Why Homelessness? Some Theory

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WHY HOMELESSNESS? SOME THEORY

I. INTRODUCTION

During the 1980's, a decade of relative prosperity, the number of people living in the streets, in subway stations, cardboard boxes, bus terminals, or severely derelict buildings probably grew substantially in most American cities. The number of people living in homeless shelters and welfare hotels definitely skyrocketed. By 1990, one person in 200 in New York City was living either on the streets or in a shelter, and each of Manhattan's major transportation terminals had the population of a medium-sized apartment building, but no apartments.

Why did this happen? In this paper I will try to give an answer. Essentially, I will show how increases in income inequality and in real interest rates -- two well-documented phenomena of the '80's -- can cause homelessness to rise. In addition, I will consider how people (altruistic people, especially) respond to homelessness and show how even optimal responses can induce greater and more persistent "homelessness" (in the American sense that includes shelter use) than none at all.
Why is homelessness so difficult a problem for economic theory that it requires a whole paper to be written about it? Subsequent papers will show that in North America the rise in homelessness has been accompanied by a rise in the rents that poor people pay, continued abandonment of residential property in some of the cities where homelessness is most severe (although at a diminished rate from the '70s), and no strong trends in the real costs of operating housing. These (stylized) facts are incompatible with rising homelessness in any standard textbook model of housing as a single homogeneous commodity (e.g., Mills and Hamilton [1990], Chapter 10).

For if homelessness resulted from a simple shock that shifted the demand schedule downward -- a poverty shock, for instance, from falling public assistance levels and deteriorating labor market rewards at the lower end of the skill distribution -- then rents should have fallen, not risen. On the other hand, a gentrification shock that moved the demand curve up could raise rents and drive poor people out of their homes, but (with homogeneous housing) could not do so without first causing a cessation of abandonment. Homelessness, rising rents, and continuing abandonment are compatible with a supply shock, but there is no independent evidence that one occurred. (These arguments are discussed in more detail in O'Flaherty [1991a].) Explaining these stylized facts requires a more complex model of the housing market. That is what this paper tries to do.
The paper is severely limited. First, as noted already, it is primarily theoretical and will not attempt to defend the stylized facts it will try to explain. Second, since these stylized facts are generally true only for North American cities, the model should be thought of primarily as a North American one. This does not mean that it is totally inapplicable to Europe; but the problems of data comparability between Europe and North America are so great that explicit comparisons are best left to subsequent work (See O'Flaherty [1991b]). Finally, it will not address the many other purported explanations of homelessness -- deinstitutionalization, for instance -- that have abounded in both journalistic and scholarly discourse. These, too, will be left to the future.

What the paper does do is present a coherent partial equilibrium model that explains not only homelessness, but also several of the other stylized facts of North American urban markets in the 1980's. I don't know any other model that does this.

The next section is a warm-up; it shows how simple changes in income distribution without any changes in true prices can explain some, but not all, of the stylized facts. Section 3 is the paper's heart, a filtering model that explains most stylized facts. Sections 4 and 5 examine extensions of the basic model: section 4 is about rent collection costs, building codes, and
traditional housing programs; and section 5 is about shelters. Section 6 concludes.

2. SIMPLE CHANGES IN INCOME DISTRIBUTION

Let there be two qualities of housing, one and two, with two better than one. Let quality zero correspond to being homeless. There is a continuum of measure one of consumers, and a nonhousing good which we use as a numeraire. Each consumer buys at most one unit of housing, and has the same utility function \( u(q,x) \), where \( q \in \{0,1,2\} \) denotes the quality of housing (or lack thereof) she consumes, and \( x \in \mathbb{R}^+ \) denotes the amount of the numeraire good she consumes.

We assume that the utility function has the following properties:

(u-1) \( u(.,.,) \) is strictly increasing in both of its arguments
(u-2) for any \( q \), \( u(q,. \) ) is twice continuously differentiable and strictly convex downward.
(u-3) for any \( x \), \( u_x(q,x) \) is nondecreasing in \( q \).

Properties (u-1) and (u-2) are standard; (u-3) essentially implies a "single-crossing property." It is satisfied, for instance, if changes in quality always add the same amount to utility; that is, if utility is additively separable in quality and nonhousing consumption. Property (u-1), though standard, is
not in this context completely innocuous; it implies that
nonhousing consumption has positive marginal utility for
homeless people.

Denote
\[ v(q, p, y) = u(q, y-p_q) \quad q \in \{0, 1, 2\}, \text{ } y \geq p_q \]
where \( p_q \) is the price of one unit of quality \( q \) housing.
Clearly, a consumer will choose that housing quality \( q \) for which
\( v(q, p, y) \) is greatest. Assume \( p_2 > p_1 > p_0 = 0 \).

Properties (u-1) through (u-3) are sufficient to assure that
higher income consumers never demand lower quality housing.
Specifically, if \( i > j \) and
\[ v(i, p, y) \geq v(j, p, y), \]
then
\[ v(i, p, y') > v(j, p, y') \]
for any
\[ y' > y. \]
This is because
\[ v(i, p, y') = v(i, p, y) + \int_y^{y'} \frac{\partial v(i, p, t)}{\partial t} \, dt \]
\[ = v(i, p, y) + \int_{y-p_i}^{y'-p_i} u_x(i, t') \, dt' \]
\[ \geq v(j, p, y) + \int_{y-p_i}^{y'-p_i} u_x(i, t') \, dt' \quad (by \ hypothesis) \]

5
Thus there is an income $y_h$ (possibly zero or infinite) such that $v(o, p, y) \geq v(1, p, y)$ for all $y \leq y_h$; and the reverse for $y \geq y_h$. Similarly, there is an income $y_1$, that divides incomes $y$ such that $v(1, p, y) \geq v(2, p, y)$ from incomes where the reverse holds. If $0 < y_h < y_1$, then consumers with income below $y_h$ will be homeless, those with income between $y_h$ and $y_1$, will demand low quality housing, and those with income $y_h$ or above will demand high quality housing. (Consumers with income exactly $y_h$, or exactly $y_1$, are indifferent between the two relevant qualities of housing; I will adopt the convention that they choose the higher quality. Since I always work with atomless income distributions, this convention will be of no significance). If $y_h > y_1$, low income consumers (those with income below $y_1$) will be homeless, and the rest of the population will demand high quality housing; no one will demand low quality housing. This could happen if $p_i$
is very close to $p_2$. We will assume, however, that low quality housing is demanded; so $y_1 > y_h$.

Let $F(.)$ be the cdf of the income distribution. Then the proportion of the population homeless is

$$h = F(y_h),$$

the proportion of the population that wants to live in low quality housing is

$$d_1 = F(y_1) - F(y_h),$$

and the proportion that wants to live in high quality housing is

$$d_2 = 1 - F(y_1).$$

In equilibrium the proportion of people desiring each type of housing and the proportion living in it must be the same.

To isolate at the simplest level the effect of a change in income inequality, assume that $p_1$ and $p_2$ are fixed (as they would be, for instance, if production technology were linear). Then $y_1$, and $y_h$ are fixed also.

To be concrete, so that changes in inequality are well defined, suppose income is normally distributed. Generalizing these results will be done shortly. Income is originally distributed normally with parameters $(\mu, \sigma)$ and the parameters change to become

$$(\mu, \sigma'), \text{ with } \sigma' > \sigma.$$  

Let $x_1^*$, and $x_2^*$ solve

$$\phi(x^*|\mu, \sigma) = \phi(x^*|\mu, \sigma')$$

where $\phi$ is the normal pdf, and let $x_1^* < x_2^*$.

As long as less than half the population was originally
homeless, such a change must increase homelessness. The new, more unequal cdf is above the old cdf everywhere below median income, and below it everywhere above median income. Hence, if \( \Phi \) is the normal cdf,

\[
h' = \Phi(y_h|\mu, \sigma') > \Phi(y_h|\mu, \sigma) = h
\]

as long as \( y_h < y \). If, in addition, \( y_1 > y \), then the quantity of high-quality housing must also increase

\[
s'_2 = 1 - \Phi(y_1|\mu, \sigma') > 1 - \Phi(y_1|\mu, \sigma) = s_2
\]

If the homeless and high quality proportions both increase, then the low quality proportion must decrease: \( s'_1 < s_1 \). To rephrase this result in the words of homeless advocates: "low income housing will decrease." This testable prediction is reasonably accurate for the U.S. in the 1980s.

There are several other interesting predictions as well: a decrease in the proportion of dilapidated housing, and an increase in average rent. If fires are more likely in low-quality housing, then fires should decrease when homelessness rises. These predictions, too, are supported empirically.

The condition \( y_1 > \mu \) is sufficient for low quality housing to decrease, but it is not necessary. Another sufficient condition is

\[
x_i' < y_h < \mu
\]

This is because

\[
s'_1 - s_1 = \int_{y_h}^{\infty} \Phi(y|\mu, \sigma') dy - \int_{y_h}^{\infty} \Phi(y|\mu, \sigma) dy
\]
and for all

\[ y \in (x_1^*, x_2^*) \]

the expression in square brackets is negative. Hence if

\[ x_1 < x_2^* \]

then \((s_1' - s_1)\) is negative from the above expression, and if

\[ y_1 > x_2^*, \text{then } y_1 > x_2^* > \mu \]

and the first condition holds. In either case, low quality housing decreases.

Even if both \(y_1 < y\) and \(y_h < x_1^*\), it is still possible for low quality housing to decrease. For if \(y_1\), is considerably above \(x_1^*\), and \(y_h\) is not too far below it, the decreases in low quality housing above \(x_1^*\), will outweigh the increases below it. Thus while it is not impossible that an increase in income inequality would increase the quantity of low income housing, many plausible scenarios can be constructed where rising inequality decreases the quantity of low income housing.

It is easy to see that the more realistic case of a lognormal income distribution presents no real problems: one need only substitute \(\exp y_h\) and \(\exp y_1\) for \(y_h\) and \(y_1\), and all the previous results go through unchanged. The experiment where the parameters \((\mu, \sigma)\) change to \((\mu, \sigma')\), however, has to be interpreted differently. Since the mean of a lognormal distribution is
\[ \exp \left( \mu + \frac{1}{2} \sigma^2 \right) \]

and the variance is

\[ \exp \left[ 2\mu + 2\sigma^2 \right] - \exp \left[ 2\mu + \sigma^2 \right] \]

the experiment is one where both the mean and the variance increase (but the median and mode stay the same); homelessness increases and in many cases low quality housing decreases.

In fact, the results can be generalized to an arbitrary income distribution with cdf \( F \). Let \( F' \) differ from \( F \) in that there exists a unique income \( \mu (F, F') \)

such that

\[ [F'(y) - F(y)] [y - \mu_{F'}(F')] > 0 \]

for all \( y > \mu (F, F') \)

and a unique income

\( x^*(F, F') < \mu(F, F') \)

such that

\[ f'(x^*(F, F')) = f(x^*(F, F')) \]

Then homelessness greater under \( F' \) than under \( F \) as long as

\[ y_h < \mu(F, F') \]

and if either

\[ y_1 > \mu(F, F') \]

or

\[ y_h > x^*(F, F') \]

low quality housing will decrease under \( F' \).
While this story is extremely simple and fairly general and can explain several of the stylized facts, it is not entirely satisfactory. Two problems stand out. First, the housing market results are extremely sensitive to the convention one uses to define "low quality." If in fact the physical world presents us with a continuum of housing of different qualities, then the model's predictions about "low quality housing" will not be robust to changes in the cutoff one uses to distinguish "low quality" from "high quality". Of course, being robust to changes in definition is not usually a desirable characteristic for theories to have (the theory of gravitation, for instance, should not continue to hold after the words "up" and "down" are interchanged). In this case, however, the model also predicts that for some definitions of "low quality", low quality housing should have increased. It is an open question whether the data support this prediction.

