K-theoretic enumerative geometry and the Hilbert scheme of points on a surface

by

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Submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2018

Abstract

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Integrals of characteristic classes of tautological sheaves on the Hilbert scheme of points on a surface frequently arise in enumerative problems. We use the K-theoretic Donaldson-Thomas theory of certain toric Calabi-Yau threefolds to study K-theoretic variants of such expressions.

We study limits of the K-theoretic Donaldson-Thomas partition function of a toric Calabi-Yau threefold under certain one-parameter subgroups called slopes, and formulate a condition under which two such limits coincide. We then explicitly compute the limits of components of the partition function under so-called preferred slopes, obtaining explicit combinatorial expressions related to the refined topological vertex of Iqbal, Kosçaz and Vafa.

Applying these results to specific Calabi-Yau threefolds, we deduce dualities satisfied by a generating function built from tautological bundles on the Hilbert scheme of points on $\mathbb{C}^2$. We then use this duality to study holomorphic Euler characteristics of exterior and symmetric powers of tautological bundles on the Hilbert scheme of points on a general surface.

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Acknowledgments

It is a pleasure to thank my thesis advisor Andrei Okounkov for being so generous with his ideas, energy, time and resources. I am very grateful for his guidance, as well as his efforts to make his research group a stimulating and fun environment to learn and do mathematics.

I thank Rahul Pandharipande for asking a question that developed into the present thesis, and Chiu-Chu Melissa Liu, Alina Marian, Davesh Maulik, Andrei Neguț, Dragos Oprea, Petr Pushkar and Andrey Smirnov for helpful discussions. I am grateful to Maria Cueto, Sachin Gautam, David Jordan, Andrei Neguț, Joshua Sussan, Valerio Toledano-Laredo and Robin Walters for their invitations to speak at their seminars. I would also like to thank Chiu-Chu Melissa Liu, Alina Marian, Dusa McDuff and Alexei Oblomkov for serving on my defense committee.

I thank the other graduate students, postdocs, and faculty at Columbia for making graduate school both enriching and enjoyable. I am also grateful to the administrative staff of Columbia's mathematics department, particularly Terrance Cope and Nathan Schweer.

Finally, I would like to thank my friends and family, both in New York and beyond, for their support and companionship.

This work was partially supported by the Department of Defense (DoD) through the National Defense Science and Engineering Graduate Fellowship (NDSEG) Program, and was completed while I was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2018 semester, partially supported by the National Science Foundation under Grant No. DMS-1440140.

CHAPTER 1

Introduction

1.1. Tautological classes on the Hilbert scheme of points on a surface

Let S be a complex smooth quasi-projective surface and let $S^{[n]}$ denote the Hilbert scheme of n points on S. The moduli space $S^{[n]}$ is a smooth quasi-projective scheme of dimension $2n$ parametrizing zero-dimensional length n subschemes $Y \subset S$; dually, the Hilbert scheme $S^{[n]}$ parametrizes ideal sheaves $\mathcal{I}_Y \subset \mathcal{O}_S$ of colength n.

Given a line bundle $\mathcal{L}$ on S, one can produce a rank n vector bundle on $\mathcal{L}^{[n]}$ on $S^{[n]}$ using the universal family $\Xi_n \subset S^{[n]} \times S$ as a correspondence. Namely, if $p_{S^{[n]}}$ and p_S are the projection maps

$$\begin{array}{ccc} & S^{[n]} \times S & \\ {\scriptstyle p_{S^{[n]}}} \swarrow & & \searrow {\scriptstyle p_S} \\ S^{[n]} & & S \end{array}$$

we set

$$\mathcal{L}^{[n]} = (p_{S^{[n]}})_*(\mathcal{O}_{\Xi_n} \otimes p_S^*\mathcal{L});$$

the vector bundle $\mathcal{L}^{[n]}$ is called a *tautological bundle*, and its fiber over a subscheme $Y \in S^{[n]}$ is

$$H^0(\mathcal{O}_Y \otimes \mathcal{L}).$$

Characteristic classes of tautological bundles often encode geometric information. For example, series of the form

$$\sum_n m^i y^j q^n \int_{S^{[n]}} c_i(\mathcal{L}^{[n]}) s_j(\mathcal{L}^{[n]}) \tag{1.1.1}$$

arise in [**MOP1**] in the study of the cohomology ring of the moduli space of $K3$ surfaces. The special case

$$\sum_{n \geq 0} q^n \int_{S^{[n]}} s_{2n}(\mathcal{L}^{[n]}) \tag{1.1.2}$$

has been the subject of recent activity. The terms of (1.1.2) count secants to the projective embedding of S given by $\mathcal{L}$ and are relevant to the computation of certain Donaldson invariants on S. The precise form of this series was conjectured by Lehn in [**L**] and was recently proved, first by Marian, Oprea and Pandharipande [**MOP1**] in the case when the canoncial bundle $\mathcal{K}_S$ is numerically trivial, and subsequently by Voisin, and Marian, Oprea and Pandharipande for general S in [**V, MOP2**].

In this thesis, we study K-theoretic variants of series like (1.1.1). Namely, given a vector bundle $\mathcal{V}$ on a variety X, set

$$\Lambda_m^\bullet \mathcal{V} = \sum_{i=0}^{\infty} (-m)^i \Lambda^i \mathcal{V} \in K(X)[[m]], \quad \mathrm{Sym}_y^\bullet \mathcal{V} = y^i \sum_{i=0}^{\infty} \mathrm{Sym}^i \mathcal{V} \in K(X)[[y]]. \tag{1.1.3}$$

A K-theoretic version of (1.1.1) is

$$\begin{aligned} &\sum_{n \geq 0} q^n \chi\left(S^{[n]}, \Lambda_m^\bullet\left(\mathcal{L}^{[n]}\right) \otimes \mathrm{Sym}_y^\bullet\left(\mathcal{L}^{[n]}\right)\right) \\ &= \sum_{i,j,n \geq 0} (-m)^i y^j q^n \chi\left(S^{[n]}, \Lambda_m^i\left(\mathcal{L}^{[n]}\right) \otimes \mathrm{Sym}_y^j\left(\mathcal{L}^{[n]}\right)\right). \end{aligned} \tag{1.1.4}$$

In [**EGL**], a general framework is developed for analyzing expressions of a form similar to (1.1.1) and (1.1.4) using algebraic cobordism. To be precise, let g_1 and g_2 be fixed power series (with coefficients in some polynomial ring) and, for a vector bundle $\mathcal{V}$ with Chern roots v_i, set

$$G_1(\mathcal{V}) = \prod_i g_1(v_i), \quad G_2(\mathcal{V}) = \prod_i g_2(v_i).$$

Let us further assume that g_1 has nonzero constant term. Then, [**EGL**, Thm 4.2] implies that the series

$$G(S, \mathcal{L}) = \sum_{z \geq 0} q^n \int_{S^{[n]}} G_1(\mathcal{L}^{[n]}) G_2(TS^{[n]}) \tag{1.1.5}$$

can be written as

$$G(S, \mathcal{L}) = \exp(c_1(\mathcal{L})^2 A_1 + c_1(\mathcal{L}) c_2(S) A_2 + c_1(S)^2 A_3 + c_2(S) A_4),$$

where the A_i are universal series; that is, they do not depend on the choice of S and $\mathcal{L}$. Series of the form (1.1.5) are therefore determined by their values when S is a toric surface. One approach to studying such values is to use equivariant localization to further reduce an *equivariant* analog of (1.1.5), where $S = \mathbb{C}^2$ and $\mathcal{L}$ is a twist of the trivial bundle by a torus character.

The formula (1.1.4) when $S = \mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[z_1, z_2]$ can be analyzed using K-theoretic equivariant localization. The scaling action of $T = \operatorname{diag}(t_1, t_2)$ on $\mathbb{C}^2$ lifts to an action on $(\mathbb{C}^2)^{[n]}$; the fixed locus $(\mathbb{C}^2)^{[n]}$ consists of monomial ideals $I_\lambda \subset \mathbb{C}[z_1, z_2]$ of colength n. These ideals are indexed by two-dimensional partitions λ of size n; where

$$I_\lambda = \operatorname{Span}\{z_1^{b_1} z_2^{b_2} | (b_1, b_2) \notin \lambda\}.$$

As a T-module, the tangent space to $I_\lambda \in (\mathbb{C}^2)^{[|n|]}$ admits a combinatorial formula in terms of arm and leg lengths; that is,

$$T_{I_\lambda}(\mathbb{C}^2)^{[n]} = \sum_{\square \in \lambda} t_1^{-l(\square)} t_2^{a(\square)+1} + t_1^{l(\square)+1} t_2^{-a(\square)}.$$

We abbreviate the T-character T_{I_λ} by T_λ.

The fibers of the tautological bundles over the T-fixed points of $(\mathbb{C}^2)^{[n]}$ are also described combinatorially; we have

$$(\mathcal{O}(l))^{[n]}|_{I_\lambda} = \sum_{\square \in \lambda} l t_1^{-b_1} t_2^{-b_2}.$$

Consequently, expressions of the form (1.1.5) may be written purely in terms of the combinatorics of partitions; for example, this approach has been used in similar settings in [**GK1, GK2**]. However, such series may be intractable from this perspective; for example, it is not known how to use this approach to study the series (1.1.2).

In this thesis, we introduce a framework to study tautological classes over the Hilbert scheme of points on $\mathbb{C}^2$. Introduce formal variables q, y, m_1, m_2 and m_3, and consider the following expression assembled from exterior and symmetric powers of tautological bundles over the Hilbert schemes $(\mathbb{C}^2)^{[n]}$.

Definition 1.1.1. *Set $F(q, m_1, m_2, m_3, y)(t_1, t_2)$ to be the series of T-equivariant Euler characteristics*

$$\sum_{n \geq 0} (-q)^n \chi\left((\mathbb{C}^2)^{[n]}, \Lambda^\bullet_{m_1} (\mathcal{O}^{[n]})^\vee \otimes \Lambda^\bullet_{m_2} (\mathcal{O}^{[n]})^\vee \otimes \Lambda^\bullet_{m_3 y} \mathcal{O}^{[n]} \otimes \operatorname{Sym}^\bullet_y \mathcal{O}^{[n]} \otimes \Lambda^n \mathcal{O}^{[n]} \right). \quad (1.1.6)$$

We often omit the dependence of F on the equivariant parameters t_1, t_2 from our notation. Writing in terms of the combinatorics of partitions, we have

$$F(q, m_1, m_2, m_3, y) = \sum_\lambda \frac{q^{|\lambda|}}{\Lambda^\bullet(T_\lambda^\vee)} \prod_\lambda \frac{(1 - m_1 t_1^{b_1} t_2^{b_2})(1 - m_2 t_1^{b_1} t_2^{b_2})(1 - m_3 y t_1^{-b_1} t_2^{-b_2})}{y - t_1^{b_1} t_2^{b_2}},$$

where, given a T-character $V = \sum_i u_i$ where each u_i is a Laurent monomial with coefficient 1, we set

$$\Lambda^\bullet V = \prod_i (1 - u_i).$$

We show that after normalization, the series F enjoys two types of symmetries: one between the "box-counting variable" q and the "Segre variable" y, and another between the "Chern variables" m_1, m_2, m_3.

Theorem 1.1.2. *We have*

$$\frac{F(q, m_1, m_2, m_3, y)}{F(q, m_1, m_2, m_3, 0)} = \frac{F(q, m_1, m_3, m_2, y)}{F(q, m_1, m_3, m_2, 0)} = \frac{F(y, m_1, m_2, m_3, q)}{F(y, m_1, m_2, m_3, 0)}. \tag{1.1.7}$$

Moreover, we show that the denominators appearing in (1.1.7) can be characterized using the plethystic exponential.

Proposition 1.1.3. *We have*

$$F(q, m_1, m_2, m_3, 0) = \exp\left(\sum_{n>0} -\frac{q^n}{n} \frac{(1 - m_1^n)(1 - m_2^n)}{(1 - t_1^{-n})(1 - t_2^{-n})} \right).$$

Note that Proposition 1.1.3 is also a consequence of [**WZ**, Thm 3.2].

We then use Theorem 1.1.2 and Proposition 1.1.3 to study the holomorphic Euler characteristics of exterior powers, and certain symmetric powers, of tautological bundles $\mathcal{L}^{[n]}$ on $S^{[n]}$ for general surfaces S. For example, we prove the following.

Corollary 1.1.4. *If $\chi(\mathcal{O}_S) = 1$, then for $n \geq k$, we have*

$$\chi(S^{[n]}, \mathrm{Sym}^k(\mathcal{L}^{[n]})) = \binom{\chi(\mathcal{L}) + k - 1}{k}.$$

Theorem 1.1.2 also implies families of nontrivial combinatorial identities. For example, it is obvious that, for $l > n$, the coefficient of $q^n m_1^l$ is 0 on the left-hand side of (1.1.7); this is not obvious on the right-hand side.

Theorem 1.1.2 and Proposition 1.1.3 are consequences of the K-theoretic enumerative geometry of particular toric Calabi-Yau threefolds; in particular, these results follow from a more general framework that produces equalities of certain Euler characteristics over moduli spaces of sheaves. We now describe this framework.

1.2. K-theoretic enumerative geometry

We deduce Theorem 1.1.2 and Proposition 1.1.3 from the equality of particular limits of the K-theoretic Donaldson-Thomas partition function of certain Calabi-Yau threefolds. Our approach is based on two observations: first, it is shown in [**NO**, Sec 8] that special limits of the K-theoretic DT partition function can be expressed in terms of the refined topological vertex of [**IKV**], and second, geometric engineering posits an equality between certain refined topological partition functions and the K-theoretic Nekrasov partition functions of certain gauge theories; some examples of this phenomenon are computed in [**IKV, Taki**]. The expression F appearing in 1.1.2 is then related to such Nekrasov functions.

Let us outline our approach in more detail.

Let X be a complex smooth threefold, and let $DT(X)$ denote the Hilbert scheme of curves in X. In [**NO**], Nekrasov and Okounkov introduce the K-theoretic Donaldson-Thomas partition function $Z_{DT}(X)$; this is defined as an Euler characteristic

$$Z_{DT}(X) = \chi(DT(X), \tilde{\mathcal{O}}^{\text{vir}})$$

of a certain modification of the virtual structure sheaf $\mathcal{O}^{\text{vir}}$ on $DT(X)$.

Our applications will be in the case when X is toric and Calabi-Yau, with torus $T = \text{diag}(t_1, t_2, t_3)$ acting so that the anti-canonical bundle $\mathcal{K}_X^\vee$ has T-weight $t_1t_2t_3$. Given a weight $w \in T^\vee$, we set $T_w = \ker(w) \subset T$. We say a weight w is *non-compact* if $(DT(X))^{T_w}$ has a non-proper component.

Set $A = \ker(t_1t_2t_3) \subset T$. Then, define a *slope* σ to be a one-parameter subgroup $\mathbb{C}^\times \to A$. Given a (multivariate) power series Z with coefficients in $\mathbb{Q}(t_k, (t_1t_2t_3)^{1/2})$ and a slope σ, we set

$$Z^\sigma = \lim_{z\to 0} Z|_{\sigma(z)t},$$

if the limit in question exists. For such a slope σ and weight w, we have $\sigma(w(t)) = z^r w(t)$ for some r; we say that w is *attracting* with respect to σ if $r > 0$, and *repelling* with respect to σ if $r < 0$.

In [**NO**, Prop 7.4], it is shown that if M_0 is compact (and $\tilde{\mathcal{O}}^{\mathrm{vir}}$ can be defined), then, for generic σ, the limit $\chi(M_0, \tilde{\mathcal{O}}^{\mathrm{vir}})^\sigma$ is independent of the choice of σ. Our first step is to formulate a version of this result for non-compact geometries. The Hilbert scheme of curves $DT(X)$ is a union of components $DT(X, \beta, n)$ indexed by the curve class β and Euler characteristic n of subschemes in that component; let M be one such component. We prove the following.

THEOREM 1.2.1. *If two generic slopes σ_1 and σ_2 share the same attracting/repelling behavior in each non-compact direction of X, then*

$$\chi(M, \tilde{\mathcal{O}}^{\mathrm{vir}})^{\sigma_1} = \chi(M, \tilde{\mathcal{O}}^{\mathrm{vir}})^{\sigma_2}.$$

When σ_1 and σ_2 are so-called *preferred slopes*, roughly speaking, those that nearly preserve a coordinate direction of each toric chart of X, Theorem 1.2.1 may be regarded as a mathematical formulation of the independence of the refined topological partition function of toric Calabi-Yau threefolds X under certain choices of preferred direction; examples are computed in [**AK, IKV**]. Note that the resulting limit

$$\chi(M, \tilde{\mathcal{O}}^{\mathrm{vir}})^{\sigma}$$

may depend on the choice of σ; this is already evident, for example, in [**IKV**, Sec. 5.3]. We remark that in the special case when the space of effective curves in X is compact, the independence of this limit of the choice of generic σ would follow from a K-theoretic version of DT/stable pairs correspondence ([**NO**, Conj 2]).

The next step is to explicitly compute certain limits of

$$Z_{DT}(X) = \sum_{\beta,n} Q^n u^\beta \chi(DT(X, \beta, n), \tilde{\mathcal{O}}^{\mathrm{vir}})$$

under certain one-parameter subgroups σ for a particular class of toric geometries X. When X is toric, the partition function $Z_{DT}(X)$ can be studied using equivariant localization. As shown in [**MNOP**], the T-fixed points of $DT(X)$ are given by configurations of three-dimensional partitions along the 1-skeleton of the toric polytope $\Delta(X)$; the partition function can be written in terms of contributions from the vertices and edges of the 1-skeleton. The edge contributions admit explicit description in terms of $(\mathbb{C}^2)^{[n]}$; we work out the contribution for the relevant edge configurations in Propositions 4.2.1 and 4.2.2.

The vertex contributions $V(\lambda, \mu, \nu)$ are more complicated series in Q, t_k and $(t_1 t_2 t_3)^{1/2}$; their cohomological analogs are studied using the topological vertex studied in [**AKMV, ORV**]. In K-theory, it is difficult to analyze $V(\lambda, \mu, \nu)$ in full generality. However, for generic σ, the resulting limit $V(\lambda, \mu, \nu)^\sigma$ is a series in Q and $(t_1 t_2 t_3)^{1/2}$; moreover, for preferred slopes σ, it was explained in [**NO**] that the limit $V(\lambda, \mu, \nu)^\sigma$ is related to the refined topological vertex of [**IKV**], and is therefore well-suited to computation. In Proposition 4.1.7, we explicate the precise relationship between such $V(\lambda, \mu, \nu)^\sigma$ and the refined topological vertex. In particular, the choice of direction of preferred slope corresponds to a choice of preferred direction of the refined topological vertex.

We remark that in the physics literature, computations using refined topological vertex computations incorporate different so-called "framing factors"; see for instance [**IKV**] and [**Taki**]. The computation using DT invariants provides one consistent way to choose such factors.

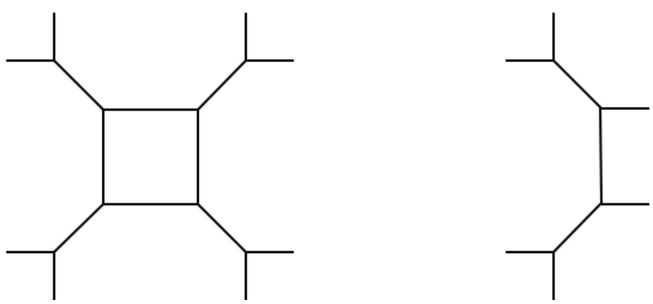

FIGURE 1.1. The toric geometries that give rise to Theorem 1.1.2 and Proposition 1.1.3, respectively

We use these vertex and edge computations to explicitly compute $Z_{DT}(X)^{\sigma_1}$ and $Z_{DT}(X)^{\sigma_2}$ for the Calabi-Yau threefolds X in Figure 1.1, and σ_1 and σ_2 are two preferred slopes satisfying the hypotheses of Theorem 1.2.1. After a certain specialization, we obtain Theorem 1.1.2 and Proposition 1.1.3.

