq 1$, $\zeta(z)$ *is continuous and finite.*

2. *(Trivial zeroes) Let* $z = -2n$ *for* $n \in \mathbb{N}$*. Then* $\zeta(z) = 0$*.*

We require the following lemmata which express the polylogarithm of an exponential function.

Lemma G.0.14. *[Wood, 1992] Let* $s \notin \mathbb{N}$, $s > 1$*. Let* $|\mu| < 2\pi$*. Then*

$$\mathrm{Li}_s\left(e^{\mu}\right) = \Gamma(1-s)(-\mu)^{s-1} + \sum_{j=0}^{\infty} \frac{\zeta(s-j)}{j!}\mu^j. \tag{G.3}$$

Lemma G.0.15. *[Wood, 1992] Let* $s \in \mathbb{N}$, $s > 1$*. Let* $|\mu| < 2\pi$*. Then*

$$\mathrm{Li}_s\left(e^{\mu}\right) = \frac{\mu^{s-1}}{(s-1)!}\left[\sum_{j=1}^{s} \frac{1}{j} - \log(-\mu)\right] + \sum_{\substack{j=0 \\ j\neq s-1}}^{\infty} \frac{\zeta(s-j)}{j!}\mu^j. \tag{G.4}$$

We now use Proposition G.0.7 through Lemma G.0.15 to prove the following result on the asymptotics of the scaled dispersion relation $M^s_\alpha(Q)$.

Proposition G.0.9. *Let* $Q \in \mathcal{B}_\alpha = \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$*, let* $C_s > 0$ *be defined in* (4.33)*, and let* $\kappa_s(\alpha) > 0$ *be defined in* (4.23)*. Then there exists a function* $f_s(Q;\alpha)$ *such that* $|f_s(Q;\alpha)| \lesssim 1$ *and*

$$M^s_\alpha(Q) = \frac{M^s(Q\alpha)}{\kappa_s(\alpha)} = |Q|^{2p} + \begin{cases} \alpha^{2-2s}\ f_s(Q;\alpha)\ |Q|^2 & : \quad s<1 \\ \frac{1}{-\log(\alpha)}\ f_s(Q;\alpha)\ \left(\frac{3}{2} - \log(|Q|)\right)\ |Q|^2 & : \quad s=1 \\ \alpha^{2s-2}\ f_s(Q;\alpha)\ |Q|^{2s} & : \quad 1<s<2 \\ (-\log(\alpha))\ \alpha^2\ f_s(Q;\alpha)\ |Q|^4 & : \quad s=2 \\ \alpha^2\ f_s(Q;\alpha)\ |Q|^4 & : \quad 2<s\leqslant\infty \end{cases}. \tag{G.5}$$

Proof of Proposition G.0.9: Let $q = Q\alpha \in \mathcal{B} = [-\pi, \pi]$. Recall that $M^s(q) = \frac{1}{C_s}\sum_{m=1}^{\infty} \frac{4\sin^2(qm/2)}{m^{1+2s}}$ for $0 < s < \infty$ and $M^s(q) = \frac{1}{C_\infty}\sum_{m=1}^{\infty} 4e^{-\gamma m}\sin^2(qm/2),\ \gamma > 0$, for $s = \infty$. Here, $C_s > 0$ is to be determined.

Case 1: Let $1 + 2s \notin \mathbb{N},\ s \neq \infty$. Expand $4\sin^2(qm/2) = 2 - e^{iqm} - e^{-iqm}$, and note that since $|q| \leqslant \pi < 2\pi$, we may apply Lemma G.0.14:

