Bordered Heegaard Floer Homology, Satellites, and Decategorification

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ABSTRACT

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We use the methods of bordered Floer homology to provide a formula for both τ and $\widehat{HFK}$ of certain satellite knots. In many cases, this formula determines the 4-ball genus of the satellite knot. In parallel, we explore the structural aspects of the bordered theory, developing the notion of an Euler characteristic for the modules associated to a bordered manifold. The Euler characteristic is an invariant of the underlying space, and shares many properties with the analogous invariants for closed 3-manifolds. We study the TQFT properties of this invariant corresponding to gluing, as well as its connections to sutured Floer homology. As one application, we show that the pairing theorem for bordered Floer homology categorifies the classical Alexander polynomial formula for satellites.

Contents

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To my Grandfather

Chapter 1

Introduction

Heegaard Floer homology is an approach, motivated by gauge theory, to studying knots, links, and 3- and 4-manifolds, developed by Ozsváth and Szabó over the last decade. To a closed 3-manifold one associates a graded chain complex $CF^\bullet$ which comes in different decorations. The chain homotopy type of this complex is a powerful diffeomorphism invariant of the manifold. A knot K in a 3-manifold Y induces a filtration on the complex $\widehat{CF}(Y)$, which in turn leads to knot Floer homology - a bigraded homology theory for knots. Amongst the many valuable properties of knot Floer homology, the simplest version, $\widehat{HFK}(Y,K)$, categorifies the Alexander polynomial, and detects the genus and fiberedness. The filtration on $\widehat{CF}(Y)$ gives rise to a concordance invariant $\tau(K)$, whose absolute value is lower bound for the 4-ball genus of K.

The ideas of Heegaard Floer homology were recently generalized by Lipshitz, Ozsváth and Thurston to 3-manifolds with boundary [LOT]. The new theory, bordered Heegaard Floer homology, provides powerful gluing techniques for computing the original Heegaard Floer invariants of closed manifolds and knots.

In this thesis, we use the methods of bordered Heegaard Floer homology to find answers to questions about knots and 3-manifolds. We concentrate on knot concordance and satellites, providing a formula for both τ and $\widehat{HFK}$ of certain satellite knots. In many cases, this formula determines the 4-ball genus of the satellite knot. In parallel, we explore the structural aspects of the bordered theory, developing the notion of an Euler characteristic for each of the two types of modules associated to a bordered manifold. The Euler characteristic is an invariant of the underlying space, and in its simplest form, it recovers information about the inclusion of the boundary into the 3-manifold. It shares many properties with the analogous invariants for closed 3-manifolds. For example, it recovers the Alexander polynomial of a knot, and its behavior under gluing implies the Alexander polynomial formula for satellites.

1.1 Knot satellites and concordance

The study of knot and link concordance began with the work of Fox and Milnor in the 60's [FM66]. A knot or link in $S^3 = \partial D^4$ is *slice* if it bounds a smoothly embedded disk in D^4. The four-ball genus of a knot is the smallest number such that the knot bounds a surface of that genus in D^4. Two knots are *concordant* if they co-bound a cylinder. Thus, a knot is slice if and only if it is concordant to the unknot. Knots modulo concordance form a group, with connect sum serving as addition. Knot concordance plays an important role in the study of 4-manifolds, both in the topological and smooth categories [FQ90].

There are some classical obstructions to sliceness coming from the knot signature and the Alexander polynomial Δ_K. Conversely, $\Delta_K = 1$ implies that K is *topologically* slice [FQ90]. On the other hand, there exist knots with trivial Alexander polynomial that are not smoothly slice - evidence of the difference between smooth and topological 4-manifold theory.

Over the last decade, Heegaard Floer homology has proven to be a powerful tool for studying knot concordance. Using the knot filtration on $\widehat{CF}(S^3)$, Ozsváth and Szabó define an integer concordance invariant $\tau(K)$, whose absolute value is a lower bound on the genus of smoothly embedded surfaces in the four-ball with boundary K [OS03c].

