Columbia University

Department of Economics
Discussion Paper Series

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Discussion Paper No.: 0304-04

Department of Economics
Columbia University
New York, NY 10027

October 2003

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October 5, 2003

Abstract

We focus on a class of Multiple Prior Models. Those characterized by nonatomic countably additive priors. Preferences generating such representations have been recently axiomatized in [17]. We argue that this is the proper setting for comparing the notions of unambiguous event given by Epstein and Zhang in [7] and by Ghirardato, Maccheroni and Marinacci in [10]. The two definitions are known to be nonequivalent. Our main result is that an event T is unambiguous in the sense of Epstein and Zhang if and only if either (i) it is unambiguous in the sense of [10]; or (ii) conditional on T, the decision maker is an expected utility maximizer. We also provide an easy operational criterion for establishing whether or not an event is unambiguous in the sense of Epstein and Zhang.

JEL Classification Number: D81

Keywords: Ambiguous events, multiple priors

1 Introduction

Since the fundamental work of Gilboa and Schmeidler [13], Multiple Prior models have become the object of a thorough investigation among decision theorists (see, for instance [5], [10], [11], [15], [16]). Moreover, an increasing number of papers have made use of such models to address, in a novel way, issues in Finance, Macroeconomics and Political Economy ([6], [9], [14]). Several reasons explain this trend. At one end of the spectrum, a recent result of Ghirardato, Maccheroni and Marinacci [10] shows that a very general model of decision making under uncertainty takes the form of a Multiple Prior model. At the other end, we find what originally motivated the work of Gilboa and Schmeidler. Multiple

[*]I am grateful to Larry Epstein, Emel Filiz, Paolo Ghirardato, Fabio Maccheroni, Massimo Marinacci and Marco Scarsini for comments and suggestions.

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Prior models deliver a distinction of the uncertainty faced by the decision maker into two parts: Ambiguity and Risk. Hence, they leave room to accommodate phenomena like hedging and Ellsberg's choices.

The ability of dealing with Ambiguity is one of the most distinguishing features between Multiple Prior models and the Savage-de Finetti theory. Probably motivated by this observation, a number of papers have abstracted from the Multiple Prior model itself and have focused directly on the idea of Ambiguity. This has given rise to several nonequivalent definitions of ambiguous event (see [7] and [12] for thorough discussions). Here, we single out two. Roughly speaking, for [10] an event A is ambiguous if a bet on A is suitable of more than one evaluation. While for [7], A is ambiguous if bets involving A are associated to Ellsberg's choices.

In this paper, we focus on a class of preferences satisfying the α-maxmin criterion. Specifically, those identified by the properties that all the priors in the representation are (a) countably additive and (b) nonatomic. A number of reasons motivate these restrictions. It is well-known that countable additivity produces remarkable properties both from a mathematical viewpoint and a decision-theoretic one. For the latter, the reader is referred to [3] and [18]. As for the nonatomicity of the measures, this is a property that is often imposed in conventional subjective expected utility theory. It occurs, for instance, in Savage. Besides these virtues, here we have yet another reason for using countable additivity in the place of finite additivity as well as for imposing the nonatomicity of the measures. The Epstein-Zhang axioms produce a countably additive nonatomic measure on the class of unambiguous sets. Elsewhere [2], we showed that the Ghirardato-Maccheroni-Marinacci definition produces such properties if and only if all the priors in the representation are both countably additive and nonatomic. Hence, the class of preferences we deal with appears to be the proper ground for comparing the two definition of ambiguous events.

The paper proceeds as follows. Section 2 is a brief description of the results contained in [17], which characterize the class of preferences we study. This section serves only to make the paper self-contained. Reader familiar with [17] should certainly skip it. Section 3 reports the two definitions of ambiguous events we study and introduces some terminology we use in the rest of the paper. Section 4 contains our results. There, we provide necessary and sufficient conditions for an event to be unambiguous in the sense of Epstein and Zhang [7]. For an α-maxmin decision maker, we show that an event T is unambiguous in the sense of Epstein and Zhang if either (i) it is unambiguous in the sense of Ghirardato, Maccheroni and Marinacci [10]; or (ii) conditional on T, the decision maker is an expected utility maximizer. Moreover, Corollary 12 provides an easy operational criterion which permits to establish whether or not a given event T is Epstein-Zhang ambiguous for an α-maxmin decision maker. An Appendix containing the proofs of the various statements completes the paper.

2 Monotone continuous, countably additive multiple priors

The setting we are going to focus on is defined by three objects. First, a collection, $F(S,\Phi)$, of mappings $S \rightarrow \Phi$, which represent the alternatives available to the decision maker. S is called the state space and Φ the prize space. Second, a fixed σ-field, Σ, of subsets of S. Third, a preference relation, $\succsim$, on $F(S,\Phi)$. Let $B(\Sigma)$ denote the set of bounded, Σ-measurable functions. Then, the preference relation $\succsim$ is said to satisfy the α-maxmin criterion if and only if there exists a functional $I : B(\Sigma) \rightarrow R$ such that for $a, b \in F(S,\Phi)$

$$a \succsim b \qquad iff \qquad I(u \circ a) \geq I(u \circ b)$$

where $u : \Phi \rightarrow R$ is a utility function on the prize space, and for every $h \in B(\Sigma)$, I is defined by

$$I(h) = \alpha \min_{P \in \mathcal{F}} \int_S h dP + (1-\alpha) \max_{P \in \mathcal{F}} \int_S h dP$$

with $\mathcal{F}$ being a set of probability measures on (S,Σ), and $\alpha \in [0,1]$.

Preferences satisfying such a criterion have been recently axiomatized in [10] and in [15]. In [17], it has been shown that whenever such preferences satisfy two additional axioms, namely $\succsim$ is (a) both upward and downward atomless and (b) satisfies the axiom of monotone continuity (see [17]), then all the measures in $\mathcal{F}$ are (i) nonatomic; (ii) countably additive; and (iii) there exists a measure $\nu \in \Delta(S)$ such that all the measures in $\mathcal{F}$ are absolutely continuous with respect to ν.

3 Ambiguous events

In this section we report the definitions of ambiguous events given by Ghirardato, Maccheroni and Marinacci [10] and by Epstein and Zhang [7], and collect some facts about the classes of unambiguous events determined by the two definitions. In addition, we introduce some terminology that we use throughout the paper.

Let $\mathcal{F}$ be the set of priors as in the previous section. Generic elements in $\mathcal{F}$ are denoted by P_f, P_g, etc.. All subsets of S which appear in the definitions below are elements of the given σ-algebra, Σ, of subsets of S.

Definition 1 (Ghirardato, Maccheroni and Marinacci [10]) *A set $T \subseteq S$ is unambiguous if and only if $P_f(T) = P_g(T)$ for all $P_f, P_g \in \mathcal{F}$.*

Note that if T is unambiguous according to this definition, so is its complement T^c.

The class of unambiguous events according to Ghirardato, Maccheroni and Marinacci [10] is a λ-system. The dependence of such a class on the set of priors is studied in [2]. There, it is also shown that there is a natural measure defined on the class of unambiguous events and that, whenever the class is nontrivial, the natural measure is nonatomic.

Throughout the paper, we refer to events which are unambiguous in the sense of Ghirardato, Maccheroni and Marinacci [10] as $\mathcal{F}$-measurable events. This terminology was introduced in [2], and motivated by the concepts developed there. Here, its virtue is that it would avoid some possible confusion between the two concepts of ambiguity we study.

The Epstein and Zhang [7] definition is sensibly different from the Ghirardato-Maccheroni-Marinacci's one. In fact, it is well-known that the two are not equivalent (see [7] and the next section). The Epstein-Zhang's definition, being formulated directly in terms of preferences, is independent of the particular model one is using. As a notable consequence, this allows one to ascertain the ambiguous/unambiguous nature of an event in any model.

Definition 2 (Epstein and Zhang [7]) *An event T is unambiguous if: (a) for all disjoint subevents A, B of T, acts h and outcomes y^*, y, z, z'*

$$\begin{pmatrix} y^* & if & s \in A \\ y & if & s \in B \\ h(s) & if & s \in T \setminus (A \cup B) \\ z & if & s \in T^c \end{pmatrix} \succeq \begin{pmatrix} y & if & s \in A \\ y^* & if & s \in B \\ h(s) & if & s \in T \setminus (A \cup B) \\ z & if & s \in T^c \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} y^* & if & s \in A \\ y & if & s \in B \\ h(s) & if & s \in T \setminus (A \cup B) \\ z' & if & s \in T^c \end{pmatrix} \succeq \begin{pmatrix} y & if & s \in A \\ y^* & if & s \in B \\ h(s) & if & s \in T \setminus (A \cup B) \\ z' & if & s \in T^c \end{pmatrix}$$

and (b) The condition obtained if T^c is everywhere replaced by T in (a) is also satisfied. Otherwise, T is ambiguous.

Just like in the Ghirardato-Maccheroni-Marinacci's case, the class of unambiguous events in the sense of Epstein and Zhang is a λ-system. Moreover, their axioms (see [7]) guarantee that such a class is nontrivial and that there exists a countably additive nonatomic measure defined on it.

In what follows, we are going to be using a definition of unambiguous events which is more permissive than the Epstein-Zhang's in that we drop the part (b) of the definition above. We do so because the following developments suggest that the more permissive definition has its own value. Below, we give necessary and sufficient conditions for an event to be unambiguous according to the more permissive definition.[1] However, as (obviously) our statements refer to an arbitrary event T, it suffices to require that the same conditions be satisfied both for T and T^c (the complement of T) to get back to the full Epstein-Zhang's setting. Because of this trivial observation, we call Epstein-Zhang's unambiguous also those events which satisfy only part (a) of the Epstein-Zhang's definition.

[1] Observe that, obviously, if T is ambiguous according to the more permissive definition, then T is ambiguous for Epstein-Zhang.

4 Results

In this section, within the context of the model of Section 2, we are going to study the relation between the two concepts of ambiguity described above.

We begin by establishing that if a decision maker satisfies the assumptions of Section 2, and (relative to such a decision maker) an event $T \subset S$ is $\mathcal{F}$-measurable (i.e., unambiguous for Ghirardato-Maccheroni-Marinacci), then T is unambiguous in the sense of Epstein and Zhang (EZ-unambiguous, for short). This is the content of Proposition 3 and Corollary 4 below.

Proposition 3 *Let $\succsim$ be a preference relation on $F(S, \Phi)$ satisfying the assumptions of Section 2 for $\alpha = 1$, and let $T \in \Sigma$. If T is $\mathcal{F}$-measurable (= unambiguous for Ghirardato-Maccheroni-Marinacci), then T is EZ-unambiguous.*

A quick look at the proof of Proposition 3 delivers, at once, the following

Corollary 4 *If a preference relation on $F(S, \Phi)$ satisfies the assumptions of Section 2 for $\alpha \in [0, 1]$, then T $\mathcal{F}$-measurable implies T EZ-unambiguous.*

In general, the converse to Proposition 3 does not hold. Both Epstein [4] and Ghirardato [8] have exhibited examples of decision makers for whom every set $A \in \Sigma$ is unambiguous in the sense of Epstein-Zhang, but the set of priors is not a singleton. The next proposition provides an example of this sort for our setting. That is, it exhibits a maxmin decision maker and a set $T \in \Sigma$ which is EZ-unambiguous but not $\mathcal{F}$-measurable.