Second, the model does not explain why rents as a proportion of income appear to have risen, especially at the low end of the income distribution. (Some questions have also been raised about this particular stylized fact -- in particular whether income is being measured properly -- but the existing consensus supports it, and I will examine it in subsequent work.) In this model, that cannot happen. Everyone with income in the appropriate range always buys the same quality of housing, and that quality of housing always costs the same.
3. PRODUCTION, FILTERING, AND MANY QUALITIES

To address these difficulties, we need to elaborate on the production side of the model, and to allow there to be many qualities of housing -- in fact, a continuum. A continuum of housing qualities is important because some economists -- Filer (1992) most notably -- have argued that regulation contributes significantly to homelessness. In doing so, we need to consider the possibility of filtering.

Let \( q \in \mathbb{R}^+ \) denote the quality of housing. Substitute for \((u-4)\) for \((u-2)\) and \((u-3)\):

\[
(u-4) \ u(.,.) \text{ is twice continuously differentiable with }
\]

\[
\begin{align*}
&u_{xx} \leq 0, \\
&u_{qq} \leq 0,
\end{align*}
\]

and

\[
u_{qx} \geq 0.
\]

Let \( p(q) \) be a function giving the price for each quality of housing. Assume that \( p(0) = 0 \) and that \( p(.) \) is strictly increasing. The price function need not be continuous.

Let

\[
\tilde{W}(q|p,y) = u(q, y - p(q))
\]

denote the semi-direct utility of quality \( q \) for a consumer with income \( y \), and

\[
\tilde{W}'(y|p) = \{q^*|\tilde{W}(q^*|p,y) \geq \tilde{W}(q|p,y) \text{ for all } q \}
\]

denote the demand quality set for income \( y \). It is easy to show
that under (u-1) and (u-4), if \( p(.) \) is linear, semi-direct
utility for every \( y \) has a unique maximum and so the demand
quality set exists and is a singleton. Moreover, whether or not
\( p(.) \) is linear, whenever semi-direct utility has a unique
interior maximum,

\[ w'(y|p) \]

is an increasing function of \( y \). Richer people demand higher
quality housing. In general, the demand set for each \( y \) will
exist and be unique as long as \( p''(q) \) is not tremendously
negative, relative to the second partials of \( u(.) \);

\[
\frac{2u_{xy}u_qu_x - u_{yy}u_x - u_{xx}u^2_q}{u_x^3} > -p''(q) \tag{1}
\]

is the second order condition for the demand set to exist and be
a singleton. Since the LHS is positive this condition will be
met if \( p'' \geq 0 \).

A person is homeless if 0 is the only element in her demand
set. Homelessness can occur two different ways. First, if \( p(.) \)
is \( C^2 \) and satifies condition (1) everywhere (so all stationary
points are maxima), then a person will be homeless iff semi-
direct utility is nonincreasing at \( q = 0 \). That is, iff

\[ p'(0) \leq \frac{u_q(0,y)}{u_x(0,y)}. \]

Since decreases in \( y \) increase \( u_x(0,y) \) (because \( u_{xx} < 0 \)) and
decrease \( u_q(0,y) \) (because \( u_{qx} > 0 \)), if a person with income \( y \) is
homeless, everyone with income \( y' < y \) will also be homeless, and
we can specify a unique \( y \) as the greatest homeless income.
Increases in \( p'(0) \) will cause increases in \( y_h \) and homelessness even if the income distribution is constant.

Second, if \( p(.) \) takes a discontinuous jump upward at \( q = 0 \), a person may be better off at \( q = 0 \) than at any other quality, even if semi-direct utility is increasing in the neighborhood of \( q = 0 \). Suppose \( p(.) \) is continuous except at zero, and let

\[
W^{**}(y|p) = \sup_{q > 0} W(q|p, y).
\]

A person will be homeless if

\[
u(0, y) > W^{**}(y|p).
\]

Let \( q^* \) denote the positive value of \( q \) that maximizes semi-direct utility in (2). Then

\[
\frac{\partial W^{**}}{\partial y} \geq u_x(q^*, y - p(q^*)) \geq u_x(0, y) = u_x(0, y) + \int_0^{q^*} u_x(q, y) \, dq \geq u_x(0, y) = \frac{\partial u(0, y)}{\partial y}.
\]

So once again, if a person with income \( y \) is homeless, so too will everyone with income \( y' < y \) be homeless.

On the production side, the essential idea is that every quality of housing produced will be produced at the lowest possible supply price. Housing of a given quality can be produced either by direct construction or by filtering, and once produced can be either maintained or not.

Let \( c(q) \) be the construction cost schedule: it costs \( c(q) \) to construct a unit of quality \( q \) housing. I assume that unit cost is independent of the number of units being constructed.

Assume

\[
(c - 1) \quad c \text{ is twice continuously differentiable on } \mathbb{R}_+. \]
c' > 0 and c'' > 0.

(c - 2) \( c(0) = 0 \), and

(c - 3) \( c(q) > 0 \) for all \( q > 0 \).

(c-1) and (c-2) are largely standard, but notice that (c-1) does not preclude a positive jump at \( q = 0 \).

Maintenance works in the following fashion: if \( m > 0 \) is being spent per unit time on maintenance, quality stays constant; if nothing is being spent on maintenance, quality deteriorates at the rate of one unit per unit time; and if \( m' < m \) is being spent, quality deteriorates at the rate of \( (1 - m'/m) \) per unit time. Arnott, Davidson, and Pines [1983] have rightly criticized Henderson [1977] for using such a specification of maintenance because it is a special case, but we are more interested here in tractability than generality. Notice that maintenance here is entirely forward-looking: rent can still be collected even if no money is being put into the current operation of the unit. We will also relax this assumption shortly.

Consider a steady-state equilibrium with perfect competition, free exit and entry, and no taxes or regulations.

There are two different kinds of steady-state equilibria. In one kind, construction is expensive relative to maintenance, and houses are maintained forever. We call this the cheap maintenance case. In the other kind of equilibrium, maintenance
maintenance case. In the other kind of equilibrium, maintenance is expensive relative to construction, and some houses are eventually abandoned. We call this the cheap construction case. To determine which case applies, we need some notation. Let
\[ \rho(q) = c(q) - mq \]
and let
\[ q^* = \text{argmin} \rho(q) \]
By (c-1) \( q^* \) is unique. Then, we will see, equilibrium will be of the cheap maintenance variety iff \( \rho(q^*) > 0 \) and \( q^* > 0 \); it will be of the cheap construction variety iff \( \rho(q^*) < 0 \). (The case \( q^* = 0 \) is essentially the same as the cheap maintenance case as is the case \( \rho(q^*) = 0 \); the case where \( q^* \) does not exist is essentially the same as cheap construction case. For brevity I will ignore these three cases.) Thus, which case applies is independent of the rate of interest or the distribution of income or demand; it depends only on the construction cost schedule and the cost of maintenance.

In both cases, houses of sufficiently high quality are treated the same way: they are constructed at that quality and maintained. Let \( r \) denote the (real) rate of interest. Then:

**Proposition 1**

If \( q > q^* \) and housing of quality \( q \) is consumed, then
\[ p(q) = rc(q) + m \]
in any steady-state equilibrium, and all houses for which \( q > q^* \) are maintained in a steady state equilibrium.
Proof

Let \( v(q) \) denote the value of a house of quality \( q \). In equilibrium, \( v(q) \leq c(q) \), for all \( q \), or construction would be unbounded. On the other hand, if demand for quality \( q \) is positive, then \( v(q') = c(q') \) for some \( q' > q \); otherwise there would be no supply at \( q \).

Suppose the hypothesized price schedule prevails. I will show this to be an equilibrium. First, given this schedule, maintenance is the optimal policy for an owner. Maintaining gives a value at \( q \) of

\[
\frac{p(q) - m}{r} = \frac{rc(q) + m - m}{r} = c(q)
\]

Alternatively, suppose the owner lets the house deteriorate for the next instant \( dt \). This saves the owner \( mdt \) in maintenance costs but reduces the value of the house by \( c'(q)dt \). Since \( q > q^* \) implies \( c'(q) > m \), letting the house deteriorate is not an optimal policy.

Since \( v(q) = c(q) \), supply can accommodate any pattern of demand. Hence the price schedule is an equilibrium.

To show that no other price schedule is an equilibrium, suppose some other price schedule prevails and demand is positive at \( q > q^* \). Then for some \( q' > q \), \( v(q') = c(q') \) and it must be optimal for the owner at \( q' \) to let the property deteriorate to \( q \). In particular the owner at \( q' \) must be able to do at least as well.
by letting the property deteriorate to \( q \) as by maintaining it at \( q' \). The value of letting the property deteriorate is

\[
D = \int_q^{q'} \bar{p}(\bar{q}) e^{-r(q' - \bar{q})} d\bar{q} + e^{-r(q' - q)} v(q)
\]

where \( \bar{p}(\cdot) \) is some alternative price schedule. But suppose that for some \( \bar{q} \in [q, q'] \)

\[
\bar{p}(\bar{q}) > r c(\bar{q}) + m
\]

Then

\[
\frac{\bar{p}(\bar{q}) - m}{r} > c(\bar{q})
\]

and so a house could be built at \( \bar{q} \) and maintained forever at positive profit. The constraint

\[
v(\bar{q}) < c(\bar{q})
\]

would be violated. Hence

\[
\bar{p}(\bar{q}) = r c(\bar{q}) + m = p(\bar{q})
\]

Similarly,

\[
v(q) < c(q)
\]

Hence

\[
D < \int_q^{q'} p(\bar{q}) e^{-r(q' - \bar{q})} d\bar{q} + e^{-r(q' - q)} c(q)
\]

which is the value of letting the house deteriorate from \( q' \) to \( q \) when the equilibrium price schedule prevails. Since maintaining at \( q' \) under the equilibrium price schedule was strictly better than letting the house deteriorate,

\[
\int_q^{q'} p(\bar{q}) e^{-r(q' - \bar{q})} d\bar{q} + e^{-r(q' - q)} c(q) < c(q')
\]

and so

---
\[ D < c(q') = v(q'). \]

So it is not optimal to let the house deteriorate to \( q \). Hence supply at \( q \) is zero, and there is no equilibrium.

Q.E.D.

The cheap construction and cheap maintenance cases differ for \( q < q^* \). We will consider each case in turn. We will assume, to make things interesting, that if \( p(q) = rc(q) + m \) for all \( q \), demand for some \( q < q^* \) is positive.

