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Learning-by-Doing and Its Implications for Economic Growth and International Trade

A Thesis

By

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INTRODUCTION

In economics, human capital is defined as the stock of physical strengths, knowledge, skills and intelligence the labor force possesses, which contributes to the efficiency of labor productivity. In other words, human capital represents the overall quality of workers involving in the production activities of an economy. Since Lewis (1954) first introduced this concept in his path-breaking work on economic growth, *Economic Development with Unlimited Supplies of Labour*, its implications for various fields of economics have been studied, among which the two most worth noting ones are perhaps economic growth and international trade.

In the field of economic growth, following the pioneering work of Schultz (1981) and Lucas (1988), Mankiw, Romer & Weil (1992) conducted an empirical study showing the limit of Solow model (Solow, 1956) on explaining the wage differentials across countries. To fix this problem, they introduced the concept of human capital with an evolutionary dynamics similar to that of technological advancement in the Romer model proposed and refined by Romer (1986), Jones (1995) and others, and successfully extended the classical Solow framework of growth theory.

The critical rule of human capital in international trade was not paid much attention until Leontief (1953) conducted an empirical validation of the classical Heckscher-Ohlin model (Jones & Neary, 1984), in which he derived a counter-intuitive conclusion that the US was actually importing capital-intensive and exporting labor-intensive goods. This so-called "Leontief Paradox" was finally resolved by Kenen (1965), who paid specific attention to the wage differential between the export and import sectors of the US: since the average wage in the export sector was higher, its workers should have higher productivity, or in modern terminologies, should have higher amount of efficient labor. By adding this additional human capital to physical capital, Kenen showed that the export sector was indeed more capital intensive. This finding was further confirmed by a follow-up study by Branson & Junz (1971) using a different approach, which treats human capital as an independent input factor parallel to physical capital. From the theoretic aspect, the most natural way to incorporate human capital into the Heckscher-Ohlin model would be simply adding skilled labor as another factor of production. However, this approach does not provide us any insight into the mechanism that determines the skilled-to-unskilled labor ratio, i.e. the incentives and process of transforming unskilled workers to the skilled ones. Motivated by this problem, Findlay & Kierzkowski (1983) developed a two-good, two-factor model that is in line with the Heckscher-Ohlin framework, in which the development of human capital is modeled as a result of education, and the skilled-to-unskilled labor ratio is determined endogenously.

Since World War II, a huge amount of work exploring the dynamic effects of international trade has been conducted by Bhagwati, Findlay, Hicks, Johnson and many other prominent authors in the literature (Findlay, 1984). This revolution has greatly expanded our understanding of the connections between growth, development and trade, and by late 1980's, a neoclassical framework that unifies the Solow-Romer growth theory and the static Heckscher-Ohlin trade theory was established, which we may loosely call "open-economy growth theory". Influenced by this movement, the study on the dynamic effects and evolution of human capital was carried on. For example, Bardhan (1970, Ch. 7) first theorized the learning-by-doing effect in order to justify the old "infant industry hypothesis" proposed by Alexander Hamilton, the first United States Secretary of the Treasury. The argument states that newly established industries in a country may not have the economies of scale compared with their foreign opponents, and it takes time for them to improve their productivity by practice. Thus, it is justifiable for the government to protect new industries from international competitions in the domestic markets, which may be implemented by tariff, quotas, etc.

This new area, although is being developed prosperously, is still a debating arena. The main reason, in my opinion, is that there is no generally accepted micro-foundation of human capital that is general enough to fit into the standard framework of growth and trade. Without this theoretic foundation, it is hard to perform further modeling work based on the empirical facts we have known, and thus is hard to yield common agreements. Intuitively, corresponding to the four broad categories of human capital defined at the beginning of this section, the development of human capital should mainly result from three sources, namely health improvement, formal education and learning-by-doing, whereas intelligence, although is also an important factor of the development of human capital, is usually omitted in detailed economic analysis since it is considered as inborn, whose causes are studied in the scope of other sciences, such as psychology.

The positive relationship between health improvement and human capital has been verified by a lot of empirical studies. For example, Strauss & Thomas (1998) studied the correlation between wage and height differences in both the US and Brazil, and showed that on average 1% difference in height is associated with 1% difference in wage in US, whereas the wage difference becomes 7.7% in Brazil. Under the assumptions that the height differences are mainly due to gene in the US and is mainly due to malnutrition in Brazil, and that wage differentials are due to differences in productivity that is in turn partially determined by health conditions, they thus concluded that there is a positive relationship between health and human capital. As for the improvement of human capital through formal education, an empirical study conducted by Hall & Jones (1999) on a dataset drawn from both developing and developed countries shows a large positive correlation between wage and years of schooling.

Intuitively, both health improvement and education require inputs of scarce resources, such as time, effort and money. Thus, theoretic models on the accumulation of human capital due to these two sources can be naturally developed under the usual cost-benefit framework in which individuals and firms make optimal investment decisions, which may then be applied to solve public policy issues. Given this straightforward and less disputable approach, most of the previous research on this topic has been focusing on health and education as well as their impacts on growth and trade. Some of these notable studies include Becker (1962), Findlay & Kierzkowski (1983), Acemoglu (1996), Bloom, Canning & Sevilla (2004), and Becker (2008).

In contrast to health and education, the effect of learning-by-doing on the development of human capital, although is confirmed by a rich profile of empirical evidences ranging from the progress ratio analysis in Dutton & Thomas (1984) to the "10, 000-hour rule" in popular culture inspired by Ericsson (2006) and to the old idiom, "practice makes it perfect", still lacks a solid theoretic foundation. This is possibly because the true mechanism of learning-by-doing at the individual level is beyond the scope of economics, and thus no intuitive approach is available to us. As a result, we are still far from a complete integration of learning-by-doing into the standard framework of growth and trade. Still, a few insightful attempts have been made. In the field of international trade, Clemhout & Wan (1970) first proposed a two-sector model in which learning-by-doing happens in both sectors and is set as a function of cumulative output of the specific industry, from which they also studied the optimal domestic price policy in the infant-industry context. Their study was then followed by Teubal (1973), who used a specific form of this model to study the classical Rybczynski theorem (Rybczynski, 1955). Finally, Young (1991) proposed a more general model assuming that a continuum of goods are being produced and consumed.

Compared with other constructions, this “Young model” has a few advantages, which make it the most compelling candidate for the integration of learning-by-doing into the standard framework of growth and trade. First, by assuming an infinite number of goods, the model not only makes it closer to the reality that a large number of goods are produced and consumed at each time, but also opens the possibility that consumers’ consumption basket is changing over time, which is what we observe in our daily life. Second, it takes into account the “spillover effects” across different industries as verified empirically by Rosenberg (1982) and Jaffe (1986), which refers to the phenomenon that the knowledge acquired by experience in one industry increases the productivity of some other industries as well. Third, it also incorporates the empirical fact that the productivity increment triggered by learning-by-doing is bounded, the so called “diminishing returns in learning-by-doing process”, as studied by Wright (1936), Hirsch (1956), Alchian (1963) and others. Fourth, the combination of a continuum of goods, the spillover effect, and the diminishing returns in learning-by-doing magically enables technical progress in the sense that the “average” technical sophistication of the goods an economy produces keeps rising.

Despite these advantages, the assumption that labor is the sole factor of production, although greatly simplifies the model without losing the key points, limits its potential of being part of the standard formulation of growth and trade theory, in which we usually assume that there are at least two factors of production, namely physical capital and labor. Moreover, the Young model does not take into account the dynamic effects of capital accumulation and population growth. Expanding the Young model from these two perspectives is a critical step on our journey that may finally lead to a unified growth and trade theory that integrates human capital.

This paper serves as an attempt to explore this very step. In section one, we construct a generalized Young model following the two perspectives discussed above. In section two, we narrow down our focus on a special functional form of this generalized model by assuming the production function of each industry is of the Cobb-Douglas type. We then use this special functional form to study its implications for economic growth in section three, and to reconstruct Young's conclusions on the dynamic effects of trade in section four. In section five, we briefly summarize our main findings in this paper, and discuss some possible extensions.

Throughout this paper, the wording "Young's paper" refers to Young (1991).

I. A GENERALIZED YOUNG MODEL

1.1 Unit Labor Requirement

Following Young's construction, we start by assuming that there is a continuum of goods, each corresponds to a specific industry, indexed along a part of the real line, $[B,\infty)$, where higher index indicates higher level of "technical sophistication", and B can be taken as any positive real number. It is assumed that the economy is populated by a large number of households, and there is perfect competition in each industry so that no economic profit is left. In the Young model, the unit labor requirement of industry s at time t, *a(s,t)*, is set to be independent of the inputs of that industry, which is not compatible with our intention to include physical capital as a factor of production. We instead assume that the unit labor requirement is also negatively related to the capital-to-labor ratio, $k_s = \frac{K_s}{L_s}$, where K_s and L_s are the corresponding physical capital and labor inputs of industry *s*. The intuition behind this assumption is that the more capital-intensive an industry is, the more resources each worker can use in its production activities, which results a faster learning progress and a higher level of labor productivity in the long-run. Moreover, for the limit cases, we make the following assumptions:

(1) $$\lim_{k_s \to \infty} a(k_s, s, t) = 0$$

(2) $$\lim_{k_s \to 0} a(k_s, s, t) = \infty$$

We adopt Young's assumption that for each s, $a(k_s,s,t)$ is not only non-increasing in t as the result of learning-by-doing, but also bounded below by the "potential unit labor requirement", $\bar{a}(k_s, s)$. I.e.