Proposition 5 *Let d be a maxmin decision maker ($\alpha = 1$) with $\mathcal{F} = co\{P_f, P_g\}$ (co denotes the convex hull). Let P_f, P_g and $T \in \Sigma$ be such that*

(a) $P_f(T) > 0$; $\quad P_g(T) > 0$;

(b) $P_f(T) \neq P_g(T)$;

(c) $\forall A \in \Sigma$,

$$\frac{P_f(A \cap T)}{P_f(T)} = \frac{P_g(A \cap T)}{P_g(T)}$$

Then, T is not $\mathcal{F}$-measurable but is EZ-unambiguous.

The crucial property of the probability measures, P_f and P_g, in the proposition is that, while $P_f(T) \neq P_g(T)$, the conditional measures on T coincide. As an example, take $S = [0, 1]$ equipped with the Lebesgue measure, and let $T = [0, 2/3]$. Let l be the density associated to the Lebesgue measure (the constant function 1), and let $m : [0, 1] \to [0, 1]$ be defined by $m(s) = 5/4$ if $s \in [0, 2/3]$ and $m(s) = 1/2$ if $s \in (2/3, 1]$. Let P_f and P_g be defined by $P_f(A) = \int_A ds$ and $P_g(A) = \int_A m(s)ds$, respectively. Then, P_f and P_g satisfy the conditions of Proposition 5.

Example 6 *The example just given can be elaborated further. Consider a countable partition of $[0, 1]$ into intervals, and a family $\{m_n\}_{n \in N}$ of densities such that each m_n is constant on the elements of the partition and $m_0 = 1$. One* *can choose the m_n's in such a way that they differ on each and any element of the partition. Let P_n be the measure on S defined by $P_n(A) = \int_A m_n(s)ds$. Now consider a maxmin decision maker for whom $\mathcal{F} = co\{P_n\}_{n\in N}$. Then, for such a decision maker, any element of the given partition of S is EZ-unambiguous but it is not $\mathcal{F}$-measurable (= unambiguous for Ghirardato, Maccheroni and Marinacci).*

The reader should observe that as far as the decision maker of the proposition is concerned only with acts with domain T, then he has just one measure space as all of his priors coincide when conditioned on T. Hence, the proposition says that a non $\mathcal{F}$-measurable event $T \in \Sigma$ can be EZ-unambiguous if, *conditional on* T, the decision maker is an expected utility maximizer.

Next we show that, in the case of a two-dimensional set of priors, these are the only possibilities. That is, if is $\mathcal{F}$ two-dimensional, then an event $T \in \Sigma$ is EZ-unambiguous if either (i) T is unambiguous in the sense of Ghirardato, Maccheroni and Marinacci; or (ii) Conditional on T, the decision maker is an expected utility maximizer. In fact, as Theorem 11 below shows, the situation is exactly the same in the general case. Nevertheless, we find it useful to begin with the two-prior case as, in such a case, the intuition is very transparent. In addition, the proof of Proposition 7 (which will be used in the proof of Theorem 10) makes it clear the crucial role played by the conditional probabilities.

Proposition 7 *Let d be a maxmin decision maker ($\alpha = 1$) with $\mathcal{F} = co\{P_f, P_g\}$ (co denotes the convex hull). Let $P_f, P_g \in \mathcal{F}$ and $T \in \Sigma$ be such that*

(a') $P_f(T) > 0$; $P_g(T) > 0$;

(b') Property (c) of the previous proposition is not satisfied.

Then, T EZ-unambiguous $\Longrightarrow$ T $\mathcal{F}$-measurable.

The remaining of the paper is devoted to establishing Theorem 10 below, which, along with Proposition 3, provides a complete characterization of the unambiguous events in the sense of Epstein and Zhang. The argument leading to Theorem 10 splits into several lemmata. Two of them, Lemmata 8 and 9 below, are recorded into the main text in order to highlight the main steps in our construction. A complete proof is in the Appendix. However, as the proof is rather lengthy, it is probably useful to precede the formal developments with an informal discussion.

A simple look at the Epstein-Zhang definition (Definition 2, above) reveals that an event T is EZ-unambiguous if for any two subsets, A and B of T, the relative likelihood of A with respect to B is invariant with respect to changes in the (constant) prize assigned on T^c. Here, we have put ourselves in the context of a Multiple Prior model. In such a context, it is immediate to derive a sufficient condition for T to be EZ-unambiguous. In fact, if all the priors rank (in terms of likelihood) all the subsets of T in exactly the same way, then T must be EZ-unambiguous. Since we are concerned with the relative likelihood of subsets of T only, the condition that all the priors rank all the subsets of T

in exactly the same way translates into a condition about conditional (on T) probabilities. As shown in [1], in the context of countably additive nonatomic priors, this condition is that all such conditional probabilities be the same. We have already seen an example of this in Proposition 5 above.

However, just a moment of thought shows that such a sufficient condition can be relaxed. To see this, consider a maxmin decision maker ($\alpha = 1$). In such a model, any act is evaluated according to a single prior, even though such a prior may vary with the act. Divide the set of priors into two subsets, $\mathcal{A}$ and $\mathcal{B}$, in such a way that all the priors in $\mathcal{A}$ rank all the subsets of T in exactly the same way. Now, if it happens that no prior in $\mathcal{B}$ ever intervenes in the evaluation of an act satisfying the Epstein-Zhang definition, then we can certainly conclude that T is EZ-unambiguous. That is, a sufficient condition for T to be EZ-unambiguous is that the decision maker's preferences over acts satisfying the Epstein-Zhang definition are entirely determined by a subset $\mathcal{A} \subset \mathcal{F}$ and all the priors in $\mathcal{A}$ have the same conditional on T.[2]

While it is straightforward to see the sufficiency of the latter condition, it is, perhaps, not so immediate to see that the condition is also necessary. As a matter of fact, Lemmata 8 and 9 as well as a big chunk of the proof of Theorem 10 are devoted to this task. The crucial part of the argument is to show that if T is EZ-unambiguous, then the decision maker's preferences over acts satisfying the Epstein-Zhang definition must be determined by a family $\mathcal{A}$ whose elements rank all the subsets of T in exactly the same way. To this end, we proceed, roughly, as follows. We assume that T is EZ-unambiguous, and suppose that, by the way of contradiction, that there exist two EZ-acts that are not ranked by means of the family $\mathcal{A}$. Then, we use the characterization provided by Lemma 9 to construct another pair of EZ-acts which are necessarily ranked accordingly to the family $\mathcal{A}$. Finally, we use these two pairs to construct a third a pair of EZ-acts, and show that the ranking of these two acts is not invariant with respect to changes in the prize on T^c.

The first step to establish the necessity of the condition described above is recorded in Lemma 8 below. The lemma delivers a first necessary condition, stated as Condition (A) right after the statement of Lemma 8.

Set $V_f(\alpha) = \int_S u[\alpha(s)]dP_f$. Suppose that there exist $P_f, P_g \in \mathcal{F}$ such that, $P_f(T) > 0$, $P_g(T) > 0$, and the conditional probabilities, $\frac{P_f(\cdot \cap T)}{P_f(T)}$ and $\frac{P_g(\cdot \cap T)}{P_g(T)}$, do not coincide. Then, just like in the proof of the previous proposition, we can construct two acts, α and β, satisfying the EZ-definition such that

$$\begin{aligned} V_g(\alpha) &> V_g(\beta) \\ V_f(\alpha) &< V_f(\beta) \end{aligned}$$

[2]The condition discussed in the text does not say that no prior in $\mathcal{B}$ ever intervenes in the evaluation of any act. Even when T is unambiguous, priors in $\mathcal{B}$ typically intervene in the evaluation of acts not satisfying the Epstein-Zhang definition. For instance, bets on a set C which intersects both T and its complement.

Given such acts, define

$$\mathcal{F}_1 = \left\{ P_{\tilde{f}} \in \mathcal{F} \mid V_{\tilde{f}}(\alpha) \geq V_{\tilde{f}}(\beta) \right\}$$
$$\mathcal{F}_2 = \left\{ P_{\tilde{f}} \in \mathcal{F} \mid V_{\tilde{f}}(\alpha) < V_{\tilde{f}}(\beta) \right\}$$

Lemma 8 *Under the above assumptions, if there exists $P_{\bar{f}} \in \mathcal{F}_2$ such that either*

$$\begin{array}{lcccc} (i) & P_{\bar{f}}(T^c) & > & P_{\tilde{g}}(T^c) & , & \forall P_{\tilde{g}} \in \mathcal{F}_1 \\ & & & \textit{or} & \\ (ii) & P_{\bar{f}}(T^c) & < & P_{\tilde{g}}(T^c) & , & \forall P_{\tilde{g}} \in \mathcal{F}_1 \end{array}$$

then, T is EZ-ambiguous.

The lemma says that if there exists at least two measures whose conditionals on T do not coincide, then T EZ-unambiguous implies that the following condition is satisfied.

Condition (A) $\forall P_f \in \mathcal{F}_2$, $\exists P_{g_1}, P_{g_2} \in \mathcal{F}_1$ such that

$$P_{g_1}(T^c) \leq P_f(T^c) \leq P_{g_2}(T^c)$$

In other words, the lemma says that Condition (A) is a necessary condition for T to be EZ-unambiguous.

In general, Condition (A) is not sufficient for T to be EZ-unambiguous. The definition of $\mathcal{F}_1$ and $\mathcal{F}_2$ in Condition (A) depends on the particular acts that we constructed in the lemma, and there might exist another pair of acts which would reveal that T is EZ-ambiguous. The next lemma strengthens Condition (A) by making sure that this does not happen. From now on, unless otherwise stated, we are going to restrict attention to events T for which $\min_{P_f \in \mathcal{F}} P_f(T) > 0$.

Lemma 9 *Let d be a maxmin decision maker. A necessary condition for T to be EZ-unambiguous is that there exists a set of priors $\mathcal{A} \subset \mathcal{F}$ with the following properties*

(i) $\exists P_{g_1}, P_{g_2} \in \mathcal{A}$ such that for any $P_f \in \mathcal{B} = \mathcal{A}^c$

$$P_{g_1}(T^c) \leq P_f(T^c) \leq P_{g_2}(T^c)$$

and,

(ii) For all $P_g, P_{g'} \in \mathcal{A}$, $\frac{P_g(\cdot \cap T)}{P_g(T)} = \frac{P_{g'}(\cdot \cap T)}{P_{g'}(T)}$.