1.3. Outline

The structure of this thesis is as follows. In Chapter 2 we define K-theoretic Donaldson-Thomas invariants of Calabi-Yau threefolds. In Chapter 3, we prove a general relationship between non-compact directions in a moduli space and the limits of Euler characteristics of coherent sheaves under slopes in order to deduce Theorem 1.2.1. In Chapter 4, we recall from [**MNOP, NO, O2**] the descriptions of the vertex and edge contributions to K-theoretic Donaldson-Thomas invariants of toric Calabi-Yau threefolds in terms of partitions, and explicitly work out combinatorial descriptions of their limits under preferred slopes. In Chapter 5, we apply these results to specific Calabi-Yau threefolds to deduce Theorem 1.1.2 and Proposition 1.1.3. Finally, in Chapter 6, we use Theorem 1.1.2 and Proposition 1.1.3 to study certain tautological classes over the Hilbert scheme of points on a general surface.

CHAPTER 2

Preliminaries

We recall the definition of K-theoretic Donaldson-Thomas invariants, and fix notation concerning partitions. None of the material in this chapter is original.

2.1. K-theoretic DT invariants

We begin by recalling the definition and construction of K-theoretic Donaldson-Thomas invariants from [**Th**], [**MNOP**], and [**NO**].

Fix a smooth complex quasi-projective threefold X; in anticipation of our applications, we assume that X is toric Calabi-Yau. While this assumption has the advantage of simplifying the exposition, it also obscures the generality in which these invariants can be formulated. In particular, a main source of inspiration for the definition of K-theoretic DT invariants is a conjectural connection betweeen K-theoretic DT invariants and curve counting in certain Calabi-Yau fivefolds (see [**NO, O2**]); such a relationship can be formulated for arbitrary smooth threefolds X.

2.2. DT moduli space

Roughly speaking, Donaldson-Thomas invariants, introduced in [**Th**], count embedded curves in X. They are computed as Euler characteristics over the Hilbert scheme $DT(X)$ of projective curves in X; the space $DT(X)$ decomposes into components $DT(X, \beta, n)$ parametrizing projective subschemes $Y \subset X$ whose components are of dimension at most 1, with $[Y] = \beta$ and $\chi(\mathcal{O}_Y) = n$.

In contrast to the Hilbert scheme of points on smooth surfaces, which are themselves smooth varieties, Hilbert schemes of curves on threefolds are badly behaved; in general, they are of unknown dimension and highly singular. In particular, the $DT(X, \beta, n)$ do not possess fundamental classes. As is the case in many enumerative problems, in cohomology, one circumvents this issue by replacing the fundamental class with a so-called *virtual fundamental class* of the expected dimension. The desired enumerative invariants are integrals over the virutal class. In [**BF**], such a class is produced for any space that possesses a perfect obstruction theory (see also [**LT**]). Here too, however, the Hilbert scheme presents difficulties: there is a standard way to associate an obstruction theory to any Hilbert scheme, but the obstruction theory associated in this way to the Hilbert schemes of curves in threefolds is not perfect.

However, it is shown in [**Th**] that $DT(X)$ admits another description as moduli of *ideal sheaves*, rank 1, torsion-free sheaves $\mathcal{I}_Y$ on X with trivial determinant. One passes from a subscheme $\mathcal{O}_Y$ to its ideal sheaf by setting

$$\mathcal{I}_Y = \ker(\mathcal{O}_X \twoheadrightarrow \mathcal{I}_Y);$$

conversely, one passes from an ideal sheaf $\mathcal{I}_Y$ to the subscheme

$$\mathcal{O}_Y = \operatorname{coker}(\mathcal{I}_Y \hookrightarrow (\mathcal{I}_Y)^{\vee\vee} \cong \mathcal{O}_X).$$

Using this description, it is shown in [**Th**] that $DT(X, \beta, n)$ possesses a perfect obstruction theory given by

$$\mathrm{Def}_{\mathcal{I}} - \mathrm{Obs}_{\mathcal{I}} = \chi(\mathcal{O}) - \chi(\mathcal{I}, \mathcal{I}).$$

When X is toric Calabi-Yau, the obstruction theory is T-equivariant and has virtual dimension 0. Moreover, for such X, Serre duality implies that the obstruction theory enjoys the following symmetry: if κ is the T-weight of the anticanonical bundle $\mathcal{K}_X^\vee$, then as T characters we have

$$\mathrm{Def}_{\mathcal{I}} = (\mathrm{Obs}_{\mathcal{I}})^\vee \cdot \kappa. \tag{2.2.1}$$

2.3. Modified virtual structure sheaf

We let $K_T(X)$ and $K_T(DT(X))$ denote the Grothendieck groups of T-equivariant coherent sheaves on X and $DT(X)$, respectively. In passing from cohomology to K-theory, the role of the virtual fundamental class is played by an equivariant K-theory class called the *virtual structure sheaf*. As with the virtual fundamental class, the virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ is constructed from a perfect obstruction theory. This construction may be found, for example, in [**BF**, 5.4], [**C-FK**], [**Lee**, Sec 2].

The definition of K-theoretic Donaldson-Thomas invariants incorporates a certain modification of the virtual structure sheaf. This modification endows the resulting invariants with certain symmetries that make for easier analysis; it also aligns such invariants with the indices of certain Dirac operators of interest to physicists (see for example, [**?**, 3.2.7]).

We let

$$T^{\mathrm{vir}} = \mathrm{Def} - \mathrm{Obs} \in K_T(DT(X))$$

denote the virtual tangent bundle over $DT(X)$ and define the virtual canonical bundle $\mathcal{K}^{\mathrm{vir}}$ to be the bundle

$$\det T^{\mathrm{vir}} = \det \mathrm{Obs} \otimes (\det \mathrm{Def})^{\vee} \in \mathrm{Pic}(DT(X)).$$

We recall the following result.

PROPOSITION 2.3.1. [**NO**, Ch. 6] *The line bundle $\mathcal{K}^{\mathrm{vir}}$ admits a square root $(\mathcal{K}^{\mathrm{vir}})^{1/2}$ in* $\mathrm{Pic}(DT(X))$. *Moreover, if $\tilde{T}$ is a minimal cover of T on which the square root $\kappa^{1/2}$ is defined, the line bundle $(\mathcal{K}^{\mathrm{vir}})^{1/2}$ carries a canonical $\tilde{T}$-equivariant structure.*

DEFINITION 2.3.2. *The* m*odified virtual structure sheaf is*

$$\tilde{\mathcal{O}}^{\mathrm{vir}} = \mathcal{O}^{\mathrm{vir}} \otimes (\mathcal{K}^{\mathrm{vir}})^{1/2}.$$

K-theoretic Donaldson-Thomas invariants are constructed as $\tilde{T}$-equivariant holomorphic Euler characteristics of $\tilde{\mathcal{O}}^{\mathrm{vir}}$. A more general variant studied in [**NO, O1, O2**] incoporates further tautological insertions, but for our purposes, it suffices to consider the K-theoretic partition function

$$Z_{DT}(X) = \sum_{\substack{\beta \in H_2(X,\mathbb{Z})^{\mathrm{eff}} \\ n \in \mathbb{Z}}} Q^n u^{\beta} \chi(DT(X,\beta,n), \tilde{\mathcal{O}}^{\mathrm{vir}}) \in \mathbb{Z}(t_k, \kappa^{1/2})((Q))[u^{\beta}].$$

Note that for fixed β, the space $DT(X,\beta,n)$ is empty for $n << 0$, so the resulting sums are formal Laurent series in Q.

From the point of view of computation, it will be convenient (but not qualitatively different) to normalize the Donaldson-Thomas partition function by the contribution from the Hilbert scheme of points on X; to this end, we define the *reduced K-theoretic DT partition function* to be

$$Z'_{DT}(X) = \frac{Z_{DT}(X)}{\sum_{n\in\mathbb{Z}} Q^n \chi(DT(X,0,n), \tilde{\mathcal{O}}^{\mathrm{vir}})}.$$

We remark that, when X is not toric Calabi-Yau, the general definition of K-theoretic DT invariants also takes as input the total space $\mathcal{L}_1 \oplus \mathcal{L}_2$ of two line bundles over X satisfying $\mathcal{L}_1 \otimes \mathcal{L}_2 \simeq \mathcal{K}_X$. The construction of $\tilde{\mathcal{O}}^{\mathrm{vir}}$ then incorporates a tautological bundle obtained from $\mathcal{L}_1$ and $\mathcal{L}_2$. In the case when X is Calabi-Yau and $\mathcal{L}_1$ and $\mathcal{L}_2$ are trivial, this more general construction specializes to our the construction of $\tilde{\mathcal{O}}^{\mathrm{vir}}$ above; moreover, in this setting the parameter Q arises as the torus weight of the line bundle $\mathcal{L}_1$.

2.3.1. Equivariant localization. We will study K-theoretic invariants using virtual equivariant localization, as developed in [**C-FK**] and [**GP**].

Before proceeding to the statement of equivariant localization, we recall from [**O1**, Sec 2] the requisite preliminary notions. Given a Laurent polynomial $p = \sum_i u_i - \sum_j v_j$ in some variables t_k where each u_i and v_j is a nonconstant Laurent monomial with coefficient 1, we define the plethystic exponential $\mathrm{Sym}^\bullet(p)$ to be the rational function

$$\mathrm{Sym}^\bullet(p) = \frac{\prod_j 1 - v_j}{\prod_i 1 - u_i} \in \mathbb{Z}(t_k). \tag{2.3.1}$$

When needed, we extend the operation $\mathrm{Sym}^\bullet$ to a suitable completion by allowing the products in the numerator and denominator of (2.3.1) to be infinite. Similarly, given a K-theory class $\mathcal{F}_1 - \mathcal{F}_2$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are T-equivariant vector bundles on a quasi-projective scheme M, we would like to define an element

$$\mathrm{Sym}^\bullet(\mathcal{F}_1 - \mathcal{F}_2) = \Big(\sum_{i\geq 0} \mathrm{Sym}^i \mathcal{F}_1\Big) \otimes \Big(\sum_{j\geq 0} \Lambda^j \mathcal{F}_2\Big)$$

of a suitable completion of $K_T(M)$; however, this operation is not always well-defined (consider, for example, $\mathcal{F}_1 = \mathcal{O}_X$). One way to circumvent this issue is to introduce a new parameter y, and, as in (1.1.3), define

$$\mathrm{Sym}^\bullet_y(\mathcal{F}_1 - \mathcal{F}_2) = \Big(\sum_{i\geq 0} y^i \mathrm{Sym}^i \mathcal{F}_1\Big) \otimes \Big(\sum_{j\geq 0} y^j \Lambda^j \mathcal{F}_2\Big) \in K_T(M)[[y]].$$

Now, suppose that M carries a T-equivariant perfect obstruction theory and that the fixed locus M^T is proper and nonempty. Given a T-equivariant coherent sheaf $\mathcal{F}$ on sheaf on M, equivariant localization asserts first that, as v_j ranges over some finite set of weights, we have

$$\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}}) \in \mathbb{Z}[t_i^{\pm 1}]\Big[\frac{1}{1-v_j}\Big],$$

and moreover that

$$\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}}) = \chi\bigg(M^T, \mathcal{F}|_{M^T} \otimes \mathrm{Sym}^\bullet_y\big((\mathcal{N}^{\mathrm{vir}}_{M/M^T})^\vee\big)\bigg)\bigg|_{y=1}, \tag{2.3.2}$$

where $\mathcal{N}^{\mathrm{vir}}_{M/X^T}$ denotes the virtual normal bundle over M^T. Henceforth, we omit y and abbreviate the right-hand side of (2.3.2) by

$$\chi\bigg(M^T, \mathcal{F}|_{M^T} \otimes \mathrm{Sym}^\bullet\big((\mathcal{N}^{\mathrm{vir}}_{M/M^T})^\vee\big)\bigg).$$

We apply this result to the particular case when M is $DT(X)$ for some Calabi-Yau threefold X, and $\mathcal{F}$ is $\tilde{\mathcal{O}}^{\mathrm{vir}}$. In this case, the fixed locus $DT(X)^{\tilde{T}}$ consists of isolated fixed points. Suppose $\mathcal{I} \in DT(X)^{\tilde{T}}$; then, as $\tilde{T}$-characters, we may write

$$T^{\mathrm{vir}}_{\mathcal{I}} = \mathrm{Def}_{\mathcal{I}} - \mathrm{Obs}_{\mathcal{I}} = \sum_i u_{\mathcal{I}_i} - v_{\mathcal{I}_i}$$

where each $u_{\mathcal{I}_i}$ and $v_{\mathcal{I}_i}$ is a Laurent monomial (with coefficient 1) in t_k and $\kappa^{1/2}$; for clarity, we omit the index $\mathcal{I}$. Then, by (2.2.1) we can reorder the u_i and v_i so that that

$$v_i = \frac{\kappa}{u_i}.$$

We may then write

$$(\mathcal{K}^{\mathrm{vir}})^{1/2}_{\mathcal{I}} = \prod_i \frac{v_i^{1/2}}{u_i^{1/2}}.$$

Applying virtual localization, we conclude that, as $\tilde{T}$ characters, we have

$$Z_{DT}(X) = \sum_{\substack{\beta \in H_2(X;\mathbb{Z})^{\text{eff}} \\ n \in \mathbb{Z}}} Q^n u^\beta \chi\Big(DT(X,\beta,n)^T, \big((\mathcal{K}^{\text{vir}})^{1/2} \otimes \text{Sym}^\bullet T^{\text{vir}}\big)\big|_{DT(X,\beta,n)^T} \Big)$$

$$= \sum_{\substack{\beta \in H_2(X;\mathbb{Z})^{\text{eff}} \\ n \in \mathbb{Z}}} Q^n u^\beta \sum_{\mathcal{I} \in DT(X,\beta,n)^T} \prod_i \frac{(\frac{\kappa}{u_i})^{1/2} - (\frac{\kappa}{u_i})^{-1/2}}{u_i^{1/2} - u_i^{-1/2}}. \tag{2.3.3}$$

Motivated by this expression, we introduce the following notation from [**O1, O2**]: given a Laurent polynomial $p = \sum_i u_i - \sum_j v_j$ where u_i and v_j are again Laurent monomials with coefficient 1, we define

$$\hat{a}(p) = \frac{\prod_j v_j^{1/2} - v_j^{-1/2}}{\prod_i u_i^{1/2} - u_i^{-1/2}} \in \mathbb{Z}(t_k^{\pm 1/2}).$$

In Chapter 6, we will also use the classical variant of K-theoretic equivariant localization where M is a smooth scheme; in this case the ordinary structure sheaves and normal bundles replace their virtual analogs.

2.4. Partitions

We fix some notation concerning partitions.

A two-dimensional partition λ is an ordered finite sequence $(\lambda_i) = (\lambda_1, \lambda_2, \cdots)$ of nonincreasing integers, or, equivalently, a collection of ordered pairs $(b_1, b_2) \in \mathbb{Z}^2_{\geq 0}$ such that if $(b_1, b_2) \in \lambda$, then for any b_1', b_2' satisfying $0 \leq b_1' \leq b_1$ and $0 \leq b_2' \leq b_2$, we also have $(b_1', b_2') \in \lambda$; the bijection between these two notions is given by

$$\{(\lambda_i)\} \mapsto \{(b_1, b_2) \mid b_2 \leq \lambda_{b_1+1} - 1\}.$$

We define the *size* $|\lambda|$ of a partition λ to be $\sum_i \lambda_i$, and we set

$$||\lambda^2|| = \sum_i |\lambda_i|^2.$$

For concision, we sometimes abbreviate an ordered pair (b_1, b_2) by the symbol $\square$; we refer to such a square as a *box* of λ. Two-dimensional partitions are often drawn as top-left justified configurations of boxes, with λ_i boxes in the ith row from the top.

We let λ^t denote the conjugate partition of λ; that is

$$\lambda^t = \{(b_2, b_1) \mid (b_1, b_2) \in \lambda\}.$$

For a box $\square = (b_1, b_2) \in \lambda$, we let $a(\square)$ and $l(\square)$ denote the arm and leg lengths of $\square$, that is

$$a(\square) = \lambda_{b_1+1} - b_2 - 1,\ l(\square) = \lambda^t_{b_2+1} - b_1 - 1.$$

When our meaning is clear, we sometimes abbreviate $a(\square)$ and $l(\square)$ by a and l respectively.

A three-dimensional partition π is a (possibly infinite) collection of points of $\mathbb{Z}^3_{\geq 0}$ such that if $(b_1, b_2, b_3) \in \pi$, and $0 \leq b'_i \leq b_i$ for $i = 1, \cdots, 3$, then $(b'_1, b'_2, b'_3) \in \pi$. Such partitions may be visualized as a collection of unit boxes in the octant $\mathbb{R}^3_{\geq 0}$ centered at the points $(b_1 + \frac{1}{2}, b_2 + \frac{1}{2}, b_3 + \frac{1}{2})$; see for example Figure 2.1. Again, when clear, we abbreviate (b_1, b_2, b_3) by the symbol $\square$.

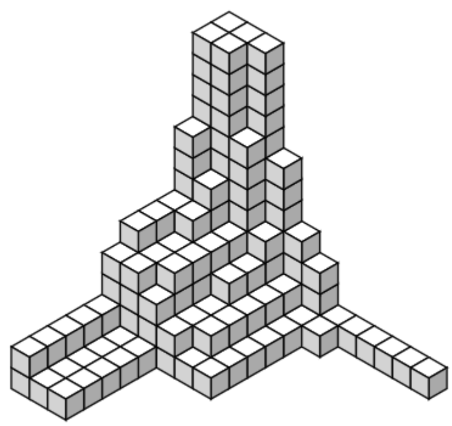

FIGURE 2.1. A three-dimensional partition π with asymptotics $\pi^{(1)} = (2, 1, 1)$, $\pi^{(2)} = (1)$ and $\pi^{(3)} = (3, 2)$

In this thesis, we restrict to the class of three-dimensional partitions π for which there exists some integer B_π such that, for any $(b_1, b_2, b_3) \in \pi$, at most one of the coordinates b_1, b_2, b_3 satisfies $b_i > B_\pi$; visually, this means the asymptotics of π along the coordinate directions of $\mathbb{R}^3$ are given by (finite) two-dimensional partitions. Figure 2.1 depicts one such partition.

Given a three-dimensional partition π corresponding to a monomial ideal let $\pi^{(k)}$ denote the two-dimensional partition describing the asymptotics of π along the k-th coordinate axis; formally, we set

$$\pi^{(1)} = \{(b_1, b_2) \mid (b, b_1, b_2) \in \pi \quad \text{for all b} \geq 0\},$$

$$\pi^{(2)} = \{(b_1, b_2) \mid (b_2, b, b_1) \in \pi \quad \text{for all b} \geq 0\},$$

$$\pi^{(3)} = \{(b_1, b_2) \mid (b_1, b_2, b) \in \pi \quad \text{for all b} \geq 0\}.$$

For example, if π denotes the three-dimensional partition in Figure 2.1, we have $\pi^{(1)} = (2, 1, 1), \pi^{(2)} = (1)$, and $\pi^{(3)} = (3, 2)$.

