Bifurcation of localized eigenstates of perturbed periodic Schrödinger operators

Iva Vukićević

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ABSTRACT

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A spatially localized initial condition for an energy-conserving wave equation with periodic coefficients disperses (spatially spreads) and decays as time advances. This dispersion is associated with the continuous spectrum of the underlying differential operator and the absence of discrete eigenvalues. The introduction of spatially localized perturbations in a periodic medium leads to "defect modes", states in which the wave is spatially localized and periodic in time. These modes are associated with eigenvalues which bifurcate from the continuous spectrum induced by the perturbation.

This thesis investigates specific families of perturbations of one-dimensional periodic Schrödinger operators and studies the resulting bifurcating eigenvalues from the unperturbed continuous spectrum. For $Q(x)$ a real-valued periodic function, the Schrödinger operator $H_Q = -\partial_x^2 + Q(x)$ has a continuous spectrum equal to the union of closed intervals, called spectral bands, separated by open spectral gaps. We find that upon the introduction of a bounded, "small", and sufficiently decaying perturbation $W(x)$, the spectrum of H_{Q+W} has discrete eigenvalues (with corresponding eigenstates which are exponentially decaying in $|x|$) which lie in the open spectral gaps of H_Q.

Our analysis covers two large classes of perturbations $W(x)$:

1. $W(x) = \lambda V(x)$, $0 < \lambda \ll 1$, and $V(x)$ sufficiently rapidly decaying as $x \to \pm\infty$;
2. $W(x) = q(x, x/\epsilon)$, $0 < \epsilon \ll 1$, where $x \mapsto q(x,y)$ is spatially localized, $q(x, y+1) = q(x,y)$ for $x \in \mathbb{R}$, and $y \mapsto q(x,y)$ has mean zero.

In Case 1. $W(x)$ corresponds to a small and localized absolute change in the medium's material properties. In Case 2. $W(x)$ corresponds to a high-contrast microstructure. $Q(x) + W(x)$ may be pointwise very large, but on average it is a small perturbation of $Q(x)$.

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To my parents,
Dragutin and Jadranka

Part I

Introduction

Chapter 1

Introduction

The Schrödinger equation is the scalar partial differential equation:

$$i\partial_t \psi = \left(-\partial_x^2 + Q(x) + W(x)\right)\psi \equiv H_{Q+W}\psi, \tag{1.1}$$

governing ψ, a complex-valued function of position x, and time t. The initial value problem for (1.1) with data $\psi(0,x) = \psi_0(x)$ is discussed in [Pazy, 1983; Reed and Simon, 1978]. Here $Q(x)$ is a bounded and smooth periodic function called the *background potential* and $W(x)$, assumed to be real-valued and sufficiently smooth, is viewed as a *perturbation* of the background potential. We shall focus on time-harmonic and spatially localized one-dimensional solutions $\psi(x,t) = e^{-iEt}u(x)$, which yield the eigenvalue problem for the time-independent Schrödinger equation:

$$H_{Q+W}u = \left(-\partial_x^2 + Q(x) + W(x)\right)u = Eu, \quad u \in L^2(\mathbb{R}). \tag{1.2}$$

Equations (1.1) and (1.2) arise in the following settings:

1. *Quantum Mechanics:* The solution $\psi(x,t)$ of equation (1.1) denotes the wave function of a quantum particle. $\int_A |\psi(x,t)|^2 dx$ is the probability that the particle is in the subset $A \subset \mathbb{R}$ at time t.

2. *Electromagnetism:* The propagation of light through an electromagnetic waveguide is governed by the Maxwell's equations. In dielectric media whose material properties (*e.g.* refractive index) depend only on variations in one or two spatial dimensions, time-harmonic transverse magnetic (TM) and transverse electric (TE) solutions can be found. In the TM

case, the electric field is governed by a scalar Helmholtz equation. The eigenvalue solutions of (1.2) can be used to construct guided TM mode solutions to Maxwell's equations that are localized in the x-direction and propagating in the z-direction. The potential $Q(x) + W(x)$ is expressible in terms of the refractive index, a property of the material of propagation. See Appendix A for more detail.

The main focus of this thesis is the study of equation (1.2) with specific families of spatially localized perturbations $W(x)$. Assume $Q(x)$ is periodic, the unperturbed operator H_Q has no discrete spectrum and only a continuous spectrum. We analyze the bifurcation of discrete eigenvalues from its continuous spectrum due to the perturbation $W(x)$. Associated with these eigenvalues are localized eigensolutions of the eigenvalue problem (1.2).

A localized perturbation of the form $W(x)$ of a background periodic potential $Q(x)$ may physically model an engineered or a random "defect" in a periodic medium, [Joannopoulos *et al.*, 2008]. Periodic media have applications in many fields, only one of which is electromagnetism (see Appendix A). In such media, a spatially localized initial condition is known to spatially disperse and decay in amplitude as time advances, [Brown *et al.*, 2012; Cuccagna, 2008; Cai, 2006; Firsova, 1996]. The introduction of a localized perturbation in a periodic medium leads to "defect modes," solutions which are spatially localized and time-periodic. In this thesis we study the emergence of these defect modes via the bifurcation of eigenvalues from the continuous spectrum induced by the perturbation $W(x)$.

Outline of the Introduction: We begin with a summary of basic spectral theory in Section 1.1. We are then set up to give a summary of the results of this thesis in Section 1.2. Section 1.3 is devoted to a discussion of results related to this thesis and Section 1.4 considers future research directions. We end with an outline and brief summary of the chapters found in this thesis in Section 1.5.

1.1 Background: Spectral Theory

In this thesis, we study detailed aspects of the spectrum of the Schrödinger operator:

$$H_{Q+W} = -\partial_x^2 + Q(x) + W(x)$$

for appropriate choices of $Q(x)$ and $W(x)$. We take some time in this section to define and discuss the notion of a spectrum of an operator and its properties. For a much more detailed and thorough discussion, consult [Reed and Simon, 1978; Hunter and Nachtergaele, 2001; Eastham, 1973], from which the following summary was modified.

Consider a Banach space $\mathcal{S}$.

Definition 1.1.1. *The* resolvent set *of a bounded operator $A : \mathcal{S} \to \mathcal{S}$, denoted by $\rho(A)$, is the set of complex numbers, μ, such that $(A - \mu I) : \mathcal{S} \to \mathcal{S}$ is one-to-one and onto. The* spectrum *of A, denoted by* $\operatorname{spec}(A)$*, is the complement of the resolvent set in $\mathbb{C}$,* $\operatorname{spec}(A) = \mathbb{C} \backslash \rho(A)$.

There are various types of elements that can belong to the spectrum of A. We separate them into two disjoint sets with the following definition.

Definition 1.1.2. *Suppose that A is a bounded linear operator acting on a Banach space $\mathcal{S}$.*

(a) The discrete spectrum *of A consists of all isolated points $\mu \in \operatorname{spec}_{\mathrm{ds}}(A)$ such that μ is an eigenvalue of A of finite algebraic multiplicity.*

(b) The essential spectrum *of A consists of all $\mu \in \operatorname{spec}_{\mathrm{ess}}(A)$ such that μ is not in the discrete spectrum of A.*

A fundamental problem is to determine the effect of perturbations of an operator on its spectrum. A basic result is Weyl's Theorem ([Reed and Simon, 1978]) on the stability of the essential spectrum.

Definition 1.1.3. *Let A be self-adjoint. An operator C is called* relatively compact *with respect to A if and only if $C(A - zI)^{-1}$ is compact for some $z \in \rho(A)$.*

We can now state Weyl's Theorem, which says that a relatively compact perturbation of an operator does not change its essential spectrum:

Theorem 1.1.4 (Weyl's Theorem, Cor 3, pg 114 in [Reed and Simon, 1978])**.** *Let A be a self-adjoint operator and let C be relatively compact with respect to A. Then* $\operatorname{spec}_{\mathrm{ess}}(A + C) = \operatorname{spec}_{\mathrm{ess}}(A)$.

In fact, Definition 1.1.2 of the spectrum of an operator can be refined. We can partition the spectrum into three disjoint sets:

Definition 1.1.5. *Suppose that A is a bounded linear operator acting on a Banach space $\mathcal{S}$.*

(a) The point spectrum *of A consists of all $\mu \in \mathrm{spec}_{\mathrm{ps}}(A)$ such that $A - \mu I$ is not one-to-one. In this case μ is called an* eigenvalue *of A.*

(b) The continuous spectrum *of A consists of all $\mu \in \mathrm{spec}_{\mathrm{cont}}(A)$ such that $A - \mu I$ is one-to-one but not onto, and* $\mathrm{ran}(A - \mu I)$ *is dense in $\mathcal{S}$.*

(c) The residual spectrum *of A consists of all $\mu \in \mathrm{spec}_{\mathrm{res}}(A)$ such that $A - \mu I$ is one-to-one but not onto, and* $\mathrm{ran}(A - \mu I)$ *is not dense in $\mathcal{S}$.*

For self-adjoint operators A, one has

Theorem 1.1.6 (Cor. 9.14 in [Hunter and Nachtergaele, 2001])**.** *If A is a self-adjoint operator, then* $\mathrm{spec}_{\mathrm{res}}(A) = \emptyset$. *Therefore,* $\mathrm{spec}(A) = \mathrm{spec}_{\mathrm{cont}}(A) \cup \mathrm{spec}_{\mathrm{ds}}(A)$.

1.1.1 Spectrum of Periodic Schrödinger Operators

A major part of this thesis considers perturbations of the second order self-adjoint differential operator (Schrödinger operator):

$$H_Q = -\partial_x^2 + Q(x).$$

Here, $Q(x)$ is a continuous, real-valued, $1-$periodic potential:

$$Q(x+1) = Q(x).$$

The spectrum of H_Q is continuous. In fact, it is the union of closed intervals called *spectral bands* [Reed and Simon, 1978; Eastham, 1973]. The complement of the spectrum is a union of open intervals called *spectral gaps*.

The spectrum is determined by the family of self-adjoint eigenvalue problems parametrized by the *quasi-momentum* $k \in (-1/2, 1/2]$:

$$H_Q u(x;k) = E\ u(x;k), \tag{1.3}$$

$$u(x+1;k) = e^{2\pi i k} u(x;k). \tag{1.4}$$

That is, we seek k-*pseudo-periodic* solutions of the eigenvalue equation. For each $k \in (-1/2, 1/2]$, the self-adjoint eigenvalue problem (1.3)-(1.4) has discrete (eigenvalue) spectrum (listed with multiplicity):

$$E_0(k) \leq E_1(k) \leq \cdots \leq E_b(k) \leq \dots \tag{1.5}$$

with corresponding k-pseudo-periodic eigenfunctions $u_b(x;k)$, $b \geq 0$. The b^{th} spectral band is given by:

$$\mathcal{B}_b = \bigcup_{k\in(-1/2,1/2]} E_b(k). \tag{1.6}$$

The spectrum of H_Q is given by:

$$\mathrm{spec}(H_Q) = \bigcup_{b\geq 0} \mathcal{B}_b = \bigcup_{b\geq 0} \bigcup_{k\in(-1/2,1/2]} E_b(k). \tag{1.7}$$

Since the boundary condition (1.4) is invariant with respect to $k \mapsto k+1$, the functions $E_b(k)$ can be extended to all $\mathbb{R}$ as periodic functions of k. The minima and maxima of $E_b(k)$ occur at $k = k_* \in \{0, 1/2\}$; see Figure 1.1. In cases where extrema border a spectral gap, we have that $\partial_k^2 E_b(k_*)$ is either strictly positive or strictly negative [Eastham, 1973; Reed and Simon, 1978]. See Appendix B for a more detailed analysis of $\mathrm{spec}(H_Q)$ and its properties.

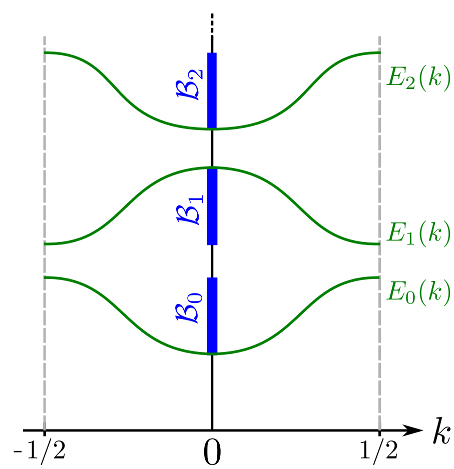


Figure 1.1: Sketch of spectrum associated with the operator H_Q. Eigenvalues, $E_b(k)$, $k \in (-1/2, 1/2]$, $b = 0, 1, 2, \dots$, are displayed in green. The continuous spectrum is in blue.

Example 1.1.7. Consider the case $Q \equiv 0$, $H_0 = -\partial_x^2$ acting on $L^\infty(\mathbb{R})$. We wish to solve the equation

$$-\partial_x^2 \psi(x) = \mu \psi(x).$$

The only bounded solutions to this problem are $\psi(x) = e^{\pm i\sqrt{\mu}x}$ for $\mu \geq 0$. Furthermore, if $\mu < 0$ then the operator $-\partial_x^2 - \mu = -\partial_x^2 + |\mu|$ is invertible on $L^2(\mathbb{R})$ (by the Fourier Transform) and hence the resolvent set $\rho(H_0) = (-\infty, 0)$. Therefore, the spectrum of H_0 is continuous and can be written explicitly as $\text{spec}(H_0) = [0, \infty)$.

We wish to show that for $W(x) \in L^2(\mathbb{R})$, one has $\text{spec}_{\text{cont}}(H_W) = \text{spec}_{\text{cont}}(H_0)$. We need to establish that $W(x)$ is a relatively compact perturbation of H_0. That is, we need to show that $W(H_0 - zI)^{-1}$ is compact for some $z \in \rho(H_0) = (-\infty, 0)$. Without-loss-of-generality, take $z = -1$ and note that from Fourier analysis we can write

$$W(x)(H_0 + I)^{-1} f(x) = W(x)(-\partial_x^2 + 1)^{-1} f(x) = \int_{-\infty}^{\infty} W(x) \ \frac{1}{2} e^{-|x-y|} f(y) dy,$$

which is an integral operator with kernel

$$k(x, y) = W(x) \ \frac{1}{2} e^{-|x-y|}.$$

An integral operator is compact if its kernel satisfies: $k \in L^2(\mathbb{R} \times \mathbb{R})$ (Thm. 9.21 in [Hunter and Nachtergaele, 2001]). Note that for $W(x) \in L^2(\mathbb{R})$, $k(x, y) \in L^2(\mathbb{R} \times \mathbb{R})$ and thus $W(H_0 + I)^{-1}$ is compact. We have shown that $W(x)$ is relatively compact with respect to H_0 and therefore, by Weyl's Theorem 1.1.4, $\text{spec}_{\text{cont}}(H_W) = \text{spec}_{\text{cont}}(H_0)$. □

Remark 1.1.8. *Consider the operator H_{Q+W}. Here, $Q(x)$ is real-valued and periodic, and $W(x) \in L^2(\mathbb{R})$. One can show by Theorem 1.1.4 that $\text{spec}_{\text{cont}}(H_{Q+W}) = \text{spec}_{\text{cont}}(H_Q)$ [Reed and Simon, 1978; Christ and Kiselev, 1998]. Therefore, the effect of a localized perturbation of the operator H_Q is to possibly introduce discrete eigenvalues into the open spectral gaps. Note that if $W \in L^2(\mathbb{R})$ is real-valued, H_{Q+W} does not have discrete eigenvalues embedded in its continuous spectrum; see [Rofe-Beketov, 1964; Gesztesy and Simon, 1993].*

1.2 Statement of Results

The main results in this thesis concern spectral analysis of operators of the form H_{Q+W}. Here, $Q(x)$ is real-valued and periodic, and $W(x)$ is sufficiently localized is space. These results are presented in Section 1.2.1. A second set of results concerns scattering properties (*eg.* transmission coefficient, local energy time-decay) of the operator H_W for a particular choice of localized potential $W(x)$. These results are presented in Section 1.2.2.

1.2.1 Spectral analysis of the operator H_{Q+W}

We are interested in analyzing the eigenvalue problem:

$$H_{Q+W}\psi(x) = (-\partial_x^2 + Q(x) + W(x))\psi(x) = E\psi(x), \quad \psi \in L^2(\mathbb{R}), \tag{1.8}$$

for specific choices of the perturbing potential $W(x)$. Here, $Q(x)$ is a real-valued periodic function. We approach the problem by first transforming (1.8) into energy space via the Gelfand-Bloch transform (see Appendix B for a presentation of Floquet-Bloch theory). Using a Lyapunov-Schmidt type reduction [Nirenberg, 2001] by projecting the resulting equation onto energies near the $(b_*)^{\text{th}}$ band edge (uppermost or lowermost), and far from this edge, we derive a closed equation for the near-band-edge energy components of the eigenfunction (see Appendix C for a discussion of the standard Lyapunov-Schmidt reduction). An appropriate rescaling of the latter equation yields an effective operator for the rescaled near-energy components of the eigenfunction of the original problem (1.8). The effective operator is of the form

$$H_{b_*,\text{eff}} = -\frac{d}{dy} A_{b_*,\text{eff}} \frac{d}{dy} - B_{b_*,\text{eff}}\ \delta(y).$$

Here, $A_{b_*,\text{eff}}$ is the inverse effective mass associated with a band edge of the $(b_*)^{\text{th}}$ spectral band of spec(H_Q), and $B_{b_*,\text{eff}}\ \delta(y)$ is an effective potential well with coefficient $B_{b_*,\text{eff}}$ depending on the perturbation $W(x)$.

We now give a brief description of our results for particular choices of $W(x)$.

1. $\mathbf{Q \equiv 0,\ H_{\lambda V} \equiv -\partial_x^2 + \lambda V(x)}$: Consider the potential $W(x) = \lambda V(x)$ such that $V(x)$ is sufficiently smooth and localized, $-\infty < \int_{\mathbb{R}} V(x)dx < 0$, and $0 < \lambda \ll 1$. We seek eigenpair solutions $(E^\lambda, \psi^\lambda)$ to the eigenvalue problem:

$$H_{\lambda V}\psi^\lambda(x) = (-\partial_x^2 + \lambda V(x))\psi^\lambda(x) = E^\lambda\psi^\lambda(x), \quad \psi^\lambda(x) \in L^2(\mathbb{R}).$$

In Chapter 2, we prove in Theorem 2.2.1 that there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, $H_{\lambda V}$ has a simple discrete eigenvalue,

$$E^\lambda = \lambda^2\mu_\star + \mathcal{O}(\lambda^3), \quad \text{for some} \quad \mu_\star < 0,$$

and corresponding eigenfunction $\psi^\lambda(x)$. The eigenfunction $\psi^\lambda(x)$ is well-approximated in $L^\infty(\mathbb{R})$ by $\psi_\star(\lambda x)$, where $(\mu_\star, \psi_\star(y))$ denotes the unique eigenpair of the effective operator:

$$H_{0,\text{eff}} = -\frac{d^2}{dy^2} + \int_{\mathbb{R}} V \times \delta(y). \tag{1.9}$$

Here, $\delta(y)$ denotes a Dirac delta mass at $y = 0$. Since $\int V < 0$, $H_{0,\text{eff}}$ has a delta function potential well. The unique eigenvalue of $H_{0,\text{eff}}$ is

$$\mu_\star = -\frac{1}{4}\left(\int_{\mathbb{R}} V(x)dx\right)^2.$$

The corresponding one-dimensional eigenspace is spanned by

$$\psi_\star(y) = \exp\left[\frac{1}{2}\left(\int V\right)|y|\right].$$

2. $\mathbf{Q \equiv 0}$, $\mathbf{H_{q_\epsilon} \equiv -\partial_x^2 + q(x, x/\epsilon)}$: Consider the potential $W(x) = q(x, x/\epsilon)$ which is spatially localized on the slow scale x, and periodic with zero mean on the fast scale $y = x/\epsilon$:

$$q(x, y+1) = q(x, y), \qquad \int_0^1 q(x, y)dy = 0. \tag{1.10}$$

By expanding with respect to the Fourier coefficients of the fast variable, one can write

$$q(x, y) = \sum_{j \neq 0} q_j(x) e^{2\pi i j y}. \tag{1.11}$$

We seek eigenpair solutions $(E^\epsilon, \psi^\epsilon)$ to the eigenvalue problem:

$$H_{q_\epsilon}\psi^\epsilon(x) = (-\partial_x^2 + q(x, x/\epsilon))\psi^\epsilon(x) = E^\epsilon\psi^\epsilon(x), \quad \psi^\epsilon(x) \in L^2(\mathbb{R}).$$

In Chapter 4, we prove in Theorem 4.2.1 that there exists $\epsilon_0 > 0$, such that for all $0 < \epsilon < \epsilon_0$, H_{q_ϵ} has a simple discrete eigenvalue,

$$E^\epsilon = \epsilon^4\mu_\star + \mathcal{O}(\lambda^{4+\alpha}), \quad \text{for some} \;\; \mu_\star < 0, \;\; \alpha > 0,$$

and corresponding eigenfunction $\psi^\epsilon(x)$. The eigenfunction $\psi^\epsilon(x)$ is well approximated in $L^\infty(\mathbb{R})$ by $\psi_\star(\epsilon^2 x)$, where $(\mu_\star, \psi_\star(y))$ denotes the unique eigenpair of the effective operator:

$$H_{0,\text{eff}} = -\frac{d^2}{dy^2} - \int_{\mathbb{R}} \Lambda_{\text{eff}} \times \delta(y), \quad \Lambda_{\text{eff}}(x) = \frac{1}{4\pi^2}\sum_{j\neq 0}\frac{1}{j^2}|q_j(x)|^2; \tag{1.12}$$

$\mu_\star$ is the unique eigenvalue of $H_{0,\text{eff}}$,

$$\mu_\star = -\frac{1}{4}\left(\int_{\mathbb{R}} \Lambda_{\text{eff}}(x)dx\right)^2,$$

with the corresponding one-dimensional eigenspace spanned by

$$\psi_\star(y) = \exp\left[\frac{1}{2}\left(\int \Lambda_{\text{eff}}\right)|y|\right].$$

3. $\mathbf{Q \not\equiv 0}$, $\mathbf{H_{Q+\lambda V}} \equiv -\partial_{\mathbf{x}}^{\mathbf{2}} + \mathbf{Q}(\mathbf{x}) + \lambda\mathbf{V}(\mathbf{x})$: Consider the potential $W(x) = \lambda V(x)$ where $V(x)$ is spatially localized and sufficiently smooth. We make no assumptions on the sign of the integral $\int_{\mathbb{R}} V(x)dx$ as we did in case 1. above ($Q \equiv 0$). We seek eigenpair solutions $(E^\lambda, \psi^\lambda)$ to the eigenvalue problem:

$$H_{Q+\lambda V}\psi^\lambda(x) = (-\partial_x^2 + Q(x) + \lambda V(x))\psi^\lambda(x) = E^\lambda\psi^\lambda(x).$$

Let $E_b(k_*),\ k_* \in \{0, 1/2\}$, denote an endpoint (uppermost or lowermost) of the b^{th} spectral band, bordering a spectral gap. In Chapter 2, we prove in Theorem 2.2.4 that under the condition:

$$\partial_k^2 E_b(k_*) \ \times \ \int_{\mathbb{R}} |u_b(x;k_*)|^2 \ V(x) \ dx \ < \ 0, \tag{1.13}$$

the following holds: There exists $\lambda_0 > 0$, such that for all $0 < \lambda < \lambda_0$, $H_{Q+\lambda V}$ has a simple discrete eigenvalue,

$$E^\lambda \ = \ E_b(k_*) \ + \ \lambda^2\mu_\star \ + \ \mathcal{O}(\lambda^{2+\alpha}), \ \text{ for some } \ \alpha > 0, \tag{1.14}$$

which bifurcates from the edge, $E_b(k_*)$ of $\mathcal{B}_b$, into a spectral gap.

1. If $\partial_k^2 E_b(k_*) > 0$ and $\int_{\mathbb{R}} |u_b(x;k_*)|^2 \ V(x) \ dx < 0$, then $\mu_\star < 0$ and E^λ lies near the lowermost edge of $\mathcal{B}_b$; see the left panel of Figure 1.2.

2. If $\partial_k^2 E_b(k_*) < 0$ and $\int_{\mathbb{R}} |u_b(x;k_*)|^2 \ V(x) \ dx > 0$, then $\mu_\star > 0$ and E^λ lies near the uppermost edge of $\mathcal{B}_b$; see the right panel of Figure 1.2.

For $0 < \lambda < \lambda_0$, $\psi^\lambda(x)$, the eigenstate corresponding to the eigenvalue E^λ, is well-approximated in L^∞ by $\psi_\star(\lambda x)$, where $\psi_\star(y)$ denotes the unique eigenstate of the effective operator:

$$H_{b,\text{eff}} \ = \ -\frac{d}{dy} \ A_{b,\text{eff}} \ \frac{d}{dy} \ + \ B_{b,\text{eff}} \times \delta(y), \tag{1.15}$$

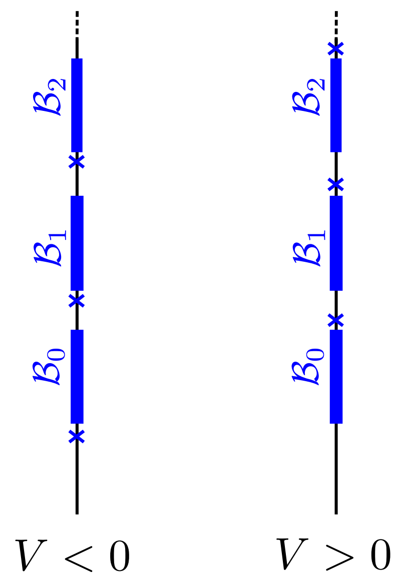


Figure 1.2: Sketch of spectrum associated with the operator $H_{Q+\lambda V}$. The left panel (resp. right) corresponds to spec$(H_{Q+\lambda V})$, where λV is small, localized negative (resp. positive).

with constant effective parameters $A_{b,\text{eff}}$ and $B_{b,\text{eff}}$. Here,

$$A_{b,\text{eff}} \;=\; \frac{1}{8\pi^2}\partial_k^2 E_b(k_*), \tag{1.16}$$

is the inverse *effective mass* associated with the spectral edge $E_b(k_*)$, and

$$B_{b,\text{eff}} \;=\; \int_{\mathbb{R}} |u_b(x;k_*)|^2 \; V(x) \; dx. \tag{1.17}$$

The unique eigenvalue of $H_{b,\text{eff}}$ is

$$\mu_\star = -\frac{B_{b,\text{eff}}^2}{4A_{b,\text{eff}}}.$$

The corresponding one-dimensional eigenspace is spanned by

$$\psi_\star(y) = \exp\left(\frac{B_{b,\text{eff}}}{2A_{b,\text{eff}}}|y|\right).$$

4. $\mathbf{Q \not\equiv 0}$, $\mathbf{H_{Q+q_\epsilon} \equiv -\partial_x^2 + Q(x) + q_\epsilon(x)}$: Consider the potential $W(x) = q_\epsilon(x)$, where $q_\epsilon(x)$ is spatially localized in x, and localized at high frequencies in the frequency space. Note, while in results 1. and 3. the perturbation $\lambda V(x)$ tends to zero in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ as $\lambda \to 0$, $q_\epsilon(x)$ tends to zero *weakly* as $\epsilon \to 0$. Equations (1.10)-(1.11), represent a family of such potentials.

We seek eigenpair solutions $(E^\epsilon, \psi^\epsilon)$ to the eigenvalue problem:

$$H_{Q+q_\epsilon}\psi^\epsilon(x) = (-\partial_x^2 + Q(x) + q_\epsilon(x))\psi^\epsilon(x) = E^\epsilon\psi^\epsilon(x), \quad \psi^\epsilon(x) \in L^2(\mathbb{R}).$$

Assume that the b^{th} band has left-endpoint $E_b(k_*)$, $k_* \in \{0, 1/2\}$, bordering a spectral gap. Then $\partial_k^2 E_b(k_*) > 0$; see the left panel of Figure 1.3. In Chapter 4, we prove in Theorem 4.2.3 that there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, H_{Q+q_ϵ} has a simple discrete eigenvalue which bifurcates from the band edge, $E_b(k_*)$ of $\mathcal{B}_b$, into a spectral gap:

$$E^\epsilon = E_b(k_*) + \epsilon^4\mu_\star + \mathcal{O}(\epsilon^{4+\alpha}), \quad \text{for some} \quad \mu_\star < 0, \ \alpha > 0. \tag{1.18}$$

see the right panel in Figure 1.3.

We show that for $0 < \epsilon < \epsilon_0$, $\psi^\epsilon(x)$, the eigenstate corresponding to the eigenvalue E^ϵ, is well approximated in L^∞ by $\psi_\star(\epsilon^2 x)$, where $\psi_\star(y)$ denotes the unique eigenstate of the effective operator:

$$H_{b,\text{eff}} \ = \ -\frac{d}{dy}\ A_{b,\text{eff}}\ \frac{d}{dy} \ - \ B_{b,\text{eff}} \times \delta(y), \tag{1.19}$$

with constant effective parameters $A_{b,\text{eff}}$ and $B_{b,\text{eff}}$. Here,

$$A_{b,\text{eff}} \ = \ \frac{1}{8\pi^2}\partial_k^2 E_b(k_*), \tag{1.20}$$

is the inverse effective mass associated the the spectral edge $E_b(k_*)$, and

$$\left| B_{b,\text{eff}} - \epsilon^{-2}\int_{\mathbb{R}} |u_b(x;k_*)|^2\ q_\epsilon(x)\overline{Q_\epsilon(x)}\ dx \right| \ \lesssim \ \mathcal{O}\left(\epsilon^{\sigma_{\text{eff}}}\right), \quad \text{for some} \quad \sigma_{\text{eff}} > 0, \tag{1.21}$$

where $Q_\epsilon(x)$ is defined by:

$$\widehat{Q}_\epsilon(\xi) = \frac{1}{4\pi^2|\xi|^2}\widehat{q}_\epsilon(x).$$

For the specific case of $q_\epsilon(x)$ as in (1.10)-(1.11), one can show that

$$B_{b,\text{eff}} \ = \ \int_0^1 \ |u_b(x;k_*)|^2\ \Lambda_{\text{eff}}(x)\ dx, \qquad \Lambda_{\text{eff}}(x) = \frac{1}{4\pi^2}\sum_{j\neq 0}\frac{1}{j^2}|q_j(x)|^2. \tag{1.22}$$

The unique eigenvalue of $H_{b,\text{eff}}$ is

$$\mu_\star = -\frac{B_{b,\text{eff}}^2}{4A_{b,\text{eff}}}.$$

The corresponding one-dimensional eigenspace is spanned by

$$\psi_\star(y) = \exp\left(-\frac{B_{b,\text{eff}}}{2A_{b,\text{eff}}}|y|\right).$$

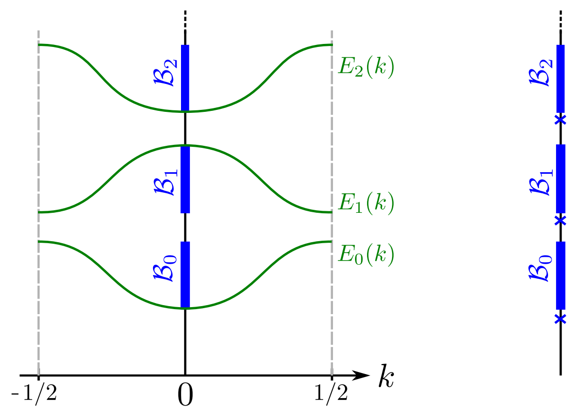


Figure 1.3: Sketch of spectra. Eigenvalues $E_b(k)$, $k \in (-1/2, 1/2]$, $b = 0, 1, 2, \ldots$, are displayed in green. The continuous spectrum, is in blue, and discrete eigenvalues are indicated through cross markers. Left panel corresponds to spec(H_Q), Q periodic. The right panel corresponds to spec(H_{Q+q_ϵ}).

1.2.2 Scattering properties of the operator $H_{q_{\rm av}+q_\epsilon}$

In Chapter 3, we prove results concerning the scattering properties of the operator:

$$H_{q_\epsilon} = -\partial_x^2 + q_{\rm av}(x) + q(x, x/\epsilon).$$

In particular we study

(a) the transmission coefficient $t^{q_\epsilon}(k)$ of H_{q_ϵ} as $\epsilon \to 0$, and

(b) the time evolution operator $e^{-itH_{q_\epsilon}}$.

Here, $q_{\rm av}(x)$ is a localized background potential and $q(x, x/\epsilon)$ is as described in equations (1.10)-(1.11): it is spatially localized on the slow scale x, and periodic with zero mean on the fast scale $y = x/\epsilon$.

Remark 1.2.1. *More generally, our results hold for potentials which are aperiodic. For example, we allow for real-valued potentials:*

$$q(x, y) \;=\; \sum_{j \neq 0} q_j(x)\; e^{2\pi i \lambda_j y}, \tag{1.23}$$

where $\{\lambda_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ *is a sequence of non-zero distinct frequencies for which there is a constant* $\theta > 0$ *such that*

$$\inf_{j\neq k} |\lambda_j - \lambda_k| \geq \theta > 0, \quad \inf_{j\in\mathbb{Z}} |\lambda_j| \geq \theta > 0. \tag{1.24}$$

The scattering problem for the Schrödinger equation with a general localized potential $V(x)$:

$$(H_V - k^2)\psi = 0, \quad H_V = -\partial_x^2 + V(x), \tag{1.25}$$

is the question of the scattering field in response to an incoming plane wave e^{ikx}:

$$\psi(x;k) = \begin{cases} e^{ikx} + r^V(k)e^{-ikx}, & x \to -\infty, \\ t^V(k)e^{ikx}, & x \to \infty. \end{cases} \tag{1.26}$$

The functions $t^V(k)$ and $r^V(k)$ are respectively called the reflection and transmission coefficients for the potential $V(x)$.

In Theorem 3.3.3, we prove that there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $t^{q_{\rm av}+q_\epsilon}(k)$, the transmission coefficient of the scattering problem (1.25)-(1.26) with potential

$$V_\epsilon(x) = q_{\rm av}(x) + q(x, x/\epsilon),$$

is well-approximated by the transmission coefficient $t^{q_{\rm av}-\epsilon^2\Lambda_{\rm eff}}(k)$ corresponding to the potential

$$V_{\epsilon,{\rm eff}}(x) = q_{\rm av}(x) - \epsilon^2\Lambda_{\rm eff}(x), \quad \Lambda_{\rm eff}(x) = \frac{1}{4\pi^2}\sum_{j\neq 0}\frac{1}{j^2}|q_j(x)|^2.$$

Specifically, we prove that for an appropriate set $K \subset \mathbb{C}$,

$$\sup_{k\in K}\left|\frac{k}{t^{q_{\rm av}-\epsilon^2\Lambda_{\rm eff}}(k)} - \frac{k}{t^{q_{\rm av}+q_\epsilon}(k)}\right| = \mathcal{O}(\epsilon^3).$$

In Corollary 3.3.6, we show that for $q_{\rm av}(x) \equiv 0$, (more generally, Corollary 3.3.8 for $q_{\rm av}(x) \not\equiv 0$) there is a universal scaled limit depending on a single parameter $\int_{\mathbb{R}} \Lambda_{\rm eff}$:

$$t^{q_\epsilon}(\epsilon^2\kappa) \to t^\star\left(\kappa; \int_{\mathbb{R}} \Lambda_{\rm eff}\right) \equiv \frac{\kappa}{\kappa - \frac{i}{2}\int_{\mathbb{R}}\Lambda_{\rm eff}} \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad \kappa \neq i\frac{\epsilon^2}{2}\left(\int_{\mathbb{R}} \Lambda_{\rm eff}\right).$$

Moreover, in Corollary 3.3.7, we prove that the transmission coefficient $t^{q_\epsilon}(k)$ has a pole at

$$k_\epsilon = i\frac{\epsilon^2}{2}\left(\int_{\mathbb{R}} \Lambda_{\rm eff}\right) + \mathcal{O}(\epsilon^3), \quad \epsilon \to 0,$$

and therefore the operator H_{q_ϵ} has a simple eigensolution $(E^\epsilon, \psi^\epsilon)$ such that

$$E^\epsilon = k_\epsilon^2 = -\frac{\epsilon^4}{4}\left(\int_{\mathbb{R}} \Lambda_{\text{eff}}\right)^2 + \mathcal{O}(\epsilon^5), \quad \epsilon \to 0,$$
$$\psi^\epsilon(x) = \mathcal{O}\left(e^{-\sqrt{|E^\epsilon|}|x|}\right), \quad |x| \gg 1.$$

See Figure 1.4 for an example of an oscillatory potential $q_\epsilon(x)$ and its effective behavior along with plots of the transmission coefficients $t^{q_\epsilon}(k)$ and $t^{-\epsilon^2\Lambda_{\text{eff}}}(k)$ for various values of ϵ.

Finally, using the results on the transmission coefficient, we obtain time decay estimates for solutions to the time-dependent Schrödinger equation,

$$i\partial_t \psi = H_{q_\epsilon}\psi, \quad \psi(0,x) = \psi_0(x).$$

In Theorem 3.5.1, for $\psi_0(x)$ sufficiently localized and in the continuous spectral part of H_{q_ϵ}, that is $\psi_0 \perp \psi^\epsilon$ in $L^2(\mathbb{R})$, ψ^ϵ the eigenfunction of H_{q_ϵ}, we prove the bound

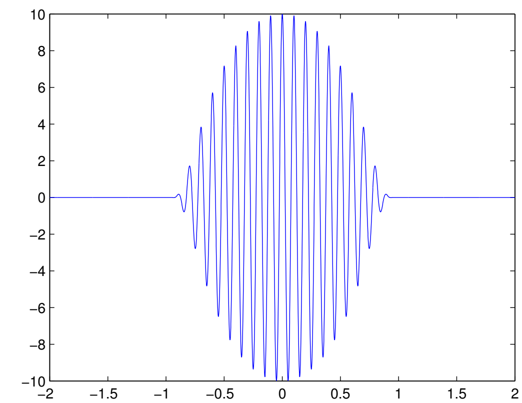
$$\left\| \left(1+|x|^3\right)^{-1} e^{-itH_{q_\epsilon}} P_\perp \psi_0 \right\|_{L^\infty(\mathbb{R})} \le |t|^{-1/2} \left(1 + \epsilon^4 \left(\int_{\mathbb{R}} \Lambda_{\text{eff}}\right)^2 |t|\right)^{-1} \left\| \left(1+|\zeta|^3\right)\psi_0(\zeta) \right\|_{L^1(\mathbb{R})}. \tag{1.27}$$

Here, $P_\perp$ denotes the projection onto the orthogonal complement of the eigenspace, span$\{\psi^\epsilon\}$, corresponding to the bifurcating eigenvalue E^ϵ. Note that for $|t| \ll 1$ the decay rate is as for the free Schrödinger evolution, $e^{-i\partial_x^2 t}$, while for $|t| \gg \epsilon^{-4}$ we have enhanced decay.

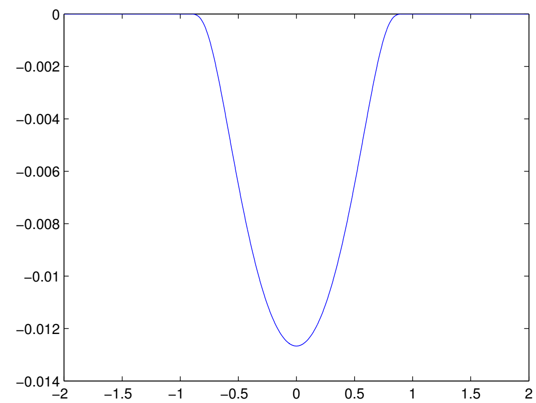
1.3 Related Previous Results

1. In [Simon, 1976], Simon analyzes the operator $H_{\lambda V} = -\Delta + \lambda V(x)$ in one and two spatial dimensions for $0 < \lambda \ll 1$ and $V(x)$ sufficiently smooth and localized. It is shown that in one dimension, $H_{\lambda V}$ has an eigenvalue of order λ^2 while in two dimensions $H_{\lambda V}$ has an exponentially small eigenvalue of order $\exp[-\alpha/\lambda^2]$, for some $\alpha > 0$. Our results in Chapter 2 (Theorem 2.2.4) generalize Simon's one-dimensional results to small perturbations $\lambda V(x)$ of a periodic background potential $Q(x) \not\equiv 0$ using a Lyapunov-Schmidt type of reduction. Furthermore, we find asymptotic expansions of the eigenfunctions corresponding to the bifurcating eigenvalues induced by the perturbation $\lambda V(x)$.

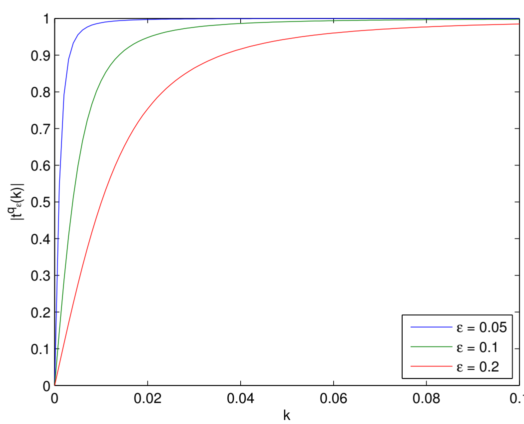
2. In [Gesztesy and Simon, 1993], Gesztesy and Simon analyze localized perturbations to the operator $H_Q = -\partial_x^2 + Q(x)$ for periodic potentials $Q(x)$. They give sufficient conditions for



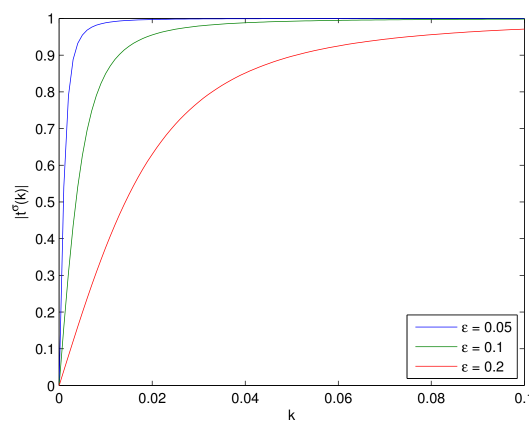
(a) $q_\epsilon(x) = \mathbf{1}_{[-1;1]}(x)10e^{\frac{-x^2}{1-x^2}}\cos(2\pi x/\epsilon)$



(b) $-\epsilon^2\Lambda_{\text{eff}}(x) = \frac{-\epsilon^2}{8\pi^2}\mathbf{1}_{[-1;1]}(x)\left(10e^{\frac{-x^2}{1-x^2}}\cos\left(\frac{2\pi x}{\epsilon}\right)\right)^2$



(c) $|t^{q_\epsilon}(k)|$, $k \in (0; 0.1)$, ϵ varying



(d) $\left|t^{-\epsilon^2\Lambda_{\text{eff}}}(k)\right|$, $k \in (0; 0.1)$, ϵ varying

Figure 1.4: Plots of potentials $q_\epsilon(x)$, (a), and the corresponding effective potential $-\epsilon^2\Lambda_{\text{eff}}(x)$, (b). Transmission coefficients $t^{q_\epsilon}(k)$, (c), and $t^{-\epsilon^2\Lambda_{\text{eff}}}(k)$, (d).

the bifurcation of eigenvalues into the gaps of the continuous spectrum of H_Q. We extend this result in Chapter 2 (Theorem 2.2.4) by giving asymptotic expressions of the bifurcating eigenvalues and their corresponding eigenfunctions.

3. In [Parzygnat *et al.*, 2010], Parzygnat *et. al.* analyze small and localized perturbations to the operator $H_Q = -\Delta + Q(x)$ in one and two spatial dimensions for periodic potentials $Q(x)$ and study the emergence of eigenvalues in the spectral gaps of H_Q. They use formal trial function arguments to show the existence of such states and, by formal asymptotic arguments, they obtain the condition (1.13) for the case of the first spectral gap. We make these results rigorous for the one-dimensional case in Chapter 2 (Theorem 2.2.4). Furthermore, we generalize to all band gaps, not just the first band gap.

4. In [Deift and Hempel, 1986], Deift and Hempel consider operators of the form $H_{Q+\lambda V} = -\Delta + Q(x) + \lambda V(x)$, where $Q(x)$ is real, periodic, and bounded, λ is constant, and $V(x)$ is real and bounded. Using the Birman-Schwinger principle, they determine requirements on λ and $V(x)$ such that $\text{spec}(H_{Q+\lambda V})$ has point spectrum components. In Chapter 2 (Theorem 2.2.4), we obtain conditions for which the one-dimensional operator $\text{spec}(H_{Q+\lambda V})$ has bifurcating eigenvalues in the spectral gaps of $\text{spec}(H_Q)$ and furthermore, find explicit asymptotic expansions for them and their corresponding eigenfunctions.

5. In [Bronski and Rapti, 2011], Bronski and Rapti consider operators of the form $H_{Q+W} = -\partial_x^2 + Q(x) + W(x)$, where $Q(x)$ is periodic and $W(x)$ is globally supported and not necessarily localized (and therefore does not necessarily satisfy Weyl's Theorem 1.1.4). Using a homotopy argument, they determine conditions on $W(x)$ under which a point spectrum is found in the gaps of $\text{spec}(H_Q)$. They also obtain results on the number of eigenvalues in the spectral gaps.

6. In [Hoefer and Weinstein, 2011], Hoefer and Weinstein consider operators of the form $H_{Q+\epsilon^2 V(\epsilon\cdot)} = -\Delta + Q(x) + \epsilon^2 V(\epsilon x)$ in one, two, and three dimensions. Here $Q(x)$ is periodic on $\mathbb{R}^d$ and $V(y)$ is localized. Using a Lyapunov-Schmidt argument, they prove that for sufficiently small $0 < \epsilon \ll 1$, there emerge eigenvalues from the bands of $\text{spec}(H_Q)$. We apply a similar Lyapunov-Schmidt technique in Chapters 2 and 4 to the operators $H_{Q+\lambda V}$ and H_{Q+q_ϵ}, respectively.

7. In [Duchêne and Weinstein, 2011], Duchêne and Weinstein consider operators of the form $H_{q_{\rm av}+q_\epsilon} = -\partial_x^2 + q_{\rm av}(x) + q(x, x/\epsilon)$, where $q_{\rm av}(x)$ is a localized background potential, $0 < \epsilon \ll 1$, and $q(x, x/\epsilon)$ is localized in the x-variable and 1-periodic in the x/ϵ-variable. Using a method developed in [Golowich and Weinstein, 2005], they derive a detailed and rigorous expansion of the transmission coefficient, $t^{q_{\rm av}+q_\epsilon}(k)$. In this work, singular potentials are also admitted. Potentials with singularities, *e.g.* jump discontinuities or Dirac delta singularities, give rise to interface-effects which require the inclusion of interface corrections, not captured by standard bulk homogenization theory, in the expansions. For generic potentials, these expansions hold for any fixed $k \in \mathbb{R}$ and $\epsilon \to 0$, but for nongeneric potentials, they require k to be bounded away from 0. In Chapter 3 (Theorem 3.3.3), we extend these results for sufficiently smooth potentials by finding an approximation for the transmission coefficient $t^{q_{\rm av}+q_\epsilon}(k)$ even for k close to zero. In particular, for nongeneric potentials $q_{\rm av}(x)$, we find that $t^{q_{\rm av}+q_\epsilon}(k)$ has a pole at $k_\star = \mathcal{O}(\epsilon^2)$ along the positive imaginary axis and therefore $H_{q_{\rm av}+q_\epsilon}$ has a eigenvalue at $k_\star^2$.

8. In [Borisov and Gadyl'shin, 2008], Borisov and Gadyl'shin consider the operator $H_\epsilon = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x) + \epsilon L_\epsilon$, where $p(x)$ and $q(x)$ are sufficiently smooth periodic functions, $p(x) \geq p_0 > 0$, and for $0 < \epsilon \ll 1$, L_ϵ is a compactly supported operator. Using spectral theoretic results, the authors determine the conditions on L_ϵ which induce the emergence of a point spectrum to the spectrum of the unperturbed operator, $\text{spec}(H_0)$. With a Birman-Schwinger like argument, they also construct the eigenpair solutions for a particular family of perturbations L_ϵ. In [Borisov, 2011], Borisov studies a two-dimensional version of the operator H_ϵ. Here L_ϵ is not necessarily compact, but it is localized and bounded uniformly with respect to ϵ. The author proves necessary and sufficient conditions for the bifurcation of eigenvalues from the continuous spectrum of H_0 and constructs an asymptotic expansion for these discrete eigensolutions. Using a Lyapunov-Schmidt like argument In Chapter 2, we study the spectrum of the related operator $H_{Q+\lambda V}$ for $Q(x)$ a periodic background potential and $V(x)$ localized with $0 < \lambda \ll 1$. Using a Lyapunov-Schmidt like argument, we prove results similar to those found in [Borisov and Gadyl'shin, 2008] except that we allow for non-compact perturbations of $Q(x)$.

   Furthermore, in [Borisov, 2007], Borisov considers operators of the form $H_{Q+q_\epsilon} = -\partial_x^2 + Q(x) + q(x, x/\epsilon)$ where $Q(x)$ is periodic and $q(x, x/\epsilon)$ is compact in the slow x-variable and periodic with zero mean in the fast x/ϵ-variable. These results extend those found in [Borisov and Gadyl'shin, 2006] where the $Q(x) \equiv 0$ case was considered. Borisov finds conditions under which eigenvalues bifurcate from the band of $\mathrm{spec}(H_Q)$ into the spectral gaps. He also finds asymptotic expansions of these eigenvalues and their corresponding eigenfunctions. In Chapter 4 (Theorem 4.2.3), we extend these results to non-compact localized perturbations of the operator H_Q which converge weakly to zero. A special case would include perturbations of the form found in [Borisov, 2007].

1.4 Future Directions

In this section, we briefly outline some possible future directions for the work presented in the thesis:

1. For the operator $H_{q_\epsilon} = -\partial_x^2 + q(x, x/\epsilon)$, we derived local time-decay estimates for localized initial conditions orthogonal to the bound state, see equation (3.17) and Chapter 3. In particular, the decay rate is $\mathcal{O}(t^{-1/2})$ for times $t \ll \epsilon^{-4}$ and $\mathcal{O}(t^{-3/2})$ for $t \geq \epsilon^{-4}$. We believe that our methods can be extended to give detailed properties of the resolvent $(-\partial_x^2 + Q(x) + \lambda V(x) - E)^{-1}$ (and $(-\partial_x^2 + Q(x) + q(x, x/\epsilon) - E)^{-1}$) and therefore the spectral measure [Reed and Simon, 1978] near the band edges. Such information could be used to derive the detailed dispersive time-decay behavior. However, the decay estimates of the type obtained in (3.17) can be expected to hold only for initial conditions which are spatially localized near band edges. Initial conditions with spectral components away from band edges can sample a regime where, for Q non-zero, the dispersion relation has higher degeneracy, yielding different (slower) dispersive time-decay [Firsova, 1996; Cai, 2006; Cuccagna, 2008].

2. It would also be of interest to extend the results obtained for the operators $H_{Q+\lambda V}$ and H_{Q+q_ϵ} in higher dimensions. In spatial dimension $d = 2$ and case $Q = 0$, Simon [Simon, 1976] proved that the bound states generated by a multiplicatively small perturbation is exponentially close to the edge of the continuous spectrum. Such results have been extended by Borisov [Borisov, 2011] in the periodic (Q nontrivial) case. Formal asymptotics were obtained by Wang *et. al.* [Wang *et al.*, 2007]. In spatial dimensions $d \geq 3$, it is well know that for sufficiently small λ, $-\Delta + \lambda V(x)$ does not have a discrete spectrum, by the Cwikel-Lieb-Rozenblum bound

[Hundertmark, 2007]. Finally, Parzygnat *et. al.* [Parzygnat *et al.*, 2010] also treat the case of dimensions $d \geq 3$, where the defect potential, $V(x)$, is localized in one or two dimensions.

3. Another problem of interest would be to consider operators of the form $H_{w_\epsilon} \equiv -\nabla \cdot w_\epsilon(x)\nabla$ for appropriate potentials $w_\epsilon(x)$. In [Figotin and Klein, 1997; Figotin and Klein, 1998], Figotin and Klein consider operators of the form $H_Q = \nabla \times \frac{1}{Q(x)}\nabla\times$ on $\mathbb{R}^n$ for periodic potentials $Q(x)$, in the context of acoustic and electromagnetic waves. They consider perturbations to the potential $Q(x)$ and show that the essential spectrum does not change while the emergence of localized modes in the spectral gaps occurs under specific conditions using a Berman-Schwinger type method.

4. In [Moskow and Vogelius, 1997], Moskow and Vogelius, and in [Santosa and Vogelius, 1993], Santosa and Vogelius, consider the operator $H_a = -\nabla \cdot a(x/\epsilon)\nabla$ where $a(y)$ is periodic and a symmetric positive definite matrix. Using standard homogenization techniques, they derive an effective expression $H_{a_{\text{eff}}} = -\nabla \cdot a_{\text{eff}}\nabla$ and expansions for the eigenvalues of the problem $H_a u = \lambda u$, $x \in \Omega \subset \mathbb{R}^d$, with the boundary condition $u = 0$ on $\delta\Omega$.

   Similarly, in [Gérard-Varet and Masmoudi, 2012], Gérard-Varet and Masmoudi consider the operator $H_a = -\nabla \cdot a(x/\epsilon)\nabla$ where $a(y)$ is periodic and sufficiently smooth. They seek to homogenize the system $H_a u = 0$, $x \in \Omega \subset \mathbb{R}^d$, $d \geq 2$, with boundary conditions $u(x) = \phi(x, x/\epsilon)$, $x \in \delta\Omega$, ϕ sufficiently smooth and periodic in the fast variable. They prove that there exists a solution u^ϵ to the above problem that is well approximated by u^0 which solves the effective problem $-\nabla \cdot a^0\nabla u = 0$, $x \in \Omega$, and $u^0(x) = \phi_\star(x)$, $x \in \delta\Omega$.

1.5 Thesis outline

This thesis is structured in four chapters, three appendices, and an extensive bibliography. Chapters 2, 3, and 4 are adapted from peer-reviewed publications and can be read independently: [Duchêne *et al.*, 2014a; Duchêne *et al.*, 2014c; Duchêne *et al.*, 2014b]. **Please note that the notation used throughout the thesis may not be consistent, but each chapter is consistent in and of itself.** Each chapter begins with a list of notation used for that section.

The summary of each chapter is given below. References are given for chapters where the initial content has been published elsewhere.

Ch. 1 The current chapter presents a brief introduction to this thesis. It introduces and motivates the spectral problem and presents a summary of the results.

Ch. 2 The analysis of localized and small in amplitude perturbations to the periodic, one-dimensional, time-independent Schrödinger operator, $H_{Q+\lambda V}$, is presented. Using a variant of the Lyapunov-Schmidt reduction technique, we prove the emergence of a discrete spectrum from the unperturbed spectrum of H_Q, which consists of only a continuous spectrum. ([Duchêne *et al.*, 2014a])

Ch. 3 We investigate scattering, localization, and dispersive time decay properties for the one-dimensional Schrödinger operator with rapidly oscillating and spatially localized potential, $H_{q_{\rm av}+q_\epsilon}$. Here $q_{\rm av}(x)$ is a spatially localized background potential and $q_\epsilon = q(x, x/\epsilon)$ where $q(x, y)$ is periodic with zero mean with respect to y. ([Duchêne *et al.*, 2014c])

Ch. 4 Using a Lyapunov-Schmidt type of reduction again, we study the effect of localized oscillatory perturbations to the periodic, one-dimensional, time-independent Schrödinger operator, H_{Q+q_ϵ}. We prove the emergence of a discrete spectrum from the unperturbed spectrum of H_Q, which consists of only a continuous spectrum. ([Duchêne *et al.*, 2014b])

Ap. A We demonstrate how solutions of the eigenvalue problem for the Schrödinger equation can be used to construct guided wave modes for Maxwell's equations.

Ap. B We review properties of Floquet-Bloch theory and derive some bounds that are necessary in the proofs in Chapters 2 and 4.

Ap. C For completeness, we present the standard Lyapunov-Schmidt reduction as described in [Nirenberg, 2001].

Part II

Main Results

Chapter 2

Small and localized perturbations of H_Q

2.1 Introduction

In this chapter, we consider the Schrödinger operator:

$$H_{Q+\lambda V} \equiv -\partial_x^2 + Q(x) + \lambda V(x), \quad 0 < \lambda \ll 1. \tag{2.1}$$

Here, $Q(x)$ is continuous, real-valued, and 1-periodic, $Q(x+1) = Q(x)$, and $\lambda V(x)$ is taken to be localized and sufficiently smooth. In particular, we consider $Q(x)$ to be a background potential and $\lambda V(x)$ a localized perturbation of the operator $H_Q \equiv -\partial_x^2 + Q(x)$. We thus begin with the analysis of the spectrum of H_Q.

As discussed in Chapter 1 (Section 1.1), the spectrum of the Schrödinger operator H_Q is continuous and is the union of closed intervals called *spectral bands* [Reed and Simon, 1978]. The complement of the spectrum is a union of open intervals called *spectral gaps*. The spectrum is determined by the family of self-adjoint eigenvalue problems parametrized by the *quasi-momentum* $k \in (-1/2, 1/2]$:

$$H_Q u(x;k) = E\ u(x;k), \tag{2.2}$$

$$u(x+1;k) = e^{2\pi i k} u(x;k). \tag{2.3}$$

That is, we seek *k-pseudo-periodic* solutions of the eigenvalue equation. For each $k \in (-1/2, 1/2]$, the self-adjoint eigenvalue problem (2.2)-(2.3) has discrete eigenvalue-spectrum (listed with multiplicity):

$$E_0(k) \leq E_1(k) \leq \cdots \leq E_b(k) \leq \ldots \tag{2.4}$$

with corresponding k-pseudo-periodic eigenfunctions $u_b(x; k)$, $b \geq 0$. The b^{th} spectral band is given by:

$$\mathcal{B}_b \ = \bigcup_{k \in (-1/2, 1/2]} E_b(k). \tag{2.5}$$

The spectrum of H_Q is given by:

$$\mathrm{spec}(H_Q) = \bigcup_{b \geq 0} \mathcal{B}_b \ = \bigcup_{b \geq 0} \ \bigcup_{k \in (-1/2, 1/2]} E_b(k). \tag{2.6}$$

Since the boundary condition (2.3) is invariant with respect to $k \mapsto k+1$, the functions $E_b(k)$ can be extended to all $\mathbb{R}$ as periodic functions of k. The minima and maxima of $E_b(k)$ occur at $k = k_* \in \{0, 1/2\}$; see Figure 2.1. In cases where extrema border a spectral gap, we have that $\partial_k^2 E_b(k_*)$ is either strictly positive or strictly negative [Eastham, 1973; Reed and Simon, 1978]; see Lemma B.1.2 in Appendix B.

Consider now the perturbed operator H_{Q+W}, where $W(x)$ is sufficiently localized in space. By Weyl's Theorem 1.1.4 on the stability of the essential spectrum, one has $\mathrm{spec}_{\mathrm{cont}}(H_{Q+W}) = \mathrm{spec}_{\mathrm{cont}}(H_Q)$ [Reed and Simon, 1978]. The effect of a localized perturbation is to possibly introduce discrete eigenvalues into the open spectral gaps. Note that in our setting, H_{Q+W} does not have discrete eigenvalues embedded in its continuous spectrum; see [Rofe-Beketov, 1964; Gesztesy and Simon, 1993].

Therefore, perturbations of the form $\lambda V(x)$ to the operator H_Q can only result in the bifurcation of localized bound states into gaps of the continuous spectrum of H_Q and $\mathrm{spec}_{\mathrm{cont}}(H_{Q+\lambda V}) = \mathrm{spec}_{\mathrm{cont}}(H_Q)$.

Before giving a summary of results, let us discuss the physical importance of the above phenomenon. In a periodic medium, a spatially localized initial condition for an energy-conserving wave equation disperses (spatially spreads) and decays in amplitude as time advances. This (Floquet-Bloch) dispersion is associated with the continuous spectrum (extended states) of the underlying differential operator and the absence of discrete eigenvalues (localized bound states) [Kuchment, 2001;

Reed and Simon, 1978]. The introduction of localized perturbations in a periodic medium leads to *defect modes*, states in which energy remains trapped and spatially localized. The process by which the system undergoes a transition from one with only propagating delocalized states to one which supports both localized and propagating states is associated with the emergence or bifurcation of discrete eigenvalues from the continuous spectrum associated with the unperturbed periodic structure.

We next turn to a summary of our results. See Theorem 2.2.1 and Theorem 2.2.4 for detailed statements.

Let $E_b(k_*),\ k_* \in \{0, 1/2\}$, denote an endpoint (uppermost or lowermost) of the b^{th} spectral band, bordering a spectral gap. We show that under the condition:

$$\partial_k^2 E_b(k_*) \ \times \ \int_{\mathbb{R}} \ |u_b(x;k_*)|^2 \ V(x) \ dx \ < \ 0, \tag{2.7}$$

the following holds: There exists a positive number, λ_0, such that for all $0 < \lambda < \lambda_0$, $H_{Q+\lambda V}$ has a simple discrete eigenvalue,

$$E^\lambda \ = \ E_b(k_*) \ + \ \lambda^2 \mu_\star \ + \ \mathcal{O}(\lambda^{2+\alpha}), \quad \text{for some} \quad \alpha > 0. \tag{2.8}$$

which bifurcates from the edge, $E_b(k_*)$ of band $\mathcal{B}_b$, into a spectral gap.

1. If $\partial_k^2 E_b(k_*) > 0$ and $\int_{\mathbb{R}} \ |u_b(x;k_*)|^2 \ V(x) \ dx < 0$, then $\mu_\star < 0$ and E^λ lies near the lowermost edge of $\mathcal{B}_b$; see the center panel of Figure 2.1.

2. If $\partial_k^2 E_b(k_*) < 0$ and $\int_{\mathbb{R}} \ |u_b(x;k_*)|^2 \ V(x) \ dx > 0$, then $\mu_\star > 0$ and E^λ lies near the uppermost edge of $\mathcal{B}_b$; see the right panel of Figure 2.1.

For $0 < \lambda < \lambda_0$, $\psi^\lambda(x)$, the eigenstate corresponding to the eigenvalue E^λ, is well-approximated in L^∞ by $\psi_\star(\lambda x)$, where $(\mu_\star, \psi_\star(y))$ denotes the unique eigenpair of the effective operator:

$$H_{b,\text{eff}} \ = \ -\frac{d}{dy} \ A_{b,\text{eff}} \ \frac{d}{dy} \ + \ B_{b,\text{eff}} \times \delta(y), \tag{2.9}$$

with constant effective parameters $A_{b,\text{eff}}$ and $B_{b,\text{eff}}$. Here,

$$A_{b,\text{eff}} \ = \ \frac{1}{8\pi^2} \partial_k^2 E_b(k_*) \tag{2.10}$$

is the inverse *effective mass* associated to the spectral edge $E_b(k_*)$,

$$B_{b,\mathrm{eff}} \ = \ \int_{\mathbb{R}} |u_b(x;k_*)|^2 \ V(x) \ dx, \tag{2.11}$$

and $\delta(y)$ denotes the Dirac delta mass at $y = 0$. The unique eigenpair $(\mu_\star, \psi_\star)$ of the eigenvalue problem $H_{b,\mathrm{eff}}\psi = \mu\psi$ can be written as:

$$\mu_\star = -\frac{B_{b,\mathrm{eff}}^2}{4A_{b,\mathrm{eff}}}, \quad \psi_\star(y) = \exp\left(\frac{B_{b,\mathrm{eff}}}{2A_{b,\mathrm{eff}}}|y|\right). \tag{2.12}$$

Remark 2.1.1. *For the case $Q \equiv 0$, $H_Q = H_0 = -\partial_x^2$ and its spectrum consists of a semi-infinite interval,* $\mathrm{spec}(H_0) = [0, \infty)$*, the union of touching bands with no finite length gaps. Furthermore,* $|u_b(x;k)| \equiv 1$*, for all $|k| \leq 1/2$ and $b \geq 0$. The only band edge which borders a gap is located at* $E_0(0) = 0$*, where we have: $k_* = 0$, $E_0(k) = 4\pi^2 k^2$ and $\partial_k^2 E_0(k_*) = 8\pi^2$. In this case, our results describe the bifurcation of an eigenvalue from the edge of the continuous spectrum of H_0 induced by a small and localized perturbation: $H_{\lambda V} = -\partial_x^2 + \lambda V(x)$, under the condition $\int_{\mathbb{R}} V < 0$. The effective operator is:*

$$H_{0,\mathrm{eff}} = -\frac{d^2}{dy^2} + \int_{\mathbb{R}} V \times \delta(y). \tag{2.13}$$

2.1.1 Outline and remarks on the proof

In Section 2.2 we give precise technical statements of our main results: Theorem 2.2.1 and Theorem 2.2.4.

Our strategy of proof is to transform the eigenvalue problem using an appropriate spectral transform (Fourier or Floquet-Bloch) to a formulation in frequency (quasi-momentum) space. Anticipating a bifurcation from the spectral edge, we express the eigenvalue problem in terms of coupled equations governing the frequency components located *near* the band edge and those which are *far* from the band edge. The precise frequency cutoff depends on the small parameter, λ. We employ a Lyapunov-Schmidt reduction strategy [Nirenberg, 2001] in which we solve for the *far*-frequency components as a functional of the *near*-frequency components. This yields a reduction to a closed *bifurcation equation* for the *near*-frequency components. In contrast to classical applications of this strategy, our reduced equation is infinite dimensional. For λ small, in an appropriate scaled limit, the bifurcation equation is asymptotically exactly solvable; it is the eigenvalue problem for the effective operator $H_{b,\mathrm{eff}}$.

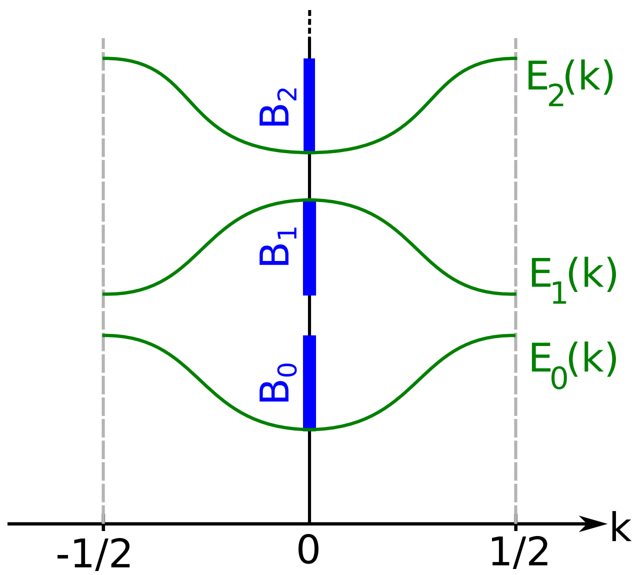


Figure 2.1: Sketch of spectra. Eigenvalues, $E_b(k), k \in (-1/2, 1/2], b = 0, 1, 2, \ldots,$ are displayed in green. The continuous spectrum, is in blue, and discrete eigenvalues are indicated through cross markers. Left panel corresponds spec(H_Q), Q periodic. The center (resp. right) panel corresponds spec$(H_{Q+\lambda V})$, where λV is small, localized negative (resp. positive).

In Section 2.2 we state the results of this chapter, Theorems 2.2.1 and 2.2.4. In Section 2.3, we prove a general technical lemma, crucial to the analyses of Sections 2.4 and 2.5, covering the kinds of bifurcation equations which arise. Finally, in Section 2.6 we give proofs, by a bootstrap method, of Corollary 2.2.3 and Corollary 2.2.6 which contain more detailed expansions and sharper error terms for the bifurcating eigenstates than those in Theorem 2.2.1 and Theorem 2.2.4.

2.1.2 Definitions and notation

We denote by C a constant, which does not depend on the small parameter, λ. It may depend on norms of $Q(x)$ and $V(x)$, which are assumed finite. $C(\zeta_1, \zeta_2, \dots)$ is a constant depending on ζ_1, ζ_2, ... We write $A \lesssim B$ if $A \leq C\ B$, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

χ and $\overline{\chi}$ are the characteristic functions defined by

$$\chi_\delta(\xi) = \chi\left(|\xi| < \delta\right) \equiv \begin{cases} 1, & |\xi| < \delta \\ 0, & |\xi| \geq \delta \end{cases}, \qquad \overline{\chi}_\delta(\xi) = \overline{\chi}(|\xi| < \delta) \equiv 1 - \chi(|\xi| < \delta)\ . \tag{2.14}$$

For $f, g \in L^2(\mathbb{R})$, the Fourier transform and its inverse are given by

$$\mathcal{F}\{f\}(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx, \qquad \mathcal{F}^{-1}\{g\}(x) \equiv \check{g}(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} g(\xi)d\xi.$$

$\mathcal{T}$ and $\mathcal{T}^{-1}$ denote the Gelfand-Bloch transform and its inverse; see Appendix B.

$L^{p,s}(\mathbb{R})$ is the space of functions $F : \mathbb{R} \to \mathbb{R}$ such that $(1 + |\cdot|^2)^{s/2} F \in L^p(\mathbb{R})$, endowed with the norm

$$\|F\|_{L^{p,s}(\mathbb{R})} \equiv \|(1 + |\cdot|^2)^{s/2} F\|_{L^p(\mathbb{R})} < \infty, \quad 1 \leq p \leq \infty. \tag{2.15}$$

$W^{k,p}(\mathbb{R})$ is the space of functions $F : \mathbb{R} \to \mathbb{R}$ such that $\partial_x^j F \in L^p(\mathbb{R})$ for $0 \leq j \leq k$, endowed with the norm

$$\|F\|_{W^{k,p}(\mathbb{R})} \equiv \sum_{j=0}^{k} \|\partial_x^j F\|_{L^p(\mathbb{R})} < \infty, \quad 1 \leq p \leq \infty.$$

2.2 Bifurcation of defect states into gaps; main results

Consider the eigenvalue problem:

$$\left(-\partial_x^2 + Q(x) + \lambda V(x)\right)\psi^\lambda \;=\; E^\lambda \psi^\lambda\ ,\ \psi \in L^2(\mathbb{R}),$$

where $Q(x)$ is continuous, $1-$periodic, $\lambda > 0$ is small, and $V(x)$ is spatially localized. Our first result concerns the case where $Q \equiv 0$:

Theorem 2.2.1 ($Q \equiv 0$)**.** *Let V be such that $\widehat{V} \in W^{1,\infty}(\mathbb{R})$; thus $\int_{\mathbb{R}}(1+|x|)|V(x)|\,dx < \infty$ suffices. Assume $\widehat{V}(0) = \int_{\mathbb{R}} V < 0$. There exists positive constants λ_0 and $C(V,\lambda_0)$, such that for all $0 < \lambda < \lambda_0$, there exists an eigenpair $(E^\lambda, \psi^\lambda)$, solution of the eigenvalue problem*

$$\left(-\partial_x^2 + \lambda V(x)\right)\psi^\lambda(x) \;=\; E^\lambda \psi^\lambda(x) \tag{2.16}$$

with negative eigenvalue of the order λ^2. Specifically,

$$\left| E^\lambda \;-\; \left[-\frac{\lambda^2}{4}\left(\int_{\mathbb{R}} V\right)^2\right]\right| \;\leq\; C\lambda^{5/2}\,, \tag{2.17}$$

$$\sup_{x\in\mathbb{R}}\left|\psi^\lambda(x) \;-\; \exp\left(\frac{\lambda}{2}\left(\int_{\mathbb{R}} V\right)|x|\right)\right| \;\leq\; C\lambda^{1/2}. \tag{2.18}$$

The eigenvalue, E^λ, is unique in the neighborhood defined by (2.17)*, and the corresponding eigenfunction, ψ, is unique up to a multiplicative constant.*

Remark 2.2.2. *Theorem 2.2.1 shows, and is essentially proved by demonstrating, that for small positive λ, the leading order behavior of the eigenstate $\left(E^\lambda, \psi^\lambda(x)\right)$ is a scaling of the unique eigenstate of the attractive Dirac delta potential:*

$$\Big(\; E^\lambda \;,\; \psi^\lambda(x) \;\Big) \;\approx\; \left(\; \lambda^2\theta_0^2 \;,\; g_0(\lambda x) \;\right)\;,$$

where $\theta_0 = -\frac{1}{2}\int_{\mathbb{R}} V > 0$ and $g_0(y) = e^{-\theta_0|y|}$ satisfy

$$\left[-\partial_y^2 \;+\; \int_{\mathbb{R}} V \;\times\; \delta(y)\right]\; g_0(y) = -\theta_0^2\; g_0(y). \tag{2.19}$$

The error bounds in Theorem 2.2.1 are not optimal. However, the bootstrap argument of Section 2.6 can be used to recover a higher order expansion on E^λ, similar to that obtained in [Simon, 1976].

Corollary 2.2.3. *Assume $(1+|x|^2)V \in L^1$, and $\widehat{V}(0) = \int_{\mathbb{R}} V(z)\,dz < 0$. Then E^λ, as defined in Theorem 2.2.1, satisfies the precise estimate:*

$$E^\lambda \;=\; -\lambda^2\;[\theta(\lambda)]^2\,,\;\textit{with}\;\theta(\lambda) \;=\; -\frac{1}{2}\int_{\mathbb{R}} V \;-\; \frac{1}{4}\,\lambda\iint_{\mathbb{R}^2} V(x)|x-y|V(y)\,dxdy \;+\;\mathcal{O}(\lambda^{3/2})\,. \tag{2.20}$$

Simon [Simon, 1976] and Klaus [Klaus, 1977] prove expansion (2.20), under the conditions: $(1+|x|)V(x) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} V \leq 0$, with the error term $o(\lambda)$. Corollary 2.2.3 gives a sharper error term under a more stringent decay condition on V. That Theorem 2.2.1 implies Corollary 2.2.3 is proved in Section 2.6.

Theorem 2.2.4 (Q non-trivial, $1-$periodic). *Let Q be continuous, $1-$periodic, and let V be such that $\int_{\mathbb{R}}(1+|x|)V(x)dx < \infty$ and $V \in L^\infty$. Let $E_{b_*} : k \in (-1/2, 1/2] \to \mathbb{R}$ denote the band dispersion function associated with the $(b_*)^{th}$ band of the continuous spectrum of $-\partial_x^2 + Q(x)$. Fix a spectral band edge of the $(b_*)^{th}$ band; thus $E_* = E_{b_*}(k_*)$, where $k_* = 0$ or $k_* = 1/2$ (see Lemma B.1.2 in Appendix B).*

Assume either

$$\partial_k^2 E_{b_*}(k_*) > 0 \quad \text{and} \quad \int_{\mathbb{R}} |u_{b_*}(x;k_*)|^2 V(x)dx < 0, \tag{2.21}$$

or

$$\partial_k^2 E_{b_*}(k_*) < 0 \quad \text{and} \quad \int_{\mathbb{R}} |u_{b_*}(x;k_*)|^2 V(x)dx > 0. \tag{2.22}$$

Then, there is a positive constants, λ_0 and $C = C(\lambda_0, V, Q)$, such that for all $\lambda < \lambda_0$, the following assertions hold:

1. *There exists an eigenpair $\left(E^\lambda, \psi^\lambda(x)\right)$ of the eigenvalue problem*

$$\left(-\partial_x^2 + Q(x) + \lambda V(x)\right)\psi^\lambda(x) \;=\; E^\lambda \psi^\lambda(x), \quad \psi^\lambda \in L^2(\mathbb{R}) \;. \tag{2.23}$$

2. *Define*

$$\alpha_0 \;\equiv\; \frac{\int_{-\infty}^{\infty} |u_{b_*}(x;k_*)|^2 V(x)dx}{\frac{1}{4\pi^2}\partial_k^2 E_{b_*}(k_*)} \;<\; 0, \tag{2.24}$$

where the inequality holds by (2.21) and (2.22). Then, E^λ and $\psi^\lambda(x)$ satisfy the following approximations:

$$\left| E^\lambda \;-\; \left(\; E_{b_*}(k_*) \;+\; \lambda^2 E_2 \;\right) \right| \;\leq\; C\lambda^{2+1/4} \;, \tag{2.25}$$

$$\sup_{x\in\mathbb{R}} \left| \;\psi^\lambda(x) \;-\; u_{b_*}(x;k_*)\exp(\lambda\alpha_0|x|)\right| \;\leq\; C\lambda^{1/4} \;, \tag{2.26}$$

where

$$E_2 \;=\; -\frac{\left|\int_{-\infty}^{\infty} |u_{b_*}(x;k_*)|^2 V(x)dx\right|^2}{\frac{1}{2\pi^2}\partial_k^2 E_{b_*}(k_*)} \;. \tag{2.27}$$

Note that the direction of bifurcation of E^λ is given by:

$$\mathrm{sgn}\,(E_2) = \; -\mathrm{sgn}\left(\partial_k^2 E_{b_*}(k_*)\right) \;.$$

3. *The eigenstate,* $(E^\lambda, \psi^\lambda)$, *is unique (up to a multiplicative constant for* ψ^λ*) in the neighborhood defined by* (2.25), (2.26).

Remark 2.2.5. *By Theorem 2.2.4, the bifurcating eigenvalue* E^λ *lies in the spectral gap of* $-\partial_x^2 + Q(x)$ *at a distance* $\mathcal{O}(\lambda^2)$ *near the spectral edge* E_**; see Figure 2.1. Moreover,* E_2 *is the unique eigenvalue and* $g_0(y) = e^{\alpha_0|y|}$ *is the unique (up to multiplication by a constant) eigenfunction of the effective (homogenized) Hamiltonian:*

$$H_{\text{eff}} \;=\; -\frac{d}{dy}\,\frac{1}{8\pi^2}\partial_k^2 E_{b_*}(k_*)\,\frac{d}{dy} \;+\; \int_{-\infty}^{\infty} |u_{b_*}(x;k_*)|^2 V(x)dx \times \delta(y).$$

The following refinement of Theorem 2.2.4 can be proved via the bootstrap argument presented in Section 2.6.

Corollary 2.2.6. *Assume* $\int_{\mathbb{R}}(1+|x|^2)V(x)dx < \infty$ *and that the hypotheses of Theorem* (2.2.4) *hold. Then,*

$$E^\lambda - E_{b_*}(k_*) \;=\; \lambda^2(E_2 + \lambda E_3) \;+\; \mathcal{O}(\lambda^{3+1/4}) = \;-\lambda^2 \frac{8\pi^2}{\partial_k^2 E_{b_*}(k_*)}\,[\Theta(\lambda)]^2\,, \tag{2.28}$$

where E_2 *is as in* (2.27),

$$E_3 \equiv \frac{-8\pi^4}{(\partial_k^2 E_{b_*}(k_*))^2}\left(\int_{-\infty}^{\infty} |u_{b_*}(x;k_*)|^2 V(x)\;dx\right) \times \left(\iint_{\mathbb{R}^2} V(x)|u_{b_*}(x;k_*)|^2|x-y||u_{b_*}(y;k_*)|^2 V(y)\;dx\;dy\right)\,,$$

and

$$\begin{aligned}\Theta(\lambda) = &-\frac{1}{2}\int_{\mathbb{R}} |u_{b_*}(x;k_*)|^2 V(x)\;dx \\ &-\frac{1}{4}\lambda\,\frac{8\pi^2}{\partial_k^2 E_{b_*}(k_*)}\iint_{\mathbb{R}^2} V(x)|u_{b_*}(x;k_*)|^2|x-y||u_{b_*}(y;k_*)|^2 V(y)\;dx\;dy + \mathcal{O}(\lambda^{1+1/4}).\end{aligned} \tag{2.29}$$

Remark 2.2.7. *For the case* $Q \equiv 0$*, the spectrum consists of only one semi-infinite band which we can label the* $b = 0$ *band. In this case,* $u_0(x; k_* = 0) = 1$ *and* $E_0(k) = 4\pi^2 k^2$*. Therefore, to leading order, relation* (2.29) *simplifies to the result of Corollary 2.2.3 and the two results are consistent.*

2.3 Key general technical results

In this section, we study the operator $\widehat{\mathcal{L}}_0[\theta]$, defined by:

$$\widehat{f}(\xi) \;\mapsto\; \widehat{\mathcal{L}}_0[\theta]\widehat{f}(\xi) \equiv \left(4\pi^2 A\xi^2 + \theta^2\right)\widehat{f}(\xi) - B\;\chi\left(|\xi| < \lambda^{-\beta}\right)\int_{\mathbb{R}} \chi\left(|\eta| < \lambda^{-\beta}\right)\;\widehat{f}(\eta)\;d\eta. \tag{2.30}$$

Here, A, B and β are fixed positive constants. The operator $\widehat{\mathcal{L}}_0[\theta]$ appears in the bifurcation equations we derived via the Lyapunov-Schmidt reduction; see Section 2.1.1.

In x-space, we have that $\mathcal{L}_0[\theta]$ is a rank one perturbation of $-A\partial_y^2+\theta^2$:

$$\mathcal{L}_0[\theta]f \;\equiv\; (-A\partial_y^2+\theta^2)f(y) \;-\; \frac{2B}{\lambda^\beta}\;\left\langle \frac{2}{\lambda^\beta}\,\mathrm{sinc}\left(\frac{2\pi}{\lambda^\beta}\,\cdot\right), f(\cdot)\right\rangle_{L^2}\;\mathrm{sinc}\left(\frac{2\pi y}{\lambda^\beta}\right), \tag{2.31}$$

where $\mathrm{sinc}(z)=\sin(z)/z$. $\mathcal{L}_0[\theta]$ is a band-limited regularization of the operator:

$$\left(\,H^{A,B} \;+\; \theta^2\right)\; f \;\equiv\; \left(-A\partial_y^2 \;-\; B\delta(y) \;+\; \theta^2\right) f\ , \tag{2.32}$$

appearing in the effective equations governing the leading order behavior of bifurcating eigenstates; see Remarks 2.2.2 and 2.2.5.

2.3.1 The operator $\widehat{\mathcal{L}}_0$

Lemma 2.3.1. *Fix constants $A>0$, $B>0$ and $\beta>0$. Define, for $\theta^2>0$, the linear operator*

$$\widehat{f}(\xi)\;\mapsto\;\widehat{\mathcal{L}}_0[\theta]\widehat{f}(\xi)\equiv\left(4\pi^2A\xi^2+\theta^2\right)\widehat{f}(\xi)-B\;\chi\left(|\xi|<\lambda^{-\beta}\right)\int_{\mathbb{R}}\chi\left(|\eta|<\lambda^{-\beta}\right)\;\widehat{f}(\eta)\;d\eta. \tag{2.33}$$

Note that $\widehat{\mathcal{L}}_0[\theta]: L^1(\mathbb{R})\to L^{1,-2}(\mathbb{R})$.