$$\begin{aligned}
C_s\ M^s(q) = 4\sum_{m=1}^{\infty}\frac{\sin^2(qm/2)}{m^{1+2s}} &= \sum_{m=1}^{\infty}\frac{2}{m^{1+2s}} - \sum_{m=1}^{\infty}\frac{e^{iqm}}{m^{1+2s}} - \sum_{m=1}^{\infty}\frac{e^{iqm}}{m^{1+2s}} \\
&= 2\zeta(1+2s) - \mathrm{Li}_{1+2s}(e^{iq}) - \mathrm{Li}_{1+2s}(e^{-iq}) \\
&= 2\zeta(1+2s) - \Gamma(-2s)(-iq)^{2s} - \sum_{j=0}^{\infty}\frac{\zeta(1+2s-j)}{j!}(-iq)^j \\
&\qquad - \Gamma(-2s)(iq)^{2s} - \sum_{j=0}^{\infty}\frac{\zeta(1+2s-j)}{j!}(iq)^j \\
&= -\Gamma(-2s)\ (e^{-i\pi s} + e^{i\pi s})\ |q|^{2s} - \sum_{j=1}^{\infty}\frac{\zeta(1+2s-j)}{j!}\ \left[(-i)^j + i^j\right]\ q^j \\
&= -2\Gamma(-2s)\ \cos(\pi s)\ |q|^{2s} + 2\sum_{j=1}^{\infty}\frac{\zeta(1+2s-2j)}{(2j)!}\ (-1)^{j+1}\ |q|^{2j}.
\end{aligned} \tag{G.6}$$

Applying the rescaling $q = Q\alpha$ gives

$$\begin{aligned}
M_\alpha^s(Q) &= \frac{1}{\kappa_s(\alpha)}M^s(Q\alpha) \\
&= -\frac{2\Gamma(-2s)\ \cos(\pi s)}{C_s\ \kappa_s(\alpha)}\ \alpha^{2s}\ |Q|^{2s} + \sum_{j=1}^{\infty}\frac{\zeta(1+2s-2j)}{C_s\ \kappa_s(\alpha)\ (2j)!}\ (-1)^{j+1}\alpha^{2j}\ |Q|^{2j}.
\end{aligned} \tag{G.7}$$

The series is absolutely convergent on $Q = q/\alpha \in \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$ due to Proposition G.0.8. Taking $C_s = -2\Gamma(-2s)\ \cos(\pi s)$, $\kappa_s(\alpha) = \alpha^{2s}$ for $s < 1$ and $C_s = \zeta(2s-1),\quad \kappa_s(\alpha) = \alpha^2$ for $s > 1$ gives (G.5) for $1 + 2s \notin \mathbb{N}, s \neq \infty$.

Case 2: Let $1 + 2s \in \mathbb{N},\ s \neq \infty$. Again expand $4\sin^2(qm/2) = 2 - e^{iqm} - e^{-iqm}$, and note that since $|q| \leqslant \pi < 2\pi$, we may apply Lemma G.0.15:

$$\begin{aligned}
M^s(q) &= 4\sum_{m=1}^{\infty} \frac{\sin^2(qm/2)}{m^{1+2s}} = \sum_{m=1}^{\infty} \frac{2}{m^{1+2s}} + -\sum_{m=1}^{\infty} \frac{e^{iqm}}{m^{1+2s}} - \sum_{m=1}^{\infty} \frac{e^{iqm}}{m^{1+2s}} \\
&= 2\zeta(1+2s) - \mathrm{Li}_{1+2s}(e^{iq}) - \mathrm{Li}_{1+2s}(e^{-iq}) \\
&= 2\zeta(1+2s) - \frac{(iq)^{2s}}{(2s)!}\left[\sum_{j=1}^{2s} \frac{1}{j} - \log(-iq)\right] - \sum_{\substack{j=0 \\ j\neq 2s}}^{\infty} \frac{\zeta(1+2s-j)}{j!}(iq)^j \\
&\qquad - \frac{(-iq)^{2s}}{(2s)!}\left[\sum_{j=1}^{2s} \frac{1}{j} - \log(iq)\right] - \sum_{\substack{j=0 \\ j\neq 2s}}^{\infty} \frac{\zeta(1+2s-j)}{j!}(-iq)^j \\
&= -\left(\frac{e^{-i\pi s} + e^{i\pi s}}{(2s)!}\ \left(\sum_{j=1}^{2s} \frac{1}{j}\right) - \frac{e^{i\pi s}}{(2s)!}\ \left[\log(|q|) - \frac{\pi i}{2}\right] - \frac{e^{-i\pi s}}{(2s)!}\ \left[\log(|q|) + \frac{\pi i}{2}\right]\right)\ |q|^{2s} \\
&\qquad - \sum_{\substack{j=1 \\ j\neq 2s}}^{\infty} \frac{\zeta(1+2s-j)}{j!}\left[(-i)^j + i^j\right] q^j \\
&= \frac{2\cos(\pi s)}{(2s)!}\left[-\left(\sum_{j=1}^{2s} \frac{1}{j}\right) + \log(|q|)\right]\ |q|^{2s} + \frac{\pi\ \cos(\pi(s-1/2))}{(2s)!}\ |q|^{2s} \\
&\qquad + 2\sum_{\substack{j=1 \\ j\neq s}}^{\infty} \frac{\zeta(1+2s-2j)}{(2j)!}\ (-1)^{j+1}\ |q|^{2j}.
\end{aligned} \tag{G.8}$$