We let $\pi\langle k\rangle$ denote the infinite leg of the partition π along the k-th coordinate axis; formally, we set

$$\pi\langle 1\rangle = \{(b, b_1, b_2) \mid (b_1, b_2) \in \pi^{(1)}, b \in \mathbb{Z}_{\geq 0}\},$$

$$\pi\langle 2\rangle = \{(b_2, b, b_1) \mid (b_1, b_2) \in \pi^{(2)}, b \in \mathbb{Z}_{\geq 0}\},$$

$$\pi\langle 3\rangle = \{(b_1, b_2, b) \mid (b_1, b_2) \in \pi^{(3)}, b \in \mathbb{Z}_{\geq 0}\}.$$

CHAPTER 3

Slope independence

In this chapter, we fix a quasi-projective scheme M with a perfect obstruction theory; examples include smooth schemes, DT moduli space, or the Pandharipande-Thomas stable pair moduli space of a threefold (see [**PT1, PT2**]).

We impose the additional assumption that M admits the action of a torus T such that M^T is proper and nonempty. Let $\mathcal{F}$ denote a T-equivariant K-theory class of coherent sheaves on M; our hypotheses enable us to calculate the T-character $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})$ using localization.

In their study of K-theoretic enumerative invariants of Calabi-Yau threefolds, one assumption in the analysis of [**NO**, Sec. 7] is that the each component of the moduli space M is proper. If κ and $\tilde{T}$ are as in the previous chapter, then the $\tilde{T}$-character $\chi(M, \mathcal{F})$ is Laurent polynomial in t_k and $\kappa^{1/2}$. Let $A \subset T$ denote the Calabi-Yau subtorus; it follows that if $\chi(M, \mathcal{F})^\sigma$ exists for generic one-parameter subgroups $\sigma : \mathbb{C}^\times \to A$, then $\chi(M, \mathcal{F})$ is a function of $\kappa^{1/2}$. In particular, the value of $\chi(M, \mathcal{F})$ is equal to the limit $\chi(M, \mathcal{F})^\sigma$ for any generic σ.

For our applications, we analayze an expression of the form $\chi(M, \mathcal{F})$ when M is the Hilbert scheme of curves in a threefold, the components of which are not generally proper. Consequently, given a $\tilde{T}$-equivariant coherent sheaf $\mathcal{F}$ on a component M of the Hilbert scheme, the Euler characteristic $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})$ is a rational function in t_k and $\kappa^{1/2}$. We characterize the possible denominators of this rational function in terms of the equivariant geometry of M.

Using our restriction on the denominators of $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})$, we show that if the limit $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})^\sigma$ exists for generic one-parameter subgroups σ landing in a subtorus of T, then the value of this limit depends only on the attracting behavior of each weight appearing in the denominator with respect to σ. One advantage to this approach is that one does not need a K-theoretic version of the Donaldson-Thomas/stable pairs correspondence in order to mathematically formulate the independence of the refined topological partition function on certain choices of preferred slope.

3.1. Denominators in localization

Given a weight $w \in T^\vee$, we let $T_w \subset T$ denote the maximal torus inside the subgroup $\ker(w)$ of T. In particular, for nonzero $n \in \mathbb{Z}$ we have $T_{w^n} = T_w$.

DEFINITION 3.1.1. *A weight $w \in T^\vee$ is said to be* compact *if the fixed locus M^{T_w} is proper, and* noncompact *otherwise.*

As M^T is compact, by equivariant localization and a variant of [**O1**, Lemma 7.1.11] we have

$$\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}}) \in \mathbb{Z}[t_k^{\pm}]\Big[\frac{1}{1-w}\Big],$$

where w ranges over all T-weights occurring in the normal bundle $\mathcal{N}^{\mathrm{vir}}_{M/M^T}$.

While many possible denominators can occur in any particular term of a localization expression for $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})$, the following result restricts that the possible denominators that occur in the total sum.

PROPOSITION 3.1.2. *The T-character $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})$ can be written as a quotient $p(t)/q(t)$ of two Laurent polynomials for a Laurent polynomial $q(t)$ of the form $\prod(1-w)$ where w are noncompact weights of M.*

PROOF. Write $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})$ as a quotient $f(t) = p(t)/q(t)$, where $q(t) = \prod(1-w)$ where each w is a weight in $\mathcal{N}_{M/M^T}$; if $p(t)$ is zero there is nothing to show.

If w is a compact weight, then equivariant localization with respect to T_w states that as T_w-characters, we have

$$\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}}) = \chi\big(M^{T_w}, \mathcal{F}|_{M^{T_w}} \otimes \mathrm{Sym}^\bullet\Big(\big(\mathcal{N}^{\mathrm{vir}}_{M/M^{T_w}}\big)^\vee\big)\Big) \in \mathbb{Z}[t_i^{\pm}]\Big[\frac{1}{1-v}\Big],$$

where v ranges over the T-weights occuring in $\mathcal{N}^{\mathrm{vir}}_{M/M^{T_w}}$. No such v vanishes on T_w, so we conclude that $f(t)$ may be written in a form whose denominator is a product of terms of the form $(1-v)$ where no v is a power of w. In particular, $f(t)$ has no poles at $1 = \zeta w$ for any root of unity ζ; the proposition follows. $\square$

For example, when M is the Hilbert scheme $DT(\mathbb{C}^3, 0, n)$ of n points on $\mathbb{C}^3$ with the standard action of a three-dimensional torus T, then $M^{T_w} = M^T$ is proper unless w is a nontrivial power of some t_k, so the only noncompact directions are of the form $t_k^{\pm}$. Nekrasov's formula ([**O1**, Thm 3.3.6]) furnishes an explicit formula for the K-theoretic DT partition function:

$$\sum_{n=0}^{\infty}(-Q)^n \chi(DT(\mathbb{C}^3, 0, n), \tilde{\mathcal{O}}^{\mathrm{vir}}) = \mathrm{Sym}^{\bullet}\left(\frac{-Q}{(1-Q\kappa^{1/2})(1-Q\kappa^{-1/2})}\prod_{k=1}^{3}\frac{\kappa^{1/2}t_k^{-1} - \kappa^{-1/2}}{1-t_k^{-1}}\right). \tag{3.1.1}$$

In particular, we see that $\chi(DT(\mathbb{C}^3, 0, n), \tilde{\mathcal{O}}^{\mathrm{vir}})$ can be written as a Laurent polynomial whose denominator is a product of terms of the form $(1 - t_k^n)$.

3.2. Limit independence

Let $A \subset T$ be a subtorus, and let $f \in \mathbb{C}(t_k)$ be a nonzero function that can be written in the form

$$\frac{p(t)}{\prod_i (1 - w_i)},$$

where $p(t)$ is a Laurent polynomial in t_k and $w_i \in T^{\vee}$ are nontrivial. A generic one parameter subgroup $\sigma : \mathbb{C}^{\times} \to A$ is one for which $w(\sigma(z)t) \neq w(t)$ for any monomial w appearing in f. Let σ be generic; for a weight w, we have $w(\sigma(z)t) = z^r w(t)$ for some $r \neq 0$. We say that σ is *attracting* with to w if $r > 0$ and is *repelling* with respect to w if $r < 0$.

Proposition 3.2.1. *If*

$$\lim_{z\to 0} f(\sigma(z)t)$$

exists for any generic one parameter subgroups $\sigma : \mathbb{C}^{\times} \to A$, then the value of this limit depends only on the attracting/repelling behavior of σ with respect to each weight w_k.

Proof. Let σ_1 and σ_2 be two generic one parameter subgroups sharing the same attracting/repelling behavior for each weight w_i. Multiplying the numerator and denominator of f by a Laurent monomial, we may assume that w_i is attracting with respect to both σ_1 and σ_2.

As σ_1 is attracting for each w_i, we may write $q(\sigma_1(z)t) = 1 + zq_1(t)$ for some $q_1 \in \mathbb{C}[t_k^{\pm}, z]$.

Via the map $\varphi : T^\vee \to A^\vee$, any monomial appearing in the numerator or denominator of f can be regarded as an A-weight. Write $p(t) = \sum u_j$ where each u_j is a monomial, and set

$$p_0(t) = \Big\{ \sum u_j \Big| \varphi(u_j) = 1 \in A^\vee \Big\}.$$

As

$$\lim_{z\to 0} f(\sigma(z)t)$$

exists for generic one parameter subgroups, the set of A-weights appearing in the numerator of f must be the same as the set of A-weights appearing in the denominator of f. Therefore, $p(t) = p_0(t) + zp_1(t)$ for some $p_1 \in \mathbb{C}[t_k^{\pm}, z]$. It follows that we have

$$\lim_{z\to 0} f(\sigma_1(z)t) = \frac{p_0(t) + zp_1(t)}{1 + zq_1(t)} = p_0(t);$$

the same is true of σ_2. □

Combining Propositions 3.1.2 and 3.2.1, we obtain the following.

THEOREM 3.2.2. *If two generic one-parameter subgroups σ_1 and σ_2 share the same attracting/repelling behavior for each non-compact weight of M, then $\chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})^{\sigma_1} = \chi(M, \mathcal{F} \otimes \mathcal{O}^{\mathrm{vir}})^{\sigma_2}$.*

CHAPTER 4

Preferred limits of the DT partition function

Let X be a toric Calabi-Yau threefold admitting the action of a torus $T = \mathrm{diag}(t_1, t_2, t_3)$ such that the anticanonical bundle $\mathcal{K}_X^\vee$ is scaled with weight κ; for this section, we assume that $\kappa = t_1 t_2 t_3$. We begin by recalling from [**MNOP, NO, O2**] how to to write the partition function

$$Z_{DT}(X) = \sum_{\substack{\beta \in H_2(X;\mathbb{Z})^{\mathrm{eff}} \\ n \in \mathbb{Z}}} Q^n u^\beta \chi(DT(X,\beta,n), \tilde{\mathcal{O}}^{\mathrm{vir}}) \in \mathbb{Z}(t_1, t_2, t_3, \kappa^{1/2})((Q))[u^\beta],$$

and its reduced analog

$$Z'_{DT}(X) = \frac{\displaystyle\sum_{\substack{\beta \in H_2(X;\mathbb{Z})^{\mathrm{eff}} \\ n \in \mathbb{Z}}} Q^n u^\beta \chi(DT(X,\beta,n), \tilde{\mathcal{O}}^{\mathrm{vir}})}{\displaystyle\sum_{n \in \mathbb{Z}} Q^n \chi(DT(X,0,n), \tilde{\mathcal{O}}^{\mathrm{vir}})}$$

in terms of combinatorial expressions associated to the vertices and edges of the toric diagram $\Delta(X)$. Using results of [**NO**, Sec 8], we explicitly compute the limits of these vertex and edge contributions under preferred slopes, and, as a result, deduce the precise between the vertex contribution to $Z_{DT}(X)$ and the refined topological vertex of [**IKV**].

The only original results in this chapter are the precise statements of Propositions 4.2.1, 4.2.2 and 4.2.5.

4.1. K-theoretic building blocks

Using equivariant localization, the partition $Z_{DT}(X)$ can be written in terms of the combinatorics of configurations of three-dimensional partitions.

4.1.1. Torus-fixed locus. The fixed locus $DT(X)^T$ consists of isolated points, with finitely many in any component $DT(X, \beta, n)$. We recall from [**MNOP**, 4.1-4.2] a combinatorial description of this fixed-point locus.

The points of X^T correspond to the vertices of the toric polytope $\Delta(X)$ and the T-fixed rational curves of X correspond to the bounded edges of $\Delta(X)$. For $x_i \in X^T$, let U_i be the toric chart centered at x_i. Suppose that $x_i, x_j \in X^T$ are connected by a T-fixed rational curve $C_{ij} \subset X$; then $U_i \cap U_j$ is the total space of the direct sum of two line bundles $\mathcal{O}(l_{ij}) \oplus \mathcal{O}(l'_{ij})$ over C_{ij}.

Choose coordinates on each U_i such that the torus T scales the coordinate directions by T-weights $w_{i_1}, w_{i_2}, w_{i_3}$; after fixing an orientation on $\Delta(X)$, these coordinates can be compatibly ordered at each fixed point. The edges in the toric polytope emanating from the fixed point p_i correspond to the axes in these coodinates. Let $z_{i_1}, z_{i_2}, z_{i_3}$ denote the corresponding coordinate functions on U_i so that $U_i = \text{Spec } \mathbb{C}[z_{i_1}, z_{i_2}, z_{i_3}]$. The torus T scales these coordinate functions by $w_{i_1}^{-1}, w_{i_2}^{-1}, w_{i_3}^{-1}$, respectively. After cyclically permuting coordinates and exchanging l_{ij} with l'_{ij}, if necessary, the weights of coordinate directions on U_i, U_j are identified by

$$w_{j_1} = w_{i_1}^{-1}, w_{j_2} = w_{i_1}^{-l'_{ij}} w_{i_3}, w_{j_3} = w_{i_1}^{-l_{ij}} w_{i_2};$$

see Figure 4.1.

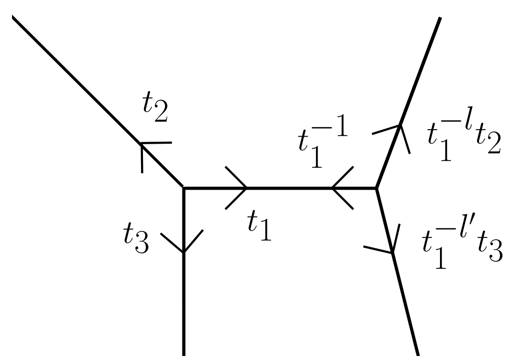

FIGURE 4.1. The T-weights along a T-fixed rational curve with normal bundle $\mathcal{O}(l) \oplus \mathcal{O}(l')$

A T-fixed ideal sheaf $\mathcal{I} \in DT(X)$ is determined by a collection of T fixed ideals $\{\mathcal{I}_i \subset \mathcal{O}_{U_i}\}$ whose restrictions $\mathcal{I}_i|_{U_i \cap U_j}$ are compatible, and glue to an ideal sheaf that cuts out a proper subscheme of X.

A T-fixed ideal

$$\mathcal{I}_i \subset U_i$$

is given by a monomial ideal

$$I_i \subset \mathbb{C}[z_{i_1}, z_{i_2}, z_{i_3}].$$

Monomial ideals are in one-to-one correspondence with three-dimensional partitions π_i, given by

$$I_i \leftrightarrow \pi_i = \{(b_1, b_2, b_3) | z_{i_1}^{b_1} z_{i_2}^{b_2} z_{i_3}^{b_3} \notin I_i\};$$

we pass freely between an ideal and its associated partition.

The condition that $\mathcal{I}$ cuts out a proper subscheme restricts the possible asymptotics of the three-dimensional partitions π_i. The asymptotics of each such π_i along each coordinate direction must be finite (as described in Section 2.4); more restrictively, the partition π_i at a given fixed point x_i must satisfy the following condition:

$$\pi_i^{(k)} \neq \varnothing \text{ only if the edge of } \Delta(X) \text{ in the direction of coordinate } z_{i_k} \text{ is bounded.} \quad (4.1.1)$$

The asymptotics of the partitions must also agree in the direction of any T-fixed rational curve. To be precise, given $x_i, x_j \in X^T$ connected by a T-fixed rational curve C_{ij}, the restrictions $\mathcal{I}_i|_{U_i \cap U_j}$ and $\mathcal{I}_j|_{U_i \cap U_j}$ are compatible if the two localizations

$$(I_i)_{z_{i_1}} \subset \mathbb{C}[z_{i_1}^{\pm 1}, z_{i_2}, z_{i_3}], \ (I_j)_{z_{i_1}^{-1}} \subset \mathbb{C}[z_{i_1}^{\pm 1}, z_{i_2}, z_{i_3}]$$

coincide. In terms of three-dimensional partitions, this compatibility translates to the condition that

$$\pi_i^{(1)} = \left(\pi_j^{(1)}\right)^t; \quad (4.1.2)$$

for example, in Figure 4.2, if the horizontal directions of π_i and π_j correspond to the coordinates z_{i_1} and z_{j_1}, and the coordinates are oriented counter-clockwise, we have $\pi_i^{(1)} = (3, 2)$ and $\pi_j^{(1)} = (2, 2, 1)$.

Conversely, any collection of three-dimensional partitions π_i satisfying (4.1.1) and (4.1.2) corresponds to a T-fixed ideal sheaf $\mathcal{I}$. It follows that a point of $DT(X)^T$ is specified by a collection $\{\lambda_{ij}, \pi_i\}$, consisting of first, a choice of two-dimensional partition λ_{ij} for

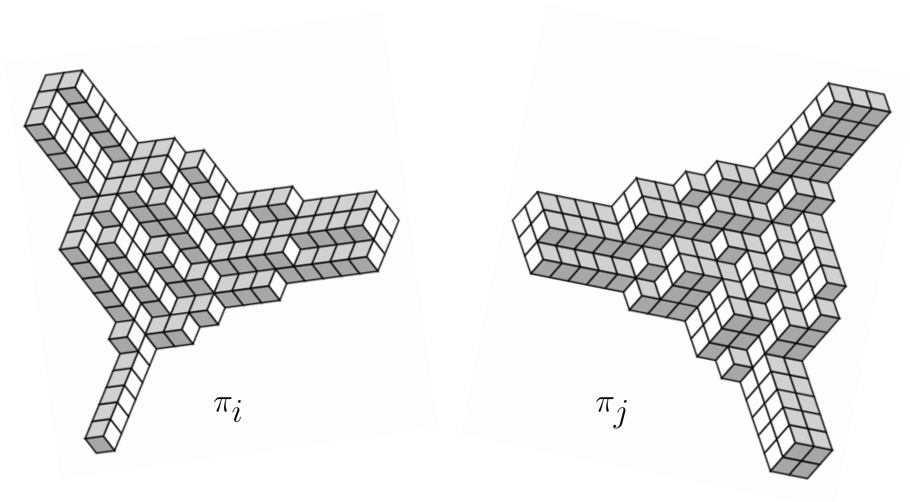

FIGURE 4.2. Two compatible partitions π_i and π_j at vertices connected by an edge of $\Delta(X)$

each bounded edge of $\Delta(X)$, and then, a choice of three-dimensional partition π_i for each vertex of $\Delta(X)$ whose asymptotics coincide with the chosen λ_{ij}; by convention we set

$$\lambda_{ji} = \lambda_{ij}^t.$$

4.1.2. Obstruction theory at fixed points. Recall from [**MNOP**, Sec. 4.6] that the T-equivariant obstruction theory of $DT(X)$ at an ideal sheaf $\mathcal{I}$ is given by

$$T_{\mathcal{I}}^{\mathrm{vir}} = \mathrm{Ext}^1(\mathcal{I}, \mathcal{I}) - \mathrm{Ext}^2(\mathcal{I}, \mathcal{I}).$$

As X is Calabi-Yau, the obstruction theory is symmetric; concretely, as T-characters we have

$$T_{\mathcal{I}}^{\mathrm{vir}} = -(T_{\mathcal{I}}^{\mathrm{vir}})^{\vee} \cdot \kappa. \tag{4.1.3}$$

The obstruction theory at torus-fixed point $\mathcal{I} \in DT(X)^T$ can be calculated from the corresponding combinatorial data $\{\lambda_{ij}, \pi_i\}$. We recall from [**MNOP**, Sec. 4.7-4.9] the resulting combinatorial procedure; this procedure is also presented in [**NO**, Sec. 8.2] and [**O2**, Sec. 6.3].

