A Note on Bivariate

Box Splines on a k-direction Mesh

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CUCS-155-84

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January 1984

This research was initiated while working at IBM Research Laboratories, Yorktown Heights, New York, Summer 1983 and was supported in part by the National Science Foundation under Grant DCR-82-14322.

Abstract.

We determine the dimension of the polynomial subspace of the linear space spanned by the translates over lattice points of a bivariate box spline on a k-direction mesh.

Keywords: Box spline, k-direction mesh.

1. Introduction.

Box splines were introduced by de Boor and De Vore, [1], and systematically studied by de Boor and Hollig in [2,3] and by Dahmen and Micchelli in [4,5,7].

For box splines on a k-direction mesh in an s-dimensional space, one is interested in the dimension of the polynomial subspace of the linear space spanned by the translates of a box spline over lattice points in Z^s, since this dimension is closely related to the rate of approximation using box splines. This problem has been solved for the case $s = 2$ and $k = 4$. For the proof see [4]. For the general case, the result has been announced in a recent paper [6]. The authors suggest to prove it by employing an induction. In this paper, we provide a proof of this result for the case of bivariate box splines (i.e., $s = 2$) and arbitrary k. Our proof does not use induction.

A k-direction mesh is a set of vectors

$$(1.1) \qquad X = \{\underbrace{v^1,\ldots,v^1}_{m_1},\ldots,\underbrace{v^k,\ldots,v^k}_{m_k}\},$$

where $v^i = (\alpha_i,\beta_i) \in \mathbb{Z}^2$, $\mathbb{Z}$ is the set of integers, and $\beta_i \geq 0$, $\alpha_i\beta_j \neq \alpha_j\beta_i$ for $i \neq j$, $m_i \geq 1$, $i,j = 1,\ldots,k$, $k \geq 2$. Let

$$(1.2) \qquad n = \Sigma_{i=1}^{k} m_i \quad \text{and} \quad d = \min_i\{n - m_i\} - 1.$$

Then there exists a unique function $B(\cdot|X)$ in R^2, called a box spline on a k-direction mesh, such that

$$(1.3) \qquad \int_{R^2} f(x,y)B((x,y)|X)\,dxdy = \int_0^1 \cdots \int_0^1 f(t_1 v^1 + \ldots + t_n v^k) \times dt_1 \ldots dt_n$$

for all $f \in C(R^2)$. The box spline $B(\cdot|X)$ is a piecewise polynomial in $C^{d-1}(R^2)$ where d is given in (1.2), and has compact support [2,3].

Let $S(X)$ be the linear span of translates of the box spline over lattice points in Z^2, i.e.,

$$(1.4) \qquad S(X) = \text{span}(\{B(\cdot-(\alpha,\beta)|X): (\alpha,\beta) \in Z^2\}).$$

We are particularly interested in the subspace $S_\pi(X)$ of polynomials in $S(X)$. Let

$$Q_\ell(x,y) = \prod_{i \neq \ell} (\alpha_i x + \beta_i y)^{m_i},$$

and let

$$(1.5) \qquad Q_\ell(D) = \prod_{i \neq \ell} \left(\alpha_i \frac{\partial}{\partial x} + \beta_i \frac{\partial}{\partial y}\right)^{m_i}.$$

It was proved [3,6,7] that $S_\pi(X)$ is of finite dimension and that

$$(1.6) \qquad S_\pi(X) = \mathcal{D}(X),$$

where

$$(1.7) \qquad \mathcal{D}(X) = \{f: Q_\ell(D)\, f=0,\ \ell=1,\ldots,k\}.$$

It was proved [3]

<u>Theorem 1.1</u>: If $\det(v^i, v^j) = 1$ for each pair of vectors in X which spans R^2, then

$$\dim S_\pi(X) = \dim \mathcal{D}(X) = A(X), \tag{1.8}$$

where $A(X)$ is the area of the support of $B(\cdot|X)$. □

A simple example is the case of a 3-direction mesh, where $X = \{\underbrace{e^1,\ldots,e^1}_{m_1}, \underbrace{e^1+e^2,\ldots,e^1+e^2}_{m_2}, \underbrace{e^2,\ldots,e^2}_{m_3}\}$, $e^1 = (1.0)$, $e^2 = (0,1)$. We have

$$\dim_\pi S\ (X) = \dim \mathcal{D}(X) = A(X) = \Sigma_{1\le i<j\le 3}\, m_i m_j.$$