Motivated by this property of τ, Hedden [Hed08] studied the behavior of this invariant under the cabling operation. Given a knot K, its (p, q)-cable, denoted $K_{p,q}$, is the satellite knot with pattern the torus knot $T_{p,q}$ and companion K. Hedden obtained upper and lower bounds for $\tau(K_{p,pn+1})$ in terms of $\tau(K)$, p, and n. For sufficiently large $|n|$, he also described the knot Floer homology of the cable in the topmost Alexander gradings. Van Cott generalized Hedden's bounds for τ to cables $K_{p,q}$ for any relatively prime p and q [Cot08].

Using bordered Floer homology, we obtain new results on the behavior of knot Floer homology and τ under cabling thin knots. Thin knots are a class of knots whose knot Floer homology is, in a certain sense, as simple as possible. This class contains alternating and quasi-alternating knots as a proper subset.

Theorem 1. *Suppose K is a thin knot. The homology $\widehat{HFK}(K_{p,pn+1})$ can be described completely in terms of $\tau(K)$, $\Delta_K(t)$, p, and n. Further,*

$$\tau(K_{p,pn+1}) = \begin{cases} p\tau(K) + \frac{np(p-1)}{2} & \text{if } \tau(K) = 0 \text{ and } n \geq 0, \text{ or if } \tau(K) > 0 \\ p\tau(K) + \frac{np(p-1)}{2} + p - 1 & \text{otherwise.} \end{cases}$$

While it is not generally true that τ of a cable depends only on τ of the pattern [Hom11], it is true when the pattern, i.e. the knot we apply the cabling to, is a thin knot. The above formula, combined with Van Cott's bounds, yields a formula for $\tau(K_{p,q})$ for any relatively prime p and q, when K is a thin knot.

Corollary 2. *Suppose K is a thin knot, and p and q are relatively prime integers, with $p > 0$.*

$$\tau(K_{p,q}) = \begin{cases} p\tau(K) + \frac{(p-1)(q-1)}{2} & \text{if } \tau(K) = 0 \text{ and } q > 0, \text{ or if } \tau(K) > 0 \\ p\tau(K) + \frac{(p-1)(q+1)}{2} & \text{otherwise.} \end{cases}$$

Note that since $K_{-p,-q} = -K_{p,q}$, where $-K_{p,q}$ is $K_{p,q}$ with reversed orientation, and since τ does not distinguish orientation, the result of Corollary 2 extends to all one-component cables of thin knots.

Theorem 1 and Corollary 2, combined with the inequality $|\tau| \leq g_4$, provide information about the four-ball genus of cables. For example:

Corollary 3. *Suppose K is a thin knot with $g_4(K) = \tau(K)$, $p > 1$ is an integer, and $q > 0$ is an integer relatively prime to p. Then $g_4(K_{p,q}) = \tau(K_{p,q})$.*

We plan to continue our work in this area and study the knot Floer homology of cables of non-thin knots, iterated cables, etc. Hom [Hom11] has used bordered Floer homology to determine τ of cables of arbitrary knots (the result depends on more than simply τ of the pattern). We would like to get additional information, such as the rank of $\widehat{HFK}$ or a full description of CFK^- for some new families of cables.

1.2 The decategorification of bordered Heegaard Floer homology

For closed manifolds, the Euler characteristic of HF^+ is the Turaev torsion, and for knots in S^3, the Euler characteristic of $\widehat{HFK}$ is the Alexander polynomial. For sutured manifolds, Juhász developed a Floer theory called sutured Floer homology [Juh06], whose Euler characteristic has been shown to be a certain Turaev-type torsion function [FJR11]. Motivated by the above, we study the Euler characteristic of bordered Floer homology, its relation to the Euler characteristics of the aforementioned Floer theories, and its behavior under gluing.