Suppose that $T = \mathrm{diag}(t_1, t_2, t_3)$ scales the coordinate axes of $\mathbb{C}^3 = \mathrm{Spec}\ \mathbb{C}[z_1, z_2, z_3]$, so T scales the coordinate z_k by t_k^{-1}. Given a partition π corresponding to a T-fixed ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^3}$, we set O_π to be the corresponding T-character, that is

$$O_\pi = \sum_{\square \in \pi} t_1^{-b_1} t_2^{-b_2} t_3^{-b_3}.$$

Then, as computed in [**MNOP**, 4.7], for π with finitely many boxes we have

$$T_\pi^{\text{vir}} = O_\pi - O_\pi^\vee t_1 t_2 t_3 - O_\pi O_\pi^\vee (1-t_1)(1-t_2)(1-t_3); \tag{4.1.4}$$

this is a Laurent polynomial in the variables t_k. For π with infinitely many boxes, we first write the infinite sum O_π as a rational function, and define the tangent space T_π^{vir} to be the *rational function* in t_k given by the same formula (4.1.4).

Given a character V of a two-dimensional torus $\operatorname{diag}(t_1, t_2)$, and torus weights w_{i_1}, w_{i_2}, we set $V(w_{i_1}, w_{i_2})$ to be the character of V under the substitution $t_1 = w_{i_1}, t_2 = w_{i_2}$, and, conversely, for an expression of the form $V(w_{i_1}, w_{i_2})$, in the special case when $w_{i_1} = t_1, w_{i_2} = t_2$ we omit the pair (t_1, t_2) from the notation. We use analogous notation for a three-dimensional torus. Then, for T-fixed ideal sheaf $\mathcal{I} \in DT(X)^T$ given by $\{\lambda_{ij}, \pi_i\}$ with weights $w_{i_1}, w_{i_2}, w_{i_3}$ as above, equivariant localization implies

$$T_{\mathcal{I}}^{\text{vir}} = \sum_i T_{\pi_i}^{\text{vir}}(w_{i_1}, w_{i_2}, w_{i_3}). \tag{4.1.5}$$

While each term in (4.1.5) is a rational function, the overall sum must be Laurent polynomial. It is explained in [**MNOP**, 4.9], [**NO**, 8.2.2-8.2.3], and [**O2**, 6.3.1] how to write $T_{\mathcal{I}}^{\text{vir}}$ as a sum of Laurent polynomials by regularizing each term of (4.1.5) by contributions associated to the asymptotics of the corresponding partition π_i. Namely, if the T-fixed rational curve C_{ij} corresponds to coordinate direction z_{i_k}, then we define

$$E_{\lambda_{ij},(l_{ij},l'_{ij})}(w_{i_1}, w_{i_2}, w_{i_3}) = \frac{T_{\lambda_{ij}}(w_{i_{k+1}}, w_{i_{k+2}})}{1 - w_{i_k}^{-1}} + \frac{T_{\lambda_{ij}^t}(w_{i_k}^{-l'_{ij}} \cdot w_{i_{k+2}}, w_{i_k}^{-l_{ij}} \cdot w_{i_{k+1}})}{1 - w_{i_k}}; \tag{4.1.6}$$

here, for $k = 1, 2, 3$, we have $w_{i_{k+3}} = w_{i_k}$. When clear from the geometry of X, we omit (l_{ij}, l'_{ij}) from the notation.

We then define

$$V_{\pi_i}(w_{i_1}, w_{i_2}, w_{i_3}) = T_{\pi_i}^{\text{vir}}(w_{i_1}, w_{i_2}, w_{i_3}) - \sum_{k=1}^{3} \frac{T_{\pi_i^{(k)}}(w_{i_{k+1}}, w_{i_{k+2}})}{1 - w_{i_k}^{-1}}. \tag{4.1.7}$$

where, in both (4.1.6) and (4.1.7) By [**MNOP**, Lemma 9], each V_{π_i} and $E_{\lambda_{ij}}$ is a Laurent polynomial in t_i. From (4.1.5), we have

$$T_{\mathcal{I}}^{\mathrm{vir}} = \sum_i V_{\pi_i}(w_{i_1}, w_{i_2}, w_{i_3}) + \sum_{ij} E_{\lambda_{ij},(l_{ij},l'_{ij})}(w_{i_1}, w_{i_2}, w_{i_3}). \tag{4.1.8}$$

It follows from (4.1.4, 4.1.6, 4.1.7) that each term of (4.1.8) enjoys the same symmetry (4.1.3); that is

$$V_{\pi_i} = -V_{\pi_i}^{\vee} \cdot \kappa,\ E_{\lambda_{ij}} = E_{\lambda_{ij}}^{\vee} \cdot \kappa. \tag{4.1.9}$$

4.1.3. Euler characteristic. In the same manner as the obstruction theory, the Euler characteristic of the subscheme cut out by an ideal sheaf can be computed in terms of edge and regularized vertex contributions using the Čech cover $\{U_i\}$; we recall the description from [**MNOP**, Lemma 5].

Set

$$\chi(\pi) = \sum_{\square \in \pi} \left(1 - \Big(\text{number of } \{\pi\langle 1\rangle, \pi\langle 2\rangle, \pi\langle 3\rangle\} \text{ that contain } \square\Big)\right). \tag{4.1.10}$$

The following reformulation of (4.1.10) will be helpful:

$$\chi(\pi) = \sum_{\{\square \in \pi \ \mid \ \square \notin \pi\langle 3\rangle\}} 1 - \sum_{\square \in \pi\langle 1\rangle} 1 - \sum_{\square \in \pi\langle 2\rangle} 1. \tag{4.1.11}$$

For a two-dimensional partition λ and integers (l, l'), set

$$\chi(\lambda, (l, l')) = \sum_{\square \in \lambda} 1 - lb_1 - l'b_2. \tag{4.1.12}$$

Then, for $\mathcal{I} \in DT(X)^T$ given by partition data $\{\pi_i, \lambda_{ij}\}$ we have

$$\chi(\mathcal{O}/\mathcal{I}) = \sum_i \chi(\pi_i) + \sum_{i,j} \chi(\lambda_{ij}, (l_{ij}, l'_{ij})). \tag{4.1.13}$$

4.1.4. Factorization of components of the partition function. Rephrasing (2.3.3), we obtain

$$Z_{DT}(X) = \sum_{\mathcal{I} \in DT(X)^T} Q^{\chi(\mathcal{O}/\mathcal{I})} u^{[\mathcal{O}/\mathcal{I}]} \hat{a}(T_{\mathcal{I}}^{\mathrm{vir}}).$$

We now write $Z_{DT}(X)$ in terms of expressions associated to the vertices and edges of $\Delta(X)$. Let $\mathcal{I} \in DT(X)^T$ be given by the configuration $\{\lambda_{ij}, \pi_i\}$. Then, by (4.1.8), we have

$$\hat{a}(T_{\mathcal{J}}^{\mathrm{vir}}) = \prod_i \hat{a}\Big(V_{\pi_i}(w_{i_1}, w_{i_2}, w_{i_3})\Big) \prod_{ij} \hat{a}\Big(E_{\lambda_{ij},(l_{ij},l'_{ij})}(w_{i_1}, w_{i_2}, w_{i_3})\Big),$$

while

$$Q^{\chi(\mathcal{O}/\mathcal{J})} = \prod_i Q^{\chi(\pi_i)} \prod_{ij} Q^{\chi(\lambda_{ij},(l_{ij},l'_{ij}))}, \quad u^{[\mathcal{O}/\mathcal{J}]} = \prod_{ij} u^{|\lambda_{ij}|[C_{ij}]}.$$

Given two-dimensional partitions $\lambda_1, \lambda_2, \lambda_3$, set $P(\lambda_1, \lambda_2, \lambda_3)$ to be the set of three-dimensional partitions π whose asymptotics are given by $\lambda_1, \lambda_2, \lambda_3$; that is

$$P(\lambda_1, \lambda_2, \lambda_3) = \Big\{\pi \text{ a three} - \text{dimensional partition} \mid \pi^{(k)} = \lambda_k,\ k \in \{1, 2, 3\}\Big\}.$$

Then, set

$$\hat{E}(\lambda, (l, l'))(w_{i_1}, w_{i_2}, w_{i_3}) = Q^{\chi(\lambda,(l,l'))} \hat{a}\Big(E_{\lambda,(l,l')}(w_{i_1}, w_{i_2}, w_{i_3})\Big),$$

and

$$\hat{V}(\lambda, \mu, \nu)(w_{i_1}, w_{i_2}, w_{i_3}) = \sum_{\pi \in P(\lambda,\mu,\nu)} Q^{\chi(\pi)} \hat{a}\Big(V_\pi(w_{i_1}, w_{i_2}, w_{i_3})\Big).$$

We may then write $Z_{DT}(X)$ as

$$\sum_{\{\lambda_{ij}\}} \Bigg(\prod_{ij} u^{|\lambda_{ij}|[C_{ij}]} \hat{E}(\lambda_{ij}, (l_{ij}, l'_{ij}))(w_{i_1}, w_{i_2}, w_{i_3}) \prod_i \hat{V}(\lambda_{i1}, \lambda_{i2}, \lambda_{i3})(w_{i_1}, w_{i_2}, w_{i_3})\Bigg), \tag{4.1.14}$$

where the sum is over all assignments of two-dimensional partitions to the bounded edges of $\Delta(X)$.

The expression $\hat{V}(\lambda, \mu, \nu)$ is called the *K-theoretic equivariant vertex.*

4.1.5. Reduced partition function. The reduced K-theoretic partition function admits a similar expression. From (4.1.14), we see

$$Z_{DT}(X)|_{u=0} = \prod_i \hat{V}(\varnothing, \varnothing, \varnothing)(w_{i_1}, w_{i_2}, w_{i_3}),$$

so that we may write $Z'_{DT}(X)$ as

$$\sum_{\{\lambda_{ij}\}} \Bigg(\prod_{ij} u^{|\lambda_{ij}|[C_{ij}]} \hat{E}(\lambda_{ij}, (l_{ij}, l'_{ij}))(w_{i_1}, w_{i_2}, w_{i_3}) \prod_i \frac{\hat{V}(\lambda_{i1}, \lambda_{i2}, \lambda_{i3})(w_{i_1}, w_{i_2}, w_{i_3})}{\hat{V}(\varnothing, \varnothing, \varnothing)(w_{i_1}, w_{i_2}, w_{i_3})}\Bigg). \tag{4.1.15}$$

The expression $\hat{V}(\varnothing,\varnothing,\varnothing)$ is completely characterized by Nekrasov's formula (3.1.1), so that the difference between $Z_{DT}(X)$ and $Z'_{DT}(X)$ is an easily understood prefactor; however, our formulas will be slightly more concise when formulated in terms of $Z'_{DT}(X)$. Consequently, we set

$$\hat{V}'(\lambda_{i1},\lambda_{i2},\lambda_{i3})(w_{i_1},w_{i_2},w_{i_3}) = \frac{\hat{V}(\lambda_{i1},\lambda_{i2},\lambda_{i3})(w_{i_1},w_{i_2},w_{i_3})}{\hat{V}(\varnothing,\varnothing,\varnothing)(w_{i_1},w_{i_2},w_{i_3})}$$

to be the vertex contribution normalized by the contribution of the Hilbert scheme of points.

4.2. Preferred limits of vertex and edge contributions

We analyze the behavior of certain edge contributions $\hat{E}(\lambda,(l,l'))$ and vertex contributions $\hat{V}(\lambda,\mu,\nu)$ under certain one-parameter subgroups.

4.2.1. Rigidity. The symmetry (4.1.9) enjoyed by the vertex and edge contributions ensures that each contribution behaves well under generic one parameter subgroups. Let $A \subset T$ be the Calabi Yau subtorus $\ker(\kappa)$.

It is explained in [**MNOP**] that no weight appearing in $T^{\mathrm{vir}}_{\mathcal{I}}$ for $\mathcal{I} \in M^T$ is a power of κ. Let w is a weight appearing in $\mathrm{Def}^{\mathrm{vir}}$, and $\sigma : \mathbb{C}^\times \to A$ be a one parameter subgroup such that $w(\sigma(z)t) = z^r w(t)$ where $r \neq 0$. Then, we have

$$\begin{aligned}\lim_{z\to 0} \frac{(\frac{\kappa}{w(\sigma(z)t)})^{1/2} - (\frac{\kappa}{w(\sigma(z)t)})^{-1/2}}{(w(\sigma(z)t))^{1/2} - (w(\sigma(z)t))^{-1/2}} &= \lim_{z\to 0} \frac{z^{-r/2}(\frac{\kappa}{w})^{1/2} - z^{r/2}(\frac{\kappa}{w})^{-1/2}}{z^{r/2}w^{1/2} - z^{-r/2}w^{-1/2}} \\ &= \begin{cases} -\kappa^{1/2} & \text{if } r>0 \\ -\kappa^{-1/2} & \text{if } r<0 \end{cases} \end{aligned} \tag{4.2.1}$$

Given a T-character V such that

$$V = -V^\vee \cdot \kappa$$

and a generic one parameter subgroup $\sigma : \mathbb{C}^\times \to A$, as in [**NO**], we set $\mathrm{ind}_\sigma(V)$ to be the number of monomial weights $w \in V$ that are attracting with respect to σ (in the sense of Section 3.2), counted with sign. It then follows from (4.2.1) that

$$(\hat{V})^\sigma = (-\kappa^{1/2})^{\mathrm{ind}_\sigma(V)}.$$

Given $(r_1, r_2, r_3) \in \mathbb{Z}$, we let $\sigma(r_1, r_2, r_3) : \mathbb{C}^\times \to A$ denote the one parameter subgroup

$$z \mapsto (z^{r_1}, z^{r_2}, z^{r_3}).$$

Set $q = -Q\kappa^{1/2}$ and $t = -Q\kappa^{-1/2}$. We separately compute the limits $E^\sigma_{\lambda_{ij},(l,l')}$ and V^σ_π as functions of q and t. These limits do not seem to admit a concise expression in terms of the partition data for arbitrary σ. However, as observed in [**NO**, Sec. 8], when one of the exponents r_k has very small magnitude compared to the others, these limits simplify considerably. Such σ are called *preferred slopes.*

4.2.2. Edge limits. For the Calabi-Yau threefolds X we consider in Section 5, and, indeed, for essentially every Calabi-Yau admitting two different choices of preferred slope, the possible normal bundles to a T-fixed rational curve $C_{ij} \subset X$ are $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}(-2)$. We compute the limits of the edge terms $\hat{E}_\lambda$ associated to these two configurations.

We first consider a T-fixed rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, as depicted in Figure 4.3.

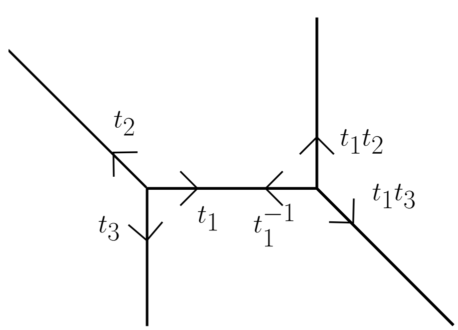

FIGURE 4.3. The weights along a T-fixed rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

We compute

$$
\begin{aligned}
E_{\lambda,(-1,-1)} &= \frac{T_\lambda(t_2,t_3)}{1-t_1^{-1}} + \frac{T_{\lambda^t}(t_1t_3,t_1t_2)}{1-t_1} \\
&= \sum_{\square\in\lambda} \frac{1-t_1^{-l(\square)+a(\square)}}{1-t_1^{-1}} t_2^{-l(\square)} t_3^{a(\square)+1} + \frac{1-t_1^{l(\square)-a(\square)}}{1-t_1^{-1}} t_2^{l(\square)+1} t_3^{-a(\square)} \\
&= \sum_{\square\in\lambda \ | \ a(\square)\le l(\square)} (1+\cdots+t_1^{-l(\square)+a(\square)}) t_2^{-l(\square)} t_3^{a(\square)+1} \\
&\quad + \sum_{\square\in\lambda \ | \ a(\square)\le l(\square)} -(t_1+\cdots+t_1^{l(\square)-a(\square)+1}) t_2^{l(\square)+1} t_3^{-a(\square)} \\
&\quad + \sum_{\square\in\lambda \ | \ a(\square)> l(\square)} -(t_1+\cdots+t_1^{-l(\square)+a(\square)}) t_2^{-l(\square)} t_3^{a(\square)+1} \\
&\quad + \sum_{\square\in\lambda \ | \ a(\square)> l(\square)} +(1+\cdots+t_1^{l(\square)-a(\square)+1}) t_2^{l(\square)+1} t_3^{-a(\square)}.
\end{aligned}
$$

We conclude the following.

$\sigma = \sigma(r_1,r_2,r_3)$	$\mathrm{ind}_\sigma(E_{\lambda,(-1,-1)})$
$r_1 >> r_2 > 0 >> r_3$	$(\|\|\lambda^t\|\|^2 - \|\|\lambda\|\|^2)/2$
$r_2 >> r_1 > 0 >> r_3$	$(\|\|\lambda^t\|\|^2 - \|\|\lambda\|\|^2)/2$
$r_1 >> r_3 > 0 >> r_2$	$(\|\|\lambda\|\|^2 - \|\|\lambda^t\|\|^2)/2$
$r_3 >> r_1 > 0 >> r_2$	$(\|\|\lambda\|\|^2 - \|\|\lambda^t\|\|^2)/2$
$r_2 >> r_3 > 0 >> r_1$	$(\|\|\lambda^t\|\|^2 - \|\|\lambda\|\|^2)/2$
$r_3 >> r_2 > 0 >> r_1$	$(\|\|\lambda\|\|^2 - \|\|\lambda^t\|\|^2)/2$

The indices for remaining preferred sigma are determined by the relation

$$\mathrm{ind}_{\sigma(-r_1,-r_2,-r_3)}(V) = -\mathrm{ind}_{\sigma(r_1,r_2,r_3)}(V).$$

By (4.1.12), we have

$$\chi(\lambda,(-1,-1)) = \frac{||\lambda||^2 + ||\lambda^t||^2}{2}.$$

We conclude the following.

Proposition 4.2.1. *The limits of the edge contribution $\hat{E}(\lambda,(-1,-1))$ under preferred slopes σ are as follows.*

$\sigma = \sigma(r_1, r_2, r_3)$	$\hat{E}(\lambda,(-1,-1))^\sigma$
$r_1 >> r_2 > 0 >> r_3$	$t^{\frac{\|\|\lambda^t\|\|^2}{2}} q^{\frac{\|\|\lambda\|\|^2}{2}}$
$r_2 >> r_1 > 0 >> r_3$	$t^{\frac{\|\|\lambda^t\|\|^2}{2}} q^{\frac{\|\|\lambda\|\|^2}{2}}$
$r_1 >> r_3 > 0 >> r_2$	$t^{\frac{\|\|\lambda\|\|^2}{2}} q^{\frac{\|\|\lambda^t\|\|^2}{2}}$
$r_3 >> r_1 > 0 >> r_2$	$t^{\frac{\|\|\lambda\|\|^2}{2}} q^{\frac{\|\|\lambda^t\|\|^2}{2}}$
$r_2 >> r_3 > 0 >> r_1$	$t^{\frac{\|\|\lambda^t\|\|^2}{2}} q^{\frac{\|\|\lambda\|\|^2}{2}}$
$r_3 >> r_2 > 0 >> r_1$	$t^{\frac{\|\|\lambda\|\|^2}{2}} q^{\frac{\|\|\lambda^t\|\|^2}{2}}$

We repeat the same procedure for T-fixed rational curve with normal bundle $\mathcal{O}\oplus\mathcal{O}(-2)$, as depicted in Figure 4.4.