**Cheap maintenance**

For \( q < q^* \), the equilibrium price schedule in the cheap maintenance case is simply a straight line from \( p(q^*) \) with a slope of \((-rm)\). Any price line with this slope has an important property: if you start at any point on such a price line, value will be the same no matter whether you maintain the house at that quality forever, or let the house deteriorate to any lower quality and maintain it at that lower quality forever. This property will be useful at several points, and we will state it as a lemma:

**Lemma 1:** Suppose \( p(q') = p(q) - rm(q-q') \) for all \( 0 < q' < q \). Then

\[ \frac{p(q) - m}{x} = \int_q^{q'} p(t) e^{-r(q-t)} dt + e^{-r(q-q')} p(q') - \frac{m}{x} \]
for all \( q' < q \).

Proof:

\[
\int_q^{q'} p(t) e^{-r(q-t)} dt = p(q) \int_q^{q'} e^{-r(q-t)} dt - r m \int_q^{q'} (q-t) e^{-r(q-t)} dt
\]

\[
= \frac{p(q)}{r} (1 - e^{-r(q-q')}) - r m \int_0^{q-q'} t e^{-rt} dt
\]

\[
= \frac{p(q)}{r} - m (1 - e^{-r(q-q')}) + e^{-r(q-q')} (q-q') m
\]

\[
= \frac{p(q)}{r} - m e^{-r(q-q')} \left[ \frac{p(q)}{r} - m (q-q') \right]
\]

\[
= \frac{p(q)}{r} - e^{-r(q-q')} \frac{p(q') - m}{r}
\]

from which the lemma follows.

Q.E.D.

Define the maintenance-case price schedule \( p^m \) as:

\[
p^m(q) = rp(q^*) + m + rmq \quad q \in (o, q^*)
\]

The basic result of this section is

**Proposition 2:** In the cheap maintenance case, the maintenance case price schedule is an equilibrium price schedule. In the steady state equilibrium with this price schedule

\[
v(q) = p(q^*) + mq \quad q \in (o, q^*)
\]

No construction occurs below \( q^* \), and all qualities that are
supplied are maintained, but the lowest quality units are neither supplied nor demanded.

Proof: First, we show that maintenance is an optimal policy for any owner. It is easy to see that for an owner who maintains forever value is

\[
p(q) \cdot m = \frac{r p(q^*) + m + rm q - m}{r} = p(q^*) + mq,
\]

as stated. This value is always strictly positive whenever \( p(q^*) > 0 \).

Suppose instead that the house is allowed to deteriorate. If the house deteriorates to any positive quality and then is maintained forever, lemma 1 shows that value is not increased over value without any deterioration. The only alternative policy that might be more profitable than eternal maintenance is to allow the house to deteriorate completely to zero, and never maintain it. We call this policy abandonment from \( q \). Such a policy would give a value of

\[
\int_0^q p^*(t) e^{-r(q-t)} dt.
\]

Since

\[ v(e) > 0 \]

lemma 1 shows that abandonment is strictly less profitable than a policy that always maintains at some sufficiently low quality \( e > 0 \).

Thus, since a policy that maintains at \( e \) has the same value as a
policy that maintains at \( q \), maintenance at \( q \) is strictly better than abandonment from \( q \). So maintenance is always an optimal policy.

Next, observe that \( v(q) < c(q) \) for all \( q < q^* \), and \( v(q) < c(q) \) for all \( q < q^* \). This is because

\[
v(q) = p(q^*) + mq = c(q^*) - m(q-q^*) + \int_q^{q^*} c'(t) dt = c(q)
\]

since \( c'(t) < m \) for \( t \in (q, q^*) \).

Thus no construction will take place at qualities below \( q^* \).

Since \( v(q^*) = c(q^*) \), an indefinite number of houses can be produced at \( q^* \), and since allowing deterioration to any positive quality is no worse than maintaining, any quantity (including zero) can be supplied at any quality. Hence supply can accommodate any pattern of demand, and \( p^m \) is an equilibrium price schedule.

Since

\[
\lim_{q \to 0} p^m(q) = r p(q^*) + m > m > 0,
\]

demand will be zero for some qualities in the neighborhood of zero, and so supply will be zero there too. Let \( q \) denote the lowest quality of housing for which demand is positive. Owners of quality \( q \) houses are indifferent between maintaining and allowing to deteriorate, and so since no one demands housing of lower quality than \( q \), they maintain. It follows that in steady state equilibrium houses at all qualities above \( q \) are also maintained; there is neither filtering nor construction.
Q.E.D.

The maintenance-case price schedule is not a unique equilibrium, simply because for qualities below q, small deviations in price will affect nothing. We can show, however, that no other equilibrium price schedule has different prices at qualities that are observed in the market.

Proposition 3: Let $\bar{p}(\cdot)$ be an equilibrium price schedule in the cheap maintenance case, and let demand and supply be positive at $q < q^*$. Then

$$p(q) = p^m(q).$$

(3) Proof: From Proposition 1,

$$\bar{p}(q^*) = p^m(q^*) = r_c(q^*) + m.$$

Next, I will show that if no deviations from $p^m(\cdot)$ have occurred in the interval $[q, q^*]$, $\bar{p}(q) = p^m(q)$. From this "induction step" and (3) will follow the proposition.

Suppose $\bar{p}(q) > p^m(q)$. Then from lemma 1 infinite profit could be made by building houses at $q^*$, letting them deteriorate to $q$, and maintaining them there forever. If $\bar{p}(q) < p^m(q)$, then no houses will be supplied because both filtering (from lemma 1) and construction are unprofitable. Hence $\bar{p}(q) = p^m(q)$ and the "induction step" is complete.
Q.E.D.

Notice that the cheap maintenance case is marked by two discontinuities. Since for
\[ q \in [0, q^*], \nu(q) < c(q), \]
c(q) must approach a positive number as q approaches zero. Since c(0) = 0 by (c.2) -- homelessness is free -- the cheap maintenance case is possible only if the construction cost schedule is discontinuous at zero. This possibility does not seem deeply offensive to physical reality. Because c(0) = 0 implies p(0) = 0, the price schedule must also be discontinuous at zero.

This price schedule discontinuity means homelessness of the second variety will occur in the cheap maintenance case. No one will want to live in \( \epsilon \) quality housing, since \( \epsilon \) quality housing is only vanishingly better than homelessness, but costs a non-vanishing amount more. In this equilibrium, certain low qualities of housing disappear from the market while people are homeless, but government interference is not the culprit.

In steady state equilibrium with cheap maintenance, prices are completely determined by supply-side considerations. Changes in income distribution cannot affect the price schedule or \( y_h \). Increasing inequality raises homelessness by increasing population below \( y_h \), but for no other reason. The relationship between income and price or quality does not change.

On the other hand, suppose that the interest rate rises.
All prices for positive qualities increase, since

\[ \frac{\partial p(q)}{\partial r} = \nu(q) > 0, \]

for all \( q > 0 \). This general price increase moves the semi-direct utility function down for every income and every positive quality of housing. At the old \( y_h \) and minimum observed quality \( q \), housing is now worse than homelessness. Homelessness increases, and the old bottom of the housing quality distribution disappears. This loss of the lowest quality housing is consistent with the observed decrease in dipalidation and in fires.

An increase in the interest rates raises the average rent of some aggregate called "low-income housing" in two ways.

First, composition changes because the lowest qualities of housing disappear. Second, price for each remaining quality increases. If "low-income housing" is defined as housing occupied by people with income below some threshold \( y \), these changes will be partially offset by a reduction in the quality demanded by everyone who remains housed.

"Low-income housing," however, usually gets defined operationally as housing with a price below some price \( p^L \) -- say, $250 a month (see, e.g., Mihaly [1991]). An interest rate increase lowers the stock of such housing in two ways: first, at the bottom, by making some very low quality housing units disappear; and second, at the top, by pushing the price \( p(q) \) of some housing qualities above \( p^L \) and thus making these qualities
disappear from the measured count of "low-income" units as well.

These two changes, however, may be partially offset by a third effect, an increase in the population inhabiting the relevant quality interval. Consider a quality interval \([q_1, q_2]\), \(q_h < q_1 < q_2\). In the original equilibrium it was occupied by all households in the income intervals \([y_1, y_2]\). Let \([y_1', y_2']\) denote the income interval that corresponds to this quality interval in the new equilibrium with a higher interest rate. The population demanding housing in the quality interval will increase if two not especially restrictive conditions are met.

(i) \(y_1' - y_2' \geq y_1 - y_2\), and

(ii) the density of income is rising in the relevant neighborhood.

Condition (ii) is likely to be satisfied in discussions of low-income housing because all the relevant incomes are below the mode of a unimodal income distribution. Condition (i) is more problematic because it depends on the exact form of the utility function, although many common utility functions (including the semilogarithmic function we use as our standing example for the cheap construction case) satisfy this condition.

The cheap maintenance case thus exhibits many features of the 80's housing market, especially for cities like Toronto and London where abandonment and demolition have been negligible. For a model with the abandonment and demolition characteristic of northeastern and midwestern American cities, we must turn to the cheap construction case.
Cheap construction

In the cheap construction case, $\rho(q') < 0$. This case is more difficult than the cheap maintenance case. I do not have a complete general solution. In this section, I will begin by outlining the most general properties a solution must have, and derive increasingly specific results under increasingly restrictive assumptions. Throughout, I will assume

P-1 The support of the income distribution is connected.

P-2 The support of the income distribution includes zero.

To summarize the results of this section: the equilibrium features a bifurcation. Above some quality $q^2 > q'$, houses are always maintained. Below some quality $q^1 < q'$, they are never maintained. Between $q_1$ and $q_2$, there is neither demand nor supply. Houses are continuously being constructed at $q_1$; these houses deteriorate until they fall to quality zero and are abandoned. Homelessness is of the first type.

This equilibrium will be described in more detail in a series of propositions. First, notice that the maintenance-case price schedule cannot be an equilibrium in the cheap construction case. This is because with the maintenance-case price schedule and $\rho(q') < 0$ the value of letting a house deteriorate to zero is always greater than the value of maintaining it. Since the value of maintaining a house at $q'$ is $c(q')$ under the maintenance-case price schedule, the value of letting a house deteriorate from $q'$ is greater than the cost of building it. Such a condition cannot obtain at equilibrium.
Proposition 4. If \( p(q^*) < 0 \),

\[
\int_b^q p^m(t)e^{-r(t-s)}\,dt > \frac{P^m(q)-m}{r}
\]

for any \( q \in (q^*, 0) \), where \( b \) is the highest quality at which \( p^m \) is zero, or zero, whichever is greater.

Proof: If \( p(q') < 0 \),

\[
p^m(0) = rP(q^*) + m < m
\]

so that for some \( s > 0 \), \( p^m(s) = m \) and so the value of maintaining forever at \( s \) is

\[
\frac{P^m(s) - m}{r} = 0
\]

However, the value of letting a property deteriorate until either its price or its quality is zero is positive. So a policy of letting a house deteriorate until it is worthless is more profitable than a policy of letting it deteriorate to \( s \) and maintaining it there forever. But by lemma 1, the latter policy has the same value as a policy of maintaining the house at \( q \) forever.

Q.E.D.

Corollary: If \( p(q') < 0 \),

\[
\frac{p(q^*)-m}{r} > c(q^*)
\]

and so the maintenance-case price schedule is not an equilibrium.

Intuitively, maintenance-case prices are too high to be used in the cheap construction case. It should not be surprising then
that the first deviation from the maintenance-case price schedule must be below it. Let $p(.)$ denote a cheap construction equilibrium price schedule.