That is, if T is EZ-unambiguous, then there must exist a subset of the priors which (i) provides both the upper and the lower bounds for $P_f(T^c)$ (and, hence, for $P_f(T)$); and (ii) is such that all the conditional probabilities on T coincide.

Note that the lemma contains as a special case the case of T being $\mathcal{F}$-measurable, which occurs for $P_{g_1}(T^c) = P_{g_2}(T^c)$. In such a case, Proposition

3 guarantees that T is EZ-unambiguous, that is the condition is also sufficient. Note, also, that when specialized to the case of two priors, the lemma provides necessary and sufficient conditions for T to be EZ-unambiguous.

We are now ready to prove Theorem 10, which gives necessary and sufficient conditions for T to be EZ-unambiguous whenever T is not $\mathcal{F}$-measurable.

Theorem 10 *Assume that T is not $\mathcal{F}$-measurable. A necessary and sufficient condition for T to be EZ-unambiguous is that there exists a unique maximal (in the sense of inclusion) set of priors $\mathcal{A} \subset \mathcal{F}$ with the following properties*

(i) $\exists P_{g_1}, P_{g_2} \in \mathcal{A}$ such that for any $P_f \in \mathcal{B} = \mathcal{A}^c$

$$P_{g_1}(T^c) \leq P_f(T^c) \leq P_{g_2}(T^c)$$

(ii) For all $P_g, P_{g'} \in \mathcal{A}$, $\frac{P_g(\cdot \cap T)}{P_g(T)} = \frac{P_{g'}(\cdot \cap T)}{P_{g'}(T)}$.

(iii) Let (α, β) be any two acts satisfying the Epstein-Zhang's definition. If for some $P_g \in \mathcal{A}$ we have $V_g(\alpha) > V_g(\beta)$, then $\alpha \succ \beta$.

The necessity of the existence of a subset of the priors, $\mathcal{A}$, satisfying conditions (i) and (ii) had already been established by Lemma 9. With respect to that, Theorem 10 adds two more conditions. That the maximal set $\mathcal{A}$ be unique, and that property (iii) be satisfied. The latter properties, combined with the continuity property of preferences we have been studying, implies that the set $\mathcal{A}$ completely determines the decision maker's ranking over acts satisfying the EZ-definition. Since, by property (ii), all the priors in $\mathcal{A}$ rank those acts in exactly the same way, and such a ranking is independent of z (the prize that the acts pay on T^c), this implies at once that the condition in the theorem is also sufficient for T to be EZ-unambiguous.

The following theorem restates our conclusions by showing that the general case displays the same properties of the two-prior one. That is, T is EZ-unambiguous if and only if T is either $\mathcal{F}$-measurable, or, *conditional on T*, the decision maker is an expected utility maximizer.

Theorem 11 *An event $T \in \Sigma$ is EZ-unambiguous if and only if either*

(i) T is unambiguous in the sense of Ghirardato, Maccheroni and Marinacci; or

(ii) Conditional on T, the decision maker is an expected utility maximizer.

Let us remark that the ability to account for the latter case is, in our opinion, a notable feature of the Epstein-Zhang's definition of unambiguous event.

The next corollary provides an easy operational criterion to establish whether or not a given event T is EZ-unambiguous.

Observe that if T is EZ-unambiguous, then $\min\limits_{P_f \in \mathcal{F}} P_f(T^c)$ and $\max\limits_{P_f \in \mathcal{F}} P_f(T^c)$ are obtained by the P_{g_1} and the P_{g_2} which appear in the statement of the theorem.

Denote by

$$\begin{aligned}\max T &= \left\{P_g \in \mathcal{F} \mid P_g(T) = \max_{P_f \in \mathcal{F}} P_f(T)\right\} \\ \min T &= \left\{P_g \in \mathcal{F} \mid P_g(T) = \min_{P_f \in \mathcal{F}} P_f(T)\right\}\end{aligned}$$

Then,

Corollary 12 *An event $T \in \Sigma$ is EZ-unambiguous if and only if either it is $\mathcal{F}$-measurable, or $\exists P_{g_1} \in \min T$ and $\exists P_{g_2} \in \max T$ such that the conditional probabilities, $\frac{P_{g_1}(\cdot \cap T)}{P_{g_1}(T)}$ and $\frac{P_{g_2}(\cdot \cap T)}{P_{g_2}(T)}$, coincide, and if $(P_f, P_{f'})$ is any other pair having such a property, then $\frac{P_f(\cdot \cap T)}{P_f(T)} = \frac{P_{g_1}(\cdot \cap T)}{P_{g_1}(T)}$.*

In other words, given a maxmin decision maker, it suffices to consider the set of priors which evaluate the bet which pays 1 on T and 0 otherwise, and the set of priors which evaluate the bet which pays 1 on T^c and 0 otherwise. If one can select two priors, one for each set, such that their conditionals on T coincide and the requirement in the final part of the corollary is satisfied, then T is EZ-unambiguous. Otherwise, T is EZ-ambiguous. The final part in the corollary is necessary to ensure that the (maximal) set of priors having properties (i) and (ii) as in Theorem 10 be, in fact, unique.

As an immediate corollary to Lemma 9, we also have

Corollary 13 *If $\mathcal{F}$ contains no two priors whose conditionals on T coincide, then T EZ-unambiguous $\Longleftrightarrow$ T $\mathcal{F}$-measurable.*

From Proposition 3 and the proof of Theorem 10, we also have

Corollary 14 *T is null in the sense of Epstein-Zhang if and only if T is $\mathcal{F}$-measurable and for any $P_f \in \mathcal{F}$, $P_f(T) = 0$.*

We conclude the paper by stating one more corollary. It is immediate that our characterization of EZ-unambiguous events holds if we consider α-maxmin expected utility decision maker with $\alpha = 0$. Using this observation, we have,

Corollary 15 *Theorem 10, and, hence, Theorem 11 hold for any α-maxmin expected utility decision maker, $\alpha \in [0, 1]$ and $\alpha \neq 1/2$.*

The case $\alpha = 1/2$ is somewhat pathological as already observed in [16], to which we refer the reader for more detail.

References

[1] Amarante, M. (2003), "On the uniqueness of convex-ranged probabilities", Columbia University, WP #0203-24.

[2] Amarante, M. (2002), "Ambiguity, Measurability and Multiple Priors", Columbia University, WP #0203-23.

[3] Arrow, K. (1970), *Essays in the Theory of Risk-Bearing* ,Amsterdam, North Holland.

[4] Epstein, L.G., private communication.

[5] Epstein, L.G. and M. Schneider (2001), "Recursive Multiple Priors", forthcoming *JET*.

[6] Epstein, L.G. and T. Wang (1994), "Intertemporal Asset Pricing under Knightian Uncertainty", *Econometrica* 62, 283-322.

[7] Epstein, L.G. and J. Zhang (2000), "Subjective Probabilities on Subjectively Unambiguous Events", *Econometrica* 69, 265-306.

[8] Ghirardato, P., private communication.

[9] Ghirardato, P. and J.N. Katz (2000), "Indecision theory: Explaining selective abstention in multiple elections", CalTech WP 1106.

[10] Ghirardato, P., F. Maccheroni and M. Marinacci (2002), "Ambiguity from the differential viewpoint", ICER WP 17.

[11] Ghirardato, P. and M. Marinacci (2001), "Risk, ambiguity, and the separation of utility and beliefs" *MOR* 26, 864-890.

[12] Ghirardato, P. and M. Marinacci (2002), "Ambiguity made precise: A comparative foundation" *JET* 102, 251-89.

[13] Gilboa, I. and D. Schmeidler (1989), "Maxmin Expected Utility with Nonunique Prior", *Journal of Mathematical Economics* 18, 141-53.

[14] Hansen, L.P., T.J. Sargent and N. Wang (2002), "Robust Permanent Income and Pricing with Filtering", Macroeconomic Dynamics 6, 363-404.

[15] Kopylov, I. (2001), "Procedural rationality in the multiple prior model", University of Rochester, mimeo.

[16] Marinacci, M. (2002), "Probabilistic Sophistication and Multiple Priors", *Econometrica* 70, 755-64.

[17] Marinacci, M., F. Maccheroni, A. Chateauneuf and J-M. Tallon (2003), "Monotone Continuous Multiple Priors", University of Turin, mimeo.

[18] Stinchcombe, M. (1997), "Countably Additive Subjective Probabilities", *RES* 64, 125-46.

APPENDIX.

Before proceeding to the proofs of the various statements, we would like to remind the reader that, in the model of Section 2, the prize space has a convex structure (see [17]), and that the utility function is determined only up to a positive affine transformation. We will use these properties several times in what follows. The main notational conventions are as follows.

If α is an act, and $P_f \in \mathcal{F}$, we denote by

$$V_f(\alpha) = \int_S u[\alpha(s)]dP_f$$

the value of the act α, when the prior P_f is used for such evaluation.

Moreover, we set

$$V(\alpha) = \min_{P_f \in \mathcal{F}} \{V_f(\alpha)\}_{P_f \in \mathcal{F}}$$

and, for maxmin decision makers (1-maxmin), we have

$$\alpha \succsim \beta \qquad iff \qquad V(\alpha) \geq V(\beta)$$

Now, let α and β be two acts satisfying the first part of Epstein-Zhang's definition, and let α' (β') be the act obtained from α (β) by changing the prize on T^c from z to z'. In our notation, it is immediate to verify that for any such acts and for any $P_f, P_g \in \mathcal{F}$, we have

$$V_f(\alpha) - V_f(\beta) = V_f(\alpha') - V_f(\beta') = [u(y^*) - u(y)][P_f(A) - P_f(B)] \quad (1)$$

$$V_g(\alpha) - V_g(\beta) = V_g(\alpha') - V_g(\beta') = [u(y^*) - u(y)][P_g(A) - P_g(B)]$$

and

$$\begin{aligned} V_f(\alpha) - V_g(\alpha) &= u(y^*)[P_f(A) - P_g(A)] + u(y)[P_f(B) - P_g(B)] \\ &\quad + w(h, f, g) + u(z)[P_f(T^c) - P_g(T^c)] \end{aligned} \quad (2)$$

where $w(h, f, g)$ is a number that depends on the priors $P_f, P_g \in \mathcal{F}$ and on the mapping h in the Epstein-Zhang's definition. Moreover, we have a similar expression for α' by replacing z with z', and for β and β' by switching y^* and y.

Proof of Proposition 3. If T is $\mathcal{F}$-measurable, then for any $P_f, P_g \in \mathcal{F}$, $P_f(T^c) = P_g(T^c)$. Hence, equation (2) implies that for any $P_f, P_g \in \mathcal{F}$, and for any acts $\alpha, \alpha', \beta, \beta'$ satisfying the Epstein-Zhang's definition, we have

$$V_f(\alpha) - V_g(\alpha) \lesseqgtr 0 \quad \Longleftrightarrow \quad V_f(\alpha') - V_g(\alpha') \lesseqgtr 0 \quad (3)$$

and similarly for β and β'.

Now, suppose that T is EZ-ambiguous.

