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ABSTRACT: We study an optimal weak collusion-proof auction in an environment where a subset (or subsets) of bidders may collude not just on their bids but also on their participation. Despite their ability to collude on participation, informational asymmetry facing the potential colluders can be exploited significantly to weaken their collusive power. The second-best outcome — i.e., the noncollusive optimum — can be made weak collusion-proof, if at least one bidder is not collusive, or there are multiple bidding rings, or the second-best outcome involves a nontrivial probability of the object not being sold. In case the second-best is not weak collusion proof, we characterize an optimal weak collusion-proof auction. This auction involves nontrivial exclusion of collusive bidders — i.e., the object is not sold to any collusive bidder with positive probability.

KEYWORDS: Collusion on participation, subgroup collusion, multiple bidding rings, an exclusion principle.

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1 Introduction

The possibility that some agents may collude is an important concern in organization design. Nowhere is the concern more pronounced than in auctions where bidders can manipulate or simply withdraw their bids to limit competition. Not surprisingly, auctions have provided the volume and prominence to the study of collusion, with evidence of collusion documented in a variety of auctions ranging from highway construction contracts (Porter and Zona (1993)), timber sales (Baldwin et al. (1997)), to school milk delivery (Pesendorfer (2000), and Porter and Zona (1999)).

Thanks to the resurgence of interest in collusion, we have by now a fairly good understanding about how bidders overcome their informational asymmetry to achieve successful collusion.[1] Less is known, except for a handful of recent research, is to what extent and how a mechanism can be designed to deal with collusion. The issue at the heart is the informational asymmetry facing potential colluders. If the colluders had complete information about one another, they could behave as if they were a single agent; the only issue then would be to enforce their agreement. Consequently, there would be no role for auction design in preventing collusion, other than to undermine the enforcement power of the colluders. If the colluders had asymmetric information, however, auction design may exploit that informational asymmetry to disrupt collusion even when they have strong enforcement power. Consequently, the presence of informational asymmetry makes the role of auction design nontrivial.

Laffont and Martimort (1997, 2001) offered a tractable modeling framework for analyzing the role of colluders' informational asymmetry in organization design. Following their framework, Che and Kim (2006) have shown that the information asymmetry can be exploited to such an extent that collusion problem can be avoided at no cost in a broad set of circumstances. Their method of collusion-proofing — namely "selling the firm" to potential colluders — requires that no coalition be formed prior to all potential colluders participating into the mechanism. While this assumption may be realistic in some circumstances, it may

[1] Graham and Marshall (1987), McAfee and McMillan (1992), Mailath and Zemsky (1991), Eso and Schummer (200?) and Marshall and Marx (2003) study collusion in one-shot auctions of various formats, while Aoyagi (2003), Blume and Heidhues (2002), Skrzypacz and Hopenhayn (2004) and Abdulkadiroğlu and Chung (2003) study collusion in repeated auctions.

not be appropriate in settings such as auctions where bidders may be intimately familiar with their opponents even before participating. In fact, an allegedly predominant form of collusion involves bidders coordinating on their participation decisions: Colluders either refuse to participate or withdraw their bids to let a designated ring member win without competition. It turns out that "selling the firm to the potential coalition" provides an ineffective protection to the auction designer from such collusion on participation.

This leads us to the main questions of the current paper: Can an auction be designed to exploit the informational asymmetry even when the agents can coordinate on their participation decisions? If so, to what extent can the collusion problem be dealt with and what is the nature of the resulting optimal collusion-proof auction? We address these questions in the single-unit auctions environment.

A few authors have studied these questions. In the context of first-price auctions, McAfee and McMillan (1992) argued that the bidders can overcome informational asymmetry completely if they can use transfers, meaning that they can in fact behave as if they are a single agent maximizing their joint interest. The same result is established in a fully general setting (i.e., without assuming a particular auction form) by Dequiedt (2005), who showed that the seller can never expect to exploit the agents' informational asymmetry in dealing with their collusion — that is, the colluding agents can behave like a single agent — if collusion is organized by a third party who can commit not only to enforce the side contract agreed upon by the agents but also to punish a unilaterally disagreeing agent to the worst possible level. This result clearly provides an upper bound on what collusive agents can accomplish in an adverse selection environment, and thus a lower bound on what the auction designer can achieve. Yet, this benchmark may not be practically the most useful. In practice, it is likely that bidders are limited in their abilities to punish a deviator. In particular, the minmax punishment assumed in Dequiedt (2005) may not be credible against a bidder who refuses to collude. In this sense, the concept of collusion-proofness underlying the negative view may be too strong.

It is thus useful to pursue a different approach — one based on a less extreme presumption of the coalition's punishment capabilities. In this regard, Laffont-Martimort's weak collusion-proofness offers a sensible alternative in the modelling of collusion. This notion presumes coalition's punishment capability to be limited to a level achievable by the standard noncooperative play. A noncooperative play provides a compelling scenario of what may occur when a collusion attempt fails. We thus explore the above questions with this notion of collusion. As will be seen below, the adoption of this weak collusion-proofness notion has a dramatic consequence: The the standard second-best outcome — i.e., the noncollusive optimum — can be made weak collusion-proof for symmetric bidders if at least one bidder is noncollusive or there are multiple bidding rings, or if the second best involves nontrivial probability of not selling to collusive bidders.

The main idea behind our collusion-proof implementation is explained as follows. As will be seen, in a collusion-proof auction, the seller cannot extract any entry fees from the coalition when the good is not allocated to its member, or else the coalition would simply boycott the auction. Further, when the good is sold to any coalition, the seller cannot price discriminate the coalition based on the types of its members. The inability to price-discriminate means that, to attain the second-best outcome, the seller may have to charge a sufficiently high (uniform) price to the coalition, sometimes even higher than their members' valuations. Why can the coalition members then not simply boycott the auction in such a case? Indeed, boycotting an auction whenever the coalition suffers ex post loss would increase the joint payoff of its members. This does not occur in our auction, however, due to the way it exploits the informational asymmetry facing the coalition members. In particular, our collusion-proof auction makes it impossible for the surplus from collusion to be shared across types to make them all better off. For instance, in our collusion-proof auction, boycotting an auction would reduce the informational rents accruing to the highest valuation bidder, thus causing the latter type bidder to object to the boycott. For this reason, an auction rule that forces collusive bidders to sustain some ex post loss can be made collusion proof, unless the amount of ex post loss is too large. In particular, the second-best is weak collusion-proof implementable if there is at least one noncollusive bidder or there are multiple bidding rings or the second best outcome involves nontrivial exclusion of bidders with low valuations. The same logic applies to the case the second-best is not weak collusion proof, implying that an optimal collusion-proof auction involves a positive probability of not selling to any collusive bidder.

Our approach is closest to Pavlov (2006), who also studies the optimal auctions that are weak collusion proof.[2] His analysis concerns only the case of grand collusion — when the

[2]Other authors have studied optimal collusion-proof mechanisms in different contexts. Quesada (2004) finds

entire bidders are involved in a single bidding ring — and focuses only on ex ante symmetric bidders.[3] The current paper goes much beyond that environment. First of all, we consider the general case in which a subset (or subsets) of bidders is (are) collusive. In fact, the most important result concerns the case in which a proper subset of bidders is collusive — i.e., at least one bidder is noncollusive or there are multiple bidding rings — in which case the second-best outcome is shown to be collusion-proof implementable. Second, we can handle the case of ex ante asymmetric bidders, at least for the case of grand collusion: We show that the second-best outcome is collusion-proof implementable, given a somewhat stronger condition than is needed for the symmetric bidders case. Third, our model of collusion differs from his in that we allow members of collusion to reallocate the good once it is sold to one of the members.

Above all, the ability to handle collusion by a subset(s) of bidders is practically important and useful. In many circumstances, not all bidders are in a position to collude. Government auctions used in defense procurement, mineral extraction, or spectrum licenses, competition often have incumbents with long history of operation competing against relative new comers. Long term interaction and shared experiences among the incumbents will put them in a better position to collude than the new comers. Likewise, in auctions for construction repairs or food service procurement, competition may involve both local and non-local providers, and the former group may be able to collude more effectively, based on their regular contacts and the trade association relationships. The problem of only a subset of bidders being collusive introduces a new challenge, since the coalition may prey on noncollusive agents as much as on the seller. Hence, a collusion-proof design must eliminate incentives for the coalition to engage in such behavior.

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an optimal collusion-proof mechanism in the LM setting where an (informed) agent proposes a side contract. In fact, she treats collusion on participation, but adopts the strong notion similar to Dequiedt (2003) where the side contract can impose maximum punishment on refusing agents. Jeon and Menicucci (2005) shows that second-best is achievable in the weak collusion proof sense, much like Che and Kim (2006), in the nonlinear pricing context. Laffont and Martimort (1998), Celik (2004) and Faure-Grimaud et al. (2004) study the optimality of delegation in the presence of collusive agents.

[3] Our results for the grand collusion case, i.e., Theorem 5, is similar to his, but the results are obtained independently. As will be apparent, the methods of analyses are quite different. Of course, the results on other subjects, particularly the subgroup collusion, are completely novel here.

The rest of the paper is organized as follows. Section 2 motivates collusion on participation and illustrates the main design features of collusion-proof auctions. Section 3 introduces an auction model and describes the second-best outcome in a collusion-free environment. Section 4 introduces a model of collusion and the notion of weak collusion-proof auctions. Section 5 identifies the properties of weak collusion-proof auctions and use them to derive a condition that is necessary and sufficient for implementing the second-best outcome in a collusion-proof fashion. In Section 6, we characterize the optimal collusion-proof auction when the second-best is not collusion-proof implementable. Section 7 concludes.

2 Illustration by an Example

Suppose a seller has an object to sell to one of three bidders, each with valuation θ_i drawn privately and uniformly from $[1, 2]$. If the seller values the object at zero, she can do no better than using a standard auction, say a second-price auction. Bidders have weak dominant strategies of bidding their valuations, and the bidder with the highest valuation wins. This generates the expected revenue of 3/2, which is the most the seller can obtain from any feasible selling mechanism.

Standard auctions are susceptible to collusion, however. Suppose bidders 1 and 2 can collude on their bids. Then, they will select, say via a knock out auction between them, the bidder with the higher valuation and have him bid his valuation and the other bid 1, or the lowest possible bid.[4] Although the noncollusive bidder (bidder 3) will continue to bid his valuation, the collusion will cause the revenue to fall; whenever the noncollusive bidder has the lowest valuation, the seller collects that amount rather than the second-highest valuation.

If collusion occurs only after the bidders participate, then Che and Kim (2006)'s mechanism (henceforth referred to as "CK mechanism") can implement the second-best revenue of $\frac{3}{2}$ for the seller. The CK mechanism effectively sells the object at a fixed price of $\frac{3}{2}$ to the two collusive bidders (or more precisely their third-party representative), who then either allocate the good between the two of them or turn around and sell the good to the third bidder at

[4]For instance, they can hold auction whereby the winner (the high bidder) pays his bid to the loser for the right to participate the official auction without any competition from the other collusive bidder. One can easily confirm existence of a symmetric monotonic equilibrium in which the high valuation bidder wins.

price, $p(\theta_3) := \mathbb{E}[\max\{\theta_1, \theta_2\} | \max\{\theta_1, \theta_2\} < \theta_3]$ if he bids θ_3 — the price he would have paid in a collusion-free environment. The CK mechanism replicates the same interim incentives as the second price auction but guarantees the seller an ex post constant revenue of $\frac{3}{2}$ whenever all bidders participate. If bidders do not collude on participating, they will indeed participate. This mechanism would not work, however, if the two bidders collude on participation. Suppose their valuations are both close to 1. Then, they will most likely sell the object to bidder 3 at a price close to $\frac{1}{2} = \mathbb{E}[\theta_3]$, so they will have to pay 1 out of their pockets. Hence, they will coordinate not to participate in such cases.

Here we suggest a different auction rule. This auction rule, which will be explained later in a greater detail, allocates the object efficiently, just like the second-price auction. The rule differs from both the second-price auction and from the CK mechanism, in the following way. Unlike the latter, bidders 1 and 2 are charged only when the object is allocated to them. Further, the joint sale price charged to them depends only on bidder 3's bid, specifically equal to $2b_3 - \frac{5}{4}$, where b_3 is bidder 3's bid. This feature makes it difficult for the coalition to manipulate the sale price, for any attempt to reduce the price cause the probability of their winning to fall. Since bidder 3 bids his valuation as a dominant strategy and the allocation is efficient, when a collusive bidder, say with valuation θ, wins, the coalition ends up paying $\theta - \frac{1}{4}$, high enough to extract the same payment from the collusive bidders as in a collusion-free environment. Since the auction leaves net surplus to the coalition, it has no incentive to boycott the auction, regardless of θ. In fact, this auction is weak collusion proof, even though bidders 1 and 2 can collude on their participation decisions. We shall revisit this example, along with others, for a more in-depth discussion.

3 Primitives

A risk-neutral seller has an object for sale. The seller's valuation of the object is normalized to zero. There are $n \geq 2$ risk-neutral buyers who each independently draw a value, θ_i, on the object from an interval $\Theta_i := [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}_+$ according to distribution F_i, which has strictly positive density f_i on the support. We assume that both

$$J_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \text{ and } K_i(\theta_i) := \theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}.$$

are strictly increasing for all $i \in N$ and all $\theta_i \in \Theta_i$. Throughout, we let $\mathbb{E}[\cdot] := \int_{\Theta}[\cdot]d(\prod_{i\in N} F_i(\theta_i))$. and $\mathbb{E}_{\tilde{\theta}_{-i}}[\cdot] := \int_{\Theta_{-i}}[\cdot]d(\prod_{j\neq i} F_j(\theta_j))$ denote expectation operators based on the prior distribution, where $\Theta := \prod_{i\in N}\Theta_i$ and $\Theta_{-i} := \prod_{j\neq i}\Theta_j$.

For a later analysis, it is analytically convenient to augment each bidder's type space to include his "participation decision" as part of each buyer's possible type. Specifically, we let $\theta_\emptyset$ denote "non-participation" or "exit" option available to each buyer, and define $\overline{\Theta}_i := \{\theta_\emptyset\}\cup\Theta_i$. We then let $\theta := (\theta_1, ..., \theta_n) \in \overline{\Theta} := \prod_{i\in N}\overline{\Theta}_i$ denote a possible profile of types in these enriched type spaces. Since we shall consider randomization in coalition members' reports over their augmented type spaces, it is convenient to consider arbitrary probability distribution, μ^C, over $\overline{\Theta}_C := \prod_{i\in C}\overline{\Theta}_i$ for any $C \subset N$ and to use $\mathbf{E}_{\mu^C}[\cdot] := \int_{\overline{\Theta}_C}[\cdot]d(\mu^C(\theta_C))$ as an expectation operator relative to μ^C.

We now describe arbitrary auction rules, and we do so in direct mechanisms. An *auction rule*, $M = (q, t)$, consists of *an allocation rule*, $q = (q_1, \cdots, q_n) : \overline{\Theta} \mapsto [0, 1]^n$ with $\sum_{i\in N} q_i(\theta) \leq 1$, $\forall\theta \in \overline{\Theta}$, and *a payment rule*, $t = (t_1, ..., t_n) : \overline{\Theta} \mapsto \mathbb{R}^n$, such that $q_i(\theta_\emptyset, \theta_{-i}) = t_i(\theta_\emptyset, \theta_{-i}) = 0, \forall i.\theta_{-i} \in \overline{\Theta}_{-i}$. An auction rule determines, for each profile of bidders' reports in $\overline{\Theta}$, a vector of probabilities for the bidders to obtain the object and a vector of expected payments they must pay, subject to the constraint that, if a bidder invokes the non-participation option, he does not receive the good and collects his reservation utility, normalized to zero. Any equilibrium arising in any auction game can be described as an auction rule in this framework, so we sometimes use an "outcome" interchangeably with an auction rule.