Applying the rescaling $q = Q\alpha$ gives

$$\begin{aligned}
M^s_\alpha(Q) &= \frac{1}{\kappa_s(\alpha)} M^s(Q\alpha) \\
&= \frac{2\cos(\pi s)}{C_s\ \kappa_s(\alpha)\ (2s)!}\left[-\left(\sum_{j=1}^{2s} \frac{1}{j}\right) + \log(\alpha) + \log(|Q|)\right]\ \alpha^{2s}\ |Q|^{2s} \\
&+ \frac{\pi\ \cos(\pi(s-1/2))}{C_s\ \kappa_s(\alpha)\ (2s)!}\alpha^{2s}\ |Q|^{2s} + 2\sum_{\substack{j=1 \\ j\neq s}}^{\infty} \frac{\zeta(1+2s-2j)}{C_s\ \kappa_s(\alpha)\ (2j)!}\ (-1)^{j+1}\alpha^{2j}\ |Q|^{2j}.
\end{aligned} \tag{G.9}$$

The infinite series is absolutely convergent on $Q = q/\alpha \in \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$ due to Proposition G.0.8. Let $C_s = -2\Gamma(-2s)\cos(\pi s)$, $\kappa_s(\alpha) = \alpha^{2s}$ for $s = 1/2$ and $C_s = \zeta(2s-1)$, $\quad \kappa_s(\alpha) = \alpha^2$ for $s > 1$. For $s = 1$, take $C_s = 1$ and $\kappa_s(\alpha) = (-\log(\alpha))\ \alpha^2$. This gives (G.5) for $1 + 2s \in \mathbb{N}, s \neq \infty$.

Remark G.0.3. *Note that one of the first two terms in* (G.9) *will be zero, depending on whether* $s \in \mathbb{N}$ *or* $s = (2k+1)/2$ *for* $k \in \mathbb{N}$.

Furthermore, in the case where $s = (2k+1)/2$ *for* $k \in \mathbb{N}$, *the series in* (G.9) *is finite. To see this, observe from Proposition G.0.8 that the "trivial zeros" of the zeta function on the real line lie occur at negative even integers. In the series above, this occurs when* $1+2s-2j \leqslant -2 \implies j \geqslant s+3/2$.

We also note that for $s = (2k+1)/2$ *for* $k \in \mathbb{N}$, *the expansion* (G.9) *is compatible with the expansion* (G.7) *where we assumed that* $1 + 2s \notin \mathbb{N}$. *To see this we use Euler's reflection formula from Proposition G.0.7 to get, for* $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{s\to\frac{2k-1}{2}} -2\,\Gamma(-2s)\,\cos(\pi s) &= \lim_{s\to\frac{2k-1}{2}} \frac{-2\pi\,\cos(\pi s)}{\sin([2s+1]\pi)\,\Gamma(2s+1)} = \lim_{s\to\frac{2k-1}{2}} \frac{2\pi\cos(\pi s)}{\sin(2\pi s)\,\Gamma(2s+1)}\\ &= \lim_{s\to\frac{2k-1}{2}} \frac{\pi}{\sin(\pi s)\,\Gamma(2s+1)} = \frac{\pi}{\sin\left(\pi(k-1/2)\right)\,\Gamma(2k)}\\ &= \frac{\pi(-1)^{k+1}}{(2k-1)!} = \left.\frac{\pi(-1)^{s-1/2}}{(2s)!}\right|_{s=\frac{2k-1}{2}}. \end{aligned} \tag{G.10}$$