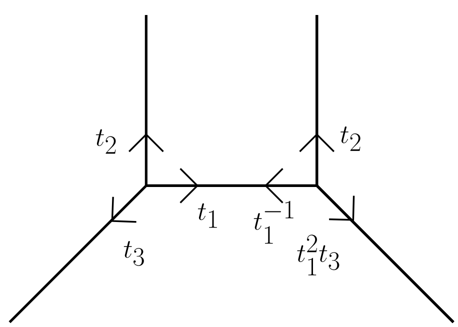

Figure 4.4. The weights along a T-fixed rational curve with normal bundle $\mathcal{O}\oplus\mathcal{O}(-2)$

We then compute

$$\begin{aligned}
E_{\lambda,(0,-2)} &= \frac{T_\lambda(t_2,t_3)}{1-t_1^{-1}} + \frac{T_{\lambda^t}(t_1^2 t_3, t_2)}{1-t_1} \\
&= \sum_{\square\in\lambda} \frac{1-t_1^{2a(\square)+1}}{1-t_1^{-1}} t_2^{-l(\square)} t_3^{a(\square)+1} + \frac{1-t_1^{-2a(\square)-1}}{1-t_1^{-1}} t_2^{l(\square)+1} t_3^{-a(\square)} \\
&= \sum_{\square\in\lambda} (1+t_1^{-1}+\cdots+t_1^{-2a(\square)}) t_2^{l(\square)+1} t_3^{-a(\square)} - (t_1+\cdots+t_1^{2a(\square)+1}) t_2^{-l(\square)} t_3^{a(\square)+1}.
\end{aligned}$$

and conclude the following.

$\sigma=\sigma(r_1,r_2,r_3)$	$\mathrm{ind}_\sigma(E_{\lambda,(0,-2)})$
$r_1>>r_2>0>>r_3$	$\lvert\lambda\rvert$
$r_2>>r_1>0>>r_3$	$\lvert\lvert\lambda\rvert\rvert^2$
$r_1>>0>r_2>>r_3$	$-\lvert\lambda\rvert$
$r_2>>0>r_1>>r_3$	$\lvert\lvert\lambda\rvert\rvert^2$

By (4.1.12), we have $\chi(\lambda,(0,-2))=||\lambda||^2$. We conclude the following.

Proposition 4.2.2. *The limits of the edge contribution* $\hat{E}(\lambda,(0,-2))$ *under preferred slopes are as follows.*

$\sigma=\sigma(r_1,r_2,r_3)$	$\hat{E}(\lambda,(0,-2))^\sigma$
$r_1>>r_2>0>>r_3$	$q^{\frac{\lvert\lvert\lambda\rvert\rvert^2+\lvert\lambda\rvert}{2}}t^{\frac{\lvert\lvert\lambda\rvert\rvert^2-\lvert\lambda\rvert}{2}}$
$r_2>>r_1>0>>r_3$	$q^{\lvert\lvert\lambda\rvert\rvert^2}$
$r_1>>0>r_2>>r_3$	$q^{\frac{\lvert\lvert\lambda\rvert\rvert^2-\lvert\lambda\rvert}{2}}t^{\frac{\lvert\lvert\lambda\rvert\rvert^2+\lvert\lambda\rvert}{2}}$
$r_2>>0>r_1>>r_3$	$q^{\lvert\lvert\lambda\rvert\rvert^2}$
$r_3>>0>r_2>>r_1$	$q^{\frac{\lvert\lvert\lambda\rvert\rvert^2-\lvert\lambda\rvert}{2}}t^{\frac{\lvert\lvert\lambda\rvert\rvert^2+\lvert\lambda\rvert}{2}}$
$r_3>>0>r_1>>r_2$	$t^{\lvert\lvert\lambda\rvert\rvert^2}$
$r_3>>r_2>0>>r_1$	$q^{\frac{\lvert\lvert\lambda\rvert\rvert^2+\lvert\lambda\rvert}{2}}t^{\frac{\lvert\lvert\lambda\rvert\rvert^2-\lvert\lambda\rvert}{2}}$
$r_3>>r_1>0>>r_2$	$t^{\lvert\lvert\lambda\rvert\rvert^2}$

4.2.3. Vertex limits. We recall from [**NO**, Sec. 8] how to write $\mathrm{ind}_\sigma(V_\pi(t_1,t_2,t_3))$ for preferred σ as a sum over the boxes of π.

Some notation is needed before we introduce this formula for the index. Fix a two-dimensional partition $\nu\subset\mathbb{Z}^2_{\geq 0}$, and let c_i^+ and c_i^- denote the ordered values of b_2-b_1 when (b_1,b_2) are among the inner and outer corners of ν, respectively; see figure 4.5

To be precise, if $n_1>\ldots>n_{d(\nu)}$ are the distinct parts occuring in ν, and if ν is the partition in which the part n_i occurs with nonzero multiplicity m_i, then for $i=0,\cdots,d(\nu)$, we set

$$c_i^+=n_{i+1}-\sum_{j=1}^{i}m_j,$$

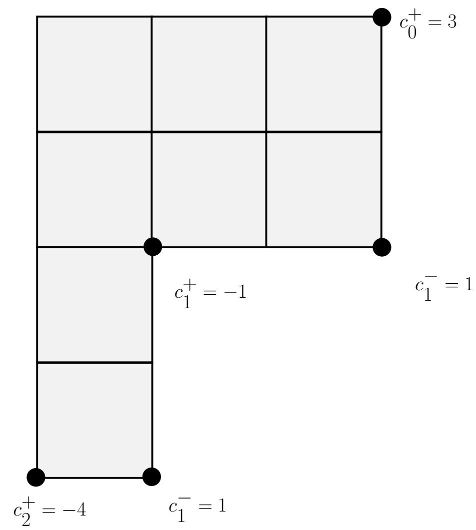

FIGURE 4.5. The two-dimensional partition $\lambda = (3, 3, 1, 1)$ with corners labeled

(here $n_{d(\nu)} + 1 = 0$), and for $i = 1, \ldots, d(\nu)$, we set

$$c_i^- = n_i - \sum_{j=1}^{i} m_j.$$

In particular, in the special case $\nu = \varnothing$, the only corner is $c_0^+ = 0$.

We then define a function $\psi_\nu : \mathbb{Z}^3_{\geq 0} \to \{\pm 1\}$ as follows:

$$\psi_\nu(b_1, b_2, b_3) = \begin{cases} 1 & \text{if } b_2 - b_1 \geq c_0^+ \\ & \text{or } c_i^- > b_2 - b_1 \geq c_i^+ \text{ for some } i \\ -1 & \text{if } c_i^+ > b_2 - b_1 \geq c_{i+1}^- \\ & \text{or } c_{d(\nu)}^+ > b_2 - b_1 \text{ for some } i \end{cases}$$

In particular, when ν is empty, we have

$$\psi_\varnothing(b_1, b_2, b_3) = \begin{cases} 1 & \text{if } b_2 - b_1 \geq 0 \\ -1 & \text{if } b_2 - b_1 < 0 \end{cases}$$

Then, [**NO**, Theorem 2] furnishes a formula for $\operatorname{ind}_\sigma(\pi)$ for preferred slopes σ.

Theorem 4.2.3. [**NO**, Theorem 2] *For $r_1 >> r_3 > 0 >> r_2$, we have*

$$\mathrm{ind}_{\sigma(r_1,r_2,r_3)}(V_\pi(t_1,t_2,t_3)) = \sum_{\{\square\in\pi \ | \ \square\notin\pi\langle 3\rangle\}} \psi_\nu(\square) - \sum_{\square\in\pi\langle 1\rangle} \psi_0(\square) - \sum_{\square\in\pi\langle 3\rangle} \psi_0(\square). \quad (4.2.2)$$

As $\psi_\nu(b_1,b_2,b_3) = \psi_0(b_1,b_2,b_3)$ when $|b_2 - b_1| >> 0$, the sum in (4.2.2) is finite. Analagous results of Theorem 4.2.3 hold for any preferred slope σ, and when t_1, t_2, t_3 are replaced by arbitrary w_{i_1}, w_{i_2} and w_{i_3} satisfying $w_{i_1}w_{i_2}w_{i_3} = \kappa$.

Note the similarity between (4.1.11) and (4.2.2). Define $\Psi_\nu : \mathbb{Z}^3_{\geq 0} \to \{q,t\}$ to be the composition of ψ_ν with the function

$$\{1,-1\} \to \{t,q\}$$

sending $1 \mapsto t$ and $-1 \mapsto q$. For $N > 0$, set

$$B_N = \{(b_1,b_2,b_3)| \ 0 \leq b_k \leq N \text{ for } k = 1,2,3\}.$$

Combining (4.1.11) and (4.2.2), we obtain the following.

Corollary 4.2.4. *Let $\sigma = (r_1,r_2,r_3)$. Then, for $r_1 >> r_3 > 0 >> r_2$ and $N >> 0$, we have*

$$Q^{\chi(\pi)}\hat{a}(V_\pi(t_1,t_2,t_3))^{\sigma(r_1,r_2,r_3)} = \frac{\prod_{\{\square\in\pi\cap B_N \ | \ \square\notin\pi\langle 3\rangle\}} \Psi_\nu(\square)}{\prod_{\square\in B_N\cap\pi\langle 1\rangle} \Psi_0(\square) \prod_{\square\in B_N\cap\pi\langle 2\rangle} \Psi_0(\square)}. \quad (4.2.3)$$

If $\pi \in P(\lambda,\mu,\nu)$, then for $N >> 0$, we have

$$\prod_{\square\in B_N\cap\pi\langle 1\rangle} \Psi_0(\square) = q^{N|\lambda|}t^{|\lambda|}t^{\frac{||\lambda^t||^2-|\lambda|}{2}},$$

and

$$\prod_{\square\in B_N\cap\pi\langle 2\rangle} \Psi_0(\square) = t^{N|\mu|}t^{|\mu|}q^{\frac{||\mu||^2-|\mu|}{2}}.$$

The limit $\hat{V}(\lambda,\mu,\nu)^\sigma$ can then be computed by summing (4.2.3) over all partitions $\pi \in P(\lambda,\mu,\nu)$. We remark that while there are no values of (r_1,r_2,r_3) and N for which (4.2.3) holds simultaneously for all $\pi \in P(\lambda,\mu,\nu)$, for any fixed $m \in \mathbb{Z}$ one may choose R_m, R'_m and $N_m \in \mathbb{Z}$ such that for any $\pi \in P(\lambda,\mu,\nu)$, $(r_1,r_2,r_3) \in \mathbb{Z}^3$ and $N > 0$ satisfying

$$\chi(\pi) \leq m, \ r_1 > R_m > R'_m > r_3 > 0 > r_2, \ \textit{and } N > N_m,$$

the equation (4.2.3) holds. In plainer language, any individual term of the limit

$$\hat{V}(\lambda,\mu,\nu)^{\sigma(r_1,r_2,r_3)}$$

stabilizes as $r_1 >> r_3 > 0 >> r_2$, and can be computed using (4.2.3).

Applying (4.2.3), when $r_1 >> r_3 > 0 >> r_2$ we have

$$\hat{V}(\lambda,\mu,\nu)^{\sigma(r_1,r_2,r_3)} = \sum_{\pi\in P(\lambda,\mu,\nu)} \lim_{N\to\infty} \left(\frac{t^{-\frac{||\lambda^t||^2-|\lambda|}{2}} q^{-\frac{||\mu||^2-|\mu|}{2}}}{q^{N|\lambda|}t^{|\lambda|}t^{N|\mu|}t^{|\mu|}} \prod_{\{\square\in\pi\cap B_N \ | \ \square\notin\pi\langle 3\rangle\}} \Psi_\nu(\square) \right). \tag{4.2.4}$$

A general framework for evaluating such sums was introduced in [**OR**]. Namely, given a partition $\pi \in P(\lambda,\mu,\nu)$, the configuration $\pi/\pi\langle 3\rangle$ can be regarded as a sequence of two-dimensional partitions whose interlacing behavior can be read off from ν. Sums of the form (4.2.4) can then be evaluated using a transfer matrix approach, where each such two-dimensional partition is assigned a weight using Ψ_ν.

This procedure is carried out in [**IKV**, Eqn (150)] and yields the answer

$$\sum_{\pi\in P(\lambda,\mu,\nu)} \lim_{N\to\infty} \frac{\prod_{\{\square\in\pi\cap B_N,\square\notin\pi\langle 3\rangle\}} \Psi_\nu(\square)}{q^{N|\lambda|}t^{|\lambda|}t^{N|\mu|}t^{|\mu|}} = \frac{t^{\frac{-|\lambda|}{2}} q^{\frac{-|\mu|}{2}} \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|}{2}} s_{\lambda/\eta}(t^{-\rho}q^{-\nu^t}) s_{\mu^t/\eta}(q^{-\rho}t^{-\nu})}{\prod_{i,j\geq 0}(1-q^i t^{j+1}) \prod_{\square\in\nu}(1-q^{l(\square)}t^{a(\square)+1})};$$

here $t^{-\rho}$ denotes the sequence

$$(t^{1/2}, t^{3/2}, t^{5/2}, \cdots),$$

the symbol $t^{-\lambda}$ denotes the sequence $(t_{\lambda_1}, t_{\lambda_2}, \cdots,)$, products of sequences are taken termwise. The symbols $s_{\lambda/\eta}$ denote skew Schur polynomials; see, for example, [**IK-P**, App. A] for a concise introduction.

After incorporating the remaining factor

$$t^{-\frac{||\lambda^t||^2-|\lambda|}{2}} q^{-\frac{||\mu||^2-|\mu|}{2}},$$

we deduce that for $r_1 >> r_3 > 0 >> r_2$, we have

$$\hat{V}(\lambda,\mu,\nu)^{\sigma(r_1,r_2,r_3)} = \frac{t^{\frac{-||\lambda^t||^2}{2}} q^{-\frac{||\mu||^2}{2}} \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|}{2}} s_{\lambda/\eta}(t^{-\rho}q^{-\nu^t}) s_{\mu^t/\eta}(q^{-\rho}t^{-\nu})}{\prod_{i,j\geq 0}(1-q^i t^{j+1}) \prod_{\square\in\nu}(1-q^{l(\square)}t^{a(\square)+1})}.$$

In particular, we have

$$\hat{V}(\varnothing,\varnothing,\varnothing)^{\sigma(r_1,r_2,r_3)} = \frac{1}{\prod_{i,j\geq 0}(1-q^i t^{j+1})}; \tag{4.2.5}$$

Note that this equality also follows directly from Nekrasov's formula (3.1.1).

In the same manner, or using symmetries of the expression $\hat{V}(\lambda,\mu,\nu)$, we can evaluate the limit for any preferred slope.

Set

$$C(\lambda,\mu,\nu)(t,q) = \frac{t^{-\frac{||\lambda^t||^2}{2}} q^{-\frac{||\mu||^2}{2}}}{\prod\limits_{\square\in\nu}(1-q^{l(\square)}t^{a(\square)+1})} \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|}{2}} s_{\lambda/\eta}(t^{-\rho}q^{-\nu^t}) s_{\mu^t/\eta}(q^{-\rho}t^{-\nu});$$

up to a prefactor, this is the refined topological vertex used in [**IKV**]. We conclude the following.

Proposition 4.2.5. *The limits of the normalized vertex contribution $\hat{V}'(\lambda,\mu,\nu)$ under preferred slopes σ are as follows.*

$\sigma = \sigma(r_1,r_2,r_3)$	$\hat{V}'(\lambda,\mu,\nu)^{\sigma}$
$r_1 >> r_3 > 0 >> r_2$	$C(\lambda,\mu,\nu)(t,q)$
$r_1 >> 0 > r_3 >> r_2$	$C(\mu^t,\lambda^t,\nu^t)(q,t)$
$r_2 >> r_3 > 0 >> r_1$	$C(\mu^t,\lambda^t,\nu^t)(t,q)$
$r_2 >> 0 > r_3 >> r_1$	$C(\lambda,\mu,\nu)(q,t)$

Note that there are four different choices of preferred slope corresponding to a given choice of preferred direction.

CHAPTER 5

Dualities for tautological classes on $(\mathbb{C}^2)^{[n]}$

In this chapter, we prove Theorem 1.1.2. For now, let X either of the toric Calabi-Yau threefolds X_1 and X_2 given by the toric diagrams with action of a three-dimensional torus T as labeled in Figure 5.1 and Kähler parameters

$$m_1,\ m_2,\ m_3,\ m_4,\ u,\ v,$$

indexing generators of $H_2(X;\mathbb{Z})^{\text{eff}}$ as labeled in Figure 5.2.

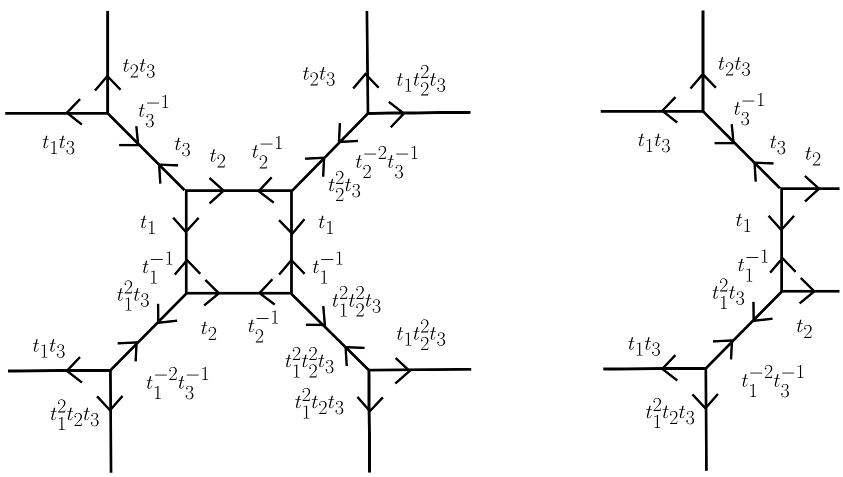

FIGURE 5.1. The Calabi-Yau threefolds X_1 and X_2, respectively, with T-weights labeled

As above, let $\tilde{T}$ be the minimal cover on which the character $(t_1t_2t_3)^{1/2}$ is defined and let $A \subset T$ be the Calabi-Yau subtorus; then $\tilde{T}$ acts on X via the map $\tilde{T} \to T$. Note that as the weight $t_1t_2t_3$ vanishes on the Calabi-Yau torus, for our purposes the distinction between $\tilde{T}$ and T is not particularly important. Any weight $w \in \tilde{T}^\vee$ for which

$$DT(X)^{\tilde{T}_w} \neq DT(X)^{\tilde{T}}$$

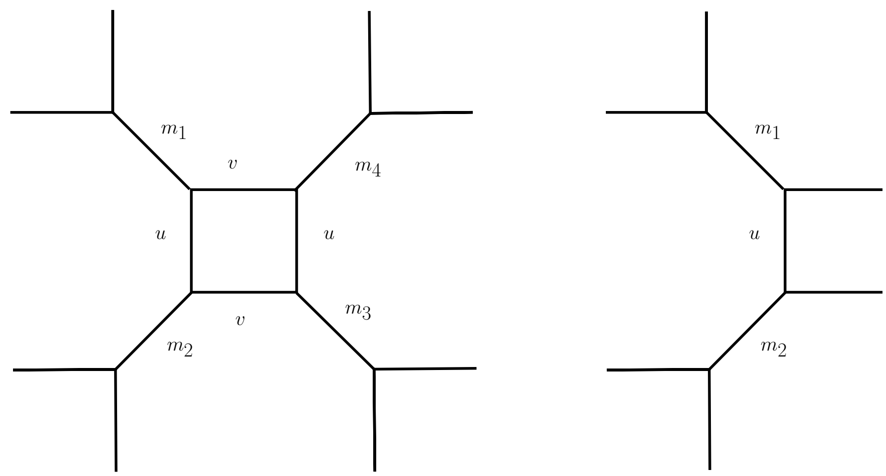

FIGURE 5.2. The Calabi-Yau threefolds X_1 and X_2, respectively, with Kähler parameters labeled

then must be of the form

$$t_1^i(t_1t_2t_3)^{j/2},\ t_2^i(t_1t_2t_3)^{j/2},\ t_3^i(t_1t_2t_3)^{j/2},\ (t_1t_2^{-1})^i(t_1t_2t_3)^{j/2}$$

for integers i, j. Among these, when w is of the form

$$t_3^i(t_1t_2t_3)^{j/2},\ (t_1t_2^{-1})^i(t_1t_2t_3)^{j/2},$$

then any curve in $DT(X)^{\tilde{T}_w}$ must be supported on a proper set given by a union of points and rational curves. For such weights w, the fixed locus $DT(X)^{\tilde{T}_w}$ is proper. We conclude that the only possible non-compact directions of $DT(X)$ are

$$t_1^{\pm 1}(t_1t_2t_3)^{j/2},\ t_2^{\pm 1}(t_1t_2t_3)^{j/2}.$$

Let M be a component of $DT(X)$, and $\sigma(r_1, r_2, r_3)$ be the one-parameter subgroup $\mathbb{C}^\times \to A$ given by

$$z \mapsto (z^{r_1}, z^{r_2}, z^{r_3}).$$

For generic (r_1, r_2, r_3), we apply Theorem 1.2.1 to deduce that the value of the limit

$$\chi(M, \tilde{\mathcal{O}}^{\mathrm{vir}})^{\sigma(r_1,r_2,r_3)}$$

depends only on the signs of r_1 and r_2.