Bordered Floer homology is a TQFT-type generalization of $\widehat{HF}$ to manifolds with boundary. To a parametrized surface we associate a differential graded algebra $\mathcal{A}(\mathcal{Z})$, where $\mathcal{Z}$ is a way to represent the surface, and to a manifold with parametrized boundary represented by $\mathcal{Z}$, we associate a left type D structure CFD over $\mathcal{A}(\mathcal{Z})$, or a right $\mathcal{A}_\infty$-module CFA over $\mathcal{A}(\mathcal{Z})$. Both structures are invariants of the manifold up to homotopy equivalence, and their derived tensor product is an invariant of the closed manifold obtained by gluing two bordered manifolds along their boundary, and recovers CF. Another variant of these structures is associated to knots in bordered 3-manifolds, and recovers CFK after gluing.

We study the Grothendieck group of the surface algebra $\mathcal{A}(\mathcal{Z})$, and prove that the image of the above structures in this group is an invariant of the bordered manifold. The difficulty in obtaining an interesting invariant lies in the fact that there is no differential $\mathbb{Z}$-grading on the algebra and modules. Instead, $\mathcal{A}(\mathcal{Z})$ is graded by a non-abelian group G, and the modules are graded by a set with a G-action. An Euler characteristic which carries no grading data loses too much information about the manifold, while one carrying the full data from G is not as easy to interpret and relate to its sisters in the closed and sutured worlds.

To obtain an invariant with integer coefficients, we define a $\mathbb{Z}/2$ differential grading m on the algebra and modules, and show that it agrees with the Maslov grading under gluing.

Theorem 4. *Let $\mathcal{Z}$ be a pointed matched circle with associated surface F of genus k. The Grothendieck group of the category of $\mathbb{Z}/2$-graded $\mathcal{A}(\mathcal{Z})$-modules is given by*

$$K_0(\mathcal{A}(\mathcal{Z})) \cong \Lambda^*(H_1(F;\mathbb{Z})) \cong \mathbb{Z}^{2^{2k}}.$$

Moreover, for an $\mathcal{A}(\mathcal{Z})$-module M, its image in this group is given by

$$[M] = \sum_{x\in\mathfrak{S}(M)} (-1)^{m(x)} h(x),$$

where h is a map from $\mathfrak{S}(M)$ to a basis for $\Lambda^*(H_1(F;\mathbb{Z}))$.

In other words, the Euler characteristic of an $\mathcal{A}(\mathcal{Z})$-module counts generators according to their grading, and the algebra action on them.

The behavior of the Euler characteristic under gluing is as one would hope. First note that the tensor product $M \boxtimes N$ is just a chain complex, and so its Euler characteristic is an integer. Thus, gluing corresponds to multiplying $[M]$ and $[N]$, and interpreting the result as an integer. Specifically, for $a, b \in K_0(\mathcal{A}(\mathcal{Z}))$, we define a product $a \cdot b \in \mathbb{Z}$.

Theorem 5. *Let M be a right $\mathcal{A}_\infty$-module over $\mathcal{A}(\mathcal{Z})$ and N a left type D structure over $\mathcal{A}(\mathcal{Z})$. The Euler characteristic of the chain complex $M \boxtimes N$ is $\chi(M \boxtimes N) = [M]\cdot[N]$. In particular, if $Y = Y_1 \cup_F Y_2$, then up to an overall sign*

$$[CFA(Y_1)]\cdot[CFD(Y_2)] = \chi(CF(Y))$$

Thus, Theorem 5 shows that the behavior of the bordered Euler characteristic is exactly as one would expect. A similar statement can be made when one of the bordered manifolds is endowed with a knot. For that purpose, we define a second grading which behaves much like the Alexander grading for knots in closed manifolds, and study the Euler characteristic in $\Lambda^*(H_1(F;\mathbb{Z})) \otimes \mathbb{Z}[t, t^{-1}]$, where t corresponds to the Alexander grading. One topological significance of this new invariant is that it recovers the Alexander polynomial. Below, a_1 and a_2 are generators of $H_1(T^2;\mathbb{Z})$ chosen based on the parametrization T^2.