(3) $$\lim_{t \to \infty} a(k_s, s, t) = \bar{a}(k_s, s)$$

Since higher index means higher level of technical sophistication, intuitively this implies that at the maximum productivity potential, higher indexed industry requires less amount of labor to produce the same amount of good. Hence, we adopt Young's assumption that $\bar{a}(k_s, s)$ is non-increasing in *s*. In addition, following the same intuition as above, we assume $\bar{a}(k_s, s)$ is non-increasing in k_s, and

(4) $$\lim_{k_s \to \infty} \bar{a}(k_s, s) = 0$$

(5) $\lim_{k_s \to 0} \bar{a}(k_s, s) = \infty$

Moreover, we assume

(6) $\lim_{s \to \infty} a(k_s, s, t) = \infty$

The intuition of this assumption, as pointed out by Young, is that although the blueprints of all goods are available at any time, the economy "must pass through a certain amount of production experience before the costs of production of advanced goods fall to acceptable levels". Finally, we assume that both $a(k_s,s,t)$ and $\bar{a}(k_s, s)$ are continuously differentiable.

As for the dynamics of $a(k_s,s,t)$, based on the assumption made in Young's paper, we further assume that its proportional change over time is also related to the amount of capital and labor inputs in the particular industry, but temporarily ignore what this specific relationship is at this point. Formally speaking,

(7)
$$\frac{\partial a(k(s,t),s,t)/\partial t}{a(k(s,t),s,t)} = -\int_B^\infty B\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right) L(v,t)\, dv$$
$$-\int_B^\infty D\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right) K(v,t) dv \ +g(K,L)(s,t)$$

In the above equation, $B\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right)$ and $D\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right)$, the so called "learning-by-doing coefficients" in Young's paper, are used to model the spillover effect, or equivalently "knowledge spillover", from industry v to industry s, as we discussed in the introduction. $k_s = k(s,t)$ and $L_s = L(s,t)$ are now allowed to change over time, which corresponds to the assumption of free capital and labor flows across industries. Moreover, we assume both $K(s,t)$ and $L(s,t)$ are differentiable, and $g: (C^1\{[B,\infty) \times [0,\infty)\})^2 \to C^1(-\infty,\infty)$ is a linear functional, where $(C^1\{[B,\infty) \times [0,\infty)\})^2$ denotes the Cartesian product of the space of continuously differentiable functions defined on $[B,\infty) \times [0,\infty)$.

We still hold Young's six assumptions on the learning-by-doing coefficients, namely:

(8) $$B\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right) = 0 \qquad \text{if } a(k_s, s, t) = \bar{a}(k_s, s)$$

, which means once the learning-by-doing potential of an industry is exhausted, it can no longer take the benefit of knowledge spillovers offered by other industries;

(9) $$B\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right) \geq 0 \qquad \forall \text{v} \in [\text{B}, \infty)$$

, which means the spillover effect is always non-negative;

(10) $$B(s, v, 1) = 0 \qquad \forall \text{s}, \text{v} \in [\text{B}, \infty)$$

, which means once the learning-by-doing potential of an industry is exhausted, it can no longer offer knowledge spillovers to any other industries;

(11) $\forall$s $\in$ [B, ∞), $\exists$ ε = ε (s) >0 s.t. $\forall$v $\in$ (s - ε, s + ε),

$$B\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right) > 0, \text{ if } a(k_s, s, t) > \bar{a}(k_s, s)$$

, which means any industry in which the learning-by-doing potential is not exhausted has strictly positive spillover effect at least on some of its "neighborhood" industries;

(12) $$Sup_s Sup_v B\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right) < \infty$$

, which means the spillover effect is bounded;

(13) $B\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right)$ is continuous in s

, which is just for the convenience of our analysis. The same set of assumptions also applies to $D\left(s, v, \frac{a(k(v,t),v,t)}{\bar{a}(k(v,t),v)}\right)$.

For convenience, starting from this point, when the context makes it clear that the word "capital" refers to physical capital other than human capital, we simply call physical capital "capital".

1.2 Goods Market

By definition, the total output of industry s at time t is

(14) $$X(s,t) = \frac{L(s,t)}{a(k(s,t),s,t)}$$

Thus, the marginal productivities of capital and labor are

(15) $$MPK(s,t) = -\frac{L(s,t)}{a^2(k(s,t),s,t)} \frac{\partial a(k(s,t),s,t)}{\partial K(s,t)}$$

(16) $$MPL(s,t) = \frac{1}{a(k(s,t),s,t)} - \frac{L(s,t)}{a^2(k(s,t),s,t)} \frac{\partial a(k(s,t),s,t)}{\partial L(s,t)}$$

1.3 Labor Market

Since in our model the productivity gain resulted from learning-by-doing is fixed in the specific industry instead of in the labor force, there is no distinction between skilled and unskilled labor, and we assume that all workers are identical. Suppose there is no friction in the labor market, then wage rate must be the same across all "active industries", i.e. industries that are still producing their specific products. We take the wage rate as numéraire, i.e. $W(t) \equiv 1$.

Finally, we assume full employment in the economy, and denote the total amount of labor contribution as $L(t) = \int_B^\infty L(s,t)ds$, whose dynamics is given exogenously.

1.4 Capital Market

For simplicity, we assume that the production function, *X(s,t)*, in each industry is homogenous of degree 1. Moreover, we assume that there is no friction in the capital market such that interest rates across all active industries are the same.

Denoting the price of good *s* at time *t* as *p(s,t)*, we have $1 \equiv W(t) = p(s,t)MPL(s,t)$. Thus, from (16) we get

(17) $$p(s,t) = \frac{a^2(k_s,s,t)}{a(k_s,s,t) - L(s,t)\frac{\partial a(k_s,s,t)}{\partial L(s,t)}}$$

Hence the interest rate can be calculated as

(18) $$r(t) = p(s,t)MPK(s,t) = \frac{L(s,t)\frac{\partial a(k(s,t),s,t)}{\partial k(s,t)}}{L(s,t)\frac{\partial a(k(s,t),s,t)}{\partial L(s,t)} - a(k(s,t),s,t)}$$

Note that in our model, although capital is assumed to be able to flow freely across all industries in order to guarantee that the interest rate is the same everywhere, the total amount of capital stock, $K(t) = \int_B^\infty K(s,t)ds$, is independent of the flows. More precisely speaking, let us assume that a constant proportion, *γ*, of returns to capital in each industry is saved, and there is a constant depreciation rate, *d*. Let *δ(s,v,t)* denote the rate of capital flow from industry *v* to industry *s* at time *t* with $\delta(s,v,t) = -\delta(v,s,t)$, then the evolution of capital stock in each industry takes the form

(19) $$\frac{\partial K(s,t)/\partial t}{K(s,t)} = \gamma r(t) - d + \frac{1}{K(s,t)}\int_B^\infty \delta(s,v,t)\, dv$$

Note that $\iint_{[B,\infty)^2} \delta(s,v,t)dvds = 0$. The evolution of capital stock at the aggregate level, according to Leibnitz Rule, is thus

(20) $$\frac{dK(t)/dt}{K(t)} = \gamma r(t) - d$$

1.5 Consumption

For simplicity, we follow Young's assumption that all households are identical and have symmetric preference over all goods. Thus the aggregate consumption behavior of the economy is equivalent to the case in which there is only one "aggregate consumer" in the economy, who contributes *L(t)* amount of labor. The instantaneous utility is

(21) $$V(t) = \int_B^\infty U\big(C(s,t)\big)ds$$

, where *C(s,t)* denotes the consumption of good *s* at time *t* by the aggregate consumer, and *U* is strictly increasing and quasi-concave with *U(0) = 0*. The aggregate consumer at time *t* thus maximizes its intertemporal utility

(22) $$E(t) = \int_t^\infty \int_B^\infty e^{-\rho(x-t)} U\big(C(s,x)\big)dsdx$$

, where $\rho \in [0,1]$ is a discount factor, subject to the instantaneous budget constraint

(23) $$\int_B^\infty p(s,t)C(s,t)ds \leq L(t) + (1-\gamma)r(t)K(t)$$

, where we assume that the remaining proportion of returns to capital left unsaved is transferred to households and then consumed.