1. *There exists a unique $\theta_0^2>0$ such that $\widehat{\mathcal{L}}_0[\theta_0]$ has a non-trivial kernel.*

2. *The "eigenvalue" θ_0^2 is the unique positive solution of*

$$1\;-\;B\int_{\mathbb{R}}\frac{\chi\left(|\xi|<\lambda^{-\beta}\right)}{4\pi^2A\xi^2+\theta_0^2}d\xi\;=\;0\ . \tag{2.34}$$

3. *The kernel of $\widehat{\mathcal{L}}_0[\theta_0]$ is given by:*

$$\mathrm{kernel}\left(\widehat{\mathcal{L}}_0[\theta_0]\right)\;=\;\mathrm{span}\left\{\widehat{f}_0(\xi)\right\},\quad\text{where}\quad\widehat{f}_0(\xi)\equiv\frac{\chi\left(|\xi|<\lambda^{-\beta}\right)}{4\pi^2A\xi^2+\theta_0^2}\ . \tag{2.35}$$

4. *$\theta_0=\theta_0(\lambda)$ can be approximated as follows:*

$$\left|\theta_0\;-\;\frac{B}{2\sqrt{A}}\right|\;\le\frac{\theta_0}{2\pi^2}\;\frac{B}{A}\;\lambda^\beta. \tag{2.36}$$

5. *Define $g(x)=\exp(\alpha_0|x|)$, with $\alpha_0=-\frac{B}{2A}<0$. Then one has*

$$\sup_{x\in\mathbb{R}}\left|\;\mathcal{F}^{-1}\left\{\widehat{f}_0\right\}(x)\;-\;\frac{1}{B}g(x)\;\right|\;\le C(A,B)\lambda^\beta. \tag{2.37}$$

Proof. First note, by rearranging terms in the equation $\widehat{\mathcal{L}}_0[\theta_0]\widehat{g} = 0$, that any element, $\widehat{g}(\xi)$, of the kernel of $\widehat{\mathcal{L}}_0[\theta]$, is a constant multiple of the function $\widehat{f}_\lambda(\xi;\theta) \equiv \chi(|\xi| < \lambda^{-\beta}) \times (4\pi^2 A\xi^2 + \theta^2)^{-1}$. Thus, if $\widehat{g}$ is non-trivial then it is strictly positive or strictly negative and therefore $\int_{\mathbb{R}} \widehat{g} \neq 0$. Next, note that a necessary condition for $\widehat{g}$ to lie in the kernel of $\widehat{\mathcal{L}}_0[\theta]$ is that equation (2.34) holds. To see this, divide the equation $\widehat{\mathcal{L}}_0[\theta_0]\widehat{g} = 0$ by $\left(4\pi^2 A\xi^2 + {\theta_0}^2\right)$, multiply by $\chi\left(|\xi| < \lambda^{-\beta}\right)$, and integrate $d\xi$ over $\mathbb{R}$. This yields:

$$\int_{-\infty}^{\infty} \widehat{g}(\xi)\ d\xi \ \times \ \left[1 \ - \ B\int_{-\infty}^{\infty} \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta^2} d\xi\right] \ = \ 0, \tag{2.38}$$

By the above discussion, if $\widehat{g}$ is non-trivial then $\int_{\mathbb{R}} \widehat{g} \neq 0$. Hence θ^2 satisfies

$$J(\theta^2) \equiv 1 \ - \ B\int_{-\infty}^{\infty} \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta^2} d\xi \ = \ 0.$$

Since $J : (0,\infty) \to \mathbb{R}$ is smooth, $J'(X) > 0$, $\lim_{X\to 0} J(X) = -\infty$ and $\lim_{X\to\infty} J(X) = 1$, the function J has a unique positive root, which we denote by θ_0^2. One can check by direct substitution and the condition $J(\theta_0^2) = 0$, that any multiple of

$$\widehat{f}_0(\xi) \ \equiv \ \widehat{f}_\lambda(\xi;\theta_0) \ = \ \chi(|\xi| < \lambda^{-\beta}) \times (4\pi^2 A\xi^2 + \theta_0^2)^{-1} \tag{2.39}$$

satisfies $\widehat{\mathcal{L}}_0[\theta_0]\widehat{f}_0(\xi) = 0$.

The approximation to $\theta_0(\lambda)$, (2.36), is obtained as follows. Let θ_0^2 denote the unique solution of $J(\theta_0^2) = 0$ and θ_0 its positive square root. Then,

$$\begin{aligned} \frac{1}{B} \ &= \ \int_{\mathbb{R}} \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2\ A\ \xi^2 + \theta_0^2}\ d\xi \ = \ \int_{\mathbb{R}} \frac{1 + \left(\chi\left(|\xi| < \lambda^{-\beta}\right) - 1\right)}{4\pi^2\ A\ \xi^2 + \theta_0^2} d\xi \\ &= \ \frac{1}{2\sqrt{A}\ \theta_0} + \int_{\mathbb{R}} \frac{\chi\left(|\xi| < \lambda^{-\beta}\right) - 1}{4\pi^2 A\xi^2 + \theta_0^2} d\xi\ . \end{aligned} \tag{2.40}$$

The last term can be bounded as follows:

$$\left|\int_{\mathbb{R}} \frac{1 - \chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta_0^2} d\xi\right| \ = \ \int_{|\xi|\geq\lambda^{-\beta}} \frac{d\xi}{4\pi^2\ A\ \xi^2 + \theta_0^2} \ \leq \ \int_{|\xi|\geq\lambda^{-\beta}} \frac{d\xi}{4\pi^2\ A\ \xi^2} \ \leq \ \frac{\lambda^\beta}{2\pi^2 A}. \tag{2.41}$$

Relations (2.40), (2.41), after rearrangement of terms, yield (2.36).

Finally, let us turn to the asymptotic expression for $\mathcal{F}^{-1}\left\{\widehat{f}_0\right\}(x)$ given in (2.37). By residue computation, one has $\widehat{g}(\xi) = \dfrac{-2\alpha_0}{4\pi^2|\xi|^2+\alpha_0^2} = \dfrac{B}{4\pi^2 A|\xi|^2+\frac{B^2}{4A}}$. It follows that

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\ \mathcal{F}^{-1}\left\{\widehat{f_0}\right\}(x)-\frac{1}{B}g(x)\ \right| &\le \left\|\ \widehat{f_0}-\frac{1}{B}\widehat{g}\ \right\|_{L^1}\\ &\le \int_{\mathbb{R}}\left|\frac{\chi\left(|\xi|<\lambda^{-\beta}\right)}{4\pi^2A\xi^2+\theta_0^2}-\frac{1}{4\pi^2A|\xi|^2+\frac{B^2}{4A}}\right|\,d\xi\\ &\le \int_{\mathbb{R}}\chi\left(|\xi|<\lambda^{-\beta}\right)\left|\frac{1}{4\pi^2A\xi^2+\theta_0^2}-\frac{1}{4\pi^2A|\xi|^2+\frac{B^2}{4A}}\right|\,d\xi+\int_{\mathbb{R}}\left|\frac{1-\chi\left(|\xi|<\lambda^{-\beta}\right)}{4\pi^2A\xi^2+\frac{B^2}{4A}}\right|\,d\xi.\end{aligned}$$

A bound on the second term follows from (2.41). The first term is easily bounded, using (2.36), by $C(A,B)\lambda^{\beta}$, with some constant $C(A,B)>0$. Estimate (2.37) follows, and the proof of Lemma 2.3.1 is now complete. □

We shall also require a result on the solvability of the inhomogeneous equation

$$\left(\widehat{\mathcal{L}}_0[\theta_0]\widehat{\varphi}\right)(\xi)=\widehat{h}(\xi), \tag{2.42}$$

where $\widehat{\mathcal{L}}_0[\theta_0]$ is defined in (2.33).

Lemma 2.3.2. *The equation* (2.42) *is solvable if and only if* $\widehat{h}$ *is such that* $\chi\left(|\xi|<\lambda^{-\beta}\right)\widehat{h}(\xi)=\widehat{h}(\xi)$ *and satisfies the orthogonality condition*

$$\left\langle\widehat{f_0},\widehat{h}\right\rangle_{L^2(\mathbb{R})}=0, \tag{2.43}$$

where $\widehat{f_0}$*, displayed in* (2.35)*, spans the kernel of* $\widehat{\mathcal{L}}_0[\theta_0]$*. In that case,*

1. *any solution of the inhomogeneous equation* (2.42) *is of the form*

$$\widehat{\varphi}(\xi)\ \equiv\ (C+\widehat{h}(\xi))\widehat{f_0}(\xi)\ \equiv\ (C+\widehat{h}(\xi))\ \frac{\chi\left(|\xi|<\lambda^{-\beta}\right)}{4\pi^2A\xi^2+\theta_0^2}, \tag{2.44}$$

   for some constant C*.*

2. *The unique solution of* (2.42) *such that* $\int_{\mathbb{R}}\widehat{\varphi}=0$ *is obtained by choosing* $C=0$*:*

$$\widehat{\varphi}(\xi)\ \equiv\ \widehat{h}(\xi)\ \widehat{f_0}(\xi). \tag{2.45}$$

Proof. The solvability condition $\chi\left(|\xi|<\lambda^{-\beta}\right)\widehat{h}(\xi)=\widehat{h}(\xi)$ is straightforward, and (2.43) is obtained by taking the inner product of (2.42) with $\widehat{f_0}$, and using that $\widehat{\mathcal{L}}_0[\theta_0]$ is symmetric, and $\widehat{\mathcal{L}}_0[\theta_0]\widehat{f_0}=0$.

To show that (2.44) solves the inhomogeneous equation (2.42) we simply insert the function (2.44) into (2.42), and use the properties: $\widehat{\mathcal{L}}_0(\theta_0)\widehat{f}_0 = 0$ and $\left\langle \widehat{f}_0, \widehat{h} \right\rangle_{L^2} = 0$. This gives

$$\left(\widehat{\mathcal{L}}_0[\theta_0]\widehat{\varphi}\right)(\xi) = (4\pi^2 A\xi^2 + \theta_0^2)\widehat{h}(\xi)\widehat{f}_0(\xi) - B\,\chi\left(|\xi| < \lambda^{-\beta}\right)\int_{-\infty}^{\infty}\widehat{h}(\eta)\widehat{f}_0(\eta)d\eta$$
$$= (4\pi^2 A\xi^2 + \theta_0^2)\,\frac{\chi\left(|\xi| < \lambda^{-\beta}\right)\widehat{h}(\xi)}{4\pi^2 A\xi^2 + \theta_0^2} \;-\; B\,\chi\left(|\xi| < \lambda^{-\beta}\right)\left\langle \widehat{f}_0, \widehat{h} \right\rangle_{L^2(\mathbb{R})} \;=\; \widehat{h}(\xi).$$

The converse clearly holds by Lemma 2.3.1, since the difference of solutions of the inhomogeneous equation solves the homogeneous equation (2.33). Finally, using the orthogonality condition $\left\langle \widehat{f}_0, \widehat{h} \right\rangle_{L^2} = 0$, one has that $\int_{\mathbb{R}} \widehat{\varphi} = C\int_{\mathbb{R}} \widehat{f}_0 = 0$ if and only if $C = 0$. $\square$

2.3.2 A perturbation result for $\widehat{\mathcal{L}}_0$

As discussed in the introduction, our strategy is to obtain a reduction of the eigenvalue problem for $H_{Q+\lambda V}$ to an eigenvalue problem (the bifurcation equation) for functions supported at energies near the band-edge. These reduced equations have a general form which we study in this section.

Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ denote Banach spaces with $\mathcal{Z}_1, \mathcal{Z}_2 \subset L^1_{\rm loc}$. Assume that for any $(f, g) \in \mathcal{Z}_1 \times \mathcal{Z}_2$,

$$|\langle f, g\rangle_{L^2}| \lesssim \|f\|_{\mathcal{Z}_2}\|g\|_{\mathcal{Z}_1}, \qquad \|fg\|_{\mathcal{Z}_2} \lesssim \|f\|_{\mathcal{Z}_2}\|g\|_{L^\infty}, \qquad \text{and} \qquad \|(1+\xi^2)^{-1}f\|_{\mathcal{Z}_2} \lesssim \|f\|_{\mathcal{Z}_1}. \quad (2.46)$$

Furthermore, we also assume that $\widehat{f}_0 \in \mathcal{Z}_1 \cap \mathcal{Z}_2$, where $\left(\theta_0^2, \widehat{f}_0\right)$ is the unique normalized solution of the homogeneous equation $\widehat{\mathcal{L}}_0[\theta]\widehat{f} = 0$; see Lemma 2.3.1.

Remark 2.3.3. *In order to prove Theorems 2.2.1 and 2.2.4, we shall apply Lemma 2.3.4, below, with*

- *Case $Q \equiv 0$: $(\mathcal{Z}_1, \mathcal{Z}_2) = \left(L^\infty, L^1\right)$; and*
- *Q non-trivial, $1-$periodic: $(\mathcal{Z}_1, \mathcal{Z}_2) = \left(L^{2,-1}, L^{2,1}\right)$, where $L^{2,s}$ is the space of locally integrable functions such that*

$$\|F\|_{L^{2,s}} \;\equiv\; \|(1+|\xi|^2)^{s/2}F\|_{L^2(\mathbb{R}_\xi)} \;<\; \infty.$$

It is straightforward to check that such spaces satisfy (2.46)*, and $\widehat{f}_0 \in \mathcal{Z}_1 \cap \mathcal{Z}_2$.*

We seek a solution of the equation:

$$\widehat{\mathcal{L}}_0[\theta]\widehat{f} \;=\; R\big(\widehat{f}\,\big)\;, \qquad (2.47)$$

where $\widehat{\mathcal{L}}_0(\theta)$ is the operator defined in (2.33) and the mapping $\widehat{f} \mapsto R(\widehat{f})$ is linear and satisfies the following properties:

Assumptions $R_{\alpha,\beta}$:

There exist constants $\alpha > 0$, $\beta > 0$ and $C_R > 0$ such that for any $\widehat{f} \in \mathcal{Z}_2$

$$\chi\left(|\xi| < \lambda^{-\beta}\right) R(\widehat{f})(\xi) \;=\; R(\widehat{f})(\xi)\ , \qquad \text{and} \qquad \left\| R(\widehat{f}) \right\|_{\mathcal{Z}_1} \;\leq\; C_R \lambda^{\alpha} \left\| \widehat{f}\, \right\|_{\mathcal{Z}_2} . \tag{2.48}$$

In the above setting we have the following

Lemma 2.3.4. *Let $(\theta_0^2, \widehat{f}_0(\xi))$ be the solution of $\widehat{\mathcal{L}}_0(\theta_0)\widehat{f}_0 = 0$, as defined in Lemma 2.3.1, where A, B and $\beta > 0$ are fixed. Let $R : \widehat{f} \in \mathcal{Z}_2 \to \mathcal{Z}_1$ be a linear mapping satisfying assumptions $R_{\alpha,\beta}$ displayed in (2.48), where $\mathcal{Z}_1, \mathcal{Z}_2$ satisfy (2.46). Then there exists $\lambda_0 > 0$ such that for any $0 < \lambda < \lambda_0$, the following holds:*

1. *There exists a unique solution $\left(\theta, \widehat{f}(\xi)\right) \in \mathbb{R}^+ \times \mathcal{Z}_2$ of the equation (2.47), such that*

$$\left\| \widehat{f} - \widehat{f}_0 \right\|_{\mathcal{Z}_2} \leq C\lambda^{\alpha}, \qquad \int_{-\infty}^{\infty} \widehat{f}(\xi) - \widehat{f}_0(\xi)\ d\xi \;=\; 0,$$

   with $C = C(A, B, C_R, \beta)$, independent of λ.

2. *Moreover, one has $\widehat{f}(\xi) \;=\; \chi\left(|\xi| < \lambda^{-\beta}\right) \widehat{f}(\xi)$ and $\left|\theta^2 - \theta_0^2\right| \leq C\lambda^{\alpha}$.*

Proof. Our strategy is to use a fixed point argument. We seek a solution (θ^2, f) to (2.47) of the form

$$\theta^2 \;\equiv\; \theta_0^2 \;+\; \theta_1^2 \qquad \text{and} \qquad \widehat{f} \;\equiv\; \widehat{f}_0 \;+\; \widehat{f}_1.$$

Clearly, any solution $\widehat{f}$ of (2.47) satisfies $\widehat{f}(\xi) = \chi\left(|\xi| < \lambda^{-\beta}\right) \widehat{f}(\xi)$. Therefore, since one has, by definition, $\widehat{f}_0(\xi) = \chi\left(|\xi| < \lambda^{-\beta}\right) \widehat{f}_0(\xi)$, it follows that $\widehat{f}_1(\xi) = \chi\left(|\xi| < \lambda^{-\beta}\right) \widehat{f}_1(\xi)$. Substitution of these expressions into (2.47) yields

$$\begin{aligned}(4\pi^2 A\xi^2 + \theta^2)\chi\left(|\xi| < \lambda^{-\beta}\right)\left(\widehat{f}_0 + \widehat{f}_1\right)(\xi)& \\ - \chi\left(|\xi| < \lambda^{-\beta}\right) B \int_{-\infty}^{\infty} \chi\left(|\eta| < \lambda^{-\beta}\right)\left(\widehat{f}_0 + \widehat{f}_1\right)(\eta) d\eta &= R\left(\widehat{f}_0 + \widehat{f}_1\right)(\xi).\end{aligned}$$

Rearranging terms yields the following equation for $\widehat{f}_1$, in which θ_1^2 is a parameter to be determined:

$$\left(\widehat{\mathcal{L}}_0[\theta_0]\widehat{f}_1\right)(\xi) = -\theta_1^2\left(\widehat{f}_0 + \widehat{f}_1\right)(\xi) + R\left(\widehat{f}_0 + \widehat{f}_1\right)(\xi). \tag{2.49}$$

By Lemma 2.3.2, (2.49) is solvable in L^2 only if the right hand side is L^2-orthogonal to $\widehat{f}_0$:

$$\left\langle \widehat{f}_0, -\theta_1^2 \left(\widehat{f}_0 + \widehat{f}_1\right) + R\left(\widehat{f}_0 + \widehat{f}_1\right)\right\rangle_{L^2} = 0.$$

Solving for θ_1^2, we obtain

$$\theta_1^2 = \frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0 + \widehat{f}_1\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} + \left\langle \widehat{f}_0, \widehat{f}_1\right\rangle_{L^2}} \ . \tag{2.50}$$

In summary, equation (2.47) can be rewritten equivalently as two coupled equations in terms of $\widehat{f}_1$ and θ_1^2: (2.49)–(2.50).

Substitution of θ_1^2 in (2.49), or equivalently projecting the right hand side of (2.49) onto the orthogonal complement of span$\{\widehat{f}_0\}$, yields the following closed equation for $\widehat{f}_1$:

$$\left(\widehat{\mathcal{L}}_0[\theta_0]\widehat{f}_1\right)(\xi) = -\frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0 + \widehat{f}_1\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} + \left\langle \widehat{f}_0, \widehat{f}_1\right\rangle_{L^2}} \left(\widehat{f}_0 + \widehat{f}_1\right)(\xi) + R\left(\widehat{f}_0 + \widehat{f}_1\right)(\xi). \tag{2.51}$$

By Lemma 2.3.2, $\widehat{f}_1$ is a solution of (2.51) with $\int_{\mathbb{R}} \widehat{f}_1 = 0$ if and only if :

$$\widehat{f}_1(\xi) \ = \ \mathcal{G}(\widehat{f}_1)(\xi), \tag{2.52}$$

where

$$\mathcal{G}(\widehat{f}_1)(\xi) \ \equiv \ \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta_0^2} \left(-\frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0 + \widehat{f}_1\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} + \left\langle \widehat{f}_0, \widehat{f}_1\right\rangle_{L^2}} \left(\widehat{f}_0 + \widehat{f}_1\right)(\xi) + R\left(\widehat{f}_0 + \widehat{f}_1\right)(\xi) \right). \tag{2.53}$$

We solve the fixed point equation (2.52) by the contraction mapping principle. Once $\widehat{f}_1$ has been obtained, θ_1^2 is determined using (2.50).

Introduce

$$\mathcal{S} \ = \ \left\{ \widehat{f} \in \mathcal{Z}_2 \ : \ \|\widehat{f}\|_{\mathcal{Z}_2} \leq C_H \lambda^{\alpha} \right\}, \ \text{ for some fixed } C_H > 0 \ . \tag{2.54}$$

Note that $\mathcal{S}$ is a closed subset of the Banach space $\mathcal{Z}_2$. We next show that there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$: $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ and $\mathcal{G}$ is a contraction mapping. As a consequence, it will follow that for $0 < \lambda < \lambda_0$, there is a unique solution $\widehat{f}_1 \in \mathcal{S}$ of the equation $\widehat{f}_1 = \mathcal{G}(\widehat{f}_1)$ and therefore of (2.51). Moreover, $\|\widehat{f}_1\|_{\mathcal{Z}_2} \lesssim \lambda^\alpha$ by definition of $\mathcal{S}$, and one can check:

$$\begin{aligned}\int_{\mathbb{R}} \widehat{f}_1 &= \int_{\mathbb{R}} \mathcal{G}(\widehat{f}_1) \\ &= \int_{\mathbb{R}} \widehat{f}_0(\xi)\left(-\frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0+\widehat{f}_1\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} + \left\langle \widehat{f}_0, \widehat{f}_1\right\rangle_{L^2}} \left(\widehat{f}_0+\widehat{f}_1\right)(\xi) + R\left(\widehat{f}_0+\widehat{f}_1\right)(\xi)\right) d\xi \\ &= 0.\end{aligned}$$

It then remains to obtain an estimate of ${\theta_1}^2 = {\theta_0}^2 - \theta^2$. From (2.50), one has

$$|\theta_1^2| \ \le\ \left|\left\langle \widehat{f}_0, R\left(\widehat{f}_0+\widehat{f}_1\right)\right\rangle_{L^2}\right| \left| \frac{1}{\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} + \left\langle \widehat{f}_0, \widehat{f}_1\right\rangle_{L^2}} \right| \ \lesssim\ \lambda^\alpha,$$

where we used (2.46) and (2.48), and the fact that for λ sufficiently small, $\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} \ge c > 0$, where c is independent of λ. Lemma 2.3.4 is proved. □

Proof that $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ is a contraction mapping: The result will follow from the two following claims, proved below:

Claim 2.3.5. *There exists $C_H = C\left(\theta_0, A, C_R, \|\widehat{f}_0\|_{\mathcal{Z}_2}\right) > 0$ such that $\|\mathcal{G}(0)\|_{\mathcal{Z}_2} \le \frac{1}{2} C_H \lambda^\alpha$.*

Claim 2.3.6. *There exists $\lambda_0 > 0$ such that if $0 \le \lambda < \lambda_0$, then $\left\|\mathcal{G}(\widehat{f}_1) - \mathcal{G}(\widehat{f}_2)\right\|_{\mathcal{Z}_2} \le \frac{1}{2}\left\|\widehat{f}_1 - \widehat{f}_2\right\|_{\mathcal{Z}_2}$.*

It follows that $\mathcal{G}$ maps $\mathcal{S} \equiv \left\{ f \in \mathcal{Z}_2 \ :\ \|\widehat{f}\|_{\mathcal{Z}_2} \le C_H \lambda^\alpha \right\}$ into $\mathcal{S}$ since

$$\left\|\mathcal{G}(f)\right\|_{\mathcal{Z}_2} \ \le\ \left\|\mathcal{G}(f) - \mathcal{G}(0)\right\|_{\mathcal{Z}_2} + \left\|\mathcal{G}(0)\right\|_{\mathcal{Z}_2} \ \le\ \frac{1}{2}\|f-0\|_{\mathcal{Z}_2} + \frac{1}{2} C_H \lambda^\alpha \ \le\ C_H \lambda^\alpha \ .$$

Therefore, by Claim 2.3.6, $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ is a contraction mapping.

Proof of Claim 2.3.5: By definition, one has

$$\mathcal{G}(0)(\xi) \ \equiv\ \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A \xi^2 + \theta_0^2} \left(-\left\langle \widehat{f}_0, R\left(\widehat{f}_0\right)\right\rangle_{L^2} \widehat{f}_0(\xi) + R\left(\widehat{f}_0\right)(\xi)\right).$$

It follows, from our assumptions (2.46) on functional spaces $(\mathcal{Z}_1, \mathcal{Z}_2)$:

$$\begin{aligned}\|\mathcal{G}(0)\|_{\mathcal{Z}_2} &\lesssim \left\| \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A \xi^2 + \theta_0^2} \right\|_{L^\infty} \left\langle \widehat{f}_0, R\left(\widehat{f}_0\right)\right\rangle_{L^2} \|\widehat{f}_0\|_{\mathcal{Z}_2} \\ &\qquad + \left\| \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)(1+\xi^2)}{4\pi^2 A \xi^2 + \theta_0^2} \right\|_{L^\infty} \left\| \frac{R\left(\widehat{f}_0\right)}{1+|\cdot|^2} \right\|_{\mathcal{Z}_2} \\ &\lesssim \left\| R\left(\widehat{f}_0\right)\right\|_{\mathcal{Z}_1} \|\widehat{f}_0\|_{\mathcal{Z}_2}^2 + \left\| R\left(\widehat{f}_0\right)\right\|_{\mathcal{Z}_1} .\end{aligned} \tag{2.55}$$

Claim 2.3.5 is now obvious, using the smallness hypothesis on the operator R, (2.48) . $\square$

Proof of Claim 2.3.6: Let us decompose the mapping $\mathcal{G}$ as follows:

$$\begin{aligned}
\mathcal{G}(\widehat{f}_1 - \widehat{f}_2) = \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta_0^2} \Bigg(&- \frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0 + \widehat{f}_1\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} + \left\langle \widehat{f}_0, \widehat{f}_1\right\rangle_{L^2}} \left(\widehat{f}_0 + \widehat{f}_1\right)(\xi) \\
&+ \frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0 + \widehat{f}_2\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0, \widehat{f}_0\right\rangle_{L^2} + \left\langle \widehat{f}_0, \widehat{f}_2\right\rangle_{L^2}} \left(\widehat{f}_0 + \widehat{f}_2\right)(\xi) \ + \ R\left(\widehat{f}_1 - \widehat{f}_2\right)(\xi) \Bigg) \\
\equiv \ \frac{S_1[\widehat{f}_1](\xi) - S_1[\widehat{f}_2](\xi)}{4\pi^2 A\xi^2 + \theta_0^2} + \frac{\chi\left(|\xi| < \lambda^{-\beta}\right) \ R\left(\widehat{f}_1 - \widehat{f}_2\right)(\xi)}{4\pi^2 A\xi^2 + \theta_0^2}&.
\end{aligned}$$

The following estimate follows from our assumptions (2.46) on the spaces $(\mathcal{Z}_1, \mathcal{Z}_2)$:

$$\begin{aligned}
\left\|\mathcal{G}(\widehat{f}_1 - \widehat{f}_2)\right\|_{\mathcal{Z}_2} &\leq \left\| \frac{S_1[\widehat{f}_1](\xi) - S_1[\widehat{f}_2](\xi)}{4\pi^2 A\xi^2 + \theta_0^2} \right\|_{\mathcal{Z}_2} + \left\| \frac{\chi\left(|\xi| < \lambda^{-\beta}\right) \ R\left(\widehat{f}_1 - \widehat{f}_2\right)(\xi)}{4\pi^2 A\xi^2 + \theta_0^2} \right\|_{\mathcal{Z}_2} \\
&\lesssim \left\| \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta_0^2} \right\|_{L^\infty} \left\| S_1[\widehat{f}_1] - S_1[\widehat{f}_2] \right\|_{\mathcal{Z}_2} \\
&\quad + \left\| \frac{\chi\left(|\xi| < \lambda^{-\beta}\right)(1+\xi^2)}{4\pi^2 A\xi^2 + \theta_0^2} \right\|_{L^\infty} \left\| \frac{R\left(\widehat{f}_1 - \widehat{f}_2\right)}{1 + |\cdot|^2} \right\|_{\mathcal{Z}_2} \\
&\lesssim \ \left\| S_1[\widehat{f}_1] - S_1[\widehat{f}_2] \right\|_{\mathcal{Z}_2} + \left\| R\left(\widehat{f}_1 - \widehat{f}_2\right) \right\|_{\mathcal{Z}_1} \ . \qquad (2.56)
\end{aligned}$$

The second term in (2.56) is estimated using assumptions $R_{\alpha,\beta}$, (2.48):

$$\left\| R\left(\widehat{f}_1 - \widehat{f}_2\right) \right\|_{\mathcal{Z}_1} \ \leq \ C_R \lambda^\alpha \left\| \widehat{f}_1 - \widehat{f}_2 \right\|_{\mathcal{Z}_2} . \qquad (2.57)$$

Let us now turn to the first term in (2.56).

$$
\begin{aligned}
S_1[\widehat{f}_1] - S_1[\widehat{f}_2] &= -\frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0+\widehat{f}_1\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0,\widehat{f}_0\right\rangle_{L^2}+\left\langle \widehat{f}_0,\widehat{f}_1\right\rangle_{L^2}}\left(\widehat{f}_0+\widehat{f}_1\right) + \frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0+\widehat{f}_2\right)\right\rangle_{L^2}}{\left\langle \widehat{f}_0,\widehat{f}_0\right\rangle_{L^2}+\left\langle \widehat{f}_0,\widehat{f}_2\right\rangle_{L^2}}\left(\widehat{f}_0+\widehat{f}_2\right) \\
&= -\ \frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_1-\widehat{f}_2\right)\right\rangle_{L^2}\left(\widehat{f}_0+\widehat{f}_1\right)}{\left\langle \widehat{f}_0,\widehat{f}_0\right\rangle_{L^2}+\left\langle \widehat{f}_0,\widehat{f}_1\right\rangle_{L^2}} \ -\ \frac{\left\langle \widehat{f}_0, R\left(\widehat{f}_0+\widehat{f}_2\right)\right\rangle_{L^2}\left(\widehat{f}_1-\widehat{f}_2\right)}{\left\langle \widehat{f}_0,\widehat{f}_0\right\rangle_{L^2}+\left\langle \widehat{f}_0,\widehat{f}_1\right\rangle_{L^2}} \\
&\qquad -\ \left\langle \widehat{f}_0, R\left(\widehat{f}_0+\widehat{f}_2\right)\right\rangle_{L^2}\left(\widehat{f}_0+\widehat{f}_2\right) \\
&\qquad\qquad \times\left(\frac{1}{\left\langle \widehat{f}_0,\widehat{f}_0\right\rangle_{L^2}+\left\langle \widehat{f}_0,\widehat{f}_1\right\rangle_{L^2}} - \frac{1}{\left\langle \widehat{f}_0,\widehat{f}_0\right\rangle_{L^2}+\left\langle \widehat{f}_0,\widehat{f}_2\right\rangle_{L^2}}\right) \\
&= I + II + III. \qquad (2.58)
\end{aligned}
$$

The result is a consequence of the following estimates:

$$
\left\langle \widehat{f}_0, g\right\rangle_{L^2} \le C\big\|\widehat{f}_0\big\|_{\mathcal{Z}_1}\|g\|_{\mathcal{Z}_2} \le C_1\|g\|_{\mathcal{Z}_2}, \quad \left\langle \widehat{f}_0, R(g)\right\rangle_{L^2} \le C\big\|\widehat{f}_0\big\|_{\mathcal{Z}_2}\|R(g)\|_{\mathcal{Z}_1} \le C_2\lambda^\alpha\|g\|_{\mathcal{Z}_2}.
$$

with $C_1 = C\big(\big\|\widehat{f}_0\big\|_{\mathcal{Z}_1}\big)$ and $C_2 = C_2\big(\big\|\widehat{f}_0\big\|_{\mathcal{Z}_2}, C_R\big)$. Using the above, one checks that for sufficiently small λ,

$$
\begin{aligned}
\|I\|_{\mathcal{Z}_2} &\lesssim C_2\lambda^\alpha\left\|\widehat{f}_1-\widehat{f}_2\right\|_{\mathcal{Z}_2}(\big\|\widehat{f}_0\big\|_{\mathcal{Z}_2}+C_H\lambda^\alpha), \\
\|II\|_{\mathcal{Z}_2} &\lesssim C_2\lambda^\alpha(\big\|\widehat{f}_0\big\|_{\mathcal{Z}_2}+C_H\lambda^\alpha)\left\|\widehat{f}_1-\widehat{f}_2\right\|_{\mathcal{Z}_2}, \\
\|III\|_{\mathcal{Z}_2} &\lesssim C_1\left\|\widehat{f}_1-\widehat{f}_2\right\|_{\mathcal{Z}_2}C_2\lambda^\alpha(\big\|\widehat{f}_0\big\|_{\mathcal{Z}_2}+C_H\lambda^\alpha)^2.
\end{aligned}
$$

Thus if $C_1\lambda^\alpha < 1/2$, one has

$$
\left\|S_1[\widehat{f}_1]-S_1[\widehat{f}_2]\right\|_{\mathcal{Z}_2} \ \le\ \|I\|_{\mathcal{Z}_2}+\|II\|_{\mathcal{Z}_2}+\|III\|_{\mathcal{Z}_2} \ \lesssim\ \lambda^\alpha\left\|\widehat{f}_1-\widehat{f}_2\right\|_{\mathcal{Z}_2}. \qquad (2.59)
$$

Plugging (2.57) and (2.59) into (2.56), it follows the existence of a constant, $C_0 > 0$, such that $\|\mathcal{G}(\widehat{f}_1)-\mathcal{G}(\widehat{f}_2)\|_{\mathcal{Z}_2} \le C_0\lambda^\alpha\|\widehat{f}_1-\widehat{f}_2\|_{\mathcal{Z}_2}$. Thus for $0<\lambda<\lambda_0\le C_0^{-\frac{1}{\alpha}}$, we obtain a contraction and Claim 2.3.6 is proved. □

2.4 Proof of Th'm 2.2.1; Edge bifurcations for $-\partial_x^2+\lambda V(x)$

In this section we prove Theorem 2.2.1, the special case: $Q\equiv 0$ of our main result, Theorem 2.2.4. In this case we study the bifurcation of solutions to the eigenvalue problem

$$
\left(-\partial_x^2+\lambda V(x)\right)\psi^\lambda(x) \ = \ E^\lambda\psi^\lambda(x), \quad \psi^\lambda\in L^2(\mathbb{R}) \qquad (2.60)
$$

into the interval $(-\infty, 0)$, the semi-infinite spectral gap of $H_0 \equiv -\partial_x^2$, for V a spatially localized potential, and $\lambda > 0$ sufficiently small. Here, the Floquet-Bloch eigenfunctions are exponentials. Hence, calculations are more straightforward and error bounds on the approximations are sharper.

2.4.1 Near and far frequency decomposition

Taking the Fourier transform of (2.60) yields

$$\left(4\pi^2\xi^2 - E^\lambda\right)\widehat{\psi^\lambda}(\xi) + \lambda\int_{\mathbb{R}} \widehat{V}\left(\xi - \zeta\right)\widehat{\psi^\lambda}(\zeta)\ d\zeta = 0. \tag{2.61}$$

We shall study (2.61) via the equivalent system for the

$$\textit{near frequency components}\text{: } \{\widehat{\psi^\lambda}(\xi) : |\xi| < \lambda^r\ \} \ \text{ and}$$
$$\textit{far frequency components}\text{: } \{\widehat{\psi^\lambda}(\xi) : |\xi| \geq \lambda^r\ \} \text{ of } \psi^\lambda.$$

Let r be a parameter, chosen to satisfy: $0 < r < 1$. Recall the cutoff functions, χ and $\overline{\chi}$, introduced in (2.14) and $1 = \chi_{\lambda^r}(\xi) + \overline{\chi}_{\lambda^r}(\xi)$. Multiplying (2.61) by this identity we get

$$\begin{aligned} 0 = \left(4\pi^2|\xi|^2 - E^\lambda\right)\left(\chi_{\lambda^r} + \overline{\chi}_{\lambda^r}\right)(\xi)\ \widehat{\psi^\lambda}(\xi) \\ + \lambda\int_{-\infty}^{\infty}\left(\chi_{\lambda^r} + \overline{\chi}_{\lambda^r}\right)(\xi)\ \widehat{V}(\xi-\zeta)\left(\chi_{\lambda^r} + \overline{\chi}_{\lambda^r}\right)(\zeta)\ \widehat{\psi^\lambda}(\zeta)d\zeta. \end{aligned}$$

Introduce notation for the near- and far-frequency components of ψ^λ:

$$\widehat{\psi}_{\text{near}}(\xi)\ \equiv\ \chi_{\lambda^r}(\xi)\ \widehat{\psi^\lambda}(\xi) \quad \text{and} \quad \widehat{\psi}_{\text{far}}(\xi) \equiv \overline{\chi}_{\lambda^r}(\xi)\ \widehat{\psi^\lambda}(\xi). \tag{2.62}$$

Then, the eigenvalue equation is equivalent to the following coupled system:

$$\left(4\pi^2|\xi|^2 - E^\lambda\right)\widehat{\psi}_{\text{near}}(\xi) + \lambda\chi_{\lambda^r}(\xi)\int_{-\infty}^{\infty}\widehat{V}(\xi-\zeta)\big(\widehat{\psi}_{\text{near}}(\zeta) + \widehat{\psi}_{\text{far}}(\zeta)\big)d\zeta = 0, \tag{2.63}$$

$$\left(4\pi^2|\xi|^2 - E^\lambda\right)\widehat{\psi}_{\text{far}}(\xi) + \lambda\overline{\chi}_{\lambda^r}(\xi)\int_{-\infty}^{\infty}\widehat{V}(\xi-\zeta)\big(\widehat{\psi}_{\text{near}}(\zeta) + \widehat{\psi}_{\text{far}}(\zeta)\big)d\zeta = 0. \tag{2.64}$$

In what follows we shall set $E^\lambda = -\lambda^2\theta^2$, where $\theta = \theta(\lambda)$ is expected to be $\mathcal{O}(1)$ as $\lambda \downarrow 0$. This anticipates that the bifurcating eigenvalue, E^λ, will be real, negative and $\mathcal{O}(\lambda^2)$.

2.4.2 Analysis of the far frequency components

We view (2.64) as an equation for $\widehat{\psi}_{\text{far}}$, depending on "parameters" $(\widehat{\psi}_{\text{near}}; \lambda)$. The following proposition studies the mapping $(\widehat{\psi}_{\text{near}}; \lambda) \mapsto \widehat{\psi}_{\text{far}}$.

Proposition 2.4.1. *Let $\widehat{\psi}_{near} \in L^1$. There exists $\lambda_0 > 0$, such that for $0 < \lambda < \lambda_0$, the following holds. Set $E^\lambda \equiv -\lambda^2\theta^2$, with $|\theta| \leq \pi\lambda^{r-1}$, $r \in (0,1)$. There is a unique solution $\widehat{\psi}_{far} = \widehat{\psi}_{far}\left[\widehat{\psi}_{near};\lambda\right]$ of the far frequency equation* (2.64)*. The mapping $(\widehat{\psi}_{near};\lambda) \mapsto \widehat{\psi}_{far}\left[\widehat{\psi}_{near};\lambda\right]$ maps $L^1(\mathbb{R}) \times \mathbb{R}$ to $L^1(\mathbb{R})$ and satisfies the bound:*

$$\left\|\widehat{\psi}_{far}\right\|_{L^1} \leq C(\|\widehat{V}\|_{L^\infty})\ \lambda^{1-r}\ \|\widehat{\psi}_{near}\|_{L^1}. \tag{2.65}$$

Proof. We seek to solve (2.64) for $\widehat{\psi}_{\rm far}$ as a functional of $\widehat{\psi}_{\rm near}$. Since $|\xi| \geq \lambda^r$, with $0 < r < 1$, and $|\theta| \leq \pi\lambda^{r-1}$, we have $\left|4\pi^2|\xi|^2 - E^\lambda\right| = \left|4\pi^2|\xi|^2 + \lambda^2\theta^2\right| \geq 3\pi^2\lambda^{2r}$, which is bounded away from zero for any fixed $\lambda > 0$. Dividing (2.64) by $4\pi^2\xi^2 - E^\lambda = 4\pi^2\xi^2 + \lambda^2\theta^2$, we obtain that (2.64) is equivalent to the equation:

$$\left(I + \widehat{\mathcal{T}}_\lambda\right)\widehat{\psi}_{\rm far}(\xi) = -\left(\widehat{\mathcal{T}}_\lambda\widehat{\psi}_{\rm near}\right)(\xi)\ , \tag{2.66}$$

where

$$\left(\widehat{\mathcal{T}}_\lambda\ \widehat{g}\right)(\xi)\ \equiv \int_\zeta \mathcal{K}_\lambda(\xi,\zeta)\ \widehat{g}(\zeta)\ d\zeta \qquad \text{and} \qquad \mathcal{K}_\lambda(\xi,\zeta)\ \equiv\ \lambda\frac{\overline{\chi}_{\lambda^r}(\xi)}{4\pi^2|\xi|^2 + \lambda^2\theta^2}\widehat{V}(\xi-\zeta)\ .$$

We next show that the integral operator $\widehat{\mathcal{T}}_\lambda$, viewed as an operator from L^1 to L^1 has small norm, for λ small. This implies the invertibility of $I + \widehat{\mathcal{T}}_\lambda$ and the assertions of Proposition 2.4.1. Let $\widehat{g} \in L^1$. One has

$$\left\|\widehat{\mathcal{T}}_\lambda\widehat{g}\right\|_{L^1}\ \leq\ C\ \lambda\int_{|\xi|\geq\lambda^r} \frac{1}{4\pi^2|\xi|^2 + \lambda^2\theta^2}\ d\xi\ \ \left\|\widehat{V}\right\|_{L^\infty}\left\|\widehat{g}\right\|_{L^1}\ \lesssim\ \lambda^{1-r}\ \left\|\widehat{V}\right\|_{L^\infty}\left\|\widehat{g}\right\|_{L^1}.$$

Thus $\widehat{\mathcal{T}}_\lambda$ is bounded from L^1 to L^1 with norm bound: $\left\|\widehat{\mathcal{T}}_\lambda\right\|_{L^1\to L^1}\ \lesssim\ \lambda^{1-r}\ \left\|\widehat{V}\right\|_{L^\infty}$. For $r \in (0,1)$, $\left\|\widehat{\mathcal{T}}_\lambda\right\|_{L^1\to L^1} \to 0$ as $\lambda \to 0$. Therefore $I + \widehat{\mathcal{T}}_\lambda$ is invertible, for λ sufficiently small. Moreover,

$$\left\|\widehat{\psi}_{\rm far}\right\|_{L^1} = \left\|\left(I + \widehat{\mathcal{T}}_\lambda\right)^{-1}\left(\widehat{\mathcal{T}}_\lambda\widehat{\psi}_{\rm near}\right)\right\|_{L^1} \leq \left\|\left(I + \widehat{\mathcal{T}}_\lambda\right)^{-1}\right\|_{L^1\to L^1}\left\|\widehat{\mathcal{T}}_\lambda\right\|_{L^1\to L^1}\left\|\widehat{\psi}_{\rm near}\right\|_{L^1},$$

which implies the bound (2.65). Proposition 2.4.1 is proved. □

2.4.3 Analysis of the near frequency components

Now that we have constructed $\widehat{\psi}_{\rm far}$ as a functional of $\widehat{\psi}_{\rm near}$ and λ (Proposition 2.4.1), it is possible to treat (2.63), for λ small, as a *closed equation* for a *low frequency projected eigenstate*, $\widehat{\psi}_{\rm near}(\xi;\lambda)$, and corresponding eigenvalue E^λ. Substitution of $\widehat{\psi}_{\rm far} = \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda]$ into (2.63) yields:

$$\left(4\pi^2|\xi|^2 - E^\lambda\right)\widehat{\psi}_{\rm near}(\xi) + \lambda\chi_{\lambda^r}(\xi)\int_\zeta \widehat{V}(\xi-\zeta)\widehat{\psi}_{\rm near}(\zeta)d\zeta + \lambda\chi_{\lambda^r}(\xi)\widehat{R}(\xi) = 0, \tag{2.67}$$

where $\widehat{R}$ is defined by

$$\widehat{R}(\xi) \equiv \int_\zeta \widehat{V}(\xi-\zeta)\ \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda](\zeta)\ d\zeta\ . \tag{2.68}$$

Recall that $\widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda]$ is in L^1, and of size $\mathcal{O}\left(\lambda^{1-r}\ \|\widehat{\psi}_{\rm near}\|_{L^1}\right)$ by Proposition 2.4.1.

Our next goal is, via appropriate expansion, reorganization and scaling, to re-express (2.67) as a simple leading order asymptotic equation plus controllable corrections. The terms in (2.67) are supported in the near (low) frequency regime. Note that for $|\xi| < \lambda^r$ and $|\zeta| < \lambda^r$ we have $|\xi-\zeta| \le |\xi| + |\zeta| < 2\lambda^r$. Taylor expansion of $\widehat{V}(\xi-\zeta)$ gives $\widehat{V}(\xi-\zeta) = \widehat{V}(0) + (\xi-\zeta)\widehat{V}'(\eta)$, for some $\eta = \eta(\zeta,\xi)$ such that $|\eta| < 2\lambda^r$. Using this expansion in the second term of (2.67) yields

$$\left(4\pi^2|\xi|^2 - E^\lambda\right)\widehat{\psi}_{\rm near}(\xi) + \lambda\chi_{\lambda^r}(\xi)\widehat{V}(0)\int_\zeta \widehat{\psi}_{\rm near}(\zeta)d\zeta = \lambda\chi_{\lambda^r}(\xi)\mathcal{R}\left[\widehat{\psi}_{\rm near};\lambda\right](\xi), \tag{2.69}$$

where $\mathcal{R}\left[\widehat{\psi}_{\rm near};\lambda\right] \ \equiv\ \mathcal{R}_1 \ +\ \mathcal{R}_2$, with

$$\mathcal{R}_1(\xi) \equiv -\widehat{R}(\xi) \ = \ -\int_\zeta \widehat{V}(\xi-\zeta)\ \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda](\zeta)\ d\zeta,$$

$$\mathcal{R}_2(\xi) \equiv -\int_\zeta (\xi-\zeta)\widehat{V}'(\eta)\widehat{\psi}_{\rm near}(\zeta)\ d\zeta\ .$$

We now introduce the scaled near-frequency Fourier component, $\widehat{\Phi}_\lambda$, by

$$\widehat{\psi}_{\rm near}(\xi;\lambda) = \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{\xi}{\lambda}\right), \tag{2.70}$$

Note that

$$\left\|\widehat{\psi}_{\rm near}(\cdot;\lambda)\right\|_{L^1} \ = \ \left\|\frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{\cdot}{\lambda}\right)\right\|_{L^1} \ = \ \left\|\widehat{\Phi}_\lambda\right\|_{L^1}. \tag{2.71}$$

We also denote $E^\lambda = -\lambda^2\theta^2$, and restrict to $\theta = \theta(\lambda)$ satisfying the constraint in the hypotheses of Proposition 2.4.1. Substitution of (2.70) into (2.69), defining $\xi' = \lambda\xi$ and dividing by λ yields the following rescaled near-frequency equation:

$$\left(4\pi^2|\xi'|^2 + \theta^2\right)\widehat{\Phi}_\lambda(\xi') + \chi_{\lambda^{r-1}}(\xi')\widehat{V}(0)\int_{\zeta'} \widehat{\Phi}_\lambda(\zeta')d\zeta' = \chi_{\lambda^{r-1}}(\xi')\mathcal{R}'\big(\widehat{\Phi}_\lambda\big)(\xi') \tag{2.72}$$

where $\mathcal{R}'\big(\widehat{\Phi}_\lambda\big)(\xi') \equiv \mathcal{R}\left[\widehat{\psi}_{\rm near};\lambda\right](\lambda\xi') \equiv \mathcal{R}_1'(\xi') + \mathcal{R}_2'(\xi')$, with

$$\mathcal{R}_1'(\xi') \ \equiv\ -\int_\zeta \widehat{V}(\lambda\xi'-\zeta)\ \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda](\zeta)\ d\zeta\ , \tag{2.73}$$

$$\mathcal{R}_2'(\xi') \ \equiv\ -\int_\zeta (\lambda\xi'-\zeta)\widehat{V}'(\eta)\widehat{\psi}_{\rm near}(\zeta)\ d\zeta \ \ = \ -\lambda\int_\zeta (\xi'-\zeta')\widehat{V}'(\eta)\widehat{\Phi}_\lambda(\zeta')d\zeta'\ . \tag{2.74}$$

Equation (2.72) is in the form of the class of equations to which Lemma 2.3.4 applies. We shall use Lemma 2.3.4 to obtain a non-trivial eigenpair solution ($\widehat{\Phi}_\lambda, \theta(\lambda)$) of (2.72). Toward verification of the hypotheses of Lemma 2.3.4, we next bound the right hand side of (2.72).

Proposition 2.4.2. *Let V be such that $\|\widehat{V}\|_{W^{1,\infty}} \equiv \|\widehat{V}\|_{L^\infty} + \|\widehat{V}'\|_{L^\infty} < \infty$. Then, the right hand side of the rescaled near-frequency equation* (2.72) *satisfies the bound*

$$\left\|\chi_{\lambda^{r-1}}(\xi)\mathcal{R}'\big(\widehat{\Phi}_\lambda\big)\right\|_{L^\infty} \le C\big(\|\widehat{V}\|_{W^{1,\infty}}\big)\ \left(\lambda^{1-r}+\lambda^r\right)\ \left\|\widehat{\Phi}_\lambda\right\|_{L^1}. \tag{2.75}$$

Proof. We proceed by estimating each term individually.
Estimation of $\mathcal{R}'_1(\xi')$, given by (2.73)*:* By Proposition 2.4.1, one has

$$\|\widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda]\|_{L^1(\mathbb{R})} \le C\big(\|\widehat{V}\|_{L^\infty}\big)\ \lambda^{1-r}\ \|\widehat{\psi}_{\rm near}\|_{L^1(\mathbb{R})}\ . \tag{2.76}$$

Plugging (2.76) into (2.73), and making use of (2.71), we have

$$\begin{aligned}\|\mathcal{R}'_1\|_{L^\infty} &= \left\|\int_{\mathbb{R}}\widehat{V}(\lambda\xi'-\zeta)\widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda](\zeta)\ d\zeta\right\|_{L^\infty_{\xi'}}\\ &\le \|\widehat{V}\|_{L^\infty}\ \|\ \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near},\lambda]\ \|_{L^1}\ \le\ C(\|\widehat{V}\|_{L^\infty})\ \lambda^{1-r}\ \|\widehat{\Phi}_\lambda\|_{L^1}\ .\end{aligned}$$

Estimation of $\mathcal{R}'_2(\xi')$, given by (2.74)*:* We have the bound

$$\|\chi_{\lambda^{r-1}}(\xi')\mathcal{R}'_2\|_{L^\infty} = \|\chi_{\lambda^{r-1}}(\xi')\int_{\zeta'}\lambda(\xi'-\zeta')\widehat{V}'(\eta)\widehat{\Phi}_\lambda(\zeta')d\zeta'\|_{L^\infty_{\xi'}} \le 2\lambda^r\|\widehat{V}'\|_{L^\infty}\|\widehat{\Phi}_\lambda\|_{L^1},$$

using that $\widehat{\Phi}_\lambda(\zeta') = \chi_{\lambda^{r-1}}(\zeta')\widehat{\Phi}_\lambda(\zeta')$, so that $|\xi'-\zeta'| \le 2\lambda^{r-1}$. Proposition 2.4.2 is proved. □

Remark 2.4.3. *We expect that by using a higher order Taylor approximation of $\widehat{V}(\xi-\zeta)$ in the second term of equation* (2.67)*, it should be possible to obtain a variant of Proposition 2.4.2 with a bound which is higher order in λ. This would require a higher order variant of Lemma 2.3.4.*

2.4.4 Completion of the proof

We now prove Theorem 2.2.1 by an application of Lemma 2.3.4 to equation (2.72), using the remainder estimate of Proposition 2.4.2.

Proof of Theorem 2.2.1. We construct ψ^λ, solution to (2.61) as $\widehat{\psi^\lambda} = \widehat{\psi}_{\rm far} + \widehat{\psi}_{\rm near}$, where $\widehat{\psi}_{\rm far}, \widehat{\psi}_{\rm near}$ satisfy (2.63)–(2.64). The far-frequency component, $\widehat{\psi}_{\rm far}$, is uniquely determined by $\widehat{\psi}_{\rm near}$ and

λ sufficiently small; see Proposition 2.4.1. Now set $\widehat{\psi}_{\rm near}(\xi) \equiv \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{\xi}{\lambda}\right)$. Since $\widehat{V} \in W^{1,\infty}$, Proposition 2.4.2 implies that the rescaled near-frequency equation (2.72) can be written as

$$\left(4\pi^2|\xi'|^2+\theta^2\right)\widehat{\Phi}_\lambda(\xi') + \chi_{\lambda^{r-1}}(\xi')\widehat{V}(0)\int_{\zeta'}\widehat{\Phi}_\lambda(\zeta')d\zeta' = \chi_{\lambda^{r-1}}(\xi')\mathcal{R}(\widehat{\Phi}_\lambda)(\xi'), \tag{2.77}$$

with $\|\mathcal{R}(u)\|_{L^\infty} \leq C\ \lambda^\alpha\ \|u\|_{L^1}$, where $\alpha = \min(1-r,r)$ and $C = C(\|\widehat{V}\|_{W^{1,\infty}})$. From now on, we set

$$r = 1/2 = \alpha$$

as this yields optimal estimates. Applying Lemma 2.3.4 to (2.77) with $A=1$, $-B = \widehat{V}(0) = \int_{\mathbb{R}} V$ (assumed to be negative), we deduce that there exists a solution $\left(\theta^2, \widehat{\Phi}_\lambda\right)$ of the rescaled near-frequency equation (2.77), satisfying

$$\|\widehat{\Phi}_\lambda - \widehat{f}_0\|_{L^1}\ \leq\ C\ \lambda^{\frac{1}{2}} \qquad \text{and} \qquad |\theta^2-\theta_0^2|\ \leq\ C\ \lambda^{\frac{1}{2}}\ . \tag{2.78}$$

Here $\left(\theta_0^2(\lambda), \widehat{f}_0\right)$ is the unique (normalized) solution of the homogeneous equation

$$\widehat{\mathcal{L}}_{0,\lambda}\left(\theta_0, \widehat{f}_0\right) = (4\pi^2\xi^2+\theta^2)\widehat{f}_0 + \chi\left(|\xi|<\lambda^{-\frac{1}{2}}\right)\widehat{V}(0)\int_{\mathbb{R}}\chi\left(|\eta|<\lambda^{-\frac{1}{2}}\right)\widehat{f}_0(\eta)d\eta = 0,$$

as described in Lemma 2.3.1. Thus $\widehat{\psi}_{\rm near}(\xi) = \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{\xi}{\lambda}\right)$ and $E^\lambda = -\lambda^2\theta^2(\lambda)$ are well-defined.

In conclusion, the eigenpair solution to (2.61) (*i.e.* (2.16)) , $(E^\lambda, \psi^\lambda)$, is uniquely determined by

$$E^\lambda\ \equiv\ -\lambda^2\theta^2(\lambda), \qquad \text{and} \qquad \psi^\lambda \equiv \mathcal{F}^{-1}(\widehat{\psi}_{\rm near}+\widehat{\psi}_{\rm far}).$$

Estimate (2.17), the small λ expansion of the eigenvalue E^λ, follows from (2.78). The approximation, (2.18), of the corresponding eigenstate, $\psi^\lambda = \psi_{\rm near}+\psi_{\rm far}$, is obtained as follows. First, by (2.78) we have

$$\left\|\widehat{\psi}_{\rm near}(\eta)\ -\ \lambda\frac{\chi_{\lambda^{1/2}}(\eta)}{4\pi^2|\eta|^2+\lambda^2\theta_0^2}\right\|_{L^1}\ =\ \left\|\widehat{\Phi}_\lambda - \widehat{f}_0\right\|_{L^1}\ \lesssim\ \lambda^{1/2}. \tag{2.79}$$

The high frequency components are small, as is seen from the following calculation:

$$\left\|\lambda\frac{\overline{\chi}_{\lambda^{1/2}}(\eta)}{4\pi^2|\eta|^2+\lambda^2\widehat{V}(0)^2}\right\|_{L^1}\ \leq\ \lambda\int_{|\eta|\geq\lambda^{1/2}}\frac{d\eta}{4\pi^2|\eta|^2} \lesssim\ \lambda^{1/2}. \tag{2.80}$$

Finally, from Proposition 2.4.1, one has (with $r=1/2$)

$$\left\|\widehat{\psi}_{\rm far}\right\|_{L^1}\ \leq\ C\ (\|\widehat{V}\|_{L^\infty})\ \lambda^{1/2}\ \|\widehat{\psi}_{\rm near}\|_{L^1}, \tag{2.81}$$

and $\left\|\widehat{\psi}_{\rm near}\right\|_{L^1} = \left\|\widehat{\Phi}_\lambda\right\|_{L^1} \to \left\|\widehat{f}_0\right\|_{L^1}$ (as $\lambda \to 0$). Altogether, (2.79), (2.80) and (2.81) yield

$$\left\|\psi^\lambda \ - \ \mathcal{F}^{-1}\left\{\lambda\frac{1}{4\pi^2|\cdot|^2+\lambda^2\theta_0^2}\right\}\right\|_{L^\infty} \ \le \ \left\|\widehat{\psi^\lambda} \ - \ \lambda\frac{1}{4\pi^2|\cdot|^2+\lambda^2\theta_0^2}\right\|_{L^1} \ \lesssim \ \lambda^{1/2}.$$

Note, by residue computation, that $\mathcal{F}^{-1}\left\{(4\pi^2|\cdot|^2+\lambda^2\theta_0^2)^{-1}\right\} \ = \ \frac{1}{2}(\lambda\theta_0)^{-1}\exp(-\lambda\theta_0|x|)$, with $\theta_0 = -\frac{1}{2}\int_{\mathbb{R}} V > 0$. Thus estimate (2.18) holds. This completes the proof of Theorem 2.2.1. $\square$

2.5 Proof of Th'm 2.2.4; Edge bifurcations of $-\partial_x^2 + Q + \lambda V$

Let $Q(x)$ denote a non-trivial, continuous, 1−periodic function, $Q(x+1) = Q(x)$. In this section we study the bifurcation of solutions to the eigenvalue problem

$$\left(-\partial_x^2 + Q(x) + \lambda V(x)\right)\psi^\lambda(x) = E^\lambda\psi^\lambda(x), \quad \psi \in L^2(\mathbb{R}) \tag{2.82}$$

into the spectral gaps of $-\partial_x^2 + Q(x)$. We proceed by the same general approach of Section 2.4. That is, by appropriate spectral localization, in this case by applying the Gelfand-Bloch transform, we reduce (2.82) to an equivalent *near-frequency* eigenvalue problem supported on frequencies lying near a spectral band edge of $-\partial_x^2 + Q(x)$.

2.5.1 Near and far frequency components

We take the Gelfand-Bloch transform of (2.82) and get

$$-\left(\partial_x + 2\pi ik\right)^2\widetilde{\psi^\lambda}(x;k) + Q(x)\widetilde{\psi^\lambda}(x;k) + \lambda\left(V\psi^\lambda\right)^\sim(x;k) = E^\lambda\widetilde{\psi^\lambda}(x;k), \tag{2.83}$$

where

$$\left(V\psi^\lambda\right)^\sim(x;k) = \sum_{n\in\mathbb{Z}} e^{2\pi inx}\left(V\psi^\lambda\right)^\wedge(k+n) = \sum_{n\in\mathbb{Z}} e^{2\pi inx}\left(\widehat{V}\star\widehat{\psi^\lambda}\right)(k+n).$$

Here, the quasi-momentum, k, varies over the interval $(-1/2, 1/2]$.

As in Section 2.4, we express ψ in terms of its near- and far-frequency components around a band edge $E_{b_*}(k_*)$, for fixed b_* and k_*:

$$\psi^\lambda = \psi_{\rm near} + \psi_{\rm far} = \mathcal{T}^{-1}\left\{\widetilde{\psi}_{\rm near}(k)p_{b_*}(x;k)\right\} + \mathcal{T}^{-1}\left\{\sum_{b=0}^{\infty}\widetilde{\psi}_{{\rm far},b}(k)p_b(x;k)\right\}, \tag{2.84}$$

where we define, for $b = 0, 1, \dots$:

$$\widetilde{\psi}_{\text{near}}(k) \equiv \chi\left(|k-k_*| < \lambda^r\right) \mathcal{T}_{b_*}\{\psi^\lambda\}(k) = \chi\left(|k-k_*| < \lambda^r\right) \left\langle p_{b_*}(\cdot,k), \widetilde{\psi^\lambda}(\cdot,k) \right\rangle_{L^2([0,1])},$$
$$\widetilde{\psi}_{\text{far},b}(k) \equiv \chi\left(|k-k_*| \geq \lambda^r \delta_{b_*,b}\right) \mathcal{T}_{b}\{\psi^\lambda\}(k) = \chi\left(|k-k_*| \geq \lambda^r \delta_{b_*,b}\right) \left\langle p_{b}(\cdot,k), \widetilde{\psi^\lambda}(\cdot,k) \right\rangle_{L^2([0,1])},$$

where $\delta_{i,j}$ denotes Kronecker's delta function and r a parameter chosen to satisfy $r > 0$. Equivalently, one has

$$\psi^\lambda(x) \;=\; \int_{-1/2}^{1/2} \left(\widetilde{\psi}_{\text{near}}(k) u_{b_*}(x;k) + \sum_{b=0}^{\infty} \widetilde{\psi}_{\text{far},b}(k) u_b(x;k) \right) \, dk.$$

Recall that $\{p_b(x;k)\}_{b\geq 0}$ form a complete orthonormal set in $L^2_{\text{per}}([0,1])$, and satisfy

$$\Big(-\left(\partial_x + 2\pi i k\right)^2 + Q(x) \Big) \, p_b(x;k) = E_b(k) p_b(x;k), \;\; x \in [0,1], \;\; p_b(x+1;k) = p_b(x;k). \tag{2.85}$$

Therefore, taking the inner product of (2.83) with $p_b(x;k)$, and using self-adjointness of the operator $-\left(\partial_x + 2\pi i k\right)^2 + Q$ as well as the identity (2.85), yields

$$\Big(E_b(k) - E^\lambda\Big) \left\langle p_b(\cdot,k), \widetilde{\psi^\lambda}(\cdot,k) \right\rangle_{L^2([0,1])} + \lambda \left\langle p_b(\cdot,k), \Big(V\psi^\lambda\Big)^\sim (\cdot,k) \right\rangle_{L^2([0,1])} = 0. \tag{2.86}$$

or equivalently, using notation (B.36),

$$\Big(E_b(k) - E^\lambda\Big) \mathcal{T}_b \left\{\psi^\lambda\right\}(k) + \lambda \mathcal{T}_b \left\{V\psi^\lambda\right\}(k) = 0. \tag{2.87}$$

We can now decompose equation (2.86) into near- and far-frequency equations, around $E_{b_*}(k_*)$, the edge of the b_*-th band of the continuous spectrum. The coupled equations for ψ_{near} and ψ_{far} read:

$$\begin{aligned} &\Big(E_{b_*}(k) - E^\lambda\Big) \chi\left(|k| < \lambda^r\right) \left\langle p_{b_*}(\cdot,k), \widetilde{\psi^\lambda}(\cdot,k) \right\rangle_{L^2([0,1])} \\ &\qquad + \lambda \chi\left(|k| < \lambda^r\right) \left\langle p_{b_*}(\cdot,k), \left[V\left(\psi_{\text{near}} + \psi_{\text{far}}\right)\right]^\sim (\cdot,k) \right\rangle_{L^2([0,1])} = 0, \end{aligned} \tag{2.88}$$

and for $b \in \mathbb{N}$:

$$\begin{aligned} &\Big(E_{b}(k) - E^\lambda\Big) \chi\left(|k| \geq \lambda^r \delta_{b_*,b}\right) \left\langle p_{b}(\cdot,k), \widetilde{\psi^\lambda}(\cdot,k) \right\rangle_{L^2([0,1])} \\ &\qquad + \;\lambda \chi\left(|k| \geq \lambda^r \delta_{b_*,b}\right) \left\langle p_{b}(\cdot,k), \left[V\left(\psi_{\text{near}} + \psi_{\text{far}}\right)\right]^\sim (\cdot,k) \right\rangle_{L^2([0,1])} = 0. \end{aligned} \tag{2.89}$$

Equivalently, we write the near and far frequency equations in the form

$$\left(E_{b_*}(k) - E^\lambda\right) \widetilde{\psi}_{\rm near}(k) + \lambda\chi\left(|k| < \lambda^r\right)\left(\mathcal{T}_{b_*}\left\{V\psi_{\rm near}\right\}(k) + \mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}(k)\right) = 0, \tag{2.90}$$

$$\left(E_b(k) - E^\lambda\right) \widetilde{\psi}_{{\rm far},b}(k) + \lambda\chi\left(|k| \geq \lambda^r\delta_{b_*,b}\right)\left(\mathcal{T}_b\left\{V\psi_{\rm near}\right\}(k) + \mathcal{T}_b\left\{V\psi_{\rm far}\right\}(k)\right) = 0. \tag{2.91}$$

Equations (2.90) and (2.91) are, for the case of non-trivial periodic potentials, $Q(x)$, the analogues of (2.63)-(2.64).

2.5.2 Analysis of the far frequency Floquet-Bloch components

In this section we study the far frequency equation (2.91). We will show that we can write it in terms of the near frequency solution and will determine a bound on the far solution in terms of the near solution. The next result is therefore the analogue of Proposition 2.4.1 and facilitates the reduction of the eigenvalue problem to a closed equation for the near-frequency components of the eigenstate.

For clarity of presentation and without any loss of generality, we assume henceforth that we are localizing near the lowermost end point of the b_*-th band and that $k_* = 0$. Thus, by Lemma B.1.2,

$$b_* \ \textit{ is even}, \quad k_* = 0, \quad \text{with} \quad E_{b_*}(0) = E_*.$$

N.B. For $k_* = 0$, note that $p_b(x;k_*) = u_b(x;k_*)$ and we use these expressions interchangeably. For $k_* = 1/2$ one has to distinguish between $p_b(x;k_*)$ and $u_b(x;k_*)$.

Proposition 2.5.1. *Assume b_* is even and consider $E_* = E_{b_*}(0)$ the lowermost edge of the b_*-th band. There exists $\lambda_0 > 0$, such that for $0 < \lambda < \lambda_0$, the following holds. Set*

$$E^\lambda = E_* - \lambda^2\theta^2, \quad \theta \leq \lambda^{r-1}\frac{1}{2}|\partial_k^2 E_{b_*}(0)|^{1/2}, \qquad 0 < r < \frac{1}{2}. \tag{2.92}$$

Then for any $\psi_{near} \in L^2(\mathbb{R})$, there is a unique solution $\psi_{far}\left[\psi_{near},\lambda\right] \in L^2(\mathbb{R})$ of the far-frequency system (2.91). The mapping $(\psi_{near};\lambda) \mapsto \psi_{far}$ maps $L^2(\mathbb{R}) \times (0,\lambda_0)$ to $H^2(\mathbb{R})$ and ψ_{far} satisfies the bound

$$\left\|\psi_{far}\left[\psi_{near};\lambda\right]\right\|_{H^2(\mathbb{R})} \ \leq \ C\left(\|V\|_{L^\infty}\right)\lambda^{1-2r}\left\|\psi_{near}\right\|_{L^2(\mathbb{R})}. \tag{2.93}$$

Remark 2.5.2. *Recall that we have assumed* $(1+|x|)V(x) \in L^1(\mathbb{R})$ *and* $V \in L^\infty$. *It is in the proof of the bound* (2.93) *that we have used* $V \in L^\infty$. *We believe it possible to work under the milder assumption* $(1+|x|)V(x) \in L^1(\mathbb{R})$. *In this case, we would bound* ψ_{far} *in* $H^1(\mathbb{R})$ *and the analysis that would follow would be a bit more technical. We leave this as an exercise.*

Proof. We begin by showing that there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, there is a constant $C_1 > 0$ such that

$$|E_{b_*}(k) - E_*| \ \geq \ C_1\lambda^{2r}, \quad \lambda^r \leq |k| \leq 1/2 \ , \tag{2.94}$$

$$|E_b(k) - E_*| \geq \ C_1, \qquad b \neq b_*, \quad |k| \leq 1/2 \ . \tag{2.95}$$

Note first that (2.95) is an immediate consequence of E_* being the endpoint of the $(b_*)^{th}$ spectral gap. To prove (2.94) recall, by Lemma B.1.2 that $E_* = E_{b_*}(0)$, an eigenvalue at the edge of a spectral gap, is simple, and $k \mapsto E_{b_*}(k) - E_*$ is continuous. Therefore, for any λ_0, such that $0 < \lambda_0 < 1/2$

$$\min_{\lambda_0 \leq |k| \leq 1/2} |E_{b_*}(k) - E_*| \geq C(\lambda_0) > 0. \tag{2.96}$$

For $|k| \leq \lambda_0$, we approximate $E_{b_*}(k)$ by a Taylor expansion. In particular, since $E_{b_*}(k)$ is smooth for k near $k_* = 0$, $\partial_k E_{b_*}(0) = 0$ and $\partial_k^2 E_{b_*}(0) \neq 0$, we have $E_{b_*}(k) - E_{b_*}(0) - \frac{1}{2}\partial_k^2 E_{b_*}(0)\, k^2 = \mathcal{O}(|k|^3)$. Therefore, we can choose $\lambda_0 > 0$ sufficiently small so that for all $\lambda \leq \lambda_0$ we have

$$|E_{b_*}(k) - E_{b_*}(0)| \geq \frac{1}{3}\left|\partial_k^2 E_{b_*}(0)\right| \lambda^{2r}, \quad \text{for all } \lambda^r \leq |k| \leq \lambda_0 \ . \tag{2.97}$$

It follows from (2.96) and (2.97) that for sufficiently small $\lambda_0 > 0$,

$$\frac{1}{2} \geq |k| \geq \lambda > 0 \quad \Longrightarrow \quad |E_{b_*}(k) - E_*| \geq \min\left\{\frac{1}{3}\left|\partial_k^2 E_{b_*}(0)\right| \lambda^{2r}, C(\lambda_0)\right\} \ .$$

Thus if $E^\lambda = E_* - \lambda^2\theta^2, \quad \theta \leq \lambda^{r-1}\frac{1}{2}|\partial_k^2 E_{b_*}(0)|^{1/2}$, then for $0 < \lambda < \lambda_0$ sufficiently small, there is a positive constant C_1, such that

$$|E_{b_*}(k) - E^\lambda| \ \geq \ C_1\lambda^{2r}. \tag{2.98}$$

By (2.94) and (2.95), the far-frequency system, (2.91), may be re-written as

$$\widetilde{\psi}_{\mathrm{far},b}(k) \ + \ \lambda\frac{\chi\left(|k| \geq \lambda^r\delta_{b_*,b}\right)}{E_b(k) - E^\lambda}\ \mathcal{T}_b\left\{V\psi_{\mathrm{far}}\right\}(k) \ = \ -\lambda\frac{\chi\left(|k| \geq \lambda^r\delta_{b_*,b}\right)}{E_b(k) - E^\lambda}\mathcal{T}_b\left\{V\psi_{\mathrm{near}}\right\}(k), \quad b \geq 0. \tag{2.99}$$

We wish to rewrite this equation in terms of $\psi_{\rm far}(x)$. In order to do so, we multiply (2.99) by $u_b(x;k) = p_b(x;k)e^{2\pi ikx}$, sum over $b \geq 0$ and integrate with respect to $k \in (-1/2, 1/2]$. This yields

$$(I + \mathcal{K}_\lambda)\, \psi_{\rm far}(x) = -\left(\mathcal{K}_\lambda \psi_{\rm near}\right)(x), \tag{2.100}$$

where we define

$$\left(\mathcal{K}_\lambda g\right)(x) \equiv \int_{-1/2}^{1/2} \sum_{b\geq 0} \lambda \frac{\chi\left(|k| \geq \lambda^r \delta_{b_*,b}\right)}{E_b(k) - E^\lambda}\ \mathcal{T}_b \left\{V\ g\right\}(k) p_b(x;k) e^{2\pi ikx}\ dk.$$

We next show that the operator $\mathcal{K}_\lambda$, viewed as an operator from L^2 to H^2 has small norm, for λ small. Let $g \in L^2$. Using Proposition B.2.1, one has

$$\begin{aligned}
\|\mathcal{K}_\lambda g\|_{H^2}^2 &\lesssim \left\|\widetilde{\mathcal{K}_\lambda g}\right\|_{\mathcal{X}^2}^2 = \int_{-1/2}^{1/2} \sum_{b\geq 0} (1+b^2)^2 \left|\mathcal{T}_b \left\{\mathcal{K}_\lambda g\right\}(k)\right|^2\ dk \\
&= \lambda^2 \int_{-1/2}^{1/2} \sum_{b\geq 0} (1+b^2)^2 \frac{\chi\left(|k| \geq \lambda^r \delta_{b_*,b}\right)}{|E_b(k) - E^\lambda|^2} \left|\mathcal{T}_b \left\{V\ g\right\}(k)\right|^2\ dk.
\end{aligned}$$

Now, by (2.98), for $|k| \geq \lambda^r$ one has $\left|E_{b_*}(k) - E^\lambda\right|^{-1} \leq C_1 \lambda^{-2r}$, and recall $0 < r < 1/2$. For $b \neq b_*$, we use Weyl asymptotics to write $\left|(E_b(k) - E^\lambda)^{-1}\right| \sim \left|(b^2 - E_*)^{-1}\right| \sim (b^2+1)^{-1}$. We therefore have

$$\begin{aligned}
\|\mathcal{K}_\lambda g\|_{H^2}^2 &\lesssim \lambda^2 \int_{-1/2}^{1/2} \sum_{b\geq 0} \left|\mathcal{T}_b \left\{V\ g\right\}(k)\right|^2\ dk\ +\ \lambda^{2-4r} \int_{-1/2}^{1/2} (1+{b_*}^2)^2 \chi\left(|k| \geq \lambda^r\right) \left|\mathcal{T}_{b_*} \left\{V\ g\right\}(k)\right|^2\ dk \\
&\lesssim \lambda^{2-4r} \left\|(V\ g)^{\sim}\right\|_{\mathcal{X}^0}^2 \lesssim \lambda^{2-4r} \|V\|_{L^\infty}^2 \|g\|_{L^2}^2.
\end{aligned}$$

Thus, since $r \in (0, 1/2)$, one can choose $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then $\|\mathcal{K}_\lambda\|_{L^2 \to H^2} < 1$. In particular, $\mathcal{K}_\lambda$ is a contraction from L^2 to L^2, and therefore $I + \mathcal{K}_\lambda$ is invertible. The existence and uniqueness of $\psi_{\rm far} \in L^2(\mathbb{R})$ solution to (2.91) is now given through (2.100). Moreover, one has

$$\begin{aligned}
\|\psi_{\rm far}\|_{H^2} &= \left\|\left(I + \mathcal{K}_\lambda\right)^{-1} \left(\mathcal{K}_\lambda \psi_{\rm near}\right)\right\|_{H^2} \leq \|\left(I + \mathcal{K}_\lambda\right)^{-1}\|_{H^2 \to H^2} \|\mathcal{K}_\lambda\|_{L^2 \to H^2} \|\psi_{\rm near}\|_{L^2} \\
&\lesssim \lambda^{1-2r} \|V\|_{L^\infty} \|\psi_{\rm near}\|_{L^2},
\end{aligned}$$

which implies the bound (2.93). The proof of Proposition 2.5.1 is complete. □

2.5.3 Analysis of the near frequency Floquet-Bloch component

With the properties of the map $\psi_{\rm near} \mapsto \psi_{\rm far}[\psi_{\rm near}, \lambda]$ now understood via Proposition 2.5.1, we now view and study (2.90) as a closed eigenvalue problem for $(E^\lambda, \psi_{\rm near})$:

$$\left(E_{b_*}(k) - E^\lambda\right) \widetilde{\psi}_{\rm near}(k) + \lambda \chi_{\lambda^r}(k) \mathcal{T}_{b_*} \left\{V \psi_{\rm near}\right\}(k) + \lambda\ \chi_{\lambda^r}(k) \mathcal{T}_{b_*} \left\{V \psi_{\rm far}\left[\psi_{\rm near}; \lambda\right]\ \right\}(k) = 0. \tag{2.101}$$

Equation (2.101) is localized in the region $|k| < \lambda^r$, $0 < r < 1/2$. By careful expansion and rescaling of (2.101) we shall obtain an equation, which at leading order in λ, is a perturbation of the general class of equations to which Lemma 2.3.1 applies. The size of the perturbation is estimated in Proposition 2.5.6 and the perturbed equation is then solved by applying Lemma 2.3.4.

In Lemmata 2.5.3, 2.5.4 and 2.5.5 we expand the first two terms in (2.101) about $k_* = 0$ using Taylor's Theorem, making explicit the leading and higher order contributions.

Lemma 2.5.3. *Denote $E^\lambda = E_* - \lambda^2\theta^2 = E_{b_*}(0) - \lambda^2\theta^2$, as in Proposition 2.5.1. There exists k' such that $|k'| < \lambda^r$, and*

$$\left(E_{b_*}(k) - E^\lambda\right)\widetilde{\psi}_{near}(k) \;=\; \left(\frac{1}{2}\partial_k^2 E_{b_*}(0)\; k^2 + \lambda^2\theta^2\right)\widetilde{\psi}_{near}(k) \;+\; \lambda R_0\left[\widetilde{\psi}_{near};\lambda\right](k,k'),$$

where

$$R_0\left[\widetilde{\psi}_{near};\lambda\right](k,k') \;=\; \frac{1}{\lambda}\,\frac{1}{4!}\; k^4\;\partial_k^4 E_{b_*}(k')\;\widetilde{\psi}_{near}(k). \tag{2.102}$$

Proof. Taylor expanding $E_{b_*}(k)$ about $k_* = 0$ to fourth order and making use of $E^\lambda = E_{b_*}(0) - \lambda^2\theta^2$ and $\partial_k^j E_{b_*}(0) = 0$ for $j = 1,\ 3$, one obtains $E_{b_*}(k) - E^\lambda = \frac{1}{2}\partial_k^2 E_{b_*}(0)k^2 + \lambda^2\theta^2 + \frac{1}{4!}\partial_k^4 E_{b_*}(k')k^4$, which is equivalent to (2.102). □

Lemma 2.5.4. *One can decompose*

$$\mathcal{T}_{b_*}\left\{V\psi_{near}\right\}(k) = \left\langle p_{b_*}(\cdot;0), p_{b_*}(\cdot;0)\mathcal{T}\left\{V\mathcal{F}^{-1}\left\{\widetilde{\psi}_{near}\right\}\right\}(\cdot;k)\right\rangle_{L^2([0,1])} + R_1\left[\widetilde{\psi}_{near};\lambda\right](k),$$

with

$$\begin{aligned} R_1\left[\widetilde{\psi}_{near};\lambda\right](k) \;&=\; \langle p_{b_*}(\cdot;0), \mathcal{T}\left\{V\mathcal{E}_1\right\}(\cdot,k)\rangle_{L^2([0,1])} \\ &\quad + \langle p_{b_*}(\cdot;k) - p_{b_*}(\cdot;0), \mathcal{T}\left\{V\psi_{near}\right\}(\cdot,k)\rangle_{L^2([0,1])}\,, \end{aligned} \tag{2.103}$$

where $\mathcal{E}_1 \equiv \mathcal{T}^{-1}\left\{\widetilde{\psi}_{near}(k)\big(p_{b_*}(x;k) - p_{b_*}(x;0)\big)\right\}$.

Proof. Let us recall that by definition (2.84), one has

$$\psi_{\text{near}}(x) = \mathcal{T}^{-1}\left\{\widetilde{\psi}_{\text{near}}(\cdot)p_{b_*}(x;\cdot)\right\}. \tag{2.104}$$

Since $\widetilde{\psi}_{\text{near}}(k) = \chi\left(|k| < \lambda^r\right)\widetilde{\psi}_{\text{near}}(k)$, we decompose:

$$\psi_{\text{near}}(x) = \mathcal{T}^{-1}\left\{\widetilde{\psi}_{\text{near}}(\cdot)p_{b_*}(x;\cdot)\right\}(x) = p_{b_*}(x;0)\mathcal{F}^{-1}\{\widetilde{\psi}_{\text{near}}\} + \mathcal{E}_1(x) \tag{2.105}$$

where

$$\mathcal{E}_1(x) \;\equiv\; \mathcal{T}^{-1}\left\{\widetilde{\psi}_{\rm near}(\cdot)\big(p_{b_*}(x;\cdot)-p_{b_*}(x;0)\big)\right\}. \tag{2.106}$$

Above, we used that $\mathcal{T}^{-1}$ commutes with multiplication by a 1- periodic function of x, and that when acting on a function which is localized near $k=0$, and which does not depend on x, $\mathcal{T}^{-1}$ is equivalent to the standard inverse Fourier transform; see Appendix B.

The proof of Lemma 2.5.4 is now straightforward. □

We next give a precise expression of the leading order term in Lemma 2.5.4.

Lemma 2.5.5. *One can decompose*

$$\begin{aligned}&\left\langle p_{b_*}(\cdot;0), p_{b_*}(\cdot;0)\mathcal{T}\left\{V\mathcal{F}^{-1}\{\widetilde{\psi}_{near}\}\right\}(\cdot;k)\right\rangle_{L^2([0,1])} \\ &\quad= \left(\int_{-\infty}^{\infty}|p_{b_*}(x;0)|^2V(x)dx\right)\int_{-\infty}^{\infty}\widetilde{\psi}_{near}(l)dl \;+\; R_2\left[\widetilde{\psi}_{near}\right](k),\end{aligned} \tag{2.107}$$

with

$$R_2\left[\widetilde{\psi}_{near}\right](k) \;=\; \int_{-\infty}^{\infty}dx\;|p_{b_*}(x;0)|^2\;V(x)\int_{-\infty}^{\infty}\big(e^{2i\pi(l-k)x}-1\big)\widetilde{\psi}_{near}(l)\;dl. \tag{2.108}$$

Proof. By the definition of $\mathcal{T}$, one has

$$\begin{aligned}\mathcal{T}\left\{V\mathcal{F}^{-1}\{\widetilde{\psi}_{\rm near}\}\right\}(x;k) &= \sum_{n\in\mathbb{Z}}e^{2\pi inx}\mathcal{F}\left\{V\mathcal{F}^{-1}\{\widetilde{\psi}_{\rm near}\}\right\}(k+n)\\ &= \sum_{n\in\mathbb{Z}}e^{2\pi inx}\int_{-\infty}^{\infty}\widehat{V}(k+n-l)\widetilde{\psi}_{\rm near}(l)\;dl.\end{aligned}$$

Since $|k|<\lambda^r$ and $\widetilde{\psi}_{\rm near}(l)$ is localized on $|l|<\lambda^r$, the leading order term is obtained when replacing $\widehat{V}(k+n-l)$ with $\widehat{V}(n)$. The first term of (2.107) now follows from the identity:

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\left\langle p_{b^*}(\cdot;0),p_{b^*}(\cdot;0)e^{2\pi in\cdot}\right\rangle_{L^2([0,1])}\widehat{V}(n) &= \int_0^1|p_{b^*}(x;0)|^2\sum_{n\in\mathbb{Z}}e^{2\pi inx}\widehat{V}(n)dx\\ &= \sum_{n\in\mathbb{Z}}\int_0^1|p_{b^*}(x;0)|^2V(x+n)dx = \int_{-\infty}^{\infty}|p_{b^*}(x;0)|^2V(x)dx.\end{aligned}$$

Here, we used the Poisson summation formula and that $x\mapsto p_{b^*}(x;0)$ is $1-$ periodic.