Case 3: Finally, let $s = \infty$. We have $M^s(q) = \frac{1}{C_\infty}\sum_{m=1}^{\infty} 4e^{-\gamma m}\sin^2(qm/2)$, $\gamma > 0$ and $q \in [-\pi,\pi]$. Note that $\partial_q^4 \sin^2(qm/2) = -\frac{1}{2}\,m^4\,\cos(qm)$. Expanding about $q = 0$, Taylor's Theorem implies that there exists $z \in [-\pi,\pi]$ such that $4\sin^2(qm/2) = m^2\,|q|^2 - 2\,m^4\,\cos(zm)\,|q|^4$. on $q \in [-\pi,\pi]$. In turn,

$$M^s(q) = \frac{1}{C_\infty}\sum_{m=1}^{\infty} 4e^{-\gamma m}\sin^2(qm/2) = \frac{1}{C_s}\sum_{m=1}^{\infty} m^2\,e^{-\gamma m}\,|q|^2 - \frac{2}{C_s}\sum_{m=1}^{\infty} m^4\,\cos(zm)\,|q|^4. \tag{G.11}$$

Therefore, for $Q \in \left[-\frac{\pi}{\alpha},\frac{\pi}{\alpha}\right]$ and $M^s_\alpha(Q) = \frac{1}{\kappa_s(\alpha)}M^s(Q\alpha)$,

$$\begin{aligned} M^s_\alpha(Q) = \frac{1}{C_s\,\kappa_s(\alpha)}&\left(\sum_{m=1}^{\infty} m^2\,e^{-\gamma m}\right)\alpha^2\,|Q|^2\\ &- \frac{2}{C_s\,\kappa_s(\alpha)}\sum_{m=1}^{\infty} m^4\,\cos(zm)\,e^{-\gamma m}\,\alpha^4\,|Q|^4. \end{aligned} \tag{G.12}$$

Observe that

$$\begin{aligned}\left(\sum_{m=1}^{\infty} m^2\,e^{-\gamma m}\right) &= \frac{\exp(\gamma)(\exp(\gamma)+1)}{(\exp(\gamma)-1)^3} = C_\infty,\\ \left|\sum_{m=1}^{\infty} m^4\,\cos(zm)\,e^{-\gamma m}\right| &\leqslant \sum_{m=1}^{\infty} m^4\,e^{-\gamma m} \lesssim 1. \end{aligned} \tag{G.13}$$

Taking $\kappa_s(\alpha) = \alpha^2$ gives (G.5) for $s = \infty$. This completes the proof of Proposition G.0.9. □

Appendix H

Exponential Decay of Solitary Waves in Fourier Space

We are interested in bounding solutions to the following two equations:

1. The Fourier transform of continuum FNLS for $d = 1, 2, 3$ and frequency $\omega = -1$, (4.28)

$$\left[1 + |Q|^{2p}\right] \; \widetilde{\psi}(Q) - \frac{1}{4\pi^2} \widetilde{\psi} * \widetilde{\psi} * \widetilde{\psi}(Q) = 0, \qquad Q \in \mathbb{R}^d. \tag{H.1}$$

   Here, $K = 1$, $C_1 \equiv -1/4\pi^2$, $\tau_1 \equiv 0$, $A \equiv \mathbb{R}^d$, and $M(Q) \equiv |Q|^{2p}$.

2. The rescaled Fourier transform of DNLS localized to the Brillouin zone for $d = 1$, (4.58):

$$[1 + M^s_\alpha(Q)] \; \widehat{\Phi}(Q) - \frac{\chi_{\mathcal{B}_\alpha}(Q)}{4\pi^2} \sum_{m=-1}^{1} e^{2\pi i \sigma m} \; \widehat{\Phi} * \widehat{\Phi} * \widehat{\Phi}(Q - 2m\pi/\alpha) = 0, \qquad Q \in \mathbb{R}. \tag{H.2}$$

   Here, $K = 3$, $C_k \equiv (-1/4\pi^2) e^{2\pi i \sigma m_k}$ for $m_k \in \{-1, 0, 1\}$, $\tau_k \equiv 2m_k\pi/\alpha$, $A \equiv \mathcal{B}_\alpha$, and $M(Q) \equiv M^s_\alpha(Q) = \frac{4}{\kappa_s(\alpha)} \sum_{m=1}^\infty \frac{\sin^2(Qm\alpha/2)}{m^{1+2s}}$.