If we let $[M, k]$ be Euler characteristic of the module M, where t is replaced by t^k, we have

Theorem 6. *Given oriented knots $K \hookrightarrow S^3$ and C in a 0-framed $S^1 \times D^2$, let $k = \#(C \cap D^2)$, i.e. let k be the homology class of C inside $S^1 \times D^2$. Let $(S^3 \backslash K, 0)$ be the 0 framed complement of K. Then*

$$[\widehat{CFA}(S^1 \times D^2, C)] \cdot [\widehat{CFD}(S^3 \setminus K, 0), k] = \chi(\widehat{CFK}(K_C)).$$

Moreover,

$$[\widehat{CFA}(S^1 \times D^2, C)] = \chi(\widehat{CFK}(C))a_1 = \Delta_C(t)a_1,$$

and

$$[\widehat{CFD}(S^3 \setminus K, 0)] = \chi(\widehat{CFK}(K))a_1 + Q_K(t)a_2 = \Delta_K(t)a_1 + Q_K(t)a_2.$$

In other words, the decategorification of bordered Floer homology in this case is precisely the classical Alexander polynomial formula for satellites

$$\Delta_C(t) \cdot \Delta_K(t^k) = \Delta_{K_C}(t).$$

We would like to study the polynomial $P_C(t)$ further to see what additional information one might gain about the satellite C.

Work of Zarev [Zar09] implies that the Euler characteristic of bordered Floer homology recovers the Euler characteristic of sutured Floer homology for the same manifold with boundary with properly chosen sutures. The Euler characteristic of sutured Floer homology has some interesting topological properties [FJR11]. We hope that since bordered Floer homology also satisfies a nice gluing formula, we can find new topological information in its Euler characteristic.

We plan to explore further the additional structure of the Grothendieck group of a surface algebra, for example, the structure arising from the action of the mapping class group, or from other types of surface cobordisms. Similar to the surgery formulae for Turaev torsion, one should be able to obtain new formulae when the gluing is along any genus surface.

Chapter 2

Background in bordered Floer homology

This Chapter is a brief introduction to bordered Floer homology – the tool for studying the knot satellites in Chapter 3, and the object of study in Chapter 4. We introduce all algebraic structures in the most general settings, and conclude this Chapter with a section on the special case of torus boundary.

2.1 The algebra

In this Section, we describe the differential graded algebra $\mathcal{A}(\mathcal{Z})$ associated to the parametrized boundary of a 3-manifold. For further details, see [LOT, Chapter 3].

Definition 2.1.1. *The* strands algebra $\mathcal{A}(n,k)$ *is a free* $\mathbb{Z}/2$*-module generated by partial permutations* $a = (S,T,\phi)$*, where* S *and* T *are* k*-element subsets of the set* $[n] := \{1,\dots,n\}$ *and* $\phi : S \to T$ *is a non-decreasing bijection. We let* $inv(a) = inv(\phi)$ *be the number of inversions of* ϕ*, i.e. the number of pairs* $i,j \in S$ *with* $i < j$ *and* $\phi(j) < \phi(i)$*. Multiplication is given by*

$$(S,T,\phi)\cdot(U,V,\psi) = \begin{cases} (S,V,\psi\circ\phi) & \text{if } T = U,\ inv(\phi)+inv(\psi) = inv(\psi\circ\phi) \\ 0 & \text{otherwise.} \end{cases}$$

See Section 3.1.1 of [LOT]. We can represent a generator (S,T,ϕ) by a strands diagram of horizontal and upward-veering strands. See Section 3.1.2. of [LOT]. The differential of (S,T,ϕ) is the sum of all possible ways to "resolve" an inversion of ϕ so that inv goes down by exactly 1. Resolving an inversion (i,j) means switching $\phi(i)$ and $\phi(j)$, which graphically can be seen as smoothing a crossing in the strands diagram.

The ring of idempotents $\mathcal{I}(n,k) \subset \mathcal{A}(n,k)$ is generated by all elements of the form $I(S) := (S,S,\mathrm{id}_S)$ where S is a k-element subset of $[n]$.