The relation (3) implies that if α is evaluated by means of a measure P_g, that is $V(\alpha) = V_g(\alpha)$, then so is α'. Hence, for T to be ambiguous it must be the case that β is evaluated by means of a measure $P_f \neq P_g$, that is $V(\beta) = V_f(\beta)$, $P_f \neq P_g$. For if not, then (by (3)) both β and β' are evaluated by means of the same measure P_g, and the reversal of preference cannot occur. Therefore, suppose that this is the case, and let P_f, P_g be such measures.

Given α, we have three possible cases corresponding to

$$V_f(\alpha) - V_g(\alpha) \lesseqqgtr 0$$

To begin, suppose that

$$V_f(\alpha) - V_g(\alpha) \geq 0 \tag{4}$$

In such a case, α is evaluated by means of P_g (by the definition of the functional V), and, by (3), so is α'. In correspondence of (4), we have either

$$V_f(\beta) - V_g(\beta) \geq 0$$

or

$$V_f(\beta) - V_g(\beta) < 0$$

The first along with (3) implies that both β and β' are evaluated by means of P_g as well. This contradicts T ambiguous because, as we have already observed, in such a case the preference reversal cannot occur. Hence, we are left with the case $V_f(\beta) - V_g(\beta) < 0$, which (along with (3)) implies that both β and β' are evaluated by means of P_f. Then, for T to be ambiguous, it must be the case that

$$\begin{array}{lccccc} (i) & \alpha & \succsim & \beta & \iff & V_g(\alpha) \geq V_f(\beta) \\ (ii) & \alpha' & \prec & \beta' & \iff & V_g(\alpha') < V_f(\beta') \end{array}$$

By (4), (i) implies

$$0 \geq V_g(\alpha) - V_f(\alpha) \geq V_f(\beta) - V_f(\alpha)$$

that is

$$V_f(\alpha) - V_f(\beta) \geq 0$$

which, by (1), implies

$$V_f(\alpha') - V_f(\beta') \geq 0$$

At the same time, (ii) implies

$$V_g(\alpha') - V_f(\alpha') < V_f(\beta') - V_f(\alpha')$$

Hence, by using (1) and (2), we have

$$V_g(\alpha) - V_f(\alpha) = V_g(\alpha') - V_f(\alpha') < V_f(\beta') - V_f(\alpha') = V_f(\beta) - V_f(\alpha) \leq 0$$

that is, $V_g(\alpha) < V_f(\beta)$, contradicting (i).

This establishes that if $V_f(\alpha) - V_g(\alpha) \geq 0$, then T $\mathcal{F}$-measurable implies T EZ-unambiguous. Along the same lines, the reader can verify the conclusion for the case $V_f(\alpha) - V_g(\alpha) < 0$. ■

Proof of Proposition 5. T is evidently non $\mathcal{F}$-measurable (by (b)). To prove that T is unambiguous, it clearly suffices to restrict attention to the two extreme points P_f and P_g.

By property (c) in the statement, $\forall A, B \subset T,\ A, B \in \Sigma$

$$P_f(A) > P_f(B) \qquad \Longrightarrow \qquad P_g(A) > P_g(B)$$

Hence, by (1)

$$\begin{array}{rcl} V_f(\alpha) - V_f(\beta) & = & V_f(\alpha') - V_f(\beta') \gtreqless 0 \\ & \Longleftrightarrow & \\ V_g(\alpha) - V_g(\beta) & = & V_g(\alpha') - V_g(\beta') \gtreqless 0 \end{array} \tag{5}$$

That is, whenever V_f prefers α to β so does V_g, and viceversa.

To show that T is unambiguous, we consider the four possible cases

$$\begin{array}{lrcll} (a) & V(\alpha) & = & V_g(\alpha); & V(\beta) = V_f(\beta) \\ (b) & V(\alpha) & = & V_f(\alpha); & V(\beta) = V_g(\beta) \\ (c) & V(\alpha) & = & V_g(\alpha); & V(\beta) = V_g(\beta) \\ (d) & V(\alpha) & = & V_f(\alpha); & V(\beta) = V_f(\beta) \end{array}$$

Case (a):

$$\begin{array}{l} \alpha \ \succsim \ \beta \qquad \Longleftrightarrow \qquad V_g(\alpha) \geq V_f(\beta) \\ \quad \Longrightarrow \quad \text{(by def of } V\text{)} \quad V_f(\alpha) \geq V_g(\alpha) \geq V_f(\beta) \\ \quad \Longrightarrow \quad V_f(\alpha) \geq V_f(\beta) \\ \quad \Longrightarrow \quad \text{(by (5))} \quad V_g(\alpha) \geq V_g(\beta) \end{array}$$

Now, for T to be ambiguous, we must have $\alpha' \prec \beta'$. Note that it cannot be the case that α' and β' are evaluated by means of the same functional, be it either V_f or V_g. For, if this were the case, (1) and the preceding would imply $\alpha' \succsim \beta'$. Hence, we are left with two possibilities:

$$\begin{array}{lrcl} (a.1) & V_g(\alpha') & < & V_f(\beta') \\ & & \Longrightarrow & \text{(by def of } V\text{)} \quad V_g(\alpha') < V_f(\beta') \leq V_g(\beta') \\ & & \Longrightarrow & V_g(\alpha') < V_g(\beta') \end{array}$$

which, by (1), contradicts $V_g(\alpha) \geq V_g(\beta)$.

$$\begin{array}{lrcl} (a.2) & V_f(\alpha') & < & V_g(\beta') \\ & & \Longrightarrow & \text{(by def of } V\text{)} \quad V_f(\alpha') < V_g(\beta') \leq V_f(\beta') \end{array}$$

which contradicts $V_f(\alpha) \geq V_f(\beta)$.

Similarly, Case (b):

$$\begin{aligned}
\alpha \quad \succsim \quad \beta \quad & \Longleftrightarrow \quad V_f(\alpha) \geq V_g(\beta) \\
& \Longrightarrow \quad \text{(by def of } V) \quad V_g(\alpha) \geq V_f(\alpha) \geq V_g(\beta) \\
& \Longrightarrow \quad \text{(by (5))} \quad V_f(\alpha) \geq V_f(\beta)
\end{aligned}$$

Just like before, we need to consider only two cases

$$\begin{aligned}
(b.1) \qquad V_f(\alpha') \quad & < \quad V_g(\beta') \\
& \Longrightarrow \quad V_f(\alpha') < V_f(\beta')
\end{aligned}$$

a contradiction.

$$\begin{aligned}
(b.2) \qquad V_g(\alpha') \quad & < \quad V_f(\beta') \\
& \Longrightarrow \quad V_g(\alpha') < V_g(\beta')
\end{aligned}$$

again, a contradiction.

We leave to the reader the easy verification of the other two cases. ∎

In order to prove Proposition 7, we are going to show that T non $\mathcal{F}$-measurable $\Longrightarrow T$ EZ-ambiguous. To do so, we are going to construct a pair of acts, α and β, satisfying Epstein-Zhang's definition which reveal that T is EZ-ambiguous. For this, we need the following

Lemma 16 *Under the hypothesis of Proposition 7, $\exists A, B \subset T$, $A, B \in \Sigma$, such that*

$$P_f(A) > P_f(B) \Longrightarrow P_g(A) < P_g(B)$$

Proof of the Lemma. Suppose not, that is $\forall A, B \subset T$, $A, B \in \Sigma$,

$$P_f(A) > P_f(B) \Longrightarrow P_g(A) \geq P_g(B)$$

This is equivalent to

$$\frac{P_f(A)}{P_f(T)} > \frac{P_f(B)}{P_f(T)} \Longrightarrow \frac{P_g(A)}{P_g(T)} \geq \frac{P_g(B)}{P_g(T)}$$

The conditional probabilities, $\frac{P_f(\cdot \cap T)}{P_f(T)}$ and $\frac{P_g(\cdot \cap T)}{P_g(T)}$ are both nonatomic, and Proposition 1 in [1] implies that they coincide, thus contradicting assumption (b'). ∎

It is immediate to check that the sets, A and B, whose existence is granted by the Lemma, can be taken so that $A \cap B = \emptyset$.

Proof of Proposition 7. Now, define α and β like in the EZ-definition by using the events A and B from the previous Lemma. First, pick y^* and y so that

$$V_g(\alpha) > V_g(\beta) \tag{6}$$

which implies

$$V_f(\alpha) < V_f(\beta) \qquad (\Longrightarrow \qquad V_f(\alpha') < V_f(\beta') \qquad) \tag{7}$$

Next, since T non $\mathcal{F}$-measurable $\Longrightarrow P_f(T^c) \neq P_g(T^c)$, by (2), we can always pick z in such a way that both

$$\begin{aligned} V_f(\alpha) - V_g(\alpha) &> 0 \\ V_f(\beta) - V_g(\beta) &> 0 \end{aligned}$$

(This can be obtained either for a certain choice of the utility function u, which is unique only to a positive affine transformation, or by offering an act which pays n times a certain given prize). In such a way, our acts α and β are such that

$$V(\alpha) = V_g(\alpha) \qquad \textit{and} \qquad V(\beta) = V_g(\beta) \tag{8}$$

By the same argument, we can find a z' such that both

$$\begin{aligned} V_f(\alpha') - V_g(\alpha') &< 0 \\ V_f(\beta') - V_g(\beta') &< 0 \end{aligned}$$

That is,

$$V(\alpha') = V_f(\alpha') \qquad \textit{and} \qquad V(\beta') = V_f(\beta') \tag{9}$$

Now, by (8), both α and β are evaluated by means of g, and (6) implies

$$\alpha \succ \beta$$

By (9), both α' and β' are evaluated by means of f, and (7) implies

$$\beta' \succ \alpha'$$

that is, T is EZ-ambiguous. ■

Proof of Lemma 8. By assumption,both $\mathcal{F}_1$ and $\mathcal{F}_2$ are non empty. Let $\alpha \succsim \beta$. If either (i) or (ii) is satisfied, we can find (see (2)) a z' such that

$$V_{\bar{f}}(\beta') - V_{\tilde{g}}(\beta') < 0 \qquad , \qquad \forall P_{\tilde{g}} \in \mathcal{F}_1$$

which implies that $V(\beta') = \inf_{P_l \in \mathcal{F}} V_l(\beta') = V_f(\beta')$ for some $P_f \in \mathcal{F}_2$.