In this section, we prove the exponential decay of solutions to an equation of the general form,

$$[1 + M(Q)] \; \Phi(Q) + \chi_A(q) \sum_{k=1}^{K} C_k \; \Phi * \Phi * \Phi(Q + \tau_k) = 0. \tag{H.3}$$

where $0 < \eta \leqslant 1$, $K \geqslant 1$ is an integer, and $C_k \in \mathbb{R}$, $\tau_k \in \mathbb{R}^d$ are constants for $1 \leqslant k \leqslant K$, $A \subset \mathbb{R}^d$ and $M(Q)$ is a continuous function which satisfies, for some constants $0 < \eta \leqslant 1$ and $D_M > 0$,

$$\frac{|Q|^\eta}{1 + M(Q)} \leqslant D_M, \qquad \text{and} \qquad |Q| \leqslant |Q + \tau_k|, \qquad \text{for} \qquad Q \in A. \tag{H.4}$$

We obtain the exponential decay

$$\|e^{\mu|Q|^\eta}\Phi(Q)\|_{L^{2,a}(\mathbb{R}_Q)} \lesssim \|\Phi\|_{L^{2,a}(\mathbb{R})}, \qquad \mu > 0. \tag{H.5}$$

We follow an approach similar to that given in [Frank and Lenzmann, 2010] and motivated by the approach to proving exponential decay estimates in [Bona and Li, 1997]. In particular, we make use of the following identity.

Lemma H.0.16. *(Abel's identity [Riordan, 1968]) For any constants $c_1, c_2 \neq 0$, we have*

$$\sum_{x=0}^{y} \begin{pmatrix} y \\ x \end{pmatrix} (x+c_1)^{x-1}\ (y-x+c_2)^{y-x-1} = \frac{c_1+c_2}{c_1 c_2}(y+c_1+c_2)^{y-1}. \tag{H.6}$$

Remark H.0.4. *We use $\|fg\|_{H^a(\mathbb{R}^d)} \lesssim \|f\|_{H^a(\mathbb{R}^d)}\ \|f\|_{H^a(\mathbb{R}^d)}$ since we seek to show $\|\widehat{\Phi^{\text{off}}} - \widehat{\Phi^{\text{on}}}\|_{L^2(\mathcal{B})} \lesssim \|\widehat{\Phi^{\text{off}}} - \widehat{\Phi^{\text{on}}}\|_{L^{2,a}(\mathbb{R}^d)}$. In [Frank and Lenzmann, 2010], Frank et. al. obtain the equivalent result in $L^1(\mathbb{R})$ using Young's inequality: $\|f_1 * f_2\|_{L^1(\mathbb{R})} \leqslant \|f_1\|_{L^1(\mathbb{R})}\ \|f_2\|_{L^1(\mathbb{R})}$. We may also perform our analysis in $L^1(\mathbb{R})$.*

We seek to show $\Phi(Q) \sim e^{-\mu|Q|^\eta},\ Q \in \mathbb{R}^d,\ \mu > 0$. To prove this, we will estimate the moments of Φ, given by $|Q|^{j\eta}\Phi(q)$, $j \in \mathbb{N}$, inductively in $L^{2,a}(\mathbb{R}^d)$, and then use these estimates to bound the Taylor expansion of $e^{\mu|Q|}$ for an appropriate choice of $C > 0$.