Definition 2.1.2. *A* pointed matched circle *is a quadruple $\mathcal{Z} = (Z,\mathbf{a},M,z)$ consisting of an oriented circle Z, a collection of $4k$ points $\mathbf{a} = \{a_1,\ldots,a_{4k}\}$ in Z, a* matching *of $\mathbf{a}$, i.e., a 2-to-1 function $M : \mathbf{a} \to [2k]$, and a basepoint $z \in Z \setminus \mathbf{a}$. We require that performing oriented surgery along the $2k$ 0-spheres $M^{-1}(i)$ yields a single circle.*

A matched circle specifies a handle decomposition of an oriented surface $F(\mathcal{Z})$ of genus k: take a 2-dimensional 0-handle with boundary Z, $2k$ oriented 1-handles attached along the pairs of matched points, and a 2-handle attached to the resulting boundary.

If we forget the matching on the circle for a moment, we can view $\mathcal{A}(4k) = \bigoplus_i \mathcal{A}(4k,i)$ as the algebra generated by the Reeb chords in $(Z \setminus z, \mathbf{a})$: We can view a set $\boldsymbol{\rho}$ of Reeb chords, no two of which share initial or final endpoints, as a strands diagram of upward-veering strands. For such a set $\boldsymbol{\rho}$, we define the *strands algebra element associated to* $\boldsymbol{\rho}$ to be the sum of all ways of consistently adding horizontal strands to the diagram for $\boldsymbol{\rho}$, and we denote this element by $a_0(\boldsymbol{\rho}) \in \mathcal{A}(4k)$. The basis over $\mathbb{Z}/2$ from Definition 2.1.1 is in this terminology the non-zero elements of the form $I(S)a_0(\boldsymbol{\rho})$, where $S \subset \mathbf{a}$.

For a subset $\mathbf{s}$ of $[2k]$, a *section* of $\mathbf{s}$ is a set $S \subset M^{-1}(\mathbf{s})$, such that M maps S bijectively to $\mathbf{s}$. To each $\mathbf{s} \subset [2k]$ we associate an idempotent in $\mathcal{A}(4k)$ given by

$$I(\mathbf{s}) = \sum_{S \text{ is a section of } \mathbf{s}} I(S).$$

Let $\mathcal{I}(\mathcal{Z})$ be the subalgebra generated by all $I(\mathbf{s})$, and let $\mathbf{I} = \sum_{\mathbf{s}} I(\mathbf{s})$.

Definition 2.1.3. *The* algebra $\mathcal{A}(\mathcal{Z})$ associated to a pointed matched circle $\mathcal{Z}$ *is the subalgebra of $\mathcal{A}(4k)$ generated (as an algebra) by $\mathcal{I}(\mathcal{Z})$ and by all $a(\boldsymbol{\rho}) := \mathbf{I}a_0(\boldsymbol{\rho})\mathbf{I}$. We refer to $a(\boldsymbol{\rho})$ as the* algebra element associated to $\boldsymbol{\rho}$.

We will view $\mathcal{A}(\mathcal{Z})$ as defined over the ground ring $\mathcal{I}(\mathcal{Z})$. Note that $\mathcal{A}(\mathcal{Z})$ decomposes as a direct sum of differential graded algebras

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=-k}^{k} \mathcal{A}(\mathcal{Z}, i),$$

where $\mathcal{A}(\mathcal{Z}, i) = \mathcal{A}(\mathcal{Z}) \cap \mathcal{A}(4k, k+i)$. Similarly, set $\mathcal{I}(\mathcal{Z}, i) = \mathcal{I}(\mathcal{Z}) \cap \mathcal{A}(4k, k+i)$.

2.2 Type D structures, $\mathcal{A}_\infty$-modules, and tensor products

We recall the definitions of the algebraic structures used in [LOT]. For a beautiful, terse description of type D structures and their basic properties, see [Zar09, Section 7.2], and for a more general and detailed description of $\mathcal{A}_\infty$ structures, see [LOT, Section 2.3].