Fix an auction rule, $M = (q, t)$. Buyer i's interim payoff when his valuation is $\theta_i \in \Theta_i$ but reports $\tilde{\theta}_i \in \overline{\Theta}_i$ is

$$u_i^M(\tilde{\theta}_i, \theta_i) := \theta_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i),$$

where $Q_i(\theta_i) := \mathbb{E}_{\theta_{-i}}[q_i(\theta)]$ and $T_i(\theta_i) := \mathbb{E}_{\theta_{-i}}[t_i(\theta)]$. Given hidden information and the availability of the non-participation option, an auction rule must be incentive compatible and individually rational to be consistent with equilibrium. We say an auction rule M is *feasible* if

$$U_i^M(\theta_i) := u_i^M(\theta_i, \theta_i) \geq u_i^M(\tilde{\theta}_i, \theta_i), \quad \forall i, \theta_i \in \Theta_i, \tilde{\theta}_i \in \overline{\Theta}_i. \qquad (IC^*)$$

Note that (IC^*) subsumes both incentive compatibility and individual rationality; for instance, with $\tilde{\theta}_i = \theta_\emptyset$, (IC^*) requires

$$U_i^M(\theta_i) \geq u_i^M(\theta_\emptyset, \theta_i) = 0. \tag{IR}$$

Let $\mathcal{M}$ denote the set of all feasible auction rules. For later analysis, the following characterization of feasible auction rules proves useful. All proofs are relegated to the Appendix C unless stated otherwise.

Lemma 0. *If $M = (q, t) \in \mathcal{M}$, then, for each $\theta_i \in \Theta_i$,*

$$U_i^M(\theta_i) = \mathbb{E}\left[K_i(\tilde{\theta}_i)q_i(\tilde{\theta})1_{\{\tilde{\theta}_i \leq \theta_i\}} + J_i(\tilde{\theta}_i)q_i(\tilde{\theta})1_{\{\tilde{\theta}_i \geq \theta_i\}} - t_i(\tilde{\theta})\right]. \tag{1}$$

Before proceeding, it is useful to consider a collusion-free environment. It is by now well known that, in such an environment, an optimal auction rule, called *second-best* or *noncollusive optimal outcome*, solves

$$[NC] \qquad \max_{M \in \mathcal{M}} \mathbb{E}\left[\sum_{i \in N} t_i(\theta)\right],$$

and its associated outcome is characterized as follows:

Theorem 0. (Myerson) *An optimal mechanism that solves $[NC]$ involves the allocation rule given by $\forall \theta \in \Theta$,*

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) > \max\{0, \max_{k \neq i} J_k(\theta_k)\}, \\ 0 & \text{otherwise,} \end{cases}$$

and the revenue equal to

$$V^* := \mathbb{E}\left[\sum_{i \in N} J_i(\theta_i) q_i^*(\theta)\right].$$

To avoid trivial cases, we assume that $V^* > \max_{i \in N} \underline{\theta}_i$. Let $\hat{\theta}_i := \min\{\theta_i \in [\underline{\theta}_i, \bar{\theta}_i] | J_i^{-1}(\theta) \geq 0\}$ denote each bidder i's threshold type. Then, the optimal mechanism allocates the good to the bidder with highest virtual valuation $J_i(\theta_i)$ as long as $\theta_i \geq \hat{\theta}_i$.

4 A Model of Collusion

Following LM and CK, we envision subsets of bidders (henceforth called *coalitions*) enforcing side contracts via uninformed representatives to influence the outcome of the auction game being played. Formally, *a coalition structure* is an arbitrary partition $\mathcal{C}$ on N whose element $C \in \mathcal{C}$ represents a coalition of bidders who "may" collude with one another. This framework encompasses a range of possibilities that include the grand coalition (i.e., $\mathcal{C} = \{N\}$), that allows for the presence of noncollusive bidders (i.e., some elements of $\mathcal{C}$ may be singleton) and/or for multiple bidding rings (i.e., $\mathcal{C}$ may include C_j, $j = 1, ..., k$ with $|C_j| \geq 2$). The coalition structure $\mathcal{C}$ is a common knowledge for all bidders in N and for the seller. The assumption that the seller knows the coalition structure, albeit not innocuous, may not be as restrictive as it may appear. For instance, our analysis would still apply if some coalition may not collude effectively. Also, the structure of potential bidding rings (who is likely to collude with whom) can be sometimes discerned from prior auction experiences and other industry observables.[5] Of course, none of these issues arise if there is only one coalition, as has been assumed in all existing papers. In this sense, the current model generalizes all existing models of collusion.

The time line is similar to that of LM and CK, except for one important difference: Coalitions are formed prior to the bidders' participation into the mechanism.

- Time line:

  - At date 0, each bidder learns his type, θ_i, drawn from Θ_i.
  - At date 1, the seller proposes an auction rule $M \in \mathcal{M}$.
  - At date 2, the (uninformed) representative of each coalition $C \in \mathcal{C}$ simultaneously proposes a collusive side contract (to be described in detail later), which each member of C accepts or rejects. If all bidders accept, then collusion is active; or else, it is inactive.
  - At date 3, each bidder, $i \in N$, chooses $\tilde{\theta}_i \in \overline{\Theta}_i$; i.e., he accepts or rejects M, and reports from Θ_i if he accepts. (If collusion is active, then the collusive bidders report according to the side contract in force.)

[5] Che and Kim (2006) accommodate, to a degree, a possible ignorance of the coalition structure. Unfortunately, with collusion on participation, WCP implementation requires a precise knowledge by the seller about the coalition structure she is faced with.

- At date 4, if collusion by a coalition is active, then the outcome of their side contract arises. If no collusion is active, then M results.

- TECHNOLOGICAL FEASIBILITY OF COLLUSION:

We assume that each coalition has at its disposal four instruments: (a) its members' participation decisions, (b) participating members' communication with the seller (e.g., bids), (c) reallocation of the good within the coalition, in case a member of that coalition receives the good, and (d) side payments that the coalition members can exchange in a balanced-budget fashion. These four instruments together encompass all possible ways in which a coalition can coordinate their members' behavior.

To formally describe possible manipulations utilizing all these instruments, fix a possible coalition $C \in \mathcal{C}$, and an auction rule $M = (q, t) \in \mathcal{M}$ the seller may propose. It is useful to think of the coalition as choosing a *manipulation*, i.e., an outcome that will emerge if its members carry out their collusive deviation from an original auction rule and the others, including noncollusive bidders and members of different coalitions, report truthfully. We say an outcome, $\tilde{M} = (\tilde{q}, \tilde{t})$ is a *manipulation of M by coalition C*, if there exists a function, $\mu^C : \Theta_C \to \Delta\overline{\Theta}_C$, that maps from their types in Θ_C into a probability distribution over $\overline{\Theta}_C$ such that, $\forall \theta \in \Theta$,

$$\sum_{i \in C} \tilde{q}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[\sum_{i \in C} q_i(\tilde{\theta}_C, \theta_{N \setminus C})], \tag{R_C^M}$$

$$\tilde{q}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[q_i(\tilde{\theta}_C, \theta_{N \setminus C})], \forall i \in N \setminus C, \tag{$R_{N\setminus C}^M$}$$

$$\mathbb{E}\left[\sum_{i \in C} \tilde{t}_i(\theta)\right] = \mathbb{E}\left[\sum_{i \in C} \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N \setminus C})]\right], \tag{BB_C^M}$$

$$\tilde{t}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N \setminus C})]. \forall i \in N \setminus C. \tag{$BB_{N\setminus C}^M$}$$

These conditions are explained as follows. First, condition (R_C^M) says that any manipulation $\tilde{M}$ by coalition C should be generated by some randomization $\mu^C(\theta_C)$ over its members' participation/reports plus some reallocation within that coalition, when bidders outside C report truthfully. More precisely, any admissible manipulation allocates the good to the coalition with the same probability as the original auction rule (chosen by the seller) under

μ^C. Condition (BB_C^M) allows the coalition members to exchange side transfers in a budget-balanced fashion. Since budget balancing is required at the ex ante level, we are even allowing for the coalition to finance (from a competitive capital market) across different realizations of its members' type profiles.[6] Conditions $(R_{N\backslash C}^M)$ and $(BB_{N\backslash C}^M)$ simply assume that bidders outside C are not colluding: there is no reallocation and no exchange of side payments among all bidders outside C and between C and $N\backslash C$. This presumption would be without any loss if $N\backslash C$ were all noncollusive. Even if $N\backslash C$ may involve some bidding rings, the above conditions are still sufficient for there to be an equilibrium with no collusion, since they ensure that the representative of each coalition offers a null side contract *when other coalitions null side contracts.*

- Incentive Feasibility of Collusion

For collusive manipulation to work, the members of the coalition must have the incentive to carry out the manipulation. We say that $\tilde{M}$ is *feasible* if it satisfies

$$U_i^{\tilde{M}}(\theta_i) \geq u_i^{\tilde{M}}(\tilde{\theta}_i, \theta_i), \quad \forall i \in C, \theta_i \in \Theta_i, \tilde{\theta}_i \in \overline{\Theta}_i. \qquad (IC_C^*)$$

and

$$U_i^{\tilde{M}}(\theta_i) \geq U_i^M(\theta_i), \quad \forall i \in C, \theta_i \in \Theta_i. \qquad (IR_C^M)$$

These conditions are explained as follows. First of all, (IC_C^*) requires the outcome resulting from collusion to be incentive compatible to all members of coalition. Since the coalition members face asymmetric information about one another, this condition must hold, regardless of the specifics of how the coalition is formed and how the members bargain over their collusive arrangement. Next, (IR_C^M) requires that each member of the coalition must do as well with the proposed manipulation as they would by vetoing that manipulation and acting noncooperatively. Clearly, what each member will get in the latter event depends on the inferences made by the members of the coalition about that agent. Condition (IR_C^M) assumes that no new inferences about the members' types are made in such an event. This "passivity" of out-of-equilibrium beliefs is an important element of LM's weak collusion-proofness notion.

[6] Our results do not change, if budget balancing is required at the *ex post* level. Clearly, our collusion proof implementation result would be stronger with the ex ante version of budget balancing, explaining our choice.

While there could be collusive agreements supported by non-passive beliefs, the passive belief restriction seems reasonable in many settings.

Lastly, note these conditions are imposed only for the coalition members. Clearly, the coalition need not care about the welfare or incentives of noncollusive agents. We turn next to the notions of collusion-proofness we focus on throughout the paper.

Definition WCP. *An auction rule $M \in \mathcal{M}$ is weak collusion-proof (henceforth, WCP), if, for each coalition $C \in \mathcal{C}$ with $|C| \geq 2$, any feasible manipulation of M by C makes no member of C strictly better off.*

To explain this notion, suppose the seller offers an auction rule M. If M is WCP, then, for each coalition C, there exists no feasible manipulation that would make some members of C strictly better, given that all other coalitions are inactive. Hence, if a third party representative of C maximizes the joint payoffs of C, just as LM presumed, then there will be an equilibrium in representatives of coalitions propose no collusive manipulations — or equivalently, they all propose null side contracts).

Our WCP notion is comparable to the WCP of LM, except for a couple of differences. First, we allow for randomization and reallocation possibilities in the collusive agents. Second, we allow for proper subsets of bidders to be collusive, which add a new wrinkle in collusion-proof auction design. In particular, it becomes important for the seller to prevent each coalition not only from extracting rents from herself but also from preying on bidders outside that coalition.

5 Collusion-Proof Implementability of the Second-Best Outcome

In this section, we characterize the properties that WCP auction rules must satisfy (Lemma 1 and 2) and use these properties to obtain a necessary condition for the second-best outcome to be WCP implementable (Theorem 1). We then show that, for symmetric bidders, the necessary condition is also sufficient for the WCP implementability of the second-best outcome.

5.1 Properties of WCP Auction Rules

Fix a coalition $C \in \mathcal{C}$, and an auction rule $M = (q, t)$ that the seller proposes. It is useful to have a few definitions. Let $q_i^C(\theta_C) := \mathbb{E}_{\tilde{\theta}_{N\setminus C}}[q_i(\theta_C, \tilde{\theta}_{N\setminus C})]$. Let $q_C(\theta) := \sum_{i\in C} q_i(\theta)$ and $Q_C(\theta_C) := \mathbb{E}_{\tilde{\theta}_{N\setminus C}}[q_C(\theta_C, \tilde{\theta}_{N\setminus C})]$ denote the probability that the auction rule allocates to good to a member of the coalition given the value profile of all bidders and that of the coalition members, respectively. Let $\mathcal{Q}_C := [0, \sup_{\theta_C \in \overline{\Theta}_C} Q_C(\theta_C)]$ be the set of all probabilities with which the coalition can secure the good to its members under M. This set contains zero since all its members can boycott the auction (i.e., $Q_C(\theta_\emptyset, \cdots, \theta_\emptyset) = 0$. This set is convex since the coalition members can randomize between boycotting and reporting some profile $\theta_C \in \Theta_C$. We then obtain our first property of WCP auction rules.

Lemma 1. *If $M = (q, t) \in \mathcal{M}$ is WCP, then for each $C \in \mathcal{C}$ there exists a convex function, $r : \mathcal{Q}_C \to \mathbb{R}_+$ with $r(0) = 0$, such that*

$$\mathbb{E}_{\tilde{\theta}_{N\setminus C}}\left[\sum_{i\in C} t_i(\theta_C, \tilde{\theta}_{N\setminus C})\right] = r(Q_C(\theta_C)), \forall \theta_C \in \overline{\Theta}_C.$$

In a WCP auction rule, the seller cannot collect any fee from a coalition when its members never obtain the good. This feature arises from the abilities by the buyers to collude on their participation decisions; were they charged positive entry fees, they could all simply refuse to participate. Similarly, the collusive bidders can never be charged different prices for the same probability of obtaining the good; or else, they could manipulate their reports (or bids) to pick the lowest price for a given probability of obtaining the good. The surplus generated from such manipulation can be shared among all coalition members via appropriate side transfers and reallocations so that (IC^*) and (IR_C^M) conditions are satisfied. Finally, the sale price is (weakly) convex in the probability of the object being allocated to any coalition member, since the coalition members can at least randomize between non-participation and any probability of allocation attainable by some reports.