For T to be unambiguous, it must be that

$$\alpha' \succsim \beta' \qquad \Longleftrightarrow \qquad V(\alpha') \geq V(\beta')$$

But, by the preceding and the definition of $V(\cdot)$, we have

$$V_f(\alpha') \geq V(\alpha') \geq V(\beta') = V_f(\beta')$$

which contradicts $f \in \mathcal{F}_2$. ■

Proof of Lemma 9 Let (α, β) be a pair of EZ-acts, and let $\alpha \succsim \beta$. Define

$$\begin{aligned}\mathcal{F}_1 &= \left\{\tilde{f} \in \mathcal{F} \mid V_{\tilde{f}}(\alpha) < V_{\tilde{f}}(\beta)\right\} \\ \mathcal{F}_2 &= \left\{\tilde{f} \in \mathcal{F} \mid V_{\tilde{f}}(\alpha) \geq V_{\tilde{f}}(\beta)\right\}\end{aligned}$$

From Lemma 8, we know that a necessary condition for T to be EZ-unambiguous is that $\forall P_f \in \mathcal{F}_1$, $\exists P_{g_1}, P_{g_2} \in \mathcal{F}_2$ such that

$$P_{g_1}(T^c) \leq P_f(T^c) \leq P_{g_2}(T^c)$$

Such a condition depends on the particular pair (α, β) we picked, and does not exclude that there exists another pair which would reveal that T is ambiguous. To avoid this, proceed as follows. Restrict attention to $\mathcal{F}_2$ and apply Lemma 8 to $\mathcal{F}_2$. That is, assume that there exist $P_{g'}$ and $P_{g''}$ in $\mathcal{F}_2$ such that their conditionals on T do not coincide. Then, we can find an EZ-pair of acts (α_2, β_2) such that $\alpha_2 \succsim \beta_2$ and

$$\begin{aligned}V_{g'}(\alpha_2) &> V_{g'}(\beta_2) \\ V_{g''}(\alpha_2) &< V_{g''}(\beta_2)\end{aligned}$$

Define

$$\begin{aligned}\mathcal{F}_3 &= \left\{P_{\tilde{f}} \in \mathcal{F}_2 \mid V_{\tilde{f}}(\alpha_2) \geq V_{\tilde{f}}(\beta_2)\right\} \\ \mathcal{F}_2^1 &= \left\{P_{\tilde{f}} \in \mathcal{F}_2 \mid V_{\tilde{f}}(\alpha_2) < V_{\tilde{f}}(\beta_2)\right\}\end{aligned}$$

Lemma 17 *A necessary condition for T to be EZ-unambiguous is that $\forall P_f \in \mathcal{F}_2^1$, $\exists P_{g_1}, P_{g_2} \in \mathcal{F}_3$ such that*

$$P_{g_1}(T^c) \leq P_f(T^c) \leq P_{g_2}(T^c)$$

Proof of the lemma. From Lemma 8, we know that if $\exists\ P_{\bar{g}} \in \mathcal{F}_2^1$ such that either

$$\begin{aligned}(i) \qquad P_{\bar{g}}(T^c) &> P_{\tilde{g}}(T^c) \qquad , \qquad \forall P_{\tilde{g}} \in \mathcal{F}_3 \\ &\textit{or} \\ (ii) \qquad P_{\bar{g}}(T^c) &< P_{\tilde{g}}(T^c) \qquad , \qquad \forall P_{\tilde{g}} \in \mathcal{F}_3\end{aligned}$$

then, T is EZ-ambiguous with respect to the family $\mathcal{F}_2$. We want to show that the consideration of the whole family $\mathcal{F}$ does not revert this conclusion, thus proving the claim. To do so, we are going to construct a pair (α_2, β_2) which reveals that T is EZ-ambiguous.

Suppose, to fix ideas, that we are in case (i) (case (ii) is, obviously, similar), and notice that, combined with the fist step in the proof of the proof of Lemma 9, this implies that $\max_{\mathcal{F}} P_f(T^c) = P_{\bar{g}}(T^c)$. Let $P_g \in \mathcal{F}_3$. Since $P_{\bar{g}} \in \mathcal{F}_2^1$, $P_{\bar{g}}$ and

P_g do not have the same conditionals. Hence, $\exists A_2, B_2 \subset T,\ A_2, B_2 \in \Sigma$, such that

$$\begin{aligned} P_{\bar{g}}(A_2) - P_{\bar{g}}(B_2) &> 0 \\ P_g(A_2) - P_g(B_2) &< 0 \end{aligned} \tag{10}$$

Pick y_2^* so that $u(y_2^*) < 0$, and pick y_2 so that $u(y_2) = -u(y_2^*)$. Hence, $[u(y_2^*) - u(y_2)] = 2u(y_2^*) < 0$. Hence, $\alpha_2 \succ \beta_2$ when using P_g, and the reverse is true when using $P_{\bar{g}}$ (see (1)). Now, pick $h \equiv 0$ in the EZ definition. From Lemma 8, we know that such a pair, (α_2, β_2), reveals that T is EZ-ambiguous with respect to the family $\mathcal{F}_2$.

We claim that (α_2, β_2) reveals that T is EZ-ambiguous with respect to the whole family $\mathcal{F}$.

Under our assumptions, $\exists z'$ such that

$$V_g(\beta_2') - V_{\bar{g}}(\beta_2') > 0 \qquad , \qquad \forall P_g \in \mathcal{F}_3$$

Now, suppose that T is not EZ-ambiguous. Then, in such a case, β_2' cannot be evaluated by $P_{\bar{g}}$ (this would contradict $\alpha_2' \prec \beta_2'$ for $P_{\bar{g}}$). Therefore, there must exist an $P_f \in \mathcal{F}_1$ such that

$$V_f(\beta_2') - V_{\bar{g}}(\beta_2') < 0$$

At the beginning of the proof of the Lemma 9, we established that

$$P_f(T^c) \leq P_{\bar{g}}(T^c) \qquad , \qquad \forall P_f \in \mathcal{F}_1$$

Hence, we have two possibilities.

$$(i) \qquad P_f(T^c) < P_{\bar{g}}(T^c) \qquad , \qquad \forall P_f \in \mathcal{F}_1$$

In such a case, we can define z' so that

$$V_f(\beta_2') - V_{\bar{g}}(\beta_2') > 0 \qquad , \qquad \forall P_f \in \mathcal{F}_1$$

which implies that β_2' is evaluated by $P_{\bar{g}}$, thus showing that T is ambiguous.

Or,

$$(ii) \qquad \exists P_f \in \mathcal{F}_1 : \qquad P_f(T^c) = P_{\bar{g}}(T^c)$$

and, for such an f, $V_f(\alpha_2') \geq V_f(\beta_2')$.

By (1), the latter condition is equivalent to

$$[u(y_2^*) - u(y_2)][P_f(A_2) - P_f(B_2)] \geq 0$$

which implies (since $[u(y_2^*) - u(y_2)] < 0$)

$$[P_f(A_2) - P_f(B_2)] \leq 0$$

Combining this with the first of (10), we have

$$\begin{array}{rcl} P_{\bar{g}}(A_2) - P_{\bar{g}}(B_2) & > & 0 \geq P_f(A_2) - P_f(B_2) \\ & \Longleftrightarrow & \\ P_f(B_2) - P_{\bar{g}}(B_2) & > & P_f(A_2) - P_{\bar{g}}(A_2) \end{array}$$

Since $u(y_2) > 0$, the latter implies

$$\begin{array}{rcl} 0 & < & u(y_2)[P_f(B_2) - P_{\bar{g}}(B_2)] - u(y_2)[P_f(A_2) - P_{\bar{g}}(A_2)] \\ & = & u(y_2)[P_f(B_2) - P_{\bar{g}}(B_2)] + u(y_2^*)[P_f(A_2) - P_{\bar{g}}(A_2)] \\ & = & (\text{ by } (2)) \qquad V_f(\beta_2') - V_{\bar{g}}(\beta_2') \end{array}$$

that is, no such a P_f can evaluate β_2'. In other words, β_2' is either evaluated by $P_{\bar{g}}$ or by some $P_{\tilde{f}}$ such that $V_{\tilde{f}}(\alpha_2) < V_{\tilde{f}}(\beta_2)$, thus contradicting T EZ-unambiguous.

This completes the proof of Lemma 17. ■

Proof of Lemma 9 (continued). We can now complete the proof of Lemma 9. It follows from Lemma 17 that, if T is EZ-unambiguous, by considering an EZ pair (α_2, β_2) like the one above, we can find a set $\mathcal{F}_3 \subseteq \mathcal{F}_2$ such that $\forall P_f \in \mathcal{F}_2^1 \cup \mathcal{F}_1$, $\exists P_{g_1}, P_{g_2} \in \mathcal{F}_3$ such that

$$P_{g_1}(T^c) \leq P_f(T^c) \leq P_{g_2}(T^c)$$

The set $\mathcal{F}_2^1 = \mathcal{F}_2 \backslash \mathcal{F}_3$ can be empty, which occurs if there exists no $P_{\tilde{f}} \in \mathcal{F}_2$ such that $V_{\tilde{f}}(\alpha) < V_{\tilde{f}}(\beta)$. In such a case, by (1), all the conditionals, $\frac{P_g(\cdot \cap T)}{P_g(T)}$, $P_g \in \mathcal{F}_2$, are the same, and the lemma is proven.

If $\mathcal{F}_2^1$ is non empty, then $\mathcal{F}_3 \subset \mathcal{F}_2$. Set $\mathcal{B}_3 = \mathcal{F}_1 \cup \mathcal{F}_2^1$, and notice that $\mathcal{F}_3 \cap \mathcal{B}_3 = \emptyset$, and $\mathcal{F}_3 \cup \mathcal{B}_3 = \mathcal{F}$.

Now that we are at $\mathcal{F}_3$, we can continue the construction so to obtain the chain $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \cdots$ (and the chain $\mathcal{B}_{n+1} \supseteq \mathcal{B}_n \supseteq \cdots$). It is clear, that the process stops only when we reach a set $\mathcal{A}$ such that for any EZ-pair, (α, β) we have $V_g(\alpha) \geq V_g(\beta)$ for all $P_g \in \mathcal{A}$. As we have seen (in the proof of Proposition 5) this occurs if and only if all the conditionals, $\frac{P_g(\cdot \cap T)}{P_g(T)}$, $P_g \in \mathcal{A}$, are the same. If such a set fails the condition

$\exists P_{g_1}, P_{g_2} \in \mathcal{A}$ such that for any $P_f \in \mathcal{B}$

$$P_{g_1}(T^c) \leq P_f(T^c) \leq P_{g_2}(T^c)$$

then, by Lemma 8, T is EZ-ambiguous. This completes the proof of Lemma 9. ■

Remark 18 *If in the above proof $\mathcal{A}$ can be taken to be a singleton, then $P_{g_1}(T^c) = P_{g_2}(T^c)$ and T is also $\mathcal{F}$-measurable. In such a case, by Proposition 3, the condition is also sufficient.*

Proof of Theorem 10. The necessity of the existence of a set of priors $\mathcal{A}$ satisfying conditions (i) and (ii) was already established in Lemma 9. Now, we establish the necessity of condition (iii). We proceed by contradiction, and divide the proof in several claims.

Let (α, β) be any two acts satisfying the Epstein-Zhang's definition. It is useful to think of such acts as functions of z, the prize that they both pay if the realized state is in T^c. So suppose that while for some $P_g \in \mathcal{A}$ (and, hence, for all $P_g \in \mathcal{A}$) $V_g(\alpha(z)) > V_g(\beta(z))$, we have $\beta(z) \succsim \alpha(z)$ (hence, $V(\beta(z)) \geq V(\alpha(z))$). Below, after Claim 11, we show that this leads to a contradiction.