**Proposition 5:** It is never the case that for some $\delta > 0$ and all $\epsilon$ satisfying $0 < \epsilon \leq \delta$,

$$p(q^*-\epsilon) > p^m(q^*-\epsilon).$$

**Proof:** Suppose not. Then by lemma 1 building a house at $q'$, letting it deteriorate to $q' - \delta$ and maintaining it there forever will give strictly greater value than maintaining it at $q'$ forever. Since maintaining it at $q'$ forever gives a value of $c(q')$, this means that the policy of deterioration to $(q' - \delta)$ will earn strictly positive profit. Strictly positive profit cannot occur in equilibrium.

Q.E.D.

So the first deviation (starting from $q'$ and working down) from the maintenance-case price schedule has to be below it. Houses of qualities on this first deviation, if they are supplied at all, cannot be maintained.

**Proposition 6:** If $p(q) < p^m(q)$ and $p(q') \leq p^m(q')$ for all $q' \in [q,q']$, the houses of quality $q$ are either not supplied or not maintained.

**Proof:** Suppose houses of quality $q$ were maintained. Their
value would be

\[
\frac{p(q) - m}{r} < \frac{p^m(q) - m}{r} = c(q^*) - (q^* - q)m < c(q),
\]

so they would not be supplied by construction. By lemma 1, since

\[
\int_q^{q^*} p(t) e^{-r(q^*-t)} dt + \frac{p(q) - m}{r} < \int_q^{q^*} p^m(t) e^{-r(q^*-t)} dt + \frac{p^m(q) - m}{r} = \frac{p(q^*) - m}{r},
\]

they would not be supplied through construction at \( q^* \) and deterioration to \( q \), either.

Q.E.D.

Next we will show that the equilibrium will bifurcate: no houses will be supplied or demanded at the top of the first deviation (again, starting from \( q^* \) and working down) from the maintenance-case price schedule. Let \( q'' \) denote the supremum of qualities with prices below \( p^m \) on the first deviation; that is, for all qualities \( q' > q'' \), \( p^m(q') = p(q') \), and \( q'' \) is the infimum of qualities with this property. We will show that \( p(.) \) is continuous at \( q'' \), that the left-hand derivative of \( p(.) \) at \( q'' \) is greater than \( rm \) and so \( p'(.) \) is discontinuous at \( q'' \), and that this discontinuity will eliminate demand for qualities in some neighborhood below \( q'' \). We will state two lemmas before proving the main proposition.

**Lemma 2**: \( p(.) \) is continuous at \( q'' \), if qualities of the
neighborhood quality $q''$ housing are observed in the market.

**Proof:** Suppose not. Then

$$\lim_{t \to q''} p(t) = p'(q') > \lim_{t \to q''} p(t),$$

since the first deviation is below the maintenance-case price schedule. Then no houses will be supplied on the lower portion, or demanded on the upper portion, of some neighborhood of $q''$. In particular, either supply or demand will be zero at $q$, depending on which limit is actually achieved.

Q.E.D.

**Lemma 3:** If qualities in the neighborhood of quality $q''$ housing are observed in the market, $p'(.)$ is not continuous at $q$, and

$$\lim_{t \to q''} p'(t) > \lim_{t \to q''} p'(t) = rm.$$

**Proof:** Clearly

$$\lim_{t \to q''} p'(t) = rm;$$

otherwise the first deviation would not be below the maintenance-case price schedule. Denote

$$\lim_{t \to q''} p'(t)$$

as $P'(q'')$.

Since by proposition 6, houses of qualities slightly below $q''$ are not maintained, we can write the value of a house of
quality \( t \) slightly below \( q'' \) as

\[
v(t) = \int_{q''-x}^{t} p(s) e^{-r(t-s)} ds + e^{-r(t-q''-x)} v(q''-x),
\]

where \( x \) is some positive number.

Then

\[
v'(t) = p(t) - rv(t)
\]
\[
v''(t) = p'(t) - rv'(t)
\]
\[
v'''(t) = p''(t) - rv''(t).
\]

As \( t \) approaches \( q \), \( p(t) \) approaches \( p^m(q'') \), by lemma 2, \( v(t) \) approaches \( v(q'') \), and \( p'(t) \) approaches \( P'(q'') \). So \( v'(t) \) approaches

\[rv(q'') + m - rv(q'') = m,
\]

and \( v''(t) \) approaches

\[P'(q'') - rm.
\]

Suppose \( P'(q'') = rm \). Then \( v''(t) \) approaches zero, and \( V'''(t) \) approaches

\[
\lim_{t \to q''} v''(t),
\]

which we denote \( P''(q'') \). Suppose \( P''(q'') > 0 \). Then for \( t \) slightly smaller than \( q'' \), \( p'(t) < rm \), and \( q'' \) is not the beginning of a downward deviation from the maintenance-case price schedule.

Suppose \( P''(q'') < 0 \). Then \( v'''(t) < 0, v''(t) = 0 \), as \( t \) approaches \( q'' \), and so for \( t \) slightly less than \( q'' \), \( v''(t) > 0 \), and thus \( v'(t) < m \). But then value at \( t \) is greater than maintenance-case value at \( t \) (since in the maintenance-case value declines at rate \( m \)) and so by lemma 1 positive profit could be made by building a house at \( q^* \), letting it deteriorate to \( t \), and
then trading for \( v(t) \).

Then suppose \( P''(q'') = 0 \). The same analysis can be repeated until at some level \( n \) \( P^{(n)}(q'') \neq 0 \). Since \( q'' \) starts a deviation from the maintenance-case price schedule, for which all higher derivatives vanish, surely such an \( n \) exists.

Q.E.D.

**Proposition 7:** No houses of qualities in some neighborhood of \( q'' \) are observed in the market in equilibrium.

**Proof:** Proof is by contradiction.

Suppose such qualities are observed in the market. Then at \( q'' \) \( p(.) \) is continuous (by lemma 2) and \( p'(.) \) discontinuous (by lemma 3). Consider semi-direct utility

\[
W(q''|p,y) = u(q', y-p(q''))
\]

and its derivative

\[
W'(q''|p,y) = u'_q - u_p'(q'').
\]

Since \( p'(.) \) jumps down at \( q'' \), \( W'(.) \) must also jump up at \( q'' \), for all \( y \). From the population assumptions, there must be some consumers for whom \( W'(.) \) crosses zero twice; for these consumers \( W(.|p,y) \) must have two local maxima, one above and one below \( q'' \).

Consider a consumer for whom \( W'(t|p,y) = 0 \) for some \( t \) very slightly below \( q'' \). Only such a consumer might possibly purchase \( t \) housing, since only local maxima are global maxima.

But for \( t \) sufficiently close to \( q'' \) such a local maximum
could not be global. \(W(.)\) will decline only a little bit before \(q\) is reached, and then it will increase for a finite stretch before the second local maximum is achieved. Since for \(t\) sufficiently close to \(q''\) the decline is arbitrarily small, the local maximum above \(q''\) must make semi-direct utility greater than the local maximum below \(q''\). Hence, there will be no demand for qualities slightly below \(q\). A similar argument establishes the same result for qualities slightly above \(q''\).

Since the assumption that all qualities in all small neighborhoods of \(q''\) are observed in the market leads to the conclusion that demand vanishes in some neighborhood of \(q''\), we have established a contradiction.

Q.E.D.

So in a steady state houses below \(q''\) cannot be supplied by filtering from above \(q''\). We will argue that houses below \(q''\) will have to be supplied by construction below \(q''\). Let \(q_1\), denote the highest quality less than \(q''\) of a house that appears on the market. Then we will show that \(p(q_1) < p''(q_1)\), and so if a house were built at \(q_1\), it would not be maintained. To do so, we will again use two lemmas.

Lemma 4: Let \(q_2\) denote the lowest quality above \(q''\) that appears on the market, and let \(y\) denote the income of the consumers who buy \(q_1\) quality housing. Then

\[
W(q_1|p,y) = W(q_2|p,y).
\]

34
Proof Suppose
\[ W(q_1|p,y) < W(q_2|p,y). \]
Then consumers of income \( y \) would not buy \( q_1 \) housing, contrary to supposition.

Suppose
\[ W(q_1|p,y) > W(q_2|p,y). \]
Then for some \( \epsilon \) sufficiently small, consumers of income \( (y + \epsilon) \) will have a local maximum at \( q' \),
\[ q_1 < q' < q'', \]
and will strictly prefer this local maximum to the local maximum above \( q'' \). Then \( q_1 \) would not be the highest quality less than \( q'' \) to appear on the market. Another contradiction.

Q.E.D.

Lemma 5: Let \( z(q|x,y') \) be the indifference curve through point \( x \), in \((q,p)\)-space for consumers with income \( y' \). That is, a consumer with income \( y' \) is indifferent among all quality-price combinations \((q, z(q|.)\)). Then \( z''(q|x,y') < 0 \).

Proof: This follows by simple algebra from u-4, the diminishing marginal rate of substitution between housing and other consumption.

Proposition 8: \( p(q_1) < p^m(q_1) \).

Proof: At \( q_2 \), the indifference curve \( z(q|(q_2, p^m(q_2)), y) \) is tangent to the maintenance-case price schedule (understanding the
price schedule \( p(q) = rc(q) + m \) for \( q \ge q' \) to be part of the maintenance-case schedule. Then at \( q_2 \), \( (p^m)'(q_2) = z'(q_2) \).

However, \( (p^m)''(q') \ge 0 \) for all \( q' \). Thus by lemma 5 for all \( q' < q_2 \),

\[
(p^m)'(q') < z'(q')
\]

and so

\[
p^m(q') > z(q')
\]

In particular

\[
p^m(q_i) > z(q_i|q_2, p^m(q_2), y) = p(q_t)
\]

where the last equality follows from lemma 4.

Q.E.D.

**Proposition 9:** If houses are constructed at \( q_1 \), they are not maintained.

**Proof:** Suppose a house were constructed and maintained at \( q_1 \). Its value would be

\[
\frac{p(q_1) - m}{r} < \frac{p^m(q_1) - m}{r} < c(q_1)
\]

where the first inequality follows from proposition 8, and the second inequality from proposition 2. Hence if a house is to be constructed at \( q_1 \) its value must be greater than the value that it would have if it were maintained forever.

Q.E.D.
For future reference, we also have

**Proposition 10:** \( p'(q_1) > r_m \).

**Proof:** From lemma 5

\[ z'(q_1 | (q_2, \ p^m(q_2)), y) > z'(q_2 | (q_2, \ p^m(q_2)), y) \geq r_m, \]

where the first inequality holds because \( q_1 < q_2 \), and the second because the local optimality of \( q_2 \) implies that the indifference curve must be tangent to the price schedule there. Similarly, the local optimality of \( q_1 \) implies that the indifference curve must be tangent to the price schedule there also,

\[ z'(q_1 | .) = p'(q_1) \]

and the proposition follows.

Q.E.D.

Since demand is zero between \( q_1 \) and \( q_2 \), by the arguments of proposition 6 no houses will be supplied by filtering to quality \( q_1 \). Hence any houses supplied to \( q_1 \) must be supplied by construction.

Since houses will be constructed and not maintained at \( q_1 \), it becomes important to study the steady-state properties of systems where houses are not maintained. Four results are relevant.