1.6 Market Clearing Conditions in Autarky

At this moment, let us suppose that the economy is in autarky. At equilibrium, the total value of consumption and investment must be equal to the total value of production remained after depreciation. Hence, we have the following general equilibrium condition:

(24) $$\int_B^\infty p(s,t)C(s,t)ds + (\gamma r(t) - d)\int_B^\infty K(s,t)ds$$
$$= \int_B^\infty p(s,t)X(s,t)ds - d\int_B^\infty K(s,t)ds$$

Moreover, in any active industry, the output is either consumed or reinvested in the same or other industries. Formally, we have the following partial equilibrium condition:

(25) $$X(s,t) = C(s,t) + \frac{\gamma r(t)}{p(s,t)} K(s,t) = C(s,t) + \gamma MPK(s,t) K(s,t)$$

1.7 Initial Conditions

To close this model, we should also specify the initial conditions. We use $S_1(t) = \{s \in [B, \infty) \mid X(s,t) > 0\}$ to denote the set of active industries at time t, and use $S_2(t) = \{s \in [B, \infty) \mid a(k_s,s,t) > \bar{a}(k_s,s)\}$ to denote the set of industries in which learning-by-doing has not exhausted at time t. We assume that $S_1(0)$, $S_2(0)$, $a(k(s,0),s,0)$, $K(s,0)$, $L(s,0)$ are all known for $\forall s \in [B, \infty)$. Finally, to ensure that all the integrations that appear in this paper make sense, we state without any justification that $S_1(t)$ and $S_2(t)$ are Lebesgue measurable for all $t \geq 0$.

II. A SPECIAL FUNCTIONAL FORM IN AUTARKY

While this generalized Young model provides us a solid foundation for further analyses of growth and trade, its generality limits its applications. To simplify the derivations while keeping enough flexibility of this model so that it can still be comfortably fitted into the general literature, we confine our focus on a special functional form of it, which assumes that the production function of each industry is of the Cobb-Douglas type, and retains some of the simplifying assumptions on learning-by-doing as proposed in Young's paper. In the following, we first summarize all these assumptions, and then solve every aspect of this model in detail.

2.1 Basic Settings

We start by assuming that the production function of each industry is of the Cobb-Douglas type:

(26) $$X(s,t) = H(s,t)K^{\alpha}(s,t)L^{1-\alpha}(s,t)$$

, where *H(s,t)* is an industry-specific parameter that may also change with respect to time, and α is the same across all industries. Let us take

(27) $$a(s,t) = h(s,t)(\frac{L(s,t)}{K(s,t)})^{\alpha}$$

, where *h(s,t) = 1 / H(s,t)*. It can be easily verified that (26) and (27) are in line with our construction in (1), (2), and (14).

We adopt the same simplifying assumption from Young that $B\left(s,v,\frac{a(v,t)}{\bar{a}(v)}\right)$ and $D\left(s,v,\frac{a(v,t)}{\bar{a}(v)}\right)$ in (7) are equal to -2. It can then be easily verified that the evolution of *h(s,t)* is given by

(28) $$\frac{\partial h(s,t)/\partial t}{h(s,t)} = \begin{cases} -2\frac{dT(t)}{dt} & if\ h(s,t) > \bar{h}(s) \\ 0 & otherwise \end{cases}$$

, where *T(t)*, the so called "learning-by-doing equation" in Young's paper, is now changed to

(29) $$\frac{dT(t)}{dt} = \int_{T(t)}^{\infty}\big(L(s,t) + K(s,t)\big)ds$$

, and $\bar{h}(s) = \bar{h}e^{-s}$ is the greatest lower bound of *h(s,t)* with $\bar{h}$ being a constant. As for the initial conditions, we adopt Young's simplifying assumption that *h(s,0)* is symmetric around *s = T(0)* such that only those industries with indexes *s > T(0)* have not exhausted their learning-by-doing potentials at the beginning. More precisely, we assume

(30)' $$h(s,0) = \begin{cases} \bar{h}e^{-s} & if\ s \leq T(0) \\ \bar{h}e^{s-2T(0)} & otherwise \end{cases}$$

Under this initial condition, it can be easily verified that at any time *t*,

(30) $$h(s,t) = \begin{cases} \bar{h}e^{-s} & if\ s \leq T(t) \\ \bar{h}e^{s-2T(t)} & otherwise \end{cases}$$

I.e. *h(s,t)* is always symmetric around *T(t)* and only those industries *s* > *T(t)* have not exhausted their learning-by-doing potentials at any time *t*. Note that the key point of our simplifying assumptions here is that by assuming a Cobb-Douglas production function, the unit labor requirement now consists of two multiplicatively separable terms: one is the capital-to-labor ratio, *k(s,t)*, and the other one is *h(s,t)*, which may be illustrated intuitively as the "pure learning progress" that is independent of the capital-to-labor ratio. This in turn allows us to define *T(t)* and $\bar{h}(s)$ in a similar manner to those in Young's paper. The proportional change of *a(s,t)* is thus

(31) $$\frac{\partial a(s,t)/\partial t}{a(s,t)} = \begin{cases} -2\frac{dT(t)}{dt} + \alpha\left(\frac{\partial L(s,t)}{\partial t} - \frac{\partial K(s,t)}{\partial t}\right) & if\ h(s,t) > \bar{h}(s) \\ \alpha\left(\frac{\partial L(s,t)}{\partial t} - \frac{\partial K(s,t)}{\partial t}\right) & otherwise \end{cases}$$

We keep Young's construction that the aggregate consumer's instantaneous utility function for a single good is

(32) $$U\big(C(s,t)\big) = \log(C(s,t) + 1)$$

Finally, we assume that the economy exhibits constant rate of population growth, i.e.

(33) $$\frac{dL(t)/dt}{L(t)} = n$$

2.2 Prices, Interest Rate, Capital-to-Labor Ratio and Total Value of Output

To begin with, from (27) we have

(34) $$\frac{\partial a(s,t)}{\partial L(s,t)} = \alpha\frac{a(s,t)}{L(s,t)}; \quad \frac{\partial a(s,t)}{\partial K(s,t)} = -\alpha\frac{a(s,t)}{K(s,t)}$$

Hence from (17) and (18):

(17)’ $$p(s,t) = \frac{a(s,t)}{1-\alpha} = \frac{1}{1-\alpha} h(s,t) (\frac{L(t)}{K(t)})^{\alpha}$$

(18)’ $$r(t) = \frac{\alpha}{1-\alpha}\frac{L(s,t)}{K(s,t)} = \frac{\alpha}{1-\alpha}\frac{L(t)}{K(t)}$$

Note that the first equality of (18)’ implies that the capital-to-labor ratio across all active industries should be the same, and thus should also be equal to the capital-to-labor ratio of the whole economy, which is where the second equalities of (17)’ and (18)’ come from. Moreover, since *h(s,t)* is symmetric around *T(t)* according to (30), it also implies that the unit labor requirement is symmetric around *T(t)*. Finally, (17)’ implies that any two goods symmetric around *T(t)* have the same price.

Let

(35) $$Y(t) = \int_B^{\infty} Y(s,t)ds = \int_B^{\infty} p(s,t)X(s,t)ds$$

denote the total value of output of the economy, where *Y(s,t) = p(s,t)X(s,t)*. Combining (14) and (17)’ yields

(35)’ $$Y(t) = \frac{L(t)}{1-\alpha}$$

2.3 Total Capital Stock and Equilibrium Level of Capital-to-Labor Ratio

Using the fact that the capital-to-labor ratio is the same across all active industries, combining (18)’ and (20) thus yields

(20)’ $$\frac{dK(t)}{dt} = \gamma\frac{\alpha}{1-\alpha}L(t) - d\cdot K(t)$$

, which is a non-linear first order ODE. We can reduce it to a second order linear homogenous ODE with constant coefficients by differentiating both sides by *t*, which yields

(20)’’ $$\frac{d^2K(t)}{dt^2} = \gamma n\frac{\alpha}{1-\alpha}L_0 e^{nt} - d\cdot\frac{dK(t)}{dt} = (n-d)\frac{dK(t)}{dt} + d\cdot nK(t)$$

It can then be easily shown that given initial conditions $K(0) = K_0$ and $L(0) = L_0$, the unique solution is

(20)'''
$$K(t) = \left(K_0 - \frac{\gamma}{d+n}\frac{\alpha}{1-\alpha}L_0\right)e^{-d\cdot t} + \frac{\gamma}{d+n}\frac{\alpha}{1-\alpha}L_0 e^{nt}$$

Let us abuse our terminology a little bit by calling $l(t) = \frac{L(t)}{K(t)}$ the "labor-to-capital ratio". Dividing both sides of (20)' by *K(t)* thus yields:

(20)''''
$$\frac{dK(t)/dt}{K(t)} = \gamma\frac{\alpha}{1-\alpha}l(t) - d$$

Intuitively, this means that if the labor-to-capital ratio is "too high", each unit of capital will have a more than "usual" amount of labor to work for it. Thus capital will accumulate faster than "usual", which finally drags down the labor-to-capital ratio until the accumulation of capital returns to some "normal" level. Similarly, if the labor-to-capital ratio is "too low", capital will grow faster than "usual" until the labor-to-capital ratio returns to the "normal" level. This suggests that there is an equilibrium level of labor-to-capital ratio that the economy will converge to in the long-run regardless of what the initial conditions are, which also corresponds to a constant growth rate of capital.

For simplicity, let us use the accent "-" to denote the value of a variable at this equilibrium level of labor-to-capital ratio (but $\bar{h}$ is an exception, which is a constant defined above). To find this equilibrium, we divide (20)''' by *L(t)* and let $t \rightarrow \infty$ to obtain

(36)
$$l(t) \equiv \bar{l} = \frac{1-\alpha}{\alpha}\frac{\mathrm{d+n}}{\gamma}$$

At this equilibrium, the growth rate of capital should be equal to the growth rate of population, *n*.