The same argument holds when $DT(X)$ is replaced by the Hilbert scheme of points $DT_0(X)$; this can also be seen directly from Nekrasov's formula, or its corollary (4.2.5). We conclude the following.

Corollary 5.0.1. *For $r_3 >> r_2 > 0 >> r_1$ and $s_2 >> 0 > s_1 >> s_3$, we have*

$$Z'_{DT}(X)^{\sigma(r_1,r_2,r_3)} = Z'_{DT}(X)^{\sigma(s_1,s_2,s_3)}. \tag{5.0.1}$$

While sufficient to establish our results for Hilbert schemes, we remark that our notion of non-compact weight does not produce the strongest expected results of the form of Corollary 5.0.1 for arbitrary toric Calabi-Yau threefolds. Namely, if one assumes the K-theoretic DT/stable pairs correspondence ([**NO**, (16)]), then non-compact weights may be replaced by weights corresponding to directions in which curves in X may escape to infinity. This newer notion makes no difference for the particular threefolds considered in this thesis, but allows one to conclude stronger versions of Corollary 5.0.1 for other examples. For example, the space of effective curves in $\mathcal{K}_{\mathbb{P}^1\times\mathbb{P}^1}$ is compact; the K-theoretic DT/PT correspondence would then imply that, for generic σ, the limit $Z'_{DT}(\mathcal{K}_{\mathbb{P}^1\times\mathbb{P}^1})^{\sigma}$ does not depend on σ.

5.1. Evaluation of limits

To apply Corollary 5.0.1, we evaluate the limits (5.0.1).

A general prescription for performing such computations inolving the topological vertex is presented in [**IK-P**], and works for the refined topological vetex as well. The strategy is to make repeated use of the following identities involving skew Schur polynomials:

$$\sum_\lambda u^{|\lambda|} s_{\lambda/\eta_1}\big((x_i)\big) s_{\lambda/\eta_2}\big((y_j)\big) = \frac{1}{\displaystyle\prod_{i,j}(1-ux_iy_j)} \sum_\lambda u^{|\lambda|} s_{\eta_1/\lambda}\big((uy_j)\big) s_{\eta_2/\lambda}\big((ux_i)\big),$$

$$\sum_\lambda u^{|\lambda|} s_{\lambda/\eta_1}\big((x_i)\big) s_{\lambda^t/\eta_2}\big((y_j)\big) = \prod_{i,j}(1+ux_iy_j) \sum_\lambda u^{|\lambda|} s_{\eta_1^t/\lambda^t}\big((uy_j)\big) s_{\eta_2^t/\lambda^t}\big((ux_i)\big);$$

these particular identities may be found, for example, in [**Taka**].

Adopting notation from [**IK-P**], for two sequences $x = (x_0, x_1, \cdots), y = (y_0, y_1, \cdots)$ and a variable m, we set

$$\{x, y\}_m = \prod_{i,j\geq 0} (1 + m x_i y_j).$$

We begin by dividing the toric 1-skeleton of X_1 into its 8 trivalent vertices and 8 bounded edges. The computations in Propositions 4.2.1, 4.2.2, and 4.2.5 enable us to evaluate the corresponding limits of (4.1.15).

Let

$$\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4 \tag{5.1.1}$$

be the two-dimensional partitions assigned to the bounded edges as indicated in Figure 5.3.

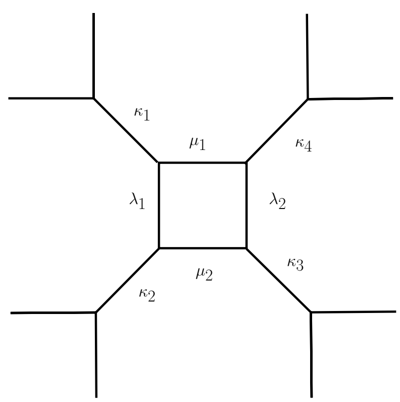

FIGURE 5.3. The Calabi-Yau threefold X_1 with two-dimensional partitions assigned to edges

Then, for $\sigma(r_1, r_2, r_3)$ with $r_3 >> r_2 > 0 >> r_1$, the corresponding contribution to the limit $Z'_{DT}(X)^{\sigma(r_1,r_2,r_3)}$ is the product of the following sixteen terms:

$$C(\kappa_1^t,\varnothing,\varnothing)(q,t)=q^{-\frac{||\kappa_1||^2}{2}}s_{\kappa_1^t}(q^{-\rho}),$$

$$m_1^{|\kappa_1|}t^{\frac{||\kappa_1^t||^2}{2}}q^{\frac{-||\kappa_1||^2}{2}},$$

$$v^{|\mu_1|}q^{||\mu_1||^2},$$

$$C(\kappa_1,\lambda_1,\mu_1^t)(t,q)=\frac{t^{\frac{-||\kappa_1^t||^2}{2}}q^{\frac{-||\lambda_1||^2}{2}}}{\prod_{\square\in\mu_1^t}1-t^{a(\square)+1}q^{l(\square)}}\sum_{\eta_1}\left(\frac{q}{t}\right)^{\frac{|\eta_1|}{2}}s_{\kappa_1/\eta_1}(t^{-\rho}q^{-\mu_1})s_{\lambda_1^t/\eta_1}(q^{-\rho}t^{-\mu_1^t}),$$

$$u^{|\lambda_1|}t^{\frac{||\lambda_1||^2+|\lambda_1|}{2}}q^{\frac{||\lambda_1||^2-|\lambda_1|}{2}},$$

$$m_2^{|\kappa_2|}t^{\frac{||\kappa_2^t||^2}{2}}q^{\frac{||\kappa_2||^2}{2}}$$

$$C(\varnothing,\kappa_2^t,\varnothing)(q,t)=t^{-\frac{||\kappa_2^t||^2}{2}}s_{\kappa_2}(t^{-\rho})$$

$$C(\lambda_1^t,\kappa_2,\mu_2)(t,q)=\frac{t^{\frac{-||\lambda_1||^2}{2}}q^{\frac{-||\kappa_2||^2}{2}}}{\prod_{\square\in\mu_2}1-t^{a(\square)+1}q^{l(\square)}}\sum_{\eta_2}\left(\frac{q}{t}\right)^{\frac{|\eta_2|}{2}}s_{\lambda_1^t/\eta_2}(t^{-\rho}q^{-\mu_2^t})s_{\kappa_2^t/\eta_2}(q^{-\rho}t^{-\mu_2}),$$

$$v^{|\mu_2|}t^{||\mu_2||^2},$$

$$m_3^{|\kappa_3|}t^{\frac{||\kappa_3||^2}{2}}q^{\frac{||\kappa_3^t||^2}{2}}$$

$$C(\kappa_3^t,\varnothing,\varnothing)(t,q)=t^{-\frac{||\kappa_3||^2}{2}}s_{\kappa_3^t}(t^{-\rho}),$$

$$C(\kappa_3,\lambda_2,\mu_2^t)(q,t)=\frac{q^{\frac{-||\kappa_3^t||^2}{2}}t^{\frac{-||\lambda_2||^2}{2}}}{\prod_{\square\in\mu_2^t}1-q^{a(\square)+1}t^{l(\square)}}\sum_{\eta_3}\left(\frac{t}{q}\right)^{\frac{|\eta_3|}{2}}s_{\kappa_3/\eta_3}(q^{-\rho}t^{-\mu_2})s_{\lambda_2^t/\eta_3}(t^{-\rho}q^{-\mu_2^t}),$$

$$u^{|\lambda_2|}t^{\frac{||\lambda_2||^2-|\lambda_2|}{2}}q^{\frac{||\lambda_2||^2+|\lambda_2|}{2}},$$

$$m_4^{|\kappa_4|}t^{\frac{||\kappa_4||^2}{2}}q^{\frac{||\kappa_4^t||^2}{2}},$$

$$C(\varnothing,\kappa_4^t,\varnothing)(t,q)=q^{-\frac{||\kappa_4^t||^2}{2}}s_{\kappa_4}(q^{-\rho}),$$

$$C(\lambda_2^t,\kappa_4,\mu_1)(q,t)=\frac{q^{\frac{-||\lambda_2||^2}{2}}t^{\frac{-||\kappa_4||^2}{2}}}{\prod_{\square\in\mu_1}1-q^{a(\square)+1}t^{l(\square)}}\sum_{\eta_4}\left(\frac{t}{q}\right)^{\frac{|\eta_4|}{2}}s_{\lambda_2^t/\eta_4}(q^{-\rho}t^{-\mu_1^t})s_{\kappa_4^t/\eta_4}(t^{-\rho}q^{-\mu_1}).$$

The limit $Z'_{DT}(X)^{\sigma(r_1,r_2,r_3)}$ will be the sum of this product over all possible choices of the two-dimensional partitions given in (5.1.1). We first sum over the partitions κ_i labeling the outer edges, yielding

$$\sum_{\kappa_1} m_1^{|\kappa_1|} s_{\kappa_1^t}(q^{-\rho}) s_{\kappa_1/\eta_1}(t^{-\rho}q^{-\mu_1}) = \{t^{-\rho}q^{-\mu_1}, q^{-\rho}\}_{m_1} \cdot s_{\eta_1^t}(m_1 q^{-\rho}),$$

$$\sum_{\kappa_2} m_2^{|\kappa_2|} s_{\kappa_2}(t^{-\rho}) s_{\kappa_2^t/\eta_2}(q^{-\rho}t^{-\mu_2}) = \{q^{-\rho}t^{-\mu_2}, t^{-\rho}\}_{m_2} \cdot s_{\eta_2^t}(m_2 t^{-\rho}),$$

$$\sum_{\kappa_3} m_3^{|\kappa_3|} s_{\kappa_3}(t^{-\rho}) s_{\kappa_3^t/\eta_3}(q^{-\rho}t^{-\mu_3}) = \{q^{-\rho}t^{-\mu_3}, t^{-\rho}\}_{m_3} \cdot s_{\eta_3^t}(m_3 t^{-\rho}),$$

and

$$\sum_{\kappa_4} m_4^{|\kappa_4|} s_{\kappa_4}(q^{-\rho}) s_{\kappa_4^t/\eta_4}(t^{-\rho}q^{-\mu_1}) = \{t^{-\rho}q^{-\mu_1}, q^{-\rho}\}_{m_4} \cdot s_{\eta_4^t}(m_4 q^{-\rho}).$$

Next, we sum over the partitions λ_i. Incorporating the $(t/q)^{|\lambda_1|/2}$ and $(q/t)^{|\lambda_2|/2}$ factor from the edge contributions into the arguments of the skew-Schur polynomials, we get

$$\sum_{\lambda_1} u^{|\lambda_1|} s_{\lambda_1^t/\eta_1}\left(\sqrt{\frac{t}{q}} q^{-\rho} t^{-\mu_1^t}\right) s_{\lambda_1^t/\eta_2}(t^{-\rho}q^{-\mu_2^t})$$
$$= \frac{\sum_{\lambda_1} u^{|\lambda_1|} s_{\eta_2/\lambda_1^t}\left(u\sqrt{\frac{t}{q}} q^{-\rho} t^{\mu_1^t}\right) s_{\eta_1/\lambda_1^t}(u t^{-\rho} q^{-\mu_2^t})}{\left\{\sqrt{\frac{t}{q}} q^{-\rho} t^{-\mu_1^t}, t^{-\rho} q^{-\mu_2^t}\right\}_{-u}},$$

and

$$\sum_{\lambda_2} u^{|\lambda_2|} s_{\lambda_2^t/\eta_3}\left(\sqrt{\frac{q}{t}} t^{-\rho} q^{-\mu_2^t}\right) s_{\lambda_2^t/\eta_4}(q^{-\rho}t^{-\mu_1^t})$$
$$= \frac{\sum_{\lambda_2} u^{|\lambda_2|} s_{\eta_4/\lambda_2^t}\left(u\sqrt{\frac{q}{t}} t^{-\rho} q^{\mu_2^t}\right) s_{\eta_3/\lambda_2^t}(u q^{-\rho} t^{-\mu_1^t})}{\left\{\sqrt{\frac{q}{t}} t^{-\rho} q^{-\mu_2^t}, q^{-\rho} t^{-\mu_1^t}\right\}_{-u}}.$$

We then sum over the ancillary partitions η_i associated to each vertex. We have

$$\sum_{\eta_1} s_{\eta_1^t}(m_1 q^{-\rho}) s_{\eta_1/\lambda_1^t}(ut^{-\rho}q^{-\mu_2^t}) = \{t^{-\rho}q^{-\mu_2^t}, q^{-\rho}\}_{m_1 u} \cdot s_{\lambda_1}(m_1 q^{-\rho}),$$

$$\sum_{\eta_2} s_{\eta_2^t}\Big(m_2\sqrt{\frac{q}{t}}t^{-\rho}\Big) s_{\eta_2/\lambda_1^t}\Big(u\sqrt{\frac{t}{q}}q^{-\rho}t^{-\mu_1^t}\Big) = \{q^{-\rho}t^{-\mu_1^t}, q^{-\rho}\}_{m_2 u} \cdot s_{\lambda_1}\Big(m_2\sqrt{\frac{q}{t}}q^{-\rho}\Big),$$

$$\sum_{\eta_3} s_{\eta_3^t}(m_3 t^{-\rho}) s_{\eta_3/\lambda_2^t}(uq^{-\rho}t^{-\mu_1^t}) = \{q^{-\rho}t^{-\mu_1^t}, t^{-\rho}\}_{m_3 u} \cdot s_{\lambda_2}(m_3 t^{-\rho}),$$

$$\sum_{\eta_4} s_{\eta_4^t}\Big(m_4\sqrt{\frac{t}{q}}q^{-\rho}\Big) s_{\eta_4/\lambda_2^t}\Big(u\sqrt{\frac{q}{t}}t^{-\rho}q^{-\mu_2^t}\Big) = \{t^{-\rho}q^{-\mu_2^t}, q^{-\rho}\}_{m_4 u} \cdot s_{\lambda_2}(m_4\sqrt{\frac{t}{q}}q^{-\rho}).$$

Finally, we once again sum over λ_1, and λ_2, yielding

$$\sum_{\lambda_1} u^{|\lambda_1|} s_{\lambda_1}(m_1 q^{-\rho}) s_{\lambda_1}\Big(m_2\sqrt{\frac{q}{t}}t^{-\rho}\Big) = \frac{1}{\Big\{\sqrt{\frac{q}{t}}t^{-\rho}, q^{-\rho}\Big\}_{-um_1m_2}}$$

and

$$\sum_{\lambda_2} u^{|\lambda_2|} s_{\lambda_2}(m_3 t^{-\rho}) s_{\lambda_2}\Big(m_4\sqrt{\frac{t}{q}}q^{-\rho}\Big) = \frac{1}{\Big\{\sqrt{\frac{t}{q}}q^{-\rho}, t^{-\rho}\Big\}_{-um_3m_4}}.$$

We conclude that for $r_3 >> r_2 > 0 >> r_1$, the series $Z'_{DT}(X_1)^{\sigma(r_1,r_2,r_3)}$ approaches

$$\sum_{\mu_1,\mu_2} v^{|\mu_1|+|\mu_2|}\frac{V_N}{V_D}, \tag{5.1.2}$$

where

$$\begin{aligned} V_N = &\{t^{-\rho}q^{-\mu_1}, q^{-\rho}\}_{m_1}\{q^{-\rho}t^{-\mu_2}, t^{-\rho}\}_{m_2}\{q^{-\rho}t^{-\mu_2}, t^{-\rho}\}_{m_3}\{t^{-\rho}q^{-\mu_1}, q^{-\rho}\}_{m_4} \\ &\cdot \{t^{-\rho}q^{-\mu_2^t}, q^{-\rho}\}_{m_1u}, \{q^{-\rho}t^{-\mu_1^t}, q^{-\rho}\}_{m_2u}, \{q^{-\rho}t^{-\mu_1^t}, t^{-\rho}\}_{m_3u}, \{t^{-\rho}q^{-\mu_2^t}, q^{-\rho}\}_{m_4u}, \end{aligned}$$

and

$$\begin{aligned} V_D = &\Big\{\sqrt{\frac{t}{q}}q^{-\rho}t^{-\mu_1^t}, t^{-\rho}q^{-\mu_2^t}\Big\}_{-u}\Big\{\sqrt{\frac{q}{t}}t^{-\rho}q^{-\mu_2^t}, q^{-\rho}t^{-\mu_1^t}\Big\}_{-u} \\ &\cdot \Big\{\sqrt{\frac{q}{t}}t^{-\rho}, q^{-\rho}\Big\}_{-um_1m_2}\Big\{\sqrt{\frac{t}{q}}q^{-\rho}, t^{-\rho}\Big\}_{-um_3m_4}. \end{aligned}$$