**Proposition 11:** In a steady state equilibrium no houses are maintained at any quality \( q' < q_1 \).

**Proof:** Suppose houses were maintained at quality \( q' < q_1 \). Then over time the number of houses of quality \( q' \) would be
increasing. This would not be a steady state.

**Proposition 12:**

\[ \lim_{t \to 0} p(t) = 0. \]

**Proof:** If no maintenance is being applied, houses will be supplied at all qualities down to zero. So demand must be positive at all qualities down to zero. Let \( \epsilon \) approach zero. If \( p(\epsilon) > 0 \), as in the cheap maintenance case, then for sufficiently small qualities demand is zero. If \( p(q') = 0 \) for some \( q' > 0 \), then there would be no demand for any quality less than \( q' \). Hence the price schedule must be continuous and upward sloping at zero.

Q.E.D.

So homelessness will be of the first variety, with semi-direct utility continuous but downward-sloping at zero.

**Proposition 13:** Let \( s(q', \tau) \) denote the supply of quality \( q' \) housing at time \( \tau \). In steady-state equilibrium, for all \( \tau \), \( s(q', \tau) \) is constant in \( q' \) for \( q' \in (0, q_i) \).

**Proof:** Consider \( (q', q'') \) both in \( (0, q_i) \) and set \( \Delta = q'' - q' > 0 \)

by construction. From no-maintenance (proposition 11),

\[ s(q'', \tau) = s(q', \tau + \Delta) \]
From the definition of a steady state.
\[ s(q', t) = s(q', t + \Delta). \]

Hence
\[ s(q', t) = s(q", t), \]
and these quantities have to be the same over time in the steady state.

Q.E.D.

If supply is constant with respect to quality, demand has to be constant too. Recall that \( F(.) \) is the cdf of the income distribution. Assume \( p(.) \) is chosen so that \( W^*(y|p) \), the demand set correspondence, is invertible, and let \( Y(.) \) denote its inverse. Specifically:
\[ Y(q') = \{y|W^*(y|p) = q'\} \]
the set of incomes whose recipients demand quality \( q' \) houses. If \( p(.) \) is continuous and differentiable, \( Y(q') \) is a function of \( p(q') \) and \( p'(q') \). Then if demand is constant over some range, \( F(Y(q')) \), considered as a function of \( q' \) has to increase at a constant rate. Thus

**Proposition 14:** Over the interval \((0, q_i)\)
\[ \frac{\partial^2 F(Y(q'))}{\partial q'} = 0, \]  
(4)
in steady state equilibrium, if all construction in this interval is at \( q_i \). If building occurs at \( q < q_i \), then (4) holds for any interval without building.
Given the income distribution, and assuming no building below $q_i$, (4) is a third-order differential equation in $p(\cdot)$, since $Y(q')$ depends on both $p(q')$ and $p'(q')$. Proposition 12 supplies one initial condition, but two initial conditions remain to be specified.

These conditions arise from consideration of $q_i$. The first is obvious: value has to equal construction cost.

**Proposition 15:**
\[ v(q_i) = \int_0^{q_i} p(t) e^{-r(q_i-t)} \, dt = c(q_i). \]

The next condition is about the price at $q_i$:

**Proposition 16:**
\[ p(q_i) = rc(q_i) + c'(q_i). \]

**Proof:** Suppose $p(q_i) > rc(q_i) + c'(q_i)$. Increasing quality slightly increases construction cost by $c'(q_i)$. It increases value by $p(q_i) - rv(q_i) = p(q_i) - rc(q_i)$, by proposition 15. Since $p(q_i) > rc(q_i) + c'(q_i)$,
\[ p(q_i) - rc(q_i) > [rc(q_i) + c'(q_i)] - rc(q_i) = c'(q_i). \]

The increase in value is greater than cost, and so construction would occur at qualities higher than $q_i$, contrary to hypothesis. (Equivalently, new houses could be built at slightly higher quality than $q_i$, and priced at less than $q_i$, but more than $rc(q_i) + c'(q_i)$. These houses would be profitable to their builders, given the price schedule at and below $q_i$, and would take all demand from $q_i$.)

Similarly, if $p(q_i) < rc(q_i) + c'(q_i)$, building a house of slightly lower quality would be more profitable than building a
house of \( q_i \), quality, and such lower quality houses could either make positive profits or undercut the price schedule.

Hence \( p(q_i) = r_c(q_i) + c'(q_i) \), as the only remaining alternative.

Q.E.D.

From the definition of \( q^* \),
\[
rc(q) + c'(q) \leq r_c(q) + m \quad \text{as} \quad q \geq q^*.
\]

From (c-1), the function \( rc + c' \) is increasing and concave upward. At \( q^* \), its slope is greater than \( rm \). For brevity we will write
\[
R(q) = rc(q) + c'(q).
\]

These properties of the \( R(.) \) function will suffice to show that \( q_i \) and \( q_2 \) bracket \( q^* \). As usual, we use three lemmas.

Lemma 6: If \( R(q_2) < z(q_i | (q_2, p^m(q_2)), y) \) then for all \( q \in [q_1, q_2] \), \( R(q) < z(q_i | (q_2, p^m(q_2)), y) \).

Proof: Let
\[
X(q) = z(q_i | .) - R(q).
\]
Then
\[
X(q_1) = 0 \\
X(q_2) > 0 \\
X''(q) = z''(q_i | .) - R''(q) < 0 \quad \text{for all} \quad q \text{ since} \ z'' < 0 \quad \text{and} \ R'' > 0.
\]

Hence if there is a stationary point for \( X \) between \( q_i \) and \( q_2 \) it is a maximum; so
\[
X(q) \geq 0 \quad \text{for} \quad q \in [q_1, q_2].
\]
Lemma 7: If \( p(q) \geq R(q) \) for all \( q \in [q_1, q_2] \), with some strict inequality, then positive profit can be made by building houses at \( q_2 \) and letting them deteriorate to \( q_1 \), and then to zero.

Proof: Note that
\[
\int_{q_1}^{q_2} R(t) e^{-r(t-t')} dt = c(q_2) - c(q_1),
\]
since \( R(t) \) is the price that keeps value, i.e.
\[
\int_0^q p(t) e^{-r(t-t')} dt,
\]
rising at the same rate as construction cost. If \( p(t) \geq R(t) \) for all \( t \), with some strict inequality
\[
\int_{q_1}^{q_2} p(t) e^{-r(t-t')} dt > c(q_2) - e^{-r(q_2-q_1)} c(q_1) = c(q_2) - v(q_2) e^{-r(q_2-q_1)}.
\]
But
\[
\int_{q_1}^{q_2} p(t) e^{-r(t-t')} dt = v(q_2) - v(q_1) e^{-r(q_2-q_1)}
\]
and so
\[
v(q_2) > c(q_2).
\]
Q.E.D.

Lemma 8: \( p^m(q_2) \leq R(q_2) \).

Proof: Suppose not. Then \( R(q_2) < z(q_2| (q_2, p^m(q_2)), y) \), since \( p^m(q_2) = z(q_2| (q_2, p^m(q_2)), y) \). From lemma 6,
\[
R(q) \leq z(q| (q_2, p^m(q_2)), y) \text{ for all } q \in [q_1, q_2] \text{ with strict}.
\]
inequality at $q_2$. Since $p(q) > z(q_{| {q}}$ ) for $q \in [q_1, q_2]$, $p(q) > R(q)$ for all $q \in [q_1, q_2]$ with strict inequality for some $q$. By lemma 7, $v(q_2) > c(q_2)$, which is impossible in equilibrium. A contradiction.

Q.E.D.

Proposition 17: $q_2 > q^*$. 

Proof: From lemma 8, $q_2$ must lie in a region where $p^m(.) < R(.)$. One such region is the region $q > q^*$; there may be another such region, where $q < q^*$ and $p^m(q) = r\rho(q^*) + m + rmq$. This region, called the lower region, is separated from $q^*$, since $R(.) < p^m(.)$ in the immediate neighborhood below $q^*$.

We will show that $q_2$ cannot lie in the lower region. Suppose $q_2$ lies in the lower region. In the lower region $R'(.) < rm$, and so $q_1 < q_2$ lies in the lower region, too. Then $R(q_1) = p(q_1) > p^m(q_1)$, which contradicts proposition 8. Hence $q_2$ does not lie in the lower region.

Q.E.D.

Since $q_1 < q^*$, $q^*$ lies in the gap $[q_1, q_2)$. While the price-schedule in the gap is of course not determined uniquely, we can summarize the results so far:

Proposition 18: For some $q_1$, an equilibrium price schedule must be of the form
\[ p(q) = p^n(q) \quad q \geq q^* \]
\[ = R(q) \quad q \leq q^* \leq q_i \]
\[ = g(q) \quad q \leq q_i \]

provided no construction takes place below \( q_i \), where \( g(q) \) solves third order differential equation (4), subject to the conditions in propositions 8, 15 and 16. No houses of qualities between \( q_i \) and \( q_2 \) appear in the market, where \( q_2 > q^* \). For qualities greater than \( q_2 \) or less than \( q_i \), no other equilibrium schedule exists.

Proposition 18 describes what an equilibrium must look like if it exists, but doesn't say whether one exists. Three kinds of problems might arise:

(a) it might be optimal for owners to maintain along the price-schedule that satisfies differential equation (4);

(b) \( q_1 \) and \( q_2 \) with the appropriate properties might not exist; or

(c) it might be optimal to build at qualities below \( q^* \).

Problem (c) is not so serious a problem as the other two; if it were to happen there still might be an equilibrium, but it would be different below \( q_i \). Problems (a) and (b) would be fatal. A fairly standard fixed point argument can be used to rule out problem (b) in general; I will omit it for the sake of brevity. Sufficient (far from necessary) conditions for avoiding (a) and (c) can also be found; we will give two such conditions below and then turn to an example that will be used for comparative dynamics purposes.

**Proposition 19:** Let \( p(.) \) be a solution to (4) satisfying
the three initial conditions. If either \( p''(q) > 0 \) or \( p''(q) < 0 \) for all \( q \in [0, q_1] \), then it is never optimal on this interval for an owner to maintain a house.