In fact, if the initial conditions of our model satisfy (36) exactly, the economy will stay precisely at the equilibrium forever in the absence of any exogenous shock. In the following, we will only conduct our analysis at this equilibrium when it is too complicated do it generally.

2.4 Consumptions

To determine which goods are consumed and how much they are consumed, we follow the same reasoning in Young's paper. Suppose a good *s* with price *p(s,t)* is consumed, then given the symmetric pattern of preference, all goods with price less than or equal to *p(s,t)* will also be consumed. Thus, given the continuity of *p(s,t)* and its symmetric and convex pattern around *T(t)* as can be seen from (17)', there will be two goods, *M* and *N*, with $M \leq N$ and symmetric around *T(t)* such that $p(M,t) = p(N,t) = Sup_s\{p(s,t) | s \in S_1(t)\}$ and C(M,t) = C(N,t) = 0. We call *M* the "lower limit good", *N* the "upper limit good", and both of them "limit goods".

At equilibrium, the marginal rate of substitution between any two goods is equal to their price ratio. Hence from (32) we have

(37) $$C(s,t) = \frac{h(M,t) - h(s,t)}{h(s,t)} \qquad \forall s \in S_1(t)$$

Thus, if we know what the limit goods are, we can then calculate the amount of consumption of each good, whereas *M* and *N* can be determined using the condition that the consumers exhaust their budget at equilibrium. Moreover, from (37) we see that consumptions are also symmetric around *T(t)*.

2.5 Technical progress

Recall our conclusions that the capital-to-labor ratio is the same across all active industries and that *h(s,t)*, the pure learning progress, is symmetric around *T(t)*. From (26) we see that

(26)’ $$MPK(s,t) = \alpha \frac{X(s,t)}{K(s,t)} = \frac{\alpha}{h(s,t)} (\frac{L(t)}{K(t)})^{1-\alpha}$$

is also symmetric around *T(t)*. Hence, for $\forall s_1$ & $s_2 \in S_1(t)$ and symmetric around *T(t)*, we must have $X(s_1,t) = \xi X(s_2,t)$ and $K(s_1,t) = \xi K(s_2,t)$, for some $\xi > 0$. Thus (26) holds for both s_1 and s_2. Given the symmetry of *C(s,t)* and *p(s,t)*, from the partial equilibrium condition (25) it can be easily verified that the only value ξ can take is 1. Hence, both *X(s,t)* and *K(s,t)* are symmetric around *T(t)*. Similarly, it can be shown that *MPL(s,t)* and *L(s,t)* are also symmetric around *T(t)*.

Using these facts, we can then derive the explicit form of *T(t)* from (20)’’’, (29) and (33):

(29)’ $$\frac{dT(t)}{dt} = \frac{1}{2}[K(t) + L(t)]$$

$$= \frac{1}{2}\left[\left(K_0 - \frac{\gamma}{d+n}\frac{\alpha}{1-\alpha}L_0\right)e^{-d\cdot t} + \left(1 + \frac{\gamma}{d+n}\frac{\alpha}{1-\alpha}\right)L_0 e^{nt}\right]$$

It can also be easily seen that at the equilibrium level of labor-to-capital ratio

(29)’’ $$\frac{d\bar{T}(t)}{dt} = \frac{1}{2}\left(1 + \frac{1}{\bar{l}}\right)L(t)$$

The above equation implies that both a higher saving rate and a higher population growth rate speed up the technical progress, which is very intuitive since both capital and labor are essential resources for learning-by-doing.

2.6 Intertemporal Welfare

In this subsection, we focus on two measures of consumers’ welfare in autarky, namely the variety and the total quantity of goods being consumed.

To simplify the calculations, we only consider the case at the equilibrium level of labor-to-capital ratio, i.e. only consider welfare in the long-run, whose analysis in the following is essentially the same as that in Young's paper. Note that at this equilibrium, from (18)' the interest rate is simply

(18)'' $$r(t) = \frac{\alpha}{1-\alpha}\bar{l}$$

Using the fact that (23) is binding at the optimum, we combine (18)'' and (23) to get

(23)' $$\int_B^\infty p(s,t)C(s,t) = \frac{1-\alpha\gamma}{1-\alpha}L(t)$$

We then combine (23)' with (17)', (30) and (37) to get

(38) $$\frac{1-\alpha\gamma}{1-\alpha}L(t) = \frac{\bar{l}^\alpha}{1-\alpha}\int_{\bar{M}(t)}^{\bar{N}(t)} h(\bar{M},t) - h(s,t)ds$$

$$= \frac{2\bar{h}\bar{l}^\alpha}{1-\alpha}\left[(\bar{\tau}(t)-1)e^{-\bar{M}(t)} + e^{-\bar{T}(t)}\right]$$

, where $\bar{\tau}(t) = \bar{T}(t) - \bar{M}(t) = \bar{N} - \bar{T}(t)$ represents the variety of goods being consumed. To eliminate $\bar{M}$, let us multiply both sides of the above equation by $e^{\bar{T}(t)}$ to obtain

(38)' $$\frac{1-\alpha\gamma}{2\bar{h}\bar{l}^\alpha}L(t)e^{\bar{T}(t)} = (\bar{\tau}(t)-1)e^{\bar{\tau}(t)} + 1$$

Differentiating both sides by t thus yields

(39) $$\frac{d\bar{\tau}(t)}{dt} = \frac{1-\alpha\gamma}{2\bar{h}\bar{l}^\alpha}L_0\frac{e^{nt}}{\bar{\tau}(t)}e^{\bar{T}(t)-\bar{\tau}(t)}\left(n + \frac{d\bar{T}(t)}{dt}\right) > 0$$

, which implies that the variety of goods being consumed increases and moves to more technically sophisticated goods as time being.

Moreover, from (37) we have

(40) $$C(T+\Delta) = e^{\tau-\Delta} - 1$$

, which implies that the consumption of any good $T+\Delta$, where $-\tau<\Delta<\tau$, increases as time being. Hence, as pointed out in Young's paper, "although as T increases some goods are no longer consumed, the consumption of symmetrical substitutes rises".

However, although the total quantity of consumption increases, the quantity of per capita consumption is still ambiguous since population also grows, which we leave for a detailed discussion in the next section.

2.7 Growth Rate

From (35)', we directly see that the growth rate of nominal GDP per capita defined conventionally is

(35)'' $$g_N(t)=\frac{dY(t)/dt}{Y(t)}-\frac{dL(t)/dt}{L(t)}=\frac{\alpha}{1-\alpha}n$$

, which is a constant and is proportional to the population growth rate. As for the growth rate of real GDP per capita, we adopt the definition in Young's paper, i.e.

(41) $$g_Y(t)=\frac{\int_B^\infty p(s,t)\partial X(s,t)/\partial t ds}{\int_B^\infty p(s,t)X(s,t)}-\frac{dL(t)/dt}{L(t)}=-(1-\alpha)\frac{\int_B^\infty \partial p(s,t)/\partial t X(s,t)ds}{L(t)}$$

$$=-\frac{\int_B^\infty \partial a(s,t)/\partial t X(s,t)ds}{L(t)}$$

, where the second equality is obtained from (35)' and Leibnitz Rule, and the third equality is obtained from (17)'. To solve (41), we expand it using (26) and (27), which yields

(41)' $$g_Y(t)=-\frac{1}{L(t)}[\int_B^\infty \frac{\partial h(s,t)/\partial t}{h(s,t)}L(s,t)ds+\alpha\int_B^\infty \frac{\partial L(s,t)}{\partial t}ds$$

$$-\alpha\int_B^\infty \frac{\partial K(s,t)}{\partial t}\frac{L(s,t)}{K(s,t)}ds]$$

Note that the first two terms can be easily solved using (28) and (29)’. As for the third term, we use again the fact that the labor-to-capital ratio is the same across all active industries and Leibnitz rule, which yields

(41)’’
$$g_Y(t) = \frac{2}{L(t)}\frac{dT(t)}{dt}\int_{T(t)}^{\infty} L(s,t)ds - \alpha n + \alpha\frac{1}{K(t)}\frac{dK(t)}{dt}$$
$$= \frac{1}{2}\big(K(t) + L(t)\big) - \alpha n + \alpha\frac{dK(t)/dt}{K(t)}$$

We can then directly plug in the corresponding formula for each term and get the explicit form of $g_Y(t)$. However, the resulting expression will be very complicated. For our specific purpose in section four, we instead only calculate its value at the equilibrium level of labor-to-capital ratio, which is

(41)’’’
$$\bar{g}_Y(t) = \frac{1+\bar{l}}{2\bar{l}}L(t)$$

Thus, the growth rate of real GDP per capita is positively correlated with the level of total population, which is in line with Young’s conclusion.