The limit $Z'_{DT}(X)^{\sigma(s_1,s_2,s_3)}$ for $s_2 >> 0 > s_1 >> s_3$ can be computed by the same procedure, yielding

$$\sum_{\lambda_1,\lambda_2} u^{|\lambda_1|+|\lambda_2|}\frac{U_N}{U_D}, \tag{5.1.3}$$

where

$$\begin{aligned} U_N =&\{q^{-\rho}t^{-\lambda_1}, t^{-\rho}\}_{m_1}\{q^{-\rho}t^{-\lambda_1}, t^{-\rho}\}_{m_2}\{t^{-\rho}q^{-\lambda_2}, q^{-\rho}\}_{m_3}\{t^{-\rho}q^{-\lambda_2}, q^{-\rho}\}_{m_4} \\ &\cdot \{q^{-\rho}t^{-\lambda_2^t}, t^{-\rho}\}_{m_1 v}, \{q^{-\rho}t^{-\lambda_2^t}, t^{-\rho}\}_{m_2 v}, \{t^{-\rho}q^{-\lambda_1^t}, q^{-\rho}\}_{m_3 v}, \{t^{-\rho}q^{-\lambda_1^t}, q^{-\rho}\}_{m_4 v}, \end{aligned}$$

and

$$\begin{aligned} U_D =&\Big\{\sqrt{\frac{q}{t}}t^{-\rho}q^{-\lambda_1^t}, q^{-\rho}t^{-\lambda_2^t}\Big\}_{-v}\Big\{\sqrt{\frac{t}{q}}q^{-\rho}t^{-\lambda_2^t}, t^{-\rho}q^{-\lambda_1^t}\Big\}_{-v} \\ &\cdot \Big\{\sqrt{\frac{t}{q}}q^{-\rho}, t^{-\rho}\Big\}_{-vm_1m_4}\Big\{\sqrt{\frac{q}{t}}t^{-\rho}, q^{-\rho}\Big\}_{-vm_2m_3}. \end{aligned}$$

Starting with the equality of (5.1.2) and (5.1.3), we perform a series of specializations and substitutions to obtain Theorem 1.1.2. First, set $m_4 = -\sqrt{q/t}$; then when λ is nonempty, we have

$$\{t^{-\rho}q^{-\lambda}, q^{-\rho}\}_{m_4} = 0.$$

Observe that

$$\{q^{-\rho}t^{-\lambda}, t^{-\rho}\}_{m_1'\sqrt{\frac{t}{q}}} = \prod_{\square\in\lambda_1}(1+m_1'q^{b_1}t^{-b_2})\prod_{i,j\geq 0}(1+m_1'q^it^{j+1}). \tag{5.1.4}$$

We substitute

$$m_1 = -m_1'\sqrt{\frac{t}{q}},\ m_2 = -m_2'\sqrt{\frac{t}{q}},\ m_3 = -m_3'\sqrt{\frac{t}{q}},\ u = u'\frac{q}{t},\ v = v'\frac{q}{t}.$$

Using (5.1.4), we may rewrite (5.1.2) as:

$$\begin{aligned} &\left(\sum_{\mu_2}(v')^{|\mu_2|}q^{|\mu_2|}t^{||\mu_2||^2-|\mu_2|}\prod_{\square\in\mu_2}\frac{(1-m_2'q^{b_1}t^{-b_2})(1-m_3'q^{b_1}t^{-b_2})(1-u'm_1'q^{-b_1}t^{b_2})}{(1-q^lt^{a+1})(1-q^{l+1}t^a)(1-u'q^{-b_1}t^{b_2})}\right) \\ &\cdot\prod_{i,j}\frac{(1-u'm_1'q^{i+1}t^j)(1-u'm_2'q^{i+1}t^j)}{(1-u'q^{i+1}t^j)(1-u'm_1'm_2'q^{i+1}t^j)} \\ &\cdot\prod_{i,j}(1-q^{i+1}t^j)(1-m_1'q^it^{j+1})(1-m_2'q^it^{j+1})(1-m_3'q^it^{j+1}) \end{aligned}$$

and we rewrite (5.1.3) as

$$\Big(\sum_{\lambda_1}(u')^{|\lambda_1|}q^{|\lambda_1|}t^{||\lambda_1||^2-|\lambda_1|}\prod_{\square\in\lambda_1}\frac{(1-m_1'q^{b_1}t^{-b_2})(1-m_2'q^{b_1}t^{-b_2})(1-v'm_3'q^{-b_1}t^{b_2})}{(1-q^lt^{a+1})(1-q^{l+1}t^a)(1-v'q^{-b_1}t^{b_2})}\Big)$$
$$\cdot\prod_{i,j}\frac{(1-v'm_2'q^{i+1}t^j)(1-v'm_3'q^{i+1}t^j)}{(1-v'q^{i+1}t^j)(1-v'm_2'm_3'q^{i+1}t^j)}$$
$$\cdot\prod_{i,j}(1-q^{i+1}t^j)(1-m_1'q^it^{j+1})(1-m_2'q^it^{j+1})(1-m_3'q^it^{j+1}).$$

Canceling the factors common to both sides, and redistributing terms, we conclude that the expression

$$\Big(\sum_{\mu_2}(v')^{|\mu_2|}\prod_{\square\in\mu_2}\frac{(1-m_2'q^{b_1}t^{-b_2})(1-m_3'q^{b_1}t^{-b_2})(1-u'm_1'q^{-b_1}t^{b_2})}{(1-q^lt^{a+1})(1-q^{-l-1}t^{-a})(u'-q^{b_1}t^{-b_2})}\Big)$$
$$\cdot\prod_{i,j}\frac{(1-v'q^{i+1}t^j)(1-v'm_2'm_3'q^{i+1}t^j)}{(1-v'm_2'q^{i+1}t^j)(1-v'm_3'q^{i+1}t^j)} \tag{5.1.5}$$

is equal to the expression

$$\Big(\sum_{\lambda_1}(u')^{|\lambda_1|}\prod_{\square\in\lambda_1}\frac{(1-m_1'q^{b_1}t^{-b_2})(1-m_2'q^{b_1}t^{-b_2})(1-v'm_3'q^{-b_1}t^{b_2})}{(1-q^lt^{a+1})(1-q^{-l-1}t^{-a})(v'-q^{b_1}t^{-b_2})}\Big)$$
$$\cdot\prod_{i,j}\frac{(1-u'q^{i+1}t^j)(1-u'm_1'm_2'q^{i+1}t^j)}{(1-u'm_1'q^{i+1}t^j)(1-u'm_2'q^{i+1}t^j)}. \tag{5.1.6}$$

By definition,

$$\prod_{i,j}\frac{(1-v'q^{i+1}t^j)(1-v'm_2'm_3'q^{i+1}t^j)}{(1-v'm_2'q^{i+1}t^j)(1-v'm_3'q^{i+1}t^j)}=\mathrm{Sym}^\bullet\Big(-v'\frac{q(1-m_2')(1-m_3')}{(1-q)(1-t)}\Big).$$

Consequently, after substituting $q=t_1, t=t_2^{-1}$, and expanding as series in t_1^{-1}, t_2^{-1}, we may rewrite (5.1.5) as

$$\frac{F(v',m_2',m_3',m_1')}{\mathrm{Sym}^\bullet\Big(-v'\frac{(1-m_2')(1-m_3')}{(1-t_1^{-1})(1-t_2^{-1})}\Big)};$$

similarly (5.1.6) may be rewritten as

$$\frac{F(u,m_1',m_2',m_3',v)}{\mathrm{Sym}^\bullet\Big(-u'\frac{(1-m_1')(1-m_2')}{(1-t_1^{-1})(1-t_2^{-1})}\Big)}.$$

For now we assume Proposition 1.1.3, which asserts that

$$\mathrm{Sym}^\bullet\bigg(- v' \frac{(1-m_2')(1-m_3')q}{(1-q)(1-t)}\bigg) = F(v', m_2', m_3', m_1', 0);$$

after relabeling variables, we obtain

$$\frac{F(v,m_2,m_3,m_1,u)}{F(v,m_2,m_3,m_1,0)} = \frac{F(u,m_1,m_2,m_3,v)}{F(u,m_1,m_2,m_3,0)}.$$

By definition, the expression $F(v, m_2, m_3, m_1, u)$ is symmetric under $m_2 \leftrightarrow m_3$. We conclude that

$$\begin{aligned}\frac{F(v,m_2,m_3,m_1,u)}{F(v,m_2,m_3,m_1,u)} &= \frac{F(u,m_1,m_2,m_3,v)}{F(u,m_1,m_2,m_3,0)}\\ &= \frac{F(u,m_2,m_1,m_3,v)}{F(u,m_2,m_1,m_3,0)}\\ &= \frac{F(v,m_1,m_3,m_2,u)}{F(v,m_1,m_3,m_2,0)}\\ &= \frac{F(v,m_3,m_1,m_2,u)}{F(v,m_3,m_1,m_2,0)}\\ &= \frac{F(u,m_2,m_3,m_1,v)}{F(u,m_2,m_3,m_1,0)},\end{aligned}$$

so that Theorem 1.1.2 will follow from Proposition 1.1.3.

Proposition 1.1.3 is then the result of the above procedure applied to the Calabi-Yau threefold X_2 with Kähler parameters as labeled in 5.2; the calculation in this case requires fewer steps. Let λ the the two-dimensional partition assigned to the bounded vertical edge of the diagram. Under the substitutions

$$m_1 = -m_1'\sqrt{\frac{t}{q}}, \quad m_2 = -m_2'\sqrt{\frac{t}{q}}, \quad u = u'\frac{q}{t}$$

the limit $Z'_{DT}(X_2)^{\sigma(r_1,r_2,r_3)}$ as $r_3 >> r_2 > 0 >> r_1$ becomes:

$$\prod_{i,j} \frac{(1-m_1'q^it^{j+1})(1-m_2'q^it^{j+1})(1-m_1'u'q^{i+1}t^j)(1-m_2'u'q^{i+1}t^j)}{(1-u'q^{i+1}t^j)(1-u'm_1'm_2'q^{i+1}t^j)},$$

while the limit $Z'_{DT}(X_2)^{\sigma(s_1,s_2,s_3)}$ as $s_2 >> 0 > s_1 >> s_3$ becomes

$$\left(\sum_\lambda (u')^{|\lambda|} q^{|\lambda|} t^{||\lambda||^2-|\lambda|} \prod_{\square\in\lambda} \frac{(1-m_1' q^{b_1} t^{-b_2})(1-m_2' q^{b_1} t^{-b_2})}{(1-q^l t^{a+1})(1-q^{l+1}t^a)}\right)$$
$$\cdot \prod_{i,j} (1-m_1' q^i t^{j+1})(1-m_2' q^i t^{j+1}).$$

Applying Corollary 5.0.1, we deduce that

$$\sum_\lambda (u')^{|\lambda|} \prod_\lambda \frac{(1-m_1' q^{b_1} t^{-b_2})(1-m_2' q^{b_1} t^{-b_2})}{(1-q^l t^{a+1})(1-q^{l+1}t^a)(-q^{b_1}t^{-b_2})} = \prod_{i,j} \frac{(1-u'm_1' q^{i+1} t^j)(1-u'm_2' q^{i+1} t^j)}{(1-u' q^{i+1} t^j)(1-u' m_1' m_2' q^{i+1} t^j)}.$$

By definition,

$$\prod_{i,j} \frac{(1-u'm_1' q^{i+1} t^j)(1-u'm_2' q^{i+1} t^j)}{(1-u' q^{i+1} t^j)(1-u' m_1' m_2' q^{i+1} t^j)} = \mathrm{Sym}^\bullet\left(u' \frac{q(1-m_1')(1-m_2')}{(1-q)(1-t)}\right),$$

after performing the same substition $q = t_1, t = t_2^{-1}$, and expanding as a series in t_1^{-1}, t_2^{-1}, we obtain Proposition 1.1.3. Theorem 1.1.2 then follows, as well.

CHAPTER 6

Tautological bundles on $S^{[n]}$

Let S be a complex projective surface and $\mathcal{L}$ be a line bundle on S. In this chapter, we use Theorem 1.1.2 and Proposition 1.1.3 to study the holomorphic Euler characteristics of certain Schur functors of the tautological bundles $\mathcal{L}^{[n]}$ on $S^{[n]}$.

Our approach has three steps. First, we use the cobordism techniques of Ellingsrud-Göttsche-Lehn to reduce to the case where S is a toric surface. We then use equivariant localization to reduce to the case where projective S is replaced by $\mathbb{C}^2$, equipped with a torus action and an *equivariant* line bundle $\mathcal{L}$; this line bundle is necessarily a trivial bundle twisted by a torus character; similar strategies have been employed in [**CO, GK1, GK2**]. Finally, we analyze this special case using the results of the previous chapter.

6.1. Reduction to toric surfaces

We introduce formal parameters m, q, and y, and define the series

$$\chi_\Lambda(S,\mathcal{L}) = \sum_{k,n\geq 0} q^n m^k \chi(S^{[n]}, \Lambda^k \mathcal{L}^{[n]}), \quad \chi_{\mathrm{Sym}}(S,\mathcal{L}) = \sum_{k,n\geq 0} q^n y^k \chi(S^{[n]}, \mathrm{Sym}^k \mathcal{L}^{[n]}).$$

For each n, the Hirzebruch-Riemann-Roch theorem implies that the q^n-coefficients of χ_Λ and χ_{Sym} can be written as integrals of expressions that factor in the Chern roots of $\mathcal{L}^{[n]}$ and $TS^{[n]}$. Namely, if $\{l_{n_i}\}$ are the Chern roots of $\mathcal{L}^{[n]}$ and $\{x_{n_j}\}$ are the Chern roots of $TS^{[n]}$, we have

$$\chi_\Lambda(S,\mathcal{L}) = \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} \prod_{i=1}^{n} (1+ye^{l_{n_i}}) \prod_{j=1}^{2n} \frac{x_{n_j}}{1-e^{-x_{n_j}}}$$

and similarly

$$\chi_{\mathrm{Sym}}(S,\mathcal{L}) = \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} \prod_{i=1}^{n} \frac{1}{1-ye^{l_{n_i}}} \prod_{j=1}^{2n} \frac{x_{n_j}}{1-e^{-x_{n_j}}}.$$

By [**EGL**, Thm 4.2], there exist universal series $A_i \in \mathbb{Q}[m,q]$, for $i = 1, \ldots, 4$ such that

$$\chi_\Lambda(S,\mathcal{L}) = \exp(c_1(\mathcal{L})^2 A_1 + c_1(\mathcal{L})c_1(S)A_2 + c_1(S)^2 A_3 + c_2(S)A_4).$$

In particular, the value of χ_Λ depends only on the numerics of S and $\mathcal{L}$. Set

$$\gamma(S,\mathcal{L}) = (c_1(\mathcal{L})^2, c_1(\mathcal{L})c_1(S), c_1(S)^2, c_2(S)) \in \mathbb{Z}^4.$$

If

$$\gamma(S,L) = \sum_i a_i \gamma(S_i,\mathcal{L}_i)$$

for $a_i \in \mathbb{Z}$, then

$$\chi_\Lambda(S,\mathcal{L}) = \prod_i (\chi_\Lambda(S_i,\mathcal{L}_i))^{a_i}. \tag{6.1.1}$$

Analagous results hold for χ_{Sym}.

We conclude that χ_Λ and χ_{Sym} can be determined for arbitrary $(S,\mathcal{L})$ by their values on any quadruple $\{(S_i,\mathcal{L}_i)\}_{i=1,\ldots,4}$ for which the vectors $\{\gamma(S_i,\mathcal{L}_i)\}$ are $\mathbb{Q}$-linearly independent. Any set of generators $\{(S_i,\mathcal{L}_i)\}$ of the algebraic cobordism ring (see [**LP**]) of surfaces equipped with a line bundle will satisfy this condition. In particular, such $(S_i,\mathcal{L}_i)$ can be chosen to be toric. We use the 4 generators

$$(\mathbb{P}^2,\mathcal{O}), (\mathbb{P}^2,\mathcal{O}(1)), (\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}), (\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1)) \tag{6.1.2}$$

given in [**EGL, Tz**]. These satisfy

$$\begin{aligned} &\gamma(\mathbb{P}^2,\mathcal{O}) = (0,0,9,3) && \gamma(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}) = (0,0,8,4) \\ &\gamma(\mathbb{P}^2,\mathcal{O}(1)) = (1,3,9,3) && \gamma(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,0)) = (0,2,8,4). \end{aligned}$$

As worked out in [**Tz**, Prop. 2.3], we have

$$\gamma(S,\mathcal{L}) = a_1\gamma(\mathbb{P}^2,\mathcal{O}) + a_2\gamma(\mathbb{P}^2,\mathcal{O}(1)) + a_3\gamma(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}) + a_4\gamma(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,0))$$

where

$$\begin{aligned} &a_1 = -c_1(\mathcal{L})^2 + 4\chi(\mathcal{O}_S) - c_2(S), && a_3 = 2c_1(\mathcal{L})^2 - 2\chi(\mathcal{O}_S) - \chi(\mathcal{L}) + c_2(S) \\ &a_2 = c_1(\mathcal{L})^2, && a_4 = -2c_1(\mathcal{L})^2 - \chi(\mathcal{O}_S) + \chi(\mathcal{L}). \end{aligned} \tag{6.1.3}$$

6.2. Reduction to equivariant $\mathbb{C}^2$

Suppose now that S is a toric surface; let T be a two-dimensional torus acting on S such that S^T consists of finitely many points and let $\mathcal{L}$ be an equivariant line bundle on S. Let $\mathcal{O}(l)$ denote the trivial bundle over $\mathbb{C}^2$ twisted by a torus weight l. We use equivariant localization to write $\chi_\Lambda(S,\mathcal{L})$ and $\chi_{\mathrm{Sym}}(S,\mathcal{L})$ in terms of equivariant Euler characteristics of the form $\chi_\Lambda(\mathbb{C}^2,\mathcal{O}(l))$ and $\chi_{\mathrm{Sym}}(\mathbb{C}^2,\mathcal{O}(l))$, respectively, computed as equivariant Euler characteristics.

The series $\chi_\Lambda(S,\mathcal{L})$ and $\chi_{\mathrm{Sym}}(S,\mathcal{L})$ can be read off from the equivariant geometry near the fixed locus S^T. Namely, let $\{s_i\}$ be the points of S^T, and let U_i be a toric chart centered at s_i. Let w_{i_1} and w_{i_2} be the tangent weights of T at s_i, and let l_i be the torus weight of the bundle $\mathcal{L}|_{U_i}$.

The T-action on S induces a T-action on $S^{[n]}$, and in particular, an action of T on $U_i^{[n]} \cong \mathbb{C}^{2[n]}$ for each U_i. The T-fixed points of $S^{[n]}$ are given by products $\prod \mathcal{I}_{\lambda^i}$ of T-fixed ideal sheaves $\mathcal{I}_{\lambda^i} \subset \mathcal{O}_{U_i}$, where $\mathcal{I}_{\lambda_i}$ is of colength $|\lambda_i|$ and $\sum|\lambda^i| = n$. The T-character of the Zariski tangent space to $\mathcal{I}_{\lambda^i}$ is $T_{\lambda^i}(w_1,w_2)$ while the restriction $\mathcal{L}|_{\mathcal{I}_{\lambda^i}}$ has torus character

$$\sum_{\square\in\lambda^i} l_i w_{i_1}^{-b_1} w_{i_2}^{-b_2}.$$

Suppose a torus $(\mathbb{C}^\times)^2$ acts on $\mathbb{C}^2$ by scaling the coordinate axes by weights w_{i_1}, w_{i_2}. Define $\chi_\Lambda(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2)$ to be the T character of

$$\sum_{n,k\geq 0} q^n y^k \chi(\mathbb{C}^{2[n]},\mathcal{O}(l))$$

and define $\chi_{\mathrm{Sym}}(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2)$ analogously. Equivariant localization implies

$$\chi_\Lambda(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2) = \sum_\lambda \frac{q^{|\lambda|}}{\Lambda^\bullet\Big(T_\lambda^\vee(w_1,w_2)\Big)} \prod_{\square\in\lambda}(1 + y w_1^{-b_1} w_2^{-b_2}).$$

By equivariant localization, we then have

$$\begin{aligned}
\chi_\Lambda(S,\mathcal{L}) &= \sum_{\{\lambda^i\}} \prod_i \frac{q^{|\lambda^i|}}{\Lambda^\bullet\Big(T^\vee_{\lambda^i}(w_{i_1},w_{i_2})\Big)} \prod_{\square\in\lambda_i} (1+yl_i w_{i_1}^{-b_1} w_{i_2}^{-b_2}) \\
&= \prod_i \sum_\lambda \frac{q^{|\lambda|}}{\Lambda^\bullet\Big(T^\vee_{\lambda^i}(w_{i_1},w_{i_2})\Big)} \prod_{\square\in\lambda} (1+yl_i w_{i_1}^{-b_1} w_{i_2}^{-b_2}) \\
&= \prod_i \chi_\Lambda(\mathbb{C}^2,\mathcal{O}(l_i))(w_{i_1},w_{i_2}).
\end{aligned}$$

The resulting identity is an equality of equivariant Euler characteristics; to recover the nonequivariant series, we specialize $t_1 = t_2 = 1$.