Since by assumption T is EZ-unambiguous, we must have that $\forall z'$, $\beta(z') \succsim \alpha(z')$ [as $\beta(z) \succsim \alpha(z) \Longrightarrow \beta(z') \succsim \alpha(z')$].

CLAIM 1: $\forall z', \exists P_{f'} \in \mathcal{B} = \mathcal{F} \backslash \mathcal{A}$ such that
(a) $V_{f'}(\beta(z')) \geq V_{f'}(\alpha(z'))$
(b) $V(\alpha(z')) = V_{f'}(\alpha(z'))$
(c) $V_{f'}(\alpha(z')) < V_g(\alpha(z'))$, $\forall P_g \in \mathcal{A}$.
Proof of CLAIM 1. If (b) is not true, then for some $P_{\tilde{g}} \in \mathcal{A}$

$$V_{\tilde{g}}(\beta(z')) \geq V(\beta(z')) \geq V(\alpha(z')) = V_{\tilde{g}}(\alpha(z'))$$

thus contradicting $P_{\tilde{g}} \in \mathcal{A}$.

If (b) is satisfied, but (a) is not, we have

$$V_{f'}(\beta(z')) \geq V(\beta(z')) \geq V(\alpha(z')) = V_{f'}(\alpha(z')) > V_{f'}(\beta(z'))$$

which is again a contradiction.

Finally, if (c) is not satisfied, we have

$$V(\beta(z')) \geq V(\alpha(z')) = V_g(\alpha(z')) > V_g(\beta(z'))$$

again a contradiction.

CLAIM 2: If T EZ-unambiguous, $\forall \varepsilon > 0$ $\exists P_{\hat{f}}, P_{\check{f}} \in \mathcal{B}$ such that

$$\begin{aligned} P_{\hat{f}}(T^c) &< P_{g_1}(T^c) + \varepsilon \\ P_{\check{f}}(T^c) &> P_{g_2}(T^c) - \varepsilon \end{aligned}$$

Proof of CLAIM 2. Suppose that either condition is not satisfied. Say the first, that is $\exists \varepsilon > 0$ such that

$$P_f(T^c) \geq P_{g_1}(T^c) + \varepsilon \qquad , \qquad \forall P_f \in \mathcal{B}$$

Then, from equation (2), $\exists z'$ such that $\forall z : u(z) \geq u(z')$

$$V_{g_1}(\alpha(z)) \leq V_f(\alpha(z)) \qquad , \qquad \forall P_f \in \mathcal{B}$$

This implies a contradiction as for such a z'

$$V(\beta(z')) \geq V(\alpha(z')) = V_{g_1}(\alpha(z')) > V_{g_1}(\beta(z'))$$

Let

$$\begin{aligned}\bar{\mathcal{B}}_\varepsilon &= \{P_f \in \mathcal{B} \mid P_f(T^c) > P_{g_2}(T^c) - \varepsilon\} \\ \underline{\mathrm{B}}_\varepsilon &= \{P_f \in \mathcal{B} \mid P_f(T^c) < P_{g_1}(T^c) + \varepsilon\}\end{aligned}$$

CLAIM 3. $\forall \varepsilon > 0$, $\exists P_{\hat{f}} \in \underline{\mathrm{B}}_\varepsilon$ and $\exists P_{\check{f}} \in \bar{\mathcal{B}}_\varepsilon$ and two prizes, z_1 and z_2, such that

$$\begin{aligned}(d) \qquad V_{\hat{f}}(\alpha(z_1)) &= V(\alpha(z_1)) \qquad ; \qquad V_{\check{f}}(\alpha(z_2)) = V(\alpha(z_2)) \\ (e) \qquad V_{\hat{f}}(\beta(z_1)) &\geq V_{\hat{f}}(\alpha(z_1)) \qquad ; \qquad V_{\check{f}}(\beta(z_2)) \geq V_{\check{f}}(\alpha(z_2))\end{aligned}$$

Proof of CLAIM 3. Fix an $\varepsilon > 0$, and consider the sets

$$\begin{aligned}\mathcal{B}_1 &= \{P_f \in \mathcal{B} \mid P_f(T^c) \geq P_{g_1}(T^c) + \varepsilon\} \\ \mathcal{B}_2 &= \{P_f \in \mathcal{B} \mid P_f(T^c) \leq P_{g_2}(T^c) - \varepsilon\}\end{aligned}$$

By equation (2), $\exists z_1$ such that $\forall z : u(z) \geq u(z_1)$

$$V_f(\alpha(z_1)) - V_{g_1}(\alpha(z_1)) \geq 0 \qquad , \qquad \forall P_f \in \mathcal{B}_1$$

and $\exists z_2$ such that $\forall z : u(z) \leq u(z_2)$

$$V_f(\alpha(z_1)) - V_{g_2}(\alpha(z_1)) \geq 0 \qquad , \qquad \forall P_f \in \mathcal{B}_2$$

Since T EZ-unambiguous and $\beta(z) \succsim \alpha(z)$ imply both $\beta(z_1) \succsim \alpha(z_1)$ and $\beta(z_2) \succsim \alpha(z_2)$, there must exist (by CLAIM 1) $P_{\hat{f}}, P_{\check{f}} \in \mathcal{B}$ such that

$$\begin{aligned}V(\alpha(z_1)) &= V_{\hat{f}}(\alpha(z_1)) < V_{g_1}(\alpha(z_1)) \\ V(\alpha(z_2)) &= V_{\check{f}}(\alpha(z_2)) < V_{g_2}(\alpha(z_2))\end{aligned}$$

and, again by CLAIM 1, $P_{\hat{f}}$ and $P_{\check{f}}$ must have the property (e) in the statement.

Finally, by combining the last four inequalities, we have

$$\begin{aligned}V_{\hat{f}}(\alpha(z_1)) &< V_f(\alpha(z_1)) \qquad , \qquad \forall P_f \in \mathcal{B}_1 \\ V_{\check{f}}(\alpha(z_2)) &< V_f(\alpha(z_2)) \qquad , \qquad \forall P_f \in \mathcal{B}_2\end{aligned}$$

Hence,

$$\begin{aligned}P_{\hat{f}} &\in \mathcal{B} \backslash \mathcal{B}_1 \qquad \Longleftrightarrow \qquad P_{\hat{f}} \in \mathrm{B}_\varepsilon \\ P_{\check{f}} &\in \mathcal{B} \backslash \mathcal{B}_2 \qquad \Longrightarrow \qquad P_{\check{f}} \in \bar{\mathcal{B}}_\varepsilon\end{aligned}$$

Summarizing,

Let $(\alpha(z), \beta(z))$ be such that $V_g(\alpha(z)) > V_g(\beta(z))$, for some $P_g \in \mathcal{A}$. Suppose that $\beta(z) \succsim \alpha(z)$. Then, T EZ-unambiguous implies that $\forall \varepsilon > 0$ there exist $P_{\hat{f}} \in \underline{\mathrm{B}}_\varepsilon$ and $P_{\check{f}} \in \bar{\mathcal{B}}_\varepsilon$ such that

(e) $V_{\hat{f}}(\beta(z)) \geq V_{\hat{f}}(\alpha(z)) \qquad ; \qquad V_{\check{f}}(\beta(z)) \geq V_{\check{f}}(\alpha(z))$

(f) There exists a prize k such that $\forall z : u(z) \geq u(k)$

$$V_{\hat{f}}(\alpha(z)) < V_{g_1}(\alpha(z)) \leq V_f(\alpha(z)) \qquad , \qquad \forall P_f : P_f(T^c) \geq P_{g_1}(T^c) + \varepsilon$$

and $\forall z : u(z) \leq -u(k)$

$$V_{\check{f}}(\alpha(z)) < V_{g_2}(\alpha(z)) \leq V_f(\alpha(z)) \qquad , \qquad \forall P_f : P_f(T^c) \leq P_{g_2}(T^c) - \varepsilon$$

CLAIM 4: There exists an EZ-act, $\iota = \iota(z)$, *defined by using the same events* as in α and β above, such that $\forall \varepsilon > 0$

(g) $\exists P_{\hat{f}} \in \underline{\mathrm{B}}_\varepsilon$ and a prize $\tilde{z}'$ such that $\forall z : u(z) \geq u(\tilde{z}')$

$$V_{\hat{f}}(\iota(z)) < V_f(\iota(z)) \qquad , \qquad \forall P_f : P_f \in \mathrm{B} \backslash \underline{\mathrm{B}}_\varepsilon \tag{11}$$

(h) $\exists P_{\check{f}} \in \bar{\mathcal{B}}_\varepsilon$ and a prize $\tilde{z}''$ such that $\forall z : u(z) \leq u(\tilde{z}'')$

$$V_{\check{f}}(\iota(z)) < V_f(\iota(z)) \qquad , \qquad \forall P_f : P_f \in \mathrm{B} \backslash \bar{\mathcal{B}}_\varepsilon \tag{12}$$

(i) Moreover, both $P_{\hat{f}}$ and $P_{\check{f}}$ can be taken so that property (e) is satisfied.

Proof of CLAIM 4. By using the same events which define α and β, define a new act, $\iota = \iota(z)$, as follows

$$\iota(z) = (x, -x, 0, z)$$

where the notation means that ι pays x on A, $-x$ on B (or, rather, a prize whose utility is $-u(x)$), a fixed prize whose utility is 0 on $T \backslash (A \cup B)$ and z on T^c.

Moreover, take x so that $u(x) < 0$.

Given $\varepsilon > 0$, pick $P_{\hat{f}} \in \underline{\mathrm{B}}_{\varepsilon/2}$. By CLAIM 3, $P_{\hat{f}}$ can be taken so that property (e) is satisfied. $\forall P_f : P_f \in \mathrm{B} \backslash \underline{\mathrm{B}}_\varepsilon$, we have

$$P_f(T^c) \geq P_{g_1}(T^c) + \varepsilon > P_{\hat{f}}(T^c) + \frac{\varepsilon}{2}$$

and (see equation (2)), we can find a $\tilde{z}'$ such that $\forall z : u(z) \geq u(\tilde{z}')$

$$V_{\hat{f}}(\iota(z)) < V_f(\iota(z)) \qquad , \qquad \forall P_f : P_f \in \mathrm{B} \backslash \underline{\mathrm{B}}_\varepsilon$$

Similarly, for the other part.