Let A be a unital differential graded algebra with differential d and multiplication μ over a base ring $\mathbf{k}$. In this thesis, $\mathbf{k}$ will always be a direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. When the algebra is $\mathcal{A}(\mathcal{Z})$, the base ring for all modules and tensor products is $\mathcal{I}(\mathcal{Z})$.

A *(right)* $\mathcal{A}_\infty$*-module* over A is a graded module M over $\mathbf{k}$, equipped with maps

$$m_i : M \otimes A^{\otimes(i-1)} \to M[2-i],$$

satisfying the compatibility conditions

$$\begin{aligned} 0 = &\sum_{i+j=n+1} m_i(m_j(\mathbf{x}, a_1, \cdots, a_{j-1}), \cdots, a_{n-1}) \\ &+ \sum_{i=1}^{n-1} m_n(\mathbf{x}, a_1, \cdots, a_{i-1}, d(a_i), \cdots, a_{n-1}) \\ &+ \sum_{i=1}^{n-2} m_{n-1}(\mathbf{x}, a_1, \cdots, a_{i-1}, (\mu(a_i, a_{i+1})), \cdots, a_{n-1}) \end{aligned}$$

and the unitality conditions $m_2(\mathbf{x}, 1) = \mathbf{x}$ and $m_i(\mathbf{x}, a_1, \cdots, a_{i-1}) = 0$ if $i > 2$ and some $a_j = 1$. We say that M is *bounded* if $m_i = 0$ for all sufficiently large i.

A *(left) type D structure* over A is a graded module N over the base ring, equipped with a homogeneous map

$$\delta : N \to (A \otimes N)[1]$$

satisfying the compatibility condition

$$(d \otimes \mathrm{id}_N) \circ \delta + (\mu \otimes \mathrm{id}_N) \circ (\mathrm{id}_A \otimes \delta) \circ \delta = 0.$$

We can define maps

$$\delta_k : N \to (A^{\otimes k} \otimes N)[k]$$

inductively by

$$\delta_k = \begin{cases} \mathrm{id}_N & \text{for } k = 0 \\ (\mathrm{id}_A \otimes \delta_{k-1}) \circ \delta & \text{for } k \geq 1 \end{cases}$$

A type D structure is said to be *bounded* if for any $\mathbf{x} \in N$, $\delta_i(\mathbf{x}) = 0$ for all sufficiently large i.

If M is a right $\mathcal{A}_\infty$-module over A and N is a left type D structure, and at least one of them is bounded, we can define the *box tensor product* $M \boxtimes N$ to be the vector space $M \otimes N$ with differential

$$\partial : M \otimes N \to M \otimes N[1]$$

defined by

$$\partial = \sum_{k=1}^{\infty} (m_k \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes \delta_{k-1}).$$

The boundedness condition guarantees that the above sum is finite. In that case $\partial^2 = 0$ and $M \boxtimes N$ is a graded chain complex.

Given two differential graded algebras, four types of bimodules can be defined in a similar way. We omit those definitions and refer the reader to [LOT10, Section 2.2.4].

2.3 Bordered three-manifolds, Heegaard diagrams, and their modules

A *bordered 3-manifold* is a triple $(Y, \mathcal{Z}, \phi)$, where Y is a compact, oriented 3-manifold with connected boundary ∂Y, $\mathcal{Z}$ is a pointed matched circle, and $\phi : F(\mathcal{Z}) \to \partial Y$ is an orientation-preserving homeomorphism. A bordered 3-manifold may be represented by a *bordered Heegaard diagram* $\mathcal{H} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$, where Σ is an oriented surface of some genus