Similarly, one has

$$\sum_{n\in\mathbb{Z}}\left\langle p_{b^*}(\cdot;0),p_{b^*}(\cdot;0)e^{2\pi in\cdot}\right\rangle_{L^2([0,1])}\widehat{V}(n+k-l) = \int_{-\infty}^{\infty}|p_{b^*}(x;0)|^2e^{2i\pi(l-k)x}V(x)dx.$$

This completes the proof of Lemma 2.5.5. □

The rescaled closed equation. Using Lemmata 2.5.3, 2.5.4 and 2.5.5, one can express the near frequency equation (2.101) as follows:

$$(\frac{1}{2}\partial_k^2 E_{b_*}(0)k^2 + \lambda^2\theta^2)\widetilde{\psi}_{\rm near}(k) \ + \ \lambda\ \chi\left(|k|<\lambda^r\right)\left(\int_{-\infty}^{\infty}|p_{b_*}(x;0)|^2 V(x)dx\right)\int_{-\infty}^{\infty}\widetilde{\psi}_{\rm near}(l)dl$$
$$= -\lambda\chi\left(|k|<\lambda^r\right)\mathcal{R}\left[\psi_{\rm near};\lambda\right](k), \tag{2.109}$$

where $\mathcal{R}\left[\psi_{\rm near};\lambda\right](k) \equiv \mathcal{T}_{b_*}\{V\psi_{\rm far}\} + R_0 + R_1 + R_2$.

Seeking to extract the dominant and higher order terms in λ, we introduce the scaled near-frequency components:

$$\widetilde{\psi}_{\rm near}(k) = \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{k}{\lambda}\right) \ = \ \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\kappa\right), \quad \text{where} \ \ k=\lambda\ \kappa\ . \tag{2.110}$$

Expressing (2.109) in terms of $\widehat{\Phi}_\lambda$ and κ we obtain, after dividing out by λ,

$$\left(\frac{1}{2}\partial_k^2 E_{b_*}(0)\kappa^2+\theta^2\right)\chi_{\lambda^{r-1}}(\kappa)\widehat{\Phi}_\lambda(\kappa) \ + \ \left(\int_{\mathbb{R}}|p_{b_*}(\cdot;0)|^2V\right)\ \chi_{\lambda^{r-1}}(\kappa)\int_{\mathbb{R}}\chi_{\lambda^{r-1}}(\eta)\widehat{\Phi}_\lambda(\eta)d\eta$$
$$= -\chi\left(|\kappa|<\lambda^{r-1}\right)\mathcal{R}\left[\psi_{\rm near};\lambda\right](\lambda\kappa) \ \equiv \ R(\widehat{\Phi}_\lambda). \tag{2.111}$$

Equation (2.111) is of the form $\widehat{\mathcal{L}}_0[\theta]\widehat{\Phi}_\lambda(\kappa) \ = \ R(\widehat{\Phi}_\lambda)$, where $\widehat{\mathcal{L}}_0[\theta]$ is given by (2.30) with parameters

$$A=\frac{1}{8\pi^2}\partial_k^2E_{b_*}(0)\ ,\ B=-\int_{\mathbb{R}}|p_{b_*}(x;0)|^2V(x)dx\ ,\ \text{and}\ \beta=1-r\ .$$

In order to solve (2.111) via Lemma 2.3.4 we need a bound on $R(\widehat{\Phi}_\lambda)$ of the form (2.48).

Proposition 2.5.6. *Assume that V is such that $(1+|\cdot|)V(\cdot)\in L^1$ and $V\in L^\infty$. Then $R(\widehat{\Phi}_\lambda)$, defined in (2.111), satisfies the bound*

$$\left\|R(\widehat{\Phi}_\lambda)\right\|_{L^{2,-1}} = \left\|\chi\left(|\cdot|<\lambda^{r-1}\right)\mathcal{R}\left[\psi_{near};\lambda\right](\lambda\cdot)\right\|_{L^{2,-1}} \le \ C\lambda^{\alpha(r)}\ \left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}. \tag{2.112}$$

where $\alpha(r)=\max\left\{\frac{1}{2}-2r, 2r, \frac{r+1}{2}\right\}$. The constant C depends on $\left\|(1+|\cdot|)V\right\|_{L^1}, \left\|V\right\|_{L^\infty}$ as well as $\sup_{|k|<\lambda^r}\|p_{b_*}(\cdot;k)\|_{L^\infty}$, $\sup_{|k|<\lambda^r}\sum_{n\in\mathbb{Z}}\left|\left\langle p_{b_*}(\cdot;k), e^{2\pi i n\cdot}\right\rangle_{L^2([0,1])}\right|$, $\sup_{|k|<\lambda^r}\left|\partial_k^4E_{b_*}(k)\right|$, $\sup_{|k|<\lambda^r}\left\|\partial_k p_{b_*}(\cdot;k)\right\|_{L^\infty}$, *and is finite by Lemmata B.1.5 and B.3.1.*

Proof of Proposition 2.5.6. Recall that $\mathcal{R}(\lambda\kappa)$, the right hand side of (2.111) has the form

$$\mathcal{R}\left[\psi_{\rm far}\left[\psi_{\rm near};\lambda\right],\psi_{\rm near};\lambda\right](\lambda\kappa) = \chi\left(|\kappa|<\lambda^{r-1}\right)\Big(\mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}(\lambda\kappa) + R_0\left[\widetilde{\psi}_{\rm near};\lambda\right](\lambda\kappa,k')$$
$$+R_1\left[\widetilde{\psi}_{\rm near};\lambda\right](\lambda\kappa) + R_2\left[\widetilde{\psi}_{\rm near}\right](\lambda\kappa)\Big) \ \equiv \ (I)\ +\ (II)\ +\ (III)\ +\ (IV)\ . \tag{2.113}$$

We proceed by estimating each of the terms: (I), (II), (III) and (IV).

(I) **Estimation of** $\chi\left(|\kappa|<\lambda^{r-1}\right)\mathcal{T}_{b_*}\left\{V\psi_{\mathbf{far}}\right\}(\lambda\kappa)$: We have

$$\left\|\chi\left(|\cdot|<\lambda^{r-1}\right)\mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}(\lambda\cdot)\right\|^2_{L^{2,-1}} = \int_{-\infty}^{\infty}\frac{\chi\left(|\kappa|<\lambda^{r-1}\right)}{1+\kappa^2}\left|\mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}(\lambda\kappa)\right|^2 d\kappa$$
$$\leq \left\|\mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}\right\|^2_{L^\infty}.$$

We now consider $\mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}(\cdot)$ in detail. By definition, one has

$$\begin{aligned}\mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}(k) &= \left\langle p_{b_*}(\cdot;k),\mathcal{T}\left\{V\psi_{\rm far}\right\}(\cdot,k)\right\rangle_{L^2([0,1])}\\ &= \left\langle p_{b_*}(\cdot;k),\sum_{n\in\mathbb{Z}}e^{2\pi in\cdot}\int_{-\infty}^{\infty}\widehat{V}(k+n-l)\widehat{\psi}_{\rm far}(l)dl\right\rangle_{L^2([0,1])}\\ &= \sum_{n\in\mathbb{Z}}\left\langle p_{b_*}(\cdot;k),e^{2\pi in\cdot}\right\rangle_{L^2([0,1])}\int_{-\infty}^{\infty}\frac{\widehat{V}(k+n-l)}{(1+|l|^2)^{1/2}}(1+|l|^2)^{1/2}\widehat{\psi}_{\rm far}(l)dl.\end{aligned}$$

Moreover,

$$\left|\int_{-\infty}^{\infty}\frac{\widehat{V}(k+n-l)}{(1+|l|^2)^{1/2}}(1+|l|^2)^{1/2}\widehat{\psi}_{\rm far}(l)dl\right| \leq \left\|\widehat{V}\right\|_{L^\infty}\left\|\psi_{\rm far}\right\|_{H^2}\lesssim\lambda^{1-2r}\left\|\psi_{\rm near}\right\|_{L^2}$$
$$\lesssim \lambda^{1-2r}\left\|\widetilde{\psi}_{\rm near}\right\|_{L^2} = \lambda^{1-2r}\lambda^{-\frac12}\left\|\widehat{\Phi}_\lambda\right\|_{L^2},$$

where we used Proposition 2.5.1, definition (2.110) and, by Proposition B.2.1,

$$\begin{aligned}\left\|\psi_{\rm near}\right\|^2_{L^2} &= \left\|\mathcal{T}^{-1}\{\widetilde{\psi}_{\rm near}(k)p_{b_*}(x;k)\}\right\|^2_{L^2} \lesssim \left\|\widetilde{\psi}_{\rm near}(k)p_{b_*}(x;k)\right\|^2_{\mathcal{X}^0}\\ &= \int_{-1/2}^{1/2}\left|\widetilde{\psi}_{\rm near}(k)\right|^2 dk = \left\|\widetilde{\psi}_{\rm near}\right\|^2_{L^2}.\end{aligned}\tag{2.114}$$

Finally, it follows

$$\left\|\mathcal{T}_{b_*}\left\{V\psi_{\rm far}\right\}\right\|_{L^\infty}\leq\lambda^{\frac12-2r}\,C\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}.\tag{2.115}$$

with $C=C\left(\left\|\widehat{V}\right\|_{L^\infty},\ \left\|V\right\|_{L^\infty},\ \sup_{|k|<\lambda^r}\sum_{n\in\mathbb{Z}}\left|\left\langle p_{b_*}(\cdot;k),e^{2\pi in\cdot}\right\rangle_{L^2([0,1])}\right|\right)$.

(II) **Estimation of** $\chi\left(|\kappa|<\lambda^{r-1}\right)R_0\left[\widetilde{\psi}_{\mathbf{near}};\lambda\right](\lambda\kappa,k')$**, given in** (2.102): We have (constants implicit)

$$\begin{aligned}\left\|\chi\left(|\cdot|<\lambda^{r-1}\right)\lambda^2(\cdot)^4\widehat{\Phi}_\lambda(\cdot)\right\|^2_{L^{2,-1}(\mathbb{R})} &= \lambda^4\int_{-\infty}^{\infty}\frac{\kappa^8}{1+\kappa^2}\chi\left(|\kappa|<\lambda^{r-1}\right)\left|\widehat{\Phi}_\lambda(\kappa)\right|^2 d\kappa\\ &= \lambda^4\int_{-\infty}^{\infty}\frac{\kappa^8}{(1+\kappa^2)^2}\chi\left(|\kappa|<\lambda^{r-1}\right)(1+\kappa^2)\left|\widehat{\Phi}_\lambda(\kappa)\right|^2 d\kappa\\ &\lesssim \lambda^4\sup_{|\kappa|<\lambda^{r-1}}\left|\frac{\kappa^8}{(1+\kappa^2)^2}\right|\left\|\widehat{\Phi}_\lambda\right\|^2_{L^{2,1}} \lesssim \lambda^{4r}\left\|\widehat{\Phi}_\lambda\right\|^2_{L^{2,1}}.\end{aligned}$$

Therefore,

$$\left\|\chi\left(|\kappa|<\lambda^{r-1}\right) R_0\left[\widetilde{\psi}_{\text{near}};\lambda\right](\lambda\kappa,k')\right\|_{L^{2,-1}} \equiv \left\|\chi\left(|\kappa|<\lambda^{r-1}\right)\frac{1}{4!}\partial_k^4 E_{b_*}(k')\lambda^2\kappa^4\widetilde{\Phi}_\lambda(\kappa)\right\|_{L^{2,-1}}$$
$$\lesssim \lambda^{2r}\sup_{|k'|<\lambda^r}|\partial_k^4 E_{b_*}(k')|\ \left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}. \tag{2.116}$$

(III) **Estimation of** $\chi\left(|\kappa|<\lambda^{r-1}\right) R_1\left[\widetilde{\psi}_{\mathbf{near}};\lambda\right](\lambda\kappa)$**, given in** (2.103)**:** Recall

$$R_1\left[\widetilde{\psi}_{\text{near}};\lambda\right](k) = \langle p_{b_*}(\cdot;0),\mathcal{T}\left\{V\mathcal{E}_1\right\}(\cdot,k)\rangle_{L^2([0,1])}$$
$$+\langle p_{b_*}(\cdot;k)-p_{b_*}(\cdot;0),\mathcal{T}\left\{V\psi_{\text{near}}\right\}(\cdot,k)\rangle_{L^2([0,1])}. \tag{2.117}$$

where $\mathcal{E}_1\equiv\mathcal{T}^{-1}\left\{\widetilde{\psi}_{\text{near}}(k)\big(p_{b_*}(x;k)-p_{b_*}(x;0)\big)\right\}$.

Let us first obtain an estimate on $\mathcal{E}_1$. Using Taylor expansion of $p_{b_*}(x;\cdot)$ around 0, one has

$$\begin{aligned}
|\mathcal{E}_1(x)| &= \left|\int_{-1/2}^{1/2} e^{2\pi ikx}\widetilde{\psi}_{\text{near}}(k)\big(p_{b_*}(x;k)-p_{b_*}(x;0)\big)\ dk\right| \\
&\le \sup_{x\in\mathbb{R},|k'|<\lambda^r}|\partial_k p_{b_*}(x;k')|\int_{-\infty}^{\infty}|k\chi\left(|k|<\lambda^r\right)\widetilde{\psi}_{\text{near}}(k)|\ dk \\
&\le \lambda\sup_{|k'|<\lambda^r}\|\partial_k p_{b_*}(\cdot;k')\|_{L^\infty}\int_{-\infty}^{\infty}|\kappa\chi\left(|\kappa|<\lambda^{r-1}\right)\widehat{\Phi}_\lambda(\kappa)|\ d\kappa \\
&\le \lambda\sup_{|k'|<\lambda^r}\|\partial_k p_{b_*}(\cdot;k')\|_{L^\infty}\left(\int_{|\kappa|<\lambda^{r-1}}\frac{\kappa^2}{1+\kappa^2}\ d\kappa\right)^{1/2}\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}} \\
&\le 2\lambda^{\frac{1+r}{2}}\sup_{|k'|<\lambda^r}\|\partial_k p_{b_*}(\cdot;k')\|_{L^\infty}\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}},
\end{aligned}$$

so that we deduce

$$\left\|\mathcal{E}_1\right\|_{L^\infty} \le 2\lambda^{\frac{1+r}{2}}\sup_{|k'|<\lambda^r}\|\partial_k p_{b_*}(x;k')\|_{L^\infty}\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}. \tag{2.118}$$

Estimation of the first term of (2.117) is as follows. One has

$$\left\|\chi\left(|\kappa|<\lambda^{r-1}\right)\langle p_{b_*}(\cdot;0),\mathcal{T}\left\{V\mathcal{E}_1\right\}(\cdot,\lambda\kappa)\rangle_{L^2([0,1])}\right\|^2_{L^{2,-1}_\kappa} =$$
$$\int_{-\infty}^{\infty}\frac{\chi\left(|\kappa|<\lambda^{r-1}\right)}{1+\kappa^2}\left|\langle p_{b_*}(\cdot;0),\mathcal{T}\left\{V\mathcal{E}_1\right\}(\cdot,\lambda\kappa)\rangle_{L^2([0,1])}\right|^2 d\kappa.$$

Turning to the integrand of the above expression, we rewrite the inner product

$$\begin{aligned}
\langle p_{b_*}(\cdot;0),\mathcal{T}\left\{V\mathcal{E}_1\right\}(\cdot,\lambda\kappa)\rangle_{L^2([0,1])} &= \int_0^1\mathcal{T}\left\{p_{b_*}(\cdot;0)\mathcal{E}_1(\cdot)V(\cdot)\right\}(x;\lambda\kappa) \\
&= \int_0^1\sum_{n\in\mathbb{Z}}e^{2\pi inx}\mathcal{F}\left\{p_{b_*}(\cdot;0)\mathcal{E}_1(\cdot)V(\cdot)\right\}(\lambda\kappa+n)\ dx \\
&= \mathcal{F}\left\{p_{b_*}(\cdot;0)\mathcal{E}_1(\cdot)V(\cdot)\right\}(\lambda\kappa),
\end{aligned}$$

where we used that $p_{b_*}(x;0)$ is $1-$periodic, so that it commutes with $\mathcal{T}$, and the Poisson summation formula. It follows that

$$\left|\langle p_{b_*}(\cdot;0), \mathcal{T}\left\{V\mathcal{E}_1\right\}(\cdot,\lambda\kappa)\rangle_{L^2([0,1])}\right| \leq \left\|p_{b_*}(\cdot;0)\mathcal{E}_1(\cdot)V(\cdot)\right\|_{L^1} \leq \left\|\mathcal{E}_1\right\|_{L^\infty}\int |p_{b_*}(x;0)||V(x)|\ dx.$$

Using (2.118), one deduces

$$\left\|\chi\left(|\kappa|<\lambda^{r-1}\right)\langle p_{b_*}(\cdot;0), \mathcal{T}\left\{V\mathcal{E}_1\right\}(\cdot,\lambda\kappa)\rangle_{L^2([0,1])}\right\|_{L^{2,-1}_\kappa} \ \leq\ C\lambda^{\frac{1+r}{2}}\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}, \tag{2.119}$$

with $C=C(\sup_{|k|<\lambda^r}\|\partial_k p_{b_*}(\cdot;k)\|_{L^\infty}, \int |p_{b_*}(x;0)||V(x)|\ dx)$.

The last term in (2.117) is estimated as follows. Note that

$$\begin{aligned}
&\left|\langle p_{b_*}(\cdot;\lambda\kappa)-p_{b_*}(\cdot;0), \mathcal{T}\left\{V\psi_{\text{near}}\right\}(\cdot,\lambda\kappa)\rangle_{L^2([0,1])}\right| \\
&\quad=\left|\int_0^1\left(p_{b_*}(x;\lambda\kappa)-p_{b_*}(x;0)\right)\sum_{n\in\mathbb{Z}}e^{2\pi inx}\mathcal{F}\{V\psi_{\text{near}}\}(\lambda\kappa+n)dx\right| \\
&\quad=\left|\int_0^1\left(p_{b_*}(x;\lambda\kappa)-p_{b_*}(x;0)\right)\sum_{n\in\mathbb{Z}}(V\psi_{\text{near}})(x+n)e^{-2\pi i(\lambda\kappa+n)x}dx\right| \\
&\quad\leq\int_{-\infty}^{\infty}\left|\left(p_{b_*}(x;\lambda\kappa)-p_{b_*}(x;0)\right)V(x)\psi_{\text{near}}(x)\right|\ dx \\
&\quad\leq\lambda\kappa\sup_{|k'|<\lambda^r}\|\partial_k p_{b_*}(\cdot;k')\|_{L^\infty}\|\psi_{\text{near}}\|_{L^\infty}\|V\|_{L^1},
\end{aligned}$$

where we used the Poisson summation formula along with the periodicity of $p_{b_*}(x;\lambda\kappa)-p_{b_*}(x;0)$ and its Taylor expansion as $|\lambda\kappa|<\lambda^r$. Now, note that

$$\|\psi_{\text{near}}\|_{L^\infty}=\|\mathcal{T}^{-1}\{\widetilde{\psi}_{\text{near}}(k)p_{b_*}(x;k)\}\|_{L^\infty}\leq\sup_{|k|<\lambda^r}\|p_{b_*}(\cdot;k)\|_{L^\infty}\int_{-\lambda^r}^{\lambda^r}|\widetilde{\psi}_{\text{near}}(l)|\ dl$$

and

$$\int_{-\infty}^{\infty}|\widetilde{\psi}_{\text{near}}(l)|\ dl\ =\ \int_{-\infty}^{\infty}|\widehat{\Phi}_\lambda(\eta)|\ d\eta\ =\ \int_{-\infty}^{\infty}\frac{1}{(1+\eta^2)^{1/2}}(1+\eta^2)^{1/2}|\widehat{\Phi}_\lambda(\eta)|\ d\eta\leq C\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}.$$

It follows

$$\begin{aligned}
&\left\|\chi\left(|\kappa|<\lambda^{r-1}\right)\langle p_{b_*}(\cdot;\lambda\kappa)-p_{b_*}(\cdot;0), \mathcal{T}\left\{V\psi_{\text{near}}\right\}(\cdot,\lambda\kappa)\rangle_{L^2([0,1])}\right\|_{L^{2,-1}_\kappa} \\
&\qquad\leq\ C\lambda\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}\left(\int\frac{\kappa^2\chi\left(|\kappa|<\lambda^{r-1}\right)}{1+\kappa^2}\right)^{1/2}\lesssim\lambda^{\frac{1+r}{2}}\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}},
\end{aligned}\tag{2.120}$$

with $C = C\left(\sup_{|k|<\lambda^r} \|p_{b_*}(\cdot;k)\|_{L^\infty}, \sup_{|k'|<\lambda^r} \|\partial_k p_{b_*}(\cdot;k')\|_{L^\infty},\ \|V\|_{L^1}\right)$.

Estimates (2.119) and (2.120) yield

$$\left\|\chi\left(|\kappa|<\lambda^{r-1}\right) R_1[\widetilde{\psi}_{\rm near}](\lambda\kappa)\right\|_{L^{2,-1}} \le C \left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}\ \lambda^{\frac{1+r}{2}}. \tag{2.121}$$

with $C = C\left(\sup_{|k|<\lambda^r} \|p_{b_*}(\cdot;k)\|_{L^\infty}, \sup_{|k|<\lambda^r} \|\partial_k p_{b_*}(\cdot;k)\|_{L^\infty},\ \|V\|_{L^1}\right)$.

(IV) **Estimation of $\chi\left(|\kappa|<\lambda^{r-1}\right) R_2[\widetilde{\psi}_{\rm \bf near}](\lambda\kappa)$, given in** (2.108)**:** Recall

$$R_2\left[\widetilde{\psi}_{\rm near}\right](k) \ = \ \int_{-\infty}^{\infty} dx\ |p_{b_*}(x;0)|^2\ V(x) \int_{-\infty}^{\infty} \left(e^{2i\pi(l-k)x}-1\right)\widetilde{\psi}_{\rm near}(l)dl.$$

We now use that $\left|e^{2i\pi(l-k)x}-1\right| \le 2\pi|l-k||x|$. It follows

$$\left|R_2\left[\widetilde{\psi}_{\rm near}\right](\lambda\kappa)\right| \ \le\ 2\pi\lambda \int_{-\infty}^{\infty} dx\ |p_{b_*}(x;0)|^2\ |x\ V(x)| \int_{-\infty}^{\infty} |\kappa-\eta||\widehat{\Phi}_\lambda(\eta)|d\eta.$$

We therefore define

$$\mathcal{I}(\kappa) \ = \ -\chi\left(|\kappa|<\lambda^{r-1}\right)\ \int_{-\infty}^{\infty} |\kappa-\eta|\chi\left(|\eta|<\lambda^{r-1}\right)|\widehat{\Phi}_\lambda(\eta)|d\eta. \tag{2.122}$$

The integral, $\mathcal{I}(\kappa)$, is bounded in $L^{2,-1}(\mathbb{R})$ as follows:

$$\begin{aligned}
\|\mathcal{I}\|^2_{L^{2,-1}} &\le \int_{-\infty}^{\infty} \frac{\chi\left(|\kappa|<\lambda^{r-1}\right)}{1+\kappa^2} \int_{|\eta|<\lambda^{r-1}} \frac{|\kappa-\eta|^2}{1+\eta^2} d\eta d\kappa \left\|\widehat{\Phi}_\lambda\right\|^2_{L^{2,1}} \\
&= \left\|\widehat{\Phi}_\lambda\right\|^2_{L^{2,1}} \int_\kappa\int_\eta \frac{|\kappa-\eta|^2}{(1+\kappa^2)(1+\eta^2)}\chi\left(|\kappa|<\lambda^{r-1}\right)\chi\left(|\eta|<\lambda^{r-1}\right) d\kappa d\eta.
\end{aligned}$$

One easily checks that

$$\int_\kappa\int_\eta \frac{|\kappa-\eta|^2}{(1+\kappa^2)(1+\eta^2)}\chi\left(|\kappa|<\lambda^{r-1}\right)\chi\left(|\eta|<\lambda^{r-1}\right) d\kappa d\eta \ \lesssim\ \lambda^{r-1},$$

so that one obtains eventually

$$\left\|\chi\left(|\kappa|<\lambda^{r-1}\right) R_2[\widetilde{\psi}_{\rm near}](\lambda\kappa)\right\|_{L^{2,-1}} \le C \left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}\ \lambda^{\frac{r+1}{2}}, \tag{2.123}$$

with $C = C\left(\sup_{|k|<\lambda^r} \|p_{b_*}(\cdot;k)\|_{L^\infty}, \left\|xV(x)\right\|_{L^1_x}\right)$.

Altogether, (2.115), (2.116), (2.121), and (2.123) yield the estimate of Proposition 2.5.6. $\square$

2.5.4 Completion of the proof of Theorem 2.2.4

We now prove Theorem 2.2.4 by an application of Lemma 2.3.4 to equation (2.111), where the remainder is estimated in Proposition 2.5.6.

Proof of Theorem 2.2.4. We seek $E^\lambda \equiv E_{b_*}(0) - \lambda^2\theta^2$ and ψ^λ of the form

$$\begin{aligned}\psi^\lambda = \psi_{\text{near}} + \psi_{\text{far}} &= \mathcal{T}^{-1}\left\{\widetilde{\psi}_{\text{near}}(k)p_{b_*(x;k)}\right\} + \mathcal{T}^{-1}\left\{\sum_{b=0}^{\infty}\widetilde{\psi}_{\text{far},b}(k)p_b(x;k)\right\}\\ &= \int_{-1/2}^{1/2}\left(\widetilde{\psi}_{\text{near}}(k)u_{b_*}(x;k) + \sum_{b=0}^{\infty}\widetilde{\psi}_{\text{far},b}(k)u_b(x;k)\right)\, dk.\end{aligned}$$

where $\widetilde{\psi}_{\text{near}}$, $\widetilde{\psi}_{\text{far}}$ satisfy equations (2.90)–(2.91); see Section 2.5.1.

By application of Proposition 2.5.1, one has that ψ_{far} is uniquely defined as a function of ψ_{near} and λ, and that $\left\|\psi_{\text{far}}[\psi_{\text{near}};\lambda]\right\|_{H^2} \le \lambda^{1-2r}\left\|\psi_{\text{near}}\right\|_{L^2}$. Then, defining $\widehat{\Phi}_\lambda$ as in (2.110), one has

$$\widetilde{\psi}_{\text{near}}(k) = \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{k}{\lambda}\right) = \frac{1}{\lambda}\widehat{\Phi}_\lambda(\kappa), \qquad k = \lambda\kappa. \tag{2.124}$$

By Proposition 2.5.6, the rescaled (from (2.90)) near-frequency equation (2.111) can be written as

$$\begin{aligned}\left(\frac{1}{2}\partial_k^2 E_{b_*}(0)\kappa^2 + \theta^2\right)\chi_{\lambda^{r-1}}(\kappa)\widehat{\Phi}_\lambda(\kappa) + \chi_{\lambda^{r-1}}(\kappa)\left(\int_{\mathbb{R}}|p_{b_*}(\cdot;0)|^2V\right)\int_{\mathbb{R}}\chi_{\lambda^{r-1}}(\eta)\widehat{\Phi}_\lambda(\eta)d\eta&\\ = -\chi\left(|\kappa| < \lambda^{r-1}\right)\mathcal{R}\left(\widehat{\Phi}_\lambda\right)(\kappa),&\end{aligned} \tag{2.125}$$

with $\left\|\mathcal{R}(\widehat{\Phi}_\lambda)\right\|_{L^{2,-1}} \le C\lambda^{\alpha(r)}\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}$, and $\alpha(r) = \max(\frac{1}{2} - 2r, 2r, \frac{r+1}{2})$.

From now on, we set $r = 1/8$, $\alpha = 1/4$, which yield optimal estimates. Applying Lemma 2.3.4 with $\beta = 1 - r = 7/8$,

$$A = \frac{1}{8\pi^2}\partial_k^2 E_{b_*}(0) \ \text{ and } \ B = -\int_{-\infty}^{\infty}|u_{b_*}(x;0)|^2V(x)dx \quad \Big(\text{assumed to be positive}\Big), \tag{2.126}$$

we deduce that there exists a solution $\left(\theta^2, \widehat{\Phi}_\lambda\right)$ of the rescaled near-frequency equation (2.125), satisfying

$$\|\widehat{\Phi}_\lambda - \widehat{f}_0\|_{L^{2,1}} \le C\,\lambda^{\frac{1}{4}} \qquad \text{and} \qquad |\theta^2 - \theta_0^2| \le C\,\lambda^{\frac{1}{4}}\,. \tag{2.127}$$

Here $\left(\theta_0^2, \widehat{f}_0\right)$ is a solution of the homogeneous equation

$$\widehat{\mathcal{L}}_{0,\lambda}\left(\theta_0, \widehat{f}_0\right) = (4\pi^2A\xi^2 + \theta^2)\widehat{f}_0 - B\,\chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right)\int_{-\infty}^{\infty}\chi\left(|\eta| < \lambda^{-\frac{7}{8}}\right)\widehat{f}_0(\eta)d\eta = 0,$$

as described in Lemma 2.3.1. Thus $\widetilde{\psi}_{\rm near}(\xi) = \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{\xi}{\lambda}\right)$ and $E^\lambda = E_{b_*}(0) - \lambda^2\theta^2(\lambda)$ are well-defined (and satisfy the Ansatz of Lemma 2.5.3), and $\widetilde{\psi}_{\rm far}$ is uniquely determined as the solution of (2.91); see Lemma 2.5.1. It follows that

$$\psi^\lambda(x) \ \equiv\ \psi_{\rm far} \ +\ \psi_{\rm near} \ \equiv\ \psi_{\rm far} \ +\ \int_{-1/2}^{1/2} \widetilde{\psi}_{\rm near}(k) u_{b_*}(x;k)\ dk \tag{2.128}$$

is well-defined.

There remains to prove estimates (2.25) and (2.26). Recalling that $E^\lambda = E_{b_*}(0) - \lambda^2\theta^2$, (2.127) implies $\left|E^\lambda - (E_{b_*}(0) - \lambda^2\theta_0^2)\right| \le C\lambda^{2+1/4}$. By Lemma 2.3.1, one has $\left|\theta_0(\lambda) - \frac{B}{2\sqrt{A}}\right| \le C(A,B)\lambda^{\frac{7}{8}}$, so that one can set

$$E_2 \ \equiv\ -\frac{B^2}{4A} \ =\ -\ \frac{\left|\int_{-\infty}^{\infty} |u_{b_*}(x;k_*)|^2 V(x) dx\right|^2}{\frac{1}{2\pi^2}\partial_k^2 E_{b_*}(k_*)};$$

and estimate (2.25) follows.

We now turn to a proof of the eigenfunction approximation (2.26). Recall

$$\begin{aligned}
\psi_{\rm near}(x) \ &\equiv\ \int_{-1/2}^{1/2} \widetilde{\psi}_{\rm near}(k) u_{b_*}(x;k) \ =\ \int_{-1/2}^{1/2} \frac{1}{\lambda}\widehat{\Phi}_\lambda\left(\frac{k}{\lambda}\right) e^{2\pi i k x} p_{b_*}(x;k)\ dk \\
&= \int_{-1/2\lambda}^{1/2\lambda} \chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right) \widehat{\Phi}_\lambda(\xi)\, e^{2\pi i\lambda\xi x} p_{b_*}(x;\lambda\xi)\ d\xi \\
&= \int_{\mathbb{R}} \chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right) \widehat{\Phi}_\lambda(\xi)\, e^{2\pi i\lambda\xi x} p_{b_*}(x;0)\ d\xi \\
&\qquad + \int_{\mathbb{R}} \chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right) \widehat{\Phi}_\lambda(\xi)\, e^{2\pi i\lambda\xi x} (\lambda\xi)\partial_k p_{b_*}(x;k')\ d\xi \\
&= u_{b_*}(x;0) \int_{\mathbb{R}} \chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right) \widehat{f_0}(\xi) e^{2\pi i\lambda\xi x}\ d\xi \\
&\qquad + u_{b_*}(x;0) \int_{\mathbb{R}} \chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right) \left(\widehat{\Phi}_\lambda - \widehat{f_0}\right)(\xi) e^{2\pi i\lambda\xi x}\ d\xi \\
&\qquad + \int_{\mathbb{R}} \chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right) \widehat{\Phi}_\lambda(\xi)\, e^{2\pi i\lambda\xi x} (\lambda\xi)\partial_k p_{b_*}(x;k')\ d\xi \\
&= I_1(x) + I_2(x) + I_3(x),
\end{aligned}$$

with $|k'| = |k'(\lambda\xi)| < \lambda^{\frac{1}{8}}$. Now, since $\chi\left(|\xi| < \lambda^{-\frac{7}{8}}\right)\widehat{f_0}(\xi) = \widehat{f_0}(\xi)$, one has

$$I_1(x) \ \equiv\ u_{b_*}(x;0)\mathcal{F}^{-1}\left\{\widehat{f_0}\right\}(\lambda x).$$

By (2.37) in Lemma 2.3.1, one has

$$\sup_{x\in\mathbb{R}}\left|\ I_1(x)\ -\ \frac{1}{B}u_{b_*}(x;0)\exp\left(-\frac{\lambda B}{2A}|x|\right)\ \right|$$
$$=\ \sup_{x\in\mathbb{R}}\left|\ u_{b_*}(x;0)\left\{\mathcal{F}^{-1}\left\{\widehat{f_0}\right\}(\lambda x)-\frac{1}{B}\exp\left(-\frac{\lambda B}{2A}|x|\right)\right\}\ \right|\le\ C\|p_{b_*}(\cdot;0)\|_{L^\infty}\ \lambda^{7/8}. \quad (2.129)$$

and $\|p_{b_*}(\cdot;0)\|_{L^\infty}$ is bounded; see Lemma B.3.1.

Let us now estimate $I_2(x)$ and $I_3(x)$. One has

$$\begin{aligned}|I_2(x)|\ &\equiv\ \left|u_{b_*}(x;0)\int_{\mathbb{R}}\chi\left(|\xi|<\lambda^{-\frac{7}{8}}\right)\left(\widehat{\Phi}_\lambda-\widehat{f_0}\right)(\xi)e^{2\pi i\lambda\xi x}\ d\xi\right|\\ &\le|p_{b_*}(x;0)|\int_{\mathbb{R}}\frac{\chi\left(|\xi|<\lambda^{-\frac{7}{8}}\right)}{(1+|\xi|^2)^{1/2}}(1+|\xi|^2)^{1/2}\left|\widehat{\Phi}_\lambda(\xi)-\widehat{f_0}(\xi)\right|\ d\xi\\ &\le\ C\|p_{b_*}(\cdot;0)\|_{L^\infty}\left\|\widehat{\Phi}_\lambda-\widehat{f_0}\right\|_{L^{2,1}}\le\ C(A,B)\|p_{b_*}(\cdot;0)\|_{L^\infty}\lambda^{1/4}, \qquad (2.130)\end{aligned}$$

where the last inequality comes from (2.127). Similarly,

$$\begin{aligned}|I_3(x)|\ &\equiv\ \left|\int_{\mathbb{R}}\chi\left(|\xi|<\lambda^{-\frac{7}{8}}\right)\widehat{\Phi}_\lambda(\xi)\,e^{2\pi i\lambda\xi x}(\lambda\xi)\partial_k p_{b_*}(x;k')\ d\xi\right|\\ &\le\lambda\sup_{|k'|<\lambda^{1-7/8}}\|\partial_k p_{b_*}(\cdot;k')\|_{L^\infty}\int_{\mathbb{R}}\chi\left(|\xi|<\lambda^{-\frac{7}{8}}\right)|\xi|\left|\widehat{\Phi}_\lambda(\xi)\right|d\xi\\ &\le\ C\sup_{|k'|<\lambda^{1/8}}\|\partial_k p_{b_*}(\cdot;k')\|_{L^\infty}\left\|\widehat{\Phi}_\lambda\right\|_{L^{2,1}}\ \lambda. \qquad (2.131)\end{aligned}$$

By (2.129), (2.130) and (2.131), one has

$$\psi_{\text{near}}\ =\ I_1(x)+I_2(x)+I_3(x)\ =\ \frac{2}{B}u_{b_*}(x;0)\exp\left(\frac{-\lambda B}{2A}|x|\right)\ +\ \psi_{\text{rem}}(x),$$

with $\|\psi_{\text{rem}}\|_{L^\infty}\lesssim\lambda^{1/4}$.

Finally, let us note that by Sobolev embeddings, one has

$$\|\psi_{\text{far}}\|_{L^\infty}\ \le\ \|\psi_{\text{far}}\|_{H^2}\le C\lambda^{1-1/4}\|\psi_{\text{near}}\|_{L^2}=C\lambda^{1/2-1/4}\|\widehat{\Phi}_\lambda\|_{L^2}\le C\lambda^{1/4},$$

where we use Proposition 2.5.1 with $r=1/8$, and (2.114).

It follows that $\psi^\lambda=\psi_{\text{near}}+\psi_{\text{far}}$ satisfies

$$\sup_{x\in\mathbb{R}}\left|\ \psi^\lambda(x)\ -\ \frac{1}{B}u_{b_*}(x;0)\exp(\lambda\alpha_0|x|)\ \right|\ \le\ C\lambda^{1/4}\ ,\quad\text{with }\alpha_0=-\frac{B}{2A}.$$

Since ψ^λ is defined up to a multiplicative constant, (2.26) holds. This completes the proof of Theorem 2.2.4. $\square$

2.6 The bootstrap: proof of Corollary 2.2.6

We give the proof of Corollary 2.2.6, on the refined expansion of the bifurcation of eigenvalues of $H_Q + \lambda V = -\partial_x^2 + Q(x) + \lambda V(x)$, for $Q(x)$ periodic. Corollary 2.2.3, in the case of $Q(x) \equiv 0$, is obtained along the same lines, using $p(x;k) = 1$, $E(k) \equiv 4\pi^2 k^2$ for $k \in \mathbb{R}$, *etc.*

Proof of Corollary 2.2.6. We know, by Theorem 2.2.4, that there exists $(\psi^\lambda, E^\lambda)$ solution of the eigenvalue problem $(H_Q + \lambda V)\psi^\lambda = E^\lambda \psi^\lambda$. Moreover, E^λ is in the gap of the continuous spectrum of $\mathrm{spec}(H_Q) = \mathrm{spec}(H_{Q+\lambda V})$, near an edge $E_* = E_{b_*}(k_*)$. In the following, we assume that $k_* = 0$ (the case where $k_* = 1/2$ can be treated using the same method).

We next seek an integral equation for ψ^λ by applying the resolvent $R_Q(E^\lambda)$ to the differential equation for ψ^λ. A construction of the resolvent kernel, $R_Q(x,y;E^\lambda)$, proceeds as follows. Recall the discriminant, $D(E)$, introduced in Appendix B.1.1 as the trace of the monodromy matrix defined by the linearly independent solutions $\phi_j(x;E), j = 1,2$: $D(E) = \phi_1(1;E) + \phi_2'(1;E)$.

Since E_* is a band edge and E^λ is in a gap, we have $D(E_*) = 2$ and $D(E^\lambda) > 2$. Therefore, there exists $\kappa = \kappa(\lambda) > 0$ with

$$E^\lambda = E(i\lambda\kappa) = E(-i\lambda\kappa), \quad D(E^\lambda) = e^{2\pi\lambda\kappa} + e^{-2\pi\lambda\kappa} > 2,$$

Additionally, we define $\psi_\pm \equiv \psi_\pm(x;E^\lambda)$, the solutions of

$$\left(-\partial_x^2 + Q(x)\right)\psi_\pm = E^\lambda \psi_\pm, \qquad \psi_\pm(x+1;E^\lambda) = e^{\pm 2\pi\lambda\kappa}\psi_\pm(x;E^\lambda).$$

More precisely, $\psi_\pm$ are defined as

$$\psi_\pm(x) \ \equiv \ p_{b_*}(x;\mp i\kappa)e^{\pm 2\pi\lambda\kappa x}, \quad \text{with} \tag{2.132}$$

$$\left(-(\partial_x - 2\pi\lambda\kappa)^2 + Q(x)\right)p_{b_*}(x;i\kappa) = \ E^\lambda p_{b_*}(x;i\kappa)\ , \qquad p_{b_*}(x+1;i\kappa) \ = \ p_{b_*}(x;i\kappa). \tag{2.133}$$

which is well-defined for λ small enough, by Theorem B.1.5.

With those definitions, the resolvent operator $R_Q(E^\lambda) = (-\partial_x^2 + Q - E^\lambda)^{-1}$ has kernel

$$R_Q(x,y;E^\lambda) = \begin{cases} \dfrac{\psi_+(x)\psi_-(y)}{W[\psi_\pm]} & \text{if } y > x, \\ \dfrac{\psi_+(y)\psi_-(x)}{W[\psi_\pm]} & \text{if } y < x. \end{cases}$$

where $W[\psi_\pm] \equiv \psi_+'(x)\psi_-(x) - \psi_+(x)\psi_-'(x)$. Thus, for any bounded function f,

$$R_Q[f](x;E^\lambda) = \int_{-\infty}^{\infty} R_Q(x,y;E^\lambda)f(y)\ dy,$$

we have $(-\partial_x^2 + Q - E^\lambda) R_Q[f](x; E^\lambda) \ = \ f$. It follows that ψ^λ satisfies the integral equation

$$\psi^\lambda(x) + \lambda \int_{\mathbb{R}} R_Q(x, y; E^\lambda) V(y) \psi^\lambda(y) \ dy \ = \ 0.$$

Multiplying by $u_{b_*}(x; 0) V(x)$ and integrating along x yields

$$\int_{\mathbb{R}} V(x) u_{b_*}(x; 0) \psi^\lambda(x) dx + \lambda \iint_{\mathbb{R}^2} u_{b_*}(x; 0) V(x) R_Q(x, y; E^\lambda) V(y) \psi^\lambda(y) \ dx \ dy \ = \ 0. \tag{2.134}$$

We will deduce from (2.134) the precise behavior of κ (and therefore $E^\lambda - E_{b_*}(0)$) as λ tends to zero, using the following

Lemma 2.6.1. *Let $E_* = E_{b_*}(0)$ be an edge of the continous spectrum, and let the hypotheses of Theorem 2.2.4 be satisfied, so that E^λ exists. Define $R_Q(x, y; E^\lambda)$ as above. Then for $\lambda > 0$ small enough, one has*

$$R_Q(x, y; E^\lambda) \ = \ \frac{u_{b_*}(x; 0) u_{b_*}(y; 0)}{2\lambda\kappa \frac{\partial_k^2 E(0)}{4\pi}} e^{-2\pi\lambda\kappa|x-y|} + R_Q^{(0)}(x, y) + \lambda\kappa R_Q^{(1)}(x, y), \tag{2.135}$$

where $R_Q^{(0)}$ is skew-symmetric: $R_Q^{(0)}(x, y) = -R_Q^{(0)}(y, x)$; and $R_Q^{(0)}, R_Q^{(1)}$ are bounded:

$$|R_Q^{(0)}(x, y)| + |R_Q^{(1)}(x, y)| \ \leq \ C e^{-2\pi\lambda\kappa|x-y|} \ \leq \ C \ ,$$

where C is a constant, uniform with respect to $\lambda\kappa$.

In order to ease the reading, we postpone the proof of this result to the end of this section, and carry on with the proof of Corollary 2.2.6. Since $u_{b_*}(x; 0)$ is uniformly bounded (see Lemma B.3.1), one has the low-order estimate

$$\left| \ R_Q(x, y; E^\lambda) \ - \ \frac{4\pi}{\partial_k^2 E(0)} \frac{u_{b_*}(x; 0) u_{b_*}(y; 0)}{2\lambda\kappa} \ \right| \ \leq \ C(1 + |x - y| + \lambda\kappa), \tag{2.136}$$

where we used $\left| \ e^{-\lambda\kappa|x-y|} \ - \ 1 \ \right| \ \leq \ C\lambda\kappa|x - y|$.

Plugging (2.136) into (2.134) and using $(1 + |x|)V \in L^1$, yields

$$\left| \ \int_{\mathbb{R}} V(x) u_{b_*}(x; 0) \psi^\lambda(x) dx + \frac{2\pi}{\kappa \partial_k^2 E(0)} \iint_{\mathbb{R}^2} u_{b_*}(x; 0)^2 V(x) u_{b_*}(y; 0) V(y) \psi^\lambda(y) \ dx \ dy \ \right| \leq \ C\lambda(1 + \lambda\kappa). \tag{2.137}$$

Now we use the fact that by Theorem 2.2.4, one has $\left\| \psi^\lambda(x) \ - \ u_{b_*}(x; 0) \exp\left(\lambda\alpha_0|x|\right) \right\|_{L^\infty} \ \lesssim \ \lambda^{1/4}$, so that $\lim\limits_{\lambda \to 0} \int V(x) u_{b_*}(x; 0) \psi^\lambda(x) dx = \int V(x) u_{b_*}(x; 0)^2 \neq 0$. It follows that for λ sufficiently small, one can divide out $\int V(x) u_{b_*}(x; 0) \psi^\lambda(x) dx$, and deduce from (2.137)

$$\left| \ \kappa + \frac{2\pi}{\partial_k^2 E(0)} \int_{\mathbb{R}} u_{b_*}(x; 0)^2 V(x) \ dx \ \right| \leq \ C\lambda\kappa(1 + \lambda\kappa),$$

from which it follows the low-order estimate of κ:

$$\left|\ \kappa + \frac{2\pi}{\partial_k^2 E(0)} \int_{\mathbb{R}} u_{b_*}(x;0)^2 V(x)\ dx\ \right| \leq\ C\lambda. \tag{2.138}$$

Let us now derive higher order estimates. For any $x, y \in \mathbb{R}^2$, $\left|e^{-2\pi\lambda\kappa|x-y|} - 1 + 2\pi\lambda\kappa|x-y|\right| \leq 4\pi^2\lambda^2\kappa^2|x-y|^2$, so that one has from Lemma 2.6.1,

$$\left|\ R_Q(x,y;E^\lambda) - \frac{2\pi}{\partial_k^2 E(0)} \frac{u_{b_*}(x;0)u_{b_*}(y;0)(1-2\pi\lambda\kappa|x-y|)}{\lambda\kappa} - R_Q^{(0)}(x,y)\ \right|\ \leq\ C\lambda(1+|x|^2+|y|^2). \tag{2.139}$$

Plugging (2.139) into (2.134), and using $(1+|x|)V \in L^1$, yields

$$\Big| \int u_{b_*}(x;0)V(x)\psi^\lambda(x)dx + \frac{2\pi}{\partial_k^2 E(0)}\frac{1}{\kappa} \iint V(x)u_{b_*}(x;0)^2(1-2\pi\lambda\kappa|x-y|)u_{b_*}(y;0)V(y)\psi^\lambda(y)\ dx\ dy$$
$$+ \frac{\lambda}{2}\iint V(x)u_{b_*}(x;0)R_Q^{(0)}(x,y)V(y)\psi^\lambda(y)\ dx\ dy\Big|\ \leq\ C\lambda^2. \tag{2.140}$$

Let us now use that by Theorem 2.2.4, $\sup_{x\in\mathbb{R}} |\psi^\lambda(x) - u_{b_*}(x;0)\exp\left(\lambda\alpha_0|x|\right)| \lesssim \lambda^{1/4}$, so $|\psi^\lambda(x) - u_{b_*}(x;0)| \leq C(\lambda^{1/4} + \lambda|x|)$. Thus (2.140) becomes

$$\Big|\left(\int u_{b_*}(\cdot;0)V\psi^\lambda\right)\left(1 + \frac{1}{\kappa}\frac{2\pi}{\partial_k^2 E(0)}\int V(x)u_{b_*}(x;0)^2\ dx\ \right)$$
$$- \lambda\frac{4\pi^2}{\partial_k^2 E(0)}\iint V(x)u_{b_*}(x;0)^2|x-y|u_{b_*}(y;0)^2V(y)\ dx\ dy$$
$$+ \frac{\lambda}{2}\iint V(x)u_{b_*}(x;0)R_Q^{(0)}(x,y)V(y)u_{b_*}(y;0)\ dx\ dy\Big|\ \leq\ C\lambda^{1+1/4}, \tag{2.141}$$

and one deduces from (2.138) that $\left|\ \kappa\left(\int u_{b_*}(\cdot;0)V\psi^\lambda\right)^{-1} + 2\pi(\partial_k^2 E(0))^{-1}\ \right| \leq C\lambda^{1/4}$. Therefore, multiplying (2.141) by $\kappa\left(\int u_{b_*}(\cdot;0)V\psi^\lambda\right)^{-1}$ yields

$$\Big|\kappa + \frac{2\pi}{\partial_k^2 E(0)}\int V(x)u_{b_*}(x;0)^2\ dx + \lambda\frac{8\pi^3}{(\partial_k^2 E(0))^2}\iint V(x)u_{b_*}(x;0)^2|x-y|u_{b_*}(y;0)^2V(y)\ dx\ dy$$
$$- \frac{\lambda}{4}\iint V(x)u_{b_*}(x;0)R_Q^{(0)}(x,y)V(y)u_{b_*}(y;0)\ dx\ dy\Big|\ \leq\ C\lambda^{1+1/4}. \tag{2.142}$$

Finally, we note that since $R_Q^{(0)}(x,y) = -R_Q^{(0)}(y,x)$ by Lemma 2.6.1, the last term in (2.142) vanishes. Thus the above estimate, together with the following Lemma, completes the proof of Corollary 2.2.6. □

Lemma 2.6.2. *Let $E_* = E_{b_*}(0)$ be an edge of the continuous spectrum, and let hypotheses of Theorem 2.2.4 be satisfied, so that E^λ exists. Then for λ small enough, one has $E^\lambda = E(i\lambda\kappa)$, and* $E^\lambda - E_* = -\frac{1}{2}\lambda^2\kappa^2\partial_k^2 E_{b_*}(0) + \mathcal{O}(\lambda^4)$.

Proof. We Taylor expand $D(E)$ about $E_* = E_{b_*}(0)$.

$$D(E) = D(E_*) + D'(E_*)(E - E_*) + \mathcal{O}\left((E - E_*)^2\right) = 2 + D'(E_*)(E - E_*) + \mathcal{O}\left((E - E_*)^2\right). \tag{2.143}$$

Let's first apply (2.143) to $E = E_{b_*}(k)$ in the spectral band. One has $D(E_{b_*}(k)) = e^{2\pi ik} + e^{-2\pi ik} = 2 - 4\pi^2 k^2 + \mathcal{O}\left(k^3\right)$. Finally, since $\partial_k E_{b_*}(0) = \partial_k^3 E_{b_*}(0) = 0$, one has $E_{b_*}(k) = E_* + \frac{1}{2}\partial_k^2 E(0)k^2 + \mathcal{O}\left(k^4\right)$. Identifying with (2.143), it follows $D'(E_*)\left(\frac{1}{2}\partial_k^2 E(0)\right) = -4\pi^2$, thus $D'(E_*) = \frac{-8\pi^2}{\partial_k^2 E_{b_*}(0)}$.

Next let's apply (2.143) to $E = E^\lambda$, recalling $D(E^\lambda) = e^{2\pi\lambda\kappa} + e^{-2\pi\lambda\kappa} = 2 + 4\pi^2\lambda^2\kappa^2 + \mathcal{O}\left(\lambda^4\kappa^4\right)$. One has from (2.25) in Theorem 2.2.4 that $E^\lambda - E_* = \mathcal{O}(\lambda^2)$, and from (2.138) that $\kappa = \mathcal{O}(1)$. Consequently, (2.143) yields

$$4\pi^2\lambda^2\kappa^2 \;=\; D'(E_*)(E^\lambda - E_*) \;+\; \mathcal{O}(\lambda^4) \;=\; \frac{-8\pi^2}{\partial_k^2 E_{b_*}(0)}(E^\lambda - E_{b_*}(0)) \;+\; \emptyset(\lambda^4).$$

Finally, we deduce $E^\lambda - E_* \;=\; -\frac{1}{2}\lambda^2\kappa^2\partial_k^2 E_{b_*}(0) \;+\; \mathcal{O}(\lambda^4)$ and the lemma is proved. □

We conclude this section by the proof of Lemma 2.6.1

Proof of Lemma 2.6.1. Let us Taylor-expand $\psi_\pm$, as defined by (2.132)–(2.133). One has $\psi_\pm(x)e^{\mp 2\pi\lambda\kappa x} \equiv p_{b_*}(x; \mp i\lambda\kappa)$, thus

$$\psi_+(x)e^{-2\pi\lambda\kappa x} \;=\; p_{b_*}(x;0) \;-\; i\lambda\kappa\partial_k p_{b_*}(x;0) \;-\; \frac{(\lambda\kappa)^2}{2}\partial_\kappa^2 p(x;0) \;+\; i\frac{(\lambda\kappa)^3}{6}\partial_\kappa^3 p(x; i\gamma_+), \tag{2.144}$$

$$\psi_-(x)e^{2\pi\lambda\kappa x} \;=\; p_{b_*}(x;0) \;+\; i\lambda\kappa\partial_k p_{b_*}(x;0) \;-\; \frac{(\lambda\kappa)^2}{2}\partial_\kappa^2 p(x;0) \;-\; i\frac{(\lambda\kappa)^3}{6}\partial_\kappa^3 p(x; i\gamma_-), \tag{2.145}$$

with $-\lambda\kappa \leq \gamma_+ \leq 0 \leq \gamma_- \leq \lambda\kappa$.

Remark 2.6.3. *Note that $\kappa \mapsto p_b(x; k_* + \kappa) \in L^2(\mathbb{R})$ is analytic in a complex neighborhood $|\kappa| < \kappa_1$. By the equation for p_b, $\kappa \mapsto p_b(x; k_* + \kappa) \in H^2(\mathbb{R})$ is analytic and thus $\partial_k^3 p_b(x;k)$ and $\partial_x\partial_k^3 p_b(x;k)$ are well-defined and uniformly bounded for k near k_* and x in any compact set.*

Since $p_{b_*}(x;0) = u_{b_*}(x;0)$, it follows

$$W[\psi_\pm]R_Q(x,y;E^\lambda) \;=\; \begin{cases} \left(u_{b_*}(x;0)u_{b_*}(y;0) + i\lambda\kappa r^{(0)}(x,y;\lambda\kappa) + (\lambda\kappa)^2 r_+^{(1)}(x,y)\right)e^{2\pi\lambda\kappa(x-y)} & \text{if } y > x, \\ \left(u_{b_*}(y;0)u_{b_*}(x;0) + i\lambda\kappa r^{(0)}(y,x;\lambda\kappa) + (\lambda\kappa)^2 r_-^{(1)}(x,y)\right)e^{2\pi\lambda\kappa(y-x)} & \text{if } y < x. \end{cases} \tag{2.146}$$

with

$$r^{(0)}(x,y;\lambda\kappa) \;\equiv\; p_{b_*}(x;0)\partial_k p_{b_*}(y;0) - \partial_k p_{b_*}(x;0)p_{b_*}(y;0) \;=\; -r^{(0)}(y,x;\lambda\kappa),$$

and $r_\pm^{(1)}(x,y)$ is bounded, uniformly with respect to $\lambda\kappa$.

Let us now turn to $W[\psi_\pm] \equiv \psi'_+(x)\psi_-(x) - \psi_+(x)\psi'_-(x)$. From (2.144)–(2.145), one has

$$W[\psi_\pm] \;=\; 2\lambda\kappa\Big(\, 2\pi p_{b_*}(x;0)^2 \; - ip_{b_*}(x;0)\partial_x\partial_k p_{b_*}(x;0) + i\big(\partial_x p_{b_*}(x;0)\big)\big(\partial_k p_{b_*}(x;0)\big)\,\Big) \\ +\; (\lambda\kappa)^3 w_r(x;\lambda\kappa),$$

with $w_r(x)$ uniformly bounded, independently of x and $\lambda\kappa$.

Since $W[\psi_\pm]$ is independent of x, one has $W[\psi_\pm] \;=\; \int_0^1 W[\psi_\pm]\; dx$ and thus

$$W[\psi_\pm] = 2\lambda\kappa\int_0^1 \Big(\, 2\pi p_{b_*}(x;0)^2 - ip_{b_*}(x;0)\partial_x\partial_k p_{b_*}(x;0) + i\big(\partial_x p_{b_*}(x;0)\big)\big(\partial_k p_{b_*}(x;0)\big)\,\Big)\; dx \\ +\; (\lambda\kappa)^3\int_0^1 w_r(x;\lambda\kappa)\; dx.$$

Using that $\int_0^1 p_{b_*}(x;0)^2\; dx \;=\; \int_0^1 u_{b_*}(x;0)^2\; dx \;=\; 1$, one deduces

$$W[\psi_\pm] \;=\; 2\lambda\kappa\Big(2\pi + 2i\int_0^1 p(x;0)\partial_x\partial_k p(x;0)\; dx\Big) \;+\; \mathcal{O}\left((\lambda\kappa)^3\right). \tag{2.147}$$

Now, let us recall that $p_{b_*}(x;i\kappa)$ satisfies (2.133). Deriving twice with respect to $k = i\kappa$, one obtains

$$\big(-(\partial_x - 2\pi\kappa)^2 + Q(x) - E(i\kappa)\big)\,\partial_k^2 p(x;i\kappa) = \; 2\partial_k E(i\kappa)\partial_k p(x;i\kappa) + \partial_k^2 E(i\kappa)p(x;i\kappa) \\ -\, 8\pi i(\partial_x - 2\pi\kappa)\partial_k p(x;i\kappa) - 8\pi^2 p(x;i\kappa)\;.$$

We now apply this identity at $\kappa = 0$, and take the inner product with $p_{b_*}(x;0)$. It follows $0 = \partial_k^2 E(0) - 8\pi i\int_0^1 p(x;0)\partial_x\partial_k p(x;0)\; dx - 8\pi^2$. Therefore, (2.147) becomes

$$W[\psi_\pm] \;=\; 2\lambda\kappa\frac{\partial_k^2 E(0)}{4\pi} \;+\; \mathcal{O}\left((\lambda\kappa)^3\right). \tag{2.148}$$

Finally, (2.146) and (2.148) clearly imply (2.135), and Lemma 2.6.1 is proved. □

Chapter 3

Scattering results for $H_{q_{\mathrm{av}}+q_\epsilon}$

3.1 Introduction

In this chapter we consider the Schrödinger operator:

$$H_{q_{\mathrm{av}}+q_\epsilon} \equiv -\partial_x^2 + q_{\mathrm{av}}(x) + q_\epsilon(x), \quad \epsilon > 0, \tag{3.1}$$

where $q_{\mathrm{av}}(x)$ is a spatially localized background average potential and $q_\epsilon(x) = q(x, x/\epsilon)$ is a potential which is spatially localized on the slow scale x, and periodic and mean zero on the fast scale $y = x/\epsilon$:

$$q(x, y+1) \ = \ q(x, y) \qquad \text{and} \qquad \int_0^1 q(x,y) \ \ \mathrm{d}y \ = \ 0. \tag{3.2}$$

By expanding with respect to the Fourier coefficients of the fast variable, one can write

$$q(x, y) \ = \ \sum_{j \neq 0} q_j(x) \ e^{2\pi i j y}. \tag{3.3}$$

More generally, our theory admits potentials which are aperiodic. For example, we allow for real-valued potentials:

$$q(x, y) \ = \ \sum_{j \neq 0} q_j(x) \ e^{2\pi i \lambda_j y}, \tag{3.4}$$

where $\{\lambda_j\}_{j \in \mathbb{Z} \setminus \{0\}}$ is a sequence of non-zero distinct frequencies for which there is a constant $\theta > 0$ such that

$$\inf_{j \neq k} |\lambda_j - \lambda_k| \geq \theta > 0, \quad \inf_{j \in \mathbb{Z}} |\lambda_j| \geq \theta > 0. \tag{3.5}$$

In Chapter 2 we studied an operator similar to (3.1) from the perspective of an eigenvalue problem. In this chapter, we take a slightly different approach and investigate scattering and localization phenomena for the operator (3.1). We find interesting and subtle low energy behavior and study its consequences for scattering, localization, and dispersive time-decay. Before stating any results, let us briefly review scattering theory.

The scattering problem for the Schrödinger equation:

$$\left(H_V \ - \ k^2\right) u \ = \ 0, \qquad H_V \ \equiv \ -\partial_x^2 \ + \ V(x), \tag{3.6}$$

is the question of the scattered field in response to an incoming plane wave e^{ikx}:

$$u(x;k) \ = \ \begin{cases} e^{ikx} \ + \ r^V(k)e^{-ikx}, & x \to -\infty, \\ t^V(k)e^{ikx}, & x \to +\infty. \end{cases} \tag{3.7}$$

Here, $t^V(k)$ and $r^V(k)$ are respectively called the reflection and transmission coefficients for the potential V; see section 3.2. Considered as a function of a complex variable k, the transmission coefficient $t^V(k)$ is meromorphic in the upper half $k-$plane, having possibly simple poles located on the positive imaginary axis. If $i\rho$, $\rho > 0$, is a pole of t^V then $E = -\rho^2$ is a discrete eigenvalue of H_V of multiplicity one.

In this chapter, we are interested in the case where $V(x)$ is spatially localized and highly oscillatory. As defined in (3.1), we consider potentials of the form:

$$V_\epsilon(x) \ = \ q_{\rm av}(x) + q(x, x/\epsilon), \quad \epsilon \ll 1 \ . \tag{3.8}$$

We ask the following question: *What are the characteristics of solutions to the scattering problem* (3.6)*,* (3.7) *in the limit as ϵ tends to zero?*

3.1.1 Motivation and statement of results

For fixed $k \neq 0$, this is the regime where averaging or homogenization theory applies; the leading order behavior in ϵ is governed by the average of V_ϵ over its fast variations. To simplify the present motivating discussion we consider the case where V_ϵ is periodic on the fast scale with vanishing mean, satisfying (3.2). Then, for any fixed $k \neq 0$, as $\epsilon \to 0$, we have

$$t^{V_\epsilon}(k) \to t^0(k) \equiv 1, \quad r^{V_\epsilon}(k) \to r^0(k) \equiv 0;$$

see [Duchêne and Weinstein, 2011], which contains very detailed asymptotic expansions of $t^{V_\epsilon}(k)$ for a general class of V_ϵ, admitting singularities. Very generally, as k tends to infinity, $t^V(k) \to 1$; the large k transmission behavior of $V_\epsilon(x)$ and its average, $q_{\rm av}(x)$, agree.

However, the low energy, $k \approx 0$, comparison between the scattering behavior for $q_{\rm av}(x) \equiv 0$ and $V_\epsilon(x)$ is far more subtle. First of all, the potential $V(x) \equiv 0$ has non-generic low energy behavior! Indeed for *generic* localized potentials, V, $\lim_{k\to 0} t^V(k) = 0$; see the discussion of and references to genericity in Section 3.2. Thus we expect (and our analysis implies for small and non-zero ϵ) that $t^{V_\epsilon}(k) \to 0$ as $k \to 0$; see Corollary 3.3.4.

It follows that the convergence of $t^{V_\epsilon}(k)$, as ϵ tends to zero, to the *homogenized transmission coefficient* $t^{q_{\rm av}}(k) \equiv t^0(k) \equiv 1$ is non-uniform in a neighborhood of $k = 0$. Figure 3.1c displays plots of $t^{V_\epsilon}(k)$ for several successively smaller values of ϵ. *Underlying this non-uniformity is a subtle behavior of* $t^{V_\epsilon}(k)$ *in the complex plane and an interesting localization phenomenon, which we now explain.*

To fix ideas, stick with the case $q_{\rm av}(x) \equiv 0$ and thus, $H_{V_\epsilon} = H_{q_\epsilon}$, with $q_\epsilon(x) \equiv q(x, x/\epsilon)$. We comment below on the case where $q_{\rm av}$ is non-zero. We clarify the nature of low energy scattering by proving that there is an *effective potential well*:

$$\sigma^\epsilon_{\rm eff}(x) \;=\; -\epsilon^2 \Lambda_{\rm eff}(x), \tag{3.9}$$

such that

$$t^{q_\epsilon}(k) \;- t^{\sigma^\epsilon_{\rm eff}}(k) \to \;0 \;\text{ as } \epsilon \to 0, \;\text{uniformly in } k \in \mathbb{R}; \tag{3.10}$$

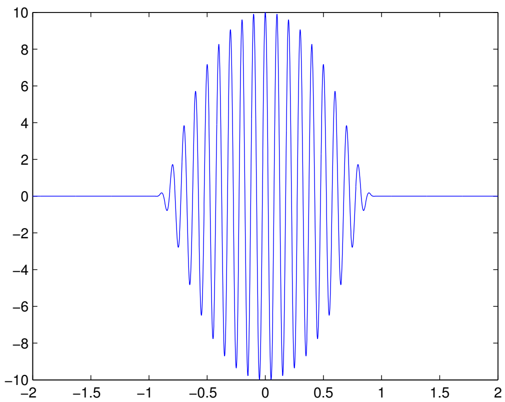
see Theorem 3.4.1, Corollary 3.4.4, and Theorem 3.3.3, proved by a "normal form" type analysis in section 3.6. Here, $\Lambda_{\rm eff}(x)$ is a positive and localized function defined in terms of the Fourier expansion of the 2-scale potential, $q(x, y)$:

$$\Lambda_{\rm eff}(x) \;=\; \frac{1}{4\pi^2} \sum_{j\neq 0} \frac{1}{\lambda_j^2} |q_j(x)|^2. \tag{3.11}$$

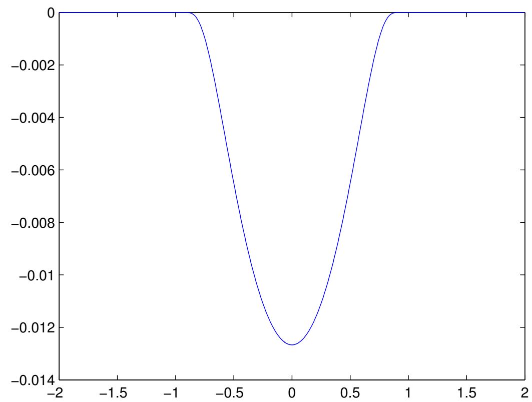
For the periodic case, $q(x, y + 1) = q(x, y)$, $\lambda_j = j,\; j \neq 0$ and $\Lambda_{\rm eff}$ is given by:

$$\Lambda_{\rm eff}(x) \;=\; \frac{1}{4\pi^2} \sum_{j\neq 0} \frac{1}{j^2} |q_j(x)|^2 \;=\; \left\langle -\partial_y^{-2} q(x, y), q(x, y) \right\rangle_{L^2(S^1_y)}. \tag{3.12}$$

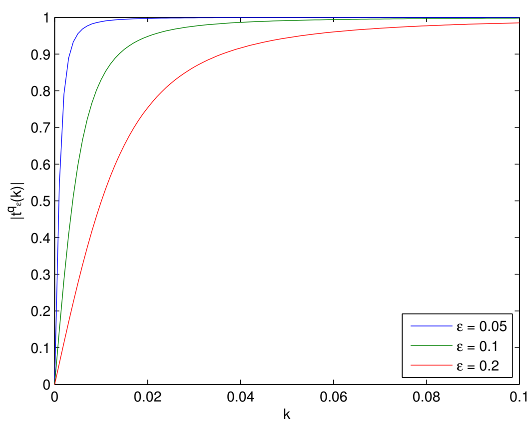
This particular choice of effective potential well is anticipated by a formal two-scale homogenization expansion. An example of a mean zero potential $V_\epsilon(x) = q_\epsilon(x) = q(x, x/\epsilon)$ and the associated effective potential is displayed in Figures 3.1a and 3.1b. A clue to the source of non-uniformity in k is offered by a result of [Simon, 1976], applied to $\sigma^\epsilon_{\rm eff}$, which implies that for ϵ small, the operator



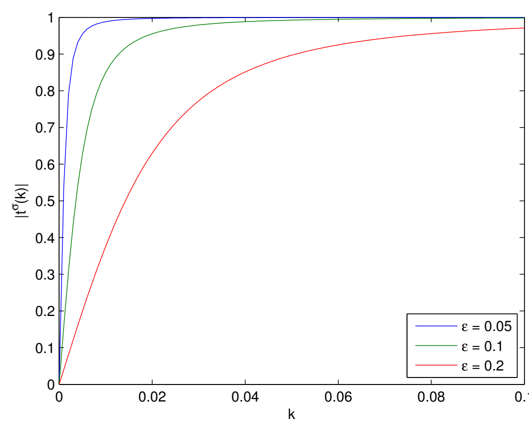
(a) $V_\epsilon(x) = \mathbf{1}_{[-1;1]}(x) 10 e^{\frac{-x^2}{1-x^2}} \cos(2\pi x/\epsilon)$



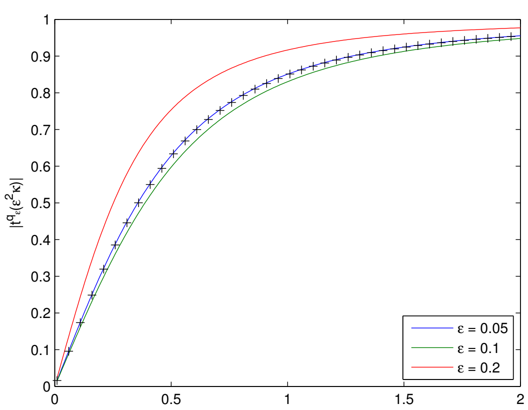
(b) $\sigma^\epsilon_{\mathrm{eff}}(x) = \frac{-\epsilon^2}{8\pi^2} \mathbf{1}_{[-1;1]}(x) \left(10 e^{\frac{-x^2}{1-x^2}} \cos\left(\frac{2\pi x}{\epsilon}\right)\right)^2$



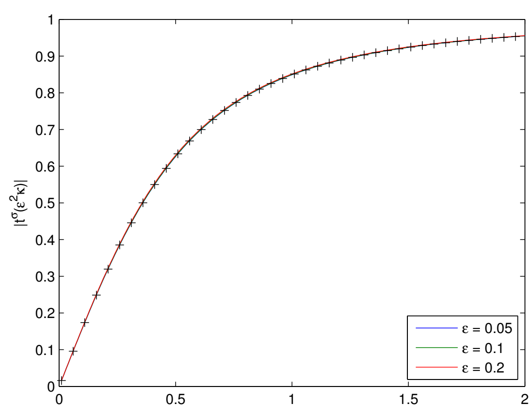
(c) $|t^{V_\epsilon}(k)|$, $k \in (0; 0.1)$, ϵ varying



(d) $\left|t^{\sigma^\epsilon_{\mathrm{eff}}}(k)\right|$, $k \in (0; 0.1)$, ϵ varying



(e) $|t^{V_\epsilon}(\epsilon^2\kappa)|$, $\kappa \in (0; 2)$, ϵ varying



(f) $|t^{\sigma^\epsilon_{\mathrm{eff}}}(\epsilon^2\kappa)|$, $\kappa \in (0; 2)$, ϵ varying

Figure 3.1: Plots of potentials $V_\epsilon(x)$, (a), and the corresponding effective potential $\sigma^\epsilon_{\mathrm{eff}}(x)$, (b). Transmission coefficients $t^{V_\epsilon}(k)$, (c), and $t^{\sigma^\epsilon_{\mathrm{eff}}}(k)$, (d). Plots (e) and (f) show convergence of scaled transmission coefficients $t^{V_\epsilon}(\epsilon^2\kappa)$ and $t^{\sigma^\epsilon_{\mathrm{eff}}}(\epsilon^2\kappa)$ to the transmission coefficient $t^{\mathrm{Dirac}}(\kappa) = \frac{\kappa}{\kappa - \frac{i}{2}\int \Lambda_{\mathrm{eff}}}$ associated with the Dirac delta potential well of mass $\int \Lambda_{\mathrm{eff}}$. The cross markers in plots (e) and

$H_{\sigma^\epsilon_{\rm eff}}$, has a single negative eigenvalue:

$$E^{\sigma^\epsilon_{\rm eff}} \ = \ -\frac{\epsilon^4}{4}\left(\int_{\mathbb{R}} \Lambda_{\rm eff}\right)^2 \ + \ \mathcal{O}(\epsilon^6). \tag{3.13}$$

Since the eigenvalues of H_V are associated with poles of $t^V(k)$ located on the positive imaginary axis (Section 3.2), the eigenvalue $E^{\sigma^\epsilon_{\rm eff}}$ is associated with a pole at:

$$k^{\sigma^\epsilon_{\rm eff}}(\epsilon) \ = i\frac{\epsilon^2}{2}\ \left(\ \int_{\mathbb{R}} \Lambda_{\rm eff}\right) \ + \ \mathcal{O}(\epsilon^4). \tag{3.14}$$

The estimates of Theorem 3.3.3 and Corollary 3.3.5, comparing $t^{q_\epsilon}(k)$ to $t^{\sigma^\epsilon_{\rm eff}}(k)$, in a complex neighborhood of $k=0$ for small ϵ, enable us to conclude, via Rouché's Theorem, that $t^{q_\epsilon}(k)$ has a pole $k^{q_\epsilon}(\epsilon) \approx k^{\sigma^\epsilon_{\rm eff}}(\epsilon)$. It follows that H_{q_ϵ} has a bound state, $u_{E_{q_\epsilon}}(x)$, with energy:

$$E^{q_\epsilon} \ = \ -\frac{\epsilon^4}{4}\left(\int_{\mathbb{R}} \Lambda_{\rm eff}\right)^2 \ + \ \mathcal{O}(\epsilon^5). \tag{3.15}$$

Moreover, $u_{E_{q_\epsilon}}(x) \ = \ \mathcal{O}\left(e^{-\sqrt{|E_{q_\epsilon}|}\ |x|}\right)$ as $|x| \to \infty$ (Corollary 3.3.7). Furthermore, by Corollary 3.3.6, there is a universal scaled limit depending on a single parameter, $\int_{\mathbb{R}} \Lambda_{\rm eff}$:

$$t^{q_\epsilon}(\epsilon^2\kappa) \ \to \ t^\star\left(\kappa; \int_{\mathbb{R}} \Lambda_{\rm eff}\right) \ \equiv \ \frac{\kappa}{\kappa - \frac{i}{2}\int_{\mathbb{R}} \Lambda_{\rm eff}} \ \text{as} \ \ \epsilon \to 0 \quad \text{for } \kappa \neq \frac{i}{2}\ \int_{\mathbb{R}} \Lambda_{\rm eff}.$$

Note that $t^\star\left(\kappa; \int_{\mathbb{R}} \Lambda_{\rm eff}\right)$ is the transmission coefficient for the Schrödinger operator with a Dirac-distribution potential well of total mass $\int_{\mathbb{R}} \Lambda_{\rm eff} > 0$:

$$H^\star \ \equiv \ -\partial_x^2 - \left(\int_{\mathbb{R}} \Lambda_{\rm eff}(\zeta) d\zeta\right) \times \delta(x).$$

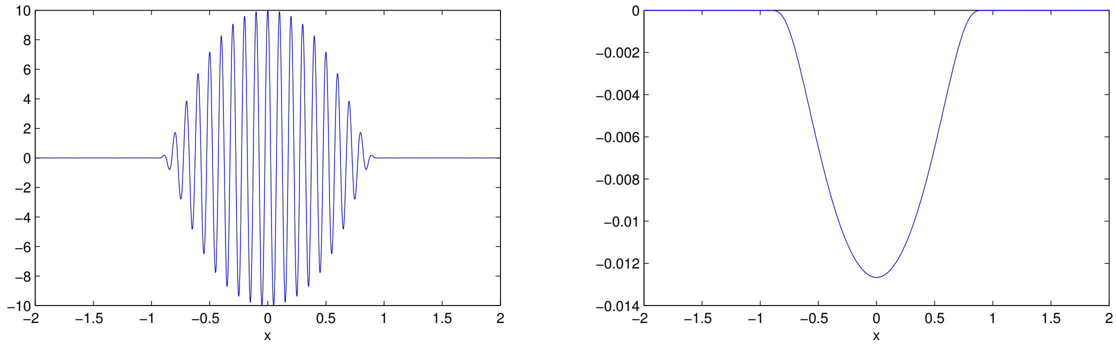
Figures 3.1e and 3.1f, as well as Figure 3.2, illustrate this behavior.

A further consequence concerns the large-time dispersive character of solutions to the time-dependent Schrödinger equation:

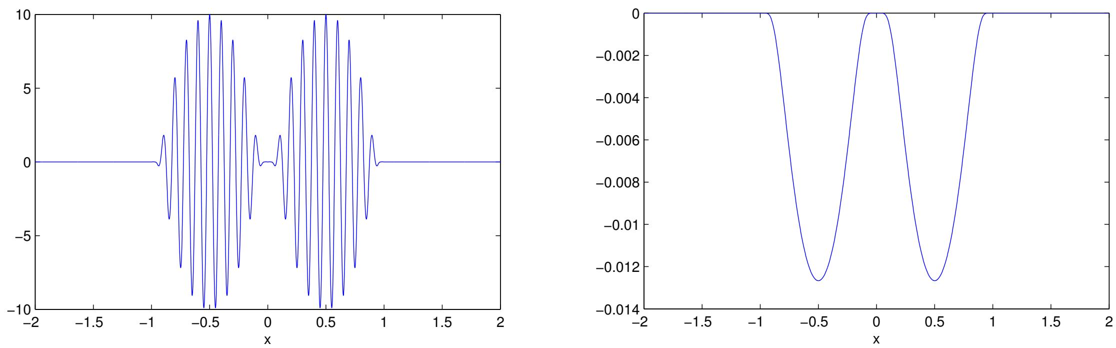
$$i\partial_t\psi \ = \ -\partial_x^2\psi + q(x, x/\epsilon)\psi, \quad \psi(0,x) = \psi_0. \tag{3.16}$$

In Theorem 3.5.1, for $\psi_0(x)$ sufficiently localized and in the continuous spectrum of H_{q_ϵ}, that is $\psi_0 \perp \psi^\epsilon$ in $L^2(\mathbb{R})$, ψ^ϵ the eigenfunction of H_{q_ϵ}, we prove the bound

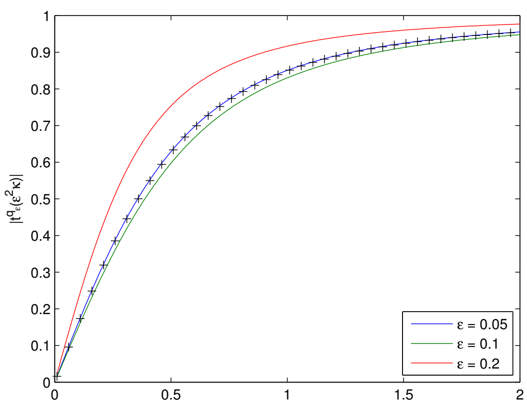
$$\left\| \left(1+|x|^3\right)^{-1} e^{-itH_{q_\epsilon}} P_\perp \psi_0 \right\|_{L^\infty(\mathbb{R})} \leq t^{-1/2}\left(1 + \epsilon^4\left(\int_{\mathbb{R}} \Lambda_{\rm eff}\right)^2 t\right)^{-1} \left\| \left(1+|\zeta|^3\right)\psi_0(\zeta)\right\|_{L^1(\mathbb{R})}. \tag{3.17}$$



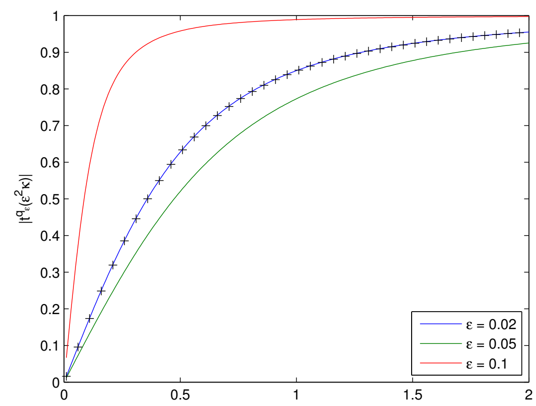
(a) $V_{1,\epsilon}(x) = \mathbf{1}_{[-1;1]}(x) 10 e^{\frac{-x^2}{1-x^2}} \cos\left(\frac{2\pi x}{\epsilon}\right)$, and effective potential, $\sigma_{1,\epsilon} \equiv -\epsilon^2 \Lambda_1$, $\epsilon = 0.1$



(b) $V_{2,\epsilon}(x) = 10 \left(\mathbf{1}_{[-1;0]}(x) e^{\frac{-(2x+1)^2}{1-(2x+1)^2}} + \mathbf{1}_{[0;1]}(x) e^{\frac{-(2x-1)^2}{1-(2x-1)^2}} \right) \cos\left(\frac{2\pi x}{\epsilon}\right)$, and effective potential $\sigma_{2,\epsilon} \equiv -\epsilon^2 \Lambda_2$, $\epsilon = 0.1$



(c) $|t^{V_{1,\epsilon}}(\epsilon^2\kappa)|$, $\kappa \in (0;2)$, ϵ varying



(d) $|t^{V_{2,\epsilon}}(\epsilon^2\kappa)|$, $\kappa \in (0;2)$, ϵ varying

Figure 3.2: Plots (a) and (b) are of two mean zero potentials, $V_{1,\epsilon}$ and $V_{2,\epsilon}$ (left), and effective potentials $\sigma^\epsilon_{1,\mathrm{eff}}$ and $\sigma^\epsilon_{2,\mathrm{eff}}$ (right). Potentials chosen so that: $\int \Lambda_{1,\mathrm{eff}} = \int \Lambda_{2,\mathrm{eff}}$. Plots (c) and (d) illustrate universality of scaled limits: $t^{V_\epsilon}(\epsilon^2\kappa)$ and $t^{\sigma^\epsilon_{\mathrm{eff}}}(\epsilon^2\kappa)$. The cross markers correspond to the scaled limit: $t^\star(\kappa) = \frac{\kappa}{\kappa - \frac{i}{2}\int \Lambda_{1,\mathrm{eff}}} = \frac{\kappa}{\kappa - \frac{i}{2}\int \Lambda_{2,\mathrm{eff}}}$

Here, $P_\perp$ denotes the projection onto the orthogonal complement of the eigenspace, $\text{span}\{\psi^\epsilon\}$, corresponding to the bifurcating eigenvalue E^ϵ. Therefore, the effect of the oscillatory perturbation on the rate of dispersion is only seen on the time scale $t \gtrsim \epsilon^{-4}$.

The above results follow from the non-generic low energy behavior of the average potential $V \equiv 0$. Thus we ask:

Question: Are there non-trivial potentials, $V(x) \equiv q_{\rm av}(x)$, with low energy behavior analogous to $V \equiv 0$, such that $V_\epsilon = q_{\rm av}(x) + q_\epsilon(x)$ exhibits similar behavior?