In general, the condition in Theorem 1.1 does not hold, as in the case of a 4-direction mesh, where

$$X = \{\underbrace{e^1,\ldots,e^1}_{m_1}, \underbrace{e^1+e^2,\ldots,e^1+e^2}_{m_2}, \underbrace{e^2,\ldots e^2}_{m_3}, \underbrace{e^2-e^1,\ldots,e^2-e^1}_{m_4}\},$$

since $\det(e^1+e^2, e^2-e^1) = 2$. We address this problem in section 2.

2. Box splines on a k-direction mesh.

For a k-direction mesh as given in (1.1), we give the dimension of $S_\pi(X)$ in Theorem 2.1.

We need the following

<u>Lemma 2.1:</u> Let $(\alpha_i, \beta_i) \in Z^2$ with $\alpha_i\beta_j \neq \alpha_j\beta_i$, $i,j = 1,\ldots,k$, and let $G_j(\lambda) = \Pi_{i\neq j}(\alpha_i+\beta_i\lambda)^{m_i}$, where $m_i \geq 1$. Then for all distinct $\lambda_0,\ldots,\lambda_{n-1}$, the following matrix is nondegenerate:

$$(2.1) \qquad M_n = \begin{pmatrix} G_1(\lambda_0) & \cdots & G_1(\lambda_{n-1}) \\ \lambda_0 G_1(\lambda_0) & \cdots & \lambda_{n-1}G_1(\lambda_{n-1}) \\ \vdots & & \vdots \\ \lambda_0^{m_1-1} G_1(\lambda_0) & \cdots & \lambda_{n-1}^{m_1-1} G_1(\lambda_{n-1}) \\ \vdots & & \vdots \\ G_k(\lambda_0) & \cdots & G_k(\lambda_{n-1}) \\ \lambda_0 G_k(\lambda_0) & \cdots & \lambda_{n-1}G_k(\lambda_{n-1}) \\ \vdots & & \vdots \\ \lambda_0^{m_k-1} G_k(\lambda_0) & \cdots & \lambda_{n-1}^{m_k-1} G_k(\lambda_{n-1}) \end{pmatrix}$$

where $n = \Sigma_{i=1}^{k} m_i$.

<u>Proof:</u> The matrix M_n is nondegenerate for any choice of distinct λ_j, $j=0,\ldots,n-1$, if and only if for any vector a, $M_n a = 0$ implies $a = 0$. Let

$$a^T = (a_{1,0}, a_{1,1}, \ldots, a_{1,m_1-1}, \ldots, a_{k,0}, a_{k,1}, \ldots, a_{k,m_k-1}),$$

and let $P_\ell(x) = a_{\ell,0} + a_{\ell,1}x + \cdots + a_{\ell,m_\ell-1}x^{m_\ell-1}$, $\ell=1,\ldots,k$. Then $\sum_{\ell=1}^{k} P_\ell(x)\, G_\ell(x)$ vanishes at all the n distinct λ_j, since $M_n a = 0$. On the other hand, $\sum_{\ell=1}^{k} P_\ell(x)\, G_\ell(x)$ is a polynomial of degree less than n, hence must be identically zero. But then since all summands except for the ℓ-th one have the factor $(\alpha_\ell+\beta_\ell x)^{m_\ell}$, the ℓ-th summand must also have it and, since G_ℓ does not have it, P_ℓ must have it, and that is possible only when $P_\ell = 0$. This shows that $a = 0$, as required. □

Remark 2.1: We choose distinct $\lambda_0,\ldots,\lambda_{n-1}$ with $\lambda_i \neq 1,0,1$, $\alpha_j + \beta_j\lambda_i \neq 0$, $j = 1,\ldots,k$, $i = 0,\ldots,n-1$, such that M_n in (2.1) is non-degenerate, and we denote this matrix with fixed λ_i as M_n^*. □