Lemma H.0.17. *Let $a > d/2$, $0 < \eta \leqslant 1$, $K \geqslant 1$ be an integer, and $C_k \in \mathbb{R},\ \tau_k \in \mathbb{R}^d$ be constants for $1 \leqslant k \leqslant K$. Let $A \subset \mathbb{R}^d$ and $M(Q) \geqslant 0$ be a continuous function which satisfy, for some constant $D_M > 0$,*

$$\frac{|Q|^\eta}{1+M(Q)} \leqslant D_M, \qquad \text{and} \qquad |Q| \leqslant |Q+\tau_k|, \qquad \text{for} \qquad Q \in A. \tag{H.7}$$

Let $\Phi \in L^{2,a}(\mathbb{R}^d)$ be the solution to

$$[1+M(Q)]\ \Phi(Q) + \chi_A(Q) \sum_{k=1}^{K} C_k\ \Phi * \Phi * \Phi(q+\tau_k) = 0. \tag{H.8}$$

Then there exists a constant $\mu = \mu\left(\|\Phi\|_{L^{2,a}(\mathbb{R}^d)}\right) > 0$ such that

$$\left\|e^{\mu|Q|^\eta}\ \Phi(Q)\right\|_{L^{2,a}(\mathbb{R}^d_Q)} \lesssim \|\Phi\|_{L^{2,a}(\mathbb{R}^d)}. \tag{H.9}$$

Proof of Lemma H.0.17: Our strategy is to estimate the moments $|Q|^{j\eta}\ \Phi(Q)$, $j \geqslant 0$. Observe that by Taylor expansion, for any $\mu > 0$,

$$\left\| e^{\mu |Q|^{\eta}}\ \Phi(Q) \right\|_{L^{2,a}(\mathbb{R}^d_Q)} \leqslant \sum_{j=0}^{\infty} \left\| \frac{\mu^j}{j!}\ |Q|^{j\eta}\ \Phi(Q)\ \right\|_{L^{2,a}(\mathbb{R}^d_Q)}. \tag{H.10}$$

We then have the following proposition.

Proposition H.0.10. *There exists a constant* $b = b\left(\|\Phi\|_{L^{2,a}(\mathbb{R}^d)} \right) > 0$ *such that* $\Phi \in L^{2,a}(\mathbb{R}^d)$ *satisfies the estimates*

$$\|\ |Q|^{j\eta}\ \Phi(Q)\ \|_{L^{2,a}(\mathbb{R}^d_Q)} \leqslant b^j\ (2j+1)^{j-1}\ \|\Phi\|_{L^{2,a}(\mathbb{R}^d)}, \quad j \in \mathbb{N}, \quad j \geqslant 0. \tag{H.11}$$

The proof of Proposition H.0.10 is found below. By Proposition H.0.10 and (H.10), for any $\mu > 0$,

$$\left\| e^{\mu |Q|^{\eta}}\ \Phi(Q) \right\|_{L^{2,a}(\mathbb{R}^d_Q)} \leqslant\ \|\Phi\|_{L^{2,a}(\mathbb{R}^d)} \sum_{j=0}^{\infty} A_j, \tag{H.12}$$

where $A_j \equiv \frac{\mu^j}{j!}\ b^j\ (2j+1)^{j-1}$, and μ is to be determined so that the sum converges. The ratio test gives

$$\lim_{j\to\infty} \frac{A_{j+1}}{A_j} = \lim_{j\to\infty} \frac{b\ \mu\ (2j+3)^j}{(j+1)\ (2j+1)^{j-1}} = 2b\mu e. \tag{H.13}$$

If $\mu < \frac{1}{2eb\left(\|\Phi\|_{L^{2,a}(\mathbb{R})}\right)}$, (H.12) is convergent, completing the proof of Lemma H.0.17. □

Proof of Proposition H.0.10: Rewrite equation (H.8) as

$$\Phi(Q) = \frac{\chi_A(Q)}{1+M(Q)} \sum_{k=1}^{K} C_k\ \Phi * \Phi * \Phi(Q+\tau_k). \tag{H.14}$$

We prove (H.11) by induction. Define

$$W(Q) \equiv\ \Phi * \Phi(Q). \tag{H.15}$$

Let $m \geqslant 1$ and multiply equation (H.14) by $|Q|^{m\eta}$ to get

$$|Q|^{m\eta}\ \Phi(Q) = \frac{\chi_A(Q)\ |Q|^{m\eta}}{1+M(Q)} \sum_{k=1}^{K} C_k\ W * \Phi(Q+\tau_k). \tag{H.16}$$

Note that (H.11) holds trivially for $j = 0$ if we choose $b > 0$. Now assume, for all $0 \leqslant j \leqslant m-1$, that there exists a constant $b > 0$ such that (H.11) holds. We will prove that (H.11) holds for $j = m$.