The next property is obtained for a class of allocation rules satisfying *monotonicity*: $q_C(\cdot)$ is nondecreasing in θ_C and, for all $i \in C$, $q_i(\cdot)$ is nondecreasing in θ_i and nonincreasing in θ_{-i}. Let $\mathcal{M}_0 \subset \mathcal{M}$ denote the set of auction rules satisfying this monotonicity. The monotonic auction rules are reasonable and account for a large class of auction allocations, including the first-best and second-best auction rules. Next, we define the average price charged to the coalition per unit probability:

$$p(\theta_C) := \begin{cases} \dfrac{r(Q_C(\theta_C))}{Q_C(\theta_C)} & \text{if } Q_C(\theta_C) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2. *If $M = (q, t) \in \mathcal{M}_0$ is WCP, then $\forall\, C \in \mathcal{C}$, $\forall\, i \in C$ and for almost every $\theta_C \in \Theta_C$,*

$$(K_i(\theta_i) - p(\theta_C))\, q_i^C(\theta_C) \geq \max_{\theta_i' \in \{\theta_\emptyset\} \cup [\underline{\theta}_i, \theta_i]} (K_i(\theta_i) - p(\theta_i', \theta_{C-i}))\, q_i^C(\theta_i', \theta_{C-i}). \tag{2}$$

This lemma captures the sense in which each coalition is able or unable to "behave like a single agent." Specifically, condition (2) resembles an incentive compatibility constraint for a "single" agent who may realize one of $|C|$ alternative values. But this resemblance is not perfect. First, that agent realizes "pseudo" value $K_i(\theta_i)$ rather than true value θ_i. Second, the agent's constraint is required only in one direction, i.e., not to under-report or withdraw from the auction. Third, the agent may not be able to shift his consumption among the alternative uses. All together, these features serve to limit the extent to which the coalition can coordinate their members' behavior. For instance, the fact that pseudo values, rather than true values, matter means that the coalition can be forced to sustain some ex post loss. Since $K_i(\theta_i) > \theta_i$, an average price $p(\theta_i) > \theta_i$ need not violate the above constraint. The coalition's limited ability to coordinate their behavior arises from the fact that any collusive defection requires a consensus from all types of bidders. Typically, different types of bidders have conflicting interests, say about consumption of the good by any particular type θ_i. Suppose $\theta_i < p(\theta_C)$, so ex post loss arises from the consumption by type θ_i. Although the coalition wishes to cancel such consumption, the highest type of bidder i would be reluctant to cancel such consumption, as long as $K_i(\theta_i) > p(\theta_C)$. But if the latter inequality fails, all types of bidder, including the highest type, will benefit from canceling the sale to type θ_i. In this sense, the pseudo value $K_i(\theta_i)$, limits the degree of ex post loss that can be feasibly imposed on the coalition without triggering its defection.

Naturally, the necessary conditions for WCP implementation (given by Lemma 1 and 2) constrain the set of circumstances in which the second-best outcome is WCP implementable. We next characterize these circumstances. To this end, fix any bidder $i \in C$ for some $C \in \mathcal{C}$

with $|C| \geq 2$. For each profile $\theta_{N\setminus C} \in \Theta_{N\setminus C}$, let

$$\phi_i(\theta_{N\setminus C}) := \inf\{\theta_i \in \Theta_i \mid J_i(\theta_i) \geq \max\{\max_{j \in N\setminus C} J_j(\theta_j), 0\}\}$$

denote the lowest type of bidder i that can obtain the good with positive probability in the second-best allocation, when bidders outside C have types $\theta_{N\setminus C}$. Consider the following condition:

Condition (SB): (i) If $\mathcal{C} = \{N\}$, then

$$K_i(\hat{\theta}_i) \geq \frac{\mathbb{E}\left[\sum_{i \in N} J_i(\theta_i) q_i^*(\theta)\right]}{\mathbb{E}\left[\sum_{i \in N} q_i^*(\theta)\right]} =: r^*, \forall i \in N.$$

(ii) If $\mathcal{C} \neq \{N\}$, then, for each $C \in \mathcal{C}$ with $|C| \geq 2$,

$$\mathbb{E}\left[\sum_{i \in C} K_i(\phi_i(\theta_{N\setminus C})) q_i^*(\theta)\right] \geq \mathbb{E}\left[\sum_{i \in C} J_i(\theta_i) q_i^*(\theta)\right].$$

This condition is explained as follows. The RHS of the inequalities represents the amount of surplus that *should* be extracted from the coalition to implement the second-best payoff for the seller. As will be shown next, the LHS of the inequalities represent the highest payment that can be charged against the coalition, given the second-best allocation, q^*. Thus, the inequalities are necessary for the second-best outcome to be WCP implementable.

Theorem 1. (Necessity) Condition (SB) *is necessary for the second-best outcome to be WCP implementable.*

5.2 WCP Implementation of the Second-Best Outcome: Symmetric Bidders

Here we show that Condition (SB) is also sufficient for the second-best outcome to be WCP implementable when bidders are symmetric. Specifically, we construct an auction rule that will WCP implement the second-best outcome, given (SB). Further, this sufficient condition will be seen to hold if either at least one bidder is noncollusive or the second-best mechanism involves a nontrivial probability of no sale.

We begin with the symmetry assumption: $F_i(\cdot) =: F(\cdot)$ for all $i \in N$, for some common cdf $F(\cdot)$ which has a positive density f. The associated virtual valuations J and K are defined analogously, and their monotonicity properties are maintained. Likewise, we let $\hat{\theta} = \inf\{\theta | J(\theta) \geq 0\}$. Condition (SB) is now more succinctly described in this environment. Define first $\theta_C^{(1)} := \max_{i \in C} \theta_i$ and $\theta_{N\backslash C}^{(1)} := \max\{\max_{i \in N\backslash C} \theta_i, \hat{\theta}\}$. (We adopt a convention that $\theta_{N\backslash C}^{(1)} := \hat{\theta}$ when $\mathcal{C} = \{N\}$.) Then, Condition (SB) simplifies to:

Condition (SB′): For each C with $|C| \geq 2$, $\mathbb{E}\left[K(\theta_{N\backslash C}^{(1)}) | \theta_C^{(1)} > \theta_{N\backslash C}^{(1)}\right] \geq \mathbb{E}\left[J(\theta_C^{(1)}) | \theta_C^{(1)} > \theta_{N\backslash C}^{(1)}\right]$.

As pointed out earlier, this condition requires a collusive bidder's valuation to be sufficiently high whenever the good is allocated to him. It turns the exact requirement is not very onerous. The condition holds if at least one buyer is noncollusive or there are more than one bidding ring, or if the cutoff threshold $\hat{\theta}$ is sufficiently high.

Lemma 3. Condition (SB′) *holds if* $\mathcal{C} \neq \{N\}$, *or if* $\mathcal{C} = \{N\}$ *and*

$$K(\hat{\theta}) \geq \mathbb{E}\left[J(\theta_N^{(1)}) | \theta_N^{(1)} > \hat{\theta}\right]. \tag{3}$$

In case all bidders are collusive (i.e., $\mathcal{C} = \{N\}$), Condition (SB′), or equivalently (3), is not trivial. For instance, if $\hat{\theta} = \underline{\theta}$, then $K(\hat{\theta}) = \underline{\theta}$, so the condition fails. Although the case of grand coalition appears distinct from the case of $\mathcal{C} \neq \{N\}$, the same logic applies. In both cases, exclusion of collusive bidders is needed for WCP implementation. This objective is accomplished in the latter case by allocating the object to noncollusive bidders or to collusive bidders in a different bidding ring, whereas in the case of grand coalition, the necessary exclusion can only be accomplished by not selling the object to any bidder.

We now construct an auction rule that WCP implements the second-best outcome when Condition (SB′) holds. Let $M^* = (q^*, t^*)$ denote the second-price auction with a reserve price $\hat{\theta}$, which implements the second-best outcome. Consider a new auction rule $\hat{M} = (q^*, \hat{t})$. This auction rule has the same allocation rule as M^*, but has a payment rule $\hat{t}(\cdot)$ that satisfies two properties: (i) For each bidder $i \in N$, $\mathbb{E}_{\tilde{\theta}_{-i}}[\hat{t}_i(\theta_i, \tilde{\theta}_{-i})] = \mathbb{E}_{\tilde{\theta}_{-i}}[\hat{t}_i(\theta_i, \tilde{\theta}_{-i})]$, so that each bidder has the same interim incentives as with M; and (ii) Each coalition $C \in \mathcal{C}$ with $|C| \geq 2$ is charged a single sale price, payable only when the object is allocated to its member. It is easy to understand the purpose of these properties. Property (i) ensures that $\hat{M}$ provides the same interim incentives to the bidders as M^*. Property (ii) ensures that each coalition cannot manipulate the sale price charged against its members.

The sale price against each coalition $C \in \mathcal{C}$ (with $|C| \geq 2$) is constructed as follows. Let $\alpha_C \in [0,1]$ be such that

$$\mathbb{E}\left[\alpha_C K(\theta_{N\backslash C}^{(1)}) + (1-\alpha_C)J(\theta_{N\backslash C}^{(1)})|\theta_C^{(1)} > \theta_{N\backslash C}^{(1)}\right] = \mathbb{E}\left[J(\theta_C^{(1)})|\theta_C^{(1)} > \theta_{N\backslash C}^{(1)}\right]. \tag{4}$$

Condition (SB′) allows such α_C to be well defined. The sale price against coalition C is then set at $H_C(\theta_{N\backslash C}^{(1)}) := \alpha_C K(\theta_{N\backslash C}^{(1)}) + (1-\alpha_C)J(\theta_{N\backslash C}^{(1)})$. Notice this sale price is defined in terms of the highest type of bidder outside C — but set above $J(\theta_{N\backslash C}^{(1)})$ just enough to extract $J(\theta_C^{(1)})$ on average from the highest valuation bidder in C.

We now set the payment rule $\hat{t}$ to satisfy both (i) and (ii). First, for each noncollusive bidder i (i.e., $\{i\} \in \mathcal{C}$), we set $\hat{t}_i(\theta) := t_i^*(\theta)$, $\forall\theta$. For each collusive bidder $i \in C$ for each $C \in \mathcal{C}$ with $|C| \geq 2$, we let, $\forall\theta$,

$$\begin{aligned} \hat{t}_i(\theta) \quad &= \quad \frac{1}{|C|}\delta_C(\theta) + \mathbb{E}_{\tilde{\theta}_{-i}}\left[t_i^*(\theta_i, \tilde{\theta}_{-i}) - \frac{1}{|C|}\delta_C(\theta_i, \tilde{\theta}_{-i})\right] \\ &\quad -\frac{1}{|C|-1}\sum_{k \in C\backslash\{i\}} \mathbb{E}_{\tilde{\theta}_{-k}}\left[t_k^*(\theta_k, \tilde{\theta}_{-k}) - \frac{1}{|C|}\delta_C(\theta_k, \tilde{\theta}_{-k})\right], \end{aligned} \tag{5}$$

where $\delta_C(\theta) := H_C(\theta_{N\backslash C}^{(1)}) \sum_{i \in C} q_i^*(\theta)$

It can be checked that $\hat{M}$ has the same interim payments as M^*: $\mathbb{E}_{\tilde{\theta}_{-i}}[\hat{t}_i(\theta_i, \tilde{\theta}_{-i})] = \mathbb{E}_{\tilde{\theta}_{-i}}[t_i^*(\theta_i, \tilde{\theta}_{-i})]$. Hence, $\hat{M}$ satisfies (IC^*) and implements V^*. In addition, $\hat{M}$ satisfies property (ii), for $\sum_{i \in C} \hat{t}_i(\theta) = \delta_C(\theta)$ for all θ. We are now ready to present the sufficiency result.

THEOREM 2. (SUFFICIENCY) *Given* CONDITION (SB′)*, the mechanism $\hat{M}$ is WCP and achieves the second-best revenue.*

Combining Lemma 3 with Theorem 2 produces one of the main results of this paper.

COROLLARY 1. *The second-best outcome is WCP implementable if $\mathcal{C} \neq \{N\}$, or if $\mathcal{C} = \{N\}$ but (3) holds.*

In words, the second-best outcome is weak collusion-proof implementable if either there exists at least one noncollusive bidder or there are multiple bidding rings. In these cases, the seller can leverage the presence of these bidders to extract sufficient rents from each coalition. If entire bidders form a bidding coalition, then no such leverage exists, but the seller can still use the threat of no sale to accomplish the same objective as long as that threat is sufficiently credible in the sense of (3).

REMARK 1. *The WCP auction can be made incentive compatible in dominant strategies for all noncollusive bidders and for all collusive bidders except for one in each coalition. This auction has all bidders bid simultaneously and allocates the object to the highest bidder (with ties broken randomly) with valuation exceeding $\hat{\theta}$. If a noncollusive bidder wins, he pays the highest losing bid or $\hat{\theta}$, whichever is higher, just like the second-price auction. For collusive bidders in any coalition C, one bidder, say $i \in C$, is designated as a ring leader. Whenever a bidder in C wins, bidder i pays to the seller $H_C(\max\{\max_{i \in N \setminus C} b_i, \hat{\theta}\})$. If bidder $j \in C \setminus \{i\}$ wins, he pays the highest losing bid to bidder i. In addition, each j pays to i some amount $\frac{1}{|C|-1}\Delta_C(b_1)$ that depends only on i's bid. Clearly, all bidders except the ring leaders have weak dominant strategies of bidding their valuations. Further, $\Delta_C(\cdot)$ can be set to make it a best response for each ring leader to bid his valuation and to satisfy interim individual rationality for all coalition members.*[7] *This auction rule implements the same sale price $H_C(\cdot)$ (using ring leaders as direct contact points), so it is WCP. In fact, if there is only one bidding coalition, this auction is WCP in a more robust sense: The seller receives the payoff of V^* in every equilibrium supported by passive beliefs. Every manipulation by the lone coalition generates the same expected payment and allocation rule as M^*, and does not affect the incentives of the noncollusive bidders. Hence, the second best is implementable even if collusion may arise*

[7]Let $\eta(b_1)$ denote the expected payment bidder 1 makes to the seller minus the expected payment other collusive bidders make to bidder 1, when bidder 1 bids b_1. Note that $\eta(b_1)$ is a constant for $b_1 < \hat{\theta}$, which we denote by η_0. Whether win or lose, each collusive bidder is required to compensate bidder 1 with $\frac{1}{|C|-1}\Delta(b_1)$, where

$$\Delta_C(b_1) := \begin{cases} \eta(b_1) - \int_{\underline{\theta}}^{b_1} \max\{s, \hat{\theta}\} dF^{n-1}(s) & \text{if } b_1 \geq \hat{\theta} \\ \eta_0 & \text{otherwise.} \end{cases}$$

Given others are bidding their valuations, bidder 1 with type θ_1 solves

$$\max_{b_1} \theta_1 \Pr\{b_1 wins\} - \eta(b_1) + \Delta_C(b_1) = \begin{cases} \theta_1 F^{n-1}(b_1) - \int_{\underline{\theta}}^{b_1} \max\{s, \hat{\theta}\} dF^{n-1}(s) & \text{if } b_1 \geq \hat{\theta} \\ 0 & \text{otherwise.} \end{cases}$$

From this, one can easily see that it is optimal to set $b_1 = \theta_1$, which leads to the same interim payoff for each θ_1 as in the second-best. Finally, one can verify that $\mathbb{E}[\Delta_C(\theta_1)] = 0$.

on the equilibrium path.[8]

The following examples illustrate two different scenarios.