The answer is yes: Such examples need to exhibit the behavior

$$|t^{q_{\rm av}}(k)| \to |t^{q_{\rm av}}(0)| \neq 0 \quad \text{as} \quad k \to 0.$$

How such non-generic behavior arises is discussed in section 3.3.2. The class of *reflectionless potentials*, for which one has $|t(k)| \equiv 1$ for all $k \in \mathbb{R}$, is a large family of such examples. Our main Theorem 3.3.3 holds for general $q_{\rm av}$, and shows that the low energy behavior is determined by the effective potential:

$$q_{\rm av}(x) + \sigma^\epsilon_{\rm eff}(x) \;=\; q_{\rm av}(x) - \epsilon^2 \Lambda_{\rm eff}(x).$$

Therefore, if $q_{\rm av}$ is a reflectionless potential, then $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$ has a pole, $k^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon)$, situated on the positive imaginary axis, and of size $\mathcal{O}(\epsilon^2)$. An application of Rouché's Theorem yields that $t^{q_{\rm av}+q_\epsilon}(k)$, has a pole near $k^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon)$ and a bound state

$$E^{q_{\rm av}+q_\epsilon}(\epsilon) \approx \; E^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon) \;=\; \left[k^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon)\right]^2 \; < 0; \quad \text{see Corollary 3.3.8.}$$

3.1.2 Outline and remarks of the proof

In Section 3.2 we review the prerequisite one-dimensional scattering theory. Section 3.3 contains statements of our main results and is structured as follows:

1. Detailed hypotheses on the class of potentials: $V_\epsilon(x) = q_{\rm av}(x) + q(x, x/\epsilon)$ are given in Hypotheses **(V)** at the beginning of Section 3.3.

2. We consider the case where $q_{\rm av}$ is generic and the case where $q_{\rm av}$ is non-generic. As indicated above, the non-generic case, *i.e.* $q_{\rm av} \equiv 0$, is of greatest interest and we emphasize this case.

3. For non-generic $q_{\rm av}$, Theorem 3.3.3 and Corollary 3.3.5 give precise estimates on the difference $t^{q_{\rm av}+q_\epsilon}(k) - t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, for k in a complex neighborhood of zero, and $\epsilon \to 0$.

4. For $q_{\mathrm{av}} = 0$, Corollary 3.3.6 gives a universal form of the scaled limit of $t^{q_{\mathrm{av}}+q_\epsilon}(\epsilon^2\kappa)$ as $\epsilon \to 0$. This limit depends on a single parameter, given by the integral of the effective potential.

5. For $q_{\mathrm{av}} = 0$, Corollary 3.3.7 states the potential $q_{\mathrm{av}} + q_\epsilon$, has a bound state with negative energy $\approx \mathcal{O}(\epsilon^4)$, near the edge of the continuous spectrum.

6. In Section 3.3.2 we present non-trivial (non-indentically zero) potentials, q_{av}, which are non-generic, for which the above results for $q_{\mathrm{av}} \equiv 0$ also apply. We work out the details for "one-soliton" potentials $q_{\mathrm{av},\rho}(x) = -2\rho^2\mathrm{sech}^2(\rho(x-x_0))$, for which $H_{q_{\mathrm{av},\rho}}$ has exactly one negative eigenvalue at $E_0(\rho) = -\rho^2$ and continuous spectrum extending from zero to positive infinity. In this example, our result shows that $H_{q_{\mathrm{av},\rho}+q_\epsilon}$ has an eigenvalue of order $\mathcal{O}(\epsilon^4)$, which bifurcates from the edge of the continuous spectrum. Specifically,
$$E^{q_{\mathrm{av}}+q_\epsilon} \approx -\frac{\epsilon^4}{4}\left(\int_{\mathbb{R}} \tanh^2(y)\ \Lambda_{\mathrm{eff}}(y)\ \ \mathrm{d}y\right)^2; \tag{3.18}$$
compare with (3.15). A second eigenvalue is $\mathcal{O}(\epsilon^2)$ distant from $E_0(\rho)$.

7. In Section 3.3.3 we deal with the relatively simple case of highly oscillatory perturbations of a generic potential, q_{av}.

In Section 3.4, we combine our precise analysis for bounded k with the relatively simple analysis when $k \in \mathbb{R}$ is bounded away from zero, and obtain control on the difference $t^{q_\epsilon}(k) - t^{\sigma^\epsilon_{\mathrm{eff}}}(k)$, uniformly for $k \in \mathbb{R}$.

In Section 3.5 our results on the high and low energy behavior of $t^{q_\epsilon}(k)$ are used to prove the local energy time-decay estimate; Theorem 3.5.1.

The proof of Theorem 3.3.3, and the emergence of the effective potential, $\sigma^\epsilon_{\mathrm{eff}}(x)$, are presented in Section 3.6. Appendix 3.A contains detailed estimates on Jost solutions for general localized potentials in an appropriate domain in the complex plane. Appendix 3.B presents a discussion of the potential $q_{\mathrm{av}}(x) + \sigma^\epsilon_{\mathrm{eff}}(x) = q_{\mathrm{av}}(x) - \epsilon^2\Lambda_{\mathrm{eff}}(x)$.

3.1.3 Definitions and notation

Various norms are introduced in the analysis of the transmission coefficient, Jost solutions *etc.* These norms involve spatial weights of the potential which are algebraic when we analyze scattering properties for $k \in \mathbb{R}$, and exponential when we consider these properties for $k \in \mathbb{C}$. Our convention throughout this chapter is that spaces with algebraic spatial weights are denoted with calligraphic upper-case letters, *e.g.* $\mathcal{W}^{k,p}_\gamma$, and spaces with exponential spatial weights are denoted with ordinary upper-case Roman letters, *e.g.* $W^{k,p}_\beta$. The parameters γ and β define the spatial weight.

We denote by $\mathcal{L}^1_\gamma(\mathbb{R})$ the space of measurable functions g such that:

$$\big|g\big|_{\mathcal{L}^1_\gamma} \ = \ \int_{\mathbb{R}} |g(x)|(1+|x|)^\gamma \ {\rm d}x \ < \ \infty.$$

The space of functions, g, whose derivatives up to order n are in $\mathcal{L}^1_\gamma$ is denoted $\mathcal{W}^{n,1}_\gamma$ and the associated norm is:

$$\big|g\big|_{\mathcal{W}^{n,1}_\gamma} \ \equiv \ \sum_{l=0}^{n} \big|\partial^l g\big|_{\mathcal{L}^1_\gamma}.$$

For a fixed $\beta > 0$, we denote by L^∞_β the space of measurable functions g defined on $\mathbb{R}$ such that:

$$\big|g\big|_{L^\infty_\beta} \ \equiv \ \big|e^{\beta\cdot} g\big|_{L^\infty} \ \equiv \ {\rm ess\ sup}_{x\in\mathbb{R}} \ e^{\beta x} \, |g(x)| \ < \ \infty.$$

$W^{n,\infty}_\beta$ denotes the space of the functions g defined on $\mathbb{R}$, whose derivatives up to order n are in L^∞_β with associated norm

$$\big|g\big|_{W^{n,\infty}_\beta} \ \equiv \ \sum_{l=0}^{n} \big|\partial^l g\big|_{L^\infty_\beta}.$$

For a function V of the form:

$$V(x,y) \ = \ q_{\rm av}(x) + q(x,y) \ = \ q_{\rm av}(x) + \sum_{j\in\mathbb{Z}\setminus\{0\}} q_j(x) \ e^{2\pi i \lambda_j y},$$

we introduce the following norms:

exponentially weighted: $\qquad \big| V \big| \ \equiv \ \big|q_{\rm av}\big|_{W^{1,\infty}_\beta} \ + \sum_{j\in\mathbb{Z}\setminus\{0\}} \big|q_j\big|_{W^{3,\infty}_\beta};$

algebraically weighted: $\qquad \big\| V \big\| \ \equiv \ \big|q_{\rm av}\big|_{\mathcal{W}^{1,1}_2} \ + \sum_{j\in\mathbb{Z}\setminus\{0\}} \big|q_j\big|_{\mathcal{W}^{3,1}_3}.$

3.2 Review of 1d scattering theory

In this section we briefly review some of the basics of scattering theory for the one-dimensional Schrödinger equation:

$$\left(-\frac{d^2}{dx^2} \ + \ V(x) \ - \ k^2\right) \ u(x;k) \ = \ 0, \tag{3.19}$$

for localized potentials, $V(x)$, assumed to satisfy

$$V \in \mathcal{L}^1_2(\mathbb{R}) \;=\; \{V : (1+|x|)^2 V(x) \in L^1(\mathbb{R})\}.$$

In particular, in section 3.2.1 we discuss the Jost solutions, $f^V_\pm(x;k)$, and the reflection and transmission coefficients, $r^V_\pm(k)$ and $t^V(k)$. An extensive discussion can be found in [Deift and Trubowitz, 1979], [Reed and Simon, 1979], [Newton, 1986]. Section 3.2.2 explains what is meant by a *generic potential*. Finally, in Section 3.2.3 we introduce some important tools enabling us to compare the transmission coefficients of two different potentials. This is based on the Volterra integral equation for the Jost solution for a potential, V, viewed as a perturbation of a second potential, W.

3.2.1 The Jost solutions, and reflection and transmission coefficients

For $k \in \mathbb{R}$, introduce $f^V_\pm(x;k)$, the unique solutions of (3.19) with

$$f^V_\pm(x;k) \;\sim\; e^{\pm ikx}, \quad \text{as} \quad x \to \pm\infty. \tag{3.20}$$

Observe from the asymptotics as $x \to \infty$, we have $\mathcal{W}[f^V_+(\cdot;k), f^V_+(\cdot;-k)] = 2ik$, where $\mathcal{W}[h_1,h_2]$ denotes the Wronskian of functions $h_1(x)$ and $h_2(x)$:

$$\mathcal{W}[h_1,h_2] = h_1(x)h_2'(x) - h_2(x)h_1'(x). \tag{3.21}$$

Therefore, for $k \in \mathbb{R}\setminus\{0\}$, the set $\{f^V_+(x;k), f^V_+(x;-k)\}$ is a linearly independent set of solutions of (3.19).

The transmission coefficients, $t^V_\pm(k)$, and the reflection coefficients $r^V_\pm(k)$ are defined via the algebraic relations, among the Jost solutions $f^V_\pm(x;k)$:

$$f^V_+(x;k) \;\equiv\; \frac{r^V_+(k)}{t^V_+(k)} f^V_-(x;k) \;+\; \frac{1}{t^V_+(k)} f^V_-(x;-k), \tag{3.22}$$

$$f^V_-(x;k) \;\equiv\; \frac{r^V_-(k)}{t^V_-(k)} f^V_+(x;k) \;+\; \frac{1}{t^V_-(k)} f^V_+(x;-k). \tag{3.23}$$

One can check that $\mathcal{W}[f^V_+, f^V_-] \;=\; -2ik[t^V_-(k)]^{-1} \;=\; -2ik[t^V_+(k)]^{-1}$, and therefore we write

$$\mathcal{W}[f^V_+, f^V_-] \;=\; -\frac{2ik}{t^V(k)}\ , \tag{3.24}$$

with $t^V(k) \;\equiv\; t^V_-(k) \;=\; t^V_+(k)$. Furthermore, one has

$$\left|t^V(k)\right|^2 \;+\; \left|r^V_\pm(k)\right|^2 \;=\; 1, \quad k \in \mathbb{R}. \tag{3.25}$$

The Jost solutions, $f^V_\pm$, and scattering coefficients, t^V and $r^V_\pm$, can be analytically extended into the upper-half complex $k-$plane. Note that if k_1 is a pole of $t^V(k)$, with $\Im(k_1) > 0$, then $\mathcal{W}[f^V_+, f^V_-](k_1) = 0$. In this case, $f^V_+(x; k_1)$ and $f^V_-(x; k_1)$ are proportional and therefore decay exponentially as $x \to \pm\infty$. Thus, k_1^2 is an L^2-eigenvalue of H_V.

If the potential $V(x)$ is exponentially decaying as x tends to infinity, then the Jost solutions can be analytically extended into the lower half complex $k-$plane. More precisely, if $V \in L^\infty_\beta$ (see Section 3.1.3), then $f^V_\pm(x; k)$ are defined for $\Im(k) > -\beta/2$ as the unique solutions of the Volterra integral equations

$$\begin{aligned} f^V_+(x;k) \ &= \ e^{ikx} \ + \ \int_x^\infty \frac{\sin(k(y-x))}{k} V(y) f^V_+(y;k)\ \mathrm{d}y, \\ f^V_-(x;k) \ &= \ e^{-ikx} \ - \ \int_{-\infty}^x \frac{\sin(k(y-x))}{k} V(y) f^V_-(y;k)\ \mathrm{d}y. \end{aligned} \tag{3.26}$$

Detailed bounds on $f^V_\pm(x; k)$ and their derivatives are presented in Appendix 3.A.

Finally, note the following consequences of $V(x)$ being real-valued, the uniqueness of the Jost solutions as defined above, and (3.22)–(3.23):

$$f^V_\pm(x; -\overline{k}) \ = \ \overline{f^V_\pm(x;k)}, \qquad t^V(-\overline{k}) \ = \ \overline{t^V(k)}, \qquad r^V_\pm(-\overline{k}) \ = \ \overline{r^V_\pm(k)}. \tag{3.27}$$

In particular, $f^V_\pm(x; 0)$, $t^V(0)$, $r^V_\pm(0)$ are real.

3.2.2 Generic and non-generic potentials

Using the decay hypotheses of potential $V \in L^\infty_\beta$ and the method of [Deift and Trubowitz, 1979], page 145, one can check that the transmission and reflection coefficients are well-defined by (3.22)–(3.23) for $|\Im(k)| < \beta/2$, and satisfy the important relations, which follow from (3.24) and (3.26):

$$\frac{1}{t^V(k)} \ = \ 1 \ - \ \frac{1}{2ik} I^V(k), \qquad \text{thus} \quad \mathcal{W}[f^V_+, f^V_-](k) \ = \ -2ik \ + \ I^V(k),$$

where $I^V(k) \ \equiv \ \int_{-\infty}^\infty V(y) e^{-iky} f^V_+(y; k)\ \mathrm{d}y$. Equivalently, one has

$$t^V(k) \ = \ -\frac{2ik}{\mathcal{W}[f^V_+, f^V_-](k)} \ = \ \frac{2ik}{2ik - I^V(k)}. \tag{3.28}$$

Recall that if $V(x) \equiv 0$, then $t^V(k) \equiv 1$. Moreover, if

$$I^V(0) \ = \ \mathcal{W}[f^V_+, f^V_-](0) \ = \ \int_{-\infty}^\infty V(y) f^V_+(y; 0)\ \mathrm{d}y \neq 0, \tag{3.29}$$

then by continuity of $t^V(k)$ and (3.28), one has

$$t^V(0) = \lim_{k\to 0} t^V(k) \;=\; 0. \tag{3.30}$$

The case where (3.29) and therefore (3.30) holds is typical. Indeed, it has been shown in Appendix 2 of [Weder, 2000] that for a dense subset of $\mathcal{L}_1^1$, one has $I^V(0) \neq 0$; see also [Deift and Trubowitz, 1979] and [Newton, 1986]. Thus we say that (3.29) and (3.30) holds *generically in the space of potentials.*

Definition 3.2.1 (Generic potentials)**.** *We say that a potential, V, is* generic *if one has $t^V(0) = 0$. Equivalently, V is generic if and only if*

$$\frac{k}{t^V(k)} \longrightarrow \frac{I^V(0)}{2i} \neq 0, \qquad \textit{as } k \to 0.$$

Note that in the non-generic case, where $\mathcal{W}[f_+^V, f_-^V](0) = 0$, we have that Jost solutions $f_\pm^V(x;k)$ satisfy $f_\pm^V(x;0) \sim 1$ as $x \to \pm\infty$ and are multiples of one another. Thus, non-genericity is equivalent to the existence of a globally bounded solution of the Schrödinger equation at zero energy. Such states are sometimes referred to as zero energy resonances. The simplest example is $V \equiv 0$ where $f_\pm^0(x;k) = e^{\pm ikx}$ and $f_\pm^0(x;0) \equiv 1$.

3.2.3 Relations between $f_\pm^V$ and $f_\pm^W$ for general V and W

Our approach is based on associating with $V_\epsilon(x) = q_{\rm av}(x) + q_\epsilon(x)$ a more accurate (than $q_{\rm av}$) minimal model or *normal form*, $V_{\epsilon,{\rm eff}}(x) = q_{\rm av}(x) + \sigma_{\rm eff}^\epsilon(x)$, of the asymptotic scattering properties for k bounded and $\epsilon \to 0$. An important tool will then be to compare the Jost solutions associated with the potential, $V = V_\epsilon$, with those of some family of potentials, $W = q_{\rm av} + \sigma$, parametrized by σ, which is to be determined. This section develops the necessary tools for this comparison.

In the Volterra equation (3.26) we write $f_\pm^V(x;k)$ as a perturbation of the states $e^{\pm ikx}$, which lie in the kernel of $-\partial_x^2 - k^2$. In the following proposition, we generalize this formula by viewing $f_\pm^V(x;k)$ as a perturbation of the Jost solutions $f_\pm^W(x;k)$ for the problem:

$$\left(-\frac{d^2}{dx^2} \;+\; W \;-\; k^2\right) u \;=\; 0.$$

Proposition 3.2.2. *Let $V,W\in L^\infty_\beta$ and and let $f^V_\pm, f^W_\pm$ denote the associated Jost solutions. Then for $|\Im(k)|<\beta/2$, one has*

$$\begin{aligned} f^V_+(x;k) &= \alpha_+[V,W]\ f^W_+(x;k) \ + \ \beta_+[V,W]\ f^W_-(x;k)\\ f^V_-(x;k) &= \alpha_-[V,W]\ f^W_+(x;k) \ + \ \beta_-[V,W]\ f^W_-(x;k), \end{aligned} \tag{3.31}$$

with $\alpha_\pm[V,W](x;k)$ and $\beta_\pm[V,W](x;k)$ defined by

$$\alpha_+[V,W] \ \equiv \ 1+\int_x^\infty \frac{f^W_-(V-W)f^V_+}{\mathcal{W}[f^W_+,f^W_-]}\ dy, \quad \beta_+[V,W] \ \equiv \ -\int_x^\infty \frac{f^W_+(V-W)f^V_+}{\mathcal{W}[f^W_+,f^W_-]}\ dy, \tag{3.32}$$

$$\alpha_-[V,W] \ \equiv \ -\int_{-\infty}^x \frac{f^W_-(V-W)f^V_-}{\mathcal{W}[f^W_+,f^W_-]}\ dy, \quad \beta_-[V,W] \ \equiv \ 1+\int_{-\infty}^x \frac{f^W_+(V-W)f^V_-}{\mathcal{W}[f^W_+,f^W_-]}\ dy. \tag{3.33}$$

Equivalently, one has the Volterra equation

$$f^V_+(x;k)=f^W_+(x;k)+\int_x^\infty \frac{f^W_+(x;k)f^W_-(y;k)-f^W_-(x;k)f^W_+(y;k)}{\mathcal{W}[f^W_+,f^W_-]}(V-W)f^V_+(y;k)\ dy, \tag{3.34}$$

$$f^V_-(x;k)=f^W_-(x;k)-\int_{-\infty}^x \frac{f^W_+(x;k)f^W_-(y;k)-f^W_-(x;k)f^W_+(y;k)}{\mathcal{W}[f^W_+,f^W_-]}(V-W)f^V_-(y;k)\ dy.$$

A very useful consequence is:

Corollary 3.2.3. *Let $V,W\in L^\infty_\beta$ and and let $f^V_\pm, f^W_\pm$ denote their respective associated Jost solutions. Then for $|\Im(k)|<\beta/2$, one has*

$$\mathcal{W}[f^V_+,f^V_-](k) \ = \ \mathcal{M}[V,W](k)\ \mathcal{W}[f^W_+,f^W_-](k)\ , \tag{3.35}$$

where $\mathcal{M}[V,W](x;k)$ is constant in x, and given by

$$\mathcal{M}[V,W](k) \ \equiv \ \alpha_+[V,W](x;k)\beta_-[V,W](x;k) \ - \ \alpha_-[V,W](x;k)\beta_+[V,W](x;k). \tag{3.36}$$

By (3.24)*, and taking the limit as $x\to-\infty$ of* (3.32) *and* (3.33) *in* (3.36)*, one has*

$$\frac{k}{t^V(k)}=\frac{k}{t^W(k)}-\frac{I^{[V,W]}(k)}{2i}, \ \ with \ \ I^{[V,W]}(k)\equiv\int_{-\infty}^\infty f^W_-(y;k)(V-W)(y)f^V_+(y;k)\ dy. \tag{3.37}$$

Remark 3.2.4. *The relation* (3.37)*, applied for $V=V_\epsilon$ and a judicious choice of W, is the point of departure for the proofs of our main results.*

Proof of Corollary 3.2.3. Equation (3.35) follows from substituting the expressions (3.31) into the definition of $\mathcal{W}[f^V_+,f^V_-]$, and using that $\alpha_+[V,W],\beta_+[V,W]$ satisfy the identity: $(\alpha_\pm)'f^W_+ + (\beta_\pm)'f^W_- = 0$; see (3.39) below.

To prove (3.37), we begin by making use of relation (3.24). One has

$$\frac{k}{t^V(k)} \;=\; -\frac{\mathcal{W}[f_+^V, f_-^V](k)}{2i}$$

We next relate $\mathcal{W}[f_+^V, f_-^V]$ to $\mathcal{W}[f_+^W, f_-^W]$ by substitution of the expressions (3.31) into the definition of $\mathcal{W}[f_+^V, f_-^V]$ and using (3.32) and (3.33) to obtain

$$\frac{k}{t^V(k)} \;=\; -\mathcal{M}[V,W](x,k)\frac{\mathcal{W}[f_+^W, f_-^W](k)}{2i} \;=\; \mathcal{M}[V,W](x,k)\frac{k}{t^W(k)}.$$

Now, since $V, W \in L^\infty_\beta$, the estimates of Lemma 3.A.2 yield

$$\lim_{x\to-\infty}\beta_+[V,W](x) \;<\; \infty, \qquad \lim_{x\to-\infty}\alpha_-[V,W](x) \;=\; 0 \quad \text{and} \quad \lim_{x\to-\infty}\beta_-[V,W](x) \;=\; 1.$$

Therefore,

$$\mathcal{M}[V,W](k) \;=\; \lim_{x\to-\infty}\alpha_+[V,W](x).$$

Therefore, one deduces from Proposition 3.2.2 that

$$\frac{k}{t^V(k)} \;=\; \frac{k}{t^W(k)}\lim_{x\to-\infty}\alpha_+[V,W] \;=\; \frac{k}{t^W(k)}\left(1+\frac{I^{[V,W]}(k)}{\mathcal{W}[f_+^W, f_-^W](k)}\right) \;=\; \frac{k}{t^W(k)} \;-\; \frac{I^{[V,W]}(k)}{2i},$$

where $I^{[V,W]}(k)$ is given in (3.37). The proof of Corollary 3.2.3 is complete. □

Proof of Proposition 3.2.2. The integral equation governing a Jost solution for the potential V may be written relative to the potential W as follows. Start with the equation for $u_\pm = f_\pm^V$ written in the form:

$$(H_W - k^2)\;u \;=\; \left(-\frac{d^2}{dx^2} \;+\; W \;-\; k^2\right)\;u \;=\; (W-V)u. \tag{3.38}$$

Treating the right hand side of (3.38) as an inhomogeneous term, we now derive an equivalent integral equations for the Jost solutions . Thus, we seek solutions $u_\pm$ of (3.38), such that $u_\pm(x;k) \;\sim\; f_\pm^V(x;k), \quad x\to\pm\infty$ of the form

$$u(x,k) \;\equiv\; \alpha(x,k)f_+^W(x,k) \;+\; \beta(x,k)f_-^W(x,k), \quad \text{with} \;\; \alpha' f_+^W \;+\; \beta' f_-^W \;=\; 0.$$

We obtain $u' \;=\; \alpha f_+^{W\prime} \;+\; \beta f_-^{W\prime}$, $u'' \;= \alpha' f_+^{W\prime} \;+\; \beta' f_-^{W\prime} \;+\; (W-k^2)u$ and eventually the following system for (α', β'):

$$\begin{cases} \alpha' f_+^W \;+\; \beta' f_-^W \;=\; 0 \\ \alpha' f_+^{W\prime} \;+\; \beta' f_-^{W\prime} \;=\; -\left(-\partial_x^2 \;+\; W \;-\; k^2\right)\;u \;=\; (V-W)u \end{cases} \tag{3.39}$$

Solving for α' and β' we have:

$$\alpha' \;=\; \frac{-f_-^W(x,k)(V(x)-W(x))u(x,k)}{\mathcal{W}[f_+^W,f_-^W](k)} \quad \text{and} \quad \beta' \;=\; \frac{f_+^W(x,k)(V(x)-W(x))u(x,k)}{\mathcal{W}[f_+^W,f_-^W](k)}.$$

The expressions for $\alpha_\pm$ and $\beta_\pm$ in (3.32) and (3.33) follow by integrating and imposing the asymptotic behavior of $u_\pm \sim f_\pm^V$ as $x \to \pm\infty$. In particular, one has $f_+^V(x;k) \sim f_+^W(x;k) \sim e^{ikx}$ when $x \to \infty$, and $f_-^V(x;k) \sim f_-^W(x;k) \sim e^{-ikx}$ when $x \to -\infty$. This completes the proof of Proposition 3.2.2. □

3.3 Convergence of $t^{q_\epsilon}(k)$ for $k \in \mathbb{C}$ and bifurcation of eigenvalues from the edge of the continuous spectrum

In this section we state our main results for the Schrödinger equation (3.6) with potential of the form:

$$V_\epsilon(x) \;=\; V(x, x/\epsilon). \tag{3.40}$$

Recall the exponentially weighted norms $\big|g\big|_{W_\beta^{n,\infty}}$ introduced in section 3.1.3. The potential $V(x,y)$ is assumed to satisfy the following precise hypotheses:

Hypotheses (V): *$V(x,y)$ is real-valued and of the form:*

$$V(x,y) \;=\; q_{\rm av}(x) + q(x,y) \;=\; q_{\rm av}(x) \;+\; \sum_{j\neq 0} \; q_j(x)\; e^{2\pi i \lambda_j y} \;. \tag{3.41}$$

There exist positive constants $\theta > 0$ and $\beta > 0$ such that the sequence of non-zero (distinct) frequencies $\{\lambda_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies

$$\inf_{j\neq k} |\lambda_j - \lambda_k| \geq \theta > 0, \qquad \inf_{j\in\mathbb{Z}\setminus\{0\}} |\lambda_j| \geq \theta > 0 \;, \tag{3.42}$$

and the coefficients $\{q_j(x)\}_{j\in\mathbb{Z}}$, satisfy the decay and regularity assumptions

$$\boldsymbol{|}\, V \,\boldsymbol{|} \;\equiv\; \big|q_{\rm av}\big|_{W_\beta^{1,\infty}} \;+\; \sum_{j\in\mathbb{Z}\setminus\{0\}} \big|q_j\big|_{W_\beta^{3,\infty}} \;<\; \infty. \tag{3.43}$$

Remark 3.3.1. *If V satisfies Hypotheses* **(V)**, *and $\sigma_{\mathit{eff}}^\epsilon$ is defined in* (3.9),(3.11), *then $V_\epsilon \in L_\beta^\infty$, $q_{\rm av} + \sigma_{\mathit{eff}}^\epsilon \in W_\beta^{1,\infty}$ and $\sigma_{\mathit{eff}}^\epsilon \in W_\beta^{3,\infty}$, and there exists $C(\,\boldsymbol{|}\, V \,\boldsymbol{|}\,)$, independent of ϵ, such that*

$$\big|V_\epsilon\big|_{L_\beta^\infty} \;\leq\; C(\,\boldsymbol{|}\, V \,\boldsymbol{|}\,), \quad \big|q_{\rm av} + \sigma_{\mathit{eff}}^\epsilon\big|_{W_\beta^{1,\infty}} \;\leq\; C(\,\boldsymbol{|}\, V \,\boldsymbol{|}\,), \quad \big|\sigma_{\mathit{eff}}^\epsilon\big|_{W_\beta^{3,\infty}} \;\leq\; \epsilon^2 C(\,\boldsymbol{|}\, V \,\boldsymbol{|}\,).$$

Our approach is to study the Jost solutions, $f^{V_\epsilon}(x;k)$, and scattering coefficients, $t^{V_\epsilon}(k)$, $r_\pm^{V_\epsilon}(k)$, for ϵ sufficiently small $\epsilon \in [0,\epsilon_0)$, and for k in a complex neighborhood of zero. More precisely, we assume

Hypotheses (K): *We assume that the wave number, k, varies in K, a compact subset of $\mathbb{C}$ such that*

- *$K \subset \{k,\ |\Im(k)| < \alpha\}$, with $0 < \alpha < \beta/2$, and β is as in Hypotheses* **(V)***;*
- *K does not contain any pole of the transmission coefficient, $t^{q_{\rm av}}(k)$.*

It follows that $t^{q_{\rm av}}(k)$ is bounded, uniformly for $k \in K$, and we define

$$M_K \ \equiv\ \max\left(1\ ,\ \sup_{k\in K} |t^{q_{\rm av}}(k)|\right)\ <\ \infty. \tag{3.44}$$

Moreover, if $K \subset \mathbb{R}$, then $M_K = 1$; see (3.25).

Remark 3.3.2. *We can relax the spatial decay assumptions of Hypotheses* **(V)***, if we restrict Hypotheses* **(K)** *to the upper-half plane $\Im(k) \geq 0$. Our methods apply and only require sufficient algebraic decay of $V(x)$. Results of this kind for $k \in \mathbb{R}$ are presented in Section 3.4.*

We now state our main theorem and its important consequences.

Theorem 3.3.3 (Convergence of the transmission coefficient)**.** *Assume $V_\epsilon(x) = V(x, x/\epsilon)$ satisfies Hypotheses* **(V)***, and $k \in K$ satisfies Hypotheses* **(K)***. Then there exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$, $t^{q_{\rm av}+q_\epsilon}(k)$, the transmission coefficient of the scattering problem* (3.6)-(3.7) *with*

$$V_\epsilon(x)\ =\ q_{\rm av}(x) + q_\epsilon(x)\ =\ q_{\rm av}(x) + q(x, x/\epsilon),$$

is uniformly approximated by the transmission coefficient, $t^{q_{\rm av}+\sigma^\epsilon_{\mathit{eff}}}(k)$, for

$$V^\epsilon_{\rm eff}(x)\ =\ q_{\rm av}(x)\ +\ \sigma^\epsilon_{\mathit{eff}}(x),$$

where $\sigma^\epsilon_{\mathit{eff}}(x)$ denotes the <u>*effective potential well*</u>*,*

$$\sigma^\epsilon_{\mathit{eff}}(x)\ \equiv\ -\epsilon^2\ \Lambda_{\rm eff}(x)\ \equiv\ -\frac{\epsilon^2}{(2\pi)^2}\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{|q_j(x)|^2}{{\lambda_j}^2}. \tag{3.45}$$

Specifically, we have the estimate

$$\sup_{k\in K}\left|\ \frac{k}{t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)}\ -\ \frac{k}{t^{q_{\rm av}+q_\epsilon}(k)}\ \right|\ \leq\ \epsilon^3\ M_K\ C(\,|\!|\!|\, V\,|\!|\!|\,, \sup_{k\in K}|k|), \tag{3.46}$$

with $C(\,|\!|\!|\, V\,|\!|\!|\,)$ a constant, independent of ϵ.

The proof of Theorem 3.3.3 is given in section 3.6; we first present its consequences. A simple outcome of (3.46) and the genericity of $q_{\rm av}+\sigma^\epsilon_{\rm eff}$ for ϵ sufficiently small (which holds for $q_{\rm av}$ generic *and* non-generic; see Corollary 3.B.2[1]) is:

Corollary 3.3.4. *Assume $V_\epsilon = q_{\rm av}+q_\epsilon$ satisfies Hypotheses* **(V)**. *We allow $q_{\rm av}$ to be either generic or non-generic in the sense of Definition 3.2.1. Then, there exists $\epsilon_0 > 0$ such that for any $0<\epsilon<\epsilon_0$, V_ϵ is generic.*

Theorem 3.3.3 holds for both generic and non-generic potentials, $q_{\rm av}$. In the following section we explore consequences for the non-generic potential, $q_{av}(x)\equiv 0$, *i.e.* $V_\epsilon(x) = q(x,x/\epsilon)$, with $\int_0^1 q(x,y)\ {\rm d}y = 0$. In particular, we explain the non-uniformity localization phenomenon discussed in the Introduction. Results for non-trivial $q_{\rm av}(x)$ are developed in sections 3.3.2 and 3.3.3.

3.3.1 Results for mean-zero oscillatory potentials: $q_{\rm av}(x)\equiv 0$

The following corollary, comparing $t^{q_\epsilon}(k)$ and $t^{\sigma^\epsilon_{\rm eff}}(k)$, is a consequence of Theorem 3.3.3, and Lemma 3.B.1.

Corollary 3.3.5. *Let $q_{\rm av}\equiv 0$, so that $V_\epsilon(x) = q_\epsilon(x) = q(x,x/\epsilon)$. Let K denote the compact set of Hypotheses* **(K)**. *There exists $\epsilon_0>0$ such that if*

$$\left|k-\frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}\right| \geq C\epsilon^\tau, \quad \tau<3, \quad k\in K, \quad 0<\epsilon<\epsilon_0, \tag{3.47}$$

then one has for $0<\epsilon<\epsilon_0$,

$$\left|\frac{t^{\sigma^\epsilon_{\it eff}}(k)}{t^{q_\epsilon}(k)}-1\right| \ = \ \mathcal{O}\big(\epsilon^{3-\tau}\big). \tag{3.48}$$

If in addition to (3.47)*, the following condition holds:*

$$\left|k-\frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}\right| \geq C|k|, \quad k\in K, \quad 0<\epsilon<\epsilon_0$$

then one has for $0<\epsilon<\epsilon_0$,

$$\left|t^{\sigma^\epsilon_{\it eff}}(k)-t^{q_\epsilon}(k)\right| \ = \ \mathcal{O}\big(\epsilon^{3-\tau}\big), \quad \textit{and} \quad \left|t^{q_\epsilon}(k) \ - \ \frac{k}{k-\frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}}\right| \ = \ \mathcal{O}\big(\epsilon^{3-\tau}\big).$$

[1]Note that in the non-generic case, the condition $\int_{\mathbb{R}}\Lambda_{\rm eff}(y)(f_-^{q_{\rm av}}(y;0))^2\ {\rm d}y\neq 0$ is always satisfied. Indeed, $f_-^{q_{\rm av}}(\cdot;0)\in\mathbb{R}$ by (3.27), and is non-zero almost everywhere on the support of $\Lambda_{\rm eff}$.

In particular, if $k=\epsilon^2\kappa$, with $\kappa\neq\kappa^\star\equiv -\frac{1}{2i}\int\Lambda_{\rm eff}$, then for $0<\epsilon<\epsilon_0$,

$$\left|t^{\sigma^\epsilon_{\rm eff}}(\epsilon^2\kappa)-t^{q_\epsilon}(\epsilon^2\kappa)\right| \;=\; \mathcal{O}\left(\frac{\epsilon\ |\kappa|}{|\kappa-\kappa^\star|^2}\right) \;=\; \mathcal{O}(\epsilon), \quad \left|t^{q_\epsilon}(\epsilon^2\kappa)-\frac{\kappa}{\kappa-\frac{i}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}}\right| \;=\; \mathcal{O}\big(\epsilon\big). \tag{3.49}$$

Proof. Corollary 3.B.2 of appendix 3.B gives

$$\frac{k}{t^{\sigma^\epsilon_{\rm eff}}(k)} \;=\; k \;-\; \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}(y)\ \ {\rm d}y \;+\; \mathcal{O}\big(\epsilon^4\big), \quad \epsilon\to 0\ , \tag{3.50}$$

uniformly for $k\in K$. By Theorem 3.3.3, one has

$$\frac{k}{t^{q_\epsilon}(k)} \;=\; k \;-\; \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}(y)\ \ {\rm d}y \;+\; \mathcal{O}\left(\epsilon^3\right),\ \text{uniformly for } k\in K\ . \tag{3.51}$$

Expansions (3.50) and (3.51) imply straightforwardly (3.48)–(3.49). □

A direct consequence of Corollary 3.3.5 and the expansion of $t^{\sigma^\epsilon_{\rm eff}}$ implied by Lemma 3.B.1, is the following result showing a universal scaled limit of t^{q_ϵ}, depending on the single parameter, $\int_{\mathbb{R}}\Lambda_{\rm eff}$.

Corollary 3.3.6 (Scaled limit of t^{q_ϵ})**.** *Let $k=\epsilon^2\kappa$, with $\kappa\neq\frac{i}{2}\ \int_{\mathbb{R}}\Lambda_{\rm eff}$. Then one has*

$$t^{q_\epsilon}(\epsilon^2\kappa)\ \to\ t^\star\left(\kappa;\int_{\mathbb{R}}\Lambda_{\rm eff}\right)\equiv\frac{\kappa}{\kappa-\frac{i}{2}\ \int_{\mathbb{R}}\Lambda_{\rm eff}}\ \text{ as }\ \epsilon\to 0, \tag{3.52}$$

where $t^\star(\kappa;m)$ is the transmission coefficient associated with the Schrödinger operator with attractive $\delta-$function potential well of total mass $m>0$:

$$H_{-m\delta}=-\partial_X^2-m\delta(X).$$

As observed in section 3.2, the poles of the transmission coefficient in the upper half $k-$plane, which must lie on the imaginary axis, correspond to the L^2 point eigenvalues. From our estimates on the transmission coefficient, $t^{q_\epsilon}(k)$, we further deduce the existence of a discrete eigenvalue near the edge of the continuous spectrum.

Corollary 3.3.7 (Edge bifurcation of point spectrum from the continuum)**.**
If ϵ if sufficiently small, then the transmission coefficient, $t^{q_\epsilon}(k)$ has a pole in the upper half plane at

$$k_\epsilon \;=\; i\ \frac{\epsilon^2}{2}\ \left(\int_{\mathbb{R}}\Lambda_{\rm eff}\right)+\mathcal{O}(\epsilon^3)\ ,\quad \epsilon\to 0\ ,$$

and therefore H_{q_ϵ} has the simple eigenpair

$$E_\epsilon = k_\epsilon^2 = -\frac{\epsilon^4}{4}\ \left(\int_{\mathbb{R}} \Lambda_{\mathrm{eff}}\ \right)^2\ +\ \mathcal{O}(\epsilon^5),\quad \epsilon \to 0\ ,$$
$$u_{E_{q_\epsilon}}(x)\ =\ \mathcal{O}\left(e^{-\sqrt{|E_{q_\epsilon}|}\ |x|}\right),\quad |x| \gg 1\ .$$

Proof of Corollary 3.3.7: Let us recall Rouché's Theorem: Let f and g denote analytic functions, defined on an open set $A \subset \mathbb{C}$. Let γ denote a simple loop within A, which is homotopic to a point. If $|g(k) - f(k)| < |f(k)|$ for all $k \in \gamma$, then f and g have the same number of roots inside γ.

Now let

$$f(k)\ \equiv\ k\ -\ \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty} \Lambda_{\mathrm{eff}}(y)\ \mathrm{d}y\ ,$$
$$g_1(k)\ =\ \frac{k}{t^{\sigma_{\mathrm{eff}}^\epsilon}(k)},\qquad g_2(k)\ =\ \frac{k}{t^{q_\epsilon}(k)},$$

and $\gamma\ =\ \{k\ :\ |k - \frac{i\epsilon^2}{2}\int \Lambda_{\mathrm{eff}}| = C\epsilon^3\} \subset K$. These functions are analytic in k; see [Deift and Trubowitz, 1979] and our previous discussion. Theorem 3.3.3 and Corollary 3.B.2 imply, respectively,

$$g_2(k) = f(k) + \mathcal{O}(\epsilon^3)\ \text{ and }\ g_1(k) = f(k) +\ \mathcal{O}(\epsilon^4).$$

Therefore, there exist constants $a_K,\ b_K$, such that for $k \in \gamma$:

$$\big|f(k)\ -\ g_1(k)\big|\ \le\ a_K\epsilon^4,\qquad \big|f(k)\ -\ g_2(k)\big|\ \le\ b_K\epsilon^3,\qquad \text{and}\qquad |f(k)| = C\epsilon^3.$$

Taking ϵ sufficiently small and choosing C sufficiently large, Rouché's Theorem implies that both g_1 and g_2, have unique roots, poles of $t^{\sigma_{\mathrm{eff}}^\epsilon}$ and t^{q_ϵ}, in the set $\{k\ :\ |k - \frac{i\epsilon^2}{2}\int \Lambda_{\mathrm{eff}}| \le C\epsilon^3\}$. By self-adjointness, these poles lie on the positive imaginary axis. Corollary 3.3.7 now follows. □

3.3.2 Non-generic and non-zero q_{av}; example of an oscillatory perturbation of a reflectionless potential

As seen above, for the case where $q_{\mathrm{av}} \equiv 0$ the transmission coefficient $t^{q_\epsilon}(k)$, does not converge to $t^0(k) \equiv 1$ uniformly in a neighborhood of $k = 0$ and the obstruction to uniform convergence is the approach, as $\epsilon \to 0$, of a pole of $t^{q_\epsilon}(k)$ toward $k = 0$. Such non-uniform convergence will occur whenever $t^{q_{\mathrm{av}}}(0) \neq 0$. By (3.28), (3.29), we can have $t^{q_{\mathrm{av}}}(0) \neq 0$ if and only if $\mathcal{W}[f_+^{q_{\mathrm{av}}}, f_-^{q_{\mathrm{av}}}](0) = 0$, the case where q_{av} is non-generic; see section 3.2.2.

One may construct non-generic potentials as follows. Let $v(x)$ denote a potential well, $v(x) \leq 0$, say a square well, having one eigenstate and which is generic, *i.e.* $\mathcal{W}[f_+^v, f_-^v](0) \neq 0$ and therefore $t^v(0) = 0$. Consider the one-parameter family of Schrödinger operators defined as $h_g = -\partial_x^2 + gv(x),\ g \geq 1$. As g increases, new eigenvalues of h_g appear as g tranverses discrete values $g_1 < g_2 < \cdots$. These eigenvalues appear via the crossing of a pole of $t^{gv}(k)$ in the lower half $k-$plane, for $g < g_N$, into the upper half plane for $g > g_N$. For g equal to one of these transition values, g_N, one has $t^{g_N v}(0) \neq 0$. Thus, $g_N v(x)$ is a non-generic potential. Our analysis gives, for $q_{\rm av}$ taken to be any such non-generic potential, a precise description of the motion of the pole of $t^{q_{\rm av}+q_\epsilon}$ as it approaches $k = 0$ for ϵ small.

The following corollary, the analogue of Corollaries 3.3.5 and 3.3.6, follows as in the case $q_{\rm av} \equiv 0$ from Theorem 3.3.3 and Lemma 3.B.1.

Corollary 3.3.8 (Oscillatory perturbation of a reflectionless potential)**.**
Let $V_\epsilon(x) = q_{\rm av} + q_\epsilon(x) = q_{\rm av} + q(x, x/\epsilon)$ satisfy Hypotheses **(V)***, let $q_{\rm av}$ be reflectionless, and finally let $k \in K$ satisfy Hypotheses* **(K)***. Assume in addition that the following condition holds,*

$$\left| \frac{k}{t^{q_{\rm av}}(k)} - \frac{i\epsilon^2}{2} \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\ \Lambda_{\rm eff}(y)\ f_+^{q_{\rm av}}(y;k)\ dy \right| \geq C \min(|k|, \epsilon^\tau),\ \ \tau < 3, \tag{3.53}$$

then one has for ϵ sufficiently small

$$\left| t^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(k) - t^{q_{\rm av}+q_\epsilon}(k) \right| \ = \ \mathcal{O}\big(\epsilon^{3-\tau}\big). \tag{3.54}$$

In particular, $k = \epsilon^2\kappa$ satisfies (3.53), and therefore, by Lemma 3.B.1, there is a universal scaled limit of $t^{q_{\rm av}+q_\epsilon}(\epsilon^2\kappa)$:

$$\begin{aligned} t^{q_{\rm av}+q_\epsilon}(\epsilon^2\kappa) \ &\to \ \frac{t^{q_{\rm av}}(0)\ \kappa}{\kappa - \frac{i}{2} t^{q_{\rm av}}(0)\ \int_{\mathbb{R}} f_-^{q_{\rm av}}(y;0)\ \Lambda_{\rm eff}(y)\ f_+^{q_{\rm av}}(y;0)\ dy} \\ &= \ \frac{t^{q_{\rm av}}(0)\ \kappa}{\kappa - \frac{i}{2}(1 + r_-^{q_{\rm av}}(0))\ \int_{\mathbb{R}} (f_-^{q_{\rm av}}(y;0))^2\ \Lambda_{\rm eff}(y)\ \ dy}, \quad \text{as} \ \ \epsilon \to 0 \end{aligned} \tag{3.55}$$

provided $\kappa \neq \kappa^\star \equiv \frac{i}{2} t^{q_{\rm av}}(0)\ \int_{\mathbb{R}} f_-^{q_{\rm av}}(y;0)\ \Lambda_{\rm eff}(y)\ f_+^{q_{\rm av}}(y;0)\ dy$. [2] *The last equality in (3.55) follows from (3.22).*

[2]Note that $\kappa^\star$ lies in the positive imaginary axis. Indeed, $f_-^{q_{\rm av}}(\cdot;0) \in \mathbb{R}$ and $r_-(0) \in \mathbb{R}$ by (3.27), and one has $r_-(0) + 1 \geq 0$, since $|r_-(0)| \leq 1$; see (3.25).

The transmission coefficient, $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$ *has a pole in the upper half plane at* $k_{q_{\rm av}+\sigma^\epsilon_{\rm eff}}$ *the solution of the implicit equation:*

$$k \;=\; i\frac{\epsilon^2}{2}\; t^{q_{\rm av}}(k)\; \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\; \Lambda_{\rm eff}(y)\; f_+^{q_{\rm av}}(y;k)\; dy \;+\; \mathcal{O}(\epsilon^4). \tag{3.56}$$

It follows that $H_{q_{\rm av}+\sigma^\epsilon_{\rm eff}}$ *has an eigenvalue at* $E^{\sigma^\epsilon_{\rm eff}} = (k_{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon))^2 < 0$*. Finally, Lemma 3.B.1 and an application of Rouché's Theorem imply that* $t^{q_{\rm av}+q_\epsilon}(k)$*, has a pole near* $k^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon)$*, on the positive imaginary axis, and a bound state*

$$E^{q_{\rm av}+q_\epsilon}(\epsilon) \approx\; E^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon) \;=\; \left[k^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\epsilon)\right]^2 \;<\; 0.$$

We now consider this result in the context of a particular family of potentials. Consider the family of operators $h(g) = -\partial_x^2 - g\,\mathrm{sech}^2(x)$. Let $g_N = N(N+1),\ N = 0,1,2,\ldots$. For $g_N \le g < g_{N+1}$, the operator $h(g)$ has precisely N- bound states. At the transition values, $h(g_N)$ has a zero energy resonance and $t^{h(g_N)}(0) \neq 0$. The family of potentials for the values $g_N,\ N = 0,1,2,\ldots$, are called *reflectionless potentials* for which $|t(k)| \;\equiv\; 1$ and $r_\pm(k) \;\equiv\; 0,\quad k \in \mathbb{R}$; see [Ablowitz and Segur, 1981]. These potentials are well-known for their role as soliton solutions of the Korteweg-de Vries equation.

Consider the case of the one-soliton potential, corresponding to $N = 1$ in the above discussion. Here,

$$V_1(x) \;=\; -2\rho^2\mathrm{sech}^2(\rho(x-x_0)), \quad \text{where } x_0 \text{ satisfies } C = 2\rho\exp(2\rho x_0).$$

In this case, the transmission coefficient satisfies

$$\frac{1}{t^{V_1}(k)} \;=\; \lim_{x\to-\infty} f_+^{V_1}(x;k)e^{-ikx} \;=\; \frac{k-i\rho}{k+i\rho}.$$

As for the Jost solutions, one has (setting $x_0 = 0$ for simplicity)

$$f_+^{V_1}(x;k) = e^{ikx}\left(1 - \frac{2i\rho}{k+i\rho}\frac{e^{-x}}{e^x+e^{-x}}\right).$$

Since the V_1 is reflectionless, one has by (3.23)

$$f_-^{V_1}(x;k) = 0 + \frac{1}{t^{V_1}(k)}f_+^{V_1}(x;-k) \;=\; \frac{1}{t^{V_1}(k)}e^{-ikx}\left(1 - \frac{2i\rho}{-k+i\rho}\frac{e^{-x}}{e^x+e^{-x}}\right).$$

In this case, there exists a pole of $t^{V_1+\sigma^\epsilon_{\rm eff}}(k)$, and similarly a pole of $t^{V_1+q_\epsilon}(k)$, located around

$$\begin{aligned} k \;&=\; i\frac{\epsilon^2}{2}\int_{-\infty}^{\infty} t^{V_1}(0)f_-^{V_1}(y;0)\Lambda_{\rm eff}(y)f_+^{V_1}(y;0)\ \mathrm{d}y \;+\; \mathcal{O}(\epsilon^3)\ , \\ &=\; i\frac{\epsilon^2}{2}\int_{-\infty}^{\infty} \tanh^2(y)\Lambda_{\rm eff}(y)\ \mathrm{d}y \;+\; \mathcal{O}(\epsilon^3)\ ,\quad \epsilon\to 0. \end{aligned}$$

Finally, $H_{V_1+q_\epsilon}$ and $H_{V_1+\sigma^\epsilon_{\rm eff}}$ have a bound state with energy

$$E \ = \ -\frac{\epsilon^4}{4}\left(\int_{\mathbb{R}} \tanh^2(y)\Lambda_{\rm eff}(y) \ \ {\rm d}y \ \right)^2 \ + \ \mathcal{O}(\epsilon^5) \ , \quad \epsilon \to 0.$$

3.3.3 Results for generic potentials, $q_{\rm av}$, and their highly oscillatory perturbations

In this section, we study the case where $q_{\rm av}$ is a generic potential in the sense of section 3.2. In this case $t^{q_{\rm av}+q_\epsilon}(k)$ converges uniformly to $t^{q_{\rm av}}(k)$ in a neighborhood of $k=0$ [Duchêne and Weinstein, 2011]. More precise information is contained in the following Corollary, a direct consequence of Lemma 3.B.1, and Theorem 3.3.3.

Corollary 3.3.9. *Let $V_\epsilon(x) = q_{\rm av}(x) + q_\epsilon(x) = q_{\rm av}(x) + q(x, x/\epsilon)$ satisfy Hypotheses* **(V)** *with $q_{\rm av}$ generic, and $k \in K$ satisfy Hypotheses* **(K)**. *Then for k and ϵ small enough, one has*

$$\left|t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)\right| \ \le \ C_0|k|, \tag{3.57}$$

$$\left|t^{q_{\rm av}+q_\epsilon}(k)\right| \ \le \ C_0|k|, \tag{3.58}$$

$$\left|t^{q_{\rm av}+q_\epsilon}(k) - t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)\right| \ \le \ C_0\epsilon^3|k|, \tag{3.59}$$

with $C_0 = C_0(M_K)$, $M_K = \max(1, \sup_{k\in K}|t^{q_{\rm av}}(k)|)$.

Proof. In the case of generic potentials, $q_{\rm av}$, we know from [Deift and Trubowitz, 1979] that there exists a constant $a_{q_{\rm av}}$ such that

$$t^{q_{\rm av}}(k) \ = \ a_{q_{\rm av}} k \ + \ o(k), \qquad \text{as } k \to 0.$$

It follows that for k sufficiently small, there exists a positive constant C_0 such that $\left|k \ (t^{q_{\rm av}}(k))^{-1}\right| \ \ge C_0 > 0$. Estimate (3.57) follows then straightforwardly from Lemma 3.B.1, when ϵ is sufficiently small. Now, applying Theorem 3.3.3, one has

$$\begin{aligned}\left|t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k) - t^{q_{\rm av}+q_\epsilon}(k)\right| &= \left|\frac{k}{t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)} - \frac{k}{t^{q_{\rm av}+q_\epsilon}(k)}\right| \left|\frac{t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k) \ t^{q_{\rm av}+q_\epsilon}(k)}{k}\right| \\ &\le \ C_0\epsilon^3\left|t^{q_{\rm av}+q_\epsilon}(k)\right|.\end{aligned}$$

Estimate (3.58) and then (3.59) follow easily. This concludes the proof. ∎

3.4 Behavior of the transmission coefficient, uniformly in $k \in \mathbb{R}$

In this section we focus on the properties of $t^{q_\epsilon}(k)$, which hold uniformly in $k \in \mathbb{R}$. The results presented in section 3.2 are valid for $k \in \mathbb{R}$, and under the less stringent condition: $V \in \mathcal{L}^1_2(\mathbb{R}) = \{V : (1+|x|)^2 V(x) \in L^1(\mathbb{R})\}$. Most of these results can be found in [Deift and Trubowitz, 1979]. Our required bounds on the Jost solutions, $f^V_\pm$ are given in Lemma 3.A.1.

Since k is constrained to the real axis, we find that we can relax the assumption of exponential decay on the potential $V_\epsilon = V(x, x/\epsilon)$.

Hypotheses (V'): *$V(x,y)$ is a real-valued potential of the form*

$$V(x,y) \;=\; q_{\rm av}(x) + q(x,y) \;=\; q_{\rm av}(x) \;+\; \sum_{j\neq 0} q_j(x)\; e^{2\pi i \lambda_j y},$$

such that the sequence of non-zero (distinct) frequencies $\{\lambda_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies (3.42)*, and the coefficients $\{q_j(x)\}_{j\in\mathbb{Z}}$, satisfy the decay and regularity assumptions*

$$\| V \| \;\equiv\; |q_{\rm av}|_{\mathcal{W}^{1,1}_2} \;+\; \sum_{j\in\mathbb{Z}\setminus\{0\}} |q_j|_{\mathcal{W}^{3,1}_3} \;<\; \infty. \tag{3.60}$$

We first investigate the difference between the transmission coefficients $t^{q_{\rm av}+q_\epsilon}(k)$ and $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, where $\sigma^\epsilon_{\rm eff}$ is defined as in Theorem 3.3.3. The proof of the following theorem is analogous to that of Theorem 3.3.3 (section 3.6). We omit the proof for the sake of brevity.

Theorem 3.4.1 (Transmission coefficient, $t^{V_\epsilon}(k)$, for $k \in \mathbb{R}$)**.** *Assume $V_\epsilon(x) = V(x, x/\epsilon)$ satisfies Hypotheses* **(V')***. Assume $k \in \mathbb{R}$, $|k| \le 1$. Then, the following holds:*

1. *There exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$, $t^{q_{\rm av}+q_\epsilon}(k)$ is uniformly approximated by the transmission coefficient, $t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)$, for $H_{q_{\rm av}+\sigma^\epsilon_{\it eff}}$. Here $\sigma^\epsilon_{\it eff}(x)$ denotes the effective potential well defined in* (3.45)*.*

   Moreover, there is a constant $C(\| V \|)$, independent of ϵ and k, such that

$$\sup_{k\in\mathbb{R},\ |k|\le 1} \left| \frac{k}{t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)} - \frac{k}{t^{q_{\rm av}+q_\epsilon}(k)} \right| \le \epsilon^3\ C(\| V \|)\ \max(1, \sup_{k\in K} |t^{q_{\rm av}}(k)|) \le \epsilon^3\ C(\| V \|). \tag{3.61}$$

2. *Assume $q_{av} \equiv 0$, so that $H_{V_\epsilon} = -\partial_x^2 + q(x, x/\epsilon)$, where $y \mapsto q(x,y)$ has mean zero. Then, applying* (3.61) *and Corollary 3.B.2 we have*

$$t^{q_\epsilon}(k) \;=\; \frac{k}{k - \frac{i}{2}\ \epsilon^2\ \int_{\mathbb{R}} \Lambda_{\rm eff} \;+\; \mathcal{O}(\epsilon^3)} \tag{3.62}$$

In the following, we are able to control the difference between $t^{q_{\rm av}+q_\epsilon}(k)$ and $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, for large real wave number, $|k|\ge 1$. This allows, in particular, control of the difference between $t^{q_{\rm av}+q_\epsilon}(k)$ and $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, when the averaged potential $q_{\rm av}\equiv 0$, *uniformly in* $k\in\mathbb{R}$.

Proposition 3.4.2. *Let $V_\epsilon \equiv V(x,x/\epsilon)\equiv q_{\rm av}+q_\epsilon$ with V satisfying Hypotheses* **(V')***, and $\sigma^\epsilon(x)$ denote any potential for which*

$$\int |\sigma^\epsilon(y)|(1+|y|)\ dy \le \epsilon^2\ C_\sigma$$

Then, for $k\in\mathbb{R}\setminus\{0\}$, one has

$$\left|\ t^{q_{\rm av}+q_\epsilon}(k) - t^{q_{\rm av}+\sigma^\epsilon}(k)\ \right|\ \le\ \epsilon^2\ |k|^{-1}\ C(\ \Vert\, V\,\Vert\ , C_\sigma)\ , \tag{3.63}$$

where $\Vert\, V\,\Vert$ is defined in (3.60).

Remark 3.4.3. *We shall apply this proposition to $\sigma^\epsilon(x)=\sigma^\epsilon_{\mathit{eff}}(x)$, for which $C_\sigma = \mathcal{O}(\ \Vert\, V\,\Vert\ )$.*

Proof. Recall the identity (3.37), relating the transmission coefficients of *any potentials* $V, W\in\mathcal{L}^1_2$:

$$\frac{k}{t^V(k)} = \frac{k}{t^W(k)} - \frac{I^{[V,W]}(k)}{2i},\ \text{ with }\ I^{[V,W]}(k)\equiv \int_{-\infty}^{\infty} f_-^W(y;k)(V-W)(y)f_+^V(y;k)\ {\rm d}y. \tag{3.64}$$

Since $t^{q_{\rm av}+q_\epsilon} - t^{q_{\rm av}+\sigma^\epsilon} = \left[t^{q_{\rm av}+q_\epsilon}-t^{q_{\rm av}}\right]\ +\ \left[t^{q_{\rm av}}-t^{q_{\rm av}+\sigma^\epsilon}\right]$, we estimate the two bracketed terms independently. We begin by comparing the transmission coefficients for $W\equiv q_{\rm av}$ and $V\ \equiv\ q_{\rm av}+\sigma^\epsilon$. We have by (3.64)

$$\frac{k}{t^{q_{\rm av}+\sigma^\epsilon}(k)} - \frac{k}{t^{q_{\rm av}}(k)} = -\frac{1}{2i} I^{[q_{\rm av}+\sigma^\epsilon, q_{\rm av}]}(k) = -\frac{1}{2i}\int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\sigma^\epsilon(y) f_+^{q_{\rm av}+\sigma^\epsilon}(y;k)\ {\rm d}y. \tag{3.65}$$

Using the estimates of Lemma 3.A.2, we obtain

$$\left|\int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\ \sigma^\epsilon(y)\ f_+^{q_{\rm av}+\sigma^\epsilon}(y;k)\ {\rm d}y\right| \le \epsilon^2 C_\sigma. \tag{3.66}$$

From (3.65) and (3.66) we have

$$\left|\ t^{q_{\rm av}+\sigma^\epsilon}(k) - t^{q_{\rm av}}(k)\ \right|\ \le\ \epsilon^2\ |k|^{-1}\ C_\sigma\ \left|t^{q_{\rm av}}(k)\ t^{q_{\rm av}+\sigma^\epsilon}(k)\right|\ . \tag{3.67}$$

Using the general relation $|t^V(k)|\le 1$, for any $k\in\mathbb{R}$, (see (3.25)), we obtain

$$\left|\ t^{q_{\rm av}+\sigma^\epsilon}(k) - t^{q_{\rm av}}(k)\ \right|\ \le\ \epsilon^2\ |k|^{-1}\ C_\sigma\ .$$

We now turn to the comparison of the transmission coefficients of $V \equiv q_{\rm av} + q_\epsilon$ and $W \equiv q_{\rm av}$. Proceeding similarly, we have

$$\frac{k}{t^{q_{\rm av}+q_\epsilon}(k)} - \frac{k}{t^{q_{\rm av}}(k)} = -\frac{1}{2i} I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k), \quad \text{where}$$
$$I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k) \equiv \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\ q_\epsilon(y)\ f_+^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y\ . \tag{3.68}$$

Two integrations by parts yield

$$\begin{aligned} I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k) &= \sum_{j\neq 0} \int_{-\infty}^{\infty} q_j(y) e^{2i\pi\lambda_j y/\epsilon} f_-^{q_{\rm av}}(y;k) f_+^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y \\ &= \sum_{j\neq 0} \left(\frac{-\epsilon}{2i\pi\lambda_j}\right)^2 \int_{-\infty}^{\infty} \partial_y^2\big(q_j(y) f_-^{q_{\rm av}}(y;k) f_+^{q_{\rm av}+q_\epsilon}(y;k)\big) e^{2i\pi\lambda_j y/\epsilon}\ {\rm d}y. \end{aligned}$$

Using the estimates of Lemma 3.A.1 and Hypotheses **(V')**, one sees that the integrand is bounded. Indeed, one has

$$\begin{aligned} \left| I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k) \right| &\leq \sum_{j\neq 0} \left(\frac{\epsilon}{2\pi\lambda_j}\right)^2 \int_{-\infty}^{\infty} \left|\partial_y^2\big(q_j(y) f_-^{q_{\rm av}}(y;k) f_+^{q_{\rm av}+q_\epsilon}(y;k)\big)\right|\ {\rm d}y \\ \leq\ \epsilon^2 C\big(|q_{\rm av}|_{\mathcal{L}^1_2}\big) &\sum_{j\neq 0} \left[\int_{-\infty}^{\infty} \left|\partial_y^2 q_j(y)\right| \frac{(1+|y|)^2}{(1+|k|)^2}\ {\rm d}y\ +\ \int_{-\infty}^{\infty} \left|\partial_y q_j(y)\right| \frac{(1+|y|)^2}{1+|k|}\ {\rm d}y \right. \\ &\left. + \int_{-\infty}^{\infty} |q_j(y)|(1+|y|)^2\ {\rm d}y\right] \leq\ \epsilon^2 C\big(|q_{\rm av}|_{\mathcal{L}^1_2}\big) \sum_{j\neq 0} |q_j|_{\mathcal{W}^{1,1}_2}. \end{aligned}$$

Arguing as in (3.67), we deduce

$$\left| t^{q_{\rm av}+q_\epsilon}(k) - t^{q_{\rm av}}(k) \right| \leq \epsilon^2\ |k|^{-1}\ C(\,\|\, V \,\|\,)\ \left|t^{q_{\rm av}}(k)\ t^{q_{\rm av}+q_\epsilon}(k)\right| \leq \epsilon^2\ |k|^{-1}\ C(\,\|\, V \,\|\,).$$

This completes the proof of Proposition 3.4.2. □

The following corollary follows from Theorem 3.4.1 and Proposition 3.4.2.

Corollary 3.4.4. *Let $V_\epsilon = q_\epsilon = q(x, x/\epsilon)$ ($q = 0$) satisfy Hypotheses* **(V')**. *Then*

$$\sup_{k\in\mathbb{R}} \left| t^{\sigma^\epsilon_{\rm eff}}(k) - t^{q_\epsilon}(k) \right| = \mathcal{O}(\epsilon), \quad \epsilon \to 0. \tag{3.69}$$

Proof. The behavior for k small is controlled as in Corollary 3.3.5. Conditions (3.47) and (3.49) hold in particular when we restrict to real wave numbers, $k \in \mathbb{R}$. Therefore, one sees from (3.50) and (3.51) that the difference between $t^{q_\epsilon}(k)$ and $t^{\sigma^\epsilon_{\rm eff}}(k)$ is small, uniformly for $|k| \leq 1$, $k \in \mathbb{R}$:

$$\sup_{k\in\mathbb{R},\ |k|\leq 1} \left| t^{\sigma^\epsilon_{\rm eff}}(k) - t^{q_\epsilon}(k) \right| \leq C\frac{\epsilon^3}{\epsilon^2 + |k|},$$

where $C = C(M_K)$, and $M_K = \max(1, \sup_{k\in\mathbb{R}} |t^0(k)|) = 1$. The difference is controlled for $|k| \geq 1$ by Proposition 3.4.2, and Corollary 3.4.4 follows. □

3.5 Detailed dispersive time decay of $\exp(-iH_{q_\epsilon} t)\,\psi_0$; the effect of a pole of $t^{q_\epsilon}(k)$ near $k = 0$

In this section we use our detailed results on $t^{q_\epsilon}(k)$ to prove time decay estimates of the Schrödinger equation:

$$i\partial_t \psi \;=\; H_V \psi \;\equiv\; -\,\partial_x^2 \psi \;+\; V(x)\psi, \quad \psi(0,x) \;=\; \psi_0\;. \tag{3.70}$$

for initial conditions ψ_0, which are orthogonal to the bound states of H_{q_ϵ}.

Let $V \in \mathcal{L}^1_1$. Then, it is known that H_V has no singular-continuous spectrum, no positive (*embedded*) eigenvalues and its absolutely-continuous spectrum is $[0,\infty)$; see, for example, [Deift and Trubowitz, 1979]. In general, H_V may have a finite number of negative eigenvalues that are simple: $E_N < \cdots < E_0 < 0$. We denote by u_j the eigenvector associated to the eigenvalue E_j, normalized to have L^2 norm equal to one. By the spectral theorem, the solution of (3.70) can be decomposed as follows:

$$\psi(x,t) \;=\; e^{-itH_V}\psi_0 \;=\; \sum_{j=0}^{N} e^{-itE_j}(\psi_0,u_j)u_j + e^{-itH_V}P_c\psi_0,$$

where P_c denotes the projection onto the continuous spectral subspace of H.
$\exp(-itH_V)P_c\psi_0$ is a *scattering state* which spatially spreads and temporally decays: $\left|e^{-itH_V}P_c\psi_0\right|_{L^\infty_x} \to 0$ as $t \to \infty$. In the case $V(x) \equiv 0$, we have $\psi(x,t) = \exp(it\partial_x^2)\psi_0 = K_t \star \psi_0$, where $|K_t(x)| \leq (4\pi t)^{-1/2}$. From this decay estimate it follows immediately that $\left|e^{-itH_0}P_c\psi_0\right|_{L^\infty_x} \;\leq\; C\,|t|^{-1/2}\left|\psi_0\right|_{L^1}$. This $|t|^{-1/2}$ decay-rate is associated with the potential $V \equiv 0$ being non-generic. For generic potentials the decay estimate is more rapid: $\left|e^{-itH_V}P_c\psi_0\right|_{L^\infty_x} \;=\; \mathcal{O}(t^{-3/2})$; see [Goldberg, 2007], [Schlag, 2009]. In [Weder, 2000], [Artbazar and Yajima, 2000] the time-decay of spatially weighted L^2 norms is studied.

Question: What is the precise behavior of the $e^{-itH_{q_\epsilon}}P_c\psi_0$, when q_ϵ is a highly oscillatory potential: $q_\epsilon(x) \equiv q(x,x/\epsilon)$? In particular, what is the influence of the low-energy bound state induced by the effective potential well (equivalently, the complex pole of $t^{q_\epsilon}(k)$ near $k=0$) on the dispersive decay properties?

Using the preceding analysis we can prove:

Theorem 3.5.1 (Dispersive decay estimate for $\exp(-iH_{q_\epsilon}t)$)**.**
Let $V_\epsilon = q_\epsilon(x) = q(x, x/\epsilon)$ satisfy Hypotheses **(V')** *with $q_{\rm av} \equiv 0$, and $\psi_0 \in \mathcal{L}^1_3$. There exists constants $C = C(\,\|\, V \,\|\,) > 0$ and $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$,*

$$\left|(1+|x|)^{-3}\left(e^{-itH_{q_\epsilon}}P_c\psi_0\right)(t,x)\right| \;\le\; C\,\frac{1}{t^{1/2}}\,\frac{1}{1+\epsilon^4\,\left(\int_{\mathbb{R}}\Lambda_{\rm eff}\right)^2\,t}\,\left|\psi_0\right|_{\mathcal{L}^1_3}\,. \tag{3.71}$$

Remark 3.5.2. *We expect that an analogous result holds with $V_\epsilon = q_{\rm av}(x) + q(x, x/\epsilon)$, where $q_{\rm av}$ is any non-generic potential.*

Remark 3.5.3. *As a consequence of our proof, a decay estimate like* (3.71) *holds in the case of small potentials: $V \equiv \lambda Q$, with $\int Q \neq 0$ and λ sufficiently small:*

$$\left|(1+|x|)^{-3}\left(e^{-itH_{\lambda Q}}P_c\psi_0\right)(t,x)\right| \;\le\; C\,\frac{1}{t^{1/2}}\,\frac{1}{1+\lambda^2\,\left(\int_{\mathbb{R}}Q\right)^2\,t}\,\left|\psi_0\right|_{\mathcal{L}^1_3}.$$

Proof of Theorem 3.5.1. We follow the method of [Goldberg, 2007], [Schlag, 2009]. In particular, the starting point of our analysis is the spectral theorem for H: $P_c\phi = \mathcal{F}^\star\mathcal{F}\phi$, with $\mathcal{F}$ and $\mathcal{F}^\star$ the distorted Fourier transform and its adjoint, bounded operators on L^2:

$$\begin{aligned}\mathcal{F} \;&:\; \phi \mapsto \mathcal{F}[\phi](k) \equiv \int_{\mathbb{R}} \phi(x) \mathcal{O}verline\Psi(x,k)\;\,{\rm d}x,\\ \mathcal{F}^\star \;&:\; \Phi \mapsto \int_{-\infty}^{+\infty}\Phi(k)\Psi(x,k)\;\,{\rm d}k\end{aligned}$$

and

$$\Psi(x;k) \;\equiv\; \frac{1}{\sqrt{2\pi}}\begin{cases} t(k)f_+^{q_\epsilon}(x;k) & k \ge 0,\\ t(-k)f_-^{q_\epsilon}(x;-k) & k < 0.\end{cases}$$

The role of the transmission coefficient, $t^{q_\epsilon}(k)$ on the time-evolution on the continuous spectral part of H_{q_ϵ} is made explicit via the representation of $\psi_c(x,t) = P_c\psi(x,t)$:

$$\begin{aligned}\psi_c(t,x) \;&\equiv\; e^{-itH_{q_\epsilon}}P_c\psi_0 \;=\; \mathcal{F}^\star e^{-itk^2}\mathcal{F}\psi_0\\ &= \frac{1}{2\pi}\int_0^\infty e^{-ik^2t}|t^{q_\epsilon}(k)|^2F(x;k)\;{\rm d}k,\end{aligned}$$

with

$$F(x;k) \;=\; \int_{-\infty}^{\infty}\left[f_+^{q_\epsilon}(x;k)\overline{f_+^{q_\epsilon}(y,k)} + f_-^{q_\epsilon}(x;k)\overline{f_-^{q_\epsilon}(y,k)}\right]\psi_0(y)\;{\rm d}y.$$

We next decompose $\psi_c(x,t)$ into its high and low frequency components, respectively, *i.e.* components respectively near and far away from the edge of the continuous spectrum. Introduce the smooth cutoff function χ defined by

$$\chi(k) \ \equiv\ 0 \quad \text{for} \quad |k| \geq 2k_0 \ , \qquad\qquad \chi(k) \ \equiv\ 1 \quad \text{for} \quad |k| \leq k_0.$$

Here, we set $k_0 = 1 + \left\| V \right\|$, motivated by the high frequency analysis of [Schlag, 2009]. Using $\chi(k)$, we decompose into high and low energy components ψ_{high} and ψ_{low}:

$$\begin{aligned}\psi_c(t,x) = &\ \psi_{\mathrm{low}}(t,x) + \psi_{\mathrm{high}}(t,x)\\ = &\int_0^\infty \chi\ e^{-ik^2t}|t^{q_\epsilon}(k)|^2\ F(x;k)\frac{\mathrm{d}k}{2\pi} + \int_0^\infty (1-\chi)e^{-ik^2t}\ |t^{q_\epsilon}(k)|^2\ F(x;k)\frac{\mathrm{d}k}{2\pi}.\end{aligned} \tag{3.72}$$

ψ_{high}, can be estimated without regard to whether or not V is generic. We refer to Proposition 3 of [Goldberg, 2007] and Theorem 3.1 of [Schlag, 2009], for the following estimate:

$$\left|(1+|x|)^{-1}\psi_{\mathrm{high}}\right|_{L^\infty_x} \ = \ \left|(1+|x|)^{-1}e^{-itH_{q_\epsilon}}(1-\chi(H))P_c\psi_0\right|_{L^\infty_x} \ \leq \ C\ |t|^{-3/2}\left|\psi_0\right|_{\mathcal{L}^1_1}, \tag{3.73}$$

where C depends on $\left|q_\epsilon\right|_{L^1_1}$ and is bounded, independent of ϵ.

To estimate the low energy component, ψ_{low}, we make use of estimates on the Jost solutions, $f^{q_\epsilon}_{\pm}(x;k)$ and use the precise behavior of $t^{q_\epsilon}(k)$ obtained in Corollary 3.4.4. We first obtain $\mathcal{O}(t^{-1/2})$-decay, uniformly for ϵ. In a second step, we obtain the precise behavior in the statement of Theorem 3.5.1, for ϵ small.

Let us decompose ψ_{low} into contributions from frequencies in the ranges:

$$0 \leq k \leq \frac{k_0}{\sqrt{t}} \quad \text{and} \quad \frac{k_0}{\sqrt{t}} \leq k \ \leq 2k_0.$$

In terms of the cutoff function, χ, we have:

$$\begin{aligned}\psi_{\mathrm{low}} \ =&\ \frac{1}{2\pi}\int_0^\infty \chi(k\sqrt{t})\chi(k)\ e^{-ik^2t}|t^{q_\epsilon}(k)|^2\ F(x;k)\ \ \mathrm{d}k\\ &+\ \frac{1}{2\pi}\int_0^\infty (1-\chi(k\sqrt{t}))\chi(k)\ e^{-ik^2t}|t^{q_\epsilon}(k)|^2\ F(x;k)\ \ \mathrm{d}k\\ =&\ \psi^{(i)}_{\mathrm{low}}(x,t)\ +\ \psi^{(ii)}_{\mathrm{low}}(x,t)\end{aligned} \tag{3.74}$$

Straightforward estimate of $\psi^{(i)}_{\mathrm{low}}$ gives:

$$\left|\psi^{(i)}_{\mathrm{low}}(x,t)\right| \leq \frac{1}{2\pi}\int_0^{2k_0/\sqrt{t}} |t^{q_\epsilon}(k)|^2\ F(x;k)\ \mathrm{d}k\ \leq\ \frac{k_0}{\pi}\ \frac{1}{t^{\frac{1}{2}}}\ \sup_{k\in\mathbb{R}}|F(x,k)|\,. \tag{3.75}$$

To estimate $\psi^{(ii)}_{\rm low}$, we integrate by parts:

$$\psi^{(ii)}_{\rm low}(x,t) \;=\; \frac{-1}{4\pi i t}\int_0^\infty e^{-ik^2t}\partial_k\left((1-\chi(k\sqrt{t}))\chi(k)k^{-1}|t^{q_\epsilon}(k)|^2F(x;k)\right)\,{\rm d}k.$$

Note that there is no boundary contribution from $k=\infty$, since $\chi(k)$ is compactly supported, and no boundary contribution from $k=0$, since $|t^{q_\epsilon}(0)|=0$; q_ϵ is generic if ϵ is small enough, by Corollary 3.3.4.

Since $\chi(x,k)\equiv 0$ for $k\geq 2k_0$ and $1-\chi(k\sqrt{t})\equiv 0$ for $k\leq k_0/\sqrt{t}$, it follows that

$$\begin{aligned}\left|\psi^{(ii)}_{\rm low}(x,t)\right| \;&\leq\; \frac{C}{t}\;\int_{k_0/\sqrt{t}}^{2k_0}\left|\;|t^{q_\epsilon}(k)|^2F(x;k)\partial_k\left[\chi(k)\frac{1-\chi(k\sqrt{t})}{2ik}\right]\;\right| \;+\; \left|\frac{\partial_k\left[|t^{q_\epsilon}(k)|^2F(x;k)\right]}{k}\right|\,{\rm d}k\\
&\leq\; \frac{C}{t}\sup_{k\in\mathbb{R}}|F(x,k)|\int_{k_0/\sqrt{t}}^{2k_0}\;\sqrt{t}\;\frac{|\chi'(k\sqrt{t})|}{k}+\frac{1}{k^2}\,{\rm d}k \;+\; \frac{C}{t}\int_{k_0/\sqrt{t}}^{2k_0}\left|\;\frac{\partial_k\left[|t^{q_\epsilon}(k)|^2F(x;k)\right]}{k}\;\right|\,{\rm d}k.\end{aligned}$$

Note that

$$\sqrt{t}\;\int_{k_0/\sqrt{t}}^{2k_0}\frac{|\chi'(k\sqrt{t})|}{k}\,{\rm d}k=\sqrt{t}\;\int_{k_0}^{2k_0\sqrt{t}}\frac{|\chi'(z)|}{z}\,dz\;=\mathcal{O}(\sqrt{t}),$$

since $\chi'(z)$ vanishes near 0 and is of compact support. Therefore,

$$\left|\psi^{(ii)}_{\rm low}(x,t)\right| \;\leq\; \frac{C(1+k_0^{-1})}{t^{\frac{1}{2}}}\sup_{k\in\mathbb{R}}|F(x,k)| \;+\; \frac{C}{t}\int_{k_0/\sqrt{t}}^{2k_0}\left|\;\frac{\partial_k\left[|t^{q_\epsilon}(k)|^2F(x;k)\right]}{k}\;\right|\,{\rm d}k. \tag{3.76}$$

The estimates (3.75) and (3.76) are bounded thanks to uniform (in ϵ) control of $t^{q_\epsilon}(k)$, $F(x;k)$ and their $k-$derivatives, which are given in (3.87) and Lemma 3.5.4, below. It follows then from (3.74) that

$$\left|(1+|x|)^{-3}\psi_{\rm low}(x,t)\right| \;\leq\; C(\;\|\,V\,\|\;)\;\frac{1}{t^{1/2}}\;\left|\psi_0\right|_{\mathcal{L}^1_3}. \tag{3.77}$$

We now refine (3.77) by carefully considering the ϵ- dependence for ϵ small at $t\gg 1$. In order to achieve a $\mathcal{O}(t^{-3/2})$ estimate, we first integrate by parts:

$$\psi_{\rm low} \;=\; \frac{-1}{4\pi i t}\int_0^\infty e^{-ik^2t}\partial_k\left(\chi(k)k^{-1}|t^{q_\epsilon}(k)|^2F(x;k)\right)\,{\rm d}k \;\equiv\; \frac{-1}{4\pi i t}\int_0^\infty e^{-ik^2t}G(x;k)\,{\rm d}k.$$

Note again, as above, that there are no boundary contributions from $k=\infty$ or, for ϵ small, from $k=0$, by genericity of q_ϵ. We now decompose $\psi_{\rm low}$ further into contributions from frequencies in the ranges: $0\leq k\leq\frac{k_0}{\sqrt{t}}$ and $\frac{k_0}{\sqrt{t}}\leq k\;\leq 2k_0$. In terms of the cutoff function, χ, we have:

$$\begin{aligned}\psi_{\rm low} \;&=\; \frac{-1}{4\pi i t}\int_0^\infty\chi(k\sqrt{t})e^{-ik^2t}G(x;k)\,{\rm d}k \;+\; \frac{-1}{4\pi i t}\int_0^\infty(1-\chi(k\sqrt{t}))e^{-ik^2t}G(x;k)\,{\rm d}k \qquad (3.78)\\
&=\;\psi^{(1)}_{\rm low}(x,t)\;+\;\psi^{(2)}_{\rm low}(x,t)\end{aligned}$$

Estimation of $\psi^{(1)}_{\rm low}$ gives:

$$\left|\psi^{(1)}_{\rm low}(x,t)\right| \le \frac{1}{4\pi t}\int_0^{2k_0/\sqrt{t}} |G(x;k)|\ {\rm d}k\ \le\ \frac{k_0}{\pi}\ \frac{1}{t^{\frac{3}{2}}}\ \sup_{k\in\mathbb{R}}|G(x;k)|\,. \tag{3.79}$$

To estimate $\psi^{(2)}_{\rm low}$, we subject it to one further integration by parts:

$$\psi^{(2)}_{\rm low}(x,t)\ =\ \frac{1}{4\pi t^2}\ \int_0^\infty e^{-ik^2t}\frac{\partial}{\partial k}\left[\frac{1-\chi(k\sqrt{t})}{2ik}G(x;k)\right]\ {\rm d}k.$$

Since $G(x;k)\equiv 0$ for $k\ge 2k_0$, it follows that

$$\begin{aligned}\left|\psi^{(2)}_{\rm low}(x,t)\right|\ &\le\ \frac{C}{t^2}\ \int_{k_0/\sqrt{t}}^{2k_0}\left|\ G(x;k)\frac{\partial}{\partial k}\left[\frac{1-\chi(k\sqrt{t})}{2ik}\right]\ \right|\ +\ \left|\frac{\partial_k G(x;k)}{k}\right|\ {\rm d}k\\ &\le\ \frac{C}{t^2}\sup_{k\in\mathbb{R}}|\ G(x;k)|\int_{k_0/\sqrt{t}}^{2k_0}\ \sqrt{t}\ \frac{|\chi'(k\sqrt{t})|}{k}+\frac{1}{k^2}\ {\rm d}k\ +\ \frac{C}{t^2}\int_{k_0/\sqrt{t}}^{2k_0}\left|\ \frac{\partial_k G(x;k)}{k}\ \right|\ {\rm d}k\end{aligned}$$

Note again that

$$\sqrt{t}\ \int_{k_0/\sqrt{t}}^{2k_0}\ \frac{|\chi'(k\sqrt{t})|}{k}\ {\rm d}k=\sqrt{t}\ \int_{k_0}^{2k_0\sqrt{t}}\ \frac{|\chi'(z)|}{z}\ dz\ =\mathcal{O}(\sqrt{t}),$$

since $\chi'(z)$ vanishes near 0 and is of compact support. Therefore,

$$\left|\psi^{(2)}_{\rm low}(x,t)\right|\ \le\ \frac{C(1+k_0^{-1})}{t^{\frac{3}{2}}}\sup_{k\in\mathbb{R}}|\ G(x;k)|\ +\ \frac{C}{t^2}\int_{k_0/\sqrt{t}}^{2k_0}\left|\ \frac{\partial_k G(x;k)}{k}\ \right|\ {\rm d}k \tag{3.80}$$

We now use the following two bounds, proved below, to complete our estimation of $\psi^{(1)}_{\rm low}(x,t)$ and $\psi^{(2)}_{\rm low}(x,t)$:

$$|G(x;k)|\le C(\,\|\,V\,\|\,)\frac{1+|x|^2}{k^2+\epsilon^4(\int\Lambda_{\rm eff})^2}\le C(\,\|\,V\,\|\,)\frac{1+|x|^2}{\epsilon^4(\int\Lambda_{\rm eff})^2}|\psi_0|_{\mathcal{L}^1_2}, \tag{3.81}$$

$$|\partial_k G(x;k)|\ \le\ C(\,\|\,V\,\|\,)\frac{1+|x|^3}{k(k^2+\epsilon^4(\int\Lambda_{\rm eff})^2)}|\psi_0|_{\mathcal{L}^1_3}. \tag{3.82}$$

Using these bounds in (3.79) and (3.80), we obtain:

$$(1+|x|^2)^{-1}\ \left|\psi^{(1)}_{\rm low}(x,t)\right|\ \le\ C(\,\|\,V\,\|\,)\ t^{-\frac{3}{2}}\ \frac{1}{\epsilon^4\left(\int_{\mathbb{R}}\Lambda_{\rm eff}\right)^2}\ |\psi_0|_{\mathcal{L}^1_2}\ ; \tag{3.83}$$

and

$$\begin{aligned}(1+|x|^3)^{-1}\ \left|\psi^{(2)}_{\rm low}(x,t)\right|\ &\le\ C(\,\|\,V\,\|\,)\ t^{-2}\ \int_{k_0/\sqrt{t}}^{2k_0}\frac{1}{k^2(k^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2}\ \ {\rm d}k\,|\psi_0|_{\mathcal{L}^1_3}\\ &\le\ C(\,\|\,V\,\|\,)\ \frac{1}{k_0\ t^{\frac{1}{2}}}\ \int_1^{2\sqrt{t}}\ \frac{1}{l^2}\ \frac{dl}{k_0^2l^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2\ t}\ \ |\psi_0|_{\mathcal{L}^1_3}\\ &\le\ C(\,\|\,V\,\|\,)\ \frac{1}{k_0\ t^{\frac{1}{2}}}\ \frac{1}{k_0^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2\ t}\ \int_1^{2\sqrt{t}}\ \frac{1}{l^2}\ \ {\rm d}l\ \ |\psi_0|_{\mathcal{L}^1_3}\\ &\le\ C(\,\|\,V\,\|\,)\ \frac{1}{k_0\ t^{\frac{1}{2}}}\ \frac{1}{k_0^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2\ t}\ \ |\psi_0|_{\mathcal{L}^1_3}\ \ .\end{aligned} \tag{3.84}$$

Finally, one has from (3.78), (3.83) and (3.84) the estimate

$$\left|(1+|x|)^{-3}\psi_{\rm low}(x,t)\right| \;\le\; C(\,\|\, V \,\|\,)\ \frac{t^{-3/2}}{\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2}\ \left|\psi_0\right|_{\mathcal{L}^1_3}. \tag{3.85}$$

Theorem 3.5.1 is a consequence of (3.73), (3.77) and (3.85).