2.8 Distributions of Capital and Labor

The only thing left unsolved in this model is perhaps how capital and labor are distributed across all active industries. To solve for these, we first notice that from (25) and (26) we have

(42)
$$\frac{1}{h(s,t)}\left(\frac{K(s,t)}{L(s,t)}\right)^{\alpha} L(s,t) = C(s,t) + \frac{\gamma r(t)}{p(s,t)}K(s,t)$$

Using the fact that the capital-to-labor ratio at any given time is the same across all industries, and the equations (17)’ and (18)’, we simplify it as

(42)’
$$\frac{1}{h(s,t)}\left(\frac{K(t)}{L(t)}\right)^{\alpha} L(s,t) = C(s,t) + \gamma\alpha\frac{K(s,t)}{h(s,t)}\left(\frac{L(t)}{K(t)}\right)^{1-\alpha}$$

In the above equation, *h(s,t)* can be determined by (30) ,*C(s,t)* can be determined by (40), and *M(t)* can be determined by solving the aggregate consumer's utility maximization problem. Moreover, the explicit form of capital-to-labor ratio is given by (20)''' and (33). Hence, from (42)' we can derive a relationship between *K(s,t)* and *L(s,t)*, i.e.

(42)'' $$L(s,t) = \left(h(M,t) - h(s,t)\right)\left(\frac{L(t)}{K(t)}\right)^{\alpha} + \gamma\alpha\frac{L(t)}{K(t)}K(s,t)$$

Combining this relationship with the fact that the capital-to-labor ratios are the same across all active industries of the economy, we can thus find the distributions of *K(s,t)* and *L(s,t)* on s. However, it can be shown that the solution of *M(t)*, although exists, cannot be written in analytic form. Thus, we leave the general solutions unsolved.

III. A REVISIT OF ECONOMIC GROWTH THEORY WITH HUMAN CAPITAL

In the previous section, we have already solved all the endogenous macro-level variables, such as total capital stock and growth rate of per capita output at equilibrium, and conducted some welfare analysis. Thus, the specific functional form we have explored so far already constitutes a model of economic growth with human capital triggered by learning-by-doing. Why are we still intended to have this "redundant" revisit?

One reason is that, the total value of output defined in the last section is not particularly suitable to evaluate the general welfare of the society in our specific model. Given that all goods are perfect substitutes of each other, as defined by (21), consumers only care about the total quantity of goods they consume, not exactly which goods they consume. Moreover, as we learned from basic economics, given a Cobb-Douglas production function and the assumption of perfect competition, labor's share of output is always $1 - \alpha$, which is a constant. Thus, given a constant labor's share of output and a constant saving rate, the consumers' total income in real term is a constant share of total output, $(1 - \alpha) + \alpha(1 - \gamma) = 1 - \alpha\gamma$, which is also how much they consume. Since the consumers only care about their total consumption that is in turn a fixed share of products they produce, the total quantity of output

(43) $$X(t) = \int_B^\infty X(s,t)ds$$

is a more precise measure of social welfare. In fact, it can be easily proved that the instantaneous utility *V(t)* increases if and only if *X(t)* increases. In contrast, *Y(t)*, the total value of output, is biased toward luxury goods.

The second reason is that, it will be interesting if we transform our model in a format similar to the Solow-Romer framework, which is the conventional way to model additional features in growth theory, and reexamine the evolution of human capital triggered by learning-by-doing, which may offer us some additional insight and guidance for future research in this area.

To begin with, we expand (43) using (26) to get

(43)' $$X(t) = \int_B^\infty X(s,t)ds = 2\left(\frac{K(t)}{L(t)}\right)^\alpha \int_{T(t)}^{N(t)} L(s,t)H(s,t)ds$$

Let us define

(44) $$H(t) = 2\int_{T(t)}^{N(t)} \frac{L(s,t)}{L(t)} H(s,t)ds = \int_B^\infty \frac{L(s,t)}{L(t)} H(s,t)ds$$

as the human capital of the economy. Note that this is simply an average of the pure learning progress in each industry $s \in S_1(t) \cap S_2(t)$ weighted by the percentage of total labor force devoted to it, which is a very intuitive and natural way to measure human capital triggered by learning-by-doing. (43)' can then be rewritten as

(43)'' $$X(t) = H(t)K^{\alpha}(t)L^{1-\alpha}(t)$$

, which is in the form of Cobb-Douglas. We also define the general price level of the economy as

(45) $$P(t) = \frac{Y(t)}{X(t)} = \frac{1}{1-\alpha}\frac{1}{H(t)}\left(\frac{L(t)}{K(t)}\right)^{\alpha}$$

That is, the general price level is negative correlated with human capital. Since we take wage rate as the numéraire, lower price means consumers are able to consume more goods, which suggests that the development of human capital triggered by learning-by-doing is the source of welfare improvement. We will see this point more clearly in the following.

To find out the dynamics of human capital, we differentiate (44) and get

(46) $$\frac{dH(t)}{dt} = \int_{M(t)}^{N(t)} \frac{d}{dt}\left(\frac{L(s,t)}{L(t)}\right)\frac{1}{h(s,t)} ds + (1 + \frac{K(t)}{L(t)}) \int_{T(t)}^{N(t)} L(s,t)\frac{1}{h(s,t)} ds$$

Intuitively, the first term measures the impact of reallocation of labor weighted by the pure learning progress, i.e. subtracting (adding) a fixed percent of labor from (to) an industry the economy is more skilled to produce has a larger negative (positive) impact on the development of human capital, whereas the second term is a measure of the "total learning progress" weighted by the amount of labor devoted to each industry s ∈ $S_1(t) \cap S_2(t)$.

However, integrating the first term by part yields

(47) $$\int_{M(t)}^{N(t)} \frac{d}{dt}\left(\frac{L(s,t)}{L(t)}\right)\frac{1}{h(s,t)} ds = \frac{d}{dt}\left(\frac{\int L(s,t)ds}{L(t)}\right)\frac{1}{h(s,t)}\bigg|_{M(t)}^{N(t)}$$

$$- \int_{M(t)}^{N(t)} \frac{d}{dt}\left(\frac{\int L(s,t)ds}{L(t)}\right)\frac{d}{ds}\left(\frac{1}{h(s,t)}\right) ds = 0$$

, where the fact that *h(N,t) = h(M,t)* is used. Thus, (46) is actually

(46)’ $$\frac{dH(t)}{dt} = (1 + \frac{K(t)}{L(t)}) \int_{T(t)}^{N(t)} L(s,t) \frac{1}{h(s,t)} ds$$

I.e. the total learning progress in all active industries in which learning-by-doing has not been exhausted is the only source that contributes to the development of human capital, and particularly, the reallocation of labor does not contributes to it.

We can then calculate the growth rate of human capital at the equilibrium level of labor-to-capital ratio. From (44) and (46)’ we have

(48) $$\frac{dH(t)/dt}{H(t)} = \frac{1+\bar{l}}{\bar{l}} L(t) \frac{\int_{T(t)}^{\infty} L(s,t)/h(s,t) ds}{\int_{B}^{\infty} L(s,t)/h(s,t) ds}$$

To bring more intuition to this equation, we notice that at equilibrium, (42)’ can be rewritten as

(42)’’ $$\frac{L(s,t)}{h(s,t)} = \frac{\bar{l}^{\alpha}}{1-\alpha\gamma} C(s,t)$$

Plugging it into (48) thus yields

(48)’ $$\frac{dH(t)/dt}{H(t)} = \frac{1+\bar{l}}{\bar{l}} L(t) \frac{\int_{T(t)}^{\infty} C(s,t) ds}{\int_{B}^{\infty} C(s,t) ds} = \frac{1+\bar{l}}{\bar{l}} L(t) \frac{\int_{T(t)}^{\infty} X(s,t) ds}{\int_{B}^{\infty} X(s,t) ds}$$

, where the second equality is derived from the fact that in any active industry, consumption is a fixed share of output, which is proved at the beginning of this section. The interpretation of the above equation is that, the growth rate of human capital is positively correlated with the proportion of goods being produced in industries in which there is still ongoing learning progress. This conclusion is very intuitive, since the more technologically sophisticated goods the economy produces, the faster it will learn from experience and thus increase its productivity.

Knowing the growth rate of human capital, we can thus calculate the growth rate of per capita total output at the equilibrium level of labor-to-capital ratio:

(49) $$g_X = \frac{dX(t)/dt}{X(t)} - \frac{dL(t)/dt}{L(t)} = \frac{1+\bar{l}}{\bar{l}} L(t) \frac{\int_{T(t)}^{\infty} X(s,t)ds}{\int_{B}^{\infty} X(s,t)ds}$$

I.e. learning-by-doing is the engine of welfare improvement in the long-run, without which the economy cannot attain a sustainable growth path. This remarkable result is in line with the classical conclusion derived from the Romer model, which claims that the ultimate driving force of economic growth is the invention of new technology, although the Romer model uses a completely different approach to tackle the mechanism of sustainable growth.

It can also be seen that both the growth rate of per capita output and that of human capital are positively correlated with both the population growth rate and the saving rate. This agrees with the similar conclusion we made earlier on the evolution of the learning-by-doing equation in autarky.

Finally, given the fact that both *L(s,t)* and *h(s,t)* are symmetric around T(t) in autarky, we have $\frac{\int_{T(t)}^{\infty} X(s,t)ds}{\int_{B}^{\infty} X(s,t)ds} = \frac{1}{2}$. Thus the precise values of (48) and (49) are

(48)'' $$\frac{dH(t)/dt}{H(t)} = \frac{1+\bar{l}}{2\bar{l}} L(t)$$

(49)' $$g_X(t) = \frac{1+\bar{l}}{2\bar{l}} L(t)$$

To close this model, we rephrase the dynamics of capital and labor:

(20)' $$\frac{dK(t)}{dt} = \gamma \frac{\alpha}{1-\alpha} L(t) - d \cdot K(t)$$

(33) $$\frac{dL(t)/dt}{L(t)} = n$$

It should be noticed that the way in which saving rate is defined here is different from that in the classical Solow model.