Example 1. ($\mathcal{C} = \{\{1, 2\}\}$) *Suppose there are two bidders each with valuation drawn uniformly from* $[0, 1]$*. According to Theorem 0, the second-best outcome allocates the object efficiently for valuation exceeding* $\hat{\theta} = \frac{1}{2}$*, and yields revenue of* $\frac{1}{2}$*. This also satisfies (3), so the second-best is WCP implementable. The WCP auction rule charges a sale price of* $r^* = \mathbb{E}[J(\theta_N^{(1)})|\theta_N^{(1)} > \hat{\theta}] = 2/3$ *to the bidders, regardless of who wins and what their bids are. Without collusion, each bidder receives the interim payoff of*

$$U^{\hat{M}}(\theta) = \begin{cases} 0 & \text{if } \theta \in [0, \frac{1}{2}) \\ \frac{1}{2}\theta^2 - \frac{1}{8} & \text{if } \theta \in [\frac{1}{2}, 1]. \end{cases}$$

Since the coalition is charged a sale price of $2/3$*, it suffers an ex post loss whenever the highest valuation is in* $[\frac{1}{2}, \frac{2}{3}]$*. Why can they not simply boycott the auction in this situation? Indeed, their joint surplus will increase by doing so. The problem, however, is that the increased surplus cannot be allocated to benefit all types; some types will be strictly worse off and thus object to that move. To illustrate, suppose indeed that the bidders boycott auction whenever no bidder has valuation exceeding* $2/3$*, and, otherwise, the high-valuation bidder consumes the object. Under this collusive arrangement, labeled* $\tilde{M}$*, each bidder's interim payoff is*

$$U^{\tilde{M}}(\theta) = \begin{cases} \frac{1}{81} & \text{if } \theta \in [0, \frac{2}{3}) \\ \frac{1}{2}\theta^2 - \frac{17}{81} & \text{if } \theta \in [\frac{2}{3}, 1]. \end{cases}$$

[8] The same does not hold if there are multiple bidding rings. If a bidding ring is expected to run a manipulation, this may affect the noncooperative payoffs of the other bidding rings in a way that may alter the feasibility of manipulations for other bidding rings.

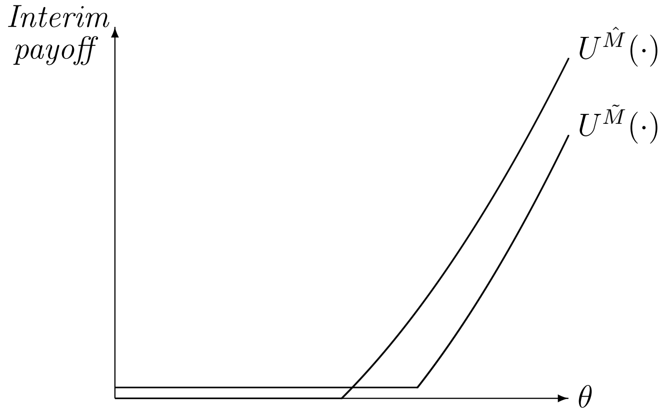

Figure 1: Comparison between interim payoffs from $\hat{M}$ and $\tilde{M}$

As can be seen from Figure 1, a bidder benefits from this collusion when his valuation is less than 0.524 *but is strictly worse off if his valuation is higher. Hence, a collusive arrangement* $\tilde{M}$ *is not feasible. (The same is true for any other feasible manipulations.) Even though the net expected surplus may rise with some collusive manipulation, incentive compatibility facing the collusive bidders limits the way surplus can be allocated across types to make them uniformly better off. In this sense, our WCP auction exploits the informational asymmetry facing the collusive bidders.*

Example 2. ($\mathcal{C} = \{\{1,2\},\{3\}\}$) *Consider the example from Section 2, where there are two collusive bidders and a noncollusive bidder, each with valuation drawn from* $[1,2]$. *Here, the presence of a noncollusive bidder is crucial for WCP implementability of the second-best outcome. (If all three bidders were collusive, the second-best is not WCP implementable, for the distribution fails (3).) Our WCP auction charges the sale price of* $H_{\{1,2\}} = 2\theta_3 - \frac{5}{4}$ *to the coalition* $\{1,2\}$. *The auction described in Remark 1 then works as follows. Given bids* (b_1, b_2, b_3), *bidder 2 pays* $\Delta(b_1) := \frac{2}{3}b_1^3 - \frac{21}{8}b_1^2 + \frac{13}{4}b_1 - \frac{5}{4}$, *win or lose, and if he wins, an additional amount equal to the highest losing bid, all to bidder 1. If bidder 1 or 2 wins, then bidder 1 pays* $2b_3 - \frac{5}{4}$ *to the seller. If bidder 3 wins, he pays the seller the highest losing bid. Clearly, bidders 2 and 3 have weak dominant strategies of bidding their valuations. To these strategies, it is a best response by bidder 1 to bid his valuation, as well. This equilibrium would generate the same second-best revenue of* $3/2$ *to the seller. Most importantly, the equilibrium is not* *susceptible to collusion by bidders 1 and 2 (in the weak collusion-proof sense). As mentioned in Section 2, the coalition has no incentive to boycott the auction; but it would benefit from pretending that the highest valuation is $\frac{\theta}{2} + \frac{5}{8}$ when it is θ. Any such manipulation will leave some types of bidder strictly worse off, and is thus not feasible.*

5.3 WCP Implementation of the Second-Best Outcome: Asymmetric Bidders

We now turn to the case of asymmetric bidders. In this case, the optimal noncollusive auction, as characterized in Theorem 0, requires bidders to be treated differently based on their ex ante distribution of types. This presents an extra challenge for the WCP implementation since, as shown in Lemma 1, the same price is charged no matter which member of the coalition receives the good. This does not mean, however, that the collusive bidders cannot be treated differently, for different interests of the heterogeneous types can be exploited to make (IR_N^M) difficult to satisfy. Indeed, we will show that the second-best outcome is WCP implementable at least with respect to the grand coalition (i.e., $\mathcal{C} = \{N\}$), under a condition that is not much stronger than Condition (SB).

Consider now a strict inequality version of (SB)-(i):

Condition (SB*): $K_i(\hat{\theta}_i) > r^* = \frac{\mathbb{E}\left[\sum_{i \in N} J_i(\theta_i) q_i^*(\theta)\right]}{\mathbb{E}\left[\sum_{i \in N} q_i^*(\theta)\right]}, \forall i \in N.$

Theorem 3. *Assume $\mathcal{C} = \{N\}$ and* Condition (SB*) *holds. Suppose also that $(J_i(\cdot) - r^*) f_i(\cdot)$ is increasing in the interval $[\underline{\theta}_i, J_i^{-1}(r^*)]$ for all $i \in N$. Then, there exists an auction rule which is WCP and achieves the second-best outcome.*

6 Optimal Collusion-Proof Auctions

What happens if the second-best outcome is not collusion-proof implementable? What will the optimal WCP auction look like in such a case? While a complete answer to this latter question is unavailable, we provide two observations that will help answer that question. First, we show that any optimal WCP auction in a general monotonic class involves "*exclusion*" —

a positive probability that the object is not sold to a collusive bidder. Second, we characterize the optimal WCP auction completely in the linear class for the case of symmetric bidders.

6.1 An Exclusion Principle

Lemma 1 implies that the seller can only charge a single sale price to collusive bidders, regardless of their types. Meanwhile, Lemma 2 says that this price cannot be too high relative to the pseudo value, $K_i(\theta_i)$, of the collusive bidder who consumes the good. Combined together, these lemmas imply that the seller must either charge a low sale price or else she should exclude types with low pseudo values from consuming the good. A similar tradeoff exists even in the collusion-free environment with respect to the setting of the reserve price, so the optimal auction rule excludes low value types of the buyers in some situations. But the (collusion-free) optimal auction does not involve any exclusion if the low types of buyers have sufficiently high valuations, i.e., when $J_i(\underline{\theta}) \geq 0 \; \forall i$. Collusion alters the tradeoff toward more exclusion, since the seller can only charge a single sale price (whereas in the collusion-free environment, the bidding competition typically generates higher payment from high valuation types beyond the reserve price). In fact, we show below that the optimal auction rule in the monotonic class must entail some exclusion of the collusive bidders.

Theorem 4 (Exclusion Principle). *Assume that there are more than one bidder $i \in C$ with $\underline{\theta}_i = \underline{\theta} := \max_{j \in C} \underline{\theta}_j$. Then, the optimal WCP mechanism in $\mathcal{M}_0$ requires that the object not be sold to any member of C with a positive probability.*

Proof. Let $M = (q, t)$ denote the optimal WCP mechanism. Suppose to the contrary that $\sum_{i \in C} q_i(\theta) = 1$ for all $\theta \in \Theta$, which implies by Lemma 1 that $\sum_{i \in C} t_i(\theta) = r$ for some r. Then, Lemma 2 requires that

$$r \leq \max_{i \in C} K_i(\underline{\theta}_i) = \underline{\theta}.$$

Thus, the revenue cannot exceed $\underline{\theta}$. We now generate a contradiction by constructing a WCP mechanism which raises a higher revenue than $\underline{\theta}$: Sell the object at a fixed price $\tilde{r}$, which is slightly greater than $\underline{\theta}$, if and only if at least one member of C has a value higher than $\tilde{r}$. This take-it-or-leave offer is clearly WCP and generates a revenue equal to $R(\tilde{r}) :=$

$\tilde{r}(1 - \prod_{i \in C} F_i(\tilde{r}))$. And $R(\tilde{r}) > R(\underline{\theta}) = \underline{\theta}$ for $\tilde{r}$ slightly above $\underline{\theta}$ since

$$\frac{d}{d\tilde{r}} R(\tilde{r})\Big|_{\tilde{r}=\underline{\theta}} = (1 - \prod_{i \in C} F_i(\underline{\theta})) - \underline{\theta} \sum_{i \in C} f_i(\underline{\theta}) \prod_{j \neq i} F_j(\underline{\theta}) = 1 > 0,$$

where the last equality holds because for each $i \in C$, there exist at least one agent $j \neq i$ for whom $F_j(\underline{\theta}) = 0$. ∎

Recall from Lemma 3 and Theorem 1 that the second-best with symmetric bidders is not WCP implementable if it requires the sale to the coalition to occur with probability 1. The above theorem establishes that *any* auction rule which allocates the object to the coalition with probability 1 can never be optimal. The idea behind this result is simple: If the seller tries to sell the object to all types, including the lowest type $\underline{\theta}$, then the (fixed) sale price can never exceed $\underline{\theta}$, as implied by Lemma 2. Instead, the seller could make a take-it-or-leave offer with a price slightly higher than $\underline{\theta}$, which is clearly WCP and will generate a higher revenue than $\underline{\theta}$.

6.2 Optimal Linear WCP Auctions for Symmetric Bidders

Here, we search for an optimal WCP auction rule when Condition (SB) fails when bidders are symmetric. This condition can only fail when all bidders are collusive (recall Lemma 3 or equivalently Corollary 1), so we focus on that case and assume $\mathcal{C} = \{N\}$. Further, we restrict our search to the class of *linear auction rules* where the aggregate payment schedule is linear in the probability of sale (to any bidder): $\sum_{i \in N} t_i(\theta) = r \cdot Q(\theta)$, for some nonnegative constant r.[9] With a linear auction rule, the seller offers a uniform price against the coalition. While this restriction is not without loss, it seems unlikely that the loss matters. Recall from Lemma 1 that the seller can only charge a single price against the coalition for each probability of sale, and that the sale price must be convex in the probability of sale. The linearity restriction simply eliminates the strictly convex portion of the sale price. The convex portion would matter only if the coalition is induced to choose a *random* purchase with a *price discount*, but this latter feature seems unlikely to be appealing to the seller (although we have

[9]Alternatively, we can restrict attention to auction rules that allocate the object efficiently among the collusive bidders. The analysis based on this restriction is available from the authors.

not ruled out this possibility). In fact, most of the plausible auction rules allocate the object deterministically as functions of bidders' types. Any such auction rules can be implemented by linear auction rules.[10]

Given the linearity restriction, the constraint in Lemma 2 implies

$$(K(\theta_i) - r)q_i(\theta) \geq 0, \forall i \in N, \forall \theta \in \Theta. \tag{6}$$

Consider the following revenue maximization program:

$$[C] \qquad \max_{(r,q)} \mathbb{E}\left[r \sum_{i \in N} q_i(\theta)\right]$$

subject to

$$\sum_{i \in N} U_i(\underline{\theta}_i) = \mathbb{E}\left[\sum_{i \in N} (J(\theta_i) - r) q_i(\theta)\right] \geq 0 \tag{IC_1^*}$$

and

$$\mathbb{E}\left[\sum_{i \in N} \min\{K(\theta_i) - r, 0\} q_i(\theta)\right] \geq 0. \tag{K}$$

The equality in (IC_1^*) follows from Lemma 0, which utilizes bidders' incentive constraints. Hence, (IC_1^*) follows from (IC^*), which is necessary for any feasible auction rule. In fact, without (K), the above maximization problem simply yields the second-best outcome. The constraint (K) follows from the coalitional incentive constraint in (6) (which in turn follows from Lemma 2). Since the program $[C]$ only imposes necessary conditions for Weak Collusion Proofness, its solution gives an upper bound for the revenue attainable by any linear WCP auction. We show next that this upper bound is attainable by a WCP auction.

THEOREM 5. *Assume $\mathcal{C} = \{N\}$ and bidders are symmetric. There exists a WCP auction rule, that implements the solution of $[C]$. Suppose Condition (SB) fails. Then, the optimal linear auction rule involves a sales price, r_0, that solves either*

$$\max_{r \in R_+} r(1 - F^n(K^{-1}(r))) \tag{7}$$

[10] As will become clear from the subsequent analysis, a solution to $[C]$ below can be obtained even with an additional constraint $\sum q_i(\theta) \in \{0, 1\}$.

or

$$\mathbb{E}[J(\theta_N^{(1)})|\theta_N^{(1)} > K^{-1}(r)] = r, \tag{8}$$

and an allocation rule $\hat{q}(\cdot)$ that satisfies

$$\hat{q}_i(\theta) = \begin{cases} 1 & \text{if } \theta_i > \max\{\max_{j \in N \setminus \{i\}} \theta_j, \hat{\theta}_0\} \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where $\hat{\theta}_0 := K^{-1}(r_0)$.

The features of optimal WCP auctions are illustrated by the next example.

Example 3. *Assume that there are two bidders whose types are uniformly distributed on interval $[m, m+1]$. Then, $J(\theta) = 2\theta - (m+1)$, $K(\theta) = 2\theta - m$, and $\hat{\theta} = \min\{\frac{m+1}{2}, m\}$. The exclusion threshold is then given by*

$$\hat{\theta}_0 = \begin{cases} \frac{m+1}{2} & \text{if } m \le \frac{7-\sqrt{33}}{2} \\ m + \frac{\sqrt{33}-5}{4} & \text{if } \frac{7-\sqrt{33}}{2} < m \le \hat{m}, \\ \frac{5m+\sqrt{m^2+12}}{6} & \text{if } m > \hat{m}, \end{cases}$$

for some $\hat{m} > \frac{7-\sqrt{33}}{2}$.[11] *Observe that the optimal WCP auction rule implements the second-best if and only if $m \le \frac{7-\sqrt{33}}{2}$. For $m > \frac{7-\sqrt{33}}{2}$, $\hat{\theta}_0 > \hat{\theta}$, so it involves more exclusion than the second-best outcome. Regardless of m, $\hat{\theta}_0 > m$, so the optimal WCP auction always involves exclusion. This is in sharp contrasts to the second-best outcome which does not involve any exclusion if $m \ge 1$.*

Remark 2. *Pavlov (2006) finds the outcome presented in Theorem 5 to be optimal in a more restricted class of WCP auction rules, namely, those that are linear* and *symmetric. (Recall the symmetry restriction is not invoked in the current paper.) In fact, he shows that there are asymmetric or nonlinear auction rules that are WCP and yield higher revenue than the optimal auction in the restricted class. However, such auction rules violate our Lemma 1 and*

[11] More precisely, $\hat{m}$ is a root of an equation,

$$(5m + \sqrt{m^2+12})(2m^2 - 9m - 48) + 6(-2m^3 + 9m^2 + 46m + 18) = 0.$$

thus are not WCP in our model. This difference arises from the fact that we allow for a reallocational ability by the coalition, which is not allowed in Pavlov (2006), at least for the main analysis.