**Proof:** Maintenance at \( q \) is optimal iff the value from maintaining at \( q \),

\[
p(q) - r\nu(q)
\]

is greater than the value of letting the house deteriorate until it is abandoned,

\[
\nu(q) = \int_0^q p(t) e^{-r(t-t)} \, dt.
\]

So maintenance is optimal iff

\[
p(q) - r\nu(q) > m.
\]

Denote

\[
x(q) = p(q) - r\nu(q) = \nu'(q).
\]

Then \( x(0) = 0 < m \), and

\[
x(q_1) = rc(q_1) + c'(q_1) - rc(q_1) = c'(q_1) < m,
\]

and

\[
x'(q) = \nu''(q)
\]

\[
x''(q) = p''(q) - r\nu''(q) = p''(q) - rx'(q).
\]

Thus it would be possible for \( x(q) > m \) for some \( q \) only if \( x(.) \) had an interior maximum on \([0, q_1]\). Let \( q' \) be such an interior maximum. Then \( x'(q') = 0 \) and \( x''(q') = p''(q') \). So if

\[
p''(.) > 0, \text{ no interior maximum exists and maintenance is never optimal.}
\]

Suppose \( p''(.) < 0 \). Let
\[ v^m(q) = \frac{p(q) - m}{r} \]

denote the value of maintenance. Clearly, \( v(q) > v^m(q) \) if \( p(q) < m \), and so suppose \( p(q) > m \). Let \( \Delta(p) \) denote the quality decrease needed to bring a line with slope \((-rm)\) down from \( p \) to \( m \):

\[
\Delta(p) = \frac{1}{rm} (p - m) = \frac{1}{m} v^m.
\]

From lemma 1

\[
v^m(q) = \int_{q - \Delta(p(q))}^{q} [p(q) - rm(q-t)] e^{-r(q-t)} dt + e^{-\Delta(p(q))} \frac{m - m}{r}
\]

\[= \int_{q - \Delta(p(q))}^{q} [p(q) - rm(q-t)] e^{-r(q-t)} dt.\]

Write

\[ D(q) = v(q) - v^m(q).\]

We need to show that \( D(.) \) is nonnegative on \([0,q_1]\). Clearly

\[ D(0) > 0, \quad \text{and} \quad D(q_1) > 0, \quad \text{and} \quad D(.) \text{ is continuous.}
\]

Rewrite \( D(.) \):

\[
D(q) = \int_0^q p(t) e^{-r(q-t)} - \int_{q - \Delta(p(q))}^{q} [p(q) - rm(q-t)] e^{-r(q-t)} dt
\]

\[= \int_{q - \Delta(p(q))}^{q} [p(t) - p(q) + rm(q-t)] e^{-r(q-t)} + \int_0^{q - \Delta(p(q))} p(t) e^{-r(q-t)} dt.\]

Then,

\[
D'(q) = -\int_{q - \Delta(p(q))}^{q} [p'(t) - rm] e^{-r(q-t)} - \frac{e^{-\Delta(p(q))}}{r} [p'(q - \Delta(p(q))) - rm] - rD(q).\]

From proposition 10, \( p'(q_1) > rm \), and so since \( p'' \leq 0 \),

\[ p'(q) > p'(q_1) > rm \quad \text{for} \quad q < q_1. \]

Hence the first two terms of \( D'(.) \) are always negative.

Suppose \( D(q') < 0 \) for some \( q' < q_1 \). Since \( D(q_1) \) is
strictly positive and \( D(.) \) is continuous, somewhere between \( q' \) and \( q_1 \), \( D \) must be both strictly positive and increasing. But since the first two terms of \( D'(.) \) are always negative, this is impossible. Hence \( D(q) > 0 \) for all \( q \in [0, q_1] \).

Q.E.D.

Next is a simple necessary condition for building not to be optimal below \( q \):

**Proposition 20:** If the lines \( R(.) \) and \( p(.) \) intersect once on the interval \((0, q)\) and either \( c'(q) \) or \( c(q) \) approaches a positive number as \( q \to 0 \), then building is optimal nowhere in this interval.

**Proof:** Since the two lines intersect once at \( q' \), say, and \( p(0) = 0 \), \( p(.) \) must be above \( R(.) \) for \( q > q' \) and below \( R \) for \( q < q' \). Let

\[
C_0 = \lim_{t \to 0} c(t).
\]

Then

\[
\delta(q) = c(q) - v(q) = C_0 + \int_0^q R(t) e^{-\epsilon(q-t)} \, dt - \int_0^q p(t) e^{-\epsilon(q-t)} \, dt
\]

\[
= C_0 + \int_0^q [R(t) - p(t)] e^{-\epsilon(q-t)} \, dt.
\]

Since \( c(q_1) = v(q_1) \) by proposition 15,

\[
C_0 = \int_0^{q_1} [p(t) - R(t)] e^{-\epsilon(q-t)} \, dt,
\]

47
and so

\[ \delta(q) = \int_{q}^{q_1} [p(t) - R(t)] e^{-r(q-t)} dt \]

\[ \delta'(q) = -[p(q) - R(q)] e^{-r(q-q)}. \]

Since \( p(t) \geq R(t) \) for \( q \geq q' \), \( \delta(q) > 0 \) for \( q > q' \).

Since \( p(t) < R(t) \) for \( q < q' \), \( \delta'(q) > 0 \) for \( q < q' \),
and since \( \delta(0) \geq 0 \), \( \delta(q) > 0 \).

Q.E.D.

An example will show that it is possible for the conditions for proposition 19 to be met, and make it easier to do comparative dynamics exercises.

Suppose

\[ u(q,x) = \ln x + q \]

and income is distributed uniformly over some sufficiently large interval. Then

\[ Y(q) = p(q) + p'(q) \]

and

\[ p(q) = Dq - K_1(1-e^q) \]

solves differential equation (4) with \( p(0) = 0 \). \( D \) and \( K_1 \) can be derived from the value-matching conditions. Since \( p''(q) = K_1 e^q \), \( p''(.) \) never changes signs. It is easy to verify that condition (1) is satisfied even if \( p'' < 0 \).

The highest income of a homeless person in the example is 48
\[ Y(0) = p'(0) = D - K_t \]

All people with lower income are homeless, even though arbitrarily cheap housing is available. \(D\) is the density: the measure of consumers per unit of quality.

We can now turn to comparative dynamics, and examine the effects of changes in income distribution and interest rates on rents and homelessness in the cheap construction case.

First consider a change in income distribution: income inequality increases because the density in the middle of the distribution goes down and the density at the two extremes goes up. Holding the price schedule constant, such a change, as we have argued in section 1, increases homelessness by increasing the population with incomes below \(Y(0)\).

What happens to the price schedule and to \(Y(0)\)? We can distinguish between two cases.

First suppose that \(q_1\) is fairly low and the change in income distribution increases the income density at every income not exceeding \(Y(q_1)\) (and possibly some greater incomes) by the same proportion \(\alpha\). In this case, the price schedule will stay the same and so \(Y(0)\) will stay the same. Homelessness will increase, as in section 2, simply because more people are poor. However, the stock of low income housing will also increase -- by the same proportion \(\alpha\) -- for any definition of low-income housing.
Proposition 21: Suppose the income density function changes from $f_1(\cdot)$ to $f_2(\cdot)$ where $f_2(y) = (1+\alpha)f_1(y)$ for all $y \leq Y(q_t)$. Then the equilibrium price schedule remains the same and the density of housing at all qualities less than or equal to $q_t$ increases.

Proof: The density of population at quality $q$ is 
\[
\frac{\partial P_i(Y(q))}{\partial q} = f_i(Y(q))Y'(q), \quad i=1,2
\]
and so equilibrium requires that this expression be identical to a constant; say $D$:
\[
f_i(Y(q))Y'(q) = D_i, \quad i=1,2
\]
for all $q \leq q_t$. The constant $D_i$ is determined by the three boundary conditions, for each income distribution. Suppose $D_2 = (1+\alpha)D_1$. Then any function $Y(.)$ that satisfies (5) for $i=1$ does so also for $i=2$; and everywhere that $D_i$ appears in the boundary conditions it can be replaced by $(1+\alpha)D_i$. The solution for price stays the same, but density increases.

Q.E.D.

The second kind of increasing income inequality is more difficult. Suppose the income distribution changes below $Y(q_t)$; relatively fewer people have incomes close to $Y(q_t)$ and relatively more have incomes close to $Y(0)$. Roughly speaking, this change causes an excess demand for low qualities of housing and an excess supply of higher qualities (near $q_t$). Prices at
and an excess supply of higher qualities (near q). Prices at the bottom increase and prices at the top decrease. The former effect increases p'(0) and hence increases Y(0) and homelessness; the latter effect increases p'(q), if q stays the same. Higher p'(q) means that the tangent indifference curve at q will be steeper and so it will cut above the rc(q)+m curve; hence equilibrium requires a smaller q. The effect of smaller q on the price schedule and homelessness is ambiguous.

These effects can be illustrated in a small example with the semilog utility function. Suppose pdf of income in the relevant range is a step function:

\[ f(y) = \begin{cases} 
 1 & y \leq y' \\
 0 & y > y'
\end{cases} \]

where y' stays in the range from Y(0) to Y(q) throughout the analysis. Since we are discussing only the bottom of the income distribution, there is no need to normalize. Then Y(q) must satisfy

\[ Y'(q) = D \quad \text{if } Y(q) \leq y' \]
\[ = D/A \quad \text{if } Y(q) > y' \]

and so the price function must be of the form

\[ p(q) = Dq - K_1(1-e^{-q}) \quad q \leq Q \]
\[ = \frac{D}{A}q + K_2 - K_3(1-e^{-q}) \quad q > Q \]

where D and the K's are constants that must be determined from boundary conditions, and
\[ Q = \frac{1}{D} [y^* - (D - K_1)] \]

is the quality corresponding to income \( y^* \). Since \( D - K_1 = Y(0) \) and by hypothesis \( y^* > Y(0) \), \( Q \) is positive.

There are two added constants to determine, but the need to keep both \( p(.) \) and \( p'(.) \) continuous at \( Q \) imposes two more boundary conditions. (A discontinuity in either \( p(.) \) or \( p'(.) \) would cause demand to vanish somewhere near \( Q \), and the equal demand equilibrium condition would be violated.)

These two continuity conditions are

\[ DQ - K_1 (1-e^{-\phi}) = \frac{D}{A} Q + K_2 - K_3 (1-e^{-\phi}) \]

\[ D - K_1 e^{-\phi} = \frac{D}{A} - K_3 e^{-\phi} \]

which imply

\[ K_2 = DQ (1 - \frac{1}{A}) (2e^{-\phi}) \]

\[ K_3 = K_1 e^{\phi} D (1 - \frac{1}{A}) , \]

and so

\[ p(q) = \frac{D}{A} (q + e^\phi (A-1) (1-e^{-\phi}) + Q (A-1) (2e^{-\phi}) - K_1 (1-e^{-\phi}) . \tag{6} \]

Note that the coefficient of \( D \) is unambiguously positive.

Now suppose \( A \) decreases: the proportion of people with incomes above \( y^* \) decreases. Since we are dealing only with low incomes, such a change is an increase in inequality. If \( D \) and \( K_1 \) stayed the same, the coefficient of \( D \) in (6) increases, and so
the equilibrium condition \( p(q_i) = R(q_i) \) would be violated. So \( D \) and \( K_1 \) must change in such a manner to reduce \( p(q_i) \). Either \( D \) must decrease or \( K_1 \) must increase or, in general, both. Since \( Y(0) = D - K_1 \), \( Y(0) \) and homelessness must increase. \( Q \) falls because of the increase in homelessness, but from the changes in \( D \) and \( K_1 \), all prices below the new \( Q \) must rise. To keep value constant, prices above \( Q \) must fall, and so \( p'(q_i) \) must rise.

Since \( D \) falls (in general), the supply of housing below any given quality falls also. Coupled with the rise in price for qualities below \( Q \), this means that low-income housing as defined by price falls for two reasons. Rates of new construction and abandonment must also fall, because \( D \) is the rate at which both happen.

Thus a reduction in the size of the "lower middle class" has much to commend it as an explanation for the '80s housing market phenomena in northeastern and midwestern U.S. cities. The intuition is fairly simple: housing for the poor is a by-product of housing for the "lower middle class". A reduction in demand by the "lower middle class" is a supply shock for the poor.