IV. THE DYNAMIC EFFECTS OF INTERNATIONAL TRADE

In this section, we will study the interactions between two countries, one of which is more technically advanced than the other, under free trade. Particular attention will be paid to the effects of free trade on technical progress and growth rate of Real GDP per capita. Moreover, we are intended to study under which conditions each of them gains from trade as well as the implications of population growth rates and saving rates of the two countries.

4.1 Basic Settings

We denote the less developed country as LDC and the developed country as DC. All variables of DC are denoted by the superscript "*". By our assumption, we have *T(0) < T*(0)*. For simplicity, we assume that both countries have the same capital share of output, α, and the same depreciation rate, *d*. But their population growth rates and saving rates need not to be the same.

Since the wage rates of LDC and DC may not be equal, we take the wage rate of LDC as the numéraire, and use *w(t)* to denote the relative wage of DC. Then for DC, (17)' and (18)' become

(17)'' $$p^*(s,t) = \frac{1}{1-\alpha}\left(\frac{L^*(t)}{K^*(t)}\right)^{\alpha} h^*(s,t)w(t)$$

(18)'' $$r^*(t) = \frac{\alpha}{1-\alpha}\frac{L^*(t)}{K^*(t)}w(t) = \frac{\alpha}{1-\alpha}\frac{L^*(t)}{K_r^*(t)}$$

, where we denote $K_r^*(t) = \frac{K^*(t)}{w(t)}$. By definition, $K_r^*(t)$ has exactly the same dynamics as (20)' and thus its solution is the same as (20)'''. Accordingly, the equilibrium level of labor-to-capital ratio of DC is

(36)' $\bar{l}^* = \frac{1-\alpha}{\alpha}\frac{n^*+d}{\gamma^*}w(t) = \bar{l}_r^* w(t)$

, where we denote $\bar{l}_r^* = \frac{1-\alpha}{\alpha}\frac{n^*+d}{\gamma^*}$, and the corresponding growth rate of capital of DC in the long-run is *n*+w(t)*. For simplicity, in the following we will only consider the case in which both countries are at their equilibrium levels of labor-to-capital ratio, and will suppress the "-" notation used earlier.

We assume that there is no capital or labor mobility, and both countries are in full employment. Under free trade, the price of each good s is the same in both countries, and each good is produced in the country that is able to offer the lowest price. Let us denote the common price of good *s* in both countries as *p(s,t)*, then from (23), the consumers' budget balancedness condition in LDC is

(50) $$\int_{S_1(t)} p(s,t)C(s,t)\,ds + \int_{S_1^*(t)} p(s,t)C(s,t)ds = L(t) + (1-\gamma)r(t)K(t)$$

, and that in DC is

(51) $$\int_{S_1(t)} p(s,t)C^*(s,t)ds + \int_{S_1^*(t)} p(s,t)C^*(s,t)ds = w(t)L^*(t) + (1-\gamma^*)r^*(t)K^*(t)$$

, from which we can determine the limit goods in both economies, *M(t)* and *M*(t)*, if w(t) is known. *w(t)* can be determined by the following trade balance condition:

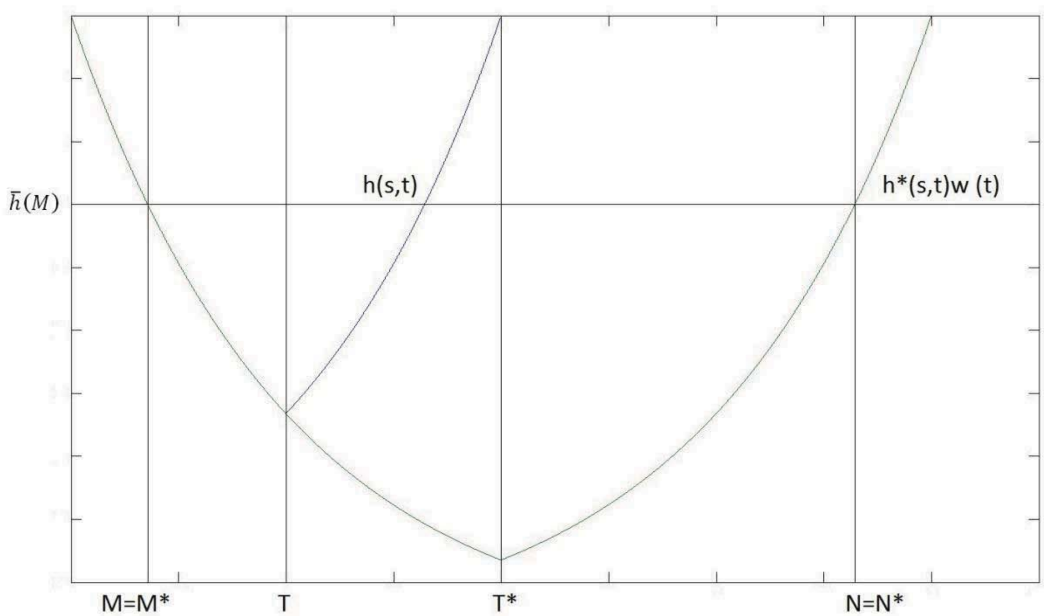
(52) $$\int_{S_1(t)} p(s,t)C^*(s,t)\,ds = \int_{S_1^*(t)} p(s,t)C(s,t)\,ds$$

4.2 Trade Patterns

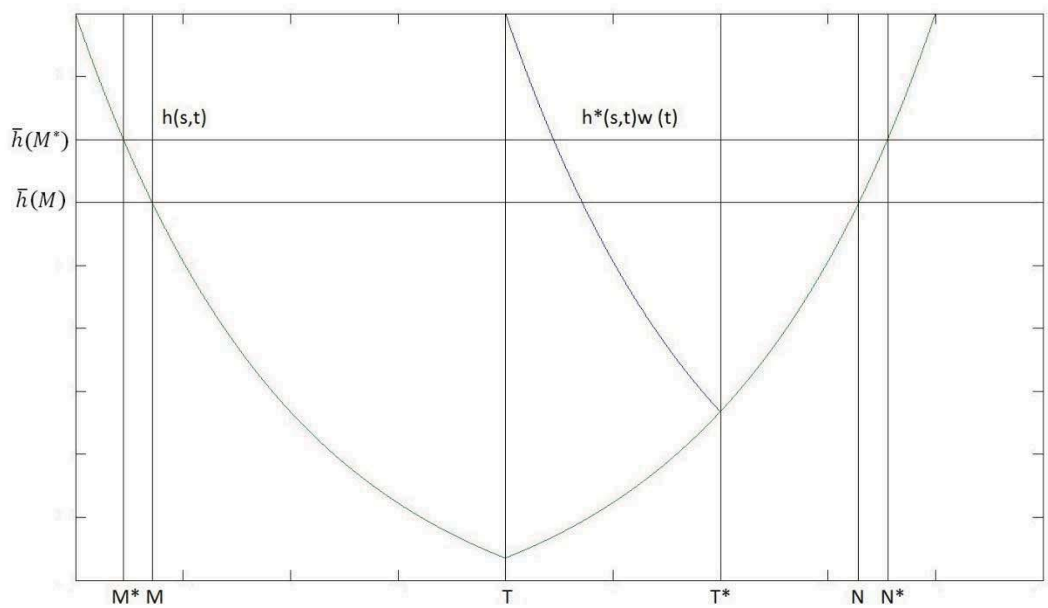
From (17)' and (17)'', it can be easily seen that for any good *s*, the best price provided by DC is not greater than that provided by LDC if and only if

(53) $$w(t)h^*(s,t) \leq \left(\frac{\bar{l}}{\bar{l}^*}\right)^{\alpha} h(s,t)$$