We are ready to prove

Theorem 2.1: Let X be a k-direction mesh. Then

(2.2) $$\dim S_\pi(X) = \Sigma_{1\leq i<j\leq k}\, m_i m_j.$$ □

Proof: In the proof we denote the differential operator $Q_\ell(D)$ by Q_ℓ, $\ell=1,\ldots,k$. Since $S_\pi(X) = \mathfrak{D}(X)$, we need only to derive the dimension of the space $\mathfrak{D}(X)$. Let Π_j be the linear space of all homogeneous polynomials of degree j. Observe that $\Pi_i \cap \Pi_j = \{0\}$ for $i \neq j$ and that $\{(x+\lambda_{0j}y)^j, (x+\lambda_{1j}y)^j, \ldots, (x+\lambda_{jj}y)^j\}$ is a basis of Π_j for arbitrary distinct λ_{ij}, $i=0,\ldots,j$, with $\lambda_{ij} \neq -1,0,1$.

Since $\mathfrak{D}(X)$ is a finite dimensional linear space of polynomials, $\mathfrak{D}(X)$ is a subspace of $\Pi_0 \oplus \cdots \oplus \Pi_N$ for sufficiently large N. Let $S_j = \mathfrak{D}(X) \cap \Pi_j$. Then $S_i \cap S_j = \{0\}$ for $i \neq j$, and therefore $S_0 \oplus \cdots \oplus S_N$ is well defined. We prove that

$$\mathfrak{D}(X) = S_0 \oplus \cdots \oplus S_N. \tag{2.3}$$

Indeed, $S_0 \oplus \cdots \oplus S_N \subseteq \mathfrak{D}(X)$ by the definition of S_j. To show that $\mathfrak{D}(X) \subseteq S_0 \oplus \cdots \oplus S_N$, take arbitrary $f \in \mathfrak{D}(X)$, and $f = \sum_{j=0}^{N} f_j$, where $f_j \in \Pi_j$. Due to (1.7), $Q_\ell f = \sum_{j=0}^{N} Q_\ell f_j = 0$, $\ell = 1,\ldots,k$. By (1.5), we know that for $i < j$ and $Q_\ell f_j \neq 0$, $\deg(Q_\ell f_i) < \deg(Q_\ell f_j)$. Thus $Q_\ell f_j = 0$, $\ell = 1,\ldots,k$, $j = 0,\ldots,N$, i.e., $f_j \in S_j$, $j = 0,\ldots,N$. This means that $\mathfrak{D}(X) \subseteq S_0 \oplus \cdots \oplus S_N$, which completes the proof of (2.3).

To derive $\dim \mathfrak{D}$ we compute $\dim S_j$, $j = 0,\ldots,N$, since $\dim \mathfrak{D}(X) = \sum_{j=0}^{N} \dim S_j$, due to (2.3). Let $f \in S_j$. Then

$$f(x,y) = \sum_{i=0}^{j} a_{ij}(x+\lambda_{ij}y)^j, \tag{2.4}$$

and

$$Q_\ell f = 0, \quad \ell = 1,\ldots,k. \tag{2.5}$$

Let $q_\ell = \deg Q_\ell = n - m_\ell$, where $n = \sum_{\ell=1}^{k} m_\ell$. Since

$$(2.6)\qquad (\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y})^m (x + \lambda y)^j = \begin{cases} 0, & \text{if } m > j, \\ j(j-1)\cdots(j-m+1)(\alpha+\beta\lambda)^m (x+\lambda y)^{j-m}, & \text{if } m \le j, \end{cases}$$

from (2.4) we have

$$(2.7)\qquad Q_\ell f = \begin{cases} 0, & \text{if } q_\ell > j, \\ \sum_{i=0}^{j} a_{ij}\, j(j-1)\cdots(j-q_\ell+1) Q_\ell(1,\lambda_{ij})(x+\lambda_{ij}y)^{j-q_\ell}, & \text{if } q_\ell \le j. \end{cases}$$

Since $S_\pi(X)$ is the polynomial subspace of the linear space spanned by the translates of a box spline for which there are only n directions, polynomials in $S_\pi(X)$ have degree less than n. Thus we only need to derive $\dim S_j$, for $j=0,1,\ldots,n-1$.