7 Conclusion

We have studied optimal weak collusion-proof auctions when a group of bidders can collude not only on their messages (e.g., "bids"), but also on their participation decisions. We have shown that, despite the strong collusive power, the asymmetric information facing the collusive bidders can be exploited to significantly weaken their collusive power, by eliminating the scope of collusive arrangements that could make all collusive bidders uniformly better off regardless of their types. We show that the second-best outcome is achievable if there is a noncollusive bidder or there are multiple bidding rings, or the outcome involves a nontrivial probability of the object not being allocated to a collusive bidder. More generally, we have shown that the optimal weak collusion-proof auction rule involves a positive probability of the object not being allocated to a collusive bidder.

There are several areas of extending/improving our WCP auction design. First, as noted in Remark 1, our auction design can provides incentives in dominant strategies up to a certain number of bidders, but it must rely on Bayesian incentives for *some* bidders. As is well known, Bayesian implementation requires bidders to have accurate assessment of the distribution of other bidders' types, which may be unrealistic in many circumstances. Second, we have assumed that the seller has accurate information about the coalition structure; i.e., which bidders may be potentially collusive with which bidders. While this assumption is not unrealistic in many situations, it would be better if the auction design need not require specific knowledge about the coalitional structure. It seems important and useful to understand how relaxing these features affects an auction designer's ability to deal with collusion.

Appendix: Proofs

Proof of Lemma 0. It suffices to show that (IC^*) implies the following: for all $i \in N$ and all $\tilde{\theta}_i \in \Theta_i$,

$$U_i^M(\theta_i) = \mathbb{E}\left[K_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \leq \theta_i\}} + J_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \geq \theta_i\}} - T_i(\tilde{\theta}_i)\right]. \tag{10}$$

Note first that a well-known necessary condition for (IC^*) is: for all $i \in N$ and all $\tilde{\theta}_i, \theta_i \in \Theta_i$,

$$U_i^M(\theta_i) - U_i^M(\tilde{\theta}_i) = \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)da. \tag{11}$$

We show that (11) implies (10). Since $U_i^M(\tilde{\theta}_i) = \tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)$, (11) becomes

$$U_i^M(\theta_i) = \tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i) + \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)da.$$

Taking expectation on both sides regarding $\tilde{\theta}_i$ yields

$$\begin{aligned}
U_i^M(\theta_i) &= \mathbb{E}[\tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)] + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)dadF_i(\tilde{\theta}_i) \\
&= \mathbb{E}[\tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)] + \int_{\underline{\theta}_i}^{\theta_i} \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)dadF_i(\tilde{\theta}_i) + \int_{\theta_i}^{\overline{\theta}} \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)dadF_i(\tilde{\theta}_i) \\
&= \mathbb{E}[\tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)] + \int_{\underline{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i)F_i(\tilde{\theta}_i)d\tilde{\theta}_i - \int_{\theta_i}^{\overline{\theta}} Q_i(\tilde{\theta}_i)(1 - F_i(\tilde{\theta}_i))d\tilde{\theta}_i \\
&= \mathbb{E}\left[K_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \leq \theta_i\}} + J_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \geq \theta_i\}} - T_i(\tilde{\theta}_i)\right],
\end{aligned}$$

where the third equality follows from the integration by parts. ▌

Proof of Lemma 1. To begin with, define $T_C(\theta_C) := \mathbb{E}_{\tilde{\theta}_{N\setminus C}}[\sum_{i\in C} t_i(\theta_C, \tilde{\theta}_{N\setminus C})]$ and $\mathcal{Q}_C^s := \{Q \in \mathcal{Q}_C | Q = Q_C(\theta_C) \text{ for some } \theta_C \in \overline{\Theta}_C\}$. Then, let us define $r : \mathcal{Q}_C \to \mathbb{R}_+$ as the greatest convex function such that for all $Q \in \mathcal{Q}_C^s$,

$$r(Q) \leq \inf\{T_C(\theta_C) | Q_C(\theta_C) = Q\}.$$

We show that $r(Q_C(\theta_C)) = T_C(\theta_C)$ for almost every θ_C. Suppose not. Then, it must be that for some $\epsilon > 0$,

$$\mathbb{E}\left[r(Q_C(\theta_C))\right] + \epsilon < \mathbb{E}\left[T_C(\theta_C)\right]. \tag{12}$$

Also, by the definition of $r(\cdot)$, it is possible to find $\mu^C : \Theta_C \to \Delta\overline{\Theta}_C$ satisfying that for all θ_C,

$$\mathbf{E}_{\mu^C(\theta_C)}\left[T_C(\tilde{\theta}_C)\right] \leq r(Q(\theta_C)) + \epsilon \text{ and } \mathbf{E}_{\mu^C(\theta_C)}[Q_C(\tilde{\theta}_C)] = Q_C(\theta_C). \tag{13}$$

We now show that M cannot be WCP with respect to C by constructing a weakly feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ of M by coalition C with which some bidder is better off than with M.

Let the coalition manipulate its type reports using $\mu^C(\cdot)$, whereafter, the object is reallocated to agent i with probability $w_i(\theta_C) := \frac{q_i^C(\theta_C)}{Q_C(\theta_C)}$ so that $\sum_{i\in C} w_i(\theta_C) = 1$, satisfying (R_C^M). Note that the interim allocation for each collusive bidder $i \in C$ is preserved since

$$\begin{aligned} \tilde{q}_i^C(\theta_C) &= \omega_i(\theta_C)\mathbb{E}_{\tilde{\theta}_{N\setminus C}}\left[\mathbf{E}_{\mu^C(\theta_C)}[q_C(\tilde{\theta}_C, \tilde{\theta}_{N\setminus C})]\right] \\ &= \omega_i(\theta_C)\mathbf{E}_{\mu^C(\theta_C)}[Q_C(\tilde{\theta}_C)] = \omega_i(\theta_C)Q_C(\theta_C) = q_i^C(\theta_C), \end{aligned} \tag{14}$$

where the second equality follows from changing the order of expectations, the third from (13), and the last from the definition of $\omega_i(\cdot)$.

Next, the coalition manipulates the transfer rule as follows: Letting $t_i^\mu(\theta) := \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N\setminus C})]$, set $\tilde{t}_j(\theta) = t_j^\mu(\theta)$ for each $j \in N\setminus C$, and for each $i \in C$,

$$\tilde{t}_i(\theta) = t_i^\mu(\theta) + \mathbb{E}_{\tilde{\theta}_{-i}}\left[t_i(\theta_i, \tilde{\theta}_{-i}) - t_i^\mu(\theta_i, \tilde{\theta}_{-i})\right] - \frac{1}{|C|-1}\sum_{j\in C\setminus\{i\}}\mathbb{E}_{\tilde{\theta}_{-j}}\left[t_j(\theta_j, \tilde{\theta}_{-j}) - t_j^\mu(\theta_j, \tilde{\theta}_{-j})\right] + \sigma_i,$$

where $\sum_{i\in C}\sigma_i = 0$. Note that $\sum_{i\in C}\tilde{t}_i(\theta) = \sum_{i\in C} t_i^\mu(\theta)$, which satisfies (BB_C^M) while $(BB_{N\setminus C}^M)$ is obviously satisfied. Also,

$$\mathbb{E}_{\tilde{\theta}_{-i}}\left[\tilde{t}_i(\theta_i, \tilde{\theta}_{-i})\right] = \mathbb{E}_{\tilde{\theta}_i}\left[t(\theta_i, \tilde{\theta}_{-i})\right] - \kappa_i, \tag{15}$$

where

$$\kappa_i := \frac{1}{|C|-1}\sum_{j\in C\setminus\{i\}}\mathbb{E}\left[t_j(\theta) - t_j^\mu(\theta)\right] - \sigma_i.$$

Then, one can choose $\sigma_i's$ so that $\kappa_i \geq 0, \forall i \in C$, since

$$\sum_{i\in C}\kappa_i = \mathbb{E}\left[\sum_{i\in C}(t_i(\theta) - t_i^\mu(\mu))\right] = \mathbb{E}\left[T_C(\theta_C) - \mathbf{E}_{\mu^C(\theta_C)}[T_C(\tilde{\theta}_C)]\right] > 0, \tag{16}$$

where the inequality follows from (12) and (13). So, (IC^*) and (IR_N^C) are satisfied for collusive bidders, due to (14), (15), and $\kappa_i \geq 0, \forall i \in C$, which means that $\tilde{M}$ is a weakly feasible manipulation of M. Also, some collusive bidder is better off than in M since $\kappa_j > 0$ for some $j \in C$.▌

Proof of Lemma 2. To begin, we adopt the convention that $\theta_\emptyset < \underline{\theta}_i$ for all $i \in N$. We observe that $Q_C(\cdot)$ and $q_i^C(\cdot)$ inherit the monotonicity of $q_C(\cdot)$ and $q_i(\cdot)$, respectively, and hence are a.e. continuous. Also, since $r(\cdot)$ is convex with $r(0) = 0$, $p(\cdot)$ is nondecreasing and hence a.e. continuous also. Suppose to the contrary that (2) does not hold for almost every type profile. Then, we can find some bidder $k \in C$ and a positive measure set $\hat{\Theta}_{C-k} \subset \Theta_{C-k}$ such that for each $\theta_{C-k} \in \hat{\Theta}_{C-k}$, there exist $\theta_k \in \Theta_k$ and $\theta'_k \in \overline{\Theta}$ satisfying

$$(K_k(\theta_k) - p(\theta_k, \theta_{C-k}))q_i^C(\theta_k, \theta_{C-k}) < (K_k(\theta_k) - p(\theta'_k, \theta_{C-k}))q_i^C(\theta'_k, \theta_{C-k}).$$

Then, the a.e. continuity of $q_i^C(\cdot)$ and $p(\cdot)$ guarantees that for each $\theta_{C-k} \in \hat{\Theta}_{C-k}$, we can find two types $\hat{\theta}_k(\theta_{C-k}) \in \overline{\Theta}$ and $\hat{\hat{\theta}}_k(\theta_{C-k}) > \hat{\theta}_k(\theta_{C-k})$ such that for all $\theta_k \in (\hat{\theta}_k(\theta_{C-k}), \hat{\hat{\theta}}_k(\theta_{C-k})]$,

$$(K_k(\theta_k) - p(\theta_k, \theta_{C-k}))q_i^C(\theta_k, \theta_{C-k}) < (K_k(\theta_k) - p(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}))q_i^C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}). \quad (17)$$

We now define

$$\hat{\Theta}_C := \{(\theta_k, \theta_{C-k}) \in \Theta | \theta_{C-k} \in \hat{\Theta}_{C-k} \text{ and } \theta_k \in (\hat{\theta}_k(\theta_{C-k}), \hat{\hat{\theta}}_k(\theta_{C-k})]\},$$

$\hat{q}_k^C(\theta_{C-k}) := q_k^C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})$, and $\hat{p}(\theta_{-k}) := p(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})$. Note that (17) holds for all $\theta_C \in \hat{\Theta}_C$.

In order to draw a contradiction, we construct a weakly feasible manipulation of M, $\tilde{M} = (\tilde{q}, \tilde{t})$, which makes bidder k better off.

Consider the following report manipulation, denoted $\mu^C : \Theta_C \to \Delta\overline{\Theta}_C$, and reallocation scheme by the coalition: if $\theta_C \notin \hat{\Theta}_C$, then report truthfully and do not perform any reallocation while if $\theta_C \in \hat{\Theta}_C$, then (i) report truthfully with probability $\frac{\sum_{i \in C\setminus\{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)}$ and, once the object is assigned, reallocate it to bidder $i \in C\setminus\{k\}$ with probability $\frac{q_i^C(\theta)}{\sum_{i \in C\setminus\{k\}} q_i^C(\theta_C)}$, (ii) report $(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})$ (or $(\theta_\emptyset, \cdots, \theta_\emptyset)$ in case $\hat{\theta}_k(\theta_{C-k}) = \theta_\emptyset$) with probability $\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k})), \theta_{C-k})}$ and, once the object is assigned, reallocate it to bidder k with probability 1, and (iii) choose $(\theta_\emptyset, \cdots, \theta_\emptyset)$ (or nonparticipation) with the remaining probability.[12] This manipulation will

[12]It is important to make sure that the probability of reporting truthfully or $(\hat{\theta}(\theta_{C-k}), \theta_{C-k})$ does not exceed

result in the following allocation probabilities: for bidder $i \in C\backslash\{k\}$,

$$\tilde{q}_i^C(\theta_C) = Q_C(\theta_C)\frac{\sum_{i\in C\backslash\{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)}\frac{q_i^C(\theta_C)}{\sum_{i\in C-k} q_i^C(\theta_C)} = q_i^C(\theta_C) \text{ if } \theta_C \in \hat{\Theta}_C,$$

and simply $\tilde{q}_i^C(\theta_C) = q_i^C(\theta_C)$ if $\theta_C \notin \hat{\Theta}_C$. Likewise, for bidder k, if $\theta_C \notin \hat{\Theta}_C$, then $\tilde{q}_k^C(\theta_C) = q_k^C(\theta_C)$, or else if $\theta_C \in \hat{\Theta}_C$, then

$$\tilde{q}_k^C(\theta_C) = Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})} = \hat{q}_k^C(\theta_{C-k}). \tag{18}$$

It can be easily verified that $\tilde{q}_k^C(\cdot,\theta_{C-k})$ is nondecreasing for every θ_{C-k}.[13] Thus, the interim allocation $\tilde{Q}_i(\theta_i) = \mathbb{E}_{\tilde{\theta}_{C-i}}[\tilde{q}_i^C(\theta_i,\tilde{\theta}_{C-i})]$ is also nondecreasing for each $i \in C$.

After the manipulation, the coalition's aggregate payment becomes

$$\begin{aligned}
&\mathbb{E}_{\tilde{\theta}_{N\backslash C}}\left[\mathbf{E}_{\mu^C(\theta_C)}[\sum_{i\in C} t_i(\tilde{\theta}_C,\tilde{\theta}_{N\backslash C})]\right]\\
&= \begin{cases} r(Q_C(\theta_C))\dfrac{\sum_{i\in C\backslash\{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)} + r(Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k}))\dfrac{\hat{q}_k^C(\theta_{C-k})}{Q(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})} & \text{if } \theta_C \in \hat{\Theta}_C \\ r(Q_C(\theta_C)) & \text{otherwise,} \end{cases}
\end{aligned}$$

which yields

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{E}_{\mu^C(\theta_C)}[\sum_{i\in C} t_i(\tilde{\theta}_C,\theta_{N\backslash C})]\right]\\
&= \mathbb{E}\left[r(Q_C(\theta_C))\right] + \mathbb{E}_{\theta_C\in\hat{\Theta}_C}\left[r(Q(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k}))\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})} - r(Q_C(\theta_C))\frac{q_k^C(\theta_C)}{Q_C(\theta_C)}\right]
\end{aligned}$$

1, for which it suffices to verify that $\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})} \le \frac{q_k^C(\theta_C)}{Q_C(\theta_C)}$. This holds trivially if $\hat{\theta}_k(\theta_{C-k}) = \theta_\emptyset$. If $\hat{\theta}_k(\theta_{C-k}) \neq \theta_\emptyset$, it holds since

$$\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})} = 1 - \frac{\sum_{i\in C\backslash\{k\}} q_i^C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k})} \le 1 - \frac{\sum_{i\in C\backslash\{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)} = \frac{q_k^C(\theta_C)}{Q_C(\theta_C)},$$

where the inequality holds since $Q_C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k}) \le Q_C(\theta_k,\theta_{C-k})$ and $q_i^C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k}) \ge q_i^C(\theta_k,\theta_{C-k})$ for all $i \neq k$, by the monotonicity of $Q_C(\cdot)$ and $q_i^C(\cdot)$.