Next, consider what happens when the interest rate rises. The cheap-maintenance results do not carry over to the cheap-construction case; with cheap-construction, higher interest rates do not ambiguously cause higher homelessness, although they might do so. Higher interest rates have two effects that work in opposite directions. Hold \( q_1 \) constant. First, higher interest makes \( p(q_i) \) = \( R(q_i) \) higher. Since \( c(q_1) \) stays the same and so
v(q_t) must stay the same, this change would decrease the price of the lowest qualities and decrease homelessness if future prices continued to be discounted the same way. The second effect is the change in the discount rate that makes future prices less valuable. Acting alone without a change in p(q_t), this would increase all prices and drive up homelessness. In general, it is impossible to say which effect will be stronger and how homelessness will respond. (A rise in interest will almost certainly increase q_t as well, since it will increase the distance from R(q_t) to the (rc(.)+m) curve; but an increase in q_t has ambiguous effects on homelessness.)

The first effect will be stronger when the percentage increase in p(q_t) is large relative to the percentage decrease in value from discounting. The former is

\[
\frac{R'(q_t)}{R(q_t)} = \frac{c(q_t)}{rc(q_t)+c'(q_t)} = [r+\gamma]^{-1}
\]

where

\[
\gamma = \frac{c'(q_t)}{c(q_t)}
\]

the percentage rate at which construction cost is increasing. Construction cost parameters have no effect on the percentage change in discounted value. Thus interest rate changes are likely to increase homelessness when \(\gamma\) is large or when \(r\) is large to begin with.

In the semilog example with a uniform income distribution, one other result is fairly immediate. Since

\[
Y(0) = p'(0) = D - K_t,
\]

\[54\]
we can rewrite price as

\[ p(q) = Y(0)(1-e^q)+D[q+e^q-1], \]

where the coefficients of both \( Y(0) \) and \( D \) are positive. Since higher interest rates make \( p(q_1) = R(q_1) \) higher, either \( Y(0) \) or \( D \) must increase. So if homelessness doesn’t increase, density must.

4. EXTENSIONS

In this section, I will examine, once again by comparing steady states, several extensions of the basic model developed in the last section. First I will look at how the model changes when a house’s owner must incur some cost every period the house is occupied, regardless of whether it is maintained or not. Then the effects of building codes will be studied. Finally, I will say a few words about public housing and vouchers; and relate the model here to the rest of the filtering literature.

Rent collection cost

In the preceding section, I assumed that an owner who decided not to maintain a house could continue to collect rents without incurring any other costs whatsoever. There are, however, at least five other types of cost that an owner who was not maintaining a house would have to incur as long as she continued to collect rent:

1. Rent collection costs. Somebody has to bill tenants, keep track of whether they pay, and take action when they don’t.
For SRO's and similar very low-income housing, these transactions costs are nontrivial.

2. Utility costs. Owners are often legally required to provide heat and sanitary facilities. While in some sense these costs might be considered maintenance (without heat, for instance, pipes will burst; without sanitary facilities, hallways will become noisome), both common language use and the law see them as "operating" expenses, rather than investments.

3. Liability costs. Owners are liable for certain injuries that occur on their property. Many purchase insurance to cover this liability; others absorb it as a cost of doing business. However they respond, liability is still a cost to the owner. Moreover the owner or his property may also sustain damage while the property is being rented.

4. Real estate taxes. Technically, in most localities owners must pay real estate taxes whether or not their properties are inhabited. However, because real estate taxes are usually assessed in rem rather than in personam (Rubin [1936]; O'Flaherty [1990]) an owner abandoning a building could (and would) cease paying them. (White [1986] shows that taxpaying will cease before rent collecting, but we will ignore this possibility in this paper.)

5. Opportunity costs. The land a house stands on could be used for some alternative purpose. An owner who continues to rent a house every year incurs an opportunity cost equal to the annuitized value of the resale price (see Brueckner [1981]).
Gentrification stories about homelessness are essentially stories about increases in opportunity costs.

Since all five of these costs must be incurred in any period in which rent is collected. I will refer to their sum as rent collection cost, in a slight abuse of terminology.

Let $H$ denote rent collection cost per unit time. For $q \geq q^*$, $H$ will simply have to be added to the equilibrium price

$$p(q) = rc(q) + m + H \quad q \geq q^*$$

and

$$v(q) = c(q).$$

Since maintaining still costs $m$ per unit time and still saves $c'(\cdot)$ in value, $q^*$ is still the right dividing line.

Consider the cheap maintenance case first. Rent collection costs simply shift the price schedule up by $H$; the value schedule remains

$$v^m(q) = \rho(q^*) + mq$$

Owners are still indifferent about maintaining because the loss in value is still $m$; $v^m(\cdot)$ still approaches a positive quantity as quality approaches zero.

The upward translation of the price schedule, of course, means greater homelessness and a higher lowest quality of housing on the market. Higher rent collection costs increase homelessness and decrease low-income housing in several dimensions.

The cheap construction case is more complicated, of course. The lowest price will have to be $H$, not zero, and the lowest
quality of housing on the market will have to be some \( q_0 > 0 \) (if
equality housing had a positive price like \( H \), demand for it
would be zero, but supply would be positive), consumers at \( q_0 \)
must be indifferent between homelessness and \( q_0 \) housing at price
\( H \) (if they strictly preferred \( q_0 \), people with slightly less
income would also prefer \( q_0 \) to homelessness, and too many people
would locate at \( q_0 \) for the equal demand condition to be met; if
they strictly preferred homelessness, they would not locate at
\( q_0 \)). The equal demand condition implies too, that these
consumers must be the ones who would "normally" occupy these
houses if \( H \) were not lowest price possible. These considerations
lead to two equations
\[
 u(y(0) - H, q_0) = u(Y(0), 0) \quad (7)
\]
\[
-p'(q_0) u_x(y(0) - H, q_0) + u_y(y(0) - H, q_0) = 0, \quad (8)
\]
where \( p'(q_0) \) should be understood as a right-hand derivative.

For each \( q_0 \) and \( H \) there is a unique \( p'(q_0) \) that solves these
two equations; \( Y(0) \) can be eliminated. In the semilog utility
element, the relationship is
\[
p'(q_0) = \frac{H}{e^{q_0} - 1}, \quad (9)
\]
so that increases in \( H \) and decreases in \( q_0 \) increase the required
\( p'(q_0) \). (Both these changes increase the average slope of the
indifference curve linking \((0, 0)\) and \((q_0, H)\), and so the slope at
\( q_0 \) -- which must be \( p'(q_0) \) -- is likely to increase.) Since
\[
Y(0) = p(q_0) + p'(q_0) = H + p'(q_0)
\]
increases in \( H \) increase homelessness if \( q_0 \) stays the same; so do
increases in \( p'(q_0) \) if \( H \) stays the same.

Given \( q_1 \) and \( q_0 \), the price schedule is determined by two price-matching conditions \( (p(q_0)=H, p(q_1)=R(q_1)) \) and the value-matching condition \( (v(q_1)=c(q_1)) \). These in turn determine \( p'(q_0) \), and equilibrium requires \( q_0 \) be chosen to equate the \( p'(q_0) \) derived from these conditions to the \( p'(q_0) \) derived from (7) and (8).

Usually the relationship between \( q_0 \) and the \( p'(q_0) \) derived from the price- and value-matching conditions will be positive, but not always so. In the semilog utility example with a uniform distribution of income, these conditions boil down to

\[
R(q_1) = D(g-(1-e^{-g}))+p'(q_0)[1-e^{-g}] \\
c(q_1) = \int_0^g [D(s-(1-e^{-s}))]+p'(q_0)[1-e^{-s}]e^{-r(s-g)}ds
\]

where \( g = q_1 - q_0 \) and \( H \) has disappeared. Increasing \( q_0 \) gives the price schedule less room in which to rise to meet \( R(q_1) \); it also gives it less room in which to accumulate \( c(q_1) \) in value. Either \( D \) or \( p'(q_0) \) must increase, and for most examples both will; but as with interest rates, some unusual examples can be constructed where \( D \) increases enough that \( p'(q_0) \) must fall.

As usual, these changes will trigger changes in \( q_1 \), but the changes in \( q_1 \) will have ambiguous effects on the variables we are interested in.

Except in these unusual examples, then, an increase in \( H \) will shift (9) upward, and this in turn will raise the
equilibrium \( q_0 \). Homelessness and \( Y(0) \) will rise as both \( H \) and \( p'(q_0) \) increase. The rise in homelessness will be accompanied by a disappearance of the lowest qualities of housing, and, usually, by an increase in the supply of the remaining low-income housing.

Moreover, when \( H \) is positive, the same disappearance of the lowest qualities of housing can also result from the kinds of shocks that simply increased homelessness when \( H \) was zero. With positive \( H \), a reduction in the size of the lower middle class or an increase in interest rates can eliminate the bottom qualities of housing from the market.

**Building codes**

The same sort of analysis can easily be extended to consider the effect of a legal prohibition of housing below some quality \( Q > 0 \). The cheap maintenance case is trivial: if the quality constraint is high enough to be binding it raises homelessness and eliminates qualities of housing; otherwise it does nothing.

In the cheap construction case, equations (7), (8), and (9) still hold, but with \( H \) (i.e., \( p(Q) \)) considered endogenous and \( Q \) considered exogenous. No major change is required in the price- and value-matching conditions. If slopes are as described in the last section, an increase in \( Q \) raises \( p(Q) \), \( p'(Q) \), and \( D \), and \( Y(0) \), and so increases homelessness.

There is, however, an important difference. If the constraint on \( Q \) is binding, then changes in income distribution and interest rates, holding \( Q \) constant, will not cause the
elimination of low qualities of housing. Prices and homelessness will increase, but the same housing qualities will continue to be available.

My preliminary analysis does not reveal any major building code changes during the late '70s and early '80s, but there is some evidence of the virtual elimination of certain qualities of low income housing, like flophouses'. If true, these two facts would indicate that rent collection costs rather than building codes are the binding constraint on the bottom quality of housing in the market. (This conclusion also accords with the observations I have heard from businessmen in the low income housing market.) Such a finding, however, does not absolve local government policy from any responsibility for the rise of homelessness, since most of components of rent collection cost depend heavily on local government actions.

**Public housing and housing vouchers**

These traditional housing policies work by changing demand from different parts of the income distribution. In the cheap-maintenance case, they decrease homelessness precisely to the extent that they house exactly the people who would have been homeless in their absence; they do not affect the price schedule. In the cheap-construction case, their impact on homelessness is extremely sensitive to the exact demands that are augmented or reduced.

Take, for instance, a public housing program. Suppose that
it is aimed towards the lower middle class (this may not be a bad
description of public housing in New York City). If the
government supplies housing to this group, it reduces demand for
housing at the top of the filtering section around $q$. The result
-- an increase in homelessness-- is precisely the same as that of
a reduction in the size of the lower middle class. On the other
hand, a public housing program aimed at people not much richer
than $Y(0)$ (probably a good description of public housing in
Newark and Chicago), will reduce demand at the bottom of the
quality distribution, and this in turn will reduce homelessness.
(That public housing grew in New York in the '70s, and '80s, and
shrank in Newark and Chicago is thus entirely consistent with the
growth of homelessness in all three cities.)