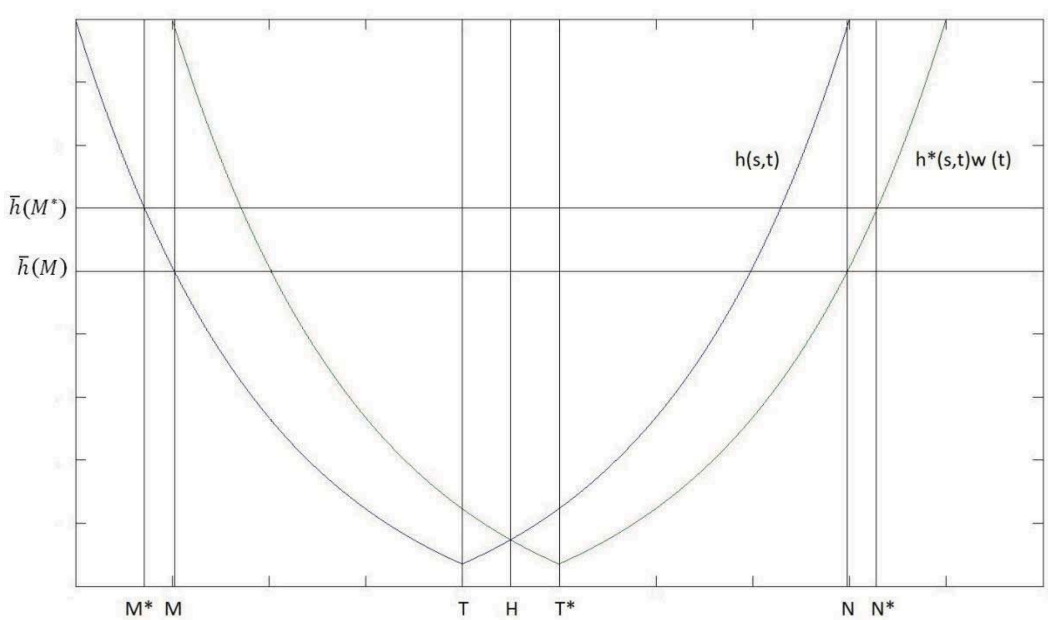
Graphically, if *T(t) < T*(t)*, then the curve of *h(s,t)* is enclosed by that of *h*(s,t)*, which corresponds to the case when $w(t) = \left(\frac{\bar{l}}{\bar{l}^*}\right)^{\alpha}$. As *w(t)* increases, the curve of *h*(s,t)w(t)* becomes thinner and moves upward. Intuitively, *w(t)* also has a maximum, above which DC will be producing nothing. When *w(t)* reaches this maximum, the curve of *h*(s,t)w(t)* is enclosed by that of *h(s,t)*. Keeping this graphical analysis in mind, we see that as long as *T(t) < T*(t)*, the static trade pattern can only be in one of five cases, which are summarized in figure 1 and the following paragraph adapted from Young's paper.



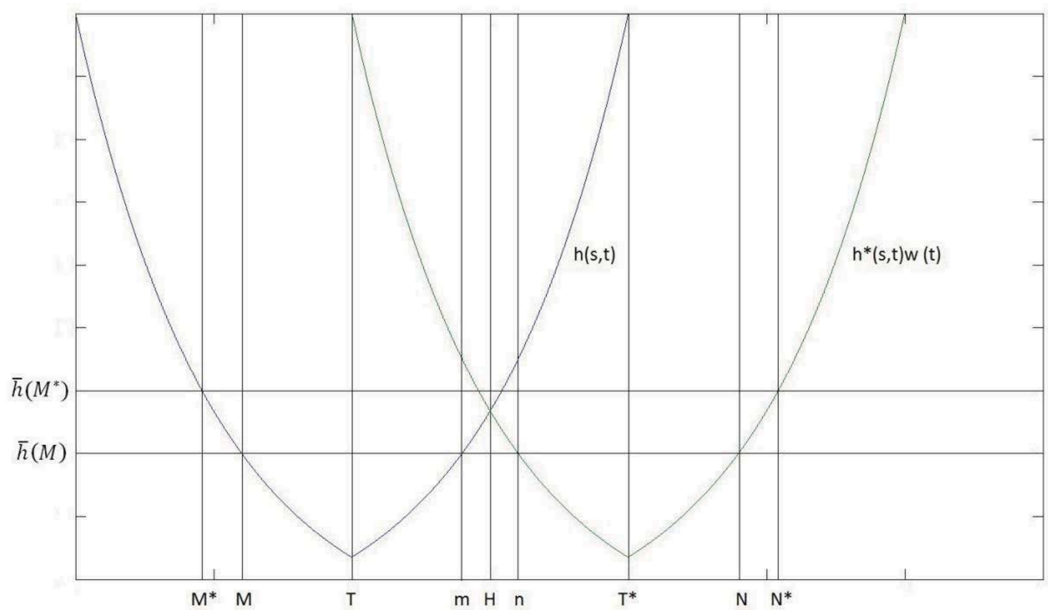
(Figure 1 – Case A)



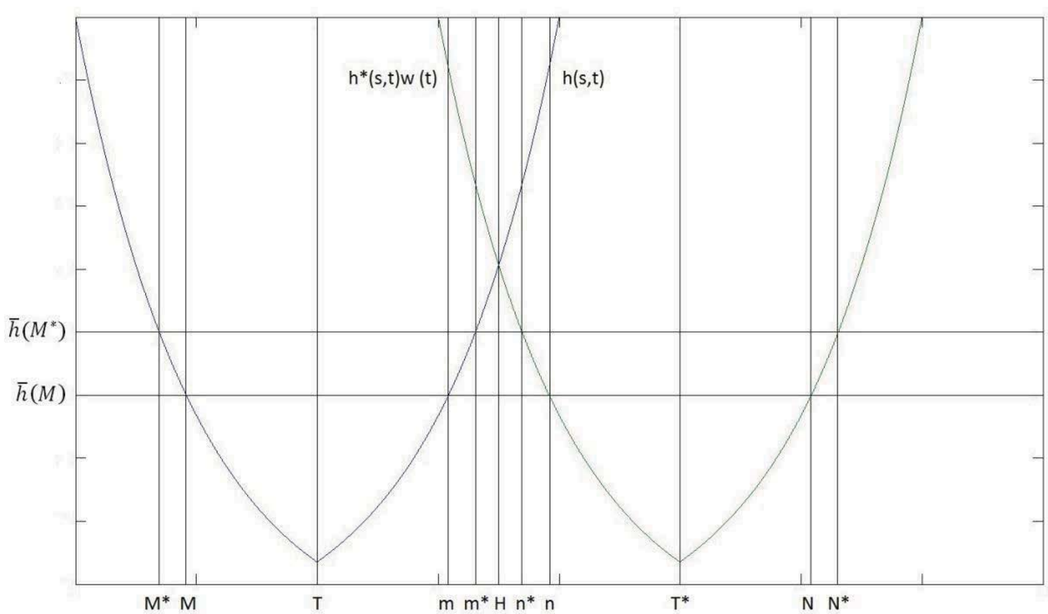
(Figure 1 – Case B)



(Figure 1 – Case C)



(Figure 1 – Case D)



(Figure 1 – Case E)

In case A, *w(t)* attains its lowest possible value, $\left(\frac{\bar{l}}{\bar{l}^*}\right)^{\alpha}$. In this situation, both countries consume the same bundle and same amount of goods. Moreover, as illustrated in the figure, DC can produce all goods being consumed in both countries, while LDC can only produce goods *s ≤ T(t)*. In case B, *w(t)* attains its highest possible value, $\left(\frac{\bar{l}}{\bar{l}^*}\right)^{\alpha} e^{2[T^*(t)-T(t)]}$. In this situation, compared with LDC, DC consumes more goods with higher variety. However, unlike case A, in this situation DC only produces goods *s ≥ T*(t)*, while LDC produces all goods being consumed. Case C, D and E are similar in the sense that $\left(\frac{\bar{l}}{\bar{l}^*}\right)^{\alpha} < w(t) < \left(\frac{\bar{l}}{\bar{l}^*}\right)^{\alpha} e^{2[T^*(t)-T(t)]}$ and that DC consumes both more and higher variety of goods than LDC. However, there are still two differences. In case C, the ranges of goods LDC and DC consume are connected invervals, namely *(M,N)* and *(M*,N*)*; in case D, LDC only consumes *(M,m)* ∩ *(n,N)*, which is unconnected; while in case E, both LDC and DC only consume unconnected ranges of goods, namely *(M,m)* ∩ *(n,N)* and *(M*,m*)* ∩ *(n*,N*)*. As for production, in both case C and D, LDC produces *(M*,H)* and DC produces *(H,N*)*; while in case E, LDC only produces *(M*,m*)* and DC only produces *(n*,N*)*.

Given different initial conditions in our dynamic settings, the actual trade pattern evolves in these five static cases. Let us define the technological gap between LDC and DC as

(54) $$G(t) = T^*(t) - T(t)$$

As discussed in Young's paper, given different population sizes of LDC and DC that are assumed to be fixed over time, there are three possible evolutionary paths of the trade pattern, and in each path, its initial case is determined by the initial values *T(0)* and *T*(0)*. The main conclusions are summarized as follows. First, if *L(0) < L*(0)*, then the only possible static trade patterns, ranked in ascending order of *G(t)* for any fixed *T(t)* and any *t ≥ 0*, are A, C, D and E, and the evolutionary path is $A \rightarrow C \rightarrow D \rightarrow E$. I.e. if the initial trade pattern is, say, case C, then it will gradually change to D, and finally change to and stay at E. Second, if $L(0) = L^*(0)$, then the only possible trade patterns are C, D and E, ranked in ascending order of $G(t)$, and the evolutionary path is $C \rightarrow D \rightarrow E$. Third, if $L(0) > L^*(0)$, the only possible trade patterns are B, C, D and E, which are again ranked in ascending order of $G(t)$, and the evolution path depends on the initial value $G(0)$. Let us denote the value of $G(0)$ that satisfies $L(0) = L^*(0)(2 + e^{2G(0)})$ as $G^*(0)$, then if $0 < G(0) < G^*(0)$, the trade pattern will first be in case B, and $G(t)$ will decrease until LDC takes over DC, otherwise the evolutionary path is $B \rightarrow C \leftarrow D \leftarrow E$.