[13]To see this, consider arbitrary θ_k and θ'_k with $\theta'_k > \theta_k$, and θ_{C-k}: (i) if $(\theta_k,\theta_{C-k}),(\theta'_k,\theta_{C-k}) \in \hat{\Theta}_C$, then $\tilde{q}_k^C(\theta_k,\theta_{C-k}) = \tilde{q}_k^C(\theta'_k,\theta_{C-k}) = \hat{q}_k^C(\theta_{C-k})$, (ii) if $(\theta_k,\theta_{C-k}) \in \hat{\Theta}_C$ and $(\theta'_k,\theta_{C-k}) \notin \hat{\Theta}_C$, then $\hat{\theta}_k(\theta_{C-k}) < \theta_k \le \hat{\hat{\theta}}_k(\theta_{C-k}) < \theta'_k$ and thus $\tilde{q}_k^C(\theta_k,\theta_{C-k}) = q_k^C(\hat{\theta}_k(\theta_{C-k}),\theta_{C-k}) \le q_k^C(\theta'_k,\theta_{C-k}) = \tilde{q}_k^C(\theta'_k,\theta_{C-k})$. And other cases can be dealt with similarly.

$$= \mathbb{E}\left[\sum_{i\in C} t_i(\theta)\right] + \mathbb{E}_{\theta_C \in \hat{\Theta}_C}\left[\hat{p}(\theta_{C-k})\hat{q}_k^C(\theta_{C-k}) - p(\theta_C)q_k^C(\theta_C)\right] \tag{19}$$

Next, $\tilde{t}(\cdot)$ is constructed as follows. For each $j \in N\backslash C$, set $\tilde{t}_j(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[t_j(\tilde{\theta}_C, \theta_{N\backslash C})]$. For each $i \in C$, we set

$$\tilde{t}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N\backslash C})] + Y_i(\theta_i) - \frac{1}{|C|-1}\sum_{j\in C\backslash\{i\}} Y_j(\theta_j) + \rho_i,$$

where

$$Y_i(\theta_i) := \theta_i \tilde{Q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \tilde{Q}_i(a)da - \mathbb{E}_{\theta_{-i}}[\mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N\backslash C})]],$$

and

$$\rho_i := \frac{1}{|C|-1}\mathbb{E}_{\theta_{-i}}\left[\sum_{j\in C\backslash\{i\}} Y_j(\theta_j)\right] - U_i^M(\underline{\theta}_i) \text{ for } i \in C\backslash\{k\}, \text{ and } \rho_k = -\sum_{i\in C\backslash\{k\}} \rho_i.$$

By construction, then $\tilde{t}$ satisfies (BB_C^M) and $(BB_{N\backslash C}^M)$.

We now complete the proof by showing that $\tilde{M}$ is a weakly feasible manipulation and makes bidder k better off. To this end, observe that for an arbitrary $\theta_k \in \Theta_k$,

$$\begin{aligned}
& U_k^{\tilde{M}}(\theta_k) + \sum_{i\in C\backslash\{k\}} U_i^{\tilde{M}}(\underline{\theta}_i) \\
= \;& \mathbb{E}\left[\left(K_k(\tilde{\theta}_k)\tilde{q}_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k<\theta_k\}} + \left(J_k(\tilde{\theta}_k)\tilde{q}_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k>\theta_k\}} + \sum_{i\in N\backslash\{k\}} J_i(\tilde{\theta}_i)q_i(\tilde{\theta}) - \sum_{i\in C}\tilde{t}_i(\tilde{\theta})\right] \\
= \;& \mathbb{E}\left[\left(K_k(\tilde{\theta}_k)q_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k<\theta_k\}} + \left(J_k(\tilde{\theta}_k)q_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k>\theta_k\}} + \sum_{i\in N\backslash\{k\}} J_i(\tilde{\theta}_i)q_i(\tilde{\theta}) - \sum_{i\in C} t_i(\tilde{\theta})\right] \\
& + \mathbb{E}_{\tilde{\theta}_C\in\hat{\Theta}_C}\Big[K_k(\tilde{\theta}_k)(\tilde{q}_k^C(\tilde{\theta}_C) - q_k^C(\tilde{\theta}_C))1_{\{\tilde{\theta}_k<\theta_k\}} + J_k(\tilde{\theta}_k)(\tilde{q}_k^C(\tilde{\theta}_C) - q_k^C(\tilde{\theta}_C))1_{\{\tilde{\theta}_k>\theta_k\}} \\
& \qquad - \left(\hat{p}(\tilde{\theta}_{C-k})\hat{q}_k^C(\tilde{\theta}_{C-k}) - p(\tilde{\theta}_C)q_k^C(\tilde{\theta}_C)\right)\Big] \\
= \;& U_k^M(\theta_k) + \sum_{i\in C\backslash\{k\}} U_i^M(\underline{\theta}_i) \\
& + \mathbb{E}_{\tilde{\theta}_C\in\hat{\Theta}_C}\Big[\left((K_k(\tilde{\theta}_k) - \hat{p}(\tilde{\theta}_{C-k}))\hat{q}_k^C(\tilde{\theta}_{C-k}) - (K_k(\tilde{\theta}_k) - p_k(\tilde{\theta}_C))q_k^C(\tilde{\theta}_C)\right)1_{\{\tilde{\theta}_k<\theta_k\}} \\
& \qquad + \left((J_k(\tilde{\theta}_k) - \hat{p}(\tilde{\theta}_{C-k}))\hat{q}_k^C(\tilde{\theta}_{C-k}) - (J_k(\tilde{\theta}_k) - p_k(\tilde{\theta}_C))q_k^C(\tilde{\theta}_C)\right)1_{\{\tilde{\theta}_k>\theta_k\}}\Big]
\end{aligned}$$

$$> \ U_k^M(\theta_k) + \sum_{i\in C\backslash\{k\}} U_i^M(\underline{\theta}_i). \tag{20}$$

The first equality follows from Lemma 0, the second from (19), the third from the rearrangement and (18), and the inequality from (17) and the fact that for all $\tilde{\theta}_C \in \hat{\Theta}_C$,

$$J_k(\tilde{\theta}_k)(\hat{q}_k^C(\tilde{\theta}_{C-k}) - q_k^C(\tilde{\theta}_C)) \geq K_k(\tilde{\theta}_k)(\hat{q}_k^C(\tilde{\theta}_{C-k}) - q_k^C(\tilde{\theta}_C)),$$

since $\hat{q}_k^C(\tilde{\theta}_{C-k}) \leq q_k^C(\tilde{\theta}_C)$ and $J_k(\tilde{\theta}_k) < K_k(\tilde{\theta}_k)$.

From the construction of $\tilde{t}(\cdot)$, one can easily verify that $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^M(\underline{\theta}_i)$, $\forall i \in C\backslash\{k\}$. Then, (20) implies $U_k^{\tilde{M}}(\theta_k) > U_k^M(\theta_k)$ for all $\theta_k \in \Theta_k$ or bidder k is better off. The construction of $\tilde{t}(\cdot)$ and monotonicity of $\tilde{Q}_i(\cdot), \forall i \in C$ guarantee that $\tilde{M}$ satisfies (IC^*) for all collusive bidders. The proof will be complete if (IR_C^M) holds for all $i \in C\backslash\{k\}$:

$$U_i^{\tilde{M}}(\theta_i) = U_i^{\tilde{M}}(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \tilde{Q}_i(\tilde{\theta})d\tilde{\theta} = U_i^M(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} Q_i(\tilde{\theta})d\tilde{\theta} = U_i^M(\theta_i), \forall \theta_i \in \Theta_i$$

since $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^M(\underline{\theta}_i)$ and $\tilde{Q}_i(\cdot) = Q_i(\cdot)$, $\forall i \in C\backslash\{k\}$. ▌

Proof of Theorem 1. Suppose an auction rule $M = (q,t) \in \mathcal{M}$ WCP implements the second-best outcome. Then, $q(\cdot) = q^*(\cdot)$ and $U_i(\underline{\theta}_i) = 0$ for all $i \in N$, which implies by Lemma 0 that for any $C \subset N$,

$$\mathbb{E}\left[\sum_{i\in C} t_i(\theta)\right] = \mathbb{E}\left[\sum_{i\in C} J_i(\theta_i)q_i^*(\theta_i)\right]. \tag{21}$$

By Lemma 1, there exists a convex function $r(\cdot)$ that represents the total payment for the coalition.

We first consider the case $\mathcal{C} = \{N\}$. Since $q_N^*(\theta) = 0$ or 1 for all $\theta \in \Theta$, Lemma 1 implies that $p(\theta) = r^*$ whenever $q_N^*(\theta) = 1$. We first prove $\hat{\theta}_i > \underline{\theta}_i$ for all $i \in N$. Suppose not. Then, there exists k such that $J_k(\underline{\theta}_k) \geq \max\{\max_{i\in N\backslash\{k\}} J_i(\underline{\theta}_i), 0\}$. It follows that $q_k^*(\underline{\theta}_1, \cdots, \underline{\theta}_n) > 0$, so $p(\underline{\theta}_1, \cdots, \underline{\theta}_n) = r^*$. Since $r^* \geq V^* > \underline{\theta}_i = K_i(\underline{\theta}_i)$, we have a contradiction to (2).

We next consider the case $\mathcal{C} \neq \{N\}$. Fix any C with $|C| \geq 2$. If no such C exists, there is no collusion, so we are done. For each bidder $i \in C$ and his type $\theta_i \in \Theta_i$, let $X_i(\theta_i) := \Pr\{\theta_{C-i} \in \Theta_{C-i} \mid J_i(\theta_i) > \max_{k\in C\backslash\{i\}} J_k(\theta_k)\}$ be the probability that i has the highest virtual value among the collusive bidders, and let $Y_i(\theta_i) := \Pr\{\theta_{N\backslash C} \in \Theta_{N\backslash C} \mid J_i(\theta_i) >$

$\max\{\max_{k\in N\setminus C} J_k(\theta_k), 0\}\}$. Letting $p_i(\theta_i) := \frac{r(Y_i(\theta_i))}{Y_i(\theta_i)}$ for each $i \in C$, Lemma 2 implies that, $\forall \theta_i \geq \hat{\theta}_i$

$$(K_i(\theta_i) - p_i(\theta_i))\, Y_i(\theta_i) \geq \max\{0, \max_{\theta_i' \in [\hat{\theta}_i, \theta_i]} (K_i(\theta_i) - p_i(\theta_i'))\, Y_i(\theta_i')\}.$$

By the envelope theorem argument, $\forall \theta_i \geq \hat{\theta}_i$,

$$(K_i(\theta_i) - p_i(\theta_i))Y_i(\theta_i) \geq (K_i(\hat{\theta}_i) - p_i(\hat{\theta}_i))Y_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da \geq \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da$$

or

$$p_i(\theta_i)Y_i(\theta_i) \leq K_i(\theta_i)Y_i(\theta_i) - \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da.$$

Thus, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i\in C} t_i(\theta)\right] &= \mathbb{E}\left[\sum_{i\in C} r(Y_i(\theta_i))X_i(\theta_i)\right] \\
&= \mathbb{E}\left[\sum_{i\in C} p_i(\theta_i)Y_i(\theta_i)X_i(\theta_i)\right] \\
&\leq \mathbb{E}\left[\sum_{i\in C}\left(K_i(\theta_i)Y_i(\theta_i) - \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da\right)X_i(\theta_i)\right]. \qquad (22)
\end{aligned}$$

Letting $Z_i(\theta_i) = \int_{\theta_i}^{\overline{\theta}_i} X_i(s)dF_i(s)$,

$$\begin{aligned}
&\mathbb{E}\left[\left(K_i(\theta_i)Y_i(\theta_i) - \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da\right)X_i(\theta_i)\right] \\
= &\int_{\hat{\theta}_i}^{\overline{\theta}_i} K_i(\theta_i)Y_i(\theta_i)X_i(\theta_i)dF_i(\theta_i) - \int_{\hat{\theta}_i}^{\overline{\theta}_i}\int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)daX_i(\theta_i)dF_i(\theta_i) \\
= &\int_{\hat{\theta}_i}^{\overline{\theta}_i} K_i(\theta_i)X_i(\theta_i)Y_i(\theta_i)dF_i(\theta_i) - \int_{\hat{\theta}_i}^{\overline{\theta}_i} K_i'(\theta_i)Y_i(\theta_i)Z_i(\theta_i)d\theta_i \\
= &\int_{\hat{\theta}_i}^{\overline{\theta}_i} K_i(\theta_i)X_i(\theta_i)Y_i(\theta_i)dF_i(\theta_i) - K_i(\theta_i)Y_i(\theta_i)Z_i(\theta_i)\Big|_{\hat{\theta}_i}^{\overline{\theta}_i} + \int_{\hat{\theta}_i}^{\overline{\theta}_i} K_i(\theta_i)\left(Y_i'(\theta_i)Z_i(\theta_i) + Y_i(\theta_i)Z_i'(\theta_i)\right)d\theta_i \\
= &K_i(\hat{\theta}_i)Y_i(\hat{\theta}_i)Z_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\overline{\theta}_i} K_i(\theta_i)Y_i'(\theta_i)Z_i(\theta_i)d\theta_i \\
= &\mathbb{E}\left[K_i(\phi_i(\theta_{N\setminus C}))q_i^*(\theta)\right].
\end{aligned}$$

The second and fourth equalities follow from integration by parts. To verify the fifth equality, note that $Y_i(\hat{\theta}_i) = \Pr\{\phi_i(\theta_{N\setminus C}) = \hat{\theta}_i\}$, $Y_i(s) = \Pr\{\phi_i(\theta_{N\setminus C}) \leq s\}$ for each $s > \hat{\theta}_i$, and

$Z_i(s) = \mathbb{E}[q_i^*(\theta)|\phi_i(\theta_{N\setminus C}) = s]$. Combine this derivation with (21) and (22) to obtain (ii) of Condition (SB). ▌