We conclude the proof by establishing (3.81)-(3.82). This requires sharp estimates on the transmission coefficient and the Jost solutions, as well as their derivatives. These estimates are given in Lemmata 3.6 and 3.9 of [Artbazar and Yajima, 2000] for any generic V sufficiently decreasing at infinity. We shall adapt the estimates to $V_\epsilon \equiv V(x,x/\epsilon)$.

The estimates concerning the Jost solutions are uniform with respect to ϵ. In particular, one has from Lemma 3.6 of [Artbazar and Yajima, 2000]:

$$\begin{aligned}
\sup_{k\in\mathbb{R}}\left|\partial_k^j\left(e^{-ikx}f_+^{V_\epsilon}(x;k)\right)\right| \;&\le\; C\big(\left|V_\epsilon\right|_{\mathbf{L}^1_3}\big)(1+\max(0,-x))^{j+1},\\
\sup_{k\in\mathbb{R}}\left|\partial_k^j\left(e^{ikx}f_-^{V_\epsilon}(x;k)\right)\right| \;&\le\; C\big(\left|V_\epsilon\right|_{\mathbf{L}^1_3}\big)(1+\max(0,x))^{j+1},\quad j=0,1,2.
\end{aligned} \tag{3.86}$$

Therefore,

$$|\partial_k^j F(x;k)| \;\le\; C(\left|V_\epsilon\right|_{\mathbf{L}^1_3})(1+|x|^{j+1})\left|\psi_0\right|_{\mathcal{L}^1_{j+1}},\qquad j=0,1,2. \tag{3.87}$$

Estimates (3.81)-(3.82) are now a direct consequence of the following Lemma, together with (3.87).

Lemma 3.5.4. *Let $V_\epsilon = V(x,x/\epsilon)$ satisfy Hypotheses* **(V')**, *with $q_{\rm av}\equiv 0$. Then for ϵ small enough, one has*

$$\left|\partial_k^j t^{V_\epsilon}(k)\right| \;\le\; C(\,\|\, V \,\|\,)\left|\frac{k^{1-j}}{k+\epsilon^2\int\Lambda_{\rm eff}}\right|\ ,$$

with $j=0,1,2$.

Proof of the Lemma. The estimate for $j=0$ is a consequence of Corollary 3.4.4 with the estimate (3.116). Estimates on the derivatives are obtained by deriving identity (3.28) with respect to k. We recall

$$t^{V_\epsilon}(k) \;=\; \frac{2ik}{2ik-I^{V_\epsilon}(k)},\qquad \text{where}\quad I^{V_\epsilon}(k) \;\equiv\; \int_{-\infty}^{\infty} V_\epsilon(y)e^{-iky}f_+^{V_\epsilon}(y;k)\,\mathrm{d}y,$$

so that

$$\partial_k t^{V_\epsilon}(k) \;=\; \frac{2i}{2ik-I^{V_\epsilon}(k)}-\frac{2ik(2i-\partial_k I^{V_\epsilon}(k))}{(2ik-I^{V_\epsilon}(k))^2} \;=\; \frac{t^{V_\epsilon}(k)}{k}-\frac{(t^{V_\epsilon}(k))^2(2i-\partial_k I^{V_\epsilon}(k))}{2ik}.$$

Using (3.86), one controls uniformly $\partial_k I^{V_\epsilon}(k)$, so that

$$\left|\partial_k t^{V_\epsilon}(k)\right| \ \leq\ \frac{|t^{V_\epsilon}(k)|}{k}(1+C|t^{V_\epsilon}(k)|)\ \leq\ C(\,\|\,V\,\|\,)\left|\frac{1}{k+\epsilon^2\int\Lambda_{\rm eff}}\right|.$$

The second derivative in k follows in the same way. □

3.6 The effective potential, $\sigma^\epsilon_{\rm eff}(x)$; proof of Theorem 3.3.3

As discussed in the introduction, for small $|k|$, $t^{q_{\rm av}+q_\epsilon}(k)$ is <u>not</u> uniformly approximated by the transmission coefficient of the homogenized (averaged) potential $q^{\rm av}(x)=\int_0^1 V(x,y)\,{\rm d}y$, for ϵ small. In this section we prove for k bounded that a uniform approximation can be achieved comparing $t^{q_{\rm av}+q_\epsilon}(k)$ to the transmission coefficient of an appropriate *effective potential well*:

$$\begin{aligned} V^{\rm eff}_\epsilon(x) \ &=\ q_{\rm av}(x)\ +\ \sigma^\epsilon_{\rm eff}(x), \quad \text{where} \\ \sigma^\epsilon_{\rm eff}(x)\ &\equiv\ -\epsilon^2\,\Lambda_{\rm eff}(x)\ \equiv\ -\frac{\epsilon^2}{(2\pi)^2}\sum_{j\neq 0}\frac{|q_j(x)|^2}{{\lambda_j}^2}. \end{aligned} \tag{3.88}$$

The point of departure for the analysis is the identity (3.37), with the choices $V=q_{\rm av}+q_\epsilon$ and $W=q_{\rm av}+\sigma$:

$$\frac{k}{t^{q_{\rm av}+q_\epsilon}(k)}\ -\ \frac{k}{t^{q_{\rm av}+\sigma}(k)}\ =\ \frac{i}{2}\ I^{[q_{\rm av}+q_\epsilon,q_{\rm av}+\sigma]}(k), \quad \text{with} \tag{3.89}$$

$$I^{[q_{\rm av}+q_\epsilon,q_{\rm av}+\sigma]}(k)\equiv\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma}(y;k)\,(\ q_\epsilon(y)-\sigma(y))\ \ f_+^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y. \tag{3.90}$$

Here, $\sigma(x)$ is unspecified and to be chosen so that $I^{[q_{\rm av}+q_\epsilon,q_{\rm av}+\sigma]}$ is sufficiently high order in ϵ. The main step in the proof is:

Proposition 3.6.1. *Let* $V_\epsilon\ \equiv\ q_{\rm av}(x)+q(x,x/\epsilon)$ *satisfy Hypotheses* **(V)**, *and* $k\in K$ *satisfy Hypotheses* **(K)**. *Define the effective potential* $\sigma^\epsilon_{\mathit{eff}}\in L^\infty_\beta$, *by the expression in* (3.88). *Then, there exists* $\epsilon_0>0$ *such that the following bound holds uniformly for* $(\epsilon,k)\in[0,\epsilon_0)\times K$*:*

$$I^{[q_{\rm av}+\sigma^\epsilon_{\mathit{eff}},q_{\rm av}+q_\epsilon]}(k)\ \leq\ \epsilon^3\ C\left(\,|\,V\,|\,,\sup_{k\in K}|k|\right)\ \max\left(1,\sup_{k\in K}|t^{q_{\rm av}}(k)|\right) \tag{3.91}$$

Theorem 3.3.3 is then a consequence of the bound (3.91), applied to the right hand side of (3.89). We now turn to derivation of the *effective potential well* $\sigma^\epsilon_{\rm eff}$, and the proof of Proposition 3.6.1.

3.6.1 The heart of the matter; derivation of the effective potential well, $\sigma^\epsilon_{\textbf{eff}}(x)$, and the proof of Proposition 3.6.1

To prove Proposition 3.6.1 we need to bound $I^{[q_{\rm av}+\sigma^\epsilon_{\rm eff},q_{\rm av}+q_\epsilon]}$, given by the integral expression in (3.90). We seek a decomposition of the integrand into oscillatory and non-oscillatory terms. Oscillatory terms can be integrated by parts to obtain bounds of high order in ϵ. Non-oscillatory terms are removed by appropriate choice of $\sigma(x)$.

We begin with $f_+^{q_{\rm av}+q_\epsilon}$. Using the Volterra equation (3.34) with $V = q_{\rm av}+q_\epsilon$ and $W = q_{\rm av}$, one has

$$f_+^{q_{\rm av}+q_\epsilon}(x;k) \;=\; f_+^{q_{\rm av}}(x;k) \;+\; J[q_{\rm av},q_\epsilon](x;k)\ , \tag{3.92}$$

where

$$J[q_{\rm av},q_\epsilon](\zeta;k) \;\equiv\; \int_\zeta^\infty q_\epsilon(y)\frac{f_+^{q_{\rm av}}(\zeta;k)f_-^{q_{\rm av}}(y;k)-f_-^{q_{\rm av}}(\zeta;k)f_+^{q_{\rm av}}(y;k)}{\mathcal{W}[f_+^{q_{\rm av}},f_-^{q_{\rm av}}]}f_+^{q_{\rm av}+q_\epsilon}(y;k)\;{\rm d}y, \tag{3.93}$$

Therefore,

$$(\ q_\epsilon(\zeta)-\sigma(\zeta)\ )\ f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) \;=\; q_\epsilon(\zeta)f_+^{q_{\rm av}}(\zeta;k) \;-\; \sigma(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)+q_\epsilon(\zeta)J[q_{\rm av},q_\epsilon](\zeta;k),$$

implying that $I^{[q_{\rm av}+\sigma,q_{\rm av}+q_\epsilon]}$, given by (3.90), can be written as

$$I^{[q_{\rm av}+\sigma,q_{\rm av}+q_\epsilon]} \;=\; \int_{-\infty}^\infty f_-^{q_{\rm av}+\sigma}(\zeta;k)\Big(q_\epsilon(\zeta)f_+^{q_{\rm av}}(\zeta;k) \;-\; \sigma(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)+q_\epsilon(\zeta)J[q_{\rm av},q_\epsilon](\zeta;k)\Big)\;{\rm d}\zeta. \tag{3.94}$$

We next show that there exists a natural choice, $\sigma=\sigma^\epsilon_{\rm eff}(x)=\mathcal{O}(\epsilon^2)$ such that the contribution of

$$-\sigma(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)+q_\epsilon(\zeta)J[q_{\rm av},q_\epsilon](\zeta;k)$$

to the integral (3.94) is of order $\mathcal{O}(\epsilon^3)$, for ϵ sufficiently small.

Lemma 3.6.2 (Cancellation Lemma). *Let $V(x,y)$ satisfy Hypotheses* **(V)**, *and $k\in K$ satisfy Hypotheses* **(K)**. *Define*

$$\sigma^\epsilon_{eff}(x) \;=\; -\frac{\epsilon^2}{(2\pi)^2}\sum_{j\neq 0}\frac{|q_j(x)|^2}{{\lambda_j}^2} \;=\; -\epsilon^2\Lambda_{\rm eff}(x). \tag{3.95}$$

Then, there exists $\epsilon_0>0$ and $C(V,K)=C(\,\big|\,V\,\big|\,,\sup_{k\in K}|k|)$ such that

$$\begin{aligned}&-\sigma^\epsilon_{eff}(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)+q_\epsilon(\zeta)J[q_{\rm av},q_\epsilon](\zeta;k)\\ &\qquad=\;\epsilon^2\sum_{j\neq 0}\tilde q_j(\zeta)e^{2i\pi\lambda_j\zeta/\epsilon}\;+\;\epsilon^2\sum_{\substack{j,l\neq 0\\ j+l\neq 0}}\tilde q_{j,l}(\zeta)e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon}\;+\;\epsilon^3 q_\epsilon(\zeta)R^\epsilon(\zeta;k),\end{aligned}$$

where the following estimate holds for any $(\epsilon,k)\in[0,\epsilon_0)\times K$:

$$\sum_{\substack{j,l\neq 0\\ j+l\neq 0}}\left(|\tilde q_{j,l}(\zeta)e^{\beta|\zeta|}|+|\tilde q'_{j,l}(\zeta)e^{\beta|\zeta|}|+|\tilde q''_{j,l}(\zeta)e^{\beta|\zeta|}|\right)\leq C(V,K),$$

$$|R^\epsilon(\zeta;k)| \ + \ \sum_{j\neq 0}\left(|\tilde q_{j}(\zeta)e^{\beta|\zeta|}|+|\tilde q'_{j}(\zeta)e^{\beta|\zeta|}|+|\tilde q''_{j}(\zeta)e^{\beta|\zeta|}|\right)\leq C(V,K)M_K(1+|\zeta|^2)e^{\alpha|\zeta|},$$

for $\beta>2\alpha$. Therefore, one has

$$\begin{aligned} I^{[q_{\rm av}+\sigma^\epsilon_{\rm eff},q_{\rm av}+q_\epsilon]}(k) \ &= \ \int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\zeta;k)\Big(q_\epsilon(\zeta)f_+^{q_{\rm av}}+\epsilon^2\sum_{j\neq 0}\tilde q_j(\zeta)e^{2i\pi\lambda_j\zeta/\epsilon}\\ &\quad+\epsilon^2\sum_{\substack{j,l\neq 0\\ j+l\neq 0}}\tilde q_{j,l}(\zeta)e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon}+\epsilon^3 q_\epsilon(\zeta)R^\epsilon(\zeta;k)\Big)\ {\rm d}y. \end{aligned} \tag{3.96}$$

Lemma 3.6.2 is proved in the next section. We first apply it to conclude the proof of Theorem 3.3.3. In succession, each term in (3.96) is controlled, for $k\in K$, by the bounds of the following:

Lemma 3.6.3. *Let $V(x,y)$ satisfy Hypotheses* **(V)**, *and $k\in K$ satisfy Hypotheses* **(K)**, *then one has*

$$\begin{aligned} \left|\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\zeta;k)\ q_\epsilon(\zeta)\ f_+^{q_{\rm av}}(\zeta;k)\ d\zeta\right| \ &\leq \ \epsilon^3\ C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|),\\ \sum_{j\neq 0}\left|\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\zeta;k)\ \tilde q_j(\zeta)\ e^{2i\pi\lambda_j\zeta/\epsilon}\ d\zeta\right| \ &\leq \ \epsilon^2\ M_K\ C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|),\\ \sum_{\substack{j,l\neq 0\\ j+l\neq 0}}\left|\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\zeta;k)\ \tilde q_{j,l}(\zeta)\ e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon}\ d\zeta\right| \ &\leq \ \epsilon^2\ C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|),\\ \left|\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\zeta;k)\ q_\epsilon(\zeta)\ R^\epsilon(\zeta;k)\ d\zeta\right| \ &\leq \ M_K\ C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|)\ , \end{aligned}$$

where $C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|)$ and $M_K=\max(1,\sup_{k\in K}|t^{q_{\rm av}}(k)|)$ are independent of $\epsilon\in[0,\epsilon_0)$.

Applying Lemma 3.6.3 to (3.96) yields the desired $\mathcal{O}(\epsilon^3)$ bound on $I^{[q_{\rm av}+\sigma^\epsilon_{\rm eff},q_{\rm av}+q_\epsilon]}(k)$. Proposition 3.6.1 and therefore Theorem 3.3.3 follow. We now turn to the proofs of Lemmata 3.6.2 and 3.6.3, in Sections 3.6.2 and 3.6.3.

3.6.2 Proof of Lemma 3.6.2

For ease of presentation, we will use the simplified notation for the expression in (3.93):

$$J[q_{\rm av},q_\epsilon](\zeta;k)\equiv\sum_{j\neq 0}\int_\zeta^\infty \mathfrak{m}(\zeta,y;k)\ q_j(y)\ e^{c\lambda_j y/\epsilon}\ f(y)\ \ {\rm d}z, \tag{3.97}$$

where $c = 2\pi i,\ f(y) = f_+^{q_{\rm av}+q_\epsilon}(y;k)$ and

$$\mathfrak{m}(\zeta, y; k) \ = \ \frac{f_+^{q_{\rm av}}(\zeta;k) f_-^{q_{\rm av}}(y;k) - f_-^{q_{\rm av}}(\zeta;k) f_+^{q_{\rm av}}(y;k)}{\mathcal{W}[f_+^{q_{\rm av}}, f_-^{q_{\rm av}}]}.$$

To make explicit the smallness of certain terms due to cancellations, we shall integrate by parts, keeping in mind that we do not control more than two derivatives of $f \equiv f_+^{q_{\rm av}+q_\epsilon}$. To evaluate boundary terms which arise, we shall use that

$$\{\mathfrak{m}(\zeta, y; k), \partial_y \mathfrak{m}(\zeta, y; k), \partial_y^2 \mathfrak{m}(\zeta, y; k)\}\big|_{y=\zeta} = \{0, 1, 0\}.$$

We now embark on the detailed expansion. From (3.97), using integration by parts, one has

$$J[q_{\rm av}, q_\epsilon](\zeta; k) \ \equiv \ \sum_j \Big(\frac{\epsilon}{c\lambda_j}\Big)^2 \Big[q_j \ f \ e^{c\lambda_j \zeta/\epsilon} \ + \ \int_\zeta^\infty \partial_y^2(\mathfrak{m} \ q_j \ f) \ e^{c\lambda_j y/\epsilon} \ {\rm d}y \Big].$$

Decompose the integrand by using: $\partial_y^2(\mathfrak{m}\ q_j\ f) = \partial_y^2(\mathfrak{m}\ q_j)\ f + 2\partial_y(\mathfrak{m}\ q_j)\ \partial_y f + \mathfrak{m}\ q_j\ \partial_y^2 f$. The first two terms can be integrated by parts once more. This gives for $j \neq 0$:

$$\begin{aligned}
\int_\zeta^\infty \partial_y^2(\mathfrak{m}\ q_j)\ f e^{c\lambda_j y/\epsilon}\ {\rm d}y \ &= \ -\frac{\epsilon}{c\lambda_j} \int_\zeta^\infty \partial_y\big(\partial_y^2(\mathfrak{m}\ q_j)\ f\big)\ e^{c\lambda_j y/\epsilon}\ {\rm d}y \ - \ 2\frac{\epsilon}{c\lambda_j} q_j'(\zeta) f(\zeta) e^{c\lambda_j \zeta/\epsilon}, \\
\int_\zeta^\infty \partial_y(\mathfrak{m}\ q_j)\ \partial_y f e^{c\lambda_j y/\epsilon}\ {\rm d}y \ &= \ -\frac{\epsilon}{c\lambda_j} \int_\zeta^\infty \partial_y\big(\partial_y(\mathfrak{m}\ q_j)\ \partial_y f\big)\ e^{c\lambda_j y/\epsilon}\ {\rm d}y \ - \ \frac{\epsilon}{c\lambda_j} q_j(\zeta)\ f'(\zeta) e^{c\lambda_j \zeta/\epsilon}.
\end{aligned}$$

As for the last term, we use the equation for the Jost solution, f, to express $\partial_y^2 f$ in terms of f: $\partial_y^2 f = \partial_y^2 f_+^{q_{\rm av}+q_\epsilon} = (q_{\rm av} + q_\epsilon - k^2) f_+^{q_{\rm av}+q_\epsilon}$. Thus we eventually obtain:

$$\begin{aligned}
J[q_{\rm av}, q_\epsilon](\zeta; k) \ &= \ \sum_{j\neq 0} \big(\frac{\epsilon}{c\lambda_j}\big)^2 \Big[q_j \ f \ e^{c\lambda_j \zeta/\epsilon} \ + \ \int_\zeta^\infty \mathfrak{m} q_j (q_{\rm av} + q_\epsilon - k^2) f \ e^{c\lambda_j y/\epsilon} \ {\rm d}y \\
&\quad + \frac{\epsilon}{c\lambda_j} \Big\{ \sum_{l,m,n} c_{lmn} \int_\zeta^\infty \Big(\ \partial^l \mathfrak{m} \ \partial^m q_j \partial^n f \ \Big) e^{c\lambda_j y/\epsilon} \ {\rm d}y \ - 2 \ (q_j f)' \ e^{c\lambda_j \zeta/\epsilon} \Big\} \Big],
\end{aligned} \tag{3.98}$$

with $0 \le l, m \le 3,\ 0 \le n \le 2$, and $c_{lmn} \in \mathbb{N}$.

We now study each of the terms of (3.98) separately, beginning with an $\mathcal{O}(\epsilon^3)$ bound on the curly bracket terms in (3.98). Using the estimates of Lemmata 3.A.2 and 3.A.3, one has for any $0 \le l, m \le 3,\ 0 \le n \le 2$,

$$\begin{aligned}
\Big| \partial_y^l \mathfrak{m}(\zeta, y; k) \partial_y^m q_j(y) \partial_y^n f_+^{q_{\rm av}+q_\epsilon}(y; k) \Big| \ &\le \ M_K \ C(1 + |k|^l)\big(1 + |y - \zeta|(1 + |y|)(1 + |\zeta|) e^{\alpha|\zeta|} e^{\alpha|y|}\big) \\
&\quad \times (1 + |k|^n)(1 + |y|) e^{\alpha|y|} \big|\partial_y^m q_j(y)\big|.
\end{aligned}$$

Therefore, the contribution to $J[q_{\rm av}, q_\epsilon]$ of the sum over all integrals in curly brackets in (3.98) is bounded by $\epsilon^3 M_K\ C\left(\left| \ V \ \right|, \sup_{k\in K} |k| \right) (1 + |\zeta|)^2 e^{\alpha|\zeta|}$, uniformly for $k \in K$. The boundary term in the curly brackets satisfy a similar bound. Its contribution is bounded by $\epsilon^3 M_K\ C\left(\left| \ V \ \right|, \sup_{k\in K} |k| \right)$.

We now turn to the first two terms, in square brackets, of (3.98). Using the Fourier decomposition of $q_\epsilon(x)$, (3.3), one sees that there are two types of terms: (a) terms where $\lambda_l = -\lambda_j$ $(l = -j)$, $q_{-j}e^{-2i\pi\lambda_j y/\epsilon}$, where no oscillations remain due to phase-cancellation, and (b) contributions from terms where $\lambda_l + \lambda_j \neq 0$, which are highly oscillatory for ϵ small. In these latter terms, an additional factor of ϵ is gained via one more integration by parts. Precisely, one has

$$\begin{aligned}\int_\zeta^\infty \mathfrak{m} q_j(q_{\rm av} + q_\epsilon - k^2) f \ e^{c\lambda_j y/\epsilon} \ {\rm d}y \ &= \ \int_\zeta^\infty \mathfrak{m} q_j q_{-j} f \ \ {\rm d}y \\ &+ \int_\zeta^\infty \mathfrak{m} q_j f \ \left((q_{\rm av} - k^2) e^{c\lambda_j y/\epsilon} + \sum_{l \notin \{0,-j\}} q_i e^{c(\lambda_l+\lambda_j)y/\epsilon} \right) {\rm d}y.\end{aligned}$$

The last terms can be integrated by parts; the resulting integral and boundary terms are estimated as above. Finally, recalling that $f = f^{q_{\rm av}+q_\epsilon}$, we obtain

$$\begin{aligned}J[q_{\rm av}, q_\epsilon](\zeta; k) \ &= \ \sum_{j\neq 0} \big(\frac{\epsilon}{c\lambda_j}\big)^2 \Big[q_j \ f^{q_{\rm av}+q_\epsilon}(\zeta; k) \ e^{c\lambda_j \zeta/\epsilon} \\ &+ \int_\zeta^\infty \mathfrak{m}(\zeta, y; k) \ q_j(y) q_{-j}(y) f^{q_{\rm av}+q_\epsilon}(y; k) \ {\rm d}y \Big] \ + \ \epsilon^3 R^\epsilon(\zeta; k), \end{aligned} \tag{3.99}$$

with $\left|R^\epsilon(\zeta;k)\right| \ \leq \ M_K \ C\Big(|q|_{W^{3,\infty}_\beta}, \sup_{k\in K} |k| \Big) \ (1 + |\zeta|^2) \ e^{\alpha|\zeta|}$.

Now multiply (3.99) by $q_\epsilon(\zeta) = \sum_{l\neq 0} q_l(\zeta) \exp(2\pi i \lambda_l \zeta/\epsilon)$ and then add the result to $-\sigma f_+^{q_{\rm av}+q_\epsilon}$ to obtain (decomposing again into non-oscillatory and highly oscillatory terms and using the notation $c = 2\pi i$):

$$\begin{aligned}& - \sigma(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta; k) + q_\epsilon(\zeta) \ J[q_{\rm av}, q_\epsilon](\zeta; k) \\ &= \left(-\sigma(\zeta) \ + \ \sum_{j\neq 0} \left(\frac{\epsilon}{c\lambda_j}\right)^2 \ q_j(\zeta) q_{-j}(\zeta) \ \right) \ f_+^{q_{\rm av}+q_\epsilon}(\zeta; k) \\ &\quad + \ \sum_{l\notin\{0,-j\}} \sum_{j\neq 0} \ \left(\frac{\epsilon}{c\lambda_j}\right)^2 \ \left[q_l q_j e^{c(\lambda_l+\lambda_j)\zeta/\epsilon} f_+^{q_{\rm av}+q_\epsilon} \right] \\ &\quad + \ \sum_{l\neq 0} \sum_{j\neq 0} \left(\frac{\epsilon}{c\lambda_j}\right)^2 \left[q_l e^{c\lambda_l \zeta/\epsilon} \int_\zeta^\infty \mathfrak{m}(\zeta, y) \ q_j(y) q_{-j}(y) f^{q_{\rm av}+q_\epsilon}(y; k) \ {\rm d}y \right] \\ &\quad + \ \epsilon^3 q_\epsilon(\zeta) R^\epsilon(\zeta; k).\end{aligned} \tag{3.100}$$

The first term on the right hand side of (3.100) is non-oscillatory in ζ for small ϵ. We remove it by choosing

$$\sigma(\zeta) \ = \ \sigma_{\rm eff}^\epsilon(\zeta) \ \equiv \ \sum_{j\neq 0} \ \left(\frac{\epsilon}{2i\pi\lambda_j}\right)^2 q_{-j}(\zeta) q_j(\zeta) \ = \ -\frac{\epsilon^2}{4\pi^2} \sum_{j\neq 0} \frac{|q_j(\zeta)|^2}{{\lambda_j}^2} \ . \tag{3.101}$$

Then

$$
\begin{aligned}
&- \sigma^\epsilon_{\rm eff}(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) + q_\epsilon(\zeta) J[q_{\rm av},q_\epsilon](\zeta;k)\\
&\qquad\qquad = \ \epsilon^2 \sum_{l\neq 0} \tilde q_l(\zeta) e^{2i\pi\lambda_l\zeta/\epsilon} \ + \ \epsilon^2 \sum_{\substack{j,l\neq 0\\ j+l\neq 0}} \tilde q_{j,l}(\zeta) e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon} \ + \ \epsilon^3 q_\epsilon(\zeta) R^\epsilon(\zeta;k),
\end{aligned}
$$

which we've written in the form of the statement of Lemma 3.6.2. Here, $\tilde q_j(\zeta)$ and $\tilde q_{j,l}(\zeta)$ are given by

$$
\tilde q_l(\zeta) \ \equiv \ q_l(\zeta) \sum_{j\neq 0} \Big(\frac{1}{2i\pi\lambda_j}\Big)^2 \int_\zeta^\infty \mathfrak{m}(\zeta,y;k) q_j q_{-j}(y) f_+^{q_{\rm av}+q_\epsilon}(y;k) \ {\rm d}y, \tag{3.102}
$$

$$
\tilde q_{j,l}(\zeta) \ \equiv \ \Big(\frac{1}{2i\pi\lambda_j}\Big)^2 q_l(\zeta) q_j(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta;k). \tag{3.103}
$$

To conclude, we verify the necessary estimates on $\tilde q_j$ and $\tilde q_{j,l}(\zeta)$, and their first and second derivatives.

As for (3.102), we use Lemmata 3.A.2 and 3.A.3, and obtain

$$
\left| \int_\zeta^\infty \mathfrak{m}(\zeta,y;k) q_j q_{-j}(y) f_+^{q_{\rm av}+q_\epsilon}(y;k) \ {\rm d}y \right| \ \leq \ M_K C(\,\|\!| V |\!\|\,, \sup_{k\in K} |k|)(1+|\zeta|^2) e^{\alpha|\zeta|}.
$$

For the derivatives, we use

$$
\begin{aligned}
\partial_\zeta \int_\zeta^\infty \mathfrak{m}(\zeta,y;k) q_j q_{-j}(y) f_+^{q_{\rm av}+q_\epsilon}(y;k) \ {\rm d}y &= \int_\zeta^\infty \partial_\zeta^2 \mathfrak{m}(\zeta,y;k) q_j q_{-j}(y) f_+^{q_{\rm av}+q_\epsilon}(y;k) \ {\rm d}y,\\
\partial_\zeta^2 \int_\zeta^\infty \mathfrak{m}(\zeta,y;k) q_j q_{-j}(y) f_+^{q_{\rm av}+q_\epsilon}(y;k) \ {\rm d}y &= \int_\zeta^\infty \partial_\zeta^2 \mathfrak{m}(\zeta,y;k) q_j q_{-j}(y) f_+^{q_{\rm av}+q_\epsilon}(y;k) \ {\rm d}y\\
&\quad - q_j q_{-j}(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta;k),
\end{aligned}
$$

so that the integrals are uniformly bounded in the same way. As these objects are multiplied by q_l, q_l' or q_l'', and since $q_l \in W^{2,\infty}_\beta$, it follows

$$
|\tilde q_l(\zeta) e^{\beta|\zeta|}| + |\tilde q_l'(\zeta) e^{\beta|\zeta|}| + |\tilde q_l''(\zeta) e^{\beta|\zeta|}| \ \leq \ M_K C\big(|q_l|_{W^{2,\infty}_\beta}, \sup_{k\in K}|k|\big)(1+|\zeta|^2) e^{\alpha|\zeta|},
$$

uniformly for $k \in K$.

As for (3.103), one has

$$
\begin{aligned}
\Big| q_l(\zeta) q_j(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) \Big| &\leq |q_l(\zeta)| |q_j f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)| \leq e^{-\beta|\zeta|} |q_l|_{L^\infty_\beta} |q_j f_+^{q_{\rm av}+q_\epsilon}(\cdot;k)|_{L^\infty}\\
&\leq C(\,\|\!| V |\!\|\,, \sup_{k\in K}|k|) |q_j|_{L^\infty_\beta} |q_l|_{L^\infty_\beta} e^{-\beta|\zeta|},
\end{aligned}
$$

where we used Lemma 3.A.2 to estimate $f_+^{q_{\rm av}+q_\epsilon}$. The first and second derivatives are bounded in the same way, and the double series converge.

This concludes the proof of the Cancellation Lemma 3.6.2.

3.6.3 Proof of Lemma 3.6.3

The last estimate of Lemma 3.6.3 follows from bounds on R^ϵ (see Lemma 3.6.2) and $f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)$ (see Lemma 3.A.2), and the decay Hypotheses **(V)** on q_ϵ. One has

$$\Big|\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)q_\epsilon(y)R^\epsilon(y;k)\ \mathrm{d}y\Big|$$
$$\leq\ M_K\ C(\,|\!|\!| V |\!|\!|\,,\sup_{k\in K}|k|)\ \int_{-\infty}^{\infty}(1+|y|)^3e^{2\alpha|y|}|q_\epsilon(y)|\ \ \mathrm{d}y\ \ \leq\ M_K\ C(\,|\!|\!| V |\!|\!|\,,\sup_{k\in K}|k|).$$

To prove the ϵ^2-smallness of the second estimate of Lemma 3.6.3, we integrate by parts:

$$\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)\tilde{q}_je^{2i\pi\lambda_j/\epsilon}\ \mathrm{d}y = \left(\frac{\epsilon}{2i\pi\lambda_j}\right)^2\int_{-\infty}^{\infty}(f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\cdot;k)\tilde{q}_j)''(y)e^{2i\pi\lambda_jy/\epsilon}\ \mathrm{d}y.$$

The estimate follows as previously from the bounds on $\tilde{q}_j$ (Lemma 3.6.2) and the ones on $f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)$ (Lemma 3.A.2), as well as the hypotheses on λ_j: (3.42) in Hypotheses **(V)**.

The third estimate follows as previously, as

$$\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)\tilde{q}_{j,l}e^{2i\pi(\lambda_j+\lambda_l)/\epsilon}\ \mathrm{d}y$$
$$= \left(\frac{\epsilon}{2i\pi(\lambda_j+\lambda_l)}\right)^2\int_{-\infty}^{\infty}(f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\cdot;k)\tilde{q}_{j,l})''(y)e^{2i\pi\lambda_jy/\epsilon}\ \mathrm{d}y.$$

The estimate follows, using now the bounds on $\tilde{q}_{j,l}$ (Lemma 3.6.2). Finally, we use three integration by parts for the first estimate of Lemma 3.6.3:

$$\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)q_j(y)f_+^{q_{\rm av}}(\cdot;k)e^{\frac{2i\pi\lambda_j}{\epsilon}}\ \mathrm{d}y$$
$$= \left(\frac{i\epsilon}{2\pi\lambda_j}\right)^3\int_{-\infty}^{\infty}(f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\cdot;k)q_jf_+^{q_{\rm av}}(\cdot;k))'''(y)e^{\frac{2i\pi\lambda_jy}{\epsilon}}\ \mathrm{d}y,$$

which is estimated using the third item of Lemma 3.A.2, and Hypotheses **(V)**.

3.7 Conclusion / Discussion

In this chapter we studied scattering, localization and dispersive time-decay properties for the one-dimensional Schrödinger equation with a rapidly oscillating and spatially localized potential,

$q_\epsilon = q(x, x/\epsilon)$, where $q(x,y)$ is periodic and mean zero with respect to y. Such potentials model a microstructured medium. Homogenization theory fails to capture the correct low-energy (k small) behavior of scattering quantities, *e.g.* the transmission coefficient, $t^{q_\epsilon}(k)$, as ϵ tends to zero. We derive an *effective potential well*, $\sigma^\epsilon_{\rm eff}(x) = -\epsilon^2\Lambda_{\rm eff}(x)$, such that $t^{q_\epsilon}(k) - t^{\sigma^\epsilon_{\rm eff}}(k)$ is small, uniformly for $k \in \mathbb{R}$ as well as in any bounded subset of a suitable complex strip. Within such a bounded subset, the scaled limit of the transmission coefficient has a universal form, depending on a single parameter, which is computable from the effective potential. A consequence is that if ϵ, the scale of oscillation of the microstructure potential, is sufficiently small, then there is a pole of the transmission coefficient (and hence of the resolvent) in the upper half plane, on the imaginary axis at a distance of order ϵ^2 from zero. It follows that the Schrödinger operator $H_{q_\epsilon} = -\partial_x^2 + q_\epsilon(x)$ has an L^2 bound state with negative energy situated a distance $\mathcal{O}(\epsilon^4)$ from the edge of the continuous spectrum. Finally, we use this detailed information to prove the local energy time-decay estimate: $\big|(1+|\cdot|)^{-3} e^{-itH_{q_\epsilon}} P_c\psi_0\big|_{L^\infty} \leq C\ t^{-1/2}\ \left(1+\epsilon^4\ \left(\int_{\mathbb{R}}\Lambda_{\rm eff}\right)^2 t\ \right)^{-1} \big|(1+|\cdot|^3)\psi_0\big|_{L^1}$, where P_c denotes the projection onto the continuous spectral part of H_{q_ϵ}.

3.A Some useful estimates used throughout the chapter

We recall that the Jost solution is defined through the Volterra equation

$$f^V_+(x;k) - e^{ikx} \ = \ \int_x^\infty \frac{\sin(k(y-x))}{2ik} V(y) f^V_+(y;k)\ {\rm d}y. \tag{3.104}$$

A detailed discussion of Jost solutions, $f_\pm(x;k)$, applying to $\Im(k) \geq 0$ can be found in [Deift and Trubowitz, 1979], where it is assumed that $V \in \mathcal{L}^1_2$. We present in the following Lemma the results holding when $k \in \mathbb{R}$, and deal with the analytic continuation in a complex strip around the real axis afterwards.

Lemma 3.A.1. *If $k \in \mathbb{R}$ and $V \in \mathcal{L}^1_2$, then one has*

$$|f^V_\pm(x;k)| \ \leq \ C(1+|k|)^{-1}(1+|x|), \tag{3.105}$$

$$|\partial_x f^V_\pm(x;k)| \ \leq \ C\frac{1+|k|(1+|x|)}{1+|k|} \ \leq \ C(1+|x|), \tag{3.106}$$

$$|\partial_x^2 f^V_\pm(x;k)| \ \leq \ |V(x)-k^2||f^V_+(x;k)| \ \leq \ C(1+|k|)(1+|x|), \tag{3.107}$$

where $C = C(|V|_{\mathcal{L}^1_2})$. *Moreover, if* $\partial_x V \in \mathcal{L}^1_2$, *then*

$$\left|\partial_x^3 f_\pm^V(x;k)\right| \;\le\; C(1+|k|^2)(1+|x|), \quad \text{with } C = C(|V|_{\mathcal{W}^{1,1}_2}).$$

Proof. As for the first two estimates, equivalent bounds are given in [Deift and Trubowitz, 1979], Lemma 1, for the function $m_\pm(x;k) \;\equiv\; f_\pm(x;k)e^{\pm ikx}$. The results for $f_\pm(x;k)$ follow straightforwardly. The last two estimates are a direct consequence of (3.104). □

If $e^{2\alpha|x|}V \in L^1$, then $f_\pm(x;k)$ has an analytic continuation to $\Im(k) > -\alpha$. Some results are presented in [Reed and Simon, 1979]. In this section we review and obtain the required extensions of these results. In order to simplify the results, we also restrict k to the complex strip $|\Im(k)| < \alpha$.

Lemma 3.A.2. *If* $|\Im(k)| < \alpha$ *and* $V \in L^\infty_\beta$, *with* $\beta > 2\alpha \ge 0$, *then one has*

$$|f_\pm^V(x;k)| \;\le\; C(1+|x|)e^{\alpha|x|}, \tag{3.108}$$

$$|\partial_x f_\pm^V(x;k)| \;\le\; C(1+|k|)(1+|x|)e^{\alpha|x|}, \tag{3.109}$$

$$|\partial_x^2 f_\pm^V(x;k)| \;\le\; |V(x)-k^2||f_+^V(x;k)| \;\le\; C(1+|k|^2)(1+|x|)e^{\alpha|x|}, \tag{3.110}$$

where $C = C(|V|_{L^\infty_\beta})$. *Moreover, if* $V \in W^{1,\infty}_\beta$, *then*

$$\left|\partial_x^3 f_\pm^V(x;k)\right| \;\le\; C(1+|k|^3)(1+|x|)e^{\alpha|x|}, \quad \text{with } C = C(|V|_{W^{1,\infty}_\beta}) \;.$$

Proof. We prove bounds for f_+^V. Analogous bounds $f_-^V(x;k)$ are similarly proved and are obtained from the above by replacing x by $-x$, and $x \ge 0$ by $-x \ge 0$ *etc.*

The estimates follow from the Volterra equation (3.104) satisfied by the Jost solutions, and make use of the following bounds: for $k \in \mathbb{C}$, and for $y \ge x$, one has

$$|\cos(k(y-x))| + |\sin(k(y-x))| \;\le\; Ce^{|\Im(k)|(y-x)} \;\le\; Ce^{\alpha|x|}e^{\alpha|y|} \;, \tag{3.111}$$

$$\frac{|\sin(k(y-x))|}{|k|} \;\le\; C\frac{y-x}{1+|k|(y-x)}e^{|\Im(k)|(y-x)} \;\le\; C(y-x)e^{\alpha|x|}e^{\alpha|y|} \;. \tag{3.112}$$

By Theorem XI.57 of [Reed and Simon, 1979], one deduces from a careful study of the iterates of the Volterra equation (3.104), that for $x \ge 0$, one has

$$|f_+^V(x;k) - e^{ikx}| \;\le\; e^{\alpha|x|}|e^{Q_k(x)}-1| \;\le\; Ce^{\alpha|x|}, \tag{3.113}$$

with $Q_k(x) \;\equiv\; \int_x^\infty \frac{4y}{1+|k|y}\; |V(y)|\; e^{2\alpha|y|}\; \mathrm{d}y$. Equation (3.108) follows for $x \ge 0$.

As for the case $x \le 0$, (3.104) yields

$$
\begin{aligned}
|f_+^V(x;k)| &= \left| e^{ikx} + \int_x^\infty \frac{\sin(k(y-x))}{k} V(y) f_+^V(y;k)\ \mathrm{d}y \right| \\
&\le e^{\alpha|x|} + \int_x^\infty (y-x) e^{\alpha|x|} e^{\alpha|y|} |V(y)||f_+^V(y;k)|\ \mathrm{d}y \\
&\le e^{\alpha|x|}\big[1 + \int_0^\infty y e^{\alpha|y|}|V(y)||f_+^V(y;k)|\ \mathrm{d}y \\
&\qquad\qquad + (-x)\int_x^\infty e^{\alpha|y|}|V(y)||f_+^V(y;k)|\ \mathrm{d}y\big] \\
&\le e^{\alpha|x|}\big[C_0 + (-x)\int_x^\infty e^{\alpha|y|}|V(y)||f_+^V(y;k)|\ \mathrm{d}y\big].
\end{aligned}
$$

We used (3.112) for the first inequality; the last inequality follows from (3.113), with $x = 0$. Therefore, one has with $g(x) \equiv \frac{|f_+^V(x;k)|}{(C_0+(-x))e^{\alpha|x|}}$,

$$
|g(x)| \le 1 + \int_x^\infty e^{\alpha|y|}|V(y)||g(y;k)|(C_0+(-y))e^{\alpha|y|}\ \mathrm{d}y\ .
$$

By Gronwall's inequality

$$
g(x) \le \exp\Big(\int_x^\infty (C_0+(-y))e^{2\alpha|y|}|V(y)|\ \mathrm{d}y\Big) \le C\big(\big|V\big|_{L^\infty_\beta}\big).
$$

Finally, one has

$$
f(x;k) \le C\big(\big|V\big|_{L^\infty_\beta}\big)(C_0+(-x))e^{\alpha|x|} \le C(1+|x|)e^{\alpha|x|},
$$

with $C = C\big(\big|V\big|_{L^\infty_\beta}\big)$. This completes the proof of (3.108).

The proof of (3.109) is similar, and obtained by differentiation and estimation of the Volterra integral equation (3.104). The bound (3.110) is a direct consequence of $\partial_x^2 f_+^V = (V-k^2)f_+^V$ and the above bounds. $\square$

Lemma 3.A.3. *Let $q_{\rm av} \in W^{1,\infty}_\beta$ and $k \in K$, satisfy Hypotheses* **(K)**. *Define*

$$
\mathfrak{m}(x,y;k) \equiv \frac{f_+^{q_{\rm av}}(x;k)f_-^{q_{\rm av}}(y;k) - f_-^{q_{\rm av}}(x;k)f_+^{q_{\rm av}}(y;k)}{W[f_+^{q_{\rm av}}, f_-^{q_{\rm av}}]}.
$$

Then one has, for $0 \le l \le 3$,

$$
|\partial_y^l \mathfrak{m}(x,y;k)| + |\partial_x^l \mathfrak{m}(x,y;k)| \le C\ M_K\ (1+|k|)^l \Big(1 + |y-x|(1+|y|)(1+|x|)e^{\alpha|x|}e^{\alpha|y|}\Big), \quad (3.114)
$$

where $C = C\big(\big|q_{\rm av}\big|_{W^{1,\infty}_\beta}\big)$, and $M_K = \max(1, \sup_{k\in K}|t^{q_{\rm av}}(k)|) < \infty$.

Restricting to $k\in\mathbb{R}$, and assuming only $q_{\rm av}\in\mathcal{W}_2^{1,1}$, one has for $0\leq l\leq 3$

$$|\partial_y^l \mathfrak{m}(x,y;k)| \ + \ |\partial_x^l \mathfrak{m}(x,y;k)| \ \leq \ C(1+|k|)^{l-2}\Big(1+|y-x|(1+|y|)(1+|x|)\Big),$$

where $C=C\big(\big|q_{\rm av}\big|_{\mathcal{W}_2^{1,1}}\big)$.

Proof. Let us start with the estimate (3.114) when $l=0$. One can always assume that $y>x$, since $\mathfrak{m}(x,y;k)=-\mathfrak{m}(y,x;k)$. Using Taylor's theorem with remainder in the integral form, one has

$$f_\pm^{q_{\rm av}}(y;k) \ = \ f_\pm^{q_{\rm av}}(x;k) \ + \ (y-x)\big(\partial_y f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=x} \ + \frac{1}{2}\ \int_x^y \big(\partial_y^2 f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=t}(y-t)\ {\rm d}t.$$

It follows that

$$\begin{aligned}\mathfrak{m}(x,y;k) \ &= \ (y-x)+\frac{1}{2}\int_x^y \frac{f_+^{q_{\rm av}}(x;k)f_-^{q_{\rm av}}(t;k)-f_-^{q_{\rm av}}(x;k)f_+^{q_{\rm av}}(t;k)}{W[f_+^{q_{\rm av}},f_-^{q_{\rm av}}]}(q_{\rm av}(t)-k^2)(y-t)\ {\rm d}t\\ &= \ (y-x)+\frac{1}{2}\int_x^y \mathfrak{m}(x,t;k)(q_{\rm av}(t)-k^2)(y-t)\ {\rm d}t.\end{aligned}$$

Therefore, one has with $g_x(y) \ \equiv \ \frac{|\mathfrak{m}(x,y;k)|}{|x-y|}$,

$$g_x(y) \ \leq \ 1 \ + \ \frac{1}{2|x-y|}\int_x^y g_x(t)|x-t||q_{\rm av}(t)-k^2||y-t|\ {\rm d}t \ \leq \ 1 \ + \ \frac{1}{2}\int_x^y g_x(t)|x-t||q_{\rm av}(t)-k^2|\ {\rm d}t,$$

since $|y-t|\leq|y-x|$ for $t\in[x,y]$. By Gronwall's inequality, one has

$$g_x(y) \ \leq \ \exp\big(\frac{1}{2}\int_x^y |x-t||q_{\rm av}(t)-k^2|\big)\ {\rm d}t \ \leq \ C\big(\big|q_{\rm av}\big|_{L^\infty_\beta}\big)e^{\frac{1}{4}k^2(y-x)^2}.$$

Therefore, we have an estimate on $|\mathfrak{m}(x,y;k)|$, uniformly for k such that $|k||x-y|\leq 1$.

When $|k||x-y|\geq 1$, one has from Lemma 3.A.2

$$\begin{aligned}|\mathfrak{m}(x,y;k)| &\leq C\frac{(1+|x|)e^{\alpha|x|}(1+|y|)e^{\alpha|y|}}{W[f_+^{q_{\rm av}},f_-^{q_{\rm av}}]}\\ &\leq CM_K(1+|x|)(1+|y|)\frac{e^{\alpha|x|}e^{\alpha|y|}}{|k|}\leq CM_K(1+|x|)(1+|y|)|x-y|e^{\alpha|x|}e^{\alpha|y|},\end{aligned}$$

where we used that $\frac{1}{W[f_+^{q_{\rm av}},f_-^{q_{\rm av}}](k)} \ = \ \frac{t^{q_{\rm av}}(k)}{-2ik}$ from (3.24), and $|t^{q_{\rm av}}(k)| \ \leq M_K$, from Hypotheses **(K)**. The estimate (3.114), when $l=0$, is now straightforward.

Let us now look at $\partial_y\mathfrak{m}(x,y;k)$. Using

$$\partial_y f_\pm^{q_{\rm av}}(y;k) \ = \ \big(\partial_y f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=x} \ + \ \int_x^y \big(\partial_y^2 f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=t}\ {\rm d}t,$$

one has the identity

$$\partial_y\mathfrak{m}(x,y;k) \ = \ 1+\int_x^y \mathfrak{m}(x,t;k)(q_{\rm av}(t)-k^2)\ {\rm d}t.$$

If $|k||x-y|\leq 1$, we use that $\mathfrak{m}(x,y;k)$ is uniformly bounded, and obtain

$$\left|\partial_y \mathfrak{m}(x,y;k)\right| \ \leq\ 1+\int_x^y |\mathfrak{m}(x,t;k)||q_{\rm av}(t)-k^2|\ {\rm d}t\ \leq\ C(1+|x-y|+|k|^2|x-y|)\ \leq\ C(1+|x-y|)(1+|k|).$$

When $|k||x-y|\geq 1$, one uses the definition of $\mathfrak{m}$ with Lemma 3.A.2, and one obtains as previously

$$|\partial_y \mathfrak{m}(x,y;k)|\ \leq\ CM_K(1+|k|)(1+|x|)(1+|y|)|x-y|e^{\alpha|x|}e^{\alpha|y|}.$$

Estimate (3.114) follows for $l=1$, using the symmetry $\mathfrak{m}(x,y;k)=-\mathfrak{m}(y,x;k)$.

Estimate (3.114) for $l=2$ is straightforward when remarking that

$$\partial_y^2\mathfrak{m}(x,y;k)\ =\ (q_{\rm av}(y)-k^2)\mathfrak{m}(x,y;k),$$

and the case $l=3$ follows in the same way.

The proof when $k\in\mathbb{R}$ and $q_{\rm av},\ \partial_x q_{\rm av}\in\mathcal{L}^1_2$ is identical, using the estimates of Lemma 3.A.1 instead of Lemma 3.A.2. Note that $M_K=1$ for $k\in\mathbb{R}$, using (3.25). □

3.B Transmission coefficient of $\sigma(x)\ \equiv\ -\epsilon^2\Lambda(x)$

In this section, we study the transmission coefficient of potentials of the form $\sigma(x)\ \equiv\ -\epsilon^2\Lambda(x)$, where $\Lambda\in L^\infty_\beta$, is independent of ϵ. We are particularly interested in the special case where $\sigma(x)$ is the effective potential

$$\sigma^\epsilon_{\rm eff}(x)\ \equiv\ -\frac{\epsilon^2}{4\pi^2}\sum_{j\neq 0}\frac{|q_j(x)|^2}{{\lambda_j}^2},$$

derived earlier.

Lemma 3.B.1 (Transmission coefficient $t^{q_{\rm av}-\epsilon^2\Lambda}(k)$)**.** *Let $q_{\rm av}$ and Λ be any functions in L^∞_β. Then, for $k\in K$ satisfying Hypotheses* **(K)***, one has*

$$\frac{k}{t^{q_{\rm av}-\epsilon^2\Lambda}(k)}\ =\ \left(\ \frac{k}{t^{q_{\rm av}}(k)}\ -\ \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\Lambda(y)f_+^{q_{\rm av}}(y;k)\ dy\ \right)\ +\ \mathcal{O}\big(\epsilon^4\big). \tag{3.115}$$

Proof. We recall the identity (3.37), satisfied by the transmission coefficient related to *any potential* $V,W\in L^\infty_\beta$:

$$\frac{k}{t^V(k)}\ =\ \frac{k}{t^W(k)}-\frac{I^{[V,W]}(k)}{2i},\quad\text{with}\quad I^{[V,W]}(k)\equiv\int_{-\infty}^{\infty} f_-^W(y;k)(V-W)(y)f_+^V(y;k)\ {\rm d}y.$$

Now, in the case where $W \equiv q_{\rm av}$ and $V \equiv q_{\rm av} - \epsilon^2\Lambda(x)$, one has

$$\frac{k}{t^{q_{\rm av}-\epsilon^2\Lambda}(k)} - \frac{k}{t^{q_{\rm av}}(k)} = -\frac{i\epsilon^2}{2} I^\epsilon(k), \quad I^\epsilon(k) \equiv \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\Lambda(y) f_+^{q_{\rm av}-\epsilon^2\Lambda}(y;k)\ {\rm d}y.$$

Then, the Volterra equation (3.34) with $V = q_{\rm av} - \epsilon^2\Lambda$ and $W = q_{\rm av}$, leads to

$$f_+^{q_{\rm av}-\epsilon^2\Lambda}(x;k) = f_+^{q_{\rm av}}(x;k) - \epsilon^2 \int_x^\infty \Lambda(y) \frac{f_+^{q_{\rm av}}(x;k) f_-^{q_{\rm av}}(y;k) - f_-^{q_{\rm av}}(x;k) f_+^{q_{\rm av}}(y;k)}{W[f_+^{q_{\rm av}}, f_-^{q_{\rm av}}]} f_+^{q_{\rm av}-\epsilon^2\Lambda}(y;k)\ {\rm d}y.$$

We can then use the estimates of Lemmata 3.A.2 and 3.A.3, so that

$$\begin{aligned} &\left| I^\epsilon(k) - \int_{-\infty}^\infty f_-^{q_{\rm av}}(y;k)\Lambda(y) f_+^{q_{\rm av}}(y;k)\ {\rm d}y \right| \\ &\quad \le C\epsilon^2 \int_{-\infty}^\infty f_-^{q_{\rm av}}(y;k)\Lambda(y) \int_y^\infty \Lambda(z)\mathfrak{m}(y,z;k) f_+^{q_{\rm av}-\epsilon^2\Lambda}(z;k)\ {\rm d}z\ {\rm d}y \\ &\quad \le \epsilon^2 M_K C, \qquad \text{uniformly for } k \in K. \end{aligned}$$

This concludes the proof. □

A simple consequence is the following

Corollary 3.B.2. *Let $q_{\rm av}$ and Λ be functions in L^∞_β. Then,*

1. *If $q_{\rm av}$ is generic, in the sense of Definition 3.2.1, then $q_{\rm av} - \epsilon^2\Lambda$ is generic for ϵ sufficiently small.*

2. *If $q_{\rm av}$ is non-generic, and $\int_{-\infty}^\infty \Lambda(y)(f_+^{q_{\rm av}}(y;0))^2\ dy \neq 0$, then $q_{\rm av} - \epsilon^2\Lambda$ is generic for ϵ sufficiently small.*

3. *If $q_{\rm av} \equiv 0$, and $k \in K$ satisfy Hypotheses* **(K)**. *Then,*

$$\frac{k}{t^{-\epsilon^2\Lambda}(k)} = k - \frac{i\epsilon^2}{2}\int_{-\infty}^\infty \Lambda(y)\ \ dy + \mathcal{O}(\epsilon^4), \tag{3.116}$$

uniformly in $k \in K$. It follows that if

$$\left| k - \frac{i\epsilon^2}{2}\int_{-\infty}^\infty \Lambda \right| \ge C\ \max(\epsilon^\tau, |k|), \qquad \text{for} \quad \tau < 4, \quad k \in K,$$

then one has

$$\left| t^{-\epsilon^2\Lambda}(k) - \frac{k}{k - \frac{i\epsilon^2}{2}\int_{-\infty}^\infty \Lambda} \right| = \mathcal{O}(\epsilon^{4-\tau}). \tag{3.117}$$

Proof. As discussed in section 3.2.2, a potential, V, is generic, if and only if its transmission coefficient satisfies $t^V(0) = 0$ or, equivalently, if $\lim_{k\to 0} \frac{k}{t^V(k)} \neq 0$. Items (1) and (2) are therefore a straightforward consequence of (3.115). As for item (3), since $q_{\mathrm{av}}(x) \equiv 0$, we have $t^{q_{\mathrm{av}}} \equiv 1$ and $f_\pm^{q_{\mathrm{av}}}(x;k) = e^{\pm ikx}$. The result follows by substitution into (3.115), and straightforward computations. $\square$

Chapter 4

Oscillatory and localized perturbations of H_Q

4.1 Introduction

In this chapter, we consider the Schrödinger operator:

$$H_{Q+\lambda V} \equiv -\partial_x^2 + Q(x) + q_\epsilon(x). \tag{4.1}$$

Here, $Q(x)$ is continuous, real-valued, and 1-periodic, $Q(x+1) = Q(x)$, and $q_\epsilon(x)$ is taken to be localized and sufficiently smooth. In particular, we consider $Q(x)$ to be a background potential and $q_\epsilon(x)$ a localized perturbation of the operator $H_Q \equiv -\partial_x^2 + Q(x)$. From this perspective, we study the spectral properties of the original operator $H_{Q+q_\epsilon(x)}$. We thus begin with the analysis of the spectrum of H_Q.

As discussed in Chapter 1 (Section 1.1), the spectrum of the Schrödinger operator H_Q is continuous and is the union of closed intervals called *spectral bands* [Reed and Simon, 1978; Eastham, 1973]. The complement of the spectrum is a union of open intervals called *spectral gaps*. The spectrum is determined by the family of self-adjoint eigenvalue problems parametrized by the *quasi-momentum* $k \in (-1/2, 1/2]$:

$$H_Q u(x;k) = E\ u(x;k), \tag{4.2}$$

$$u(x+1;k) = e^{2\pi i k} u(x;k). \tag{4.3}$$

That is, we seek *k-pseudo-periodic* solutions of the eigenvalue equation. For each $k \in (-1/2, 1/2]$, the self-adjoint eigenvalue problem (2.2)-(2.3) has discrete eigenvalue-spectrum (listed with multiplicity):

$$E_0(k) \leq E_1(k) \leq \dots \leq E_b(k) \leq \dots \tag{4.4}$$

with corresponding k-pseudo-periodic eigenfunctions $u_b(x;k)$, $b \geq 0$. The b^{th} spectral band is given by:

$$\mathcal{B}_b = \bigcup_{k\in(-1/2,1/2]} E_b(k). \tag{4.5}$$

The spectrum of H_Q is given by:

$$\mathrm{spec}(H_Q) = \bigcup_{b\geq 0} \mathcal{B}_b = \bigcup_{b\geq 0} \bigcup_{k\in(-1/2,1/2]} E_b(k). \tag{4.6}$$

Since the boundary condition (4.3) is invariant with respect to $k \mapsto k+1$, the functions $E_b(k)$ can be extended to all $\mathbb{R}$ as periodic functions of k. The minima and maxima of $E_b(k)$ occur at $k = k_* \in \{0, 1/2\}$; see Figure 4.1. In cases where extrema border a spectral gap, we have that $\partial_k^2 E_b(k_*)$ is either strictly positive or strictly negative [Eastham, 1973; Reed and Simon, 1978]; see Lemma B.1.2 in Appendix B.

Consider now the perturbed operator H_{Q+W}, where $W(x)$ is sufficiently localized in space. By Weyl's Theorem 1.1.4 on the stability of the essential spectrum, one has $\mathrm{spec}_{\mathrm{cont}}(H_{Q+W}) = \mathrm{spec}_{\mathrm{cont}}(H_Q)$ [Reed and Simon, 1978]. The effect of a localized perturbation is to possibly introduce discrete eigenvalues into the open spectral gaps. Note that in our setting, H_{Q+W} does not have discrete eigenvalues embedded in its continuous spectrum; see [Rofe-Beketov, 1964; Gesztesy and Simon, 1993].

Therefore, perturbations of the form $q_\epsilon(x)$ to the operator H_Q can only result in the bifurcation of localized bound states into gaps of the continuous spectrum of H_Q and $\mathrm{spec}_{\mathrm{cont}}(H_{Q+q_\epsilon(x)}) = \mathrm{spec}_{\mathrm{cont}}(H_Q)$.

Before giving a summary of results, let us discuss the physical importance of the above phenomenon. In a periodic medium, a spatially localized initial condition for an energy-conserving wave equation disperses (spatially spreads) and decays in amplitude as time advances. This (Floquet-Bloch) dispersion is associated with the continuous spectrum (extended states) of the underlying differential operator and the absence of discrete eigenvalues (localized bound states) [Kuchment, 2001;

Reed and Simon, 1978]. The introduction of localized perturbations in a periodic medium leads to *defect modes*, states in which energy remains trapped and spatially localized. The process by which the system undergoes a transition from one with only propagating delocalized states to one which supports both localized and propagating states is associated with the emergence or bifurcation of discrete eigenvalues from the continuous spectrum associated with the unperturbed periodic structure.

We next turn to a summary of our results. See Theorem 4.2.1 and Theorem 4.2.3 for detailed statements.

Assume that the b^{th} band has left-endpoint $E_b(k_*)$, $k_* \in \{0, 1/2\}$, bordering a spectral gap. Then $\partial_k^2 E_b(k_*) > 0$; see the left panel of Figure 4.1. We prove in Theorem 4.2.3 that there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, H_{Q+q_ϵ} has a simple discrete eigenvalue which bifurcates from the band edge, $E_b(k_*)$ of $\mathcal{B}_b$, into a spectral gap:

$$E^\epsilon = E_b(k_*) + \epsilon^4 \mu_\star + \mathcal{O}(\epsilon^{4+\alpha}), \quad \text{for some } \ \mu_\star < 0, \ \alpha > 0. \tag{4.7}$$

see the right panel in Figure 4.1.

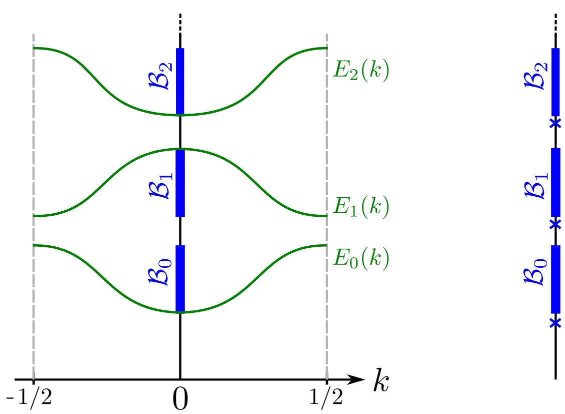


Figure 4.1: Sketch of spectra. Eigenvalues $E_b(k)$, $k \in (-1/2, 1/2]$, $b = 0, 1, 2, \ldots$, are displayed in green. The continuous spectrum, is in blue, and discrete eigenvalues are indicated through cross markers. Left panel corresponds to spec(H_Q), Q periodic. The right panel corresponds to spec(H_{Q+q_ϵ}).

We show that for $0 < \epsilon < \epsilon_0$, $\psi^\epsilon(x)$, the eigenstate corresponding to the eigenvalue E^ϵ, is well approximated in L^∞ by $\psi_\star(\epsilon^2 x)$, where $\psi_\star(y)$ denotes the unique eigenstate of the effective operator:

$$H_{b,\text{eff}} \ = \ -\frac{d}{dy}\ A_{b,\text{eff}}\ \frac{d}{dy}\ \ -\ \ B_{b,\text{eff}} \times \delta(y), \tag{4.8}$$

with constant effective parameters $A_{b,\text{eff}}$ and $B_{b,\text{eff}}$. Here,

$$A_{b,\text{eff}} \ = \ \frac{1}{8\pi^2}\partial_k^2 E_b(k_*), \tag{4.9}$$

is the inverse effective mass associated the the spectral edge $E_b(k_*)$, and

$$\left| B_{b,\text{eff}} - \epsilon^{-2}\int_{\mathbb{R}} |u_b(x;k_*)|^2\ q_\epsilon(x)\overline{Q_\epsilon(x)}\ dx \right| \ \lesssim\ \mathcal{O}\left(\epsilon^{\sigma_{\text{eff}}}\right), \quad \text{for some} \ \ \sigma_{\text{eff}} > 0, \tag{4.10}$$

where $Q_\epsilon(x)$ is defined by:

$$\widehat{Q}_\epsilon(\xi) = \frac{1}{4\pi^2|\xi|^2}\widehat{q}_\epsilon(x).$$

Consider the specific case of $q_\epsilon(x) = q(x, x/\epsilon)$ which is spatially localized on the slow scale x, and periodic with zero mean on the fast scale $y = x/\epsilon$:

$$q(x, y+1) = q(x,y), \qquad \int_0^1 q(x,y)dy = 0. \tag{4.11}$$

By expanding with respect to the Fourier coefficients of the fast variable, one can write

$$q(x,y) = \sum_{j\neq 0} q_j(x)e^{2\pi i j y}. \tag{4.12}$$

One then has:

Theorem 4.1.1. *Let $E_\star = E_b(k_*)$, $k_* \in \{0, 1/2\}$ denote the lower edge of the $b^{th}-$ spectral band and assume that this point borders a spectral gap; see the left panel of Figure 4.1. Assume q_ϵ satisfies* (4.11)-(4.12) *and $q_j(x)$ decays sufficiently rapidly as $x \to \infty$ and $j \to \infty$.*

Let $A_{b_*,\text{eff}}$ and $B_{b_*,\text{eff}}$ denote the effective-medium parameters

$$A_{b_*,\text{eff}} \ = \ \frac{1}{8\pi^2}\partial_k^2 E_{b_*}(k_*) \qquad \textit{(inverse effective mass)} \tag{4.13}$$

$$B_{b_*,\text{eff}} \ = \ \int_{\mathbb{R}} \ |u_{b_*}(x;k_*)|^2 \ \sum_{j\neq 0}\ \frac{1}{(2\pi j)^2}\ |q_j(x)|^2\ dx. \tag{4.14}$$

Then, there exists $\epsilon_0 > 0$ and $\sigma_1, \sigma_2 > 0$, such that for all $0 < \epsilon < \epsilon_0$ the following holds:

H_{Q+q_ϵ} has a simple discrete eigenvalue, $E^\epsilon < E_\star$ (see the right panel in Figure 4.1);

$$E^\epsilon = E_\star + \epsilon^4 E_2 + \mathcal{O}(\epsilon^{4+\sigma_1}); \tag{4.15}$$

with corresponding localized eigenfunction, ψ^ϵ:

$$\sup_{x\in\mathbb{R}} \left|\psi^\epsilon(x) - u_{b_*}(x;k_*) g_0(\epsilon^2 x)\right| \leq C\epsilon^{\sigma_2}. \tag{4.16}$$

Here,

$$g_0(y) = \exp\left(-\frac{B_{b_*,\mathrm{eff}}}{2A_{b_*,\mathrm{eff}}}|y|\right), \qquad E_2 \ = \ -\frac{B^2_{b_*,\mathrm{eff}}}{4A_{b_*,\mathrm{eff}}} < 0 \quad ,$$

is the unique and simple eigenpair of the effective operator

$$H_{b_*,\mathrm{eff}} \ = \ -\frac{d}{dy}\ A_{b_*,\mathrm{eff}}\ \frac{d}{dy} \ - \ B_{b_*,\mathrm{eff}} \times \delta(y) \ , \tag{4.17}$$

where $\delta(y)$ denotes the Dirac delta mass at $y = 0$.

4.1.1 Outline and remarks on the proof

In Section 4.2 we give precise technical statements of our main results: Theorem 4.2.1 and Theorem 4.2.3.

Our strategy of proof is to transform the eigenvalue problem using an appropriate spectral transform (Fourier or Floquet-Bloch) to a formulation in frequency (quasi-momentum) space. Anticipating a bifurcation from the spectral edge, we express the eigenvalue problem in terms of coupled equations governing the frequency components located *near* the band edge and those which are *far* from the band edge. The precise frequency cutoff depends on the small parameter, ϵ. We employ a Lyapunov-Schmidt reduction strategy [Nirenberg, 2001] in which we solve for the *far*-frequency components as a functional of the *near*-frequency components. This yields a reduction to a closed *bifurcation equation* for the *near*-frequency components. In contrast to classical applications of this strategy, our reduced equation is infinite dimensional. For ϵ small, in an appropriate scaled limit, the bifurcation equation is asymptotically exactly solvable; it is the eigenvalue problem for the effective operator $H_{b,\mathrm{eff}}$.

Section 4.3 reviews general technical results on a class of band-limited Schrödinger operators, derived in [Duchêne *et al.*, 2014a] and proved in detail in Section 2.3, which are applied in Sections 4.5 and 4.6. The strategy of the proof is explained in Section 4.4. Finally Appendix 4.A gives detailed proofs of bounds used in Section 4.6 and Appendix 4.B has a detailed analysis and calculation of the effective potential for the particular case of the localized and oscillatory potential $q_\epsilon(x)$ as stated in Theorem 4.1.1. Furthermore, Appendix B.3 summarizes and proves bounds relating to the Floquet-Bloch states used in Section 4.6

4.1.2 Definitions and notation

We denote by C a constant, which does not depend on the small parameter, ϵ. It may depend on norms of $Q(x)$ and $q_\epsilon(x)$, which are assumed finite. $C(\zeta_1, \zeta_2, \dots)$ is a constant depending on the parameters ζ_1, ζ_2, ... We write $A \lesssim B$ if $A \le C\ B$, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

The methods of this paper employ spectral localization relative to the background operator $-\partial_x^2 + Q(x)$, where $Q(x)$ is one-periodic. For the case, $Q \equiv 0$, we use the classical Fourier transform and for $Q(x)$ a non-trivial periodic potential, we use the spectral decomposition of $L^2(\mathbb{R})$ in terms of *Floquet-Bloch* states; see Appendix B. The notations and conventions we use are similar to those used in [Hoefer and Weinstein, 2011].

1. For $f, g \in L^2(\mathbb{R})$, the Fourier transform and its inverse are given by
$$\mathcal{F}\{f\}(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x) dx, \qquad \mathcal{F}^{-1}\{g\}(x) \equiv \check{g}(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} g(\xi) d\xi.$$
2. $\mathcal{T}$ and $\mathcal{T}^{-1}$ denote the Gelfand-Bloch transform and its inverse. We use the following notation for the Gelfand-Bloch transform of a function: $\mathcal{T}\{f\}(x;k) \equiv \widetilde{f}(x;s)$; see Appendix B. Note that we will also use the notation $\widetilde{f}(k)$ in Section 4.6 to represent the projection of $\widetilde{f}(x;s)$ onto a particular Bloch function $p_b(x;k)$, for fixed b.
3. χ and $\overline{\chi}$ are the characteristic functions defined by
$$\chi\left(|\xi| < \delta\right) \equiv \begin{cases} 1, & |\xi| < \delta \\ 0, & |\xi| \ge \delta \end{cases}, \qquad \overline{\chi}(|\xi| < \delta) \equiv 1 - \chi(|\xi| < \delta) \ \equiv \ \begin{cases} 0, & |\xi| < \delta \\ 1, & |\xi| \ge \delta \end{cases}$$
We also use the notation, for a parameter $\lambda > 0$,
$$\chi_\lambda(\xi) = \chi(|\xi| < \delta)\ , \qquad \overline{\chi}_\lambda(\xi) = \overline{\chi}(|\xi| < \delta).$$
4. $L^{2,s}(\mathbb{R})$ is the space of functions $F : \mathbb{R} \to \mathbb{R}$ such that $(1 + |x|^2)^{s/2} F \in L^2(\mathbb{R}_x)$, endowed with the norm
$$\left\|F\right\|_{L^{2,s}(\mathbb{R})} \equiv \left\|(1 + |x|^2)^{s/2} F\right\|_{L^2(\mathbb{R}_x)} < \infty. \tag{4.18}$$

5. $W^{k,\infty}(\mathbb{R})$ is the space of functions $F : \mathbb{R} \to \mathbb{R}$ such that $\partial_x^j F \in L^\infty(\mathbb{R})$ for $0 \le j \le k$, endowed with the norm
$$\|F\|_{W^{k,\infty}(\mathbb{R})} \equiv \sum_{j=0}^{k} \|\partial_x^j F\|_{L^\infty(\mathbb{R})} < \infty.$$

4.2 Bifurcation of defect states into gaps; main results

In this section we state our main results on the eigenvalue problem

$$(-\partial_x^2 + Q(x) + q_\epsilon(x))\psi^\epsilon(x) = E^\epsilon \psi^\epsilon(x), \ \ \psi \in L^2 \ , \tag{4.19}$$

where $Q(x)$ is one-periodic and $q_\epsilon(x)$ a real-valued, localized at high frequencies and decreasing at infinity (precise hypotheses are specified below).

Our first result is for the case where $Q(x) \equiv 0$.

Theorem 4.2.1. *Assume that $q_\epsilon(x)$ satisfies the following, for ϵ sufficiently small:*

(H1a) there exists $0 < \mathcal{C}_0 < \infty$, independent of ϵ, such that

$$\|\widehat{q}_\epsilon\|_{L^1} + \|\widehat{q}_\epsilon\|_{L^\infty} + \|\widehat{q}_\epsilon{}'\|_{L^\infty} \le \mathcal{C}_0, \tag{4.20}$$

(H1b) there exists $N \ge 4$ and $0 < \mathcal{C}_N < \infty$, independent of ϵ, such that

$$\sup_{\xi \in [\frac{-1}{2\epsilon}, \frac{1}{2\epsilon}]} |\widehat{q}_\epsilon(\xi)| \le \epsilon^N \mathcal{C}_N \tag{4.21}$$

(H2) there exists $0 < B_{\text{eff}}, \mathcal{C}_{\text{eff}}, \sigma_{\text{eff}} < \infty$, independent of ϵ, such that

$$\left| \epsilon^{-2} \int_{\mathbb{R}} \frac{1}{4\pi^2 \zeta^2 + 1} \widehat{q}_\epsilon(-\zeta) \widehat{q}_\epsilon(\zeta) d\zeta \ - \ B_{\text{eff}} \right| \ \le \ \mathcal{C}_{\text{eff}} \epsilon^{\sigma_{\text{eff}}}. \tag{4.22}$$

Then, there exist positive constants ϵ_0, C, depending only on the above parameters, such that the following holds. For all $0 < \epsilon < \epsilon_0$, there exists an eigenpair $(E^\epsilon, \psi^\epsilon)$, for the eigenvalue problem

$$\left(-\partial_x^2 + q_\epsilon(x)\right) \psi^\epsilon(x) = E^\epsilon \psi^\epsilon(x), \quad \psi^\epsilon \in L^2 \tag{4.23}$$

with E^ϵ strictly negative and of the order ϵ^4. Moreover, $\psi^\epsilon \in L^\infty$ and we have

$$\left| E^\epsilon + \frac{\epsilon^4 B_{\text{eff}}^2}{4} \right| \le C \epsilon^{4+\sigma}, \tag{4.24}$$

$$\sup_{x \in \mathbb{R}} \left| \psi^\epsilon(x) - \exp\left(-\frac{\epsilon^2 B_{\text{eff}}}{2} |x| \right) \right| \le C \epsilon^\sigma, \tag{4.25}$$

where $\sigma = \min\{1, \sigma_{\text{eff}}\}$. The eigenvalue E^ϵ is unique in the neighborhood defined by (4.24), and the corresponding eigenfunction, ψ^ϵ, is unique up to a multiplicative constant.

See Remark 4.2.5 for a discussion of Hypothesis (H2).

Remark 4.2.2. *Theorem 4.2.1 shows, and is essentially proved by demonstrating, that for small positive ϵ, the leading order behavior of the eigenstate $(E^\epsilon, \psi^\epsilon(x))$ is a scaling of the unique eigenstate of the one-dimensional Schrödinger equation with the attractive Dirac delta potential of mass B_{eff}:*

$$(E^\epsilon, \psi^\epsilon(x)) \approx \left(-\epsilon^4\theta_0^2, g_0(\epsilon^2 x)\right),$$

where $\theta_0 = B_{\text{eff}}/2 > 0$ and $g_0(y) = e^{-\theta_0|y|}$ satisfy

$$\left[-\partial_y^2 - B_{\text{eff}}\ \delta(y)\right]\ g_0(y) = -\theta_0^2 g_0(y).$$

We now turn to the more general case where $Q(x)$ may be a non-trivial periodic background.

Theorem 4.2.3. *Assume Q is one-periodic and satisfies:*

(HQ) $Q \in W^{4,\infty}_{\text{per}}$, so that one has (see Lemma B.3.1) the estimate

$$\forall b \geq 0, \quad \forall k \in [-1/2, 1/2], \quad \forall x \in [-1/2, 1/2], \quad |\partial_x^\alpha p_b(x;k)| \ \leq \ (1+|b|^\alpha)C_\alpha, \qquad (4.26)$$

with $\alpha = 0, \dots, 6$.

Set $E_* = E_{b_*}(k_*)$, the lower endpoint of the $(b_*)^{th}$ band, and assume that the band borders on a spectral gap. Thus $k_* = 0$ or $1/2$ and $\partial_k^2 E_{b_*}(k_*) > 0$; see Lemma B.1.2.

Assume $q_\epsilon(x)$ is localized at high frequencies in the sense that:

(H1'a) there exists $0 < \mathcal{C}_0 < \infty$, independent of ϵ, such that

$$\left\|\widehat{q}_\epsilon\right\|_{L^1} + \left\|\widehat{q}_\epsilon\right\|_{L^\infty} \leq \mathcal{C}_0; \qquad (4.27)$$

(H1'b) for any $0 \leq \beta \leq 6$ and $0 < \mathcal{C}_\beta < \infty$ independent of ϵ,

$$\left(\int_{-1/(2\epsilon)}^{1/(2\epsilon)} |\widehat{q}_\epsilon(\xi)|^2\ d\xi\right)^{1/2} \lesssim \epsilon^\beta \mathcal{C}_\beta. \qquad (4.28)$$

Furthermore, assume $q_\epsilon(x)$ is such that

(H2') there exists $0 < B_{b_*,\mathrm{eff}}, \mathcal{C}_{\mathrm{eff}}, \sigma_{\mathrm{eff}} < \infty$, independent of ϵ, such that

$$\left| \epsilon^{-2} \int_{\mathbb{R}} |u_{b_*}(x;k_*)|^2 q_\epsilon(x) \overline{Q_\epsilon(x)} \; - \; B_{b_*,\mathrm{eff}} \right| \le \mathcal{C}_{\mathrm{eff}} \epsilon^{\sigma_{\mathrm{eff}}}, \tag{4.29}$$

where $Q_\epsilon(x)$ is defined by $\widehat{Q_\epsilon}(\xi) = \frac{1}{1+4\pi^2|\xi|^2}\widehat{q_\epsilon}(\xi)$.

Then there are positive constants ϵ_0, C and σ, depending only on the above parameters, such that the following assertions hold:

1. *For for all $0 < \epsilon < \epsilon_0$, there exists an eigenpair $(E^\epsilon, \psi^\epsilon(x))$ of the eigenvalue problem*

$$\left(-\partial_x^2 + Q(x) + q_\epsilon(x)\right) \psi^\epsilon(x) = E^\epsilon \psi^\epsilon(x), \quad \psi^\epsilon \in L^2(\mathbb{R}). \tag{4.30}$$

   with eigenvalue E^ϵ in the spectral gap, at a distance $\mathcal{O}(\epsilon^4)$ from the band edge, $E_\star$.

2. *Specifically, for $\sigma = \min\{1/6, \sigma_{\mathrm{eff}}\}$ where σ_{eff} is defined in (4.29): E^ϵ and $\psi^\epsilon(x)$ satisfy the following approximations:*

$$\left|E^\epsilon - \left(E_\star + \epsilon^4 E_2\right)\right| \le C\epsilon^{4+\sigma} \tag{4.31}$$

$$\sup_{x\in\mathbb{R}} \left|\psi^\epsilon(x) - u_{b_*}(x;k_*) \exp(\epsilon^2 \alpha_0 |x|)\right| \le C\epsilon^\sigma, \tag{4.32}$$

   where $E_2 < 0$ and $\alpha_0 < 0$ are given by the expressions:

$$E_2 = -\frac{B_{b_*,\mathrm{eff}}^2}{\frac{1}{2\pi^2}\partial_k^2 E_{b_*}(k_*)} < 0 \quad \textit{and} \quad \alpha_0 = -\frac{B_{b_*,\mathrm{eff}}}{\frac{1}{4\pi^2}\partial_k^2 E_{b_*}(k_*)} < 0.$$

3. *The eigenvalue, E^ϵ, is unique in the neighborhood defined in (4.31), and the corresponding eigenfunction, ψ^ϵ, is unique up to a multiplicative constant.*

Remark 4.2.4. *By Theorem 4.2.3, the bifurcating eigenvalue E^ϵ lies in a spectral gap of $-\partial_x^2 + Q(x)$ at a distance $\mathcal{O}(\epsilon^4)$ near the spectral edge $E_\star$; see Figure 4.1. Moreover, E_2 is the unique eigenvalue and $g_0(y) = e^{\alpha_0|y|}$ is the unique (up to a multiplicative constant) eigenfunction of the effective (homogenized) Hamiltonian:*

$$H_{\mathrm{eff}} \;=\; -\frac{d}{dy}\, A_{b_*,\mathrm{eff}}\, \frac{d}{dy} \;-\; B_{b_*,\mathrm{eff}} \;\times\; \delta(y), \tag{4.33}$$

where $A_{b_*,\mathrm{eff}} = \frac{1}{8\pi^2}\partial_x^2 E_{b_*}(k_*)$ represents the inverse effective mass.

Remark 4.2.5. *Assumptions* (4.20)–(4.22) *as well as* (4.27)–(4.29) *hold in particular for potentials of the form* $q_\epsilon(x) = q(x, x/\epsilon)$, *with*

$$q(x,y) = \sum_{j\neq 0} e^{2\pi i j y} q_j(x),$$

under reasonable assumptions on the decay of $q_j(x)$ *when* $|x| \to \infty$ *and* $j \to \infty$. *In that case, one has*

$$B_{\text{eff}} = (1/4\pi^2) \sum_{j\neq 0} j^{-2} \int_{\mathbb{R}} |q_j(x)|^2 \ dx;$$

and

$$B_{b_*,\text{eff}} = (1/4\pi^2) \sum_{j\neq 0} j^{-2} \int_{\mathbb{R}} |u_{b_*}(x;k_*)|^2 \, |q_j(x)|^2 \ dx;$$

see Appendix 4.B.