It is evident that all these conclusions cited above from Young's paper remain valid in the extended model we have developed so far as long as we confine our study at the equilibrium levels of labor-to-capital ratio for both countries, except for only one difference. In our model, given the possibility of population growth in both countries, the relative magnitude between $L(t)$ and $L^*(t)$ may not be fixed over time, which opens the possibility that the trade pattern switches between different evolutionary paths.

4.3 Technical Progress

As demonstrated above, under free trade each country need not to produce the whole set of goods being consumed either domestically or internationally. As a result, for each country the distributions of capital and labor are no longer symmetric around the learning-by-doing equation, $T(t)$ or $T^*(t)$, and thus (29)' is no longer valid. To find out the technical progress of each country under free trade, we refer back to (29), which implies that for each country at the equilibrium level of labor-to-capital ratio, the rate of change of the learning-by-doing equation is solely proportional to the amount of labor force devoted to industries in which learning-by-doing has not been exhausted. Also recall from section three that for each good being produced, there is always a fixed proportion of the amount of production being consumed either domestically or internationall, which is $1-\alpha\gamma$ if it is produced in LDC, and is $1-\alpha\gamma^*$ if it is produced in DC. We can combine this fact with the budget balancedness conditions (50) and (51) to calculate the amount of labor devoted to industries with on going learning-by-doing progress, and then find out the evolution of learning-by-doing equation for each country, as discussed above, as well as the technical progress in each type of static trade pattern.

The results are summarized as follows. Since the tricks used to simplify the calculations are similar to those implicitely used in Young's paper, we ignore the details of the derivations. Having said that, it should be noticed that the solutions listed below are different from the corresponding ones in Young's paper. In case A,

(55)
$$\begin{cases} \frac{dT(t)}{dt} = 0 \\ \frac{dT^*(t)}{dt} = \frac{1}{2}\left(1+\frac{1}{\bar{l}}\right)\left(\frac{1-\alpha\gamma}{1-\alpha\gamma^*}L(t)+L^*(t)\right) \end{cases}$$

In case B,

(56)
$$\begin{cases} \frac{dT(t)}{dt} = \frac{1}{2}\left(1+\frac{1}{\bar{l}}\right)L(t) - \frac{1}{2}\left(1+\frac{1}{\bar{l}^*}\right)\frac{1-\alpha\gamma^*}{1-\alpha\gamma}L^*(t)w(t) \\ \frac{dT^*(t)}{dt} = \left(1+\frac{1}{\bar{l}^*}\right)L^*(t) \end{cases}$$

In case C and D,

(57)
$$\begin{cases} \frac{1}{2}\left(1+\frac{1}{\bar{l}}\right)L(t) > \frac{dT(t)}{dt} > 0 \\ \frac{dT^*(t)}{dt} > \frac{1}{2}\left(1+\frac{1}{\bar{l}^*}\right)L^*(t) \end{cases}$$

In case E,

(58) $$\begin{cases} \frac{dT(t)}{dt} = \frac{1}{2}\left(1 + \frac{1}{\bar{l}}\right) L(t) \\ \frac{dT^*(t)}{dt} = \frac{1}{2}\left(1 + \frac{1}{\bar{l}^*}\right) L^*(t) \end{cases}$$

Comparing with the autarky situation as illustrated by (29)'', we see that under free trade the technical progress of LDC evolves at most as fast as the speed in autarky, and that of DC evolves at least as fast as the speed in autarky. This corresponds to the conclusion in Young's paper that "the DC experiences dynamic gains from trade, while the LDC experiences dynamic losses". The intuition is straightforward. Under free trade, since LDC has comparative advantage in producing simple goods, it produces more simple goods in exchange for advanced goods. This leaves LDC less opportunities to learn to produce new things itself, which results less dynamic gains of learning-by-doing compared with that in autarky. The situation in DC under free trade is just the opposite.

The dynamics of the technological gap, *G(t)*, is more complicated than the discussions in Young's paper, as now the rate of change of *G(t)* depends not only on *L(t)* and *L*(t)*, but also on the saving rates and population growth rates of the two countries. Nevertheless, the ideas to find it remain the same, and the results have already been briefly summarized in the previous subsection on trade patterns.

From (36), (36)' and (55) – (58), it is apparent that for any country in any static trade pattern, increasing either its own population growth rate or its own saving rate speeds up, or at least does not slow down its technical progress. Recall that we have made a similar conclusion in the autarky settings. We now see that it also applies to the free trade settings.

4.4 Growth Rate

To calculate the growth rates defined in the sense of (41), we first notice that from (17)', (26) and (29)

(29)'''
$$\begin{cases} \frac{dT(t)}{dt} = (1-\alpha)\left(1+\frac{1}{\bar{l}}\right)\int_{T(t)}^{\infty} p(s,t)X(s,t)ds \\ \frac{dT^*(t)}{dt} = (1-\alpha)\left(1+\frac{1}{\bar{l}^*}\right)w(t)\int_{T^*(t)}^{\infty} p(s,t)X^*(s,t)ds \end{cases}$$

From this formula, it can be seen that

(41)''''
$$\begin{cases} g_Y(t) = 2\frac{\bar{l}}{1+\bar{l}}\frac{(dT(t)/dt)^2}{L(t)} \\ g_Y^*(t) = 2\frac{\bar{l}^*}{1+\bar{l}^*}\frac{(dT^*(t)/dt)^2}{L^*(t)} \end{cases}$$

, which holds in both autarky and free trade situations. Recall from the last subsection that under free trade, $\frac{dT(t)}{dt}$ becomes smaller and $\frac{dT^*(t)}{dt}$ becomes larger than their values in autarky as long as DC is not overtaken by LDC. Thus, in addition to its effect on the technical progress of each country, free trade also slows down the growth rate of LDC and rise up that of DC as long as DC is not overtaken by LDC. Moreover, from (55) – (58) it can be seen that the growth rate of one country is positively related to both its population growth rate its saving rate, which, again, also applies to the autarky settings as we have shown earlier.

4.5 Intertemporal Welfare

Given that the solutions (55) – (58) follow the same patterns as the original Young model even though the parameters are different, it is not hard to verify that the welfare analysis in the free trade settings in Young's paper also applies to our model as long as we ignore the dynamic changes of *L(t)* and *L*(t)*. The conclusions are summarized as follows from Young's paper. Having all the initial conditions the same, compared with aukarky, free trade: (1) improves the intertemporal utility of DC if DC will never be overtaken by LDC; (2) improves the intertemporal untility of DC even if DC will be overtaken by LDC, provided that either ρ, the discount factor, or *G(0)* are sufficiently large; (3) improves the intertemporal utility of LDC, provided that *L(t)* is sufficiently small relative to *L*(t)*; (4) reduces the intertemporal utility of LDC, provided that *L(t)* is sufficiently large relative to *L*(t)* and LDC is unable to overtake DC; (5) reduces the intertemporal utility of LDC when LDC is able to overtake DC, provided that either (a) ρ or *G(0)* is sufficiently large and the initial trade pattern is case B, or (b) ρ is sufficiently small, the initial trade pattern is case C, D, or E, and the time spent in catching up DC is sufficiently large.

Although a general discussion on the effects of population growth rates and saving rates on the intertemporal welfare of each country is impossible, from the above conclusions we can still establish some immediate, easy-to-prove results. First, a sufficiently large decrease in n or increase in n^* such that LDC is not able to overtake DC improves the intertemporal utility of LDC and reduces that of DC, provided that ρ is sufficiently large. Second, an increase in γ^* improves the intertemporal utilities of both LDC and DC. Third, an increase in γ improves the intertemporal utility of DC, and also improves that of LDC if it is not able to overtake DC.

V. SUMMARY AND EXTENSIONS

In this paper, we have successfully generalized the Young model of learning-by-doing from two perspectives. First, in addition to labor, we introduced capital as another factor of production. Second, we enabled capital accumulation and population growth. Applying this model to the studies of economic growth and international trade, we have made the following conclusions. First, learning-by-doing is the source of a strictly positive balanced growth path of an economy in autarky. Second, in both the autarky and free trade situations, an increase in the population growth rate or the saving rate speeds up both the country's growth rate of per capita output and its technical progress in the long-run. Third, under free trade between two countries, the less developed one experiences both slower technical progress and slower growth rate of per capita output, while the situation is just the opposite in the advanced country. Fourth, the dynamic gains from trade as well as the effects of changes of population growth rates and saving rates vary with respect to the initial conditions, which are summarized in detail.

Note that, however, the assumption that all industries have the same capital share of output, although greatly simplifies the calculations, are not very realistic, let alone that modern trade theory is built largely on the differentials of capital intensiveness across different industries, which limits its ability to be integrated into the literature of international trade to form a unified modeling framework. This problem may be addressed, say, by assuming that the capital share of output is an increasing and quasi-concave function of the industry's technical level, which approaches to 1 as the technical level tends to infinity, although a careful justification and construction of the micro-foundation of such an assumption is still needed.

Finally, since some of the main conclusions of this paper suggest that compared with free trade, the less developed country may be better off by staying in aukarky, studying the dynamic effects of common types of trade barriers, such as tariffs and quotas, will be quite interesting.

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