Observe that by (H.7),

$$\frac{\chi_A(Q)\ |Q|^{m\eta}}{1+M(Q)} \leqslant |Q|^{(m-1)\eta}\ \frac{\chi_A(Q)\ |Q|^{\eta}}{1+M(Q)} = |Q|^{(m-1)\eta}\ \chi_A(Q)\ D_M \leqslant D_M\ |Q+\tau_k|^{(m-1)\eta}. \tag{H.17}$$

Next, by Lemma A.0.5 and $|\widetilde{Q}|^{m-1} \leqslant \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix} |\widetilde{Q}-\xi|^{l}\ |\xi|^{m-l-1}$,

$$\begin{aligned} |\widetilde{Q}|^{(m-1)\eta} &\leqslant \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix}^{\eta} |\widetilde{Q}-\zeta|^{l\eta}\ |\zeta|^{(m-l-1)\eta} \\ &\leqslant \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix} |\widetilde{Q}-\zeta|^{l\eta}\ |\zeta|^{(m-l-1)\eta}. \end{aligned} \tag{H.18}$$

Combining (H.18) (for $\widetilde{Q} \equiv Q+\tau_k$) with (H.16) and (H.17), we have

$$\begin{aligned} |Q|^{m\eta}\ |\Phi(Q)| &\leqslant D_M \sum_{k=1}^{K} |C_k|\ |Q+\tau_k|^{(m-1)\eta} \left| \int_{\mathbb{R}^d} W(Q+\tau_k-\xi)\Phi(\xi)d\xi \right| \\ &\leqslant D_M \sum_{k=1}^{K} |C_k| \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix} \left(|\xi|^{l\eta}\ |W|\right) *^{\xi} \left(|\xi|^{(m-l-1)\eta}\ |\Phi|\right)(Q+\tau_k), \end{aligned} \tag{H.19}$$

and therefore by the inductive hypothesis (H.11) (which holds for $0 \leqslant m-l-1 \leqslant m-1$), we have

$$\begin{aligned} &\|\ |Q|^{m\eta}\ \Phi(Q)\ \|_{L^{2,a}(\mathbb{R}^d_Q)} \\ &\leqslant K\ D_a\ D_M \max_{k=1,\dots,K}\{|C_k|\} \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix} \left\| |\xi|^{l\eta}\ W(\xi)\ \right\|_{L^{2,a}(\mathbb{R}^d_\xi)} \left\| |\xi|^{(m-l-1)\eta}\ \Phi(\xi)\ \right\|_{L^{2,a}(\mathbb{R}^d_\xi)} \\ &\leqslant K\ D_a\ D_M \max_{k=1,\dots,K}\{|C_k|\} \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix} b_1\ b_2^{m-l-1}\ [2(m-l-1)+1]^{m-l-2} \\ &\qquad \cdot \left\| |\xi|^{l\eta}\ W(\xi)\ \right\|_{L^{2,a}(\mathbb{R}^d_\xi)}\ \|\Phi\|_{L^{2,a}(\mathbb{R}^d)}. \end{aligned} \tag{H.20}$$

Next, we must estimate $|\xi|^{l\eta}\ W(\xi)$. Equations (H.15) and (H.18) give

$$|\xi|^{l\eta}\ |W(\xi)| \leqslant \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} \left(|\zeta|^{k\eta}|\Phi|\right) *^{\zeta} \left(|\zeta|^{(l-k)\eta}\ |\Phi|\right)(\xi), \tag{H.21}$$

which in turn gives, by the inductive hypothesis (H.11) (for $0 \leqslant k \leqslant l \leqslant m-1$ and $0 \leqslant l-k \leqslant m-1$) and Abel's identity H.0.16,