4.3 Key general technical results

In this section, we state results concerning the operator $\widehat{\mathcal{L}}_{0,\epsilon}[\theta]$, defined by:

$$\widehat{f}(\xi) \ \mapsto \ \widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f}(\xi) \equiv \left(4\pi^2 A\xi^2 + \theta^2\right) \widehat{f}(\xi) - B \ \chi\left(|\xi| < \epsilon^{-\beta}\right) \int_{\mathbb{R}} \chi\left(|\eta| < \epsilon^{-\beta}\right) \ \widehat{f}(\eta) \ d\eta. \tag{4.34}$$

Here, A, B and β are fixed positive constants. The operator $\widehat{\mathcal{L}}_{0,\epsilon}[\theta]$ appears in the bifurcation equations we derive via the Lyapunov-Schmidt reduction; see Section 4.4.

In x-space, we have that $\mathcal{L}_{0,\epsilon}[\theta]$ is a rank one perturbation of $-A\partial_y^2 + \theta^2$:

$$\mathcal{L}_{0,\epsilon}[\theta] f \ \equiv \ (-A\partial_y^2 + \theta^2) f(y) \ - \ \frac{2B}{\epsilon^\beta} \ \left\langle \frac{2}{\epsilon^\beta} \operatorname{sinc}\left(\frac{2\pi}{\epsilon^\beta} \ z\right), f(z) \right\rangle_{L^2(\mathbb{R}_z)} \ \operatorname{sinc}\left(\frac{2\pi y}{\epsilon^\beta}\right), \tag{4.35}$$

where $\operatorname{sinc}(z) = \sin(z)/z$. $\mathcal{L}_{0,\epsilon}[\theta]$ is a band-limited regularization of the operator:

$$\left(\ H^{A,B} \ + \ \theta^2 \right) \ f \ \equiv \ \left(-A\partial_y^2 \ - \ B\delta(y) \ + \ \theta^2 \right) f \ , \tag{4.36}$$

appearing in the effective equations governing the leading order behavior of bifurcating eigenstates; see Remarks 4.2.2 and 4.2.4.

We now state two technical lemmas concerning the operator $\widehat{L}_{0,\epsilon}[\theta]$. Lemma 4.3.1 is proved in [Duchêne *et al.*, 2014a, Lemma 4.1]. Lemma 4.3.2, which concerns solvability of the inhomogeneous equation (4.43) below, has the same conclusion as Lemma [Duchêne *et al.*, 2014a, Lemma 4.4] but is stated with one more condition, (4.45), on R_ϵ. The arguments presented in [Duchêne *et al.*, 2014a] are easily adapted to yield Lemma 4.3.2.

Lemma 4.3.1. *Fix constants $A > 0$, $B > 0$ and $\beta > 0$. Define, for $\theta^2 > 0$, the linear operator*

$$\widehat{f}(\xi) \;\mapsto\; \widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f}(\xi) \equiv \left(4\pi^2 A\xi^2 + \theta^2\right)\widehat{f}(\xi) - B\;\chi\left(|\xi| < \epsilon^{-\beta}\right)\int_{\mathbb{R}} \chi\left(|\eta| < \epsilon^{-\beta}\right)\;\widehat{f}(\eta)\;d\eta. \quad (4.37)$$

Note that $\widehat{\mathcal{L}}_{0,\epsilon}[\theta] : L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R})$. There exists a unique $\theta^2_{0,\epsilon} > 0$ such that:

1. *$\widehat{\mathcal{L}}_{0,\epsilon}[\theta_{0,\epsilon}]$ has a non-trivial kernel.*

2. *The "eigenvalue" $\theta^2_{0,\epsilon}$ is the unique positive solution of*

$$1 \;-\; B\int_{\mathbb{R}} \frac{\chi\left(|\xi| < \epsilon^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta^2_{0,\epsilon}} d\xi \;=\; 0\;. \quad (4.38)$$

3. *The kernel of $\widehat{\mathcal{L}}_{0,\epsilon}[\theta_{0,\epsilon}]$ is given by:*

$$\text{kernel}\left(\widehat{\mathcal{L}}_{0,\epsilon}[\theta_{0,\epsilon}]\right) \;=\; \text{span}\left\{\widehat{f}_{0,\epsilon}(\xi)\right\}, \quad \text{where} \quad \widehat{f}_{0,\epsilon}(\xi) \equiv \frac{\chi\left(|\xi| < \epsilon^{-\beta}\right)}{4\pi^2 A\xi^2 + \theta^2_{0,\epsilon}}\;. \quad (4.39)$$

4. *$\theta_{0,\epsilon}$ can be approximated as follows:*

$$\left|\frac{1}{\theta_{0,\epsilon}} \;-\; \frac{2\sqrt{A}}{B}\right| \;\le\; \frac{1}{\pi^2\sqrt{A}}\;\epsilon^{\beta}. \quad (4.40)$$

5. *One has*

$$\sup_{x\in\mathbb{R}}\left|\;\mathcal{F}^{-1}\left\{\widehat{f}_{0,\epsilon}\right\}(x) \;-\; \frac{2}{B}\exp\left(-\frac{B}{2A}|x|\right)\;\right| \;\le C(A,B)\epsilon^{\beta}. \quad (4.41)$$

The following result concerns solutions to perturbations of $\widehat{\mathcal{L}}_{0,\epsilon}$. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ denote Banach spaces with $\mathcal{Z}_1, \mathcal{Z}_2 \subset L^1_{\text{loc}}$. Assume that for any $(f,g) \in \mathcal{Z}_1 \times \mathcal{Z}_2$,

$$|\langle f,g\rangle_{L^2}| \lesssim \big\|f\big\|_{\mathcal{Z}_2}\big\|g\big\|_{\mathcal{Z}_1}, \qquad \big\|fg\big\|_{\mathcal{Z}_2} \lesssim \big\|f\big\|_{\mathcal{Z}_2}\big\|g\big\|_{L^\infty}, \qquad \text{and} \qquad \big\|(1+\xi^2)^{-1}f\big\|_{\mathcal{Z}_2} \lesssim \big\|f\big\|_{\mathcal{Z}_1}. \quad (4.42)$$

Furthermore, we also assume that $\widehat{f}_{0,\epsilon} \in \mathcal{Z}_1 \cap \mathcal{Z}_2$, where $\left(\theta^2_{0,\epsilon}, \widehat{f}_{0,\epsilon}\right)$ is the unique normalized solution of the homogeneous equation $\widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f} = 0$; see Lemma 4.3.1.

We seek a solution of the equation:

$$\widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f} \;=\; R_\epsilon[\theta]\widehat{f}\;, \quad (4.43)$$

where $\widehat{\mathcal{L}}_{0,\epsilon}(\theta)$ is the operator defined in (4.37) and the mapping $\widehat{f} \mapsto R_\epsilon[\theta]\widehat{f}$ is linear and satisfies the following properties:

Assumptions on R_ϵ: There exist constants $\alpha, \beta, t_-, t_+, C_{R_\epsilon} > 0$ such that for ϵ sufficiently small

- for any $\widehat{f} \in \mathcal{Z}_2$, and $0 < t_- < \theta^2 < t_+ < \infty$,

$$\chi\Big(|\xi| < \epsilon^{-\beta}\Big)\big(R_\epsilon[\theta]\widehat{f}\,\big)(\xi) \;=\; \big(R_\epsilon[\theta]\widehat{f}\,\big)(\xi)\ , \qquad \text{and} \qquad \big\|R_\epsilon[\theta]\widehat{f}\,\big\|_{\mathcal{Z}_1} \;\le\; C_{R_\epsilon}\epsilon^{\alpha}\big\|\widehat{f}\,\big\|_{\mathcal{Z}_2}\ . \tag{4.44}$$

- for any $\widehat{f} \in \mathcal{Z}_2$, and $0 < t_- < \theta_1^2, \theta_2^2 < t_+ < \infty$,

$$\big\|R_\epsilon[\theta_1]\widehat{f} - R_\epsilon[\theta_2]\widehat{f}\,\big\|_{\mathcal{Z}_1} \;\le\; C_{R_\epsilon}\epsilon^{\alpha}|\theta_1^2 - \theta_2^2|\big\|\widehat{f}\,\big\|_{\mathcal{Z}_2}\ . \tag{4.45}$$

In the above setting we have the following

Lemma 4.3.2. *Let $(\theta_{0,\epsilon}^2, \widehat{f}_{0,\epsilon}(\xi))$ be the solution of $\widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f} = 0$, as defined in Lemma 4.3.1, where A, B and $\beta > 0$ are fixed. Let $R_\epsilon[\theta] : \widehat{f} \in \mathcal{Z}_2 \to \mathcal{Z}_1$ be a linear mapping satisfying the assumptions displayed in* (4.44)-(4.45)*, where $\mathcal{Z}_1, \mathcal{Z}_2$ satisfy* (4.42)*. Then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the following hold:*

1. *There exists a unique solution $\Big(\theta_\epsilon, \widehat{f}_\epsilon(\xi)\Big) \in \mathbb{R}^+ \times \mathcal{Z}_2$ of the equation* (4.43)*, such that*

$$\big\|\widehat{f}_\epsilon - \widehat{f}_{0,\epsilon}\big\|_{\mathcal{Z}_2} \le C\epsilon^{\alpha}\ , \qquad \text{and} \qquad \int_{-\infty}^{\infty} \widehat{f}_\epsilon(\xi) - \widehat{f}_{0,\epsilon}(\xi)\ d\xi \;=\; 0,$$

   with $C = C(A, B, C_{R_\epsilon}, \beta)$, independent of ϵ.

2. *Moreover, one has*

$$\widehat{f}_\epsilon(\xi) \;=\; \chi\Big(|\xi| < \epsilon^{-\beta}\Big)\,\widehat{f}_\epsilon(\xi)\ , \qquad \text{and} \qquad \big|\theta_\epsilon^2 - \theta_{0,\epsilon}^2\big| \le C\epsilon^{\alpha}\ .$$

Remark 4.3.3. *To prove Theorems 4.2.1 and 4.2.3, we shall apply Lemma 4.3.2 to the operators:*

1. *$-\partial_x^2 + q_\epsilon(x)$ with $(\mathcal{Z}_1, \mathcal{Z}_2) = \big(L^\infty, L^1\big)$, and*

2. *$-\partial_x^2 + Q(x) + q_\epsilon(x)$ with $(\mathcal{Z}_1, \mathcal{Z}_2) = \big(L^{2,-1}, L^{2,1}\big)$, where $L^{2,s}$ is the space of locally integrable functions such that*

$$\big\|F\big\|_{L^{2,s}} \;\equiv\; \big\|(1 + |\xi|^2)^{s/2}F\big\|_{L^2(\mathbb{R}_\xi)} \;<\; \infty.$$

It is straightforward to check that such spaces satisfy (4.42)*, and $\widehat{f}_{0,\epsilon} \in \mathcal{Z}_1 \cap \mathcal{Z}_2$.*

4.4 Strategy

The strategy we take in Sections 4.5 and 4.6 is to reduce the eigenvalue problem

$$H_{Q+q_\epsilon}\psi = E\psi \tag{4.46}$$

to a homogenized and band-limited Schrödinger equation of the form (4.43). We assume (E, ψ) solves the eigenvalue problem (4.46) and show by a long, formal, and reversible calculation that the rescaled near energy components of $\psi(x)$, $\widehat{\Phi}(\xi)$, and rescaled energies of E, θ^2, satisfy an equation of the form (4.43), namely

$$\mathcal{L}_{0,\epsilon}[\theta]\widehat{\Phi} = \mathcal{R}_\epsilon[\theta]\widehat{\Phi}. \tag{4.47}$$

We then apply Lemma 4.3.2 to construct solutions $(\theta_\epsilon^2, \widehat{\Phi}_\epsilon)$ to (4.47).

The reduction of (4.46) to (4.47) for the case $Q \equiv 0$ is achieved in Proposition 4.5.4, and that for $Q \not\equiv 0$ is achieved in Proposition 4.6.7. In Sections 4.5.4 and 4.6.4 the solution of the original eigenvalue problem, (4.46), is reconstructed from the solutions to (4.47).

In particular, we find the that eigenvalue problems $H_{Q+q_\epsilon}\psi = E\psi$ with $Q \equiv 0$ and $Q \not\equiv 0$ have a bifurcating branch of eigenstates such that, for $\sigma > 0$,

$$\begin{aligned} &\epsilon \mapsto \epsilon^4 E_2 + \mathcal{O}(\epsilon^{4+\sigma}), \quad \text{for } Q \equiv 0, \text{ and} \\ &\epsilon \mapsto E_{b_*} + \epsilon^4 E_2 + \mathcal{O}(\epsilon^{4+\sigma}), \quad \text{for } Q \not\equiv 0, \end{aligned}$$

where $E_2 < 0$ and E_{b_*} is the lower edge of the $(b_*)^{\text{th}}$ spectral band of the eigenvalue problem $H_Q u = Eu$.

4.5 Proof of Th'm 4.2.1; Edge bifurcations for $-\partial_x^2 + q_\epsilon(x)$

In this section we study the bifurcation of solutions to the eigenvalue problem

$$\left(-\partial_x^2 + q_\epsilon(x)\right)\psi(x) = E\psi(x), \qquad \psi \in L^2(\mathbb{R}), \tag{4.48}$$

into the interval $(-\infty, 0)$, the semi-infinite spectral gap of $H_0 \equiv -\partial_x^2$, for q_ϵ localized at high frequencies and decaying as $|x| \to \infty$.

We prove Theorem 4.2.1, which may be seen as a particular case of our main result, Theorem 4.2.3. In this case $Q \equiv 0$ and thus the Floquet-Bloch eigenfunctions are explicit exponentials, making calculations more straightforward and error bounds on the approximations sharper. Section 4.6 will present a more general argument for the $Q \neq 0$ case.

We will begin by transforming equation (4.48) into frequency space in Section 4.5.1, which we will divide into a coupled system of equations, one pertaining to energies near the expected bifurcation point, and the other of the energies far from the bifurcating points. Then, in Sections 4.5.2 and 4.5.3 we will study each part of the system in detail to finally complete the proof of Theorem 4.2.1 in Section 4.5.4.

4.5.1 Near and far energy components

Anticipating that the bifurcating eigenvalue, E, will be real, negative and of size $\approx \epsilon^4$ ([Duchêne *et al.*, 2014c]) we set

$$E \equiv -\epsilon^4\theta^2, \quad 0 < t_- \leq \theta^2 \leq t_+ < \infty, \tag{4.49}$$

where t_- and t_+ are constants of $\mathcal{O}(1)$ and independent of ϵ. We expect, and eventually prove, $\theta \to \theta_{\text{eff}}$ as $\epsilon \to 0$, with $0 < \theta_{\text{eff}} < \infty$.

Taking the Fourier transform of (4.48) yields

$$(4\pi^2\xi^2 + \epsilon^4\theta^2)\widehat{\psi}(\xi) + \int_\zeta \widehat{q}_\epsilon(\xi - \zeta)\widehat{\psi}(\zeta)d\zeta = 0. \tag{4.50}$$

We wish to study (4.50) as a coupled system of equations via the

$$\begin{aligned} &\textit{near energy components}: \{\widehat{\psi}(\xi)\ :\ |\xi| < \epsilon^r\} \text{ and} \\ &\textit{far energy components}: \{\widehat{\psi}(\xi)\ :\ |\xi| \geq \epsilon^r\} \text{ of } \widehat{\psi}. \end{aligned}$$

Let r be a positive parameter, $r > 0$, to be specified. We denote χ the cut-off function:

$$\chi(\xi) \equiv 1,\ |\xi| < 1 \text{ and } \chi(\xi) \equiv 0,\ |\xi| \geq 1.$$

We also set

$$\overline{\chi}(\xi) \equiv 1 - \chi(\xi) \text{ and } \chi_{\epsilon^r}(\xi) \equiv \chi(\epsilon^{-r}\xi).$$

Introduce notation for near and far energy components of $\widehat{\psi}$:

$$\widehat{\psi}_{\text{near}}(\xi) \equiv \chi_{\epsilon^r}(\xi)\widehat{\psi}(\xi) \quad \text{and} \quad \widehat{\psi}_{\text{far}}(\xi) \equiv \overline{\chi}_{\epsilon^r}(\xi)\widehat{\psi}(\xi). \tag{4.51}$$

The eigenvalue equation (4.50) is equivalent to the following coupled system of equations for the near and far energy components:

$$\left(4\pi^2\xi^2 + \epsilon^4\theta^2\right)\widehat{\psi}_{\text{near}}(\xi) + \chi_{\epsilon^r}(\xi)\int_\zeta \widehat{q}_\epsilon(\xi-\zeta)\left(\widehat{\psi}_{\text{near}}(\zeta) + \widehat{\psi}_{\text{far}}(\zeta)\right)d\zeta = 0, \tag{4.52}$$

$$\left(4\pi^2\xi^2 + \epsilon^4\theta^2\right)\widehat{\psi}_{\text{far}}(\xi) + \overline{\chi}_{\epsilon^r}(\xi)\int_\zeta \widehat{q}_\epsilon(\xi-\zeta)\left(\widehat{\psi}_{\text{near}}(\zeta) + \widehat{\psi}_{\text{far}}(\zeta)\right)d\zeta = 0. \tag{4.53}$$

The analysis of the far energy equation (4.53) and near energy equation (4.52) relies heavily on some smallness induced by the assumption that $\widehat{q}_\epsilon$ is localized at high frequencies, and that we encapsulate in the following Lemma.

Lemma 4.5.1. *For every $\epsilon > 0$, let $f_\epsilon, g_\epsilon \in L^1(\mathbb{R}) \cap L^\infty(|\xi| \geq \frac{1}{4\epsilon})$. Then, for $\widehat{q}_\epsilon \in L^1(\mathbb{R})$, one has*

$$\sup_{|\xi|\leq\frac{1}{4\epsilon}}\left|\int_{\mathbb{R}} g_\epsilon(\zeta)\widehat{q}_\epsilon(\xi-\zeta)d\zeta\right| \leq \sup_{|\xi|\leq\frac{1}{2\epsilon}}|\widehat{q}_\epsilon(\xi)|\big\|g_\epsilon\big\|_{L^1(\mathbb{R})} + \big\|\widehat{q}_\epsilon\big\|_{L^1(\mathbb{R})}\sup_{|\zeta|\geq\frac{1}{4\epsilon}}|g_\epsilon(\zeta)|, \tag{4.54}$$

$$\begin{aligned}\left|\iint_{\mathbb{R}^2} f_\epsilon(\xi)g_\epsilon(\zeta)\widehat{q}_\epsilon(\xi-\zeta)d\zeta d\xi\right| &\leq \sup_{|\xi|\leq\frac{1}{2\epsilon}}|\widehat{q}_\epsilon(\xi)|\big\|f_\epsilon\big\|_{L^1(\mathbb{R})}\big\|g_\epsilon\big\|_{L^1(\mathbb{R})} \\ &+ \big\|\widehat{q}_\epsilon\big\|_{L^1(\mathbb{R})}\left(\sup_{|\zeta|\geq\frac{1}{4\epsilon}}|f_\epsilon(\zeta)|\big\|g_\epsilon\big\|_{L^1(\mathbb{R})} + \big\|f_\epsilon\big\|_{L^1(\mathbb{R})}\sup_{|\zeta|\geq\frac{1}{4\epsilon}}|g_\epsilon(\zeta)|\right).\end{aligned} \tag{4.55}$$

Proof. We start with the proof of estimate (4.54). Assume $|\xi| \leq \frac{1}{4\epsilon}$. We decompose the integration domain into $|\zeta| \leq \frac{1}{4\epsilon}$ and $|\zeta| \geq \frac{1}{4\epsilon}$. For $|\zeta| \leq \frac{1}{4\epsilon}$ and $|\xi| \leq \frac{1}{4\epsilon}$, we have $|\xi-\zeta| \leq |\xi| + |\zeta| \leq \frac{1}{2\epsilon}$, and therefore

$$\sup_{|\xi|\leq\frac{1}{4\epsilon}}\int_{|\zeta|\leq\frac{1}{4\epsilon}}|g_\epsilon(\zeta)\widehat{q}_\epsilon(\xi-\zeta)|d\zeta \leq \sup_{|\xi-\zeta|\leq\frac{1}{2\epsilon}}|\widehat{q}_\epsilon(\xi-\zeta)|\big\|g_\epsilon\big\|_{L^1}. \tag{4.56}$$

The integral over $|\zeta| \geq \frac{1}{4\epsilon}$ is estimated as follows,

$$\int_{|\zeta|\geq\frac{1}{4\epsilon}}|g_\epsilon(\zeta)\widehat{q}_\epsilon(\xi-\zeta)|d\zeta \leq \big\|\widehat{q}_\epsilon\big\|_{L^1}\sup_{|\zeta|\geq\frac{1}{4\epsilon}}|g_\epsilon(\zeta)|. \tag{4.57}$$

The bound (4.54) now follows from (4.56) and (4.57).

To prove estimate (4.55), we decompose the integration domain into

$$D_1 \equiv \left\{(\zeta,\xi),\ |\zeta-\xi| \leq \frac{1}{2\epsilon}\right\}, \quad D_2 \equiv \left\{(\zeta,\xi),\ |\zeta-\xi| \geq \frac{1}{2\epsilon}\right\}.$$

The contribution from D_1 is controlled by the bound:

$$\left|\iint_{D_1} f_\epsilon(\xi)g_\epsilon(\zeta)\widehat{q}_\epsilon(\xi-\zeta)d\zeta d\xi\right| \leq \sup_{|\xi-\zeta|\leq\frac{1}{2\epsilon}}|\widehat{q}_\epsilon(\xi-\zeta)|\big\|f_\epsilon\big\|_{L^1}\big\|g_\epsilon\big\|_{L^1}. \tag{4.58}$$

For $(\zeta,\xi) \in D_2$, we have that either $|\zeta| \geq \frac{1}{4\epsilon}$ or $|\xi| \geq \frac{1}{4\epsilon}$. Assume $|\xi| \geq \frac{1}{4\epsilon}$; the case $|\zeta| \geq \frac{1}{4\epsilon}$ is treated symmetrically. One has

$$\left| \int_{\mathbb{R}} d\zeta g_\epsilon(\zeta) \int_{|\xi|\geq \frac{1}{4\epsilon}} d\xi f_\epsilon(\xi) \widehat{q}_\epsilon(\xi-\zeta) \right| \lesssim \|g_\epsilon\|_{L^1} \sup_{|\xi|\geq\frac{1}{4\epsilon}} |f_\epsilon(\xi)| \|\widehat{q}_\epsilon\|_{L^1}. \tag{4.59}$$

It follows that

$$\left| \iint_{D_2} f_\epsilon(\xi) g_\epsilon(\zeta) \widehat{q}_\epsilon(\xi-\zeta) d\zeta d\xi \right| \leq \|\widehat{q}_\epsilon\|_{L^1} \left(\|g_\epsilon\|_{L^1} \sup_{|\xi|\geq\frac{1}{4\epsilon}} |f_\epsilon(\xi)| + \|f_\epsilon\|_{L^1} \sup_{|\zeta|\geq\frac{1}{4\epsilon}} |f_\epsilon(\zeta)| \right).$$

The bound (4.55), and therefore Lemma 4.5.1, now follows from (4.58) and (4.59). □

4.5.2 Analysis of the far energy components

We view (4.53) as an equation for $\widehat{\psi}_{\text{far}}$ depending on "parameters" $(\widehat{\psi}_{\text{near}}, E; \epsilon)$. The following proposition studies the mapping $(\widehat{\psi}_{\text{near}}, E; \epsilon) \mapsto \widehat{\psi}_{\text{far}}[\widehat{\psi}_{\text{near}}, E; \epsilon]$.

Proposition 4.5.2. *Fix $r \in (0,2)$, and assume $E \equiv -\epsilon^4\theta^2$ with $\theta \in \mathbb{R}$. Let $\widehat{\psi}_{near} \in L^1$, and q_ϵ satisfying (4.20) and (4.21) of Theorem 4.2.1 with $N > 2r$. There exists ϵ_0 such that for $0 < \epsilon < \epsilon_0$ the following holds.*

There is a unique solution $\widehat{\psi}_{far}[\widehat{\psi}_{near}, \theta^2; \epsilon]$ of the far energy equation (4.53). Moreover, for any $(\theta^2, \epsilon) \in \mathbb{R} \times (0, \epsilon_0)$, the mapping

$$\widehat{\psi}_{near} \mapsto \widehat{\psi}_{far}[\widehat{\psi}_{near}, \theta^2; \epsilon]$$

is a linear mapping from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$ and satisfies the bound

$$\|\widehat{\psi}_{far}\|_{L^1} \leq C(\mathcal{C}_0, \mathcal{C}_N)\left(\epsilon^{N-2r} + \epsilon^{2-r}\right)\|\widehat{\psi}_{near}\|_{L^1} \tag{4.60}$$

Proof. We seek to solve (4.53) for $\widehat{\psi}_{\text{far}}$ as a functional of $\widehat{\psi}_{\text{near}}$. First note that since $\theta \in \mathbb{R}$, one has for $|\xi| \geq \epsilon^r$, $|4\pi^2\xi^2 + \epsilon^4\theta^2| \geq 4\pi^2\epsilon^{2r}$ is bounded away from zero for any fixed $\epsilon > 0$. Dividing (4.53) by $4\pi^2\xi^2 - E = 4\pi^2\xi^2 + \epsilon^4\theta^2$ and rearranging terms we obtain

$$\widehat{\psi}_{\text{far}}(\xi) = -\frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2 + \epsilon^4\theta^2} \int_\zeta \widehat{q}_\epsilon(\xi-\zeta)\left(\widehat{\psi}_{\text{near}}(\zeta) + \widehat{\psi}_{\text{far}}(\zeta)\right) d\zeta.$$

Iterating the equation, we have

$$\begin{aligned}\widehat{\psi}_{\text{far}}(\xi) = -\frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2 + \epsilon^4\theta^2} \int_{\mathbb{R}} d\zeta \widehat{q}_\epsilon(\xi-\zeta) \Bigg(& \widehat{\psi}_{\text{near}}(\zeta) \\ & - \frac{\overline{\chi}_{\epsilon^r}(\zeta)}{4\pi^2\zeta^2 + \epsilon^4\theta^2} \int_{\mathbb{R}} d\eta \widehat{q}_\epsilon(\zeta-\eta)\left(\widehat{\psi}_{\text{near}}(\eta) + \widehat{\psi}_{\text{far}}(\eta)\right)\Bigg),\end{aligned}$$

which we can write as

$$\left(I - \widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon\right) \widehat{\psi}_{\mathrm{far}} = -\widehat{\mathcal{T}}_\epsilon \widehat{\psi}_{\mathrm{near}} + \widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon \widehat{\psi}_{\mathrm{near}}. \tag{4.61}$$

Here $\widehat{\mathcal{T}}_\epsilon$ is the integral operator defined by

$$\left(\widehat{\mathcal{T}}_\epsilon \widehat{f}\right)(\xi) \equiv \int_\zeta \mathcal{K}_\epsilon(\xi,\zeta)\widehat{f}(\zeta)d\zeta \quad \text{and} \quad \mathcal{K}_\epsilon(\xi,\zeta) \equiv \frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2 + \epsilon^4\theta^2}\widehat{q}_\epsilon(\xi - \zeta).$$

We will show that the operator $(I - \widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)$ is invertible as an operator from L^1 to itself, using that $\big\|\widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon\big\|_{L^1\to L^1}$ is small when ϵ is small. Indeed, one has for $\widehat{h} \in L^1$,

$$\begin{aligned}
\big\|(\widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)\widehat{h}\big\|_{L^1} &\leq \int_{\mathbb{R}} d\xi \frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2 + \epsilon^4\theta^2} \int_{\mathbb{R}} d\zeta |\widehat{q}_\epsilon(\xi - \zeta)| \frac{\overline{\chi}_{\epsilon^r}(\zeta)}{4\pi^2\zeta^2 + \epsilon^4\theta^2} \int_{\mathbb{R}} d\eta |\widehat{q}_\epsilon(\zeta - \eta)||\widehat{h}(\eta)| \\
&= \int_{\mathbb{R}} d\eta |\widehat{h}(\eta)| \iint_{\mathbb{R}^2} d\xi d\zeta \frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2 + \epsilon^4\theta^2} \frac{|\widehat{q}_\epsilon(\zeta - \eta)|\overline{\chi}_{\epsilon^r}(\zeta)}{4\pi^2\zeta^2 + \epsilon^4\theta^2} |\widehat{q}_\epsilon(\xi - \zeta)| d\zeta.
\end{aligned}$$

Defining $f_\epsilon \equiv \frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2+\epsilon^4\theta^2}$ and $g_\epsilon \equiv \frac{|\widehat{q}_\epsilon(\zeta-\eta)|\overline{\chi}_{\epsilon^r}(\zeta)}{4\pi^2\zeta^2+\epsilon^4\theta^2}$, we can apply estimate (4.55) from Lemma 4.5.1, and hypothesis (H1b), *i.e.* bound (4.21) on $\widehat{q}_\epsilon$, to conclude

$$\begin{aligned}
\big\|(\widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)\widehat{h}\big\|_{L^1} &\leq \big\|\widehat{h}\big\|_{L^1} \Bigg[\sup_{|\xi| \leq \frac{1}{2\epsilon}} |\widehat{q}_\epsilon(\xi)| \big\|f_\epsilon\big\|_{L^1(\mathbb{R})} \big\|g_\epsilon\big\|_{L^1(\mathbb{R})} \\
&\qquad + \big\|\widehat{q}_\epsilon\big\|_{L^1(\mathbb{R})} \left(\sup_{|\zeta| \geq \frac{1}{4\epsilon}} |f_\epsilon(\zeta)| \big\|g_\epsilon\big\|_{L^1(\mathbb{R})} + \big\|f_\epsilon\big\|_{L^1(\mathbb{R})} \sup_{|\zeta| \geq \frac{1}{4\epsilon}} |g_\epsilon(\zeta)| \right) \Bigg] \\
&\leq C(\mathcal{C}_0, \mathcal{C}_N)(\epsilon^{N-2r} + \epsilon^{2-r}) \big\|\widehat{h}\big\|_{L^1}.
\end{aligned} \tag{4.62}$$

The final inequality above comes from noting

$$\sup_{|\zeta| \geq \frac{1}{4\epsilon}} |f_\epsilon(\zeta)| \leq C\epsilon^2, \qquad \big\|f_\epsilon\big\|_{L^1} \leq C\epsilon^{-r},$$

$$\sup_{|\zeta| \geq \frac{1}{4\epsilon}} |g_\epsilon(\zeta)| \leq C(\big\|\widehat{q}_\epsilon\big\|_{L^\infty})\epsilon^2, \qquad \big\|g_\epsilon\big\|_{L^1} \leq C(\big\|\widehat{q}_\epsilon\big\|_{L^\infty})\epsilon^{-r}.$$

It follows that if $r \in (0, 2)$ and $N > 2r$, there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, then one has $\big\|\widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon\big\|_{L^1\to L^1} \leq \frac{1}{2}$ and thus $(I - \widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)$ is invertible as an operator from L^1 to L^1, with bound:

$$\big\|(I - \widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)^{-1}\big\|_{L^1\to L^1} \leq 2.$$

We now estimate the right-hand side of (4.61) in L^1, which concludes the proof of Proposition 4.5.2.

First, one has immediately from (4.62) that

$$\left\|(\widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)\widehat{\psi}_{\rm near}\right\|_{L^1} \leq C(\mathcal{C}_0, \mathcal{C}_N)\left(\epsilon^{N-2r} + \epsilon^{2-r}\right)\left\|\widehat{\psi}_{\rm near}\right\|_{L^1}.$$

Then, since $\widehat{\psi}_{\rm near}(\zeta) = \chi_{\epsilon^r}(\zeta)\widehat{\psi}_{\rm near}(\zeta)$, one has

$$\begin{aligned}\left\|\widehat{\mathcal{T}}_\epsilon\widehat{\psi}_{\rm near}\right\|_{L^1} &\leq \int_{\mathbb{R}} d\xi \frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2 + \epsilon^4\theta^2}\int_{\mathbb{R}} d\zeta |\widehat{q}_\epsilon(\xi-\zeta)|\chi_{\epsilon^r}(\zeta)|\widehat{\psi}_{\rm near}(\zeta)| \\ &= \int_{\mathbb{R}} d\zeta \chi_{\epsilon^r}(\zeta)|\widehat{\psi}_{\rm near}(\zeta)|\int_{\mathbb{R}} d\xi \frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2 + \epsilon^4\theta^2}|\widehat{q}_\epsilon(\xi-\zeta)|.\end{aligned}$$

Defining $g_\epsilon = \frac{\overline{\chi}_{\epsilon^r}(\xi)}{4\pi^2\xi^2+\epsilon^4\theta^2}$, we can apply estimate (4.54) from Lemma 4.5.1, hypothesis (H1b), *i.e.* bound (4.21) on $\widehat{q}_\epsilon$, and using that $\epsilon^r \leq \frac{1}{4\epsilon}$ for ϵ sufficiently small, we conclude

$$\begin{aligned}\left\|\widehat{\mathcal{T}}_\epsilon\widehat{\psi}_{\rm near}\right\|_{L^1} &\leq \left\|\widehat{\psi}_{\rm near}\right\|_{L^1}\left[\sup_{|\xi|\leq\frac{1}{2\epsilon}}|\widehat{q}_\epsilon(\xi)|\left\|g_\epsilon\right\|_{L^1(\mathbb{R})} + \left\|\widehat{q}_\epsilon\right\|_{L^1(\mathbb{R})}\sup_{|\zeta|\geq\frac{1}{4\epsilon}}|g_\epsilon(\zeta)|\right] \\ &\leq C(\mathcal{C}_0, \mathcal{C}_N)(\epsilon^{N-r} + \epsilon^2)\left\|\widehat{\psi}_{\rm near}\right\|_{L^1}.\end{aligned}$$

The final inequality above comes from noting

$$\sup_{|\zeta|\geq\frac{1}{4\epsilon}}|g_\epsilon(\zeta)| \leq C\epsilon^2, \qquad \left\|g_\epsilon\right\|_{L^1} \leq C\epsilon^{-r}.$$

Altogether, we proved

$$\begin{aligned}\left\|\widehat{\psi}_{\rm far}\right\|_{L^1} &\leq \left\|(I - \widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)^{-1}\right\|_{L^1\to L^1}\Big(\left\|\widehat{\mathcal{T}}_\epsilon\widehat{\psi}_{\rm near}\right\|_{L^1} + \left\|(\widehat{\mathcal{T}}_\epsilon \circ \widehat{\mathcal{T}}_\epsilon)\widehat{\psi}_{\rm near}\right\|_{L^1}\Big) \\ &\leq C(\mathcal{C}_0, \mathcal{C}_N)\left(\epsilon^{N-2r} + \epsilon^{2-r}\right)\left\|\widehat{\psi}_{\rm near}\right\|_{L^1}.\end{aligned}$$

This completes the proof of Proposition 4.5.2. □

4.5.3 Analysis of the near energy components

By Proposition 4.5.2, we have $\widehat{\psi}_{\rm far} = \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near}, \theta^2; \epsilon]$, where we recall assumption (4.49): $E \equiv -\epsilon^4\theta^2$ with $0 < t_- \leq \theta^2 \leq t_+ < \infty$. Substitution into the near energy equation (4.52), we obtain a closed equation for $\widehat{\psi}_{\rm near}(\xi)$:

$$\left(4\pi^2\xi^2 + \epsilon^4\theta^2\right)\widehat{\psi}_{\rm near}(\xi) + \chi_{\epsilon^r}(\xi)\int_\zeta \widehat{q}_\epsilon(\xi-\zeta)\left(\widehat{\psi}_{\rm near}(\zeta) + \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near}, \theta^2; \epsilon](\zeta)\right)d\zeta = 0. \qquad (4.63)$$

The following Proposition reveals the leading order terms in (4.63).

Proposition 4.5.3. *Set $r \in (0,2)$. Assume that q_ϵ satisfies (4.20) and (4.21) of Theorem 4.2.1 with $N \geq 2, N > 2r$, and $E = -\epsilon^4\theta^2$, with $0 < t_- \leq \theta^2 \leq t_+ < \infty$. Then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ one can write (4.63) as*

$$\left(4\pi^2\xi^2 + \epsilon^4\theta^2\right)\widehat{\psi}_{near}(\xi) - \chi_{\epsilon^r}(\xi)\epsilon^2 B_{\rm eff}\int_{\mathbb{R}}\widehat{\psi}_{near}(\eta)d\eta = -\chi_{\epsilon^r}(\xi)\left(\mathcal{R}[\theta]\widehat{\psi}_{near}\right)(\xi), \tag{4.64}$$

where the remainder term, $\mathcal{R}$, satisfies the bound

$$\left\|\mathcal{R}\left[\theta\right]\widehat{\psi}_{near}\right\|_{L^\infty} \leq C(\mathcal{C}_0, \mathcal{C}_N, \mathcal{C}_{\rm eff}, t_-, t_+)\left\|\widehat{\psi}_{near}\right\|_{L^1} \times K_\epsilon, \tag{4.65}$$

with $K_\epsilon = \left(\epsilon^{N-2} + \epsilon^2\right)\left(\epsilon^{N-2r} + \epsilon^{2-r}\right) + \epsilon^r \times \left(\epsilon^{N-2} + \epsilon^2\right) + \epsilon^{2N} + \epsilon^{2N-2} + \epsilon^{2+\sigma_{\rm eff}}$.

Proof. Using equations (4.52) and (4.53) to iterate once the near energy equation (4.63) and interchanging the order of integration, we obtain

$$\begin{aligned}
\left(4\pi^2\xi^2 + \epsilon^4\theta^2\right)\widehat{\psi}_{\rm near}(\xi) &= -\chi_{\epsilon^r}(\xi)\int_\zeta \widehat{q}_\epsilon(\xi-\zeta)\left(\widehat{\psi}_{\rm near}(\zeta) + \widehat{\psi}_{\rm far}(\zeta)\right)d\zeta \\
&= \chi_{\epsilon^r}(\xi)\int_\zeta \widehat{q}_\epsilon(\xi-\zeta)\left[\frac{\chi_{\epsilon^r}(\zeta)}{4\pi^2\zeta^2+\epsilon^4\theta^2}\int_{\eta_1}\widehat{q}_\epsilon(\zeta-\eta_1)\left(\widehat{\psi}_{\rm near}(\eta_1) + \widehat{\psi}_{\rm far}(\eta_1)\right)d\eta_1\right. \\
&\qquad\left. + \frac{\overline{\chi}_{\epsilon^r}(\zeta)}{4\pi^2\zeta^2+\epsilon^4\theta^2}\int_{\eta_2}\widehat{q}_\epsilon(\zeta-\eta_2)\left(\widehat{\psi}_{\rm near}(\eta_2) + \widehat{\psi}_{\rm far}(\eta_2)\right)d\eta_2\right]d\zeta \\
&= \chi_{\epsilon^r}(\xi)\left[\int_\eta \widehat{\psi}_{\rm near}(\eta)d\eta\int_\zeta \frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}\widehat{q}_\epsilon(\xi-\zeta)\widehat{q}_\epsilon(\zeta-\eta)d\zeta\right. \\
&\qquad\left. + \int_\eta \widehat{\psi}_{\rm far}(\eta)d\eta\int_\zeta \frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}\widehat{q}_\epsilon(\xi-\zeta)\widehat{q}_\epsilon(\zeta-\eta)d\zeta\right].
\end{aligned}$$

We rewrite this equation as

$$(4\pi^2\xi^2 + \epsilon^4\theta^2)\widehat{\psi}_{\rm near}(\xi) = \left(\mathcal{Q}[\theta]\widehat{\psi}_{\rm near}\right)(\xi) + \left(\mathcal{Q}[\theta]\widehat{\psi}_{\rm far}\right)(\xi), \tag{4.66}$$

where we recall the mapping $(\widehat{\psi}_{\rm near}, \theta^2;\epsilon) \mapsto \widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near}, \theta^2;\epsilon]$, and denote

$$\left(\mathcal{Q}[\theta]\widehat{\psi}\right)(\xi) \equiv \chi_{\epsilon^r}(\xi)\int_\eta \widehat{\psi}(\eta)\int_\zeta \frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}\widehat{q}_\epsilon(\xi-\zeta)\widehat{q}_\epsilon(\zeta-\eta)d\zeta d\eta.$$

In what follows, we first show that the contribution of $\mathcal{Q}[\theta]\widehat{\psi}_{\rm far}$ is small, and then extract the leading order term from $\mathcal{Q}[\theta]\widehat{\psi}_{\rm near}$.

L^∞ bound of $\mathcal{Q}[\theta]\widehat{\psi}_{far}$. First note

$$\left\|\mathcal{Q}[\theta]\widehat{\psi}_{\rm far}\right\|_{L^\infty} \leq \sup_{|\xi|\leq\epsilon^r}\left|\sup_{\eta\in\mathbb{R}}\int_\zeta \frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}\widehat{q}_\epsilon(\xi-\zeta)\widehat{q}_\epsilon(\zeta-\eta)d\zeta\right|\ \left\|\widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near}, E;\epsilon]\right\|_{L^1}.$$

Using $\epsilon^r \leq \frac{1}{4\epsilon}$ for ϵ sufficiently small, we can bound the factor multiplying $\left\|\widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near}, E; \epsilon]\right\|_{L^1}$ using estimate (4.54) of Lemma 4.5.1 with the choice $g_\epsilon(\zeta) = \frac{\sup_{\eta\in\mathbb{R}} \widehat{q}_\epsilon(\zeta-\eta)}{4\pi^2\zeta^2+\epsilon^4\theta^2}$ and applying hypothesis (H1b), *i.e.* bound (4.21) of $\widehat{q}_\epsilon$. Noting that

$$\sup_{|\zeta|\geq\frac{1}{4\epsilon}} |g_\epsilon(\zeta)| \leq C(\|\widehat{q}_\epsilon\|_{L^\infty})\epsilon^2, \qquad \|g_\epsilon\|_{L^1} \leq C(\|\widehat{q}_\epsilon\|_{L^\infty})\epsilon^{-2}|\theta|^{-1},$$

one has the bound

$$\begin{aligned} \left\|\mathcal{Q}[\theta]\widehat{\psi}_{\rm far}\right\|_{L^\infty} &\leq C(\mathcal{C}_0, \mathcal{C}_N)\left(|\theta|^{-1}\epsilon^{N-2} + \epsilon^2\right)\left\|\widehat{\psi}_{\rm far}[\widehat{\psi}_{\rm near}, E; \epsilon]\right\|_{L^1} \\ &\leq C(\mathcal{C}_0, \mathcal{C}_N, t_-)\left(\epsilon^{N-2} + \epsilon^2\right)\left(\epsilon^{N-2r} + \epsilon^{2-r}\right)\left\|\widehat{\psi}_{\rm near}\right\|_{L^1}, \end{aligned} \tag{4.67}$$

where the last estimate follows from Proposition 4.5.2, and $0 < t_- < \theta^2 < t_+ < \infty$.

Leading order expansion of $\mathcal{Q}[\theta]\widehat{\psi}_{near}$. Let us first recall that $\widehat{\psi}_{\rm near}(\eta) = \chi_{\epsilon^r}(\eta)\widehat{\psi}_{\rm near}(\eta)$, and consequently rewrite

$$\left(\mathcal{Q}[\theta]\widehat{\psi}_{\rm near}\right)(\xi) = \int_{\mathbb{R}} d\eta\widehat{\psi}_{\rm near}(\eta) \times \chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\mathfrak{q}(\xi,\eta), \tag{4.68}$$

with

$$\mathfrak{q}(\xi,\eta) = \int_{\mathbb{R}} \frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}\widehat{q}_\epsilon(\xi-\zeta)\widehat{q}_\epsilon(\zeta-\eta)d\zeta.$$

Our aim is to expand the pointwise first order term (in ϵ) of $\mathfrak{q}(\xi,\eta)$ for $(\xi,\eta) \in [-\epsilon^r, \epsilon^r]^2$. We write

$$\begin{aligned} \left(\mathcal{Q}[\theta]\widehat{\psi}_{\rm near}\right)(\xi) &= \int_{\mathbb{R}} d\eta\widehat{\psi}_{\rm near}(\eta) \times \chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\mathfrak{q}(0,0) \\ &\quad + \int_{\mathbb{R}} d\eta\widehat{\psi}_{\rm near}(\eta) \times \chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\left[\mathfrak{q}(\xi,\eta) - \mathfrak{q}(0,0)\right] \\ &= \int_{\mathbb{R}} d\eta\widehat{\psi}_{\rm near}(\eta) \times \chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\left[\int_{\mathbb{R}} \frac{\widehat{q}_\epsilon(-\zeta)\widehat{q}_\epsilon(\zeta)}{4\pi^2\zeta^2+1}d\zeta\right] \\ &\quad + \int_{\mathbb{R}} d\eta\widehat{\psi}_{\rm near}(\eta) \times \chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\left[\int_{\mathbb{R}} \frac{\widehat{q}_\epsilon(-\zeta)\widehat{q}_\epsilon(\zeta)}{4\pi^2\zeta^2+\epsilon^4\theta^2}d\zeta - \int_{\mathbb{R}} \frac{\widehat{q}_\epsilon(-\zeta)\widehat{q}_\epsilon(\zeta)}{4\pi^2\zeta^2+1}d\zeta\right] \end{aligned} \tag{4.69}$$

$$+ \int_{\mathbb{R}} d\eta\widehat{\psi}_{\rm near}(\eta) \times \chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\left[\mathfrak{q}(\xi,\eta) - \mathfrak{q}(0,0)\right]. \tag{4.70}$$

We will now bound the last two terms in the above sum. Firstly, using the Mean Value Theorem, one has

$$\sup_{(\xi,\eta)\in[-\epsilon^r,\epsilon^r]^2} \left|\mathfrak{q}(\xi,\eta) - \mathfrak{q}(0,0)\right| \lesssim \epsilon^r \sup_{(\xi,\eta)\in[-\epsilon^r,\epsilon^r]^2} \left(\left|\frac{d}{d\xi}\mathfrak{q}(\xi,\eta)\right| + \left|\frac{d}{d\eta}\mathfrak{q}(\xi,\eta)\right|\right).$$

Using the symmetry properties of $\mathfrak{q}$, it suffices to estimate

$$\left|\frac{d}{d\eta}\mathfrak{q}(\xi,\eta)\right| \leq \int_{\mathbb{R}} \frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}|\widehat{q_\epsilon}(\xi-\zeta)||\widehat{q_\epsilon}'(\zeta-\eta)|d\zeta.$$

Using estimate (4.54) in Lemma 4.5.1 with $g_\epsilon(\zeta) = \frac{|\widehat{q_\epsilon}'(\zeta-\eta)|}{4\pi^2\zeta^2+\epsilon^4\theta^2}$, and hypothesis (H1b), *i.e.* bound (4.21) on $\widehat{q_\epsilon}$, one obtains

$$\begin{aligned}\left|\frac{d}{d\eta}\mathfrak{q}(\xi,\eta)\right| &\leq \sup_{|\xi|\leq\frac{1}{2\epsilon}} |\widehat{q_\epsilon}(\xi)| \, \|g_\epsilon\|_{L^1(\mathbb{R})} + \|\widehat{q_\epsilon}\|_{L^1(\mathbb{R})} \sup_{|\zeta|\geq\frac{1}{4\epsilon}} |g_\epsilon(\zeta)| \\ &\leq C(\mathcal{C}_0,\mathcal{C}_N)\left(|\theta|^{-1}\epsilon^{N-2}+\epsilon^2\right),\end{aligned}$$

where we note that

$$\sup_{|\zeta|\geq\frac{1}{4\epsilon}} |g_\epsilon(\zeta)| \leq C(\|\widehat{q}_\epsilon'\|_{L^\infty(\mathbb{R})})\epsilon^2, \qquad \|g_\epsilon\|_{L^1(\mathbb{R})} \leq C(\|\widehat{q}_\epsilon'\|_{L^\infty(\mathbb{R})})\epsilon^{-2}|\theta|^{-1}.$$

Therefore, term (4.70) can be bounded as

$$\left|\int_{\mathbb{R}} d\eta\widehat{\psi}_{\text{near}}(\eta)\times\chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\left[\mathfrak{q}(\xi,\eta)-\mathfrak{q}(0,0)\right]\right| \leq C(\mathcal{C}_0,\mathcal{C}_N)\ \epsilon^r\times\left(|\theta|^{-1}\epsilon^{N-2}+\epsilon^2\right)\left\|\widehat{\psi}_{\text{near}}\right\|_{L^1}. \tag{4.71}$$

As a second step, we study term (4.69). In particular, we bound the integral

$$\int_{\mathbb{R}} \widehat{q_\epsilon}(-\zeta)\widehat{q_\epsilon}(\zeta)\left[\frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}-\frac{1}{4\pi^2\zeta^2+1}\right]d\zeta.$$

To do so, we consider the above integral under two domains: $|\zeta|\leq\frac{1}{4\epsilon}$ and $|\zeta|>\frac{1}{4\epsilon}$. Notice that since q_ϵ satisfies hypothesis (H1b), *i.e.* bound (4.21), one has

$$\begin{aligned}\int_{|\zeta|\leq 1/(4\epsilon)} \frac{1}{4\pi^2\zeta^2+1}\widehat{q_\epsilon}(-\zeta)\widehat{q_\epsilon}(\zeta)d\zeta &\leq \mathcal{C}_N^2\epsilon^{2N} \\ \int_{|\zeta|\leq 1/(4\epsilon)} \frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}\widehat{q_\epsilon}(-\zeta)\widehat{q_\epsilon}(\zeta)d\zeta &\leq \mathcal{C}_N^2|\theta|^{-1}\epsilon^{2N-2}.\end{aligned}$$

Furthermore,

$$\int_{|\zeta|\geq 1/(4\epsilon)} \left|\frac{1}{4\pi^2\zeta^2+1}-\frac{1}{4\pi^2\zeta^2+\epsilon^4\theta^2}\right| |\widehat{q_\epsilon}(-\zeta)\widehat{q_\epsilon}(\zeta)|d\zeta \leq C(\epsilon^4\theta^2)\epsilon^4\|\widehat{q_\epsilon}\|_{L^1}\|\widehat{q_\epsilon}\|_{L^\infty},$$

and we conclude

$$\begin{aligned}&\left|\int_{\mathbb{R}} d\eta\widehat{\psi}_{\text{near}}(\eta)\times\chi_{\epsilon^r}(\xi)\chi_{\epsilon^r}(\eta)\left[\int_{\mathbb{R}}\frac{\widehat{q_\epsilon}(-\zeta)\widehat{q_\epsilon}(\zeta)}{4\pi^2\zeta^2+\epsilon^4\theta^2}d\zeta-\int_{\mathbb{R}}\frac{\widehat{q_\epsilon}(-\zeta)\widehat{q_\epsilon}(\zeta)}{4\pi^2\zeta^2+1}d\zeta\right]\right| \\ &\qquad\leq C(\mathcal{C}_0,\mathcal{C}_N)\left(\epsilon^{2N}+|\theta|^{-1}\epsilon^{2N-2}+C(\epsilon^4\theta^2)\epsilon^4\right)\left\|\widehat{\psi}_{\text{near}}\right\|_{L^1}.\end{aligned} \tag{4.72}$$

Altogether, plugging estimates (4.71) and (4.72) into $\mathcal{Q}[\theta]\psi_{\text{near}}$ as defined in (4.68), yields

$$\big(\mathcal{Q}[\theta]\widehat{\psi}_{\text{near}}\big)(\xi) = \chi_{\epsilon^r}(\xi)\left(\int_{\mathbb{R}} \frac{\widehat{q}_\epsilon(-\zeta)\widehat{q}_\epsilon(\zeta)}{4\pi^2\zeta^2+1}d\zeta\right) \times \int_{\mathbb{R}} \widehat{\psi}_{\text{near}}(\eta)d\eta + \chi_{\epsilon^r}(\xi)\big(\mathcal{R}_1[\theta]\widehat{\psi}_{\text{near}}\big)(\xi), \tag{4.73}$$

where the remainder $\mathcal{R}_1$ satisfies the bound

$$\left\|\mathcal{R}_1[\theta]\widehat{\psi}_{\text{near}}\right\|_{L^\infty} \leq C(\mathcal{C}_0, \mathcal{C}_N)\left(\epsilon^r \times \left(|\theta|^{-1}\epsilon^{N-2}+\epsilon^2\right) + \epsilon^{2N} + |\theta|^{-1}\epsilon^{2N-2} + C(\epsilon^4\theta^2)\epsilon^4\right)\left\|\widehat{\psi}_{\text{near}}\right\|_{L^1}.$$

Furthermore, by Hypothesis (H2), expression (4.22), we can write

$$\left|\int_{\mathbb{R}} \frac{\widehat{q}_\epsilon(-\zeta)\widehat{q}_\epsilon(\zeta)}{4\pi^2\zeta^2+1}d\zeta \ - \ \epsilon^2 B_{\text{eff}}\right| \ \leq \ \mathcal{C}_{\text{eff}}\epsilon^{2+\sigma_{\text{eff}}}.$$

Therefore, we can rewrite (4.73) as

$$\big(\mathcal{Q}[\theta]\widehat{\psi}_{\text{near}}\big)(\xi) = \chi_{\epsilon^r}(\xi)\epsilon^2 B_{\text{eff}} \int_{\mathbb{R}} \widehat{\psi}_{\text{near}}(\eta)d\eta + \chi_{\epsilon^r}(\xi)\big(\mathcal{R}_2[\theta]\widehat{\psi}_{\text{near}}\big)(\xi), \tag{4.74}$$

where the remainder $\mathcal{R}_2$ now satisfies the bound

$$\left\|\mathcal{R}_2[\theta]\widehat{\psi}_{\text{near}}\right\|_{L^\infty} \leq C(\mathcal{C}_0, \mathcal{C}_N, \mathcal{C}_{\text{eff}}, t_-, t_+)\left(\epsilon^r \times \left(\epsilon^{N-2}+\epsilon^2\right) + \epsilon^{2N} + \epsilon^{2N-2} + \epsilon^{2+\sigma_{\text{eff}}}\right)\left\|\widehat{\psi}_{\text{near}}\right\|_{L^1}, \tag{4.75}$$

where we used $0 < t_- < \theta^2 < t_+ < \infty$.

We conclude the proof of Proposition 4.5.3 by plugging in expression (4.74) and estimates (4.67) and (4.75) into (4.66). □

Rescaling the equation. We now proceed with the analysis of the near equation with the rescaling ξ and $\widehat{\psi}_{\text{near}}$ in such a way as to balance both terms on the left hand side of (4.64). Thus we define

$$\widehat{\psi}_{\text{near}}(\xi) = \frac{1}{\epsilon^2}\widehat{\Phi}\left(\frac{\xi}{\epsilon^2}\right) = \frac{1}{\epsilon^2}\widehat{\Phi}(\xi'), \quad \xi = \epsilon^2\xi' \ .$$

Note that

$$\left\|\widehat{\psi}_{\text{near}}\right\|_{L^1} = \left\|\widehat{\Phi}\right\|_{L^1}.$$

Equation (4.64) then becomes, after dividing out by ϵ^2,

$$\left(4\pi^2\xi'^2+\theta^2\right)\widehat{\Phi}(\xi') - \chi_{\epsilon^{r-2}}(\xi')B_{\text{eff}}\int_{\zeta'} \widehat{\Phi}(\zeta')d\zeta' = -\epsilon^{-2}\chi_{\epsilon^{r-2}}(\xi')\left(\mathcal{R}[\theta]\left\{\frac{1}{\epsilon^2}\widehat{\Phi}(\frac{\cdot}{\epsilon^2})\right\}\right)(\epsilon^2\xi').$$

By estimate (4.65) and choosing carefully the parameters r and N, we can ensure that the right hand side is small. The following Proposition summarizes our result, with $N = 4$ and $r = 1$.

Proposition 4.5.4. *Assume that the assumptions of Proposition 4.5.3 hold with $r = 1$ and $N = 4$. Then one has*

$$\left(4\pi^2\xi'^2 + \theta^2\right)\widehat{\Phi}(\xi') - \chi_{_{\epsilon^{-1}}}(\xi')B_{\rm eff}\int_\zeta \widehat{\Phi}(\zeta')d\zeta' = \chi_{_{\epsilon^{-1}}}(\xi')\big(\widetilde{\mathcal{R}}[\theta]\widehat{\Phi}\big)(\xi'), \tag{4.76}$$

where $\widetilde{\mathcal{R}}[\theta]\widehat{\Phi} : \xi' \mapsto \epsilon^{-2}\chi_{_{\epsilon^{-1}}}(\xi')\Big(\mathcal{R}[\theta]\{\frac{1}{\epsilon^2}\widehat{\Phi}(\frac{\cdot}{\epsilon^2})\}\Big)(\epsilon^2\xi')$ *satisfies the bound, for* $\sigma = \min\{1, \sigma_{\rm eff}\}$,

$$\big\|\chi_{_{\epsilon^{-1}}}(\xi)\big(\widetilde{\mathcal{R}}[\theta]\widehat{\Phi}\big)(\xi)\big\|_{L^\infty(\mathbb{R}_\xi)} \ \le\ \epsilon^\sigma\ C(\mathcal{C}_0, \mathcal{C}_N, \mathcal{C}_{\rm eff}, t_-, t_+)\big\|\widehat{\Phi}\big\|_{L^1}. \tag{4.77}$$

4.5.4 Conclusion of proof of Theorem 4.2.1

Proposition 4.5.4 is a formal reduction of the eigenvalue problem

$$(-\partial_x^2 + q_\epsilon)\psi = E\psi, \quad \psi \in L^2(\mathbb{R}), \tag{4.78}$$

for $(E^\epsilon, \psi^\epsilon)$ to an equation for $(\theta_\epsilon^2, \widehat{\Phi}_\epsilon)$ of the form:

$$\widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f} = (4\pi^2\xi'^2 + \theta^2)\widehat{f} - \chi\left(|\xi'| < \epsilon^{-1}\right)B_{\rm eff}\int_{\mathbb{R}} \widehat{f}(\zeta')d\zeta' = \chi\left(|\xi'| < \epsilon^{-1}\right)\big(\widetilde{\mathcal{R}}[\theta]\widehat{f}\big)(\xi'); \tag{4.79}$$

(see (4.76)) where Φ_ϵ is the rescaled near-energy component of ψ_ϵ. We now apply Lemma 4.3.2 to obtain a solution of (4.79). We then construct the solution $(E^\epsilon, \psi^\epsilon)$ of the full eigenvalue problem (4.78). This will conclude the proof of Theorem 4.2.1.

We apply Lemma 4.3.2 to equation (4.79) with $A = 1$ and $B = B_{\rm eff} > 0$, and $R_\epsilon = \widetilde{\mathcal{R}}$. By Proposition 4.5.4, R_ϵ satisfies assumption (4.44) with $\beta = 1$ and $\alpha = \sigma = \min\{1, \sigma_{\rm eff}\}$. Following the steps of its proof, and using

$$\left|\frac{1}{4\pi^2\zeta^2 + \epsilon^4\theta_1^2} - \frac{1}{4\pi^2\zeta^2 + \epsilon^4\theta_2^2}\right| = \frac{\epsilon^4|\theta_1^2 - \theta_2^2|}{(4\pi^2\zeta^2 + \epsilon^4\theta_1^2)(4\pi^2\zeta^2 + \epsilon^4\theta_1^2)} \le \frac{|\theta_1^2 - \theta_2^2|}{\theta_2^2}\frac{1}{4\pi^2\zeta^2 + \epsilon^4\theta_1^2},$$

one easily checks that assumption (4.45) also holds.

By Lemma 4.3.2 there exists a solution $(\theta_\epsilon^2, \widehat{\Phi}_\epsilon)$ of (4.79), satisfying

$$\big\|\widehat{\Phi}_\epsilon - \widehat{f}_{0,\epsilon}\big\|_{L^1} \ \le\ C\ \epsilon^\sigma \qquad \text{and} \qquad |\theta_\epsilon^2 - \theta_{0,\epsilon}^2| \ \le\ C\ \epsilon^\sigma\ . \tag{4.80}$$

Here $(\theta_{0,\epsilon}^2, \widehat{f}_{0,\epsilon})$ is the unique (normalized) solution of the homogeneous equation

$$\widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f} = (4\pi^2\xi'^2 + \theta^2)\widehat{f} - \chi\left(|\xi'| < \epsilon^{-1}\right)B_{\rm eff}\int_{\mathbb{R}} \chi\left(|\zeta'| < \epsilon^{-1}\right)\widehat{f}(\zeta')d\zeta' = 0,$$

as described in Lemma 4.3.1. Specifically,

$$\widehat{f}_{0,\epsilon}(\xi) = \frac{\chi(|\xi| < \epsilon^{-1})}{4\pi^2\xi^2 + \theta_{0,\epsilon}^2}, \quad \text{and} \quad \theta_{0,\epsilon}^2 = \frac{B_{\text{eff}}^2}{4} + \mathcal{O}(\epsilon). \tag{4.81}$$

We next construct the eigenpair solution $(E^\epsilon, \psi^\epsilon)$ of the Schrödinger equation (4.23). Define, using Proposition 4.5.2,

$$\psi^\epsilon = \psi^\epsilon_{\text{near}} + \psi^\epsilon_{\text{far}}, \qquad E^\epsilon = -\epsilon^4\theta_\epsilon^2,$$
$$\text{where} \;\; \widehat{\psi}^\epsilon_{\text{near}}(\xi) = \frac{1}{\epsilon^2}\widehat{\Phi}_\epsilon\left(\frac{\xi}{\epsilon^2}\right), \quad \text{and} \quad \widehat{\psi}^\epsilon_{\text{far}}(\xi) = \widehat{\psi}_{\text{far}}[\widehat{\psi}^\epsilon_{\text{near}}, E^\epsilon; \epsilon](\xi).$$

Then $(E^\epsilon, \psi^\epsilon)$ is a solution of the eigenvalue problem (4.78). Indeed, the steps proceeding from (4.78) to (4.79) are reversible solutions of $\psi^\epsilon \in \mathcal{Z} = \{f \in C(\mathbb{R}),\ \widehat{f} \in L^1(\mathbb{R})\}$ of (4.78), respectively, solutions $\Phi_\epsilon \in \mathcal{Z}$ of (4.79).

We now prove the estimates (4.24) and (4.25). Estimate (4.24), the small ϵ expansion of the eigenvalue E^ϵ, follows from (4.80), (4.81) and the triangle inequality. Specifically, since we defined $E^\epsilon = -\epsilon^4\theta_\epsilon^2$, we have

$$\left|E^\epsilon + \epsilon^4\frac{B_{\text{eff}}^2}{4}\right| = \epsilon^4\left|\theta_\epsilon^2 - \frac{B_{\text{eff}}^2}{4}\right| \le \epsilon^4\left(\left|\theta_\epsilon^2 - \theta_{0,\epsilon}^2\right| + \left|\theta_{0,\epsilon}^2 - \frac{B_{\text{eff}}^2}{4}\right|\right) \le C\ \epsilon^{4+\sigma}.$$

The approximation, (4.25), of the corresponding eigenstate, $\psi^\epsilon = \psi_{\text{near}} + \psi_{\text{far}}$, is obtained as follows. One has, by triangular inequality,

$$\sup_{x\in\mathbb{R}}\left|\psi^\epsilon(x) - \frac{2}{B_{\text{eff}}}\exp\left(-\epsilon^2\frac{B_{\text{eff}}}{2}|x|\right)\right| \le \sup_{x\in\mathbb{R}}\left|\psi^\epsilon_{\text{near}}(x) - \frac{2}{B_{\text{eff}}}\exp\left(-\epsilon^2\frac{B_{\text{eff}}}{2}|x|\right)\right| + \left\|\psi^\epsilon_{\text{far}}\right\|_{L^\infty}. \tag{4.82}$$

We will look at the bounds in (4.82) separately.

Recall,

$$\begin{aligned}
\psi_{\text{near}}(x) &= \int_{\mathbb{R}} \widehat{\psi}^\epsilon_{\text{near}}(\xi)e^{2\pi i x\xi}\ d\xi = \int_{\mathbb{R}} \frac{1}{\epsilon^2}\widehat{\Phi}_\epsilon\left(\frac{\xi}{\epsilon^2}\right)e^{2\pi i x\xi}\ d\xi \\
&= \int_{\mathbb{R}} \widehat{\Phi}_\epsilon(\eta)e^{2\pi i\epsilon^2 x\eta}\ d\eta \\
&= \int_{\mathbb{R}} \widehat{f}_{0,\epsilon}(\eta)e^{2\pi i\epsilon^2 x\eta}\ d\eta + \int_{\mathbb{R}}\left(\widehat{\Phi}_\epsilon(\eta) - \widehat{f}_{0,\epsilon}(\eta)\right)e^{2\pi i\epsilon^2 x\eta}\ d\eta \\
&= I_1(x) + I_2(x).
\end{aligned}$$

For $A = 1$ and $B = B_{\text{eff}} > 0$, one has from estimate (4.41) in Lemma 4.3.2,

$$I_1(x) = \mathcal{F}^{-1}\left\{\widehat{f}_{0,\epsilon}\right\}(\epsilon^2 x) = \frac{2}{B_{\text{eff}}}\exp\left(-\epsilon^2\frac{B_{\text{eff}}}{2}|x|\right) + \mathcal{O}(\epsilon). \tag{4.83}$$

Using the first bound in (4.80), one has

$$\left\|I_2\right\|_{L^\infty} = \sup_{x\in\mathbb{R}} \left| \int_{\mathbb{R}} \left(\widehat{\Phi}_\epsilon(\eta) - \widehat{f}_{0,\epsilon}(\eta) \right) e^{2\pi i \epsilon^2 x\eta}\, d\eta \right| \leq \left\| \widehat{\Phi}_\epsilon - \widehat{f}_{0,\epsilon} \right\|_{L^1} \leq C\, \epsilon^\sigma. \tag{4.84}$$

From estimate (4.83)-(4.84), we can write

$$\sup_{x\in\mathbb{R}} \left| \psi_{\text{near}}(x) - \frac{2}{B_{\text{eff}}} \exp\left(-\epsilon^2 \frac{B_{\text{eff}}}{2}|x| \right) \right| \lesssim \epsilon^\sigma. \tag{4.85}$$

To bound the second norm in (4.82), we note that from Proposition 4.5.2 with $N = 4$ and $r = 1$, one has

$$\left\| \psi^\epsilon_{\text{far}} \right\|_{L^1} \ \leq \ \left\| \widehat{\psi}^\epsilon_{\text{far}}[\widehat{\psi}^\epsilon_{\text{near}}, E^\epsilon; \epsilon] \right\|_{L^1} \ \lesssim \ \epsilon \ \left\| \widehat{\psi}^\epsilon_{\text{near}} \right\|_{L^1} \lesssim \epsilon, \tag{4.86}$$

since $\left\|\widehat{\psi}^\epsilon_{\text{near}}\right\|_{L^1} = \left\|\widehat{\Phi}_\epsilon\right\|_{L^1} \to \left\|\widehat{f}_{0,\epsilon}\right\|_{L^1} = \mathcal{O}(1)$ (as $\epsilon \to 0$).

Since ψ^ϵ is a unique solution of (4.78) up to a multiplicative constant, we can conclude from (4.82) and the estimates (4.85)-(4.86), that

$$\sup_{x\in\mathbb{R}} \left| \psi^\epsilon(x) - \exp\left(-\epsilon^2 \frac{B_{\text{eff}}}{2}|x| \right) \right| \leq C\, \epsilon^\sigma, \qquad \sigma = \min\{1, \sigma_{\text{eff}}\}.$$

This completes the proof of Theorem 4.2.1.

Remark 4.5.5. *Note that above we conclude that $\psi^\epsilon, \Phi_\epsilon \in \mathcal{Z} = \{f \in C(\mathbb{R}), \widehat{f} \in L^1(\mathbb{R})\}$ while we are in fact studying the eigenvalue problem (4.78) with $\psi^\epsilon \in L^2(\mathbb{R})$. Notice that, by definition, $\widehat{\psi}^\epsilon$ is solution to*

$$(4\pi^2\xi^2 - E^\epsilon)\widehat{\psi}^\epsilon(\xi) + \int_\zeta \widehat{q}_\epsilon(\xi - \zeta)\widehat{\psi}^\epsilon(\zeta)d\zeta = 0, \quad \widehat{\psi}^\epsilon \in L^1(\mathbb{R})$$

and therefore satisfies the following inequality (recall $E^\epsilon < 0$):

$$|\widehat{\psi}^\epsilon(\xi)| \leq \frac{1}{4\pi^2\xi^2 - E^\epsilon} \left\|\widehat{q}_\epsilon\right\|_{L^\infty} \left\|\widehat{\psi}^\epsilon\right\|_{L^1}.$$

One deduces immediately $\psi^\epsilon \in L^2(\mathbb{R})$.

4.6 Proof of Th'm 4.2.3; Edge bifurcations for $-\partial_x^2 + Q(x) + q_\epsilon(x)$

We now prove Theorem 4.2.3 concerning solutions of the eigenvalue problem

$$\left(-\partial_x^2 + Q(x)\right)\psi(x) + q_\epsilon(x)\psi(x) = E\psi(x), \quad \psi \in L^2(\mathbb{R}). \tag{4.87}$$

Here Q is one-periodic and satisfies Hypothesis (HQ), *i.e.* assumption (4.27); and q_ϵ is localized at high frequencies, and decaying as $|x| \to \infty$ in the sense of Hypothesis (H1'a-b), *i.e.* assumptions (4.27) and (4.28), and satisfies additionally Hypothesis (H2'), *i.e.* assumption (4.29). Without loss of generality, we assume thereafter $C_0 \leq \cdots \leq C_6$ and $\mathcal{C}_1 \leq \cdots \leq \mathcal{C}_6$.

Following the analysis of Section 4.5, we divide the problem into a coupled system for a "far-energy" component and a "near-energy" component (here, "near" refers to E being close to $E_{b_*}(k_*)$ a lowermost endpoint of a spectral band of $-\partial_x^2 + Q(x)$ bordering a gap. See our discussion of the strategy in Section 4.4.

In order to spectrally localize we use the Gelfand-Bloch transform, introduced in Section B.2. For fixed $k_* \in \{0, 1/2\}$ and $b_* \in \mathbb{N}$, we define

$$\psi = \psi_{\rm near} + \psi_{\rm far} = \mathcal{T}^{-1}\left\{\widetilde{\psi}_{\rm near}(k) p_{b_*}(x;k)\right\} + \mathcal{T}^{-1}\left\{\sum_{b=0}^{\infty} \widetilde{\psi}_{{\rm far},b}(k) p_b(x;k)\right\}, \tag{4.88}$$

with

$$\widetilde{\psi}_{\rm near}(k) \equiv \chi\left(|k-k_*| < \epsilon^r\right) \mathcal{T}_{b_*}\{\psi\}(k) = \chi\left(|k-k_*| < \epsilon^r\right) \left\langle p_{b_*}(x,k), \widetilde{\psi}(x,k)\right\rangle_{L^2_{\rm per}([0,1]_x)},$$
$$\widetilde{\psi}_{{\rm far},b}(k) \equiv \chi\left(|k-k_*| \geq \epsilon^r \delta_{b_*,b}\right) \mathcal{T}_{b_*}\{\psi\}(k) = \chi\left(|k-k_*| \geq \epsilon^r \delta_{b_*,b}\right) \left\langle p_b(x,k), \widetilde{\psi}(x,k)\right\rangle_{L^2_{\rm per}([0,1]_x)},$$

and where $\delta_{i,j}$ denotes Kronecker's delta function. Equivalently, one has

$$\psi(x) \ = \ \int_{-1/2}^{1/2} \left(\widetilde{\psi}_{\rm near}(k) u_{b_*}(x;k) + \sum_{b=0}^{\infty} \widetilde{\psi}_{{\rm far},b}(k) u_b(x;k)\right) \, dk.$$

In Section 4.6.1 we introduce the coupled system of equations, equivalent to (4.87), in terms of $\psi_{\rm far}$ and $\psi_{\rm near}$. In Sections 4.6.2 and 4.6.3 we analyse the far and near energy components, respectively, in more detail. Finally, in Section 4.6.4 we complete the proof of Theorem 4.2.3.

For clarity of presentation and without any loss of generality, we assume henceforth that we are localizing near the lowermost endpoint of the $(b_*)^{\rm th}$ band and that $k_* = 0$. Thus, by Lemma B.1.2,

$$b_* \ \ \textit{is even}, \quad \textit{thus} \ \ k_* = 0, \quad \textit{and} \ \ E_{b_*}(0) \equiv E_\star.$$

N.B. For $k_* = 0$, note that $p_b(x;k_*) = u_b(x;k_*)$ and we use these expressions interchangeably. For $k_* = 1/2$ one has to distinguish between $p_b(x;k_*)$ and $u_b(x;k_*)$.

4.6.1 Near and far energy components

We first take the Gelfand-Bloch transform of (4.87). We obtain

$$-\left(\partial_x+2\pi ik\right)^2\widetilde{\psi}(x;k)+Q(x)\widetilde{\psi}(x;k)+\left(q_\epsilon\psi\right)^{\sim}(x;k)=E\widetilde{\psi}(x;k). \tag{4.89}$$

Recall that $\{p_b(x;k)\}_{b\geq 0}$ form a complete orthonormal set in $L^2_{\rm per}([0,1]_x)$, and satisfy

$$\Big(-\left(\partial_x+2\pi ik\right)^2+Q(x)\Big)\,p_b(x;k)=E_b(k)p_b(x;k),\quad p_b(x+1;k)\;=\;p_b(x;k),\;x\in\mathbb{R}\;. \tag{4.90}$$

Taking the inner product of (4.89) with $p_b(x;k)$, and using self-adjointness of $-\left(\partial_x+2\pi ik\right)^2+Q$ and (4.90), it follows $\psi\in L^2(\mathbb{R})$ satisfies (4.87) if and only if

$$\left(E_b(k)-E\right)\left\langle p_b(x,k),\widetilde{\psi}(x,k)\right\rangle_{L^2_{\rm per}([0,1]_x)}+\left\langle p_b(x,k),\left(q_\epsilon\psi\right)^{\sim}(x,k)\right\rangle_{L^2_{\rm per}([0,1]_x)}=0 \tag{4.91}$$

for all $b\in\mathbb{N}$ and $k\in(-1/2,1/2]$. Equivalently, using notation (B.36),

$$\left(E_b(k)-E\right)\mathcal{T}_b\left\{\psi\right\}(k)+\mathcal{T}_b\left\{q_\epsilon\psi\right\}(k)=0,\quad\forall b\in\mathbb{N},k\in(-1/2,1/2]. \tag{4.92}$$

We now decompose (4.92) into near- and far-energy equations relative to the band edge $E_{b_*}(k_*)$. In the notation introduced in (4.88):

$$\left(E_{b_*}(k)-E\right)\widetilde{\psi}_{\rm near}(k)+\chi\left(|k|<\epsilon^r\right)\left(\mathcal{T}_{b_*}\left\{q_\epsilon\psi_{\rm near}\right\}(k)+\mathcal{T}_{b_*}\left\{q_\epsilon\psi_{\rm far}\right\}(k)\right)=0, \tag{4.93}$$

$$\left(E_b(k)-E\right)\widetilde{\psi}_{{\rm far},b}(k)+\chi\left(|k|\geq\epsilon^r\delta_{b_*,b}\right)\left(\mathcal{T}_b\left\{q_\epsilon\psi_{\rm near}\right\}(k)+\mathcal{T}_b\left\{q_\epsilon\psi_{\rm far}\right\}(k)\right)=0. \tag{4.94}$$

Equations (4.93) and (4.94) are, for the case of non-trivial periodic potentials, $Q(x)$, the analogue of (4.52)-(4.53).

4.6.2 Analysis of the far energy components

We view the system for $\{\widetilde{\psi}_{{\rm far},b}(k)\}_{b\geq 0}$ as depending on "parameters" $(\psi_{\rm near},E;\epsilon)$ and construct the mapping $(\psi_{\rm near},E;\epsilon)\mapsto\psi_{\rm far}[\psi_{\rm near},E;\epsilon]$ in the following proposition.

Proposition 4.6.1. *Assume b_* is even and consider $E_\star=E_{b_*}(0)$ the lowermost edge of the $(b_*)^{th}$ band and at the boundary of an open gap in the spectrum. Let $E<E_\star$ vary over a subset of the gap which is uniformly bounded away from the $(b_*-1)^{\rm st}$ band (note: E may be arbitrarily close to*

$E_\star$). Assume $q_\epsilon \in L^2 \cap L^\infty$ is bounded and localized at high frequencies in the sense of (4.27),(4.28) *with $\beta \geq 2$. Let $\psi_{near} = \mathcal{T}^{-1}\left\{\widetilde{\psi}_{near}(k) p_{b_*}(x;k)\right\}$ with*

$$\widetilde{\psi}_{near}(k) = \chi\left(|k| < \epsilon^r\right) \widetilde{\psi}_{near}(k) \in L^2\left(\left(-\frac{1}{2\epsilon}, \frac{1}{2\epsilon}\;\right]\right). \tag{4.95}$$

Then for any $0 < r < 1/2$, there exists $\epsilon_0 > 0$, such that for $0 < \epsilon < \epsilon_0$, the following holds.

There is a unique solution $\left\{\widetilde{\psi}_{far,b}(k)\right\}_{b\geq 0}$, and $\psi_{far} = \mathcal{T}^{-1}\left\{\sum_{b\geq 0} \widetilde{\psi}_{far,b}(k) p_b(x;k)\right\} \in L^2(\mathbb{R})$ of the far-frequency system (4.94) *. For any E, ϵ as above, the mapping $\widetilde{\psi}_{near} \mapsto \psi_{far}\left[\psi_{near}, E; \epsilon\right]$ is a linear mapping from $L^2([-\epsilon^r, \epsilon^r])$ to $L^2(\mathbb{R})$, and satisfies the bound*

$$\left\|\psi_{far}\left[\psi_{near}, E; \epsilon\right]\right\|_{L^2} \leq C(\mathcal{C}_0, \mathcal{C}_\beta, b_*)\epsilon^{2-2r}\left\|\widetilde{\psi}_{near}\right\|_{L^2}. \tag{4.96}$$

Moreover, for any $s \in (\frac{1}{2}, \frac{3}{2})$, and for ϵ sufficiently small, one has $\psi_{far}\big[\psi_{near}, E; \epsilon\big] \in H^s(\mathbb{R})$ and

$$\left\|\psi_{far}\left[\psi_{near}, E; \epsilon\right]\right\|_{H^s} \leq C(\mathcal{C}_0, \mathcal{C}_\beta, b_*, s)\epsilon^{2-\max\{2r,s\}}\left\|\widetilde{\psi}_{near}\right\|_{L^2}. \tag{4.97}$$

Proof. We begin by showing that there is a constant $0 < C_1 < \infty$, independent of ϵ, such that

$$|E_{b_*}(k) - E| \geq C_1 \epsilon^{2r}, \quad \epsilon^r \leq |k| \leq 1/2, \tag{4.98}$$

$$|E_b(k) - E| \geq C_1, \qquad b \neq b_*, \quad |k| \leq 1/2. \tag{4.99}$$

Note first that (4.99) is an immediate consequence of the assumption on E. To prove (4.98) recall, by Lemma B.1.2 that $E_\star = E_{b_*}(0)$, an eigenvalue at the edge of a spectral gap, is simple, and $k \to E_{b_*}(k) - E_\star$ is continuous. Therefore, for any k_1, such that $0 < k_1 \leq 1/2$,

$$\min_{k_1 \leq |k| \leq 1/2} |E_{b_*}(k) - E_\star| \geq C(k_1) > 0. \tag{4.100}$$

For $|k| \leq k_1$, we approximate $E_{b_*}(k)$ by Taylor expansion. In particular, since $E_{b_*}(k)$ is smooth for k near $k_* = 0$, $\partial_k E_{b_*}(0) = 0$ and $\partial_k^2 E_{b_*}(0) \neq 0$, we have $|E_{b_*}(k) - E_{b_*}(0) - \frac{1}{2}\partial_k^2 E_{b_*}(0) k^2| \leq \mathcal{C}|k|^3$. Therefore, we can choose $0 < k_1 \leq \frac{1}{6\mathcal{C}}|\partial_k^2 E_{b_*}(0)|$, so that for all $\epsilon^r \leq k_1$ we have

$$\min_{\epsilon^r \leq |k| \leq k_1} |E_{b_*}(k) - E_\star| \geq \frac{1}{3}\left|\partial_k^2 E_{b_*}(0)\right| \epsilon^{2r}. \tag{4.101}$$

Finally, notice that since $E < E_\star$, and $E_\star$ is the lowermost edge of the $(b_*)^{th}$ band, we have $|E_b(k) - E| \geq |E_b(k) - E_\star|$, and therefore (4.99) follows from (4.100) and (4.101).

Thanks to the above, we can rewrite the far-frequency system, (4.94), as

$$\widetilde{\psi}_{\text{far},b}(k) + \frac{\chi\left(|k| \geq \epsilon^r \delta_{b_*,b}\right)}{E_b(k) - E} \mathcal{T}_b \left\{ q_\epsilon \left(\psi_{\text{near}} + \psi_{\text{far}} \right) \right\}(k) = 0, \quad b \geq 0, k \in (-1/2, 1/2]. \tag{4.102}$$

Multiplying (4.102) by $u_b(x;k) = p_b(x;k)e^{2\pi ikx}$, summing over $b \geq 0$ and integrating with respect to $k \in (-1/2, 1/2]$ yields (by (B.37))

$$\left(Id + \mathcal{K}_\epsilon\right) \psi_{\text{far}}(x) = -\left(\mathcal{K}_\epsilon \psi_{\text{near}}\right)(x), \tag{4.103}$$

where we define

$$\left(\mathcal{K}_\epsilon g\right)(x) \equiv \int_{-1/2}^{1/2} \sum_{b \geq 0} \frac{\chi\left(|k| \geq \epsilon^r \delta_{b_*,b}\right)}{E_b(k) - E} \mathcal{T}_b \left\{ q_\epsilon g \right\}(k) p_b(x;k) e^{2\pi ikx} \, dk. \tag{4.104}$$

Thus we need to solve equation (4.103). As in Proposition 4.5.2, it is not clear that $(Id + \mathcal{K}_\epsilon)$ is invertible. However, by bound (4.111) (with $s = 0$) of Lemma 4.6.2, stated and proved just below, one has that for $0 < r < 1/2$, one can chose ϵ small enough so that the operator norm $\left\| \mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon \right\|_{L^2 \to L^2} \leq 1/2$. Therefore, $(Id - \mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon)$ (as an operator from L^2 to L^2) is invertible.

The solution to (4.103) is therefore uniquely defined as

$$\psi_{\text{far}}(x) = -\left(Id - \mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon\right)^{-1} \left(Id - \mathcal{K}_\epsilon\right) \left(\mathcal{K}_\epsilon \psi_{\text{near}}\right)(x). \tag{4.105}$$

Indeed, it is clear that, if it exists, ψ_{far} satisfying (4.103) is uniquely defined by (4.105) (after multiplying the equation by $(Id - \mathcal{K}_\epsilon)$). Conversely, when multiplying (4.105) by $(Id + \mathcal{K}_\epsilon)$, and since $(Id + \mathcal{K}_\epsilon)$ and $(Id - \mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon)^{-1}$ commute, then ψ_{far}, as defined by (4.105), solves (4.103).