Proof of Lemma 3. First, we prove that Condition (SB') holds for any C with $2 \leq |C| < n$. To this end, observe that

$$\begin{aligned}
& \mathbb{E}\left[K(\theta^{(1)}_{N\setminus C})1_{\{\theta^{(1)}_C > \theta^{(1)}_{N\setminus C}\}}\right] \\
= \;& K(\hat{\theta})(1-F^{|C|}(\hat{\theta}))F^{N-|C|}(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}}\left(\theta + \frac{F(\theta)}{f(\theta)}\right)(1-F^{|C|}(\theta))dF^{N-|C|}(\theta) \\
= \;& K(\hat{\theta})(1-F^{|C|}(\hat{\theta}))F^{N-|C|}(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}}\theta(1-F^{|C|}(\theta))dF^{N-|C|}(\theta) \\
& \qquad + \int_{\hat{\theta}}^{\overline{\theta}}(N-|C|)(1-F^{|C|}(\theta))F^{N-|C|}(\theta)d\theta. \qquad (23)
\end{aligned}$$

Observe also that

$$\begin{aligned}
& \mathbb{E}\left[J(\theta^{(1)}_C)1_{\{\theta^{(1)}_C > \theta^{(1)}_{N\setminus C}\}}\right] = \int_{\hat{\theta}}^{\overline{\theta}}\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)F^{N-|C|}(\theta)dF^{|C|}(\theta) \\
= \;& -(1-F^{|C|}(\theta))\theta F^{N-|C|}(\theta)\Big|_{\hat{\theta}}^{\overline{\theta}} + \int_{\hat{\theta}}^{\overline{\theta}}(1-F^{|C|}(\theta))d\left(\theta F^{N-|C|}(\theta)\right) - \int_{\hat{\theta}}^{\overline{\theta}}|C|(1-F(\theta))F^{N-1}(\theta)d\theta \\
= \;& \hat{\theta}(1-F^{|C|}(\hat{\theta}))F^{N-|C|}(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}}(1-F^{|C|}(\theta))F^{N-|C|}(\theta)d\theta \\
& \qquad + \int_{\hat{\theta}}^{\overline{\theta}}\theta(1-F^{|C|}(\theta))dF^{N-|C|}(\theta) - \int_{\hat{\theta}}^{\overline{\theta}}|C|(1-F(\theta))F^{N-1}(\theta)d\theta,
\end{aligned}$$

where the second equality follows from integration by parts. Subtracting this expression from (23) yields

$$\begin{aligned}
& \mathbb{E}\left[\sum_{i\in C}\left(K_i(\phi_i(\theta_{N\setminus C})) - J_i(\theta_i)\right)q_i^*(\theta)\right] \\
= \;& \mathbb{E}\left[\left(K(\theta^{(1)}_{N\setminus C}) - J(\theta^{(1)}_C)\right)1_{\{\theta^{(1)}_C > \theta^{(1)}_{N\setminus C}\}}\right] \\
= \;& (K(\hat{\theta}) - \hat{\theta})(1-F^{|C|}(\hat{\theta}))F^{N-|C|}(\hat{\theta}) \\
& \qquad + \int_{\hat{\theta}}^{\overline{\theta}}\left[(N-|C|-1)(1-F^{|C|}(\theta))F^{N-|C|}(\theta) + |C|(1-F(\theta))F^{N-1}(\theta)\right]d\theta > 0,
\end{aligned}$$

satisfying Condition (SB).

In case $C = N$, (3) is just a restatement of Condition (SB) with $\theta^{(1)}_{N\setminus C} = \hat{\theta}$. ▌

Proof of Theorem 2. Since $\hat{M}$ implements V^*, it suffices to prove that $\hat{M}$ is WCP. To this end, consider any $C \in \mathcal{C}$ with $|C| \geq 2$. Suppose all bidders outside C report truthfully, but coalition C contemplates a manipulation of $\hat{M}$, $\tilde{M} = (\tilde{q}, \tilde{t})$, that satisfies (IC_C^*) and $(IR_C^{\hat{M}})$. Then, there exists a function $\mu^C : \Theta_C \mapsto \Delta\Theta_C$ such that $(R_C^{\hat{M}}), (R_{N\backslash C}^{\hat{M}}), (BB_C^{\hat{M}})$ and $(BB_{N\backslash C}^{\hat{M}})$ hold.

We first prove that $\tilde{q}(\theta) = q^*(\theta)$ for almost every θ. To this end, suppose this is not the case. Then,

$$
\begin{aligned}
& \alpha_C(\sum_{i\in C} U_i^{\tilde{M}}(\overline{\theta})) + (1-\alpha_C)(\sum_{i\in C} U_i^{\tilde{M}}(\underline{\theta})) \\
= \;& \mathbb{E}\left[\sum_{i\in C} H_C(\theta_i)\tilde{q}_i(\theta) - \sum_{i\in C}\tilde{t}_i(\theta)\right] \\
= \;& \mathbb{E}\left[\sum_{i\in C} H_C(\theta_i)\tilde{q}_i(\theta) - \sum_{i\in C}\mathbf{E}_{\mu^C(\theta_C)}[\hat{t}_i(\tilde{\theta}_C, \theta_{N\backslash C})]\right] \\
= \;& \mathbb{E}\left[\sum_{i\in C} H_C(\theta_i)\tilde{q}_i(\theta) - \mathbf{E}_{\mu^C(\theta_C)}[\delta_C(\tilde{\theta}_C, \theta_{N\backslash C})\right] \\
= \;& \mathbb{E}\left[\sum_{i\in C} H_C(\theta_i)\tilde{q}_i(\theta) - H_C(\theta_{N\backslash C}^{(1)})\mathbf{E}_{\mu^C(\theta_C)}[\sum_{i\in C} q_i^*(\tilde{\theta}_C, \theta_{N\backslash C})]\right] \\
= \;& \mathbb{E}\left[\sum_{i\in C}[H_C(\theta_i) - H_C(\theta_{N\backslash C}^{(1)})]\tilde{q}_i(\theta)\right] \\
< \;& \mathbb{E}\left[\sum_{i\in C}[H_C(\theta_i) - H_C(\theta_{N\backslash C}^{(1)})]q_i^*(\theta)\right] \qquad (24) \\
= \;& \alpha_C(\sum_{i\in C} U_i^{\hat{M}}(\overline{\theta})) + (1-\alpha_C)(\sum_{i\in C} U_i^{\hat{M}}(\underline{\theta})).
\end{aligned}
$$

The first equality follows from Lemma 0, the second from equation $(BB_C^{\hat{M}})$, the third from the construction of $\hat{t}_i(\cdot)$ for $i \in C$, the fourth from the definition of $\delta_C(\cdot)$, the fifth from $(R_C^{\hat{M}})$, and the last equality from the above string of equalities repeated in the reverse order. Lastly, the strict inequality follows from the definition of α_C and the strict monotonicity of $H_C(\cdot)$. To see this, we compare the LHS and RHS of the inequality (24) at the ex-post level: (i) if $\theta_k > \max\{\max_{i\in N\backslash\{k\}}\theta_i, \hat{\theta}\}$ for some $k \in C$, then $q_k^*(\theta) = 1 \neq \tilde{q}_k(\theta)$ implies that

$$
LHS = \sum_{i\in C}(H_C(\theta_i) - H_C(\theta_{N\backslash C}^{(1)}))\tilde{q}_i(\theta_i) < H_C(\theta_k) - H_C(\theta_{N\backslash C}^{(1)}) = RHS,
$$

(ii) if $\theta_k > \max\{\max_{i\in N\setminus\{k\}} \theta_i, \hat{\theta}\}$ for some $k \in N\setminus C$, then any manipulated allocation different from $q^*(\cdot)$ implies $\tilde{q}_k(\theta) < 1$ and $\tilde{q}_{k'}(\theta) > 0$ for some $k' \in C$,[14] and thus

$$LHS = \sum_{i\in C}(H_C(\theta_i) - H_C(\theta^{(1)}_{N\setminus C}))\tilde{q}_i(\theta_i) = \sum_{i\in C}(H_C(\theta_i) - H_C(\theta_k))\tilde{q}_i(\theta_i) < 0 = RHS,$$

(iii) if $\max_{i\in N}\theta_i < \hat{\theta}$, then $\tilde{q}(\theta) \neq q^*(\theta) = \mathbf{0}$ implies that the LHS is negative while the RHS is zero. In sum, the LHS of (24) is always less than the RHS, which means that $\tilde{M}$ worsens the (interim) payoff of either the highest type or the lowest type of at least one collusive bidder. This contradicts that $\tilde{M}$ satisfies $(IR_C^{\hat{M}})$. We have thus proven that $\tilde{q}(\theta) = q^*(\theta)$ for almost every θ.

It follows from this result that the gross surplus realized within C from $\tilde{M}$ is the same as from $\hat{M}$, and, combined with (4), that the coalition pays the same expected payments with $\tilde{M}$ as with $\hat{M}$. Hence, the net total expected payoff accruing to C from $\tilde{M}$ is the same as from $\hat{M}$. Together with $(IR_C^{\hat{M}})$, this implies that no bidder of C is strictly better off from manipulation $\tilde{M}$. Since this is true for all feasible manipulation of $\hat{M}$, we conclude that $\hat{M}$ is WCP. ▌

Proof of Theorem 3. Define $\hat{T}(\cdot)$ and $\hat{t}(\cdot)$ in the same way as in (33) and (34), except that $\hat{q}$ and r_0 are replaced with q^* and r^*, respectively. Then, the argument that $\hat{M} = (q^*, \hat{t})$ satisfies (IC^*), and implements the second-best outcome without collusion, is similar to that in the proof of Proposition 5.

To prove that $\hat{M}$ is WCP consists of several steps.

STEP 1. *Suppose that a feasible manipulation, $M = (q, t)$, of $\hat{M}$ (by N) satisfies $U_i^M(\underline{\theta}_i) > U_i^{\hat{M}}(\underline{\theta}_i)$ for some $i \in N$. Then, there exists another feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ that satisfies*

$$U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N, \quad \textit{and} \;\; \mathbb{E}\left[\sum_{i\in N} U_i^{\tilde{M}}(\theta_i)\right] > \mathbb{E}\left[\sum_{i\in N} U_i^{\hat{M}}(\theta_i)\right]. \tag{25}$$

[14] This follows from the fact that noncollusive bidders always report truthfully so collusive bidders can change the allocation only by announcing that one of them has at least $\theta_k > \hat{\theta}$, and getting themselves allocated the object.

Proof. Consider a bidder k for whom $U_k^M(\underline{\theta}_k) > U_k^{\hat{M}}(\underline{\theta}_k)$. We construct a mechanism $\tilde{M}$ which satisfies (IC^*), $(IR_N^{\hat{M}})$, and $U_k^{\tilde{M}}(\underline{\theta}_k) = U_k^{\hat{M}}(\underline{\theta}_k)$.

We first construct an 'auxiliary' mechanism M' which will be used to construct $\tilde{M}$. Let us begin by defining $\Theta' \subset \Theta$ as

$$\Theta' := \{\theta \in \Theta | \theta_k \in [K_k^{-1}(r^*), \hat{\theta}_k) \text{ and } \theta_i < \hat{\theta}_i, \forall i \neq k\},$$

which must have a positive measure due to Condition (SB^*). The allocation rule is constructed as

$$q'(\theta) = (q'_k(\theta), q'_{-k}(\theta)) := \begin{cases} (1, \mathbf{0}) & \text{if } \theta \in \Theta', \\ (q^*_k(\theta), q^*_{-k}(\theta)) & \text{otherwise.} \end{cases}$$

Clearly, $q'(\cdot)$ results in a nondecreasing interim allocation probability for each agent. Next, let us construct the transfer rules, $T'_i(\cdot)$ and $t'_i(\cdot)$, as in (33) and (34) except that $\hat{q}(\cdot)$ is now replaced by $q'(\cdot)$ with ρ_i's to be determined later. Then, $M' = (q', t')$ satisfies (IC).[15] Note that M' can be obtained by manipulating $\hat{M}$ in the following way: if $\theta \in \Theta'$, then bidder i report some $\theta'_i > \hat{\theta}_i$ and others report truthfully, and if $\theta \notin \Theta'$, then all bidders report truthfully. Also, we have

$$\begin{aligned}
& U_k^{M'}(\hat{\theta}_k) + \sum_{i \in N \setminus \{k\}} U_i^{M'}(\underline{\theta}_i) \\
= \;& \mathbb{E}\left[K_k(\theta_k) q'_k(\theta) 1_{\{\theta_k \leq \hat{\theta}_k\}}\right] + \mathbb{E}\left[J_k(\theta_k) q'_k(\theta) 1_{\{\theta_k \geq \hat{\theta}_k\}}\right] + \mathbb{E}\left[\sum_{i \in N \setminus \{k\}} J_i(\theta_i) q'_i(\theta)\right] - \mathbb{E}\left[\sum_{i \in N} t'_i(\theta)\right] \\
= \;& \mathbb{E}\left[K_k(\theta_k) q'_k(\theta) 1_{\{\theta_k \leq \hat{\theta}_k\}}\right] + \mathbb{E}\left[\sum_{i \in N} J_i(\theta_i) q'_i(\theta) 1_{\{\theta_i \geq \hat{\theta}_i\}}\right] - \mathbb{E}\left[r^* \sum_{i \in N} q'_i(\theta)\right] \\
= \;& \mathbb{E}\left[(K_k(\theta_k) - r^*) q'_k(\theta) 1_{\{\theta_k \leq \hat{\theta}_k\}}\right] + \mathbb{E}\left[\sum_{i \in N} (J_i(\theta_i) - r^*) q'_i(\theta) 1_{\{\theta_i \geq \hat{t}_i\}}\right] \\
= \;& \mathbb{E}\left[(K_k(\theta_k) - r^*) 1_{\{\theta \in \Theta'\}}\right] + \mathbb{E}\left[\sum_{i \in N} (J_i(\theta_i) - r^*) q^*_i(\theta)\right] \\
> \;& \mathbb{E}\left[\sum_{i \in N} (J_i(\theta_i) - r^*) q^*_i(\theta)\right] = \sum_{i \in N} U_i^{\hat{M}}(\underline{\theta}_i) = U_k^{\hat{M}}(\hat{\theta}_k) + \sum_{i \in N \setminus \{k\}} U_i^{\hat{M}}(\underline{\theta}_i),
\end{aligned}$$

[15] Note that M' need not satisfy (IR) since it is just an auxiliary mechanism used to construct $\tilde{M}$.

where the inequality follows since $K_k(\theta_k) > r^*$ for $\theta \in \Theta'$. Thus, we can pick $\rho = (\rho_1, \ldots, \rho_n)$ with $\sum_{i \in N} \rho_i = 0$ such that $U_k^{M'}(\hat{\theta}_k) > U_k^{\hat{M}}(\hat{\theta}_k)$, and $U_i^{M'}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i)$ for each $i \neq k$.