$$\begin{aligned}
\left\| |\xi|^{l\eta} \, W(\xi) \right\|_{L^{2,a}(\mathbb{R}^d_\xi)} &\leqslant D_a \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} \left\| |\zeta|^{k\eta} \, \Phi(\zeta) \right\|_{L^{2,a}(\mathbb{R}^d_\zeta)} \left\| |\zeta|^{(l-k)\eta} \, \Phi(\zeta) \right\|_{L^{2,a}(\mathbb{R}^d_\zeta)} \\
&\leqslant D_a \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} b^k \, b^{l-k} \, (2k+1)^{k-1} \, [2(l-k)+1]^{l-k-1} \, \|\Phi\|^2_{L^{2,a}(\mathbb{R}^d)} \\
&= D_a \, \|\Phi\|^2_{L^{2,a}(\mathbb{R}^d)} \, b^l \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} (2k+1)^{k-1} \, [2(l-k)+1]^{l-k-1} \\
&= D_a \, \|\Phi\|^2_{L^{2,a}(\mathbb{R}^d)} \, b^l \, 2^{l-2} \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} (k+1/2)^{k-1} \, [l-k+1/2]^{l-k-1} \\
&= D_a \, \|\Phi\|^2_{L^{2,a}(\mathbb{R}^d)} \, b^l \, 2^l \, (l+1)^{l-1} = 2 \, D_a \, b^l \, \|\Phi\|^2_{L^{2,a}(\mathbb{R}^d)} \, (2l+2)^{l-1}. \qquad \text{(H.22)}
\end{aligned}$$

Combining (H.22) with (H.20), along with Abel's identity again, gives

$$\begin{aligned}
\| \, |Q|^{m\eta} \, \Phi(Q) \, \|_{L^{2,a}(\mathbb{R}^d_Q)} &\leqslant 2 \, K \, D_M \, D_a^2 \max_{k=1,\dots,K} \{|C_k|\} \, \|\Phi\|^3_{L^{2,a}(\mathbb{R}^d)} \\
&\quad \cdot \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix} b^{m-l-1} \, b_2^l \, (2l+2)^{l-1} \, [2(m-l-1)+1]^{m-l-2} \\
&= 2 \, K \, D_M \, D_a^2 \max_{k=1,\dots,K} \{|C_k|\} \, \|\Phi\|^3_{L^{2,a}(\mathbb{R}^d)} \, b^{m-1} \, 2^{m-3} \\
&\quad \cdot \sum_{l=0}^{m-1} \begin{pmatrix} m-1 \\ l \end{pmatrix} (l+1)^{l-1} \, [(m-l-1)+1/2]^{m-l-2} \\
&= 2 \, K \, D_M \, D_a^2 \max_{k=1,\dots,K} \{|C_k|\} \, \|\Phi\|^3_{L^{2,a}(\mathbb{R}^d)} \, b^{m-1} \, 2^{m-3} \, 3 \, (m-1+3/2)^{m-2} \\
&= 3 \, K \, D_M \, D_a^2 \max_{k=1,\dots,K} \{|C_k|\} \, \|\Phi\|^3_{L^{2,a}(\mathbb{R}^d)} \, b^{m-1} \, (2m+1)^{m-2}. \qquad \text{(H.23)}
\end{aligned}$$

To complete the proof that (H.11) is satisfied for $j = m \geqslant 1$, we need to show that this quantity is bounded by $b^m \, (2m+1)^{m-1} \, \|\Phi\|_{L^{2,a}(\mathbb{R}^d)}$. It will then follow that

$$\||Q|^{m\eta} \, \Phi(q)\|_{L^{2,a}(\mathbb{R}^d_Q)} \leqslant b^m (2m+1)^{m-1} \, \|\Phi\|_{L^{2,a}(\mathbb{R}^d)} \, . \qquad \text{(H.24)}$$

Condition (H.24) is satisfied it for any $m \geqslant 1$, b_2, dependent on $\|\Phi\|_{L^{2,a}(\mathbb{R}^d)}$, satisfies

$$3 \, K \, D_M \, D_a^2 \max_{k=1,\dots,K} \{|C_k|\} \, \|\Phi\|^2_{L^{2,a}(\mathbb{R}^d)} \leqslant b \, (2m+1), \quad \forall m \geqslant 1. \qquad \text{(H.25)}$$

It suffices to satisfy (H.25) for $m = 1$ by choosing $b > 0$ sufficiently large. Recall that $b > 0$ satisfies (H.11) trivially for $j = 0$. This completes the proof of Proposition H.0.10. $\square$