Thus ψ_{far} is uniquely defined from $\psi_{\text{near}} \in L^2$ (or, equivalently, $\widetilde{\psi}_{\text{near}}(k) \in L^2$; see (4.109) below). $\widetilde{\psi}_{\text{far},b} = \mathcal{T}_b\{\psi_{\text{far}}\}$ is then easily seen to satisfy (4.94), by (4.113). This concludes the first part of the proposition.

We now turn to estimates (4.96)-(4.97). By bound (4.111) of Lemma 4.6.2, for any $s \in \{0\} \cup (\frac{1}{2}, \frac{3}{2})$, one can choose $0 < \epsilon < \epsilon_0$ small enough so that $\left\| \mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon \right\|_{H^s \to H^s} \leq 1/2$, and therefore

$$\left\| \left(Id - \mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon\right)^{-1} \right\|_{H^s \to H^s} \leq 2. \tag{4.106}$$

Moreover, by estimates (4.110) and (4.112) of Lemma 4.6.2, one has for any $0 \leq s < \frac{3}{2}$,

$$\left\| \mathcal{K}_\epsilon \psi_{\text{near}} \right\|_{H^s}^2 \leq \mathcal{C} \left(\epsilon^{4-2s} \left\| \psi_{\text{near}} \right\|_{H^0}^2 + \epsilon^{4-4r} \left\| \psi_{\text{near}} \right\|_{H^2}^2 \right), \tag{4.107}$$

$$\left\| (\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon) \psi_{\text{near}} \right\|_{H^s}^2 \leq \mathcal{C} \left(\epsilon^{4-4r} \left\| \psi_{\text{near}} \right\|_{H^0}^2 + \epsilon^{8-2s-4r} \left\| \psi_{\text{near}} \right\|_{H^2}^2 \right). \tag{4.108}$$

Finally, we remark that by definition (4.95) and Proposition B.2.1, one has for any $s \geq 0$

$$\left\|\psi_{\rm near}\right\|_{H^s}^2 \approx \int_{-1/2}^{1/2} (1+|b_*|^2)^s |\widetilde{\psi}_{\rm near}(k)|^2 \; dk \approx \left\|\widetilde{\psi}_{\rm near}\right\|_{L^2}^2. \tag{4.109}$$

It is now straightforward to obtain (4.96)-(4.97), applying the estimates (4.106)–(4.108) and (4.109) to (4.105). This completes the proof of Proposition 4.6.1. □

To complete this argument we now prove:

Lemma 4.6.2. *Let $Q, q_\epsilon, r, \epsilon$ and E be as in Proposition 4.6.1. Then, for $0 \leq s < \frac{3}{2}$, the operator $\mathcal{K}_\epsilon : H^s(\mathbb{R}) \to H^s(\mathbb{R})$, defined by*

$$\left(\mathcal{K}_\epsilon g\right)(x) \equiv \int_{-1/2}^{1/2} \sum_{b\geq 0} \frac{\chi\left(|k| \geq \epsilon^r \delta_{b_*,b}\right)}{E_b(k) - E} \mathcal{T}_b\left\{q_\epsilon g\right\}(k) p_b(x;k) e^{2\pi i k x} \; dk$$

satisfies the bounds,

$$\left\|\mathcal{K}_\epsilon g\right\|_{H^s}^2 \leq \mathcal{C}\, \left(\epsilon^{4-2s} \left\|g\right\|_{H^0}^2 + \epsilon^{4-4r}\left\|g\right\|_{H^2}^2\right), \qquad 0 \leq s < \frac{3}{2} \tag{4.110}$$

$$\left\|(\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon) g\right\|_{H^s}^2 \leq \mathcal{C}\; \epsilon^{4-8r} \left\|g\right\|_{H^s}^2 \qquad s = 0 \text{ or } \frac{1}{2} < s < \frac{3}{2} \tag{4.111}$$

$$\left\|(\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon) g\right\|_{H^s}^2 \leq \mathcal{C}\, \left(\epsilon^{4-4r} \left\|g\right\|_{H^0}^2 + \epsilon^{8-2s-4r}\left\|g\right\|_{H^2}^2\right), \qquad 0 \leq s < \frac{3}{2} \tag{4.112}$$

with $\mathcal{C} = C(\mathcal{C}_0, \mathcal{C}_\beta, b_*)$ a constant.

Proof. In this proof, we will make repeated use of Proposition B.2.1. We first note that we can write, by (B.36) and since $\{p_b(x;k)\}_{b\geq 0}$ is orthonormal in $L^2([0,1])$ for each fixed $k \in (-1/2, 1/2]$,

$$\mathcal{T}_b\left(\mathcal{K}_\epsilon g\right)(k) = \frac{\chi\left(|k| \geq \epsilon^r \delta_{b_*,b}\right)}{E_b(k) - E} \mathcal{T}_b\left\{q_\epsilon g\right\}(k). \tag{4.113}$$

Therefore,

$$\begin{aligned}\left\|\widetilde{\mathcal{K}_\epsilon g}\right\|_{\mathcal{X}^s}^2 &= \int_{-1/2}^{1/2} \sum_{b\geq 0} (1+b^2)^s \left|\mathcal{T}_b\left\{\mathcal{K}_\epsilon g\right\}(k)\right|^2 \; dk \\ &= \int_{-1/2}^{1/2} \sum_{b\geq 0} (1+b^2)^s \frac{\chi(|k| \geq \epsilon^r \delta_{b_*,b})}{|E_b(k) - E|^2} \left|\mathcal{T}_b\left\{q_\epsilon g\right\}(k)\right|^2 \; dk.\end{aligned}$$

Using bounds (4.98) and (4.99), as well as Weyl's asymptotics (Lemma B.1.4), we have

$$\left\|\widetilde{\mathcal{K}_\epsilon g}\right\|_{\mathcal{X}^s}^2 \lesssim \sum_{b\geq 0} \int_{-1/2}^{1/2} \frac{1}{(1+b^2)^{2-s}} \left|\mathcal{T}_b\left\{q_\epsilon g\right\}(k)\right|^2 \; dk + \epsilon^{-4r} \int_{-1/2}^{1/2} \left|\mathcal{T}_{b_*}\left\{q_\epsilon g\right\}(k)\right|^2 \; dk, \tag{4.114}$$

which will be used several times in the proof.

First we estimate the right hand side of (4.114) as

$$\big\|\widetilde{\mathcal{K}_\epsilon g}\big\|_{\mathcal{X}^s}^2 \lesssim \epsilon^{-4r}\big\|\widetilde{q_\epsilon g}\big\|_{\mathcal{X}^{-(2-s)}}^2, \tag{4.115}$$

and use (4.163) in Lemma 4.A.1 with $\delta = 2-s > \frac{1}{2}$ to obtain

$$\big\|\mathcal{K}_\epsilon g\big\|_{H^s} \leq \mathcal{C}\ \epsilon^{2-s-2r}\big\|g\big\|_{H^{2-s}}, \quad 0 \leq s < \frac{3}{2}. \tag{4.116}$$

In order to obtain (4.111), we iterate (4.116), and deduce

$$\big\|\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon g\big\|_{H^s} \leq \mathcal{C}\ \epsilon^{2-s-2r}\big\|\mathcal{K}_\epsilon g\big\|_{H^{2-s}} \leq \mathcal{C}\ \epsilon^{2-4r}\big\|g\big\|_{H^s}, \quad \frac{1}{2} \leq s < \frac{3}{2}. \tag{4.117}$$

As for the case $s=0$, we use first (4.116), then (4.115)

$$\big\|\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon g\big\|_{L^2} \leq \mathcal{C}\ \epsilon^{2-2r}\big\|\mathcal{K}_\epsilon g\big\|_{H^2} \leq \mathcal{C}\ \epsilon^{2-4r}\big\|\widetilde{q_\epsilon g}\big\|_{\mathcal{X}^0}^2 \leq \mathcal{C}\ \epsilon^{2-4r}\big\|g\big\|_{L^2}^2, \tag{4.118}$$

and finally $\big\|\widetilde{q_\epsilon g}\big\|_{\mathcal{X}^0} \approx \big\|q_\epsilon g\big\|_{L^2} \leq \big\|q_\epsilon\big\|_{L^\infty}\big\|g\big\|_{L^2}$ yields the desired result.

Let us now turn to estimates (4.110) and (4.112). First we estimate the right hand side of (4.114) as

$$\big\|\widetilde{\mathcal{K}_\epsilon g}\big\|_{\mathcal{X}^s}^2 \lesssim \big\|\widetilde{q_\epsilon g}\big\|_{\mathcal{X}^{-(2-s)}}^2 + \epsilon^{-4r}\big\|\widetilde{q_\epsilon g}\big\|_{\mathcal{X}^{-2}}^2. \tag{4.119}$$

Using (4.164) in Lemma 4.A.1 on both terms of the right-hand side with (respectively) $\delta = 2-s > \frac{1}{2}$ and $\delta = 2$ yields

$$\big\|\mathcal{K}_\epsilon g\big\|_{H^s}^2 \leq \mathcal{C}\ \left(\epsilon^{2(2-s)}\big\|g\big\|_{L^2}^2 + \epsilon^{4-4r}\big\|g\big\|_{H^2}^2\right), \quad 0 \leq s < \frac{3}{2}. \tag{4.120}$$

This completes the proof of bound (4.110). Note that in order to estimate (4.119) when $s=2$, we can use $\big\|\widetilde{q_\epsilon g}\big\|_{\mathcal{X}^0} \approx \big\|q_\epsilon g\big\|_{L^2} \leq \big\|q_\epsilon\big\|_{L^\infty}\big\|g\big\|_{L^2}$ and (4.163) in Lemma 4.A.1 with $\delta = 2$ to deduce

$$\big\|\mathcal{K}_\epsilon g\big\|_{H^2}^2 \leq \mathcal{C}\ \left(\big\|g\big\|_{L^2}^2 + \epsilon^{4-4r}\big\|g\big\|_{H^2}^2\right). \tag{4.121}$$

We now turn to the operator $\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon$. We apply the now proven bound (4.110) to get

$$\big\|(\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon)g\big\|_{H^s}^2 \leq \mathcal{C}\ \left(\epsilon^{2(2-s)}\big\|\mathcal{K}_\epsilon g\big\|_{H^0}^2 + \epsilon^{4-4r}\big\|\mathcal{K}_\epsilon g\big\|_{H^2}^2\right), \quad 0 \leq s < \frac{3}{2}. \tag{4.122}$$

Using again (4.120) with $s=0$ to bound $\big\|\mathcal{K}_\epsilon g\big\|_{H^0}$, and (4.121) to bound $\big\|\mathcal{K}_\epsilon g\big\|_{H^2}$, we conclude

$$\big\|(\mathcal{K}_\epsilon \circ \mathcal{K}_\epsilon)g\big\|_{H^s}^2 \leq \mathcal{C}\ \left(\left(\epsilon^{8-2s} + \epsilon^{4-4r}\right)\big\|g\big\|_{H^0}^2 + \epsilon^{8-2s-4r}\big\|g\big\|_{H^2}^2\right). \tag{4.123}$$

This proves bound (4.112), and completes the proof of Lemma 4.6.2. $\square$

4.6.3 Analysis of the near energy components

In this section we study the near equation (4.93), that we recall:

$$\left(E_{b_*}(k) - E\right) \widetilde{\psi}_{\rm near}(k) + \chi\left(|k| < \epsilon^r\right) \left(\mathcal{T}_{b_*}\left\{q_\epsilon \psi_{\rm near}\right\}(k) + \mathcal{T}_{b_*}\left\{q_\epsilon \psi_{\rm far}\right\}(k)\right) = 0,$$

with the aim of extracting its leading order expression. We also make the following ansatz:

$$E = E_\star - \epsilon^4\theta^2, \quad 0 < t_- \le \theta^2 \le t_+ < \infty, \quad E_\star \equiv E_{b_*}(0), \tag{4.124}$$

where t_-^{-1} and t_+ are independent of ϵ. Recall also that, by definition,

$$\psi_{\rm near} + \psi_{\rm far} = \psi \quad \text{with} \quad \forall b \in \mathbb{N}, \quad \left(E_b(k) - E\right) \mathcal{T}_b\left\{\psi\right\}(k) + \mathcal{T}_b\left\{q_\epsilon \psi\right\}(k) = 0. \tag{4.125}$$

Therefore iterating (4.125) using (B.39) in Proposition B.2.3, we can write

$$\begin{aligned}
\mathcal{T}_{b_*}\{q_\epsilon\psi\}(k) &= \int_0^1 \overline{p_{b_*}(x;k)}\widetilde{q_\epsilon\psi}(x;k)\ dx \\
&= \int_0^1 \overline{p_{b_*}(x;k)} \int_{-1/2}^{1/2} \widetilde{q}_\epsilon(x;k-l)\widetilde{\psi}(x;l)\ dl\ dx \\
&= \int_0^1 \overline{p_{b_*}(x;k)} \int_{-1/2}^{1/2} \widetilde{q}_\epsilon(x;k-l)\sum_{a\ge0} p_a(x;l)\mathcal{T}_a\{\psi\}(l)\ dl\ dx \\
&= -\int_0^1 \overline{p_{b_*}(x;k)} \int_{-1/2}^{1/2} \widetilde{q}_\epsilon(x;k-l)\sum_{a\ge0} p_a(x;l)\frac{1}{E_a(l)-E_\star+\epsilon^4\theta^2}\mathcal{T}_a\{q_\epsilon\psi\}(l)\ dl\ dx.
\end{aligned}$$

We then use Fubini's theorem and rewrite the near equation (4.93) as

$$(E_{b_*}(k) - E_\star + \epsilon^4\theta^2)\widetilde{\psi}_{\rm near}(k) + \chi(|k| < \epsilon^r)\left(\mathfrak{I}[\theta]\psi_{\rm near} \ + \ \mathfrak{I}[\theta]\psi_{\rm far}\right)(k) \ = \ 0, \tag{4.126}$$

with the notation

$$\big(\mathfrak{I}[\theta]\psi\big)(k) \ = \ -\int_{-1/2}^{1/2}\sum_{a\ge0}\mathcal{T}_a\{q_\epsilon\psi\}(l)\ \frac{1}{E_a(l)-E_\star+\epsilon^4\theta^2}\ I_{b_*,a}[q_\epsilon](k;l)\ dl, \tag{4.127}$$

where we define for $a \ge 0$, $b \ge 0$:

$$I_{b,a}[q_\epsilon](k;l) \ \equiv \ \int_0^1 \overline{p_b(x;k)}\widetilde{q}_\epsilon(x;k-l)p_a(x;l)\ dx. \tag{4.128}$$

Note that (4.126) is the analogue of (4.66) for the case $Q \equiv 0$.

Proposition 4.6.3. *Let q_ϵ be such that $\widehat{q_\epsilon} \in L^1 \cap L^\infty$ and assume q_ϵ is concentrated at high frequencies in the sense of* (4.28)*. Assume $Q \in W^{N,\infty}$ with N sufficiently large, so that* (4.26) *applies. Then for ϵ sufficiently small we can write the near energy equation* (4.126) *as*

$$(E_{b_*}(k) - E_\star + \epsilon^4\theta^2)\widetilde{\psi}_{near}(k) - \chi(|k| < \epsilon^r)\epsilon^2 B_{b_*,\text{eff}} \times \int_{-1/2}^{1/2} \widetilde{\psi}_{near}(s)\ ds$$
$$= \chi(|k| < \epsilon^r)\big(R[\theta]\widetilde{\psi}_{near}\big)(k), \quad (4.129)$$

where $B_{b_*,\text{eff}}$ is as defined in Hypothesis (H2'), equation (4.29)*, and $R[\theta]\psi_{near}$, defined in* (4.131)*, satisfies the bound*

$$\big\|R[\theta]\widetilde{\psi}_{near}\big\|_{L^\infty} \le \mathcal{C}\epsilon^{2+\sigma_{near}}\big\|\widetilde{\psi}_{near}\big\|_{L^1} + \mathcal{C}\epsilon^{3+\sigma_{far}}\big\|\widetilde{\psi}_{near}\big\|_{L^2}. \quad (4.130)$$

Here, $\sigma_{near} = \min\{1/2, r, \sigma_{\text{eff}}\}$, with σ_{eff} defined in (4.29)*, $\sigma_{far} = 1/2 - 2r$, and $\mathcal{C} = C(C_6, \mathcal{C}_6, \mathcal{C}_0, b_*)$ is a constant.*

Proof. Adding and subtracting the anticipated dominant contribution to $\mathfrak{I}[\theta]\psi_{\text{near}}$, we may rewrite the near-energy equation (4.126) as

$$(E_{b_*}(k) - E_\star + \epsilon^4\theta^2)\widetilde{\psi}_{\text{near}}(k) - \chi(|k| < \epsilon^r)\epsilon^2 B_{b_*,\text{eff}} \times \int_{-1/2}^{1/2} \widetilde{\psi}_{\text{near}}(s)\ ds$$
$$= -\chi(|k| < \epsilon^r)\left[\epsilon^2 B_{b_*,\text{eff}} \times \int_{-1/2}^{1/2} \widetilde{\psi}_{\text{near}}(s)\ ds + \big(\mathfrak{I}[\theta]\psi_{\text{near}}\big)(k)\right] - \chi(|k| < \epsilon^r)\big(\mathfrak{I}[\theta]\psi_{\text{far}}\big)(k)$$
$$\equiv \chi(|k| < \epsilon^r)\big(R_1[\theta]\widetilde{\psi}_{\text{near}}\big)(k) + \chi(|k| < \epsilon^r)\big(R_2[\theta]\widetilde{\psi}_{\text{near}}\big)(k) \equiv \chi(|k| < \epsilon^r)\big(R[\theta]\widetilde{\psi}_{\text{near}}\big)(k) \quad (4.131)$$

The proof of the bound (4.130) is given in Lemmata 4.6.4, 4.6.5, and 4.6.6 below. □

Lemma 4.6.4. *Under the assumptions of Proposition 4.6.3, there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, one has*

$$\big\|\chi(|k| < \epsilon^r)\big(R_2[\theta]\widetilde{\psi}_{near}]\big)(k)\big\|_{L^\infty(\mathbb{R}_k)} = \big\|\chi(|k| < \epsilon^r)\big(\mathfrak{I}[\theta]\psi_{far}\big)(k)\big\|_{L^\infty(\mathbb{R}_k)} \ \le\ \mathcal{C}\epsilon^{2+3/2-2r}\big\|\widetilde{\psi}_{near}\big\|_{L^2},$$

with $\mathcal{C} = C(\mathcal{C}_6, C_6, \mathcal{C}_0, b_*)$.

Proof. Using Cauchy-Schwarz inequality in (4.127), one has

$$\big|\mathfrak{I}[\theta]\psi_{\text{far}}\big|(k) \le \left(\int_{-1/2}^{1/2} \sum_{a\ge 0} \left|\frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l) - E_\star + \epsilon^4\theta^2}\right|^2 dl\right)^{1/2} \times \left(\int_{-1/2}^{1/2} \sum_{a\ge 0} \big|\mathcal{T}_a\{q_\epsilon\ \psi_{\text{far}}\}(l)\big|^2\ dl\right)^{1/2} \quad (4.132)$$

The second factor of (4.132) is estimated as follows, using Proposition 4.6.1 and Hypothesis (H1'a), estimate (4.27),

$$\left(\int_{-1/2}^{1/2}\sum_{a\geq 0}\left|\mathcal{T}_a\{q_\epsilon\ \psi_{\rm far}\}(l)\right|^2\,dl\right)^{1/2}\approx\left\|q_\epsilon\psi_{\rm far}\right\|_{L^2}\leq\left\|q_\epsilon\right\|_{L^\infty}\left\|\psi_{\rm far}\right\|_{L^2}\lesssim\ \mathcal{C}_0\epsilon^{2-2r}\left\|\widetilde{\psi}_{\rm near}\right\|_{L^2}.$$

As for the first term of (4.132), we treat differently the cases $a=b_*$, $a\neq b_*$ and $a\leq a_*^\epsilon$ and $a>a_*^\epsilon$, where $a_*^\epsilon=\max\{a\geq 0 \text{ such that } \sqrt{E_a(k)}<\pi/(3\epsilon)\}$.

Case $a=b_*$. By (4.124), $|E_{b_*}(k)-E_\star+\epsilon^4\theta^2|\geq t_-\epsilon^4$. Together with estimate (4.173) of Lemma 4.A.3 with $a=b_*$, we can bound

$$\int_{-1/2}^{1/2}\left|\frac{I_{b_*,b_*}[q_\epsilon](k;l)}{E_a(l)-E_\star+\epsilon^4\theta^2}\right|^2 dl\leq C\epsilon^{-8}\int_{-1/2}^{1/2}|I_{b_*,b_*}[q_\epsilon](k;l)|^2\,dl\leq C(1+|b_*|^N)^2\epsilon^{2N-8}.$$

Case $0\leq a\leq a_*^\epsilon$, $a\neq b_*$. By Weyl's asymptotics (Lemma B.1.4), one has $\frac{1}{E_a(l)-E}\lesssim 1/(a^2+1)$ for $a\neq b_*$. Therefore, applying estimate (4.174) of Lemma 4.A.3, one has the bound

$$\begin{aligned}\sum_{0\leq a\leq a_*^\epsilon,a\neq b_*}\int_{-1/2}^{1/2}\left|\frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l)-E_\star+\epsilon^4\theta^2}\right|^2 dl&\lesssim\sum_{0\leq a\leq a_*^\epsilon,a\neq b_*}\frac{1}{(a^2+1)^2}\int_{-1/2}^{1/2}|I_{b_*,a}[q_\epsilon](k;l)|^2\,dl\\&\lesssim\epsilon^3(1+b_*^2)^2.\end{aligned}$$

Case $a>a_*^\epsilon$, $a\neq b_*$. In this case, one has $\frac{1}{E_a(l)-E}\lesssim\epsilon^2$, and therefore

$$\sum_{a>a_*}\int_{-1/2}^{1/2}\left|\frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l)-E_\star+\epsilon^4\theta^2}\right|^2 dl\lesssim\epsilon^4\sum_{a>a_*}\int_{-1/2}^{1/2}|I_{b_*,a}[q_\epsilon](k;l)|^2\,dl\lesssim\epsilon^4,$$

where we used estimate (4.170) of Lemma 4.A.2.

Thus

$$\left\|\chi(|k|<\epsilon^r)\big(\mathfrak{I}[\theta]\psi_{\rm far}\big)(k)\right\|_{L^\infty(\mathbb{R}_k)}\lesssim\epsilon^{2-2r}\left(\epsilon^{2N-8}+\epsilon^3\right)^{1/2}\left\|\widetilde{\psi}_{\rm near}\right\|_{L^2}.$$

Choosing $N=11/2$ and defining $\sigma_{\rm far}\equiv 1/2-2r$, we obtain

$$\left|\chi(|k|<\epsilon^r)\big(R_2[\theta]\widetilde{\psi}_{\rm near}\big)(k)\right|=\left|\chi(|k|<\epsilon^r)\big(\mathfrak{I}[\theta]\psi_{\rm far}\big)(k)\right|\leq\mathcal{C}\epsilon^{3+\sigma_{\rm far}}\left\|\widetilde{\psi}_{\rm near}\right\|_{L^2},$$

with $\mathcal{C}=C(\mathcal{C}_6,C_6,\mathcal{C}_0,b_*)$ which completes the proof of Lemma 4.6.4. $\square$

To complete the proof of Proposition 4.6.3, it suffices to bound $\chi(|k|<\epsilon^r)\big(R_1[\theta]\widetilde{\psi}\big)(k)$, defined in (4.131). The asserted bound in (4.130) is a consequence of the following two Lemmata and the triangle inequality.

Lemma 4.6.5. *Under the assumptions of Proposition 4.6.3, one has*

$$\sup_{|k|<\epsilon^r}\left|\big(\Im[\theta]\psi_{near}\big)(k)+\int_{-1/2}^{1/2}ds\ \widetilde{\psi}_{near}(s)\int_{-1/2}^{1/2}dl\sum_{a\geq 0}I_{b_*,a}[Q_\epsilon](k;l)I_{a,b_*}[q_\epsilon](l;s)\right|$$
$$\lesssim \mathcal{C}\ \epsilon^{5/2}\big\|\widetilde{\psi}_{near}\big\|_{L^1}, \quad (4.133)$$

where

$$Q_\epsilon(\xi)\equiv\frac{\widehat{q_\epsilon}(\xi)}{1+4\pi^2|\xi|^2}. \quad (4.134)$$

Here $\mathcal{C}=C(C_4,\mathcal{C}_4,\mathcal{C}_0,b_*)$ is a constant.

Lemma 4.6.6. *Under the assumptions of Proposition 4.6.3, one has*

$$\sup_{|k|<\epsilon^r,|s|<\epsilon^r}\left|\int_{-1/2}^{1/2}dl\sum_{a\geq 0}I_{b_*,a}[Q_\epsilon](k;l)I_{a,b_*}[q_\epsilon](l;s)-\epsilon^2B_{b_*,\mathrm{eff}}\right|$$
$$\leq C(\mathcal{C}_2,\mathcal{C}_0,b_*)\ (\epsilon^{2+r}+\epsilon^{2+\sigma_{\mathrm{eff}}}). \quad (4.135)$$

Here, $B_{b_*,\mathrm{eff}}$ is defined in (4.29) *and $C(\mathcal{C}_2,\mathcal{C}_0,b_*)$ is a constant.*

The proofs of Lemma 4.6.5 and Lemma 4.6.6 appear at the end of this section.

Rescaling the equation. The next step consists in rescaling the equation so as to balance terms on the left hand side of (4.129). We therefore define

$$k=\epsilon^2\kappa,\quad \widetilde{\psi}_{\mathrm{near}}(k)=\frac{1}{\epsilon^2}\widehat{\Phi}\left(\frac{k}{\epsilon^2}\right)=\frac{1}{\epsilon^2}\chi(|\kappa|<\epsilon^{r-2})\widehat{\Phi}(\kappa). \quad (4.136)$$

Note also that one has the following estimates:

$$\big\|\widetilde{\psi}_{\mathrm{near}}\big\|_{L^1}\lesssim\big\|\widehat{\Phi}\big\|_{L^{2,1}}\qquad\text{and}\qquad\big\|\widetilde{\psi}_{\mathrm{near}}\big\|_{L^2}\lesssim\epsilon^{-1}\big\|\widehat{\Phi}\big\|_{L^{2,1}}. \quad (4.137)$$

These follow from the definition $\big\|\widehat{\Phi}\big\|_{L^{2,1}}^2\equiv\int_{-\infty}^{\infty}(1+|\kappa|^2)|\widehat{\Phi}(\kappa)|^2\,d\kappa$, and the following bounds:

$$\big\|\widetilde{\psi}_{\mathrm{near}}\big\|_{L^1}=\int_{-\infty}^{\infty}\left|\chi(|k|<\epsilon^r)\widetilde{\psi}_{\mathrm{near}}(k)\right|\,dk=\int_{-\infty}^{\infty}\left|\chi(|\kappa|<\epsilon^{r-2})\widehat{\Phi}(\kappa)\right|\,d\kappa\leq\big\|\widehat{\Phi}\big\|_{L^{2,1}}$$
$$\big\|\widetilde{\psi}_{\mathrm{near}}\big\|_{L^2}^2=\int_{-\infty}^{\infty}\left|\chi(|k|<\epsilon^r)\widetilde{\psi}_{\mathrm{near}}(k)\right|^2\,dk=\epsilon^{-2}\big\|\widehat{\Phi}\big\|_{L^2}^2\leq\epsilon^{-2}\big\|\widehat{\Phi}\big\|_{L^{2,1}}^2.$$

The next proposition extracts the leading order terms in (4.129), in terms of the variable κ and unknown $\widehat{\Phi}$.

Proposition 4.6.7. *Assume the hypotheses of Proposition 4.6.3 hold. Then, the rescaled near-energy components solve the equation:*

$$\left(\frac{1}{2}\partial_k^2 E_{b_*}(0)\kappa^2+\theta^2\right)\widehat{\Phi}(\kappa)-\chi(|\kappa|<\epsilon^{r-2})B_{b_*,\mathrm{eff}}\times\int_{-\infty}^{\infty}\widehat{\Phi}(\xi)\,d\xi=\big(R_\flat[\theta]\widehat{\Phi}\big)(\kappa), \tag{4.138}$$

where $R_\flat[\theta]\widehat{\Phi}$ satisfies the estimate

$$\big\|R_\flat[\theta]\widehat{\Phi}\big\|_{L^{2,-1}}\leq\mathcal{C}\left(\epsilon^{2r}+\epsilon^{\sigma_{near}}+\epsilon^{\sigma_{far}}\right)\big\|\widehat{\Phi}_\epsilon\big\|_{L^{2,1}}. \tag{4.139}$$

Here $\mathcal{C}=C(\mathcal{C}_6,C_6,\mathcal{C}_0,b_*,\sup_{|k'|<\epsilon^r}|E_{b_*}^{(4)}(k')|)$ is a constant, and we recall that $\sigma_{near}=\min\{1/2,r,\sigma_{\mathrm{eff}}\}$ and $\sigma_{far}=1/2-2r$ (see Proposition 4.6.3).

Proof. Substituting the rescalings (4.136) into (4.129) and dividing by ϵ^4 yields:

$$\epsilon^{-4}(E_{b_*}(\epsilon^2\kappa)-E_\star+\epsilon^4\theta^2)\widehat{\Phi}_\epsilon(\kappa)-\chi(|\kappa|<\epsilon^{r-2})B_{b_*,\mathrm{eff}}\times\int_{-\infty}^{\infty}\widehat{\Phi}(\xi)d\xi$$
$$=\chi(|\kappa|<\epsilon^{r-2})\epsilon^{-2}\big(R[\theta]\widetilde{\psi}_{\mathrm{near}}\big)(\epsilon^2\kappa). \tag{4.140}$$

The estimate on $R[\theta]\widetilde{\psi}_{\mathrm{near}}$ in (4.130), together with (4.137), yields immediately

$$\begin{aligned}\Big\|\epsilon^{-2}\big(R[\theta]\widetilde{\psi}_{\mathrm{near}}\big)(\epsilon^2\kappa)\Big\|_{L^{2,-1}(\mathbb{R}_\kappa)}&\leq C\big\|\epsilon^{-2}R[\theta]\widetilde{\psi}_{\mathrm{near}}\big\|_{L^\infty} &(4.141)\\ &\leq\mathcal{C}\epsilon^{\sigma_{\mathrm{near}}}\big\|\widetilde{\psi}_{\mathrm{near}}\big\|_{L^1}+\mathcal{C}\epsilon^{1+\sigma_{\mathrm{far}}}\big\|\psi_{\mathrm{near}}\big\|_{L^2}\\ &\leq\mathcal{C}\left(\epsilon^{\sigma_{\mathrm{near}}}+\epsilon^{\sigma_{\mathrm{far}}}\right)\big\|\widehat{\Phi}\big\|_{L^{2,1}}, &(4.142)\end{aligned}$$

There remains to expand $\epsilon^{-4}(E_{b_*}(\epsilon^2\kappa)-E_\star+\epsilon^4\theta^2)$. Taylor expansion of $E_{b_*}(\epsilon^2\kappa)$ about $\kappa=0$ to fourth order yields

$$E_{b_*}(\epsilon^2\kappa)=E_{b_*}(0)+(\epsilon^2\kappa)E'_{b_*}(0)+\frac{\epsilon^4\kappa^2}{2}E''_{b_*}(0)+\frac{\epsilon^6\kappa^3}{6}E^{(3)}_{b_*}(0)+\frac{\epsilon^8\kappa^4}{24}4E^{(4)}_{b_*}(k'),$$

where k' is such that $|k'|<|\epsilon^2\kappa|\leq\epsilon^r$. Since $E^{(j)}_{b_*}(0)=0$ for $j=1,\ 3$, we obtain

$$E_{b_*}(\epsilon^2\kappa)-E_\star+\epsilon^4\theta^2=\frac{\epsilon^4\kappa^2}{2}E^{(2)}_{b_*}(0)+\epsilon^4\theta^2+\frac{\epsilon^8\kappa^4}{24}\partial_k^4E^{(4)}_{b_*}(k'). \tag{4.143}$$

Therefore, provided $\sup_{|k'|<\epsilon^r}\left|E^{(4)}_{b_*}(k')\right|<\infty$, one has

$$\left\|\left(\epsilon^{-4}(E_{b_*}(\epsilon^2\kappa)-E_\star+\epsilon^4\theta^2)-\left(\frac{\kappa^2}{2}E^{(2)}_{b_*}(0)+\theta^2\right)\right)\widehat{\Phi}(\kappa)\right\|_{L^{2,-1}(\mathbb{R}_\kappa)}$$
$$\lesssim\epsilon^4\big\|\widehat{\Phi}\big\|_{L^{2,1}}\left(\sup_{\kappa\in[-\epsilon^{r-2},\epsilon^{r-2}]}\frac{\kappa^8}{(1+|\kappa|^2)^2}\right)^{1/2}\lesssim\epsilon^{2r}\big\|\widehat{\Phi}\big\|_{L^{2,1}}. \tag{4.144}$$

Plugging estimates (4.142) and (4.144) into (4.140) immediately yields (4.138) with bound (4.139) on $R_\flat[\theta]\widehat{\Phi}$. This completes the proof of Proposition 4.6.7. $\square$

We now give the proofs of Lemmata 4.6.5 and 4.6.6.

Proof of Lemma 4.6.5. Let us first manipulate $\mathfrak{I}[\theta]\psi_{\rm near}$. Using Proposition B.2.3 and definition $\psi_{\rm near}(x) = \int_{-1/2}^{1/2} e^{2\pi i y s}\widetilde{\psi}_{\rm near}(s) p_{b_*}(x;s) ds$, one has

$$\begin{aligned}
-\big(\mathfrak{I}[\theta]\psi_{\rm near}\big)(k) &= \int_{-1/2}^{1/2} dl \sum_{a\geq 0} \mathcal{T}_a\{q_\epsilon\ \psi_{\rm near}\}(l) \frac{1}{E_a(l)-E_\star+\epsilon^4\theta^2} I_{b_*,a}[q_\epsilon](k,l) \\
&= \int_{-1/2}^{1/2} dl \sum_{a\geq 0} \left(\int_0^1 dy\ \overline{p_a(y;l)} \int_{-1/2}^{1/2} ds\ \widetilde{q}_\epsilon(y;l-s) p_{b_*}(y;s)\widetilde{\psi}_{\rm near}(s) \right) \\
&\qquad \times \frac{I_{b_*,a}[q_\epsilon](k,l)}{E_a(l)-E_\star+\epsilon^4\theta^2} \\
&= \int_{-1/2}^{1/2} ds\ \widetilde{\psi}_{\rm near}(s) \int_{-1/2}^{1/2} dl \sum_{a\geq 0} \frac{I_{b_*,a}[q_\epsilon](k,l)}{E_a(l)-E_\star+\epsilon^4\theta^2} \\
&\qquad \times \left(\int_0^1 dy\ \overline{p_a(y;l)} \widetilde{q}_\epsilon(y;l-s) p_{b_*}(y;s) \right) \\
&= \int_{-1/2}^{1/2} ds\ \widetilde{\psi}_{\rm near}(s) \int_{-1/2}^{1/2} dl \sum_{a\geq 0} \frac{I_{b_*,a}[q_\epsilon](k;l) I_{a,b_*}[q_\epsilon](l;s)}{E_a(l)-E_\star+\epsilon^4\theta^2}. \qquad (4.145)
\end{aligned}$$

Note that (4.145) is the analogue of (4.68) in the case $Q\equiv 0$.

Our aim is now to prove that, to leading order, as $\epsilon\to 0$:

$$\frac{I_{b_*,a}[q_\epsilon](k;l) I_{a,b_*}[q_\epsilon](l;s)}{E_a(l)-E_\star+\epsilon^4\theta^2} \approx I_{b_*,a}[Q_\epsilon](k;l) I_{a,b_*}[q_\epsilon](l;s), \quad \widehat{Q_\epsilon}(\xi) \equiv \frac{\widehat{q_\epsilon}(\xi)}{1+4\pi^2|\xi|^2}.$$

To this end, we proceed in a manner similar to the proof of Lemma 4.6.4. Decompose the sum over a into the cases: $a=b_*$, $a\neq b_*$ and $a\leq a_*^\epsilon$, and $a>a_*^\epsilon$, where

$$a_*^\epsilon \equiv \max\{a\geq 0 \text{ such that } \sqrt{E_a(k)} < \pi/(3\epsilon)\}.$$

By Weyl's asymptotics (Lemma B.1.4), one has $\sqrt{E_a(k)}\approx a$ and therefore $a_*\approx 1/\epsilon$.

Let us first notice that $Q_\epsilon\in L^2$ and clearly satisfies (4.28). Therefore, the bounds of Lemma 4.A.3 apply with q_ϵ replaced by Q_ϵ. Moreover, one has $\big\|Q_\epsilon\big\|_{H^2} \leq \big\|q_\epsilon\big\|_{L^2}$, thus (4.175), in particular, applies.

Case $a = b_*$. We use that $\frac{1}{|E_{b_*(l)} - E_\star + \epsilon^4\theta^2|} \leq t_-^{-1}\epsilon^{-4}$. By the Cauchy-Schwarz inequality, the triangle inequality, and (4.173) (noting that $I_{b_*,b_*}[q_\epsilon](k;l) = \overline{I_{b_*,b_*}[q_\epsilon](k;l)}$), one has

$$\int_{-1/2}^{1/2} dl \left| \left[\frac{I_{b_*,b_*}[q_\epsilon](k;l)}{E_{b_*}(l) - E_\star + \epsilon^4\theta^2} - I_{b_*,b_*}[Q_\epsilon](k;l) \right] I_{b_*,b_*}[q_\epsilon](l;s) \right| \leq \mathcal{C}\epsilon^{2N-4}, \tag{4.146}$$

with $\mathcal{C} = C(C_{N+1/2}, \mathcal{C}_{N+1/2}, \|q_\epsilon\|_{L^2}, b_*)$, uniformly with $k, s \in (-1/2, 1/2]$.

Case $a \leq a_*^\epsilon$, $a \neq b_*$. We now use estimate (4.174) for the contribution of $I_{b_*,a}[q_\epsilon](k;l)$, and estimate (4.175) for the contribution of $I_{b_*,a}[Q_\epsilon](k;l)$: It follows

$$\int_{-1/2}^{1/2} \left| I_{b_*,a}[q_\epsilon](k;l) \right|^2 \, dl \leq C(C_2, \mathcal{C}_2, \|q_\epsilon\|_{L^2}, b_*)\epsilon^3$$
$$\int_{-1/2}^{1/2} \left| I_{b_*,a}[Q_\epsilon](k;l) \right|^2 \, dl \leq C(C_4, \mathcal{C}_4, \|q_\epsilon\|_{L^2}, b_*)\epsilon^7.$$

Similar estimates apply of course to $I_{a,b_*}[q_\epsilon](l;s) = \overline{I_{b_*,a}[q_\epsilon](s;l)}$. By Weyl's asymptotics (Lemma B.1.4), one has $\frac{1}{|E_a(l) - E|} \lesssim \frac{1}{1+|a|^2}$ and $a_* \approx 1/\epsilon$. Using the triangle inequality and the Cauchy-Schwarz inequality, it follows

$$\sum_{a \neq b_*, a \leq a_*} \int_{-1/2}^{1/2} dl \left| \left[\frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l) - E_\star + \epsilon^4\theta^2} - I_{b_*,a}[Q_\epsilon](k;l) \right] I_{a,b_*}[q_\epsilon](l;s) \right| \leq \mathcal{C}\epsilon^3, \tag{4.147}$$

with $\mathcal{C} = C(C_4, \mathcal{C}_4, \|q_\epsilon\|_{L^2}, b_*)$, uniformly with $k, s \in (-1/2, 1/2]$.

Case $a > a_*$. Let us study in detail

$$I_{b_*,a}[Q_\epsilon](k;l) = \int_0^1 \overline{p_{b_*}(x;k)} \left(\sum_{n \in \mathbb{Z}} e^{2\pi i n x} \frac{\widehat{q_\epsilon}(k-l+n)}{1 + 4\pi^2(k-l+n)^2} \right) p_a(x;l) \, dx.$$

By Lemma B.3.2, there exists $B^+_{a,b_*}(x;k,l)$ and $B^-_{a,b_*}(x;k,l)$ such that

$$\overline{p_{b_*}(x;k)} p_a(x;l) e^{2\pi i(l-k)} = B^+_{a,b_*}(x;k,l) e^{ix\sqrt{E_a(l)}} + B^-_{a,b_*}(x;k,l) e^{-ix\sqrt{E_a(l)}},$$

and $B^\pm_{a,b_*}(x;k,l)$ satisfies

$$\|B^\pm_{a,b_*}(\cdot;k,l)\|_{W^{2,\infty}_{\mathrm{per}}} \leq C(\|Q\|_{W^{1,\infty}_{\mathrm{per}}}, b_*) \quad \text{and} \quad \|\partial_x B^\pm_{a,b_*}(\cdot;k,l)\|_{L^\infty} \leq \frac{C(\|Q\|_{W^{1,\infty}_{\mathrm{per}}}, b_*)}{1+|a|},$$

uniformly with respect to a, k, l.

After integrating twice by parts, one has (here and thereafter, we abuse notations and write $F_\pm$ for $F_+ + F_-$)

$$I_{b_*,a}[Q_\epsilon](k;l) = \frac{-1}{E_a(l)} \int_0^1 \partial_x^2 \left\{ B^\pm_{a,b_*}(x;k,l) \left(\sum_{n\in\mathbb{Z}} e^{2\pi i(n+k-l)x} \frac{\widehat{q_\epsilon}(k-l+n)}{1+4\pi^2(k-l+n)^2} \right) \right\} e^{\pm ix\sqrt{E_a(l)}} \, dx.$$

We then make use of the identity

$$-\partial_x^2\left\{\frac{B^{\pm}_{a,b_*}(x;k,l)e^{2\pi i(n+k-l)x}}{1+4\pi^2(k-l+n)^2}\right\}$$
$$=\left(B^{\pm}_{a,b_*}(x;k,l)-\frac{B^{\pm}_{a,b_*}+\partial_x^2B^{\pm}_{a,b_*}(x;k,l)+4\pi i(n+k-l)\partial_xB^{\pm}_{a,b_*}(x;k,l)}{1+4\pi^2(k-l+n)^2}\right)e^{2\pi i(n+k-l)x},$$

and deduce from the above estimates

$$I_{b_*,a}[Q_\epsilon](k;l)=\frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l)}+\frac{1}{E_a(l)}\int_0^1\sum_{n\in\mathbb{Z}}J_n(x;k,l)\widehat{q_\epsilon}(k-l+n)\ dx,$$

with

$$\big\|J_n(\cdot;k,l)\big\|_{L^\infty_{\rm per}}\leq C(\big\|Q\big\|_{W^{1,\infty}_{\rm per}},b_*)\times\Big(\frac{1}{1+4\pi^2(k-l+n)^2}+\frac{1}{\sqrt{E_a(l)}(1+2\pi|k-l+n|)}\Big),\quad(4.148)$$

uniformly with respect to a,k,l and n.

In order to estimate the latter, we decompose the sum over $|n|<1/(3\epsilon)$, and $|n|\geq 1/(3\epsilon)$. For the former, we have, thanks to assumption (4.28) and the Cauchy-Schwarz inequality,

$$\int_{-1/2}^{1/2}\left|\sum_{|n|<1/(3\epsilon)}\Big(\frac{1}{1+4\pi^2(k-l+n)^2}+\frac{1}{\sqrt{E_a(l)}(1+2\pi|k-l+n|)}\Big)\widehat{q_\epsilon}(k-l+n)\right|^2\ dl$$
$$\lesssim\int_{-1/2}^{1/2}\sum_{|n|<1/(3\epsilon)}|\widehat{q_\epsilon}(k-l+n)|^2\lesssim\big(C_\beta\epsilon^\beta\big)^2.$$

For the latter, one has

$$\int_{-1/2}^{1/2}\left|\sum_{|n|\geq1/(3\epsilon)}\Big(\frac{1}{1+4\pi^2(k-l+n)^2}+\frac{1}{\sqrt{E_a(l)}(1+2\pi|k-l+n|)}\Big)\widehat{q_\epsilon}(k-l+n)\right|^2\ dl$$
$$\lesssim\big\|q_\epsilon\big\|_{L^2}\int_{-1/2}^{1/2}\sum_{|n|\geq1/(3\epsilon)}\left|\frac{1}{1+4\pi^2n^2}+\frac{1}{\sqrt{E_a(l)}(1+2\pi|n|)}\right|^2\lesssim\big\|q_\epsilon\big\|_{L^2}\times\big(\epsilon^3+\frac{\epsilon}{E_a(l)}\big).$$

Altogether, we conclude that

$$\int_{-1/2}^{1/2}\left|\frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l)-E_\star+\epsilon^4\theta^2}-I_{b_*,a}[Q_\epsilon](k;l)\right|^2\ dl\leq\mathcal{C}\frac{1}{E_a(l)^2}\Big(C_\beta^2\epsilon^{2\beta}+\epsilon^3+\frac{\epsilon}{E_a(l)}+\frac{1}{E_a(l)^2}\Big),$$

with $\mathcal{C}=C(\mathcal{C}_\beta,\big\|Q\big\|_{W^{1,\infty}_{\rm per}},b_*)$.

Finally, summing over $a > a_*$ (and recalling that, by Weyl's asymptotics, $a_* \approx 1/\epsilon$ and $E_a(l) \lesssim |a|^2$), one has

$$\sum_{a>a_*} \int_{-1/2}^{1/2} \left| \frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l) - E_\star + \epsilon^4\theta^2} - I_{b_*,a}[Q_\epsilon](k;l) \right|^2 \, dl \leq \mathcal{C}\epsilon^5,$$

with $\mathcal{C} = C(\mathcal{C}_\beta, \|Q\|_{W^{1,\infty}_{\text{per}}}, b_*)$, uniformly with $k \in [-1/2, 1/2]$. Using Cauchy-Schwarz inequality and (4.172) of Lemma 4.A.3, we have

$$\sum_{a>a_*} \int_{-1/2}^{1/2} dl \left| \big(\frac{I_{b_*,a}[q_\epsilon](k;l)}{E_a(l) - E_\star + \epsilon^4\theta^2} - I_{b_*,a}[Q_\epsilon](k;l) \big) I_{a,b_*}[q_\epsilon](l;s) \right| \leq \mathcal{C}\epsilon^{5/2}. \tag{4.149}$$

Bounds (4.146) with $N = 7/2$, (4.147), and (4.149) imply (4.133) and complete the proof of Lemma 4.6.5. □

Proof of Lemma 4.6.6. Using the identity (4.168) of Lemma 4.A.2, one can write

$$\int_{-1/2}^{1/2} dl \sum_{a\geq 0} I_{b_*,a}[Q_\epsilon](k;l) I_{a,b_*}[q_\epsilon](l;s) = \int_{-1/2}^{1/2} dl \sum_{a\geq 0} \overline{\mathcal{T}_a\{u_{b_*}(\cdot;k)Q_\epsilon(\cdot)\}}(l) \mathcal{T}_a\{u_{b_*}(\cdot;s)q_\epsilon(\cdot)\}(l).$$

Expanding the term $\overline{\mathcal{T}_a\{u_{b_*}(\cdot;k)Q_\epsilon(\cdot)\}}$ via the definition (B.36), one has

$$\int_{-1/2}^{1/2} dl \sum_{a\geq 0} \overline{\mathcal{T}_a\{u_{b_*}(\cdot;k)Q_\epsilon(\cdot)\}}(l) \mathcal{T}_a\{u_{b_*}(\cdot;s)q_\epsilon(\cdot)\}(l)$$
$$= \int_{-1/2}^{1/2} dl \sum_{a\geq 0} \int_{\mathbb{R}} dx u_a(x;l) \overline{u_{b_*}(x;k)Q_\epsilon(x)} \mathcal{T}_a\{u_{b_*}(\cdot;s)q_\epsilon(\cdot)\}(l).$$

Finally, using the completeness of the Bloch functions, (B.37), one has

$$\int_{-1/2}^{1/2} dl \sum_{a\geq 0} \int_{\mathbb{R}} dx u_a(x;l) \overline{u_{b_*}(x;k)Q_\epsilon(x)} \mathcal{T}_a\{u_{b_*}(\cdot;s)q_\epsilon(\cdot)\}(l)$$
$$= \int_{-\infty}^{\infty} dx \overline{u_{b_*}(x;k)} u_{b_*}(x;s) \overline{Q_\epsilon(x)} q_\epsilon(x).$$

Therefore, we can write

$$\int_{-1/2}^{1/2} dl \sum_{a\geq 0} I_{b_*,a}[Q_\epsilon](k;l) I_{a,b_*}[q_\epsilon](l;s) = \int_{-\infty}^{\infty} dx |u_{b_*}(x;0)|^2 \overline{Q_\epsilon(x)} q_\epsilon(x)$$
$$+ \int_{-\infty}^{\infty} dx \overline{Q_\epsilon(x)} q_\epsilon(x) \left[\overline{u_{b_*}(x;k)} u_{b_*}(x;s) - \overline{u_{b_*}(x;0)} u_{b_*}(x;0) \right]. \tag{4.150}$$

Recalling that $\widetilde{\psi}_{\text{near}}(s) = \widetilde{\psi}_{\text{near}}(s)\chi_{\epsilon^r}(s)$, we bound the term (4.150) by Taylor expanding about $s = 0$ and $k = 0$. From Lemma B.3.1, one has for any $k, s \in [-\epsilon^r, \epsilon^r]$,

$$\left| \int_{-\infty}^{\infty} dx \overline{Q_\epsilon(x)} q_\epsilon(x) (\overline{u_{b_*}(x;k)} u_{b_*}(x;s) - \overline{u_{b_*}(x;0)} u_{b_*}(x;0)) \right| \leq \epsilon^r C(\|Q\|_{L^\infty_{\text{per}}}, b_*) \int_{-\infty}^{\infty} dx |\overline{Q_\epsilon}(x) q_\epsilon(x)|.$$

One checks using Hypothesis (H1'b), (4.28), that

$$\int_{-\infty}^{\infty} dx|\overline{Q_\epsilon}(x)q_\epsilon(x)| \leq \|q_\epsilon\|_{L^2}\|Q_\epsilon\|_{L^2} \leq \epsilon^2 C(\mathcal{C}_2)\|q_\epsilon\|_{L^2}^2,$$

to write

$$\begin{aligned}\int_{-1/2}^{1/2} dl \sum_{a\geq 0} I_{b_*,a}[Q_\epsilon](k;l)I_{a,b_*}[q_\epsilon](l;s) &= \int_{-\infty}^{\infty} dx|u_{b_*}(x;0)|^2\overline{Q_\epsilon(x)}q_\epsilon(x) + \mathcal{O}(\epsilon^{2+r}) \\ &= \epsilon^2 B_{\text{eff}} + \mathcal{O}(\epsilon^{2+r}) + \mathcal{O}(\epsilon^{2+\text{eff}}),\end{aligned}$$

by the definition of B_{eff} in Hypothesis (H2'), (4.29). This completes the proof of Lemma 4.6.6. □

4.6.4 Conclusion of the proof of Th'm 4.2.3

Proposition 4.6.7 is a formal reduction of the eigenvalue problem

$$(-\partial_x^2 + Q(x) + q_\epsilon(x))\psi = E\psi, \quad \psi \in H^2(\mathbb{R}), \tag{4.151}$$

for $(E^\epsilon, \psi^\epsilon)$ to an equation for $(\theta_\epsilon^2, \Phi_\epsilon)$ of the form:

$$\left(\frac{1}{2}\partial_k^2 E_{b_*}(0)\kappa^2 + \theta^2\right)\widehat{\Phi}(\kappa) - \chi(|\kappa| < \epsilon^{r-2})B_{b_*,\text{eff}} \times \int_{-\infty}^{\infty} \widehat{\Phi}(\xi)\ d\xi = R_\flat[\theta]\widehat{\Phi}(\kappa); \tag{4.152}$$

(see (4.138)) where Φ_ϵ is the rescaled near-energy component of ψ^ϵ. We now apply Lemma 4.3.2 to obtain a solution of (4.152). We then construct the solution $(E^\epsilon, \psi^\epsilon)$ of the full eigenvalue problem (4.151). This will conclude the proof of Theorem 4.2.3.

We apply Lemma 4.3.2 to equation (4.152), with $A = \frac{1}{8\pi^2}\partial_k^2 E_{b_*}(k_*)$ and $B = B_{b_*,\text{eff}}$ and $R_\epsilon = R_\flat$. By Proposition 4.6.7, R_ϵ satisfies assumption (4.44) with $\beta = 2-r$, $\alpha = \sigma_1 = \min\{\sigma_{\text{eff}}, r, 1/2 - 2r\}$. Following the steps of its proof, and using

$$\begin{aligned}\left|\frac{1}{E_a(l) - E_\star + \epsilon^4\theta_1^2} - \frac{1}{E_a(l) - E_\star + \epsilon^4\theta_2^2}\right| &= \frac{\epsilon^4|\theta_1^2 - \theta_2^2|}{(E_a(l) - E_\star + \epsilon^4\theta_1^2)(E_a(l) - E_\star + \epsilon^4\theta_1^2)} \\ &\leq \frac{|\theta_1^2 - \theta_2^2|}{\theta_2^2}\frac{1}{E_a(l) - E_\star + \epsilon^4\theta_1^2},\end{aligned}$$

one easily checks that assumption (4.45) also holds.

Thus by Lemma 4.3.2 there exists a solution $\left(\theta_\epsilon^2, \widehat{\Phi}_\epsilon\right)$ of (4.152), satisfying

$$\|\widehat{\Phi}_\epsilon - \widehat{f}_{0,\epsilon}\|_{L^{2,1}} \ \leq\ C\ \epsilon^{\sigma_1} \qquad \text{and} \qquad |\theta_\epsilon^2 - \theta_{0,\epsilon}^2| \ \leq\ C\ \epsilon^{\sigma_1}\ . \tag{4.153}$$

Here $\left(\theta_{0,\epsilon}^2, \widehat{f}_{0,\epsilon}\right)$ is the unique (normalized) solution of the homogeneous equation

$$\widehat{\mathcal{L}}_{0,\epsilon}[\theta]\widehat{f}(\xi) = (\frac{1}{2}\partial_k^2 E_{b_*}(k_*)\xi^2 + \theta^2)\widehat{f}(\xi) + \chi\left(|\xi| < \epsilon^{r-2}\right) B_{b_*,\mathrm{eff}} \int_{\mathbb{R}} \chi\left(|\eta| < \epsilon^{r-2}\right) \widehat{f}(\eta) d\eta = 0,$$

as described in Lemma 4.3.1. Specifically,

$$\widehat{f}_{0,\epsilon}(\xi) = \frac{\chi(|\xi| < \epsilon^{r-2})}{\frac{1}{2}\partial_k^2 E_{b_*}(k_*)\xi^2 + \theta_{0,\epsilon}^2}, \quad \text{and} \quad \theta_{0,\epsilon}^2 = \frac{B_{b_*,\mathrm{eff}}^2}{\frac{1}{2\pi^2}\partial_k^2 E_{b_*}(k_*)} + \mathcal{O}(\epsilon^{2-r}). \tag{4.154}$$

We next construct the eigenpair solution $(E^\epsilon, \psi^\epsilon)$ of the Schrödinger equation (4.30). Define, using Proposition 4.6.1,

$$\psi^\epsilon \equiv \psi_{\mathrm{near}}^\epsilon + \psi_{\mathrm{far}}^\epsilon, \qquad E^\epsilon \equiv E_{b_*}(k_*) - \epsilon^4 \theta_\epsilon^2,$$
$$\text{where} \quad \widehat{\psi}_{\mathrm{near}}^\epsilon(\xi) = \frac{1}{\epsilon^2}\widehat{\Phi}_\epsilon\left(\frac{\xi}{\epsilon^2}\right) \quad \text{and} \quad \psi_{\mathrm{far}}^\epsilon(\xi) = \psi_{\mathrm{far}}[\psi_{\mathrm{near}}^\epsilon, E^\epsilon; \epsilon](\xi).$$

Then $(E^\epsilon, \psi^\epsilon)$ is a solution of the eigenvalue problem (4.151). Indeed, the steps proceeding from (4.151) to (4.152) are reversible for solutions $\psi^\epsilon \in H^1(\mathbb{R})$ of (4.151), respectively, solutions $\Phi_\epsilon \in H^1(\mathbb{R})$ of (4.152).

We now prove the estimates (4.31) and (4.32). By (4.40) in Lemma 4.3.1, (4.153) and recalling $E^\epsilon = E_{b_*}(k_*) - \epsilon^4\theta_\epsilon^2$, one has

$$\begin{aligned}
\left| E^\epsilon - \left(E_{b_*}(k_*) - \epsilon^4 \frac{B_{b_*,\mathrm{eff}}^2}{\frac{1}{2\pi^2}\partial_k^2 E_{b_*}(k_*)} \right) \right| &= \epsilon^4 \left| \frac{B_{b_*,\mathrm{eff}}^2}{\frac{1}{2\pi^2}\partial_k^2 E_{b_*}(k_*)} - \theta_\epsilon^2 \right| \\
&\leq \epsilon^4 \left| \frac{B_{b_*,\mathrm{eff}}^2}{\frac{1}{2\pi^2}\partial_k^2 E_{b_*}(k_*)} - \theta_{0,\epsilon}^2 \right| + \epsilon^4 \left| \theta_{0,\epsilon}^2 - \theta_\epsilon^2 \right| \lesssim \epsilon^{4+\sigma_1}.
\end{aligned}$$

This shows estimate (4.31), the small ϵ expansion of the eigenvalue E^ϵ.

The approximation, (4.32), of the corresponding eigenstate, $\psi^\epsilon = \psi_{\mathrm{near}}^\epsilon + \psi_{\mathrm{far}}^\epsilon$, is obtained as follows. For $A = \frac{1}{8\pi^2}\partial_k^2 E_{b_*}(k_*)$ and $B = B_{b_*,\mathrm{eff}}$, one has

$$\begin{aligned}
\left\| \psi^\epsilon(x) - u_{b_*}(x;0)\frac{2}{B}\exp\left(-\epsilon^2 \frac{B}{2A}|x|\right) \right\|_{L^\infty} &= \left\| \psi_{\mathrm{near}}^\epsilon(x) + \psi_{\mathrm{far}}^\epsilon(x) - u_{b_*}(x;0)\frac{2}{B}\exp\left(-\epsilon^2 \frac{B}{2A}|x|\right) \right\|_{L^\infty} \\
&\leq \left\| \psi_{\mathrm{near}}^\epsilon(x) - u_{b_*}(x;0)\frac{2}{B}\exp\left(-\epsilon^2 \frac{B}{2A}|x|\right) \right\|_{L^\infty} \\
&\quad + \left\| \psi_{\mathrm{far}}^\epsilon(x) \right\|_{L^\infty}
\end{aligned} \tag{4.155}$$

We look at each of the norms in (4.155) separately.

Recall,

$$
\begin{aligned}
\psi^{\epsilon}_{\text{near}}(x) &= \int_{-1/2}^{1/2} \chi(|k|<\epsilon^r)\widetilde{\psi}_{\text{near}}(k) u_{b_*}(x;k)\ dk \\
&= \int_{-1/2}^{1/2} \chi(|k|<\epsilon^r)\frac{1}{\epsilon^2}\widehat{\Phi}_\epsilon\left(\frac{k}{\epsilon^2}\right) e^{2\pi i k x} p_{b_*}(x;k)\ dk \\
&= \int_{\mathbb{R}} \chi(|\xi|<\epsilon^{r-2})\widehat{\Phi}_\epsilon(\xi) e^{2\pi i\epsilon^2\xi x} p_{b_*}(x;\epsilon^2\xi)\ d\xi \\
&= u_{b_*}(x;0)\int_{\mathbb{R}} \chi(|\xi|<\epsilon^{r-2})\widehat{f}_{0,\epsilon}(\xi) e^{2\pi i\epsilon^2\xi x}\ d\xi \\
&\qquad + u_{b_*}(x;0)\int_{\mathbb{R}} \chi(|\xi|<\epsilon^{r-2})\left(\widehat{\Phi}_\epsilon(\xi)-\widehat{f}_{0,\epsilon}(\xi)\right) e^{2\pi i\epsilon^2\xi x}\ d\xi \\
&\qquad + \int_{\mathbb{R}} \chi(|\xi|<\epsilon^{r-2})\widehat{\Phi}_\epsilon(\xi) e^{2\pi i\epsilon^2\xi x}(p_{b_*}(x;\epsilon^2\xi)-p_{b_*}(x;0))\ d\xi \\
&= I_1(x)+I_2(x)+I_3(x).
\end{aligned}
$$

We study each of these pieces in more detail. By (4.154), $\chi(|\xi|<\epsilon^{r-2})\widehat{f}_{0,\epsilon}(\xi)=\widehat{f}_{0,\epsilon}(\xi)$. Therefore, for $A=\frac{1}{8\pi^2}\partial_k^2 E_{b_*}(k_*)$ and $B=B_{b_*,\text{eff}}$, one has from estimate (4.41) in Lemma 4.3.1,

$$
I_1(x) = u_{b_*}(x;0)\mathcal{F}^{-1}\left\{\widehat{f}_{0,\epsilon}\right\}(\epsilon^2 x) = u_{b_*}(x;0)\frac{2}{B}\exp\left(-\epsilon^2\frac{B}{2A}|x|\right)+\mathcal{O}(\epsilon^{2-r}). \tag{4.156}
$$

Using the first bound of (4.153), one has

$$
\begin{aligned}
\left\|I_2\right\|_{L^\infty} &= \sup_{x\in\mathbb{R}}\left|u_{b_*}(x;0)\int_{\mathbb{R}} \chi(|\xi|<\epsilon^{r-2})\left(\widehat{\Phi}_\epsilon(\xi)-\widehat{f}_{0,\epsilon}(\xi)\right) e^{2\pi i\epsilon^2\xi x}\ d\xi\right| \\
&\le C\ \|u_{b_*}(x;0)\|_{L^\infty(\mathbb{R}_x)}\left\|\widehat{\Phi}_\epsilon-\widehat{f}_{0,\epsilon}\right\|_{L^{2,1}} \le C\ \epsilon^{\sigma_1}.
\end{aligned} \tag{4.157}
$$

Similarly,

$$
\begin{aligned}
\left\|I_3\right\|_{L^\infty} &= \sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}} \chi(|\xi|<\epsilon^{r-2})\widehat{\Phi}_\epsilon(\xi) e^{2\pi i\epsilon^2\xi x}(p_{b_*}(x;\epsilon^2\xi)-p_{b_*}(x;0))\ d\xi\right| \\
&\le C\ \epsilon^2 \sup_{|k'|\le\epsilon^r}\|\partial_k p_{b_*}(x;k')\|_{L^\infty(\mathbb{R}_x)}\int_{\mathbb{R}} \chi(|\xi|<\epsilon^{r-2})|\xi||\widehat{\Phi}_\epsilon(\xi)|\ d\xi \\
&\le C\ \epsilon^2\left\|\widehat{\Phi}_\epsilon\right\|_{L^{2,1}},
\end{aligned} \tag{4.158}
$$

where we used that $\partial_k p_{b_*}(x;k')$ is well-defined and bounded by Lemma B.1.5. Finally, notice that $\left\|\widehat{\Phi}_\epsilon\right\|_{L^{2,1}} \to \left\|\widehat{f}_{0,\epsilon}\right\|_{L^{2,1}}$ as $\epsilon\to 0$, and $\left\|\widehat{f}_{0,\epsilon}\right\|_{L^{2,1}}$ is bounded uniformly with respect to ϵ. Therefore, from estimates (4.156)-(4.158), and noting that $\min\{\sigma_1, 2, 2-r\}=\sigma_1$, we can write

$$
\psi^{\epsilon}_{\text{near}}(x) = u_{b_*}(x;0)\frac{2}{B}\exp\left(-\epsilon^2\frac{B}{2A}|x|\right)+\psi^{\epsilon}_{\text{near,rem}}(x), \quad \left\|\psi^{\epsilon}_{\text{near,rem}}\right\|_{L^\infty} \le C\ \epsilon^{\sigma_1}. \tag{4.159}
$$

We return to the first norm in (4.155):

$$\begin{aligned}&\left\|\psi^{\epsilon}_{\text{near}}(x) - u_{b_*}(x;0)\exp\left(-\epsilon^2\frac{B}{2A}|x|\right)\right\|_{L^\infty}\\&\qquad = \left\|u_{b_*}(x;0)\frac{2}{B}\exp\left(-\epsilon^2\frac{B}{2A}|x|\right) + \psi^{\epsilon}_{\text{near,rem}}(x) - u_{b_*}(x;0)\frac{2}{B}\exp\left(-\epsilon^2\frac{B}{2A}|x|\right)\right\|_{L^\infty}\\&\qquad \le C\ \epsilon^{\sigma_1}.\end{aligned} \tag{4.160}$$

The second norm in (4.155) can be bound using (4.96) in Proposition 4.6.1 and (4.137):

$$\left\|\psi^{\epsilon}_{\text{far}}(x)\right\|_{L^\infty} \lesssim \left\|\psi_{\text{far}}\right\|_{H^s} \lesssim \epsilon^{2-s-2r}\left\|\widetilde{\psi}_{\text{near}}\right\|_{L^2} \lesssim \epsilon^{1-\max\{s,2r\}}\left\|\widehat{\Phi}_\epsilon\right\|_{L^{2,1}} \lesssim \epsilon^{1-\max\{s,2r\}}, \tag{4.161}$$

where we again note that $\left\|\widehat{\Phi}_\epsilon\right\|_{L^{2,1}} \to \left\|\widehat{f}_{0,\epsilon}\right\|_{L^{2,1}}$ as $\epsilon \to 0$, and $\left\|\widehat{f}_{0,\epsilon}\right\|_{L^{2,1}}$ is bounded uniformly with respect to ϵ.

Since ψ^ϵ is a unique solution of (4.151) up to a multiplicative constant, we can conclude from (4.155) and the estimates (4.160)-(4.161), that

$$\left\|\psi^{\epsilon}(x) - u_{b_*}(x;0)\exp\left(-\epsilon^2\frac{B}{2A}|x|\right)\right\|_{L^\infty} \le C\ \epsilon^{\sigma_2}, \qquad \sigma_2 = \min\{\sigma_{\text{eff}}, r, 1-\max\{s,2r\}\}.$$

This completes the proof of Theorem 4.2.3, with the choice $r = 1/6$ and $s = 2/3$.

4.A Bounds used in Section 4.6

To study the near- and far-energy equations (4.93) and (4.94), we will make use of the following Lemmata.

Lemma 4.A.1. *Let $q_\epsilon \in L^2 \cap L^\infty$ and assume q_ϵ is concentrated at high frequencies in the sense of (4.28): There exists $\beta \ge 2$ and a constant $\mathcal{C}_\beta$ such that*

$$\left(\int_{-\frac{1}{2\epsilon}}^{\frac{1}{2\epsilon}} |\widehat{q}_\epsilon(\xi)|^2\, d\xi\right)^{1/2} \lesssim \mathcal{C}_\beta \epsilon^\beta, \quad \text{for} \quad 0 < \epsilon \ll 1. \tag{4.162}$$

Then, for any $\psi \in H^\delta$ with $\frac{1}{2} < \delta \le 2$, we have

$$\|\widetilde{q_\epsilon\psi}\|^2_{\mathcal{X}^{-\delta}} \le C(\left\|q_\epsilon\right\|_{L^2}, \left\|q_\epsilon\right\|_{L^\infty}, \mathcal{C}_\beta)\ \epsilon^{2\delta}\|\widetilde{\psi}\|^2_{\mathcal{X}^\delta}. \tag{4.163}$$

If, moreover, $\psi \in H^2$, then we have

$$\|\widetilde{q_\epsilon\psi}\|^2_{\mathcal{X}^{-\delta}} \le C(\left\|q_\epsilon\right\|_{L^2}, \left\|q_\epsilon\right\|_{L^\infty}, \mathcal{C}_\beta)\ \left(\epsilon^{2\delta}\|\widetilde{\psi}\|^2_{\mathcal{X}^0} + \epsilon^4\ \|\widetilde{\psi}\|^2_{\mathcal{X}^2}\right). \tag{4.164}$$

Proof. The norm $\mathcal{X}^s$ is defined in (B.38). By Proposition B.2.1, one has

$$\|\widetilde{q_\epsilon \psi}\|^2_{\mathcal{X}^{-\delta}} \lesssim \|q_\epsilon \psi\|^2_{H^{-\delta}} \lesssim \int_\xi \frac{1}{(1+|\xi|^2)^\delta} \left|\widehat{q_\epsilon \psi}(\xi)\right|^2 \, d\xi.$$

Bound the above by estimating the integral separately over the ranges: $|\xi| > \frac{1}{4\epsilon}$ and $|\xi| \leq \frac{1}{4\epsilon}$. For $|\xi| > \frac{1}{4\epsilon}$, one has

$$\int_{|\xi|>\frac{1}{4\epsilon}} \left| \frac{1}{(1+|\xi|^2)^\delta} \left|\widehat{q_\epsilon \psi}(\xi)\right|^2 \right| \, d\xi \;\lesssim \epsilon^{2\delta} \|\widehat{q_\epsilon \psi}(\xi)\|^2_{L^2} \lesssim \epsilon^{2\delta} \|q_\epsilon \psi\|^2_{L^2} \lesssim \epsilon^{2\delta} \|q_\epsilon\|^2_{L^\infty} \|\psi\|^2_{L^2}. \tag{4.165}$$

For $|\xi| \leq \frac{1}{4\epsilon}$, we begin with a pointwise bound of $\widehat{q_\epsilon \psi}(\xi)$.

$$\begin{aligned} \widehat{q_\epsilon \psi}(\xi) &= \int_\zeta \widehat{q_\epsilon}(\zeta - \xi) \widehat{\psi}(\zeta) \, d\zeta \\ &= \int_{|\zeta|<1/(4\epsilon)} \widehat{q_\epsilon}(\zeta - \xi) \widehat{\psi}(\zeta) \, d\zeta \; + \; \int_{|\zeta|\geq 1/(4\epsilon)} \widehat{q_\epsilon}(\zeta - \xi) \widehat{\psi}(\zeta) \, d\zeta. \end{aligned}$$

Since q_ϵ satisfies (4.162), one has for any $\gamma \in [0, 2]$

$$\begin{aligned} \sup_{|\xi|<\frac{1}{4\epsilon}} \left|\widehat{q_\epsilon \psi}(\xi)\right| &\leq \epsilon^\beta \mathcal{C}_\beta \|\psi\|_{L^2} \; + \; \int_{|\zeta|\geq 1/(4\epsilon)} \frac{\widehat{q_\epsilon}(\zeta - \xi)}{(1+|\zeta|^2)^{\gamma/2}} (1+|\zeta|^2)^{\gamma/2} \widehat{\psi}(\zeta) \, d\zeta \\ &\leq \epsilon^\beta \mathcal{C}_\beta \|\psi\|_{L^2} + \epsilon^\gamma \|q_\epsilon\|_{L^2} \|\psi\|_{H^\gamma}. \end{aligned}$$

Since $\beta \geq 2$ and $\delta > \frac{1}{2}$, one deduces

$$\int_{|\xi|<\frac{1}{4\epsilon}} \left| \frac{1}{(1+|\xi|^2)^\delta} \left|\widehat{q_\epsilon \psi}(\xi)\right|^2 \right| \, d\xi \;\lesssim \sup_{|\xi|<\frac{1}{4\epsilon}} \left|\widehat{q_\epsilon \psi}(\xi)\right|^2 \lesssim C(\|q_\epsilon\|_{L^2}, \mathcal{C}_\beta) (\epsilon^4 \|\psi\|^2_{L^2} + \epsilon^{2\gamma} \|\psi\|^2_{H^\gamma}). \tag{4.166}$$

Estimate (4.163) follows from (4.165) and (4.166) with $\gamma = \delta$. Estimate (4.164) follows from (4.165) and (4.166) with $\gamma = 2$. Lemma 4.A.1 is proved. $\square$

We now turn to the study of

$$I_{a,b}[q_\epsilon](k;l) \;\equiv\; \int_0^1 \overline{p_a(x;k)} \widetilde{q}_\epsilon(x; k-l) p_b(x;l) \, dx, \tag{4.167}$$

Lemma 4.A.2. *Let $q_\epsilon \in L^2(\mathbb{R})$ and Q be continuous. Then for any $k, l \in (-1/2, 1/2]$, one has*

$$I_{a,b}[q_\epsilon](k;l) = \mathcal{T}_a\{u_b(\cdot;l) q_\epsilon(\cdot)\}(k) = \overline{\mathcal{T}_b\{u_a(\cdot;k) q_\epsilon(\cdot)\}(l)}. \tag{4.168}$$

For each fixed $b \geq 0$, we have the bounds:

$$\sum_{a\geq 0} \int_{-1/2}^{1/2} |I_{a,b}[q_\epsilon](k;l)|^2 \; dk \;\leq\; C(\|Q\|_{L^\infty_{\mathrm{per}}}) \|q_\epsilon\|^2_{L^2} \tag{4.169}$$

and

$$\sum_{a\geq 0}\int_{-1/2}^{1/2} |I_{b,a}[q_\epsilon](k;l)|^2 \ dl \ \leq \ C(\|Q\|_{L^\infty_{\rm per}})\|q_\epsilon\|_{L^2}^2. \tag{4.170}$$

Note: The order of (a,b) *with respect to the variable of integration is important.*

Proof. Note that, by definition (B.34), we can write

$$\widetilde{q_\epsilon}(x;k-l) = \mathcal{T}\left\{q_\epsilon(x)\right\}(x;k-l) = \mathcal{T}\left\{e^{2\pi i l x}q_\epsilon(x)\right\}(x;k).$$

Furthermore, one has

$$\begin{aligned}\widetilde{q_\epsilon}(x;k-l)p_b(x;l) &= \mathcal{T}\left\{e^{2\pi i l x}q_\epsilon(x)\right\}(x;k)p_b(x;l)\\ &= \mathcal{T}\left\{e^{2\pi i l x}p_b(x;l)q_\epsilon(x)\right\}(x;k)\\ &= \mathcal{T}\left\{u_b(x;l)q_\epsilon(x)\right\}(x;k).\end{aligned}$$

Therefore,

$$\begin{aligned}I_{a,b}[q_\epsilon](k;l) &= \int_0^1 \overline{p_a(x;k)}\widetilde{q_\epsilon}(x;k-l)p_b(x;l)\ dx\\ &= \int_0^1 \overline{p_a(x;k)}\mathcal{T}\left\{u_b(x;l)q_\epsilon(x)\right\}(x;k)\ dx\\ &= \mathcal{T}_a\{u_b(x;l)q_\epsilon(x)\}(k),\end{aligned} \tag{4.171}$$

which implies (4.168).

We now complete the proofs of the bounds (4.169)-(4.170). On the one hand,

$$\|u_b(x;l)q_\epsilon(x)\|_{L^2(\mathbb{R}_x)}^2 \lesssim \sup_{x,l}|u_b(x;l)|^2\,\|q_\epsilon\|_{L^2}^2 \lesssim \leq\ C(\|Q\|_{L^\infty_{\rm per}})\|q_\epsilon\|_{L^2}^2,$$

where we used Lemma B.3.1. On the other hand, by Proposition B.2.1,

$$\begin{aligned}\|u_b(x;l)q_\epsilon(x)\|_{L^2(\mathbb{R}_x)}^2 &\approx \|\mathcal{T}\{u_b(\cdot;l)q_\epsilon(\cdot)\}\|_{\mathcal{X}^0}^2\\ &\equiv \sum_{a\geq 0}\int_{-1/2}^{1/2} dk\,|\mathcal{T}_a\{u_b(\cdot;l)q_\epsilon(\cdot)\}(k)|^2 = \sum_{a\geq 0}\int_{-1/2}^{1/2} dk\,|I_{a,b}[q_\epsilon](k;l)|^2,\end{aligned}$$

where we used (4.171). This implies (4.169). The bound (4.170) follows by applying similar arguments to $I_{b,a}[q_\epsilon](k;l) = \overline{\mathcal{T}_a\{u_b(\cdot;k)q_\epsilon(\cdot)\}(l)}$. $\square$

Lemma 4.A.3. *Set $N \geq 0$. Assume that $q_\epsilon \in L^2$ and assume q_ϵ is concentrated at high frequencies in the sense of (4.28) with $\beta \geq N + 1/2$. Assume Q is such that (4.26) holds with $\alpha \in \mathbb{N}, \alpha \geq N + 1/2$. Let $k \in (-1/2, 1/2]$, and $a, b \geq 0$. Then*

1.
$$\int_{-1/2}^{1/2} \left|I_{b,a}[q_\epsilon](k;l)\right|^2 \; dl \leq C\big(C_0, \left\|q_\epsilon\right\|_{L^2}\big), \tag{4.172}$$

and

$$\int_{-1/2}^{1/2} \left|I_{b,a}[q_\epsilon](k;l)\right|^2 \; dl \leq \mathcal{C}\epsilon^{2N}(1 + |a|^{2\alpha})(1 + |b|^{2\alpha}), \tag{4.173}$$

where $\mathcal{C} = C(\mathcal{C}_\beta,\ C_\alpha,\ \left\|q_\epsilon\right\|_{L^2})$.

2. *If $\alpha \geq 2$ and a is such that $\sqrt{E_a(k)} < \pi/(3\epsilon)$ ($a \lesssim \epsilon^{-1}$), then*

$$\int_{-1/2}^{1/2} \left|I_{b,a}[q_\epsilon](k;l)\right|^2 \; dl \leq \mathcal{C}\epsilon^{3}(1 + |b|^{4}), \tag{4.174}$$

where $\mathcal{C} = C(\mathcal{C}_2,\ C_2,\ \left\|q_\epsilon\right\|_{L^2})$.

Furthermore, if $q_\epsilon \in H^2$, then

$$\int_{-1/2}^{1/2} \left|I_{b,a}[q_\epsilon](k;l)\right|^2 \; dl \leq \mathcal{C}\epsilon^{7}(1 + |b|^{4}), \tag{4.175}$$

where $\mathcal{C} = C(\mathcal{C}_4,\ C_4,\ \left\|q_\epsilon\right\|_{H^2})$.

All of these estimates are uniform in a, b, k, ϵ and hold, symmetrically, for $\int_{-1/2}^{1/2} \left|I_{a,b}[q_\epsilon](k;l)\right|^2 \; dk$.

Proof. Estimate (4.172) is a straightforward consequence of Lemma 4.A.2.

Let us turn to (4.173). Fix $k \in (-1/2, 1/2]$. We recall that $\widetilde{q_\epsilon}(x; k-l) = \sum_{n\in\mathbb{Z}} e^{2\pi inx}\widehat{q_\epsilon}(k-l+n)$, therefore

$$\begin{aligned}
I_{b,a}[q_\epsilon](k;l) &= \int_0^1 \overline{p_b(x;k)} \left(\sum_{n\in\mathbb{Z}} e^{2\pi inx}\widehat{q_\epsilon}(k-l+n)\right) p_a(x;l)\; dx \\
&= \int_0^1 \overline{p_b(x;k)} \left(\sum_{|n|<1/(3\epsilon)} e^{2\pi inx}\widehat{q_\epsilon}(k-l+n)\right) p_a(x;l)\; dx \\
&\qquad\qquad + \int_0^1 \overline{p_b(x;k)} \sum_{|n|\geq 1/(3\epsilon)} \left(e^{2\pi inx}\widehat{q_\epsilon}(k-l+n)\right) p_a(x;l)\; dx \\
&\equiv I^{(i)}_{b,a}[q_\epsilon](k;l) + I^{(ii)}_{b,a}[q_\epsilon](k;l).
\end{aligned}$$

We now bound $\int_{-1/2}^{1/2} \left| I^{(i)}_{b,a}[q_\epsilon](k;l) \right|^2 \ dl$.

$$\int_{-1/2}^{1/2} \left| I^{(i)}_{b,a}[q_\epsilon](k;l) \right|^2 \ dl = \int_{-1/2}^{1/2} \left| \int_0^1 \overline{p_b(x;k)} \left(\sum_{|n|<1/(3\epsilon)} e^{2\pi inx} \widehat{q}_\epsilon(k-l+n) \right) p_a(x;l) \ dx \right|^2 \ dl$$
$$\leq \sup_{l,x} \left| \overline{p_b(x;k)} p_a(x;l) \right|^2 \int_{-1/2}^{1/2} \left| \sum_{|n|<1/(3\epsilon)} \widehat{q}_\epsilon(k-l+n) \right|^2 \ dl.$$

By Cauchy-Schwarz inequality, one has

$$\left| \sum_{|n|<1/(3\epsilon)} 1 \times \widehat{q}_\epsilon(k-l+n) \right|^2 \leq \big(\frac{2}{3\epsilon} + 1 \big) \sum_{|n|<1/(3\epsilon)} |\widehat{q}_\epsilon(k-l+n)|^2.$$

Therefore, by assumption (4.28), one has

$$\int_{-1/2}^{1/2} \left| I^{(i)}_{b,a}[q_\epsilon](k;l) \right|^2 \ dl \leq C(C_0)\epsilon^{-1} \sum_{|n|<1/(3\epsilon)} \int_{-1/2}^{1/2} |\widehat{q}_\epsilon(k-l+n)|^2 \ dl \qquad [l' = n-l]$$
$$\leq C(C_0)\epsilon^{-1} \int_{-1/(2\epsilon)}^{1/(2\epsilon)} |\widehat{q}_\epsilon(k+l')|^2 \ dl' \ \leq \ C(C_0, \mathcal{C}_\beta)\epsilon^{2\beta-1},$$

which concludes the first part of the estimate.

Turning to $\int_{-1/2}^{1/2} \left| I^{(ii)}_{b,a}[q_\epsilon](k;l) \right|^2 \ dl$, we have for any $\alpha \in \mathbb{N}$,

$$\int_{-1/2}^{1/2} \left| I^{(ii)}_{b,a}[q_\epsilon](k;l) \right|^2 \ dl = \int_{-1/2}^{1/2} \left| \int_0^1 \overline{p_b(x;k)} \left(\sum_{|n|\geq 1/(3\epsilon)} e^{2\pi inx} \widehat{q}_\epsilon(k-l+n) \right) p_a(x;l) \ dx \right|^2 \ dl$$
$$= \int_{-1/2}^{1/2} \left| \sum_{|n|\geq 1/(3\epsilon)} \widehat{q}_\epsilon(k-l+n) \int_0^1 \frac{1}{(2\pi in)^\alpha} \ e^{2\pi inx} \ \partial_x^\alpha \left(\overline{p_b(x;k)} p_a(x;l) \right) \ dx \right|^2 \ dl.$$

By Cauchy-Schwarz inequality, one has for $\alpha \geq 1$:

$$\left| \sum_{|n|\geq 1/(3\epsilon)} \frac{1}{n^\alpha} \times \widehat{q}_\epsilon(k-l+n) \right|^2 \leq \big(\sum_{|n|\geq 1/(3\epsilon)} \frac{1}{n^{2\alpha}} \big) \big(\sum_{|n|\geq 1/(3\epsilon)} |\widehat{q}_\epsilon(k-l+n)|^2 \big)$$
$$\lesssim \epsilon^{2\alpha-1} \sum_{|n|\geq 1/(3\epsilon)} |\widehat{q}_\epsilon(k-l+n)|^2.$$

It follows, using (4.26)

$$\int_{-1/2}^{1/2} \left| I^{(ii)}_{b,a}[q_\epsilon](k;l) \right|^2 \ dl \lesssim C_\alpha^2 (1+|a|^\alpha)^2 (1+|b|^\alpha)^2 \epsilon^{2\alpha-1} \sum_{|n|\geq 1/(3\epsilon)} \int_{-1/2}^{1/2} |\widehat{q}_\epsilon(k-l+n)|^2 \ dl \qquad [l' = n-l]$$
$$\lesssim C_\alpha^2 (1+|a|^\alpha)^2 (1+|b|^\alpha)^2 \epsilon^{2\alpha-1} \big\| q_\epsilon \big\|^2_{L^2}.$$

Estimate (4.173) follows with $\alpha, \beta \geq N + 1/2$.