For such ρ, we have

$$\begin{aligned} U_k^{M'}(\underline{\theta}_k) &= -\sum_{i \in N \backslash \{k\}} U_i^{M'}(\underline{\theta}_i) + \mathbb{E}\left[\sum_{i \in N} (J(\theta_i) - r^*) q_i'(\theta)\right] \\ &= -\sum_{i \in N \backslash \{k\}} U_i^{\hat{M}}(\underline{\theta}_i) + \mathbb{E}\left[\sum_{i \in N} (J_i(\theta_i) - r^*) q_i^*(\theta)\right] + \mathbb{E}\left[(J_k(\theta_k) - r^*) 1_{\{\theta \in \Theta'\}}\right] \\ &< -\sum_{i \in N \backslash \{k\}} U_i^{\hat{M}}(\underline{\theta}_i) + \mathbb{E}\left[\sum_{i \in N} (J_i(\theta_i) - r^*) q_i^*(\theta)\right] = U_k^{\hat{M}}(\underline{\theta}_k), \end{aligned}$$

where the inequality follows since $J_k(\theta_k) \leq 0 < r^*$ for $\theta \in \Theta'$. In sum, under M', the agents' payoffs satisfy

$$\begin{gathered} U_i^{M'}(\theta_i) = U_i^{\hat{M}}(\theta_i), \ \forall \theta_i, \forall i \neq k \\ U_k^{M'}(\underline{\theta}_k) < U_k^{\hat{M}}(\underline{\theta}_k) \text{ and } U_k^{M'}(\theta_k) > U_k^{\hat{M}}(\theta_k) \text{ if } \theta_k \geq \hat{\theta}_k. \end{gathered} \tag{26}$$

Finally, we construct $\tilde{M}$ satisfying the desired properties. For dosing so, consider a linear combination of M and M', denoted $M^\lambda := \lambda M + (1-\lambda)M' = (\lambda q + (1-\lambda)q', \lambda t + (1-\lambda)t')$ for $\lambda \in [0.1]$. Note that for any λ, M^λ satisfies (IC) since both M and M' satisfy (IC). Note also that M^λ is a manipulation of $\hat{M}$ since both M and M' are manipulations of $\hat{M}$. Letting $U_i^\lambda(\cdot) := U_i^{M^\lambda}(\cdot)$, (26) implies

$$U_i^\lambda(\theta_i) = \lambda U_i^M(\theta_i) + (1-\lambda) U_i^{\hat{M}}(\theta_i) \geq U_i^{\hat{M}}(\theta_i), \ \forall \lambda, \forall i \neq k, \tag{27}$$

$$U_k^\lambda(\theta_k) = \lambda U_k^M(\theta_k) + (1-\lambda) U_k^{M'}(\theta_k) > U_k^{\hat{M}}(\theta_k), \ \forall \lambda < 1, \forall \theta_k \geq \hat{\theta}_k, \tag{28}$$

$$U_k^0(\underline{\theta}_k) = U_k^{M'}(\underline{\theta}_k) < U_k^{\hat{M}}(\underline{\theta}_k) \text{ and } U_k^1(\underline{\theta}_k) = U_k^M(\underline{\theta}_k) > U_k^{\hat{M}}(\underline{\theta}_k). \tag{29}$$

From (29) and the linearity of $U_k^\lambda(\cdot)$ regarding λ, there exists $\tilde{\lambda} \in (0,1)$ satisfying $U_k^{\tilde{\lambda}}(\underline{\theta}_k) = U_k^{\hat{M}}(\underline{\theta}_k)$, which implies

$$U_k^{\tilde{\lambda}}(\theta_k) \geq U_k^{\tilde{\lambda}}(\underline{\theta}_k) = U_k^{\hat{M}}(\underline{\theta}_k) = U_k^{\hat{M}}(\theta_k) \text{ for } \theta_k < \hat{\theta}_k. \tag{30}$$

Letting $\tilde{M} \equiv M^{\tilde{\lambda}}$, $\tilde{M}$ satisfies $(IR_N^{\hat{M}})$ due to (27), (28), and (30).

If there is any other agent i for whom $U_i^{\tilde{M}}(\underline{\theta}_i) > U_i^{\hat{M}}(\underline{\theta}_i)$, then we can start from $\tilde{M}$ constructed above and repeat the same procedure as above to construct another $\tilde{M}$ under which $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i)$. To repeat in this fashion will yield $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i)$ for all $i \in N$, establishing the first equation of (25). The second equation follows immediately from $(IR_N^{\hat{M}})$ and (28). ▌

To state the next step, we define $\Theta^* := \{\theta \in \Theta | \max_{i\in N} J_i(\theta_i) \geq r^*\}$.

STEP 2. *For any feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ of $\hat{M}$ by N, we have*

$$\sum_{i\in N} \int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \geq 0. \tag{31}$$

The inequality holds strictly unless $\tilde{q}(\theta) = q^*(\theta)$ for almost all $\theta \in \Theta^*$

Proof. Consider another allocation rule, $\bar{q}(\cdot)$, with $\bar{q}_i(\theta) = \tilde{q}_i(\theta)$ if $\theta_i \geq J_i^{-1}(r^*)$ and $\bar{q}_i(\theta) = q_i^*(\theta)$ otherwise, and let $\bar{Q}_i(\theta_i) := \mathbb{E}_{\theta_{-i}}[\bar{q}(\theta_i, \theta_{-i})]$, for each $i \in N$. (Whether $\bar{Q}_i(\cdot)$ is monotonic or whether $\bar{q}_i(\cdot)$ is implementable is irrelevant for the subsequent argument.) Then, it holds that

$$\begin{aligned}
& \sum_{i\in N} \int_{J_i^{-1}(r^*)}^{\bar{\theta}_i} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\
= & \sum_{i\in N} \int_{\underline{\theta}_i}^{\bar{\theta}_i} (J_i(\theta_i) - r^*)(\bar{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\
= & \ \mathbb{E}\left[\sum_{i\in N} (J_i(\theta_i) - r^*)(\bar{q}_i(\theta) - q_i^*(\theta))\right] \\
= & \ \mathbb{E}_{\theta\in\Theta^*}\left[\sum_{i\in N} (J_i(\theta_i) - r^*)(\bar{q}_i(\theta) - q_i^*(\theta))\right] \\
= & \ \mathbb{E}_{\theta\in\Theta^*}\left[\sum_{i\in N} (J_i(\theta_i) - r^*)\bar{q}_i(\theta)\right] - \mathbb{E}_{\theta\in\Theta^*}\left[\max_{i\in N} J_i(\theta_i) - r^*\right] \leq 0,
\end{aligned}$$

where the inequality follows from the definition of $q^*(\cdot)$ and becomes strict unless $\tilde{q}(\theta) = q^*(\theta)$ for almost all $\theta \in \Theta^*$. Thus, we have

$$\begin{aligned}
0 \ \leq & \ \sum_{i\in N} [\tilde{U}_i(\underline{\theta}_i) - \hat{U}_i(\underline{\theta}_i)] - \sum_{i\in N} \int_{J_i^{-1}(r^*)}^{\bar{\theta}_i} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\
= & \ \sum_{i\in N} \int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i,
\end{aligned}$$

with the inequality being strict unless $\tilde{q}(\theta) = \hat{q}(\theta)$ for almost all $\theta \in \Theta^*$. ▌

Step 3. *For any feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ of $\hat{M}$ that satisfies $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N$, we have*

$$\int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \leq 0, \forall i \in N. \tag{32}$$

The inequality holds strictly unless $\tilde{Q}_i(\theta_i) = Q_i^*(\theta_i)$ for all $\theta_i \leq J_i^{-1}(r^*)$.

Proof. It follows from the assumption on $\tilde{M}$ that for all $i \in N$ and all $\theta_i \in \Theta_i$,

$$\begin{aligned} X_i(\theta_i) := \int_{\underline{\theta}_i}^{\theta_i} [\tilde{Q}_i(a) - Q_i^*(a)] da &= U_i^{\tilde{M}}(\theta_i) - U_i^{\tilde{M}}(\underline{\theta}_i) - [U_i^{\hat{M}}(\theta_i) - U_i^{\hat{M}}(\underline{\theta}_i)] \\ &= U_i^{\tilde{M}}(\theta_i) - U_i^{\hat{M}}(\theta_i) \geq 0. \end{aligned}$$

Then, the integration by parts yields

$$\begin{aligned} &\int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\ =\ & (J_i(\theta_i) - r^*) f_i(\theta_i) X_i(\theta_i)\Big|_{\underline{\theta}_i}^{J_i^{-1}(r^*)} - \int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} X_i(\theta_i) d\left[(J_i(\theta_i) - r^*) f_i(\theta_i)\right] \\ =\ & -\int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} X_i(\theta_i) d\left[(J_i(\theta_i) - r^*) f_i(\theta_i)\right] \leq 0, \end{aligned}$$

since $(J_i(\cdot) - r^*) f_i(\cdot)$ is increasing. The inequality is strict unless $X_i(\theta_i) = 0$ for all $\theta_i \leq J_i^{-1}(r^*)$, that is $\tilde{Q}_i(\theta_i) = Q_i^*(\theta_i)$ for all $\theta_i \leq J_i^{-1}(r^*)$. ▌

Step 4. *$\hat{M}$ is WCP.*

Proof. Consider any feasible manipulation $M = (q, t)$. We claim that $U_i^M(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N$. Suppose not. By Step 1, we can find another feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ satisfying (25). Then, Steps 2 and 3 produce a contradiction unless both (31) and (32) hold as equality. But the latter fact implies that $\tilde{q}(\theta) = q^*(\theta)$ for all $\theta \in \Theta^*$ and $\tilde{Q}_i(\theta_i) = Q_i^*(\theta_i)$ for all $\theta_i \leq J_i^{-1}(r^*)$ and all $i \in N$. Thus, $\tilde{Q}_i(\cdot) = Q_i^*(\cdot)$ for all $i \in N$, which implies that for all $i \in N$ and θ_i,

$$U_i^{\tilde{M}}(\theta_i) = U_i^{\tilde{M}}(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \tilde{Q}_i(a) da = U_i^{\hat{M}}(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} Q_i^*(a) da = U_i^{\hat{M}}(\theta_i),$$

which contradicts the second equation of (25). Thus, it must be that $U_i^M(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N$. Again, Steps 2 and 3 imply that $\tilde{Q}_i(\cdot) = Q_i^*(\cdot)$ for all $i \in N$. But then, M yield the same interim payoffs as $\hat{M}$ to the bidders, proving that $\hat{M}$ is WCP. ▌

▌

Proof of Theorem 5. Suppose a pair, r_0 and $\hat{q}(\cdot)$ solves $[C]$. We construct an auction rule $\hat{M} = (\hat{q}, \hat{t})$ that implements the same payoff as the solution of $[C]$ for the seller and is WCP. To this end, we define

$$\hat{T}_i(\theta_i) \;\; := \;\; \theta_i \mathbb{E}_{\tilde{\theta}_{-i}}\left[\hat{q}_i(\theta_i, \tilde{\theta}_{-i})\right] - \int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\tilde{\theta}_{-i}}\left[\hat{q}_i(a, \tilde{\theta}_{-i})\right] da \tag{33}$$

We then define the transfer rule $\hat{t}$ such that

$$\begin{aligned}\hat{t}_i(\theta) := \frac{1}{n} r_0 \sum_{j \in N} \hat{q}_j(\theta) + \left(\hat{T}_i(\theta_i) - \frac{1}{n} r_0 \mathbb{E}_{\tilde{\theta}_{-i}}\left[\sum_{j\in N} \hat{q}_j(\theta_i, \tilde{\theta}_{-i})\right]\right) \\ - \frac{1}{n-1} \sum_{k \in N\setminus\{i\}} \left(\hat{T}_k(\theta_k) - \frac{1}{n} r_0 \mathbb{E}_{\tilde{\theta}_{-k}}\left[\sum_{j \in N} \hat{q}_j(\theta_k, \tilde{\theta}_{-k})\right]\right) + \rho_i,\end{aligned} \tag{34}$$

where $\rho_i \in \mathbb{R}$ with $\sum_{i\in N} \rho_i = 0$. Since $\sum_{i \in N} \hat{t}_i(\theta) = r_0 \sum_{i\in N} \hat{q}_i(\theta)$ and (IC_1^*) implies

$$\sum_{i \in N} U_i^{\hat{M}}(\underline{\theta}_i) = \mathbb{E}\left[\sum_{i\in N} J_i(\theta_i)\hat{q}_i(\theta) - r_0 \sum_{i \in N} \hat{q}_i(\theta)\right] \geq 0,$$

we can choose ρ_i's so that each bidder's participation constraint is satisfied. Observe also that, for all $\theta_i' \in \Theta_i$,

$$\mathbb{E}_{\theta_{-i}}\left[\hat{t}_i(\theta_i', \theta_{-i})\right] = \hat{T}_i(\theta_i') + c_i,$$

for some constant c_i, implying that (IC^*) is satisfied. Therefore, $\hat{M}$ implements the solution of $[C]$.

We next prove that $\hat{M}$ is WCP. Consider any feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ of $\hat{M}$. Then, we have

$$\sum_{i\in N} U_i^{\tilde{M}}(\overline{\theta}) \;\; = \;\; \mathbb{E}\left[\sum_{i \in N} K(\theta_i)\tilde{q}_i(\theta) - \sum_{i\in N} \tilde{t}_i(\theta)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{i\in N}(K(\theta_i)-r_0)\tilde{q}_i(\theta)]\right] \\
&\leq \mathbb{E}\left[\sum_{i\in N}(K(\theta_i)-r_0)\hat{q}_i(\theta)\right] = \sum_{i\in N} U_i^{\hat{M}}(\overline{\theta}),
\end{aligned}
$$

where the inequality follows from the definition of $\hat{q}(\cdot)$ and becomes strict unless $\tilde{q}(\cdot) = \hat{q}(\cdot)$. Thus, $(IR_N^{\hat{M}})$ requires $\tilde{q}(\cdot) = \hat{q}(\cdot)$, so

$$
\mathbb{E}\left[\sum_{i\in N}\tilde{t}_i(\theta)\right] = \mathbb{E}\left[r_0\sum_{i\in N}\tilde{q}_i(\theta)\right] = \mathbb{E}\left[r_0\sum_{i\in N}\hat{q}_i(\theta)\right] = \mathbb{E}\left[\sum_{i\in N}\hat{t}_i(\theta)\right],
$$

which, along with $(IR_N^{\hat{M}})$, implies that $U_i^{\tilde{M}}(\cdot) = U_i^{\hat{M}}(\cdot)$, $\forall i$. We thus conclude that $\hat{M}$ is WCP.

We now characterize a solution to $[C]$, assuming that Condition (SB) does not hold. We first show that the solution involves the allocation rule of the form described in (9), whatever the value of r_0 is. Let λ_R and λ_K denote the Lagrangian (nonnegative) multipliers for the constraints (IC_1^*) and (K), respectively. Then, the Lagrangian for the problem $[C]$ is written as

$$
\mathbb{E}\left[\sum_{i\in N}\Big(r+\lambda_R(J(\theta_i)-r)+\lambda_K\min\{K(\theta_i)-r,0\}\Big)q_i(\theta)\right].
$$

Since the maximand is symmetric across bidders and linearly increasing with each q_i, the optimal allocation should follow the efficient cutoff rule: Namely, there exists a threshold value $\bar{\theta}$ such that the rule allocates the object to a bidder whose type is highest and above $\bar{\theta}$. Next observe the constraint (K) must be binding at the solution; or else, the solution corresponds to the second-best outcome. This yields a contradiction, since the solution is WCP implementable and the second-best outcome cannot be WCP implementable without Condition (SB). Therefore, (K) is binding, from which it follows that $\bar{\theta} = K^{-1}(r_0)$.

The optimal sale price r_0 depends on whether (IC_1^*) is binding or not. If (IC_1^*) is not binding, then given the efficient cutoff rule as in (9), r_0 must satisfy (7). Meanwhile, (IC_1^*) is slack only if

$$
\mathbb{E}[J(\theta_N^{(1)})|\theta_N^{(1)} > K^{-1}(r_0)] > r_0.
$$

If this inequality does not hold at the level solving (7), then (IC_1^*) must be binding, so r_0 must satisfy (8). ∎

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