Again, analogous results hold for χ_{Sym}.

6.3. Exterior powers

Using Proposition 1.1.3, we compute $\chi_\Lambda(S,\mathcal{L})$ for an arbitrary surface.

Corollary 6.3.1. *For a projective surface S with line bundle $\mathcal{L}$, we have*

$$\chi(S^{[n]},\Lambda^k\mathcal{L}^{[n]}) = \binom{n-k+\chi(\mathcal{O}_S)-1}{n-k}\binom{\chi(L)}{k}.$$

We remark that this result is a specialization of [**Sc1**][Thm 5.2.1], which specifies the dimension of each of the individual cohomology groups $H^i(S^{[n]},\Lambda^k\mathcal{L}^{[n]})$.

Proof. Recall the definition of the series F from Section 1.1.2. We have

$$\chi_\Lambda(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2) = F(-qml,-\frac{1}{ml},0,0,0)(w_1,w_2).$$

Using Proposition 1.1.3, we compute

$$\begin{aligned}
F(-qml,-\frac{1}{ml},0,0,0)(w_1,w_2) &= \exp\Big(\sum_{n>0} -\frac{(-qml)^n}{n}\frac{1-(\frac{-1}{ml})^n}{(1-w_1^{-n})(1-w_2^{-n})}\Big) \\
&= \exp\Big(\sum_{n>0}\frac{q^n-(-qml)^n}{n(1-w_1^{-n})(1-w_2^{-n})}\Big) \\
&= \mathrm{Sym}^\bullet\Big(\frac{q-qml}{(1-w_1^{-1})(1-w_2^{-1})}\Big).
\end{aligned}$$

Section 6.1 implies that it suffices to check the corollary for toric surfaces. We use the notation from earlier: namely, let S be toric, admitting the action of a two-dimensional torus T with fixed points $\{s_i\}$ and tangent weights w_{i_1}, w_{i_2}, and let $\mathcal{L}$ be an equivariant line bundle such that the fiber $\mathcal{L}|_{s_i}$ has weight l_i. Then, as equivariant series, we have

$$\begin{aligned}
\chi_\Lambda(S,\mathcal{L}) &= \prod_i \chi_\Lambda(\mathbb{C}^2, \mathcal{O}(l_i))(w_{i_1}, w_{i_2}) \\
&= \prod_i \mathrm{Sym}^\bullet\Big(\frac{q - qml_i}{(1-w_{i_1}^{-1})(1-w_{i_2}^{-1})}\Big) \\
&= \mathrm{Sym}^\bullet\Big(\sum_i \frac{q - ml_i}{(1-w_{i_1}^{-1})(1-w_{i_2}^{-1})}\Big) \qquad (6.3.1)
\end{aligned}$$

But, by equivariant localization, after specializing to $t_1 = t_2 = 1$ the expression (6.3.1) becomes

$$\mathrm{Sym}^\bullet\Big(q\chi(\mathcal{O}_S) - qm\chi(\mathcal{L})\Big) = \frac{(1+qm)^{\chi(\mathcal{L})}}{(1-q)^{\chi(\mathcal{O}_S)}},$$

implying the corollary. □

6.4. Symmetric powers

In contrast to exterior powers, there does not seem to be a reasonable closed form for an arbitrary symmetric power of a tautological bundle. However, Theorem 1.1.2 and Proposition 1.1.3 can still be used to study symmetric powers of tautological bundles. We begin by using Theorem 1.1.2 to rewrite $\chi_{\mathrm{Sym}}(S,\mathcal{L})$ in a form where the Segre variable y enters the denominator in a controllable manner.

Note that the series $\chi_{\mathrm{Sym}}(\mathbb{C}^2, \mathcal{O}(l))(w_1, w_2)$ can be obtained by extracting all terms of the series $F(q,m,0,0,yl)(w_1,w_2)$ of equal q-degree and m-degree. By Theorem 1.1.2 we have

$$F(q,m,0,0,yl)(w_1,w_2) = \frac{F(q,m,0,0,0)(w_1,w_2)}{F(yl,0,0,m,0)(w_1,w_2)} \cdot F(yl,0,0,m,q)(w_1,w_2). \qquad (6.4.1)$$

The first term on the right hand side can be described using the plethystic exponential; Proposition 1.1.3 implies

$$F(q,m,0,0,0)(w_1,w_2) = \mathrm{Sym}^\bullet\Big(\frac{qm - q}{(1-w_1^{-1})(1-w_2^{-1})}\Big) \qquad (6.4.2)$$

$$F(yl,0,0,m,0)(w_1,w_2) = \mathrm{Sym}^\bullet\Big(\frac{-yl}{(1-w_1^{-1})(1-w_2^{-1})}\Big). \qquad (6.4.3)$$

By definition,

$$F(yl,0,0,m,q)(w_1,w_2)=\sum_\lambda \frac{(yl)^{|\lambda|}}{\Lambda^\bullet\Big(T_\lambda^\vee(w_1,w_2)\Big)}\prod_{\square\in\lambda}\frac{1-mqw_1^{-b_1}w_2^{-b_2}}{q-w_1^{b_1}w_2^{b_2}}.$$

Let $G(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2)$ denote the series obtained by first extracting the terms of equal q-degree and m-degree, and then setting $m=1$; we have

$$G(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2)=\sum_\lambda \frac{(yl)^{|\lambda|}}{\Lambda^\bullet\Big(T_\lambda^\vee(w_1,w_2)\Big)}\prod_{\square\in\lambda}\frac{1-qw_1^{-b_1}w_2^{-b_2}}{-w_1^{b_1}w_2^{b_2}}.$$

Consequently, extracting terms of both sides of (6.4.1) of equal q-degree and m-degree, and then setting $m=1$ yields

$$\chi_{\mathrm{Sym}}(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2)=\mathrm{Sym}^\bullet\Big(\frac{q+yl}{(1-w_1^{-1})(1-w_2^{-1})}\Big)G(\mathbb{C}^2,\mathcal{O}(l))(w_1,w_2).$$

Suppose S is toric with equivariant line bundle $\mathcal{L}$ and let $\{s_i\},\{w_{i_1},w_{i_2}\}$ and $\{l_i\}$ be the data associated to the fixed-points as above. As T-equivariant Euler characteristics, we have

$$\begin{aligned}\chi_{\mathrm{Sym}}(S,\mathcal{L})&=\prod_i \chi_{\mathrm{Sym}}(\mathbb{C}^2,\mathcal{O}(l_i))(w_{i_1},w_{i_2})\\&=\prod_i \mathrm{Sym}^\bullet\Big(\frac{q+yl_i}{(1-w_{i_1}^{-1})(1-w_{i_2}^{-1})}\Big)G(\mathbb{C}^2,\mathcal{O}(l_i))(w_{i_1},w_{i_2}).\\&=\mathrm{Sym}^\bullet\Big(\sum_i\frac{q+yl_i}{(1-w_{i_1}^{-1})(1-w_{i_2}^{-1})}\Big)\prod_i G(\mathbb{C}^2,\mathcal{O}(l_i))(w_{i_1},w_{i_2}).\end{aligned}$$

Set

$$G(S,\mathcal{L})=\Big(\prod_i G(\mathbb{C}^2,\mathcal{O}(l_i))(w_{i_1},w_{i_2})\Big)\Big|_{t_1=t_2=1}.$$

By equivariant localization we have

$$\begin{aligned}\Big(\sum_i\frac{q}{(1-w_{i_1}^{-1})(1-w_{i_2}^{-1})}\Big)\Big|_{t_1=t_2=1}&=q\chi(\mathcal{O}_S),\\ \Big(\sum_i\frac{yl_i}{(1-w_{i_1}^{-1})(1-w_{i_2}^{-1})}\Big)\Big|_{t_1=t_2=1}&=y\chi(\mathcal{L});\end{aligned}$$

we conclude that

$$\chi_{\mathrm{Sym}}(S,\mathcal{L})=\frac{1}{(1-q)^{\chi(\mathcal{O}_S)}(1-y)^{\chi(\mathcal{L})}}G(S,\mathcal{L}). \tag{6.4.4}$$

We use (6.4.4) to compute certain symmetric powers.

In the special case when $\chi(\mathcal{O}_S) = 1$, we have the following stability result.

Corollary 6.4.1. *Let S be a projective surface with $\chi(\mathcal{O}_S) = 1$, and let $\mathcal{L}$ be a line bundle on S. Then, for $n \geq k$, we have*

$$\chi(S^{[n]}, \mathrm{Sym}^k \mathcal{L}^{[n]}) = \binom{\chi(\mathcal{L}) + k - 1}{k}.$$

Proof. It suffices to check the result for toric S and equivariant $\mathcal{L}$. As $\chi(\mathcal{O}_S) = 1$, by (6.4.4) we have

$$\chi_{\mathrm{Sym}}(S, \mathcal{L}) = \frac{1}{(1-q)(1-y)^{\chi(\mathcal{L})}} G(S, \mathcal{L}).$$

Now, fix $k \geq 0$, and set $G_k(S, \mathcal{L})$ to be the sum of all terms of $G(S, \mathcal{L})$ of y-degree less than or equal to k. As any monomial in $G(S, \mathcal{L})$ has q-degree no larger than y-degree, we may write

$$G_k(S, \mathcal{L}) = \sum_{j=0}^{k} g_j(S, \mathcal{L}) q^j,$$

where $g_j(S, \mathcal{L}) \in \mathbb{Z}[y](t_1, t_2)$. Then, for any $n \geq k$, the $q^n y^k$-term of $\chi_{\mathrm{Sym}}(S, \mathcal{L})$ is the y^k term of the expression

$$\frac{1}{(1-y)^{\chi(\mathcal{L})}} \sum_{j=0}^{k} g_j(S, \mathcal{L}) = (1-y)^{\chi(\mathcal{L})} G_k(S, \mathcal{L})|_{q=1}.$$

But, by the definition, we have $G_k(S, \mathcal{L})|_{q=1} = 1$; we conclude that $\chi(S^{[n]}, \mathrm{Sym}^k \mathcal{L}^{[n]})$ is the y^k-coefficient of $1/(1-y)^{\chi(\mathcal{L})}$, implying the corollary. □

When $\chi(\mathcal{O}_S) \neq 1$, there does not appear to be a concise formula for $\chi(S^{[n]}, \mathrm{Sym}^k \mathcal{L}^{[n]})$. Nonetheless, the argument from Corollary 6.4.1 implies that for for fixed k, the

$$\sum_{n=1}^{\infty} q^n y^k \chi(S^{[n]}, \mathrm{Sym}^k \mathcal{L}^{[n]})$$

is determined by $G_k(S, \mathcal{L})$. We conclude the following.

Corollary 6.4.2. *The series $\sum_{n=1}^{\infty} q^n y^k \chi(S^{[n]}, \mathrm{Sym}^k \mathcal{L}^{[n]})$ is determined by its first k coefficients.*

We conclude with a computation of the series

$$\sum_{n=0}^{\infty} q^n y^k \chi(S^{[n]}, \mathrm{Sym}^k \mathcal{L}^{[n]})$$

for $k \leq 3$. In the same manner, one can determine an expression for this series for any fixed value of k.

Consider the 4 toric surface and bundle pairs from (6.1.2). Let $(\mathbb{C}^\times)^2$ act on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot [z_0, z_1, z_2] = [z_0 : t_1 z_2 : t_2 z_2],$$

and on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$(t_1, t_2) \cdot ([x_0 : x_1], [y_0 : y_1]) = ([t_1 x_0 : x_1], [t_2 y_0 : y_1]);$$

these actions lift to the bundles $\mathcal{O}$ and $\mathcal{O}(1)$ on $\mathbb{P}^2$ and $\mathcal{O}$ and $\mathcal{O}(1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The T-fixed points $\{s_i\}$, corresponding tangent weights $\{w_{i_1}, w_{i_2}\}$ and bundle weights $\{l_i\}$ for these surface and bundle pairs are given in the following table.

S	$\mathcal{L}$	$\{s_i\}$	$\{(w_{i_1}, w_{i_2})\}$	$\{l_i\}$
$\mathbb{P}^2$	$\mathcal{O}$	$\{[1:0:0], [0:1:0], [0:0:1]\}$	$\{(t_1, t_2), (t_1^{-1}, t_1^{-1} t_2), (t_2^{-1}, t_1 t_2^{-1})\}$	$\{1, 1, 1\}$
$\mathbb{P}^2$	$\mathcal{O}(1)$	$\{[1:0:0], [0:1:0], [0:0:1]\}$	$\{(t_1, t_2), (t_1^{-1}, t_1^{-1} t_2), (t_2^{-1}, t_1 t_2^{-1})\}$	$\{1, t_1^{-1}, t_2^{-1}\}$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathcal{O}$	$\{([1:0],[1:0]), ([1:0],[0:1]), ([0:1],[1:0]), ([0:1],[0:1])\}$	$\{(t_1^{-1}, t_2^{-1}), (t_1^{-1}, t_2), (t_1, t_2^{-1}), (t_1, t_2)\}$	$\{1, 1, 1, 1\}$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathcal{O}(1,0)$	$\{([1:0],[1:0]), ([1:0],[0:1]), ([0:1],[1:0]), ([0:1],[0:1])\}$	$\{(t_1^{-1}, t_2^{-1}), (t_1^{-1}, t_2), (t_1, t_2^{-1}), (t_1, t_2)\}$	$\{t_1^{-1}, t_1^{-1}, 1, 1\}$

Using these weights, we can compute $G(S, \mathcal{L})$ up to any given degree in y. For example, computing up to degree 4, we obtain

$$G_4(\mathbb{P}^2, \mathcal{O}) = 1 - y + qy + q^2y^3 - q^3y^3 + 11q^3y^4 - 11q^4y^4$$

$$G_4(\mathbb{P}^2, \mathcal{O}(1)) = 1 - 3y + 3qy + 3y^2 - 3qy^2 - y^3 + qy^3 - 6q^3y^4 + 6q^4y^4$$

$$G_4(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}) = 1 - y + qy + 2q^2y^3 - 2q^3y^3 + 5q^2y^4 - 8q^3y^4 + 3q^4y^4$$

$$G_4(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0)) = 1 - 2y + 2qy + y^2 - qy^2 + 4q^2y^3 - 4q^3y^3 + 15q^2y^4 - 51q^3y^4 + 36q^4y^4.$$

Let S be an arbitrary projective surface with line bundle $\mathcal{L}$. Recall the integers a_i from (6.1.3) defined in terms of the numerics of $(S, \mathcal{L})$. By (6.1.1), we have

$$\chi_{\mathrm{Sym}}(S, \mathcal{L}) = \frac{(G(\mathbb{P}^2, \mathcal{O}))^{a_1}(G(\mathbb{P}^2, \mathcal{O}(1)))^{a_2}(G(\mathbb{P}^1 \times \mathbb{P}1, \mathcal{O}))^{a_3}(G(\mathbb{P}^1 \times \mathbb{P}1, \mathcal{O}(1,0)))^{a_4}}{(1-q)^{\chi(\mathcal{O}_S)}(1-y)^{\chi(\mathcal{L})}}.$$

Expanding up to degree 3 in y, we obtain

$$\sum_{n=0}^{\infty} q^n \chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}},$$

$$\sum_{n=0}^{\infty} q^n \chi(S^{[n]}, \mathcal{L}^{[n]}) = \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}} \chi(\mathcal{L})q,$$

$$\sum_{n=0}^{\infty} q^n \chi(S^{[n]}, \mathrm{Sym}^2 \mathcal{L}^{[n]}) = \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}} \left(\chi(\mathcal{L}^{\otimes 2})q + \left(\binom{\chi(\mathcal{L})+1}{2} - \chi(\mathcal{L}^{\otimes 2}) \right) q^2 \right),$$

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \chi(S^{[n]}, \mathrm{Sym}^3 \mathcal{L}^{[n]}) = \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}} \Big(&\chi(\mathcal{L}^{\otimes 3})q \\ &+ \left(\chi(\mathcal{L}^{\otimes 2})\chi(\mathcal{L}) - \chi(\mathcal{L}^{\otimes 3}) - \chi(T_S^\vee \otimes \mathcal{L}^{\otimes 3}) \right) q^2 \\ &+ \left(\binom{\chi(\mathcal{L})+2}{3} - \chi(\mathcal{L}^{\otimes 2})\chi(\mathcal{L}) + \chi(T_S^\vee \otimes \mathcal{L}^{\otimes 3}) \right) q^3 \Big),\end{aligned}$$

where we haved used the following consequences of the Hirzebruch-Riemann-Roch theorem:

$$\chi(\mathcal{L}^{\otimes k}) = \chi(\mathcal{O}_S) + \frac{1}{2}(k^2 c_1(\mathcal{L})^2 + k c_1(\mathcal{L}) c_1(S))$$

and

$$\chi(T_S^\vee \otimes \mathcal{L}^{\otimes 3}) = 2\chi(\mathcal{L}^{\otimes 3}) - c_2(S) - 3c_1(\mathcal{L})c_1(S).$$

We then conclude the following.

Corollary 6.4.3. *We have*

$$\chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = \binom{\chi(\mathcal{O}_S) + n - 1}{n} \tag{6.4.5}$$

$$\chi(S^{[n]}, \mathcal{L}^{[n]}) = \binom{\chi(\mathcal{O}_S) + n - 2}{n-1}\chi(\mathcal{L}) \tag{6.4.6}$$

$$\chi(S^{[n]}, \mathrm{Sym}^2\mathcal{L}^{[n]}) = \binom{\chi(\mathcal{O}_S) + n - 3}{n-1}\chi(\mathcal{L}^{\otimes 2}) + \binom{\chi(\mathcal{O}_S) + n - 3}{n-2}\binom{\chi(\mathcal{L}) + 1}{2} \tag{6.4.7}$$

$$\begin{aligned}\chi(S^{[n]}, \mathrm{Sym}^3\mathcal{L}^{[n]}) = &\binom{\chi(\mathcal{O}_S) + n - 3}{n-1}\chi(\mathcal{L}^{\otimes 3}) \\ &+ \binom{\chi(\mathcal{O}_S) + n - 4}{n-2}\big(\chi(\mathcal{L}^{\otimes 2})\chi(\mathcal{L}) - \chi(T_S^\vee \otimes \mathcal{L}^{\otimes 3})\big) \\ &+ \binom{\chi(\mathcal{O}_S) + n - 4}{n-3}\binom{\chi(\mathcal{L}) + 2}{3}.\end{aligned} \tag{6.4.8}$$

We remark that equation (6.4.6) is [**EGL**, Prop 5.6(ii)], and (6.4.7, 6.4.8) are consequences of [**Sc2**, Thm 5.25]. In the cases when $n = 2$ or 3, equation (6.4.7) is also a consequence of [**Da**, Thm 1.1, 1.2].

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