In order to obtain (4.174), we shall use the above estimate concerning $\int_{-1/2}^{1/2} \left| I_{b,a}^{(i)}[q_\epsilon](k;l) \right|^2 \, dl$ (with $\beta = 2$), and the refined analysis of Lemma B.3.2 below for $\int_{-1/2}^{1/2} \left| I_{b,a}^{(ii)}[q_\epsilon](k;l) \right|^2 \, dl$. More precisely,

$$\int_{-1/2}^{1/2} \left| I_{b,a}^{(ii)}[q_\epsilon](k;l) \right|^2 \, dl = \int_{-1/2}^{1/2} \left| \int_0^1 \overline{p_b(x;k)} \left(\sum_{|n| \geq 1/(3\epsilon)} e^{2\pi inx} \widehat{q_\epsilon}(k-l+n) \right) p_a(x;l) \, dx \right|^2 \, dl$$
$$= \int_{-1/2}^{1/2} \left| \int_0^1 \overline{p_b(x;k)} \sum_{|n| \geq 1/(3\epsilon)} \widehat{q_\epsilon}(k-l+n) \right.$$
$$\left. \left(A_a^+(x;l) e^{ix(\sqrt{E_a(l)} - 2\pi l + 2\pi n)} + A_a^-(x;l) e^{-ix(\sqrt{E_a(l)} + 2\pi l - 2\pi n)} \right) \, dx \right|^2 \, dl.$$

One deduces, after integrating by parts twice and using Lemma B.3.2 (since $Q \in W^{1,\infty}_{\rm per}(\mathbb{R})$),

$$\int_{-1/2}^{1/2} \left| I_{b,a}^{(ii)}[q_\epsilon](k;l) \right|^2 \, dl \lesssim C_2^2 (1+|b|^2)^2 \int_{-1/2}^{1/2} \left| \sum_{n \geq 1/(3\epsilon)} \widehat{q_\epsilon}(k-l+n) \frac{1}{(2\pi n - n_0)^2} \right|^2 \, dl,$$

with $n_0 = \sqrt{E_a(l)} - 2\pi l < \pi(1/(3\epsilon)+1)$. Once again, by Cauchy-Schwarz inequality, and since $|n| \geq 1/(3\epsilon)$, one deduces

$$\int_{-1/2}^{1/2} \left| I_{b,a}^{(ii)}[q_\epsilon](k;l) \right|^2 \, dl \leq \epsilon^3 C_2^2 (1+|b|^2)^2 \big\| q_\epsilon \big\|_{L^2}^2,$$

and (4.174) is proved.

Estimate (4.175) is proved similarly, but using that $q_\epsilon \in H^2$ implies

$$\int_{-1/2}^{1/2} \left| \sum_{n \geq 1/(3\epsilon)} \widehat{q_\epsilon}(k-l+n) \frac{1}{(2\pi n - n_0)^2} \right|^2 \, dl$$
$$= \int_{-1/2}^{1/2} \left| \sum_{n \geq 1/(3\epsilon)} (1+|k-l+n|^2) \widehat{q_\epsilon}(k-l+n) \frac{1}{(1+|k-l+n|^2)(2\pi n - n_0)^2} \right|^2 \, dl$$
$$\lesssim \epsilon^7 \big\| q_\epsilon \big\|_{H^2}^2.$$

Estimate (4.175) follows as above, and Lemma 4.A.3 is proved. $\square$

4.B Proof of Theorem 4.1.1

In Theorems 4.2.1 and 4.2.3, we have shown that the bifurcation of localized states into the spectral gaps is controlled by an effective operator of the form

$$-A_{\text{eff}}\ \partial_y^2 - B_{\text{eff}}\ \delta(y), \quad A_{\text{eff}} > 0,\ B_{\text{eff}} > 0,$$

see Remarks 4.2.2 and 4.2.4. In this section we compute B_{eff} for the particular case of $q_\epsilon(x) = q(x, x/\epsilon)$ that is 1-periodic and mean zero in the fast variable:

$$q(x, y+1) = q(x, y), \quad \int_0^1 q(x, y)dy = 0. \tag{4.176}$$

In other words, on can expand $q_\epsilon(x) = \sum_{j\neq 0} q_j(x)e^{2i\pi jx/\epsilon}$. This yields the results claimed in Theorem 4.1.1.

We recall that $Q_\epsilon(x)$ is defined with $\widehat{Q_\epsilon}(\xi) = \frac{\widehat{q_\epsilon}(\xi)}{1+4\pi^2|\xi|^2}$.

Lemma 4.B.1. *Assume that q_ϵ is as defined in (4.176); that is $q_\epsilon(x) = \sum_{j\neq 0} q_j(x)e^{2i\pi jx/\epsilon}$, with $\sup_{j\neq 0} \|(1+|\xi|^3)\widehat{q_j}(\xi)\|_{L^\infty(\mathbb{R}_\xi)} \leq C < \infty$ and $\sum_{j\neq 0} \|(1+|\xi|^2)\widehat{q_j}(\xi)\|_{L^2(\mathbb{R}_\xi)} \leq C < \infty$. Then one has*

$$\left\| Q_\epsilon(x) - \sum_{j\neq 0} \frac{\epsilon^2}{4\pi^2|j|^2} q_j(x)e^{2i\pi jx/\epsilon} \right\|_{L^2} \lesssim \epsilon^3, \tag{4.177}$$

from which we deduce, for any $f \in W^{1,\infty}(\mathbb{R})$,

$$\left| \int_{-\infty}^{\infty} f(x)q_\epsilon(x)\overline{Q_\epsilon(x)}\ dx \ - \ \epsilon^2 \sum_{m\neq 0} \frac{1}{4\pi^2 m^2} \int_{-\infty}^{\infty} f(x)|q_m|^2(x)\ dx \right| \lesssim \epsilon^3. \tag{4.178}$$

The case $f(x) \equiv 1$ corresponds to the case $Q \equiv 0$ (Theorem 4.2.1) and $f(x) = |u_{b_*}(x;0)|^2$ to the case $Q \not\equiv 0$ (Theorem 4.2.3). In particular, since $\int_{-\infty}^{\infty} q_\epsilon(x)\overline{Q_\epsilon(x)} = \int_{-\infty}^{\infty} \widehat{q_\epsilon}(\zeta)\overline{\widehat{Q}_\epsilon(\zeta)}\ d\zeta$, one has

$$\left| \int_{-\infty}^{\infty} \frac{\widehat{q_\epsilon}(\zeta)\widehat{q_\epsilon}(-\zeta)}{1+4\pi^2|\zeta|^2}\ d\zeta \ - \ \epsilon^2 \sum_{m\neq 0} \frac{1}{4\pi^2 m^2} \int_{-\infty}^{\infty} |q_m|^2(x)\ dx \right| \lesssim \epsilon^3.$$

Remark 4.B.2. *For simplicity we have focused on the case where $y \mapsto q(x,y)$ is periodic on $\mathbb{R}$. Our arguments apply to the more general case of perturbations for which $y \mapsto q(x,y)$ is almost periodic of the form:*

$$q_\epsilon(x) = \sum_{j\neq 0} q_j(x)e^{2\pi i\lambda_j \frac{x}{\epsilon}}, \quad \inf_{j\neq l} |\lambda_j - \lambda_l| \geq \theta > 0, \quad \inf_{j\neq 0} |\lambda_j| \geq \theta > 0$$

where $\theta > 0$ is a constant. Indeed, in [Duchêne et al.*, 2014c] we obtained results in this more general setting for the special case $Q \equiv 0$, by one-dimensional scattering methods.*

Proof of Lemma 4.B.1. Since $q_\epsilon(x) = \sum_{j\neq 0} q_j(x) e^{2\pi i j x/\epsilon}$, one has $\widehat{q_\epsilon}(\xi) = \sum_{j\neq 0} \widehat{q_j}(\xi - j/\epsilon)$. Therefore,

$$\widehat{Q_\epsilon}(\xi) = \sum_{j\neq 0} \frac{\widehat{q_j}(\xi - j/\epsilon)}{1+4\pi^2\xi^2}.$$

Similarly, denoting $Q_\epsilon^\dagger(x) = \epsilon^2 \sum_{j\neq 0} \frac{q_j(x)}{4\pi^2 j^2} e^{2\pi i j x/\epsilon}$, one has

$$\widehat{Q_\epsilon^\dagger}(\xi) = \epsilon^2 \sum_{j\neq 0} \frac{1}{4\pi^2 j^2} \widehat{q_j}(\xi - j/\epsilon).$$

Defining $\mathcal{R}_\epsilon(x) \equiv Q_\epsilon(x) - Q_\epsilon^\dagger(x)$, one has by Parseval's identity,

$$\begin{aligned}\|\mathcal{R}_\epsilon\|_{L^2}^2 = \left\|\widehat{\mathcal{R}_\epsilon}\right\|_{L^2}^2 &= \int_{-\infty}^{\infty} \left| \sum_{j\neq 0} \widehat{q_j}(\xi - j/\epsilon) \left[\frac{1}{1+4\pi^2\xi^2} - \frac{\epsilon^2}{4\pi^2 j^2} \right] \right|^2 d\xi \\ &= \epsilon^4 \int_{-\infty}^{\infty} \left| \sum_{j\neq 0} \frac{1}{4\pi^2 j^2} \widehat{q_j}(\xi - j/\epsilon) \left[\frac{1+4\pi^2(\xi^2 - j^2/\epsilon^2)}{1+4\pi^2\xi^2} \right] \right|^2 d\xi.\end{aligned}$$

We consider the above integral over two domains: $|\xi| \le 1/(2\epsilon)$ and $|\xi| > 1/(2\epsilon)$. For $|\xi| \le 1/(2\epsilon)$, one has $|1+4\pi^2(\xi^2 - j^2/\epsilon^2)| \lesssim j^2/\epsilon^2$. By assumption, $\sup_{j\neq 0} \|(1+|\xi|^3)\widehat{q_j}(\xi)\|_{L^\infty(\mathbb{R}_\xi)} \le C < \infty$. Therefore,

$$\begin{aligned}&\epsilon^4 \int_{|\xi|\le 1/(2\epsilon)} \left| \sum_{j\neq 0} \frac{1}{4\pi^2 j^2} \widehat{q_j}(\xi - j/\epsilon) \left[\frac{1+4\pi^2(\xi^2 - j^2/\epsilon^2)}{1+4\pi^2\xi^2} \right] \right|^2 d\xi \\ &\qquad \lesssim \int_{|\xi|\le 1/(2\epsilon)} \left| \frac{1}{1+4\pi^2\xi^2} \right|^2 \left| \sum_{j\neq 0} \widehat{q_j}(\xi - j/\epsilon) \right|^2 d\xi \\ &\qquad = \int_{|\xi|\le 1/(2\epsilon)} \left| \frac{1}{1+4\pi^2\xi^2} \right|^2 \left| \sum_{j\neq 0} \frac{1}{1+|\xi - j/\epsilon|^3} (1+|\xi - j/\epsilon|^3) \widehat{q_j}(\xi - j/\epsilon) \right|^2 d\xi \\ &\qquad \lesssim \sup_{j\neq 0} \|(1+|\xi|^3)\widehat{q_j}(\xi)\|_{L^\infty(\mathbb{R}_\xi)}^2 \int_{|\xi|\le 1/(2\epsilon)} \left| \frac{1}{1+4\pi^2\xi^2} \right|^2 \left| \sum_{j\neq 0} \frac{1}{1+|\xi - j/\epsilon|^3} \right|^2 d\xi.\end{aligned}$$

For $|\xi| \le 1/(2\epsilon)$ and $j \neq 0$, one has $|\xi - j/\epsilon| \ge C\ |j|/\epsilon$ and thus $\sum_{j\neq 0} \frac{1}{j^2} \frac{1}{1+|\xi - j/\epsilon|^3} \lesssim \epsilon^3 \sum_{j\neq 0} |j|^{-3}$, which implies

$$\epsilon^4 \int_{|\xi|\le 1/(2\epsilon)} \left| \sum_{j\neq 0} \frac{1}{4\pi^2 j^2} \widehat{q_j}(\xi - j/\epsilon) \left[\frac{1+4\pi^2(\xi^2 - j^2/\epsilon^2)}{1+4\pi^2\xi^2} \right] \right|^2 d\xi \lesssim \epsilon^6 \sup_{j\neq 0} \|(1+|\xi|^3)\widehat{q_j}(\xi)\|^2_{L^\infty(\mathbb{R}_\xi)}. \tag{4.179}$$

For $|\xi| > 1/(2\epsilon)$, one has $|1+4\pi^2\xi^2| \gtrsim \epsilon^{-2}$ which implies that $1/|1+4\pi^2\xi^2| \lesssim \epsilon^2$. Therefore,

$$\begin{aligned} &\epsilon^4 \int_{|\xi|> 1/(2\epsilon)} \left| \sum_{j\neq 0} \frac{1}{4\pi^2 j^2} \widehat{q_j}(\xi - j/\epsilon) \left[\frac{1+4\pi^2(\xi^2 - j^2/\epsilon^2)}{1+4\pi^2\xi^2} \right] \right|^2 d\xi \\ &\qquad \lesssim \epsilon^8 \int_{|\xi|> 1/(2\epsilon)} \left| \sum_{j\neq 0} \frac{1}{4\pi^2 j^2} (1+|\xi - j/\epsilon|^2) \widehat{q_j}(\xi - j/\epsilon) \left[\frac{1+4\pi^2(\xi^2 - j^2/\epsilon^2)}{1+|\xi - j/\epsilon|^2} \right] \right|^2 d\xi. \end{aligned}$$

Now, one has

$$\frac{1+4\pi^2(\xi^2 - j^2/\epsilon^2)}{1+|\xi - j/\epsilon|^2} \lesssim \frac{1+(\xi - j/\epsilon)(\xi + j/\epsilon)}{1+|\xi - j/\epsilon|^2} \le \left| \frac{j}{\epsilon} \right| + 1.$$

Therefore, one obtains

$$\epsilon^4 \int_{|\xi|> 1/(2\epsilon)} \left| \sum_{j\neq 0} \frac{1}{4\pi^2 j^2} \widehat{q_j}(\xi - j/\epsilon) \left[\frac{1+4\pi^2(\xi^2 - j^2/\epsilon^2)}{1+4\pi^2\xi^2} \right] \right|^2 d\xi \lesssim \epsilon^6 \left(\sum_{j\neq 0} \|(1+|\xi|^2)\widehat{q_j}(\xi)\|_{L^2(\mathbb{R}_\xi)} \right)^2. \tag{4.180}$$

Bounds (4.179) and (4.180) complete the proof of estimate (4.177).

Estimate (4.178) is deduced as follows. By the triangle inequality, one has

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} f(x) q_\epsilon(x) \overline{Q_\epsilon(x)} \, dx \; - \; \epsilon^2 \sum_{m\neq 0} \frac{1}{4\pi^2 m^2} \int_{-\infty}^{\infty} f(x) |q_m|^2(x) \, dx \right| \\ &\qquad \le \left| \int_{-\infty}^{\infty} dx f(x) q_\epsilon(x) \big(\overline{Q_\epsilon(x)} \; - \; \overline{Q_\epsilon^\dagger(x)} \big) \right| \\ &\qquad\qquad + \left| \int_{-\infty}^{\infty} dx f(x) \left(q_\epsilon(x) \overline{Q_\epsilon^\dagger(x)} \; - \; \epsilon^2 \sum_{m\neq 0} \frac{1}{4\pi^2 m^2} |q_m|^2(x) \right) \right|. \end{aligned}$$

As a straightforward consequence of Cauchy-Schwarz inequality, one has

$$\left| \int_{-\infty}^{\infty} dx f(x) q_\epsilon(x) \big(\overline{Q_\epsilon(x)} \; - \; \overline{Q_\epsilon^\dagger(x)} \big) \right| \lesssim \|f\|_{L^\infty} \|q_\epsilon\|_{L^2} \|Q_\epsilon - Q_\epsilon^\dagger\|_{L^2} \lesssim \epsilon^3.$$

There remains to show that

$$\left| \int_{-\infty}^{\infty} dx f(x) \left(q_\epsilon(x) \overline{Q_\epsilon^\dagger(x)} \; - \; \epsilon^2 \sum_{m\neq 0} \frac{1}{4\pi^2 m^2} |q_m|^2(x) \right) \right| \lesssim \epsilon^3.$$

We write

$$q_\epsilon(x)\overline{Q_\epsilon^\dagger(x)} = \sum_{j\neq 0}\sum_{l\neq 0} \frac{\epsilon^2}{4\pi^2 l^2} q_j(x)\overline{q_l}(x)e^{2i\pi(j-l)x/\epsilon}.$$

One recovers the desired expression from the diagonal terms $j = l(= m)$, and note that if $j \neq l$, then one has

$$\begin{aligned}\left|\int_{-\infty}^{\infty} dx f(x)\frac{\epsilon^2}{4\pi^2 l^2} q_j(x)\overline{q_l}(x)e^{2i\pi(j-l)x/\epsilon}\right| &\lesssim \frac{\epsilon^3}{8\pi^3|l^2(j-l)|}\int_{-\infty}^{\infty} dx \partial_x\big(f(x)q_j(x)\overline{q_l}(x)\big)e^{2i\pi(j-l)x/\epsilon} \\ &\lesssim \epsilon^3\big\|f\big\|_{W^{1,\infty}}\frac{\big\|q_j\big\|_{W^{1,2}}\big\|q_l\big\|_{W^{1,2}}}{|l^2(j-l)|}.\end{aligned}$$

The proof is now complete, since $\big\|q_j\big\|_{W^{1,2}} \lesssim \big\|(1+|\xi|)\widehat{q_j}(\xi)\big\|_{L^2(\mathbb{R}_\xi)}$, and by assumption, one has $\sum_{j\neq 0}\big\|(1+|\xi|^2)\widehat{q_j}(\xi)\big\|_{L^2(\mathbb{R}_\xi)} < \infty$. □

Part III

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Part IV

Appendices

Appendix A

Maxwell to Schrödinger

Maxwell's equations are a set of coupled vector partial differential equations that describe how electric and magnetic fields are generated and altered by each other. In this appendix, we derive the Schrödinger equation from a specific set of solutions to the Maxwell's equations.

A.1 Maxwell's equations and TM wave solutions

In this section, we analyse a specific subset of solutions to the Maxwell's equations. Begin first by studying a non-magnetic material, ie. the magnetic permeability is constant, $\mu = \mu_0$, with electric permittivity depending on the x_1 and x_2 directions, $\epsilon = \epsilon(x_1, x_2)$. This is the physical setting up to first order corrections for a photonic crystal; see [Joannopoulos *et al.*, 2008] for more detail. The Maxwell's equations are:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\mu_0 \partial_t \mathbf{H}, \qquad \nabla \cdot \epsilon(x_1, x_2)\mathbf{E} = 0, \\ \nabla \times \mathbf{H} &= \epsilon(x_1, x_2)\partial_t \mathbf{E}, \qquad \nabla \cdot \mathbf{H} = 0, \end{aligned} \tag{A.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively. See Figure A.1 for an example of an idealized photonic crystal depending only on the x_2 direction.

The specific solutions that we seek to the Maxwell's equations are those which propagate in the (x_1, x_2)-plane only, implying that the electromagnetic field is x_3-independent. It can be shown that such solutions come in two groups (see Chapter 7 of [Bao *et al.*, 2001] for more detail):

- Transverse electric (TE) polarized solutions in which the magnetic field is directed along the

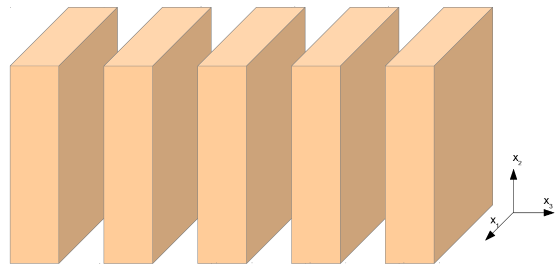

Figure A.1: Idealized sketch of a periodic photonic crystal with electric permittivity depending only on one direction, $\epsilon = \epsilon(x_2)$, and extending to "infinity" in the x_1 and x_3 directions.

x_3-axis and the electric field is normal to this axis,

$$\mathbf{H} = (0, 0, H) \quad \mathbf{E} = (E_1, E_2, 0); \tag{A.2}$$

and

- Transverse magnetic (TM) polarized solutions in which the electric field is directed along the x_3-axis and the magnetic field is normal to this axis,

$$\mathbf{H} = (H_1, H_2, 0) \quad \mathbf{E} = (0, 0, E). \tag{A.3}$$

From here on out we consider TM polarized solutions to the Maxwell's equations. Taking the curl ($\nabla\times$) of the first Maxwell's equation and applying the vector identity $\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \Delta\mathbf{V}$ to $\mathbf{E}$, we have from the description of the curl of the magnetic field in (A.1),

$$\nabla(\nabla \cdot \mathbf{E}) - \Delta\mathbf{E} = -\mu_0\epsilon(x_1, x_2)\partial_t^2\mathbf{E}. \tag{A.4}$$

Since we are assuming that $\mathbf{E} = (0, 0, E)$, the Maxwell's equation $\nabla \cdot \epsilon(x_1, x_2)\mathbf{E} = 0$ implies that $\partial_{x_3}^2 E = 0$. Therefore, $\nabla \cdot \mathbf{E} = 0$, $\Delta\mathbf{E} = (0, 0, \Delta E)$, and $\partial_t^2\mathbf{E} = (0, 0, \partial_t^2 E)$, with which equation (A.4) becomes

$$\Delta E(t, x_1, x_2) - \mu_0\epsilon(x_1, x_2)\partial_t^2 E(t, x_1, x_2) = 0, \tag{A.5}$$

where $\Delta \equiv \partial_{x_1}^2 + \partial_{x_2}^2$. Note that this is a scalar equation with two spatial variables and one time variable.

We rewrite (A.4) slightly by defining $1/c(x_1, x_2) = \sqrt{\mu_0\epsilon(x_1, x_2)}$, where $c(x_1, x_2)$ is the speed of light through the medium described by the material tensors μ_0 and $\epsilon(x_1, x_2)$. We thus get the more familiar wave equation:

$$\Delta E(t, x_1, x_2) - \frac{1}{c(x_1, x_2)}\partial_t^2 E(t, x_1, x_2) = 0. \tag{A.6}$$

We summarize the above derivation of the wave equation: Beginning with the Maxwell's equations in a non-magnetic medium, assume that the electric permittivity depends only on the (x_1, x_2) variables. Considering propagating waves in the (x_1, x_2)-plane only, we have two sets of possible solutions - TE or TM polarized solutions. Seeking TM polarized solutions to the Maxwell's equations, we find that the electric field must satisfy the wave equation (A.6). After solving for the electric field we can find the magnetic field using the Maxwell's equations.

A.2 Time harmonic solutions to the wave equation

In this section we study time harmonic solutions to the wave equation (A.6). We first rename the spatial variables, $x_1 = z$ and $x_2 = x$ to get the equation

$$\left(\partial_x^2 + \partial_z^2\right) E(x, z, t) - \frac{1}{c^2(x, z)}\partial_t^2 E(x, z, t) = 0, \tag{A.7}$$

where $c(x, z)$ is the speed of light in the material of propagation. We seek time harmonic solutions

$$E(x, z, t) = e^{i\omega t}u(x, z), \tag{A.8}$$

which, upon substitution into (A.7), yield the Helmholtz equation for $u(x, z)$,

$$\left(\partial_x^2 + \partial_z^2\right) u(x, z) + \frac{\omega^2}{c^2(x, z)}u(x, z) = 0.$$

Labelling the speed of light in a vacuum by c_0, we define

$$\frac{\omega^2}{c^2(x, z)} = \frac{\omega^2}{c_0^2}\frac{c_0^2}{c^2(x, z)} \equiv \nu^2 n^2(x, z), \quad \text{where} \quad \nu \equiv \frac{\omega}{c_0} \text{ and } n(x, z) \equiv \frac{c_0}{c(x, z)}. \tag{A.9}$$

Note that since $c_0 \geq c(x, z)$ for any material, $n(x, z) \geq 1$; we have equality, $n(x, z) = 1$, only in a vacuum. The function $n(x, z)$ is called the refractive index of the material.

We conclude by saying that time harmonic solutions of the wave equation yield the Helmholtz equation,

$$\left(\partial_x^2 + \partial_z^2\right) u(x, z) + \nu^2 n^2(x, z)u(x, z) = 0. \tag{A.10}$$

A.3 Derivation of the Schrödinger equation

In the two following subsections we will derive the Schrödinger equation from the Helmholtz equation in two different cases. In A.3.1 we will assume that the refractive index depends on the x-direction only and seek solutions propagating in the z-direction. In A.1 we will go back to a refractive index depending on both x and z and seek z-propagating solutions by applying the paraxial approximation. The latter section is more physically intuitive while the former is more mathematically straightforward.

A.3.1 x-dependent refractive index and the time-independent Schrödinger equation

In this section we will link particular solutions of the Helmholtz equation with those of the time-independent Schrödinger equation.

Assuming that the refractive index depends on x only, the Helmholtz equation is

$$\left(\partial_x^2 + \partial_z^2\right) u(x,z) + \nu^2 n^2(x) u(x,z) = 0. \tag{A.11}$$

Begin by nondimensionalizing (A.11) and consider a change of variables by rescaling $x' = \nu x$ and $z' = \nu z$. This gives the equation

$$\left(\partial_{x'}^2 + \partial_{z'}^2\right) u\left(\frac{x'}{\nu}, \frac{z'}{\nu}\right) + n^2\left(\frac{x'}{\nu}\right) u\left(\frac{x'}{\nu}, \frac{z'}{\nu}\right) = 0. \tag{A.12}$$

Assume z-propagating solutions u of the form $u\left(\frac{x'}{\nu}, \frac{z'}{\nu}\right) = e^{ik_z z'}\Psi(x')$. With this ansatz, the Helmholtz equation (A.12) becomes

$$\partial_{x'}^2 \Psi(x') - k_z^2 \Psi(x') + n^2\left(\frac{x'}{\nu}\right) \Psi(x') = 0. \tag{A.13}$$

Furthermore, assume a background refractive index, n_0, so that we can write the equation for Ψ,

$$-\partial_{x'}^2 \Psi(x') + \left(n_0^2 - n^2\left(\frac{x'}{\nu}\right)\right) \Psi(x') = \left(-k_z^2 + n_0^2\right) \Psi(x'). \tag{A.14}$$

This is the time-independent Schrödinger equation and can be rewritten as

$$-\partial_{x'}^2 \Psi(x') + V(x')\Psi(x') = E\Psi(x'), \tag{A.15}$$

with potential $V(x')$ and energy parameter E given by

$$V(x') = n_0^2 - n^2\left(\frac{x'}{\nu}\right),$$
$$E = -k_z^2 + n_0^2.$$

We conclude that if E is an eigenvalue of (A.15) for which $k_z^2 \equiv -E+n_0^2 > 0$ with corresponding localized eigenstate $\Psi(x')$, then $k_z = \pm\sqrt{-E+n_0^2}$ is real and

$$u(x,z) = e^{\pm i\sqrt{-E+n_0^2}\ \nu x}\Psi(\nu x) \tag{A.16}$$

is a propagating in z, localized in x, guided mode for the Helmholtz equation (A.25). From these solutions we can solve for the electric field of the system described in Section A.1, which has the form

$$E(x,z,t) = e^{i\omega t}e^{\pm i\sqrt{-E+n_0^2}\ \nu x}\Psi(\nu x). \tag{A.17}$$

and then the magnetic field as well. Therefore, solving equation (A.15) is of great interest in applications.

Remark A.3.1. *We use the formulation* (A.15) *of the Schrödinger equation in Chapters 2 and 4 with special coices of the material of propagation described by $V(x')$.*

A.3.2 (x,z)-dependent refractive index and the time-dependent Schrödinger Equation

In this section we will study a refractive index that depends both on x and z. In order to derive the time-dependent Schrödinger equation, we will use the paraxial approximation.

The Helmholtz equation in this setting is

$$\left(\partial_x^2 + \partial_z^2\right)u(x,z) + \nu^2 n^2(x,z)u(x,z) = 0. \tag{A.18}$$

We seek solutions to equation (A.18) which are paraxial waves travelling in the z-direction. In order to gain an intuitive understanding of what a paraxial wave is, let us consider a constant background medium of propagation, $n(x,z) \equiv n_0$. Define the solutions $u(x,z)$ as

$$u(x,z) = A(x,z)e^{i\nu n_0 z}, \tag{A.19}$$

which propagate with wave length $\lambda = 2\pi/\nu n_0$ and have amplitude $|A|$. Mathematically, one defines a paraxial wave such that the amplitude satisfies

$$|\partial_z^2 A| \ll \nu n_0 |\partial_z A| \ll (\nu n_0)^2 |A|, \tag{A.20}$$

but we would like to go into detail to understand what this means physically.

We define a paraxial wave by

Definition A.3.2. *Paraxial waves have wavefront normals which are paraxial rays, that is rays which make small angles,* θ*, with their axis of propagation. Small in the sense that we can approximate* $\sin(\theta) \approx \theta$*; see Figure A.2.*

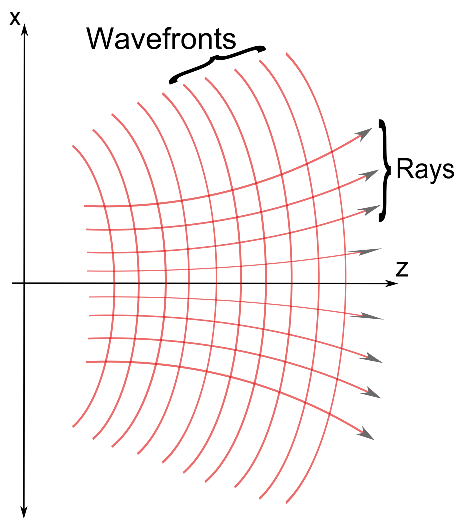

Figure A.2: Sketch of paraxial wavefronts and their respective rays. Note that the rays make small angles with respect to the z-axis, the axis of propagation.

One way to envision what a paraxial wave looks like is to take a plane wave propagating in the z-direction, $u(x, z) = Ae^{i\nu n_0 z}$, and modulate the envelope A by making it a slowly varying function of position, $A(x, z)$. By "slowly varying" we specifically mean that $A(x, z)$ changes only slightly with respect to the direction of wave propagation, z, within the distance of the wavelength, $\lambda = 2\pi/\nu n_0$. See Figure A.3. We thus consider solutions of the form (A.19) with slowly varying envelope and derive the mathematical requirements (A.20).

To see the consequences of the "slow varying" assumption, let us consider $z \in [z_0 - \lambda, z_0 + \lambda]$. We first note that the change in A within this region will be much less than the value of A itself, ie.

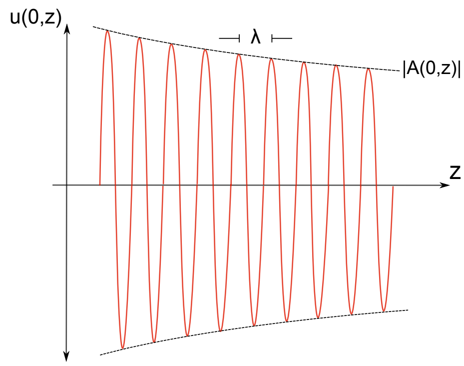

Figure A.3: Profile of a paraxial wave propagating in the z-direction. Note that the amplitude, $|A(0,z)|$, is slowly varying within a λ-distance with respect to the z-axis, the axis of propagation.

$\Delta A \ll |A|$. Second, since A does not change drastically within this interval, we can Taylor expand in the z-variable about the point z_0. Keeping x constant, we get

$$A(z) = A(z_0) + \partial_z A(z_0)|z - z_0| + \frac{1}{2}\partial_z^2 A(z_0)|z' - z_b|^2,$$

for some z' such that $|z' - z_0| < \lambda$ and $z \in [z_0 - \lambda, z_0 + \lambda]$. We rewrite this as

$$A(z) - A(z_0) = \partial_z A(z_0)|z - z_0| + \frac{1}{2}\partial_z^2 A(z_0)|z' - z_0|^2.$$

For $z \in [z_0 - \lambda, z_0 + \lambda]$, we can conclude

$$\Delta A \geq |A(z) - A(z_0)| \geq |\partial_z A|\, \lambda - \frac{1}{2}\left|\partial_z^2 A\right| \lambda^2.$$

Since we have $\Delta A \ll |A|$, we conclude that

$$\left|\frac{\partial A}{\partial z}\right| \ll \frac{|A|}{\lambda} + \mathcal{O}(\lambda) \approx \nu n_0 |A|.$$

The derivative $\partial_z A$ must also vary slowly within the distance λ, ie. $\Delta\, |\partial_z A| \ll |\partial_z A|$. Therefore, by the same argument applied to $\partial_z A$ instead of A, we have

$$\left|\partial_z^2 A\right| \ll \nu n_0\, |\partial_z A| \ll (\nu n_0)^2 |A|. \tag{A.21}$$

This last set of inequalities is relation (A.20) and is what is often stated in mathematical texts as the paraxial approximation.

Experimental set-up. There is an alternative way of thinking about the requirement (A.21) in terms of an experimental set-up. Namely, we want to interpret the first bound, $\left|\partial_z^2 A\right| \ll \nu n_0 \left|\partial_z A\right|$, in terms of physical quantities. The amplitude A and the parameter ν intrinsically depend on an incoming wave or beam of light, so we wish to non-dimensionalize this relation in terms of beam parameters. (Recall, $\nu = \omega/c_0$ where ω is the time harmonic frequency which is determined by the incoming wave to the system.)

Consider an incoming beam travelling along the z-axis with radial symmetry in the x, y-plane, i.e. instead of (A.19) consider a wave function of the form

$$u(r,z) = e^{i\nu n_0 z} A(r,z), \qquad r = \sqrt{x^2+y^2}, \tag{A.22}$$

where we now generalize to three dimensions. We non-dimensionalize r and z by

$$r' = \frac{r}{r_0}, \qquad z' = \frac{z}{2L_{DF}}, \tag{A.23}$$

where r_0 is the initial beam width and $L_{DF} = kr_0^2$ is the diffraction length. If we want (A.21) to hold, we would require

$$\left(\frac{1}{2r_0\nu n_0}\right)^2 \left|\partial_{z'}^2 A\right| \ll \left|\partial_{z'} A\right|. \tag{A.24}$$

If we have no assumptions on the envelope A, we would require $(1/2r_0\nu n_0)^2 \ll 1$. This means that the initial beam width r_0 is much larger than the wavelength λ, which is a typical physical set-up in an experimental setting when studying wave propagation through various mediums.

Note that this is equivalent to assuming that the amplitude, A, changes very slowly within a wavelength, λ, distance.

A.3.3 Paraxial Helmholtz equation

We begin the study of the paraxial Helmholtz equation by first considering the Helmholtz equation in a medium of constant refractive index n_0,

$$\Delta u + \nu^2 n_0^2 u = 0, \tag{A.25}$$

where we define the Lapalcian operator by $\Delta \equiv \; \partial_x^2 + \; \partial_z^2$. We seek paraxial wave solutions of equation (A.25) so we assume solutions of the form

$$u(x,z) = e^{i\nu n_0 z} A(x,z), \tag{A.26}$$

where A is as described above. Substituting (A.26) into (A.25) gives the equation

$$\partial_x^2 A + \partial_z^2 A + 2i\nu n_0 \partial_z A = 0.$$

By relation (A.20) we can assume that

$$|\partial_z^2 A| \ll |2\nu n_0 \partial_z A|, \tag{A.27}$$

and so we can discard the $\partial_z^2 A$ term. This gives the paraxial Helmholtz equation defined by

$$-\partial_x^2 A = 2i\nu n_0 \partial_z A, \tag{A.28}$$

which the envelope A must satisfy.

To non-dimensionalize this system, we set $x' = \nu x$ and $z' = \nu z$. This gives

$$\partial_x = \frac{\partial \tilde{x}}{\partial x}\partial_{\tilde{x}} = \nu \partial_{\tilde{x}}, \quad \partial_x^2 = \nu^2 \partial_{\tilde{x}}^2.$$

Therefore, defining $\Psi(x', z') \equiv A\left(\frac{x'}{\nu}, \frac{z'}{\nu}\right)$, the non-dimensional paraxial Helmholtz equation is

$$2in_0 \partial_{z'} \Psi(x', z') = -\partial_{x'}^2 \Psi(x', z') \tag{A.29}$$

We note that this is very similar to the time dependent Schrödinger equation with the main difference being that z' replaces the time variable t.

A.3.4 Variable refractive index, $n(x, z)$

We now consider the two-dimensional Helmholtz equation with a non-constant refractive index,

$$\left(\partial_x^2 + \partial_z^2\right) u(x, z) + \nu^2 n^2(x, z) u(x, z) = 0. \tag{A.30}$$

By again assuming paraxial solutions of the form (A.26), namely $u(x, z) = e^{i\nu n_0 z} A(x, z)$, for some background refractive index n_0, we conclude that the envelope A must satisfy the paraxial Helmholtz equation

$$\partial_x^2 A(x, z) + 2i\nu n_0 \partial_z A + \nu^2 (n^2(x, z) - n_0^2) A(x, z) = 0. \tag{A.31}$$

We non-dimensionalize again by setting $x' = \nu x$ and $z' = \nu z$. Defining $\Psi(x', z') \equiv A\left(\frac{x'}{\nu}, \frac{z'}{\nu}\right)$, equation (A.31) becomes the time-dependent Schrödinger equation

$$2in_0 \partial_{z'} \Psi(x', z') = -\partial_{x'}^2 \Psi(x', z') + V(x', z') \Psi(x', z'), \tag{A.32}$$

where $V(x', z')$ is the potential defined as

$$V(x', z') \equiv n_0^2 - n^2\left(\frac{x'}{\nu}, \frac{z'}{\nu}\right). \tag{A.33}$$

Remark A.3.3. *The time-dependent Schrödinger equation* (A.32) *can be solved with various methods depending on the structure of the potential* $V(x', z')$*. It will have localized in* x *solutions under special conditions, which is what we study in Chapter 3 with* V *a highly oscillatory potential representing a material that has steep changes in its refractive index with respect to a small spatial scale.*

Appendix B

Floquet-Bloch Theory

In this appendix we summarize basic results on the spectral theory of Schrödinger operators with periodic potentials defined on $\mathbb{R}$. For a detailed discussion, see for example, [Eastham, 1973; Reed and Simon, 1978; Magnus and Winkler, 1979].

In this section, we will make repeated use of the Poisson Summation Formula.

Theorem B.0.4. *Let f and $\widehat{f}$ be such that $f \in L^1(\mathbb{R})$ is continuous and, for some $\delta > 0$, $(1+|\cdot|)^\delta f(\cdot) \in L^1(\mathbb{R})$; [Stein and Weiss, 1971]. Then,*

$$\sum_{\nu\in\mathbb{Z}} f(x+\nu) = \sum_{\nu\in\mathbb{Z}} e^{2\pi i\nu x}\widehat{f}(\nu). \tag{B.1}$$

B.1 Floquet-Bloch states

We seek solutions of the $k-$pseudo-periodic eigenvalue problem

$$H_Q u(x) = (-\partial_x^2 + Q(x))u(x) = Eu(x) \ , \quad u(x+1) = e^{2\pi ik}u(x) \ , \tag{B.2}$$

for $k \in (-1/2, 1/2]$, the *Brillouin zone*. Setting $u(x;k) = e^{2\pi ikx}p(x;k)$, we equivalently seek eigenfunction solutions of the periodic elliptic boundary value problem:

$$\left(-(\partial_x + 2\pi ik)^2 + Q(x)\right)p(x;k) = E(k)p(x;k), \quad p(x+1;k) = p(x;k) \tag{B.3}$$

for each $k \in (-1/2, 1/2]$. The eigenvalue problem (B.3) has a discrete set of eigenpairs $\{p_b(x;k), E_b(k)\}_{b\geq 0}$ satisfying

$$E_0(k) \leq E_1(k) \leq \cdots \leq E_b(k) \leq \ldots,$$

such that (B.2) has corresponding eigenfunctions

$$u_b(x;k) = e^{2\pi ikx}p_b(x;k), \quad p_b(x+1;k) = p_b(x;k), \quad b \geq 0.$$

The b^{th} spectral band of (B.2) is given by:

$$\mathcal{B}_b = \bigcup_{k\in(-1/2,1/2]} E_b(k). \tag{B.4}$$

The full spectrum of H_Q is continuous and is given by:

$$\text{spec}(H_Q) = \bigcup_{b\geq 0} \mathcal{B}_b = \bigcup_{b\geq 0} \bigcup_{k\in(-1/2,1/2]} E_b(k). \tag{B.5}$$

The functions $p_b(x;k)$ may be chosen so that $\{p_b(x;k)\}_{b\geq 0}$ is, for each fixed $k \in (-1/2, 1/2]$, a complete orthonormal set in $L^2_{\text{per}}([0,1])$. It can be shown that the set of Floquet-Bloch states $\{u_b(x;k) \equiv e^{2\pi ikx}p_b(x;k),\ b \in \mathbb{N},\ -1/2 < k \leq 1/2\}$ is complete in $L^2(\mathbb{R})$, *i.e.* for any $f \in L^2(\mathbb{R})$,

$$f(x) \ - \sum_{0\leq b\leq N} \int_{-1/2}^{1/2} \langle u_b(\cdot,k), f\rangle_{L^2(\mathbb{R})} u_b(x;k)\ dk \to \ 0$$

in $L^2(\mathbb{R})$ as $N \uparrow \infty$.

Definition B.1.1. *We say there is a spectral gap between the b^{th} and $(b+1)^{st}$ bands if*

$$\sup_{|k|\leq 1/2} |E_b(k)| \ < \ \inf_{|k|\leq 1/2} |E_{b+1}(k)| \ .$$

Our study of eigenvalue bifurcation from the band edge $E_\star \equiv E_{b_*}(k_*)$, $k_* \in \{0, 1/2\}$, into a spectral gap, requires regularity and detailed properties of $E_b(k)$ near its edges. These are summarized in the following sections. For more detail of these results see [Eastham, 1973].

B.1.1 General properties of $E_b(k)$ and derivatives $\partial_k^j E_b(k_*)$, where $E_b(k_*)$ is the endpoint of a spectral band

For each band $\mathcal{B}_b$, we analyse $E_b(k)$ along the band edges. Begin by considering, for E fixed,

$$\left(-\partial_x^2 + Q(x)\right)\psi(x;E) = E\psi(x;E), \quad Q(x+1) = Q(x), \tag{B.6}$$

with solutions which satisfy

$$\psi(x+1;E) = \rho\psi(x;E) \qquad \rho \in \mathbb{C}.$$

Let $\phi_1(x;E)$ and $\phi_2(x;E)$ be two linearly independent solutions of (B.6) such that

$$\begin{aligned} \phi_1(0;E) &= 1, & \phi_2(0;E) &= 0, \\ \phi_1'(0;E) &= 0, & \phi_2'(0;E) &= 1. \end{aligned}$$

The functions $\phi_1(x+1;E)$ and $\phi_2(x+1;E)$ are two other linearly independent solutions to (B.6), so that we can write

$$\phi_1(x+1;E) = A_{11}\phi_1(x;E) + A_{12}\phi_2(x;E), \tag{B.7}$$

$$\phi_2(x+1;E) = A_{21}\phi_1(x;E) + A_{22}\phi_2(x;E). \tag{B.8}$$

Note that the matrix (A_{ij}) is nonsingular. In general, every solution of (B.6) has the form

$$\psi(x;E) = c_1\phi_1(x;E) + c_2\phi_2(x;E). \tag{B.9}$$

As we are specifically interested in solutions which satisfy $\psi(x+1;E) = \rho\psi(x;E)$, one has the following identity

$$\begin{aligned} \psi(x+1;E) = \rho\psi(x;E) &\Leftrightarrow c_1(\phi_1(x+1;E) - \rho\phi_1(x;E)) + c_2(\phi_2(x+1;E) - \rho\phi_2(x;E)) = 0 \\ &\Leftrightarrow (c_1(A_{11}-\rho) + c_2A_{21})\,\phi_1(x;E) + (c_1A_{12} + c_2(A_{22}-\rho))\,\phi_2(x;E) = 0 \\ &\Rightarrow \begin{cases} c_1(A_{11}-\rho) + c_2A_{21} = 0, \\ c_1A_{12} + c_2(A_{22}-\rho) = 0. \end{cases} \end{aligned} \tag{B.10}$$

The solvability condition (B.10) is satisfied for nontrivial c_1 and c_2 if

$$\det(A-\rho I) = 0, \quad \text{i.e.} \quad \rho^2 - (A_{11}+A_{22})\,\rho + \det(A) = 0. \tag{B.11}$$

Using that the Wronskian, $W[\phi_1,\phi_2](x;E) \equiv \phi_1(x;E)\phi_2'(x;E) - \phi_1'(x;E)\phi_2(x;E)$, is constant with respect to x, one has

$$\det(A) = W[\phi_1,\phi_2](1;E) = W[\phi_1,\phi_2](0;E) = 1.$$

Therefore ρ must satisfy $\rho^2 - D(E)\rho + 1 = 0$, where we define the discriminant

$$D(E) \;\equiv\; A_{11} + A_{22} \;=\; \phi_1(1;E) + \phi_2'(1;E)\ . \tag{B.12}$$

We note that the two solutions of the equation $\rho^2 - D(E)\rho + 1 = 0$ satisfy $|\rho| \leq 1$ if and only if the discriminant $|D(E)| \leq 2$. In that case, one can write $\rho = e^{\pm 2\pi i k}$, with $k \in (-1/2, 1/2]$, and

$$D(E) = 2\cos(2\pi k). \tag{B.13}$$

As $|\rho| = 1$, $\psi(x; E)$ is a bounded solution to (B.6), and $E = E_b(k)$ is in the continuous spectrum of $H_Q \equiv -\frac{d^2}{dx^2} + Q$. More precisely, for $E = E_b(k)$, one has

$$\psi(x; E_b(k)) \;=\; u_b(x; k) \;=\; e^{2\pi i k x} p_b(x; k), \quad p_b(x+1; k) = p_b(x; k),$$

where $\{E_b(k), p_b(x; k)\}_{b \geq 0}$ is the eigenpair solution to (B.3).

We are now set up to state some properties associated with the stability bands.

Lemma B.1.2. *Assume $E_b(k_*)$ is an endpoint of a spectral band of $-\partial_x^2 + Q(x)$, which borders on a spectral gap; see Definition B.1.1. Then $k_* \in \{0, 1/2\}$ and the following results hold:*

1. *$E_b(k_*)$ is a simple eigenvalue of the eigenvalue problem* (B.2).
2. *b even: $E_b(0)$ corresponds to the* left (lowermost) *end point of the band,*

   $E_b(1/2)$ corresponds to the right (uppermost) *end point.*

   b odd: $E_b(0)$ corresponds to the right (uppermost) *end point of the band,*

   $E_b(1/2)$ corresponds to the left (lowermost) *end point.*
3. *$\partial_k E_b(k_*) = 0$, $\partial_k^3 E_b(k_*) = 0$;*
4. *b even: $\partial_k^2 E_b(0) > 0$, $\partial_k^2 E_b(1/2) < 0$;*

   b odd: $\partial_k^2 E_b(0) < 0$, $\partial_k^2 E_b(1/2) > 0$;

The proof of Lemma B.1.2 is a consequence of the following result, concerning the problem (B.6), and which is proved in the first two chapters of [Eastham, 1973] and part I of [Magnus and Winkler, 1979].

Theorem B.1.3. *Consider the equation* (B.6)*, and define $D(E)$ with* (B.12)*. Denote the edges of the stability bands as*

$$G_0 < F_0 \leq F_1 < G_1 \leq G_2 < F_2 \leq F_3 < G_3 \ldots$$

Then the following facts hold (see Figure B.1 for an illustration):

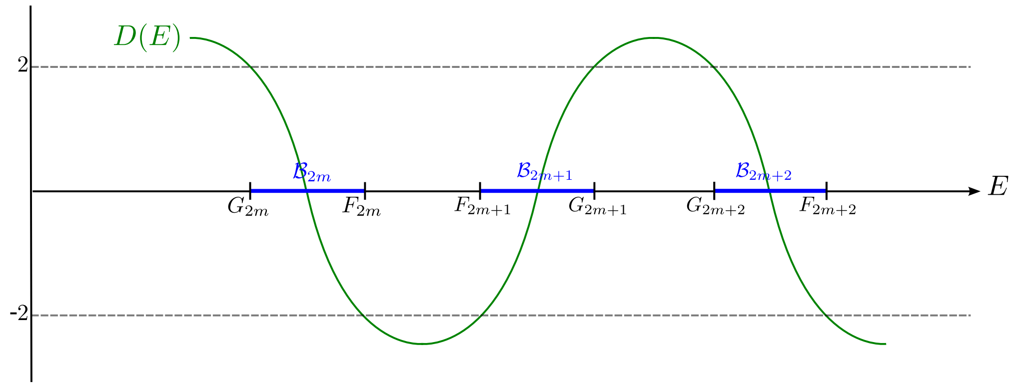

Figure B.1: Sketch of the discriminant, $D(E)$, and stability bands $\mathcal{B}_b = [G_b, F_b]$.

I *In the interval* $[G_{2m}, F_{2m}]$*,* $D(E)$ *decreases from* 2 *to* -2*.*

I′ *In the interval* (G_{2m}, F_{2m})*,* $D'(E) < 0$*.*

II *In the interval* $[F_{2m+1}, G_{2m+1}]$*,* $D(E)$ *increases from* -2 *to* 2*.*

II′ *In the interval* (F_{2m+1}, G_{2m+1})*,* $D'(E) > 0$*.*

III *In* $(-\infty, G_0)$ *and* (G_{2m+1}, G_{2m+2})*,* $D(E) > 2$*.*

IV *In* (F_{2m}, F_{2m+1})*,* $D(F) < -2$*.*

V $D(E) = \pm 2$ *and* $D'(E) = 0$ *if and only if* E *is a double eigenvalue. Furthermore,* $D''(E) < 0$ *if* $D(E) = 2$ *and* $D''(E) > 0$ *if* $D(E) = -2$*.*

Proof of Lemma B.1.2. Let us recall that one has from (B.13) that the discriminant satisfies $D(E_b(k)) = 2\cos(2\pi k)$. It follows that as k increases continuously from 0 to 1/2, $D(E)$ decreases continuously from 2 to -2. Therefore by I and II, $E_{2m}(k)$ increases continuously from G_{2m} to F_{2m} as k increases continuously from 0 to 1/2, and as k decreases continuously from 0 to $-1/2$. Similarly, $E_{2m+1}(k)$ decreases continuously from G_{2m+1} to F_{2m+1} as k increases continuously from 0 to 1/2, and as k decreases continuously from 0 to $-1/2$. This proves claim *2*.

We now turn to part *1*. Let $E_b(k_*)$ correspond to a band edge, that is if there exists a gap between the b^{th} band and the closest consecutive one. Without loss of generality, we assume $E_b(k_*)$ to be the lowermost edge of an even band, for example G_{2m} in Figure B.1. Therefore, for any $\delta > 0$

sufficiently small,

$$D(E_b(k_*) - \delta) > 2 \quad \text{and} \quad D(E_b(k_*) + \delta) < 2. \tag{B.14}$$

Assume for the sake of contradiction that $E_b(k_*)$ is a double eigenvalue, which means, by part V of Theorem B.1.3, that $D'(E_b(k_*)) = 0$ and $D''(E_b(k_*)) < 0$. Now, Taylor expand the discriminant about $E_b(k_*)$,

$$\begin{aligned} D(E) &= D(E_b(k_*)) + D'(E_b(k_*))(E - E_b(k_*)) + \frac{1}{2}D''(E_b(k_*))(E - E_b(k_*))^2 + \mathcal{O}\left((E - E_b(k_*))^3\right) \\ &= -2 + \frac{1}{2}D''(E_b(k_*))(E - E_b(k_*))^2 + \mathcal{O}\left((E - E_b(k_*))^3\right). \end{aligned}$$

Since $D''(E_b(k_*)) < 0$, we have $D(E_b(k_*) - \delta) \approx 2 + (1/2)D''(E_b(k_*))\delta^2 < 2$, which is a contradiction of (B.14). Therefore part *1* is proven and we have that at the band edges, $E_b(k_*)$, the derivative of the discriminant is nonzero,

$$\frac{dD}{dE}(E_b(k_*)) \neq 0. \tag{B.15}$$

To see the first identity in part *3*, note that differentiating $D(E_b(k)) = 2\cos(2\pi k)$ with respect to k yields $-4\pi \sin(2\pi k) = \frac{dD}{dE}(E_b(k)) \times \frac{dE_b}{dk}(k)$. Using (B.15), we conclude that $\frac{dE_b}{dk}(k) = 0$ if and only if $k = 0$ or $k = 1/2$.

To prove part *4*, we differentiate $D(E_b(k))$ twice with respect to k and evaluate at k_*:

$$-8\pi^2 \cos(2\pi k_*) = \frac{d^2}{dk^2}(D \circ E_b)(k_*) = D''(E_b(k_*))\left(\frac{dE_b}{dk}\right)^2 (k_*) + D'(E_b(k_*))\frac{d^2 E_b}{dk^2}(k_*).$$

Therefore, by I$'$, II$'$, and (B.15) we conclude *4*.

Similarly, to show the second identity of part *3*, we differentiate once more $D(E_b(k))$ with respect to k:

$$16\pi^3 \sin(2\pi k) = D'(E_b(k))\frac{d^3 E_b}{dk^3}(k) + 3D''(E_b(k))\frac{d^2 E_b}{dk^2}\frac{d\ E_b}{dk}(k) + D'''(E_b(k))\left(\frac{dE_b}{dk}\right)^3 (k).$$

Evaluated at k_*, we have $0 = D'(E_b(k))\frac{d^3 E_b}{dk^3}(k)$, which concludes the proof of Lemma B.1.2 once we again note (B.15). $\square$

We conclude this section by recalling Weyl's asymptotics (see [Courant and Hilbert, 1953; Eastham, 1973])

Lemma B.1.4. *There exists $C_1, C_2 > 0$ such that for any $b \in \mathbb{N}$ and $k \in (-1/2, 1/2]$,*

$$\pi^2 b^2 - C_1 \leq E_b(k) \leq \pi^2(b+1)^2 + C_2. \tag{B.16}$$

B.1.2 Regularity of $k \mapsto E_b(k)$ and $k \mapsto u_b(x;k)$

In this section we give a self-contained discussion of the regularity with respect to k of the Floquet-Bloch eigenvalues and eigenstates.

Consider the $k-$pseudo-periodic eigenvalue problem for each $k \in (-1/2, 1/2]$:

$$\left(-\partial_x^2 + Q(x)\right) u(x;k) = \ Eu(x;k), \quad u(x+1;k) = e^{2\pi i k} u(x;k) \tag{B.17}$$

Introducing the Floquet-Bloch phase explicitly via $u(x;k) = e^{2\pi i k x}\, p(x;k)$, we obtain the equivalent formulation

$$H_Q(k)p(x;k) \ = \left(-(\partial_x + 2\pi i k)^2 + Q(x)\right) p(x;k) = \ Ep(x;k), \quad p(x+1;k) = p(x;k)\ . \tag{B.18}$$

For each $k \in (-1/2, 1/2]$, the eigenvalue problem (B.18) (equivalently (B.17)) has a discrete sequence of eigenvalues $E_0(k) \le E_1(k) \le E_2(k) \le \cdots \le E_n(k) \le \cdots$.

It can be proved, using the min-max characterization of eigenvalues of a self-adjoint operator that the maps $k \mapsto E_b(k)$, $b = 0, 1, \ldots$, are locally Lipschitz continuous. A proof based on standard perturbation follows from results in [Reed and Simon, 1978]. An elementary proof is given in Appendix A of [Fefferman and Weinstein, 2014].

Throughout this work, we require a Taylor expansion of the $E_b(k)$ near $k = k_*$, for which $E_b(k_*)$ is the endpoint of a spectral band, which borders on a spectral gap. By part *5* of Lemma B.1.2, the eigenvalue $E_b(k_*)$ is simple. We prove the following

Theorem B.1.5. *Suppose E_* is the endpoint of a spectral band of $-\partial_x^2 + Q(x)$, which borders on a gap. Thus, $E_* = E_b(k_*)$ for $k_* \in \{0, 1/2\}$ and the corresponding eigenspace of solutions to* (B.18) *has dimension equal to* 1*. We denote the normalized eigenfunction by $p(x;k_*)$;*

$$\int_0^1 |p(y;k_*)|^2 dy \ = \ 1.$$

Then, there exists $\rho > 0$ such that for all complex k in a complex disc centered at k_*, $B_\rho(k_*) = \{k \in \mathbb{C} : |k - k_*| < \rho\}$, the following holds:

1. *$k \mapsto E_b(k)$ is analytic on B_ρ.*

2. *There is a map $k \mapsto p_b(x;k)$, such that any eigenvector corresponding to $E_b(k)$ is a multiple of $p_b(x;k)$, where $H_Q(k)p_b(x;k) = E_b(k)p_b(x;k)$.*

3. Moreover, we can choose $k \mapsto p_b(x;k),\ k \in B_\rho$ to be analytic and such that

$$\int_0^1 |p_b(x;k)|^2 dx \ = \ 1.$$

Proof. Let $k = k_* + \kappa$, where κ will be chosen to be sufficiently small. The periodic eigenvalue problem (B.18) may be rewritten as

$$H_Q(k_*)p(x;k_*+\kappa) \ - \ \left(4\pi i\kappa(\partial_x + 2\pi i k_*) - 4\pi^2\kappa^2\right) p(x;k_*+\kappa) \ = \ E\ p(x;k_*+\kappa), \tag{B.19}$$

$$p(x+1;k_*+\kappa) = p(x;k_*+\kappa)\ ,\ x \in \mathbb{R}\ . \tag{B.20}$$

We seek an eigen-solution of (B.19)-(B.20) in the form

$$p(x;k_*+\kappa) \ = \ p(x;k_*) + \eta(x;\kappa), \tag{B.21}$$

$$E(k_*+\kappa) \ = \ E_* + \mu(\kappa), \tag{B.22}$$

where we assume that $\eta(\cdot;\kappa) \perp p(\cdot;\kappa)$. Substitution into (B.19)-(B.20) yields the following equation for $\eta(x;\mu,\kappa)$:

$$\left(H_Q(k_*) - E_*\right)\eta \ - \ \left(4\pi i\kappa(\partial_x + 2\pi i k_\star) - 4\pi^2\kappa^2 + \mu\right)\eta = \left(4\pi i\kappa(\partial_x + 2\pi i k_\star) - 4\pi^2\kappa^2 + \mu\right) p(\cdot;k_*)\ . \tag{B.23}$$

Now, introduce the projection operators

$$\Pi f = \langle p(\cdot;k_*), f\rangle\, p(x;k_*), \quad \text{and} \quad \Pi_\perp = I - \Pi.$$

Applying $\Pi_\perp$ to (B.23) yields

$$\left(H_Q(k_*) - E_*\right)\eta \ - \ \Pi_\perp\left(4\pi i\kappa(\partial_x + 2\pi i k_\star) - 4\pi^2\kappa^2 + \mu\right)\eta \ = \ 4\pi i\kappa\ \Pi_\perp\partial_x p(\cdot;k_*) = 4\pi i\kappa\partial_x p(\cdot;k_*). \tag{B.24}$$

Next, applying Π to (B.23), *i.e.* taking the inner product of (B.23) with $p(\cdot;k_*)$, yields

$$\mu - 8\pi^2 k_*\kappa - 4\pi^2\kappa^2 \ + \ 4\pi i\ \kappa\ \langle p(\cdot;k_*), \partial_x\eta(\cdot;\mu,\kappa)\rangle \ = \ 0. \tag{B.25}$$

We shall now solve (B.23) for η, substitute the result into (B.25) and obtain a closed equation for the eigenvalue correction $\mu = \mu(\kappa)$. Let

$$\mu \ = \ \kappa\mu_1, \quad \eta = \ \kappa\eta_1. \tag{B.26}$$

Equations (B.24) and (B.25) become

$$\left(H_Q(k_*) - E_*\right)\eta_1 \;-\; \kappa\Pi_\perp\left(4\pi i(\partial_x + 2\pi i k_\star) - 4\pi^2\kappa + \mu_1\right)\eta_1 \;=\; 4\pi i\Pi_\perp\partial_x p(\cdot;k_*), \tag{B.27}$$

$$\mu_1 - 8\pi^2 k_* - 4\pi^2\kappa \;+\; 4\pi i\kappa\left\langle p(\cdot;k_*), \partial_x\eta_1(\cdot)\right\rangle \;=\; 0. \tag{B.28}$$

Let $R(E_*)\Pi_\perp = (H_Q(k_*) - E_*)^{-1}\Pi_\perp$. Then,

$$\eta_1(x;\mu_1,\kappa) \;=\; 4\pi i\left(I - \kappa R(E_*)\Pi_\perp\left(4\pi i(\partial_x + 2\pi i k_\star) - 4\pi^2\kappa + \mu_1\right)\right)^{-1} \; R(E_*)\;\Pi_\perp\partial_x p(\cdot;k_*)\;, \tag{B.29}$$

where we take $|\kappa| < \rho$, with ρ chosen so that the Neumann series for the operator on the right hand side of (B.29) converges. Note that the mapping

$$(\mu_1,\kappa) \mapsto \eta_1(x;\mu_1,\kappa)$$

is an analytic map from $\{(\mu_1,\kappa) : |\mu_1| < 1,\; |\kappa| < \rho'\}$ to $H^2_{\rm per}(\mathbb{R})$.

Subsitution of (B.29) into (B.28) gives the scalar equation

$$\mathcal{G}(\mu_1,\kappa) \;=\; 0\;, \tag{B.30}$$

where

$$\mathcal{G}(\mu_1,\kappa) \;=\; \mu_1 - 8\pi^2 k_* - 4\pi^2\kappa \;+\; 4\pi i\kappa\left\langle p(\cdot;k_*), \partial_x\eta_1(\cdot;\mu_1,\kappa)\right\rangle\;. \tag{B.31}$$

We now claim that (B.30) can be solved for $\mu_1 = \mu_1(\kappa)$, which is defined and analytic for $|\kappa| < \rho'$, where $0 < \rho' \le \rho$. If this claim is valid, then $\eta_1(x;\mu_1(\kappa),\kappa)$ is well-defined and analytic in κ for $|\kappa| < \rho'$ and finally

$$p(x;k_* + \kappa) \;=\; p(x;k_*) + \kappa\eta_1(x;\mu_1(\kappa),\kappa) \tag{B.32}$$

$$E(k_* + \kappa) \;=\; E_* + \kappa\mu_1(\kappa) \tag{B.33}$$

are defined and analytic Floquet-Bloch eigensolutions for $|\kappa| < \rho'$.

Now (B.30) is easily solved for $\mu_1 = \mu_1(\kappa)$ via the Implicit Function Theorem. Indeed, we have $\mathcal{G}(8\pi^2 k_*, 0) = 0$ and $\partial_{\mu_1}\mathcal{G}(\mu_1,\kappa)|_{(8\pi^2 k_*,0)} = 1 \neq 0$. This completes the proof of Theorem B.1.5. $\square$

B.2 The Gelfand-Bloch transform

Given $f \in L^2(\mathbb{R})$, we introduce the transform $\mathcal{T}$ and its inverse as follows

$$\mathcal{T}\{f(\cdot)\} = \widetilde{f}(x;k) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x}\widehat{f}(k+n), \qquad \mathcal{T}^{-1}\{\widetilde{f}(x;\cdot)\}(x) = \int_{-1/2}^{1/2} e^{2\pi i x k}\widetilde{f}(x;k)dk. \tag{B.34}$$

One can check that $\mathcal{T}^{-1}\mathcal{T} = Id$.

Let

$$u(x;k) \ = \ e^{2\pi i k x} p(x;k) \tag{B.35}$$

denote a Floquet-Bloch mode. Then, by the Poisson summation formula (B.1), we have that

$$\langle u(\cdot,k), f\rangle_{L^2(\mathbb{R})} \ = \ \langle p(\cdot,k), \widetilde{f}(\cdot;k)\rangle_{L^2_{\rm per}([0,1])} \ .$$

Define

$$\mathcal{T}_b\{f\}(k) \equiv \langle p_b(\cdot;k), \widetilde{f}(\cdot;k)\rangle_{L^2_{\rm per}([0,1])} \equiv \int_0^1 \overline{p_b(x;k)}\widetilde{f}(x;k)dx. \tag{B.36}$$

We recall that by completeness of the $\{p_b(x;k)\}_{b\geq 0}$, the spectral decomposition of $f \in L^2(\mathbb{R})$ in terms of Floquet-Bloch states is

$$\widetilde{f}(x;k) \ = \ \sum_{b\geq 0} \mathcal{T}_b\{f\}(k)p_b(x;k) \ , \quad f(x) \ = \ \sum_{b\geq 0}\int_{-1/2}^{1/2} \mathcal{T}_b\{f\}(k)u_b(x;k)dk. \tag{B.37}$$

Recall the Sobolev space, H^s, the space of functions with square integrable derivatives up to order s. It is natural to construct the following $\mathcal{X}^s$ norm in terms of Floquet-Bloch states:

$$\|\widetilde{\phi}\|^2_{\mathcal{X}^s} \equiv \int_{-1/2}^{1/2}\sum_{b\geq 0}\left(1+|b|^2\right)^s|\mathcal{T}_b\{\phi\}(k)|^2dk. \tag{B.38}$$

Proposition B.2.1. *$H^s(\mathbb{R})$ is isomorphic to $\mathcal{X}^s$ for $s \geq 0$. Moreover, there exist positive constants C_1, C_2 such that for all $\phi \in H^s(\mathbb{R})$, we have $C_1\|\phi\|_{H^s(\mathbb{R})} \leq \|\widetilde{\phi}\|_{\mathcal{X}^s} \leq C_2\|\phi\|_{H^s(\mathbb{R})}$.*

Proof. Since $E_0(0) = \inf \operatorname{spec}(-\partial_x^2 + Q)$, then $L_0 \equiv -\partial_x^2 + Q - E_0(0)$ is a non-negative operator and $H^s(\mathbb{R})$ has the equivalent norm defined by $\|\phi\|_{H^s} \sim \|(I+L_0)^{s/2}\phi\|_{L^2}$. Using orthogonality, it follows that

$$\begin{aligned}\|\phi\|^2_{H^s} \sim \|(I+L_0)^{s/2}\phi\|^2_{L^2} \ &= \ \sum_{b\geq 0}\int_{-1/2}^{1/2}|\mathcal{T}_b\{\phi\}(k)|^2|1+E_b(k)-E_0(0)|^s dk \\ &\approx \sum_{b\geq 0}\left(1+|b|^2\right)^s\int_{-1/2}^{1/2}|\mathcal{T}_b\{\phi\}(k)|^2dk \ \equiv \|\widetilde{\phi}\|^2_{\mathcal{X}^s}.\end{aligned}$$

The approximation in the last line follows from the Weyl asymptotics $E_b(k) \sim b^2$; see, for example, [Courant and Hilbert, 1953]. This completes the proof of Proposition B.2.1. □

Definition B.2.2 (Bloch Convolution). *We define the Bloch convolution of two Bloch transformed functions as*

$$\left(\widetilde{f} \star_B \widetilde{g}\right)(x;k) \equiv \int_{-1/2}^{1/2} \widetilde{f}(x;k-l)\widetilde{g}(x;l)\ dl. \tag{B.39}$$

When $|k-l| > 1/2$, we note that by definition of the Bloch transform we can write for any $c \in \mathbb{Z}$, $\widetilde{f}(x;k+c) = e^{-2\pi ixc}\widetilde{f}(x;k)$.

Proposition B.2.3 (Bloch Convolution). *The Bloch transform of a product of two functions can be written as a Bloch convolution,*

$$\widetilde{(fg)}(x;k) = \int_{-1/2}^{1/2} \widetilde{f}(x;k-l)\widetilde{g}(x;l)\ dl. \tag{B.40}$$

Proof. We will use the Poisson Summation Formula (B.1) to rewrite the right the right hand side of (B.40):

$$\begin{aligned}
\int_{-1/2}^{1/2} \widetilde{f}(x;k-l)\widetilde{g}(x;l)\ dl &= \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} e^{2\pi inx}\widehat{f}(k-l+n) \sum_{m\in\mathbb{Z}} e^{2\pi imx}\widehat{g}(l+m)\ dl \\
&= \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} e^{-2\pi i(k-l)(n+x)} f(n+x) \sum_{m\in\mathbb{Z}} e^{-2\pi il(m+x)} g(m+x)\ dl \\
&= \sum_{n\in\mathbb{Z}} e^{-2\pi ik(n+x)} f(n+x) \sum_{m\in\mathbb{Z}} g(m+x) \int_{-1/2}^{1/2} e^{-2\pi il(m-n)}\ dl \\
&= \sum_{n\in\mathbb{Z}} e^{-2\pi ik(n+x)} f(n+x)g(n+x) \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi inx}\widehat{fg}(k+n) \\
&= \widetilde{(fg)}(x;k),
\end{aligned}$$

which completes the proof of Proposition B.2.3. □

B.3 Information on the Floquet-Bloch states

In this section, we collect some information on the Floquet-Bloch states.

These results will be based on the following identity, which follows from the variations of constants formula (see [Eastham, 1973])

$$u_b(x;k) = u_b(0;k)\cos(\sqrt{E_b(k)}\ x) + \partial_x u_b(0;k)\frac{\sin(\sqrt{E_b(k)}\ x)}{\sqrt{E_b(k)}} + \int_0^x \frac{\sin(\sqrt{E_b(k)}\ (x-y))}{\sqrt{E_b(k)}} Q(y) u_b(y;k)\ dy. \quad \text{(B.41)}$$

Of course, this identity makes sense only for $E_b(k) > 0$. However, since $Q \in L^\infty_{\text{per}}$, one can always introduce a large enough constant C such that $Q + C \geq 1$ and therefore $\text{spec}(H_{Q+C}) \subset [1, +\infty)$, and the above identity holds replacing $\sqrt{E_b(k)}$ with $\sqrt{E_b(k)+C}$ and Q with $Q+C$. *In this section, without loss of generality, we assume* $\inf \text{spec}(H_Q) \geq 1$*. In particular, we have*

$$E_b(k) \geq 1, \quad k \in (-1/2, 1/2], \quad b \geq 0.$$

Lemma B.3.1. *Assume that Q is continuous, one-periodic. Then,*

$$\sup_{k\in(-\frac{1}{2},\frac{1}{2}]} \left\| u_b(x;k) \right\|_{L^\infty(\mathbb{R}_x)} \ \leq\ C\left(\left\|Q\right\|_{L^\infty_{\text{per}}} \right)\ , \quad \text{(B.42)}$$

$$\sup_{k\in(-\frac{1}{2},\frac{1}{2}]} \left(\left\| \partial_x u_b(x;k) \right\|_{L^\infty(\mathbb{R}_x)} + \left\| \partial_x u_b(x;k) \right\|_{L^2([0,1]_x)} \right) \ \leq\ C\left(\left\|Q\right\|_{L^\infty_{\text{per}}} \right)\ (1+|b|)\ , \quad \text{(B.43)}$$

uniformly for $b \geq 0$.

If moreover $Q \in W^{N-2,\infty}_{\text{per}}(\mathbb{R})$ with $N \geq 2$; then one has

$$\sup_{k\in(-\frac{1}{2},\frac{1}{2}]} \left(\left\| \partial_x^N u_b(x;k) \right\|_{L^\infty(\mathbb{R}_x)} + \left\| \partial_x^N u_b(x;k) \right\|_{L^2([0,1]_x)} \right) \ \leq\ C\left(\left\|Q\right\|_{W^{N-2,\infty}_{\text{per}}} \right)\ \left(1+|b|^N\right)\ . \quad \text{(B.44)}$$

Proof. We first bound $|u_b(0;k)|$ and $|\partial_x u_b(0;k)|$. We will use these bounds to prove the estimate claimed in Lemma B.3.1 by applying them to $u_b(x;k)$ as expressed in (B.41).

First, integrate (B.41) against $\cos(\sqrt{E_b(k)}\ x)$ over the integration domain $[0, \pi/\sqrt{E_b(k)}]$.

$$\int_0^{\frac{\pi}{\sqrt{E_b(k)}}} \cos(\sqrt{E_b(k)}\ x) u_b(x;k)\ dx = u_b(0;k)\int_0^{\frac{\pi}{\sqrt{E_b(k)}}} \cos^2(\sqrt{E_b(k)}\ x)\ dx + \partial_x u_b(0;k)\int_0^{\frac{\pi}{\sqrt{E_b(k)}}} \frac{\cos(\sqrt{E_b(k)}\ x)\sin(\sqrt{E_b(k)}\ x)}{\sqrt{E_b(k)}} + \int_0^{\frac{\pi}{\sqrt{E_b(k)}}} \cos(\sqrt{E_b(k)}\ x) \int_0^x \frac{\sin(\sqrt{E_b(k)}\ (x-y))}{\sqrt{E_b(k)}} Q(y) u_b(y;k)\ dydx.$$

Since $\pi/\sqrt{E_b(k)} \lesssim 1$ and $|\sin(x)/x| \leq 1$ for any $x \in \mathbb{R}$, we deduce

$$|u_b(0;k)| \leq C(\|Q\|_{L^\infty_{\rm per}}) \, \|u_b(x;k)\|_{L^2([0,1]_x)} \, . \tag{B.45}$$

Similarly, integrate (B.41) against $\sin(\sqrt{E_b(k)}\ x)$ over $x \in [0, \pi/\sqrt{E_b(k)}]$, and deduce

$$|\partial_x u_b(0;k)| \lesssim \sqrt{E_b(k)} C(\|Q\|_{L^\infty_{\rm per}}) \, \|u_b(x;k)\|_{L^2([0,1]_x)} \, . \tag{B.46}$$

Plugging (B.45)-(B.46) back into (B.41) yields (B.42).

Differentiating (B.41) yields

$$\begin{aligned}\partial_x u_b(x;k) = -\sqrt{E_b(k)} u_b(0;k) \sin(\sqrt{E_b(k)}\ x) + \partial_x u_b(0;k) \cos(\sqrt{E_b(k)}\ x)& \\ + \int_0^x \cos(\sqrt{E_b(k)}\ (x-y)) Q(y) u_b(y;k)\ dy&.\end{aligned}$$

Estimate (B.43) then follows from the bounds (B.45)-(B.46), as well as Weyl's asymptotics (see Lemma B.1.4): $\sqrt{E_b(k)} \approx b$.

Estimate (B.44) is then obtained by induction on N and differentiating $N-2$ times the identity

$$-\partial_x^2 u_b(x;k) + Q(x) u_b(x;k) = E_b(k) u_b(x;k).$$

This completes the proof of Lemma B.3.1. $\square$

We now give more precise asymptotics for $u_b(x;k)$ when b is large.

Lemma B.3.2. *Assume that Q is continuous, one-periodic and $Q \in W^{1,\infty}(\mathbb{R})$. Then one can write $p_b(x;k) = e^{-2\pi i k x} u_b(x;k)$ as*

$$p_b(x;k) = A_b^+(x;k) e^{ix(\sqrt{E_b(k)} - 2\pi k)} + A_b^-(x;k) e^{ix(-\sqrt{E_b(k)} - 2\pi k)}$$

with

$$\begin{aligned}\sup_{k \in (-\frac12, \frac12]} \left\| A_b^\pm(x;k) \right\|_{W^{2,\infty}([0,1]_x)} &\leq C\left(\|Q\|_{W^{1,\infty}_{\rm per}} \right), \\ \sup_{k \in (-\frac12, \frac12]} \left\| \partial_x A_b^\pm(x;k) \right\|_{L^\infty([0,1]_x)} &\leq C\left(\|Q\|_{L^\infty_{\rm per}} \right) \frac{1}{1+|b|},\end{aligned} \tag{B.47}$$

uniformly for $b \geq 0$.

Proof. Following (B.41), we set

$$A_b^+(x;k) = \frac{1}{2}u_b(0;k) + \frac{1}{2i}\frac{\partial_x u_b(0;k)}{\sqrt{E_b(k)}} + \frac{1}{2i\sqrt{E_b(k)}}\int_0^x e^{-y\sqrt{E_b(k)}}Q(y)u_b(y;k)\ dy,$$
$$A_b^-(x;k) = \frac{1}{2}u_b(0;k) - \frac{1}{2i}\frac{\partial_x u_b(0;k)}{\sqrt{E_b(k)}} - \frac{1}{2i\sqrt{E_b(k)}}\int_0^x e^{y\sqrt{E_b(k)}}Q(y)u_b(y;k)\ dy.$$

Using Weyl's asymptotics (Lemma B.1.4), we can approximate $\sqrt{E_b(k)} \approx b$. Therefore, by bounds (B.45)-(B.46),

$$\sup_{k\in(-1/2,1/2]} \left\| A_b^\pm(x;k) \right\|_{L^\infty([0,1]_x)} \leq C.$$

Differentiating $A_b^\pm(x;k)$ once with respect to $x \in [0,1]$ yields

$$\partial_x A_b^\pm(x;k) = \pm\frac{1}{2i\sqrt{E_b(k)}}e^{\mp x\sqrt{E_b(k)}}Q(x)u_b(x;k).$$

By Weyl's asymptotics and result (B.42) of Lemma B.3.1, one has

$$\sup_{k\in(-1/2,1/2]} \left\| \partial_x A_b^\pm(x;k) \right\|_{L^\infty([0,1]_x)} \leq C\left(\|Q\|_{L^\infty_{\text{per}}}\right)\ \frac{1}{1+|b|}.$$

Furthermore, differentiating once more the above identity yields

$$\partial_x^2 A_b^\pm(x;k) = e^{\mp x\sqrt{E_b(k)}}\left(-\frac{1}{2i}Q(x)u_b(x;k) \pm \frac{Q'(x)u_b(x;k) + Q(x)\partial_x u_b(x;k)}{2i\sqrt{E_b(k)}}\right).$$

Once more, using Weyl's asymptotics along with results (B.42)-(B.43) from Lemma B.3.1, one has

$$\sup_{k\in(-1/2,1/2]} \left\| \partial_x^2 A_b^\pm(x;k) \right\|_{L^\infty([0,1]_x)} \leq C\left(\|Q\|_{W^{1,\infty}_{\text{per}}}\right).$$

This concludes the estimates in (B.47) and the proof of Lemma B.3.2 is now complete. $\square$

Appendix C

The Lyapunov-Schmidt Procedure

The Lyapunov-Schmidt procedure [Nirenberg, 2001] is a method used in bifurcation theory to reduce infinite dimensional equations acting on a Banach space to finite dimensional equations. Is is often used when the Implicit Function Theorem (stated below as Theorem C.0.3) can not be applied, for example for nonlinear equations.

Given Banach spaces X, Λ, and Y, where Λ is a parameter space, and, for $p \geq 1$, a C^p map $f(x, \lambda)$ of a neighborhood of (x_0, λ_0) in $X \times \Lambda$ into Y such that $f(x_0, \lambda_0) = 0$, we wish to study the solutions near (x_0, λ_0) of

$$f(x, \lambda) = 0. \tag{C.1}$$

The standard approach to this problem is the Implicit Function Theorem [Nirenberg, 2001]:

Theorem C.0.3 (Implicit Function Theorem)**.** *Let X, Y, and Z be Banach spaces and f a continuous mapping of an open set $U \subset X \times Y \to Z$. Assume that f has a Frechet derivative with respect to x, $f_x(x, y)$, which is continuous in U. Let $(x_0, y_0) \in U$ and $f(x_0, y_0) = 0$. If $A = f_x(x_0, y_0)$ is an isomorphism of X onto Z, then:*

1. *There is a ball $\{y : \|y - y_0\| < r\} = B_r(y_0)$ and a unique continuous map $u : B_r(y_0) \to X$ such that $u(y_0) = x_0$ and $f(u(y), y) \equiv 0$.*

2. *If f is of class C^1, then $u(y)$ is of class C^1 and*

$$u_y(y) = -[f_x(u(y), y)]^{-1} \circ f_y(u(y), y).$$

3. *$u_y(y)$ belongs to C^p if f is in C^p, $p \geq 1$.*

If we have $f_x(x,\lambda)$ to be an invertible operator, then by Theorem C.0.3, there exists a solution $x(\lambda)$ such that $f(x(\lambda),\lambda)=0$ within some neighborhood of λ_0. However, if $f_x(x,\lambda)$ is not invertible, we can apply the Lyapunov-Schmidt reduction which we will now explain.

We make the following assumptions:

(A1) $f_x(x_0,\lambda_0)$ is a Fredholm operator.

(A2) $\ker(f_x(x_0,\lambda_0)) = X_1$ is finite dimensional.

(A3) $\mathrm{ran}(f_x(x_0,\lambda_0)) = Y_1$ is a closed linear subspace of Y of finite co-dimension.

We now split Y into a direct sum, $Y = Y_1 \oplus Y_2$ where $\dim Y_2 < \infty$ by assumption $(A3)$ and define Q to be the projection onto Y_1. Similarly, split X into a direct sum $X = X_1 \oplus X_2$.

Applying the projection operators Q and its complement, $(I-Q)$, on to the original problem, (C.1), and rewrite it as a coupled system of equations,

$$Qf(x,\lambda) = 0, \tag{C.2}$$

$$(I-Q)f(x,\lambda) = 0. \tag{C.3}$$

For $x_1 \in X_1$ and $x_2 \in X_2$, equation (C.2) becomes

$$Qf(x_1+x_2,\lambda) = 0. \tag{C.4}$$

Consider (C.4) as $Qf(x_1+x_2,\lambda) = g(x_2,\widetilde{y}) \ : \ X_2 \times \widetilde{Y} \to Y_1$, where $\widetilde{Y} = X_1 \times \Lambda$. We can apply Theorem (C.0.3) to solve for $g(x_2,\widetilde{y}) = 0$ with respect to x_2 and conclude that there exists a unique $x_2(x_1,\lambda)$ such that

$$Qf(x_1 + x_2(x_1,\lambda),\lambda) = 0.$$

Substituting $x_2(x_1,\lambda)$ into equation (C.3), we have a closed equation in terms of x_1,

$$(I-Q)f(x_1 + x_2(x_1,\lambda),\lambda) = 0. \tag{C.5}$$

Furthermore, since $\mathrm{ran}(I-Q) = Y_2$ is finite dimensional, the operator in equation (C.5) is also finite dimensional.

Remark C.0.4. *Our methods in Chapters 2 and 4 were motivated by the standard Lyapunov-Schmidt procedure, but do not actually follow the above outline as our system equivalent to* (C.5) *is not finite dimensional.*