From (2.7) we have

$$(2.8)\qquad \begin{cases} Q_\ell f = 0, & \text{if } q_\ell \ge j+1, \\ Q_\ell f = \sum_{i=0}^{j} a_{ij}\, j(j-1)\cdots(j-q_\ell+1) Q_\ell(1,\lambda_{ij})(x+\lambda_{ij}y)^{j-q_\ell}, & \text{if } q_\ell \le j. \end{cases}$$

From (2.5) and (2.8), we have a system of equation in a_{ij}:

$$(2.9) \qquad \sum_{i=0}^{j} a_{ij}\, Q_\ell(1,\lambda_{ij})\, \lambda_{ij}^{r} = 0, \quad r=0,\ldots,j-q_\ell; \quad q_\ell \le j,$$

and the coefficient matrix M_j of (2.9) consists of blocks $B_{\ell,j}$:

$$B_{\ell,j} = \begin{pmatrix} Q_\ell(1,\lambda_{0j}) & \cdots\cdots & Q_\ell(1,\lambda_{jj}) \\ \lambda_{0j}Q_\ell(1,\lambda_{0j}) & \cdots\cdots & \lambda_{jj}Q_\ell(1,\lambda_{jj}) \\ \vdots & & \vdots \\ \lambda_{0j}^{j-q_\ell}\, Q_\ell(1,\lambda_{0j}) & \cdots\cdots & \lambda_{jj}^{j-q_\ell}\, Q_\ell(1,\lambda_{jj}) \end{pmatrix}$$

Since $j \le n-1$, $j-q_\ell = j - (n-m_\ell) = m_\ell - (n-j) \le m_\ell - 1$, M_j is a submatrix of M_n^* in Remark 2.1, and is contained in $\sum_{\ell=1}^{k} (j+1-q_\ell)_+$ rows. Notice that $j+1 \ge \sum_{\ell=1}^{k} (j+1-q_\ell)_+$, for $j \le n-1$, since we get equality when $j + 1 = n$. Since M_n^* is non-degenerate, we can find $j+1$ columns, such that the $\sum_{\ell=1}^{k} (j+1-q_\ell)_+$ by $(j+1)$ submatrix of M_n^*, corresponding to M_j, is of rank $\sum_{\ell=1}^{k} (j+1-q_\ell)_+$. Use the $j+1$ λ_i's in M_n^*, corresponding to the $j+1$ chosen columns as $\lambda_{0j},\ldots,\lambda_{jj}$

in (2.4) and (2.9), and M_j is obviously of rank $\sum_{\ell=1}^{k}(j+1-q_\ell)_+$. Since the number of a_{ij}'s, j+1, is no less than $\sum_{\ell=1}^{k}(j+1-q_\ell)_+$, the number of equations in (2.9), and the coefficient matrix of (2.9), M_j, is non-degenerate, the solution space of (2.9) is of dimension

$(j+1) - \sum_{\ell=1}^{k}(j+1-q_\ell)_+$, which is the dimension of S_j. So

$$\sum_{j=0}^{n-1} \dim S_j = \sum_{j=0}^{n-1} [(j+1) - \sum_{\ell=1}^{k}(j+1-q_\ell)_+]$$

$$= \sum_{j=1}^{n} j - \sum_{\ell=1}^{k} \sum_{j=1}^{m_\ell} j = \sum_{1\le i<j\le k} m_i m_j ,$$

since

$$\sum_{j=0}^{n-1}(j+1-q_\ell)_+ = 1 + \cdots + n-q_\ell \quad \text{and} \quad n-q_\ell = m_\ell . \qquad \square$$

Acknowledgements.

I am grateful to Professors C. de Boor, J.F. Traub and G.W. Wasilkowski for their advice and valuable comments. My special thanks are to Professor Jia Rong-qing who has significantly shortened the proof of Lemma 2.1.

Dr. C. Micchelli introduced me to this problem for the case of a four-direction mesh.

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