Vouchers effectively move people from one "income" to
another, and so a voucher program affects demand at two
qualities, at least. The possibilities grow. A voucher program
that moved people from homelessness or very near it to very near
$q_1$, would clearly decrease homelessness; a program that moved
people from near $q_1$ to over $q_2$ would increase homelessness; and a
program that just shuffled people around near $q_1$ would do almost
nothing. The initial income distribution can matter here, too,
since a voucher program could be relieving or exacerbating a
bottleneck in the flow of houses downward to abandonment.
Administration of voucher programs also matters. The section 8
program places maintenance requirements on units being paid for
with vouchers. Maintenance requirements would do more than shift
Relation with other literature

Since Sweeney [1974] first rigorously developed and solved a model of filtering in the housing market, many writers have used models like the one in this paper. Arnott [1987, pp. 971-981] provides an excellent survey of many of these papers.

The original Sweeney paper is probably closest to the work here in that it models both supply and demand. However, there is no explicit provision for homelessness, the maintenance technology dooms every house to deteriorate eventually, and quality is a discrete variable rather than a continuous one. Discrete quality makes it difficult to investigate seriously how the lowest quality of housing in the market is determined -- too much depends on step size.

Braid [1981, 1984] studies a continuum of qualities, but focuses on the demand side. He also rules out homelessness explicitly. Henderson [1977] and Arnott, Davidson, and Pines [1983] look at continuous qualities, but they concentrate on the supply side, maintenance decisions especially. Since homelessness is a demand side phenomenon, supply side models can at best give hints about reasons for homelessness.

Arnott, Braid, Davidson and Pines[1986] integrate supply and demand, with continuous qualities, but assume that all consumers have the same income. Once again, interesting homelessness is impossible to study in this model.
Both Arnott, Davidson, and Pines and O'Flaherty [1993] explicitly consider the possibility of rebuilding, which this paper ignores except in the calculation of rent collection costs. Private rebuilding on land where buildings have been abandoned has been quite rare in the US cities -- my 1993 paper tries to answer why -- and it seems easier to ignore the phenomenon altogether here rather than construct an elaborate model in which it would end up not happening.

My 1993 paper also differs from this work in incorporating uncertainty. The important lesson from uncertainty is that in the 1970s, homelessness may have been "artificially" low relative to fundamentals. In that paper, houses deteriorate until they are rebuilt; the lowest quality on the market depends on the opportunity cost of rebuilding. Uncertainty introduces an option value for low quality housing; as long as the option is not exercised by rebuilding, the owner can wait for a more profitable time to rebuild. During bad times, this option value drives down the lowest quality available in the market: owners are waiting for good times to return. In this model, homelessness rises when good times return, owners exercise their rebuilding options, and low qualities of housing disappear from the market. This sort of model may explain part of the explosiveness of the rise of homelessness in many American cities, and part of the American association between homelessness and prosperity.
5. SHELTERS

When the government and concerned citizens find people sleeping on the streets, they usually set up shelters (or hostels, in the British and Canadian usage). Because shelter beds are reasonably easy to count, while people sleeping on the street are almost impossible, numerical studies of homelessness are almost always studies of shelters. This can lead to some biases: for instance, wealthier cities with more valuable real estate might be expected to establish more shelters simply because charity and unobstructed streets are both normal goods, and the presence of homeless people causes larger absolute reductions in the value of more expensive property.

The most important question about shelters is whether and how they affect people who would not be homeless if shelters did not exist. That government and charitable activities alter behavior is, of course, no reason to condemn them: the provision of tap water clearly lowers the demand for bottled water, and most users of tap water would shift to bottled water rather than die of thirst if the city went out of the water business. But still, for understanding changes in the number of people staying in shelters we need to know the effect of shelters.

In this section, therefore, we will look first at how shelter provision alters equilibrium conditions in the housing market model we have been working with, and then we will look at shelter provision when people differ in both tastes and income.
Finally we will endogenize shelter provision and show how hysteresis alters the time profile of measured homelessness.

**Homogeneous tastes**

Incorporating shelters into the housing market model of the last two sections is fairly easy; it just involves shifting the "zero" on the demand side. Let government or charities provide shelters of quality $S > 0$ for free to anyone who wishes to use them. While "right to shelter" is far from a universal legal doctrine, rationing of shelter beds does not seem to be a widespread phenomenon. At various times in various cities, people have been turned away from shelters, but in most cities at most times shelters have vacancies. Capacity is usually not a binding constraint, especially for a city's shelter system as a whole.

In the cheap-maintenance case, shelters must cause some low qualities of housing to disappear, and the former residents of that housing to enter shelters, as long as a building code constraint is not binding. This happens even if $S$ is below the quality of any housing on the market when shelters are introduced. The previous lowest quality on the market made its consumers indifferent between homelessness and buying that quality at the supply price. If shelters are available at the same price as street homelessness but are more attractive, then the lowest quality of housing available in the market has a higher hurdle to surpass. The lowest quality of housing rises
faster than the quality of shelters S rises, because the price schedule is upward-sloping.

For example, with semilog utility, Y(0) and q_m, the minimum quality on the market, are determined jointly by

\[
\ln \{Y(0) - p^a(q_m)\} + q_m = \ln Y(0) + S
\]

\[
im = e^{(x-s)y(0)}
\]

which makes the Y(0)-indifference curve through (S,0) tangent to the price-schedule at q_m. These imply

\[
im (e^{q_m-s} - 1) = p^a(q_m)
\]

An equal change in q_m and S leaves the left-hand side of this expression unchanged, but increases the right-hand side (which does not depend on S). So q_m must rise still more in order for equilibrium to be restored.

With cheap construction and rent collection costs the picture is much the same, although there is the further possibility of an increase in p'(q_0), which would increase homelessness further. With no rent collection costs, shelters at quality S will necessarily eliminate demand for all housing below S, and if p'(s) is greater the p'(0) (because the horizon for price-rise and value recoupment is shorter), shelter residency will be greater than homelessness for this reason as well.

Thus unless a building code constraint is binding, shelters will always contain people who would not have been homeless if shelters were not there. To the extent that shelters are
established for eleemosynary purposes this result is not necessarily to be decried. If \( S \) is low the people who enter shelters would have been very badly off even if they were housed. Since (almost) everyone who enters a shelter becomes better off by doing so, shelters may still be helping a deserving class. To the extent that shelters are established for street cleaning purposes, however, the extra people are a burden.

At its root, contamination is a moral hazard problem. If income could be observed costlessly, there would be no need for shelters: those with income below \( Y(0) \) could be paid the difference between their income and \( Y(0) \) (or a greater amount if higher quality housing were deemed desirable). Income, however, as we are using it here, is an abstraction for resources, and resources at the bottom of the social scale (as at the top) are essentially unobservable. Good family connections, for instance, are a key resource for the very poor, and they cannot be easily and verifiably observed. Most shelters do not have income eligibility requirements; they rely on self selection. A cash distribution scheme could not rely on self selection.

The emergence of bigger and better and better-publicized shelters during the 1980's is therefore one more reason why measured homelessness grew; it seems likely to have contributed especially to the explosive nature of that growth. To say this is not to argue that homelessness exists just because a bunch of soft-hearted and weak-minded liberals set up shelters. There are enough other reasons for homelessness to have risen in the United
States in the 1980's. It is to assert, rather, that certain intermediate levels of homelessness are unlikely to be observed in equilibrium.

Tooth decay is a good analogy. In the presence of dentists, small cavities do not persist in equilibrium. Dentists drill them and make them big in order to put fillings in. Dentists don't cause cavities, but the way they treat them makes them bigger. Whether or not this treatment method is optimal is a separate issue from what it does to the size of cavities; dentists seem to think their method is pretty good.

**Heterogeneous tastes**

Heterogeneous tastes make moral hazard problems bigger because they add another unobservable. There is good reason to believe tastes are heterogeneous: some people sleep on the streets even when shelters have vacancies. Thus even if shelter operators could observe income perfectly and restrict admission to people below some income level, they still could not restrict their shelters only to those who would otherwise be on the street. The same is true for cash assistance.

When the homeless have heterogeneous tastes, the quality of shelter $S$ becomes a more interesting variable for shelter operators who are motivated primarily by street-cleaning concerns. With homogeneous tastes, such operators would set $S$ as low as possible, and could still be assured that no one would stay on the street. With heterogeneous tastes, this strategy
won't work; very few street people would be encouraged inside unless the shelter quality was pretty good, and if the shelter quality was pretty good, many people who would not think of living on the streets would come to the shelter. Depending on the distribution of tastes and the cost of shelter, it is conceivable that street-cleaning shelters could have higher quality than eleemosynary shelters.

**Hysteresis**

While most of this paper has been concerned with comparative steady states, some problems of adjustment also deserve mention. The main one is hysteresis. Decreasing homelessness is much more difficult than increasing it.

On the level of the individual homeless person, hysteresis works in several different ways. There are fixed costs both to entering homelessness and to leaving it. Entering homelessness may involve loss of possessions, an investment in "learning the ropes" and in community [see e.g., for a discussion of the Manhattan Bridge homeless community, Dordick[1993]]. It also implies enhanced hazards of physical and mental illness. Leaving homelessness requires search costs and often a deposit. The conditions that are bad enough to make someone become homeless are considerably worse than the conditions good enough to make someone leave homelessness.

Individual-level hysteresis is compounded by shelter-level hysteresis. Setting up a shelter has several obvious fixed
costs—learning about homelessness and shelter operation, searching for and renovating a building, getting licenses. Another set up cost is the loss of information—once you set up a shelter you no longer can tell how many people would be on the street if the shelter weren’t there. Shelters won’t be shut down just because conditions are as good as they were when the shelters were set up.

Because of contamination, the two levels of hysteresis compound each other. If homelessness has been high, more shelters are open, more people occupy them, and more people stay homeless—which in turn causes more shelters to stay open. Under proper conditions, homelessness can decrease, but the process is a slow one.

6. CONCLUSION

To summarize:

1. Rising inequality with fixed prices can explain rising homelessness, but cannot explain why housed poor people would pay more for housing.

2. With filtering, rising inequality can also explain these higher prices, as well as decreasing abandonment and a smaller stock of low income housing.

3. Rising real interest rates can explain higher homelessness, higher prices, and less-housing available below a fixed rent, but have difficulty fully explaining the fall in
abandonment.

4. A rise in rent collection costs, if it occurred, would have many of the same effects as a rise in real interest rates. So would a rise in building code standards.

5. If a rent collection cost constraint is binding in determining the lowest quality of housing available, increases in inequality or in real interest rates will eliminate some qualities of housing from the market. This will not happen if a building code constraint is binding.

6. The impact of traditional housing programs is very sensitive to the exact place on the income distribution where they operate.

7. Unless a building code constraint is binding, shelters will always serve some people who would not otherwise be on the street and will always cause the loss of some qualities of housing. If consumers have heterogeneous tastes, this conclusion is even stronger and will hold even if a building code constraint is binding.

8. Homelessness is marked by hysteresis. Once homelessness has gone up, external conditions have to become considerably better than they were originally for homelessness to return to where it was originally.

Understanding homelessness is thus a task well within the capabilities of current economic analysis. Figuring out what to do about homelessness is harder.
REFERENCES


---, R. Braid, R. Davidson, and D. Pines [1986]: "A general equilibrium spatial model of housing quality and quantity, mimeo.


Dordick, G [1993]: Dissertation in progress. Columbia
University, Department of Sociology.


Rubin, E. [1936]: "Collection of delinquent property taxes by action in personam," Law and Contemporary Problems 3, 416-4