CLAIM 5: Let $\iota(z)$ be the act defined in the previous claim. $\exists z', z''$ such that $\forall P_{\hat{f}}, P_{\check{f}}$ satisfying the properties of CLAIM 4, we have

(j) $\forall z : u(z) \geq u(z')$

$$V_{g_1}(\iota(z)) < V_{\hat{f}}(\iota(z))$$

(k) $\forall z : u(z) \leq u(z'')$

$$V_{g_2}(\iota(z)) < V_{\check{f}}(\iota(z))$$

Proof of CLAIM 5. By assumption for some $P_g \in \mathcal{A}$, and, hence, for all $P_g \in \mathcal{A}$ (see equation (1))

$$V_g(\alpha(z)) - V_g(\beta(z)) = [u(y^*) - u(y)][P_g(A) - P_g(B)] > 0$$

Without loss, assume $[u(y^*) - u(y)] > 0$. Hence, by equation (1), for all the $P_{\hat{f}}$'s satisfying the properties of CLAIM 4, we must have

$$[P_g(A) - P_g(B)] > 0 \geq [P_{\hat{f}}(A) - P_{\hat{f}}(B)] \tag{13}$$

and, similarly, for $P_{\check{f}}$ in the place of $P_{\hat{f}}$.

Next, observe that (see equation (2))

$$\begin{aligned} V_{\check{f}}(\iota(z'')) - V_{g_2}(\iota(z'')) &= u(x)\left\{[P_{\check{f}}(A) - P_{\check{f}}(B)] - [P_{g_2}(A) - P_{g_2}(B)]\right\} \\ &\quad + u(z)[P_{\check{f}}(T^c) - P_{g_2}(T^c)] \end{aligned}$$

By inequality (13) and $u(x) < 0$, the first addendum on the RHS is > 0. Hence, for any z such that $u(z) \leq 0$, we have

$$V_{\check{f}}(\iota(z)) > V_{g_2}(\iota(z))$$

Similarly, for any z such that $u(z) \geq 0$, we have

$$V_{\hat{f}}(\iota(z)) > V_{g_1}(\iota(z))$$

By combining the inequalities in CLAIM 5 with (11) and (12), we have that $\forall \varepsilon > 0$, there exist prizes z' and z'' such that

(l) $\forall z : u(z) \geq u(z')$

$$V_{g_1}(\iota(z)) < V_{\hat{f}}(\iota(z)) \leq V_f(\iota(z))$$

for all $P_{\hat{f}}$ satisfing the properties of CLAIM 4, and $\forall P_f : P_f \in \mathrm{B} \backslash \underline{\mathrm{B}}_\varepsilon$.

(m) $\forall z : u(z) \leq u(z'')$

$$V_{g_2}(\iota(z)) < V_{\check{f}}(\iota(z)) \leq V_f(\iota(z))$$

for all $P_{\check{f}}$ satisfying the properties of CLAIM 4, and $\forall P_f : P_f \in \mathrm{B} \backslash \bar{\mathrm{B}}_\varepsilon$.

Now, observe that (l) implies that $\forall z : u(z) \geq u(z')$, $\iota(z)$ is evaluated either by V_{g_1} or by some $V_{\gamma_1} \in \underline{\mathrm{B}}_\varepsilon$. By (l) and inequality (13), such a $V_{\gamma_1} \in \underline{\mathrm{B}}_\varepsilon$ has necessarily the property that

$$V_{\gamma_1}(\alpha(z)) > V_{\gamma_1}(\beta(z))$$

as for $u(z) \geq 0$, $V_{\gamma_1}(\iota(z)) \leq V_{g_1}(\iota(z))$ requires

$$u(x)\left\{[P_{\gamma_1}(A) - P_{\gamma_1}(B)] - [P_{g_1}(A) - P_{g_1}(B)]\right\} \leq 0$$

which implies $[P_{\gamma_1}(A) - P_{\gamma_1}(B)] > 0$.

Moreover, since (if it exists) $P_{\gamma_1} \in \underline{\mathrm{B}}_\varepsilon$, we have that either $P_{\gamma_1}(T^c) > P_{g_1}(T^c)$ or $P_{\gamma_1}(T^c) = P_{g_1}(T^c)$.

In the first case, we can take z' so that $\forall z : u(z) \geq u(z')$

$$(n) \qquad V_{\gamma_1}(\iota(z)) > V_{g_1}(\iota(z))$$

In the second case, we can take z' so that $\forall z : u(z) \geq u(z')$

$$(o) \qquad V_{\gamma_1}(\alpha(z)) < V_f(\alpha(z)) \quad , \quad \forall P_f : P_f \in \mathrm{B} \backslash \underline{\mathrm{B}}_\varepsilon$$

Similarly, (m) implies that $\forall z : u(z) \leq u(z'')$, $\iota(z)$ is evaluated either by V_{g_2} or by some $V_{\gamma_2} \in \bar{\mathrm{B}}_\varepsilon$ with the property that

$$V_{\gamma_2}(\alpha(z)) > V_{\gamma_2}(\beta(z))$$

and we can take z'' in such a way that either

$$(p) \qquad V_{\gamma_2}(\iota(z)) > V_{g_2}(\iota(z))$$

(if $P_{\gamma_2}(T^c) < P_{g_2}(T^c)$) or

$$(q) \qquad V_{\gamma_2}(\alpha(z)) < V_f(\alpha(z)) \quad , \quad \forall P_f : P_f \in \mathrm{B} \backslash \bar{\mathrm{B}}_\varepsilon$$

(if $P_{\gamma_2}(T^c) = P_{g_2}(T^c)$)

Let us denote by V_{δ_1} the functional which evaluates $\iota(z)$ for $z : u(z) \geq u(z')$, with the understanding that V_{δ_1} is either V_{g_1} or V_{γ_1}. The notation V_{δ_2} has analogous meaning.

CLAIM 6: There exists a prize z^* such that, for all $P_{\hat{f}}$'s satisfying CLAIM 4 and for all $P_f \in \mathrm{B} \backslash \underline{\mathrm{B}}_\varepsilon$, all the following inequalities hold for $z : u(z) \geq u(z^*)$

$$V_{\hat{f}}(\alpha(z)) < V_{\delta_1}(\alpha(z)) \leq V_f(\alpha(z))$$

$$V_{\delta_1}(\iota(z)) < V_{\hat{f}}(\iota(z)) \leq V_f(\iota(z))$$

Proof of CLAIM 6. It follows at once from property (f) following CLAIM 3 and properties (l), (n) and (o) following CLAIM 5.

In particular, we have

$$V(\alpha(z^*)) = V_{\hat{f}}(\alpha(z^*))$$

for some $P_{\hat{f}}$ satisfying CLAIM 4, and

$$V(\iota(z^*)) = V_{\delta_1}(\iota(z^*))$$

where V_{δ_1} is such that

$$V_{\delta_1}(\alpha(z)) > V_{\delta_1}(\beta(z))$$

Now, define a new act, $\theta_a(z)$, which is a convex combination of $\alpha(z)$ and $\iota(z)$. That is, $\theta_a(z) = a\alpha(z) + (1-a)\iota(z)$, $a \in (0,1)$. Observe that, since $\alpha(z)$ and $\iota(z)$ use the same events, $\theta_a(z)$ is an EZ-act which uses the same events.

In addition, denote by $\breve{\theta}_a(z)$ $[\breve{\iota}(z)]$ the act obtained from $\theta_a(z)$ $[\iota(z)]$ by switching the prizes between A and B. Notice that $\breve{\theta}_a(z) = a\beta(z) + (1-a)\breve{\iota}(z)$.

By using equation (1), one sees that

$$V_g(\theta_a(z)) - V_g(\breve{\theta}_a(z)) = [a(u(y^*) - u(y)) + (1-a)2u(x)][P_g(A) - P_g(B)] \quad (14)$$

and we have analogous expressions for $P_{\hat{f}}$ or $P_{\check{f}}$ replacing P_g.

Recall (proofs of CLAIMs 4 and 5) that $(u(y^*)-u(y)) > 0$ and that $u(x) < 0$.

Take x such that $[a(u(y^*) - u(y)) + (1-a)2u(x)] > 0$ (while still $u(x) < 0$).

For later reference (CLAIM 9), observe that $[a(u(y^*)-u(y))+(1-a)2u(x)] > 0$ implies

$$a > \frac{-2u(x)}{u(y^*) - u(y) - 2u(x)} = a^*$$

and observe that $a^* > 0$ and that a^* can be made arbitrarily small as x is matter of our own choice. Then, by (13)

$$V_g(\theta_a(z)) > V_g(\breve{\theta}_a(z)) \quad (15)$$

and, again by inequality (13), we have that

$$\begin{aligned} V_{\hat{f}}(\theta_a(z)) &\leq V_{\hat{f}}(\breve{\theta}_a(z)) \quad (16) \\ V_{\check{f}}(\theta_a(z)) &\leq V_{\check{f}}(\breve{\theta}_a(z)) \end{aligned}$$

for all $P_{\hat{f}}$,$P_{\check{f}}$ satisfying CLAIM 4.

CLAIM 7: $\forall a \in (0,1)$

$$V_{\hat{f}}(\theta_a(z^*)) \leq V_f(\theta_a(z^*))$$

for all $P_{\hat{f}}$'s satisfying CLAIM 4 and for all $P_f \in \mathrm{B}\backslash\underline{\mathrm{B}}_{\varepsilon}$.

Proof of CLAIM 7. Since, for each P_f, V_f is linear, this follows at once from CLAIM 6.

CLAIM 8: $\exists a_1 \in (0,1)$ such that $\forall a \in [0, a_1]$ and for all $P_{\hat{f}}$'s satisfying CLAIM 4

$$V_{\delta_1}(\theta_a(z^*)) \leq V_{\hat{f}}(\theta_a(z^*))$$

Proof of CLAIM 8.

$$
\begin{array}{rcl}
V_{\delta_1}(\theta_a(z^*)) & \leq & V_{\hat{f}}(\theta_a(z^*)) \\
& \Longleftrightarrow & \\
V_{\delta_1}(a\alpha(z^*)+(1-a)\iota(z^*)) & \leq & V_{\hat{f}}(a\alpha(z^*)+(1-a)\iota(z^*)) \\
& \Longleftrightarrow & \\
aV_{\delta_1}(\alpha(z^*))+(1-a)V_{\delta_1}(\iota(z^*)) & \leq & aV_{\hat{f}}(\alpha(z^*))+(1-a)V_{\hat{f}}(\iota(z^*))
\end{array}
$$

Hence, the statement follows immediately from CLAIM 6.

CLAIM 9: $\forall a \in (a^*, a_1)$, $\theta_a(z^*) \succ \breve{\theta}_a(z^*)$.

Proof of CLAIM 9. By CLAIM 8, $V_{\delta_1}(\theta_a(z^*)) \leq V_{\hat{f}}(\theta_a(z^*))$ for any $P_{\hat{f}}$ satisfying CLAIM 4. Hence, if $V(\theta_a(z^*)) = V_\delta(\theta_a(z^*))$ for some $P_\delta \in \mathcal{F}$, we necessarily have $V_\delta(\alpha(z^*)) > V_\delta(\beta(z^*))$. In fact, it is clear that P_δ cannot be in $\mathrm{B} \backslash \underline{\mathrm{B}}_\varepsilon$ as, if this were the case, we could always redefine z^* in a way that V_δ does not evaluate $\theta_a(z^*)$. Hence, it must be that $P_\delta \in \underline{\mathrm{B}}_\varepsilon$. Moreover, it cannot be that $V_\delta(\alpha(z^*)) \leq V_\delta(\beta(z^*))$ because in such a case P_δ would satisfy CLAIM 4, thus contradicting CLAIM 8. We, then, conclude that $V_\delta(\alpha(z^*)) > V_\delta(\beta(z^*))$, and, hence, $[P_\delta(A) - P_\delta(B)] > 0$. The latter implies $V_\delta(\theta_a(z^*)) > V_\delta(\breve{\theta}_a(z^*))$ (see (14) for an analogous expression, or, directly equation (1)). Hence,

$$
\begin{array}{rcll}
V(\theta_a(z^*)) & = & V_\delta(\theta_a(z^*)) & \\
& > & V_\delta(\breve{\theta}_a(z^*)) & \text{(by } a > a^*) \\
& \geq & V(\breve{\theta}_a(z^*)) &
\end{array}
$$

CLAIM 10: $\exists \bar{z}$ such that $\forall z : u(z) \leq u(\bar{z})$ all the following inequalities are true

$$V_{\check{f}}(\breve{\theta}_a(z)) < V_f(\breve{\theta}_a(z))$$

$$V_{\delta_2}(\breve{\theta}_a(z)) < V_f(\breve{\theta}_a(z))$$

for all $P_{\check{f}}$ satisfying CLAIM 4 and for all $P_f \in \mathrm{B} \backslash \bar{\mathrm{B}}_\varepsilon$.

Proof of CLAIM 10. The first follows from the fact that the $P_{\check{f}}$'s satisfying CLAIM 4 are in $\bar{\mathrm{B}}_{\varepsilon/2}$. The second follows from equation (2) and from the properties (p) and (q) following CLAIM 5.

All the properties we have shown so far hold $\forall \varepsilon > 0$. Next, we choose ε so that it belongs to a certain neighborhood of 0.

For any z and for any $P_{\check{f}}$ satisfying CLAIM 4 (see equation (2)), we have

$$
\begin{array}{rcl}
V_{\check{f}}(\breve{\iota}(z)) - V_{g_2}(\breve{\iota}(z)) & = & -u(x)\left\{[P_{\check{f}}(A) - P_{\check{f}}(B)] - [P_{g_2}(A) - P_{g_2}(B)]\right\} \\
& & +u(z)[P_{\check{f}}(T^c) - P_{g_2}(T^c)]
\end{array}
$$

By (13) and $u(x) < 0$, the first addendum is < 0. Hence, the whole expression is < 0 if either $P_{\check{f}}(T^c) = P_{g_2}(T^c)$, or if $u(z) > \frac{u(x)\left\{[P_{\check{f}}(A)-P_{\check{f}}(B)]-[P_{g_2}(A)-P_{g_2}(B)]\right\}}{[P_{\check{f}}(T^c)-P_{g_2}(T^c)]}$ whenever $[P_{\check{f}}(T^c) - P_{g_2}(T^c)] < 0$.

Since, by CLAIM 2, $\forall \varepsilon > 0$ there exists a $P_{\check{f}}$ satisfying all the above properties, we can now choose ε in such a way that for some $u(z^{**}) \leq u(\bar{z})$, we have $u(z^{**}) > \frac{u(x)\left\{[P_{\check{f}}(A)-P_{\check{f}}(B)]-[P_{g_2}(A)-P_{g_2}(B)]\right\}}{[P_{\check{f}}(T^c)-P_{g_2}(T^c)]}$. At a such z^{**}, we have

$$V_{\check{f}}(\check{\imath}(z^{**})) - V_{g_2}(\check{\imath}(z^{**})) < 0 \tag{17}$$

Clearly, the same is true for V_{γ_2} in the place of V_{g_2}.

CLAIM 11: $\exists a_2 \in (0,1)$ such that $\forall a \in [0, a_2]$

$$V_{\check{f}}(\breve{\theta}_a(z^{**})) \leq V_{\delta_2}(\breve{\theta}_a(z^{**}))$$

Proof of CLAIM 11.

$$\begin{array}{rcl} V_{\check{f}}(\breve{\theta}_a(z^{**})) & \leq & V_{\delta_2}(\breve{\theta}_a(z^{**})) \\ & \Longleftrightarrow & \\ aV_{\check{f}}(\beta(z^{**})) + (1-a)V_{\check{f}}(\check{\imath}(z^{**})) & \leq & aV_{\delta_2}(\beta(z^{**})) + (1-a)V_{\delta_2}(\check{\imath}(z^{**})) \\ & \Longleftrightarrow & \\ (1-a)\left\{V_{\check{f}}(\check{\imath}(z^{**})) - V_{\delta_2}(\check{\imath}(z^{**}))\right\} & \leq & a\left\{V_{\delta_2}(\beta(z^{**})) - V_{\check{f}}(\beta(z^{**}))\right\} \end{array}$$

Hence, the statement follows from (17).

Notice that, by (17), the inequality in CLAIM 11 holds for any V_δ such that $V_\delta(\alpha(z^{**})) > V_\delta(\beta(z^{**}))$. Combined with CLAIM 10, this implies

$$V(\breve{\theta}_a(z^{**})) = V_{\check{f}}(\breve{\theta}_a(z^{**}))$$

for some $P_{\check{f}}$ satisfying CLAIM 4.

Now, with all the above facts available, we can complete the proof that if T is EZ-unambiguous, then the set of priors $\mathcal{A}$ satisfying conditions (i) and (ii) has to satisfy condition (iii), also.

Recall that a^* can be made arbitrarily small. This guarantees that $(a^*, a_1) \cap [0, a_2]$ is nonempty. Take $a \in (a^*, a_1) \cap [0, a_2]$. Then, CLAIM 11 implies

$$\begin{array}{rcll} V(\breve{\theta}_a(z^{**})) & = & V_{\check{f}}(\breve{\theta}_a(z^{**})) & \\ & \geq & V_{\check{f}}(\theta_a(z^{**})) & \text{(by inequality (16))} \\ & \geq & V(\theta_a(z^{**})) & \end{array}$$

Hence, $\breve{\theta}_a(z^{**}) \succsim \theta_a(z^{**})$.

Since, by assumption T is EZ-unambiguous, $\breve{\theta}_a(z^{**}) \succsim \theta_a(z^{**}) \Longrightarrow \breve{\theta}_a(z^*) \succsim \theta_a(z^*)$, but this contradicts CLAIM 9.

Now, we are going to show that if T is EZ-unambiguous, then there exists a unique maximal (in the sense of inclusion) set of priors, $\mathcal{A}$, satisfying (i), (ii) and (iii).

Suppose that there exist two subsets, $\mathcal{A}_1$ and $\mathcal{A}_2$, with $\mathcal{A}_1 \neq \mathcal{A}_2$ satisfying (i) to (iii), and assume that they are both maximal. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are associated to the same conditional probability on T, then $\mathcal{A}_1 \cup \mathcal{A}_2$ satisfies (i) to (iii), thus contradicting the maximality of $\mathcal{A}_1$ and $\mathcal{A}_2$. Hence, assume that there exist $P_g \in \mathcal{A}_1$ and $P_f \in \mathcal{A}_2$ such that their conditionals on T do not coincide. Then, (by Lemma 16) there exists an EZ-pair, (α, β), such that $V_g(\alpha) > V_g(\beta)$ and $V_f(\alpha) < V_f(\beta)$. But, since both $\mathcal{A}_1$ and $\mathcal{A}_2$ have property (iii), we have

$$\begin{aligned} V_g(\alpha) &> V_g(\beta) \Longrightarrow \alpha \succ \beta \\ V_f(\alpha) &< V_f(\beta) \Longrightarrow \beta \succ \alpha \end{aligned}$$

a contradiction.

We now show the sufficiency of the condition.

Let $\mathcal{A}$ be the maximal set of priors satisfying (i) to (iii). We already know that all the $P_g \in \mathcal{A}$ produce the same ranking for all EZ-acts, and that if $P_g \in \mathcal{A}$ and $V_g(\alpha) > V_g(\beta)$, then $\alpha \succ \beta$. Hence, since $\forall z'$, $V_g(\alpha) > V_g(\beta) \Longrightarrow V_g(\alpha') > V_g(\beta')$, we have $\alpha \succ \beta \Longrightarrow \alpha' \succ \beta'$.

To complete the proof, we have only to show that

$$V_g(\alpha) = V_g(\beta) \Longrightarrow \alpha \sim \beta$$

Suppose not. For instance,

$$V_g(\alpha) = V_g(\beta) \qquad \textit{but} \qquad \alpha \prec \beta$$

Consider a family $\{\gamma_n\}_{n \in N}$ of EZ-acts defined by

$$\gamma_n = (1 - a_n)\alpha + a_n[\sup_{s \in S} \alpha(s) + x]$$

where $\{a_n\}_{n \in N} \subset [0, 1]$, $a_n \neq 0$ and $a_n \to 0$, x is a constant act such that $u(x) = 0$ and $\sup_{s \in S} \alpha(s)$ is defined by means of the (induced) preference relation over the prize space.

For any n and any $P_g \in \mathcal{A}$, we have $V_g(\gamma_n) > V_g(\alpha)$, which implies (by (iii)) $\gamma_n \succ \alpha$. Moreover, since $V_g(\alpha) = V_g(\beta)$, we also have $V_g(\gamma_n) > V_g(\beta)$, which implies (again by (iii)) $\gamma_n \succ \beta$, $\forall n \in N$.

Continuity of the class of preferences we have studying (see [17]) along with $\gamma_n \succ \beta$, $\forall n \in N$, then imply $\alpha = \lim_{a_n \to 0} \gamma_n \succsim \beta$, which contradicts $\alpha \prec \beta$.

Hence, $\mathcal{A}$ determines the ranking of all (T-based) EZ-acts, thus implying that T is EZ-unambiguous. ■

In the next proof, we refer to acts satisfying the EZ-definition as to T-based EZ-acts.

Proof of Theorem 11. By Theorem 10, if T is EZ-unambiguous, either T is $\mathcal{F}$-measurable or the conditions in Theorem 10 are satisfied. In the latter case, the family $\mathcal{A}$ of Theorem 10 determines the ranking of all T-based EZ-acts. EZ-acts conditional on T can be identified to a subset of the T-based EZ-acts, those for which $u(z) \equiv 0$. As such, the ordering of the EZ-acts conditional on T is entirely determined by the family $\mathcal{A}$. Such an ordering corresponds to the one defined by the conditional probabilities associated to the members of $\mathcal{A}$, which are all the same. Finally, notice that the class of EZ-acts conditional on T contains all the simple acts. Hence, by using the utility function, it can be embedded in the space of all simple functions $S \to R$. Then, the statement follows by applying the Monotone Convergence Theorem.

Sufficiency follows immediately by using the above reasoning from Theorem 10 and Proposition 3. ■