g with one boundary component, $\boldsymbol{\beta}$ is a set of pairwise-disjoint, homologically independent circles in $\mathrm{Int}(\Sigma)$, $\boldsymbol{\alpha}$ is a $(g+k)$-tuple of pairwise-disjoint curves in Σ, split into $g-k$ circles in $\mathrm{Int}(\Sigma)$, and $2k$ arcs with boundary on $\partial\Sigma$, so that they are all homologically independent in $H_1(\Sigma, \partial\Sigma)$, and z is a point on $(\partial\Sigma) \setminus (\boldsymbol{\alpha} \cap \partial\Sigma)$. The boundary $\partial\mathcal{H}$ of the Heegaard diagram has the structure of a pointed matched circle, where two points are matched if they belong to the same α-arc. We can see how a bordered Heegaard diagram $\mathcal{H}$ specifies a bordered manifold in the following way. Thicken up the surface to $\Sigma \times [0,1]$, and attach a three-dimensional two-handle to each circle $\alpha_i \times \{0\}$, and a three-dimensional two-handle to each $\beta_i \times \{1\}$. Call the result Y, and let ϕ be the natural identification of $F(\partial\mathcal{H})$ with ∂Y. Then $(Y, \partial\mathcal{H}, \phi)$ is the bordered 3-manifold for $\mathcal{H}$.

To a bordered Heegaard diagram $(\mathcal{H}, z) = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, we associate either a left type D structure $\widehat{CFD}(\mathcal{H}, z)$ over $\mathcal{A}(-\partial\mathcal{H})$, or a right $\mathcal{A}_\infty$-module $\widehat{CFA}(\mathcal{H}, z)$ over $\mathcal{A}(\partial\mathcal{H})$. Similarly, we can represent a knot in a bordered 3-manifold by a doubly-pointed bordered Heegaard diagram $(\mathcal{H}, z, w) = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$, where z and w are in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, and $z \in \partial\mathcal{H}$. To this diagram we can associate a right $\mathcal{A}_\infty$-module $CFA^-(\mathcal{H}, z, w)$, this time over $\mathbb{F}_2[U]$, where a holomorphic curve passing through w with multiplicity n contributes U^n to the multiplication. Setting $U = 0$ gives $\widehat{CFA}(\mathcal{H}, z, w)$, where we count only holomorphic curves that do not cross w.

Now we define the above modules. A *generator* of a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of genus g is a g-element subset $\mathbf{x} = \{x_1, \ldots, x_g\}$ of $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$, such that there is exactly one point of $\mathbf{x}$ on each β-circle, exactly one point on each α-circle, and at most one point on each α-arc. Let $\mathfrak{S}(\mathcal{H})$ denote the set of generators. Given $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, let $o(\mathbf{x}) \subset [2k]$ denote the set of α-arcs occupied by $\mathbf{x}$, and let $\overline{o}(\mathbf{x}) = [2k] \setminus o(\mathbf{x})$ denote the set of unoccupied arcs.

Fix generators $\mathbf{x}$ and $\mathbf{y}$, and let I be the interval $[0,1]$. Let $\pi_2(\mathbf{x}, \mathbf{y})$, the *homology classes from* $\mathbf{x}$ *to* $\mathbf{y}$, denote the elements of

$$H_2(\Sigma \times I \times I, ((\boldsymbol{\alpha} \times \{1\} \cup \boldsymbol{\beta} \times \{0\} \cup (\partial\Sigma \setminus z) \times I) \times I) \cup (\mathbf{x} \times I \times \{0\}) \cup (\mathbf{y} \times I \times \{1\}))$$

which map to the relative fundamental class of $\mathbf{x} \times I \cup \mathbf{y} \times I$ under the composition of the boundary homomorphism and collapsing the rest of the boundary.

A homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ is determined by its *domain*, the projection of B to $H_2(\Sigma, \boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \partial\Sigma)$. We can interpret the domain of B as a linear combination of the components, or *regions*, of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$.

Concatenation at $\mathbf{y} \times I$, which corresponds to addition of domains, gives a product $* : \pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{w}) \to \pi_2(\mathbf{x}, \mathbf{w})$. This operation turns $\pi_2(\mathbf{x}, \mathbf{x})$ into a group called the group of *periodic domains*, which is naturally isomorphic to $H_2(Y, \partial Y)$.

Let $X(\mathcal{H})$ be the $\mathbb{F}_2$ vector space spanned by $\mathfrak{S}(\mathcal{H})$. Define $I_D(\mathbf{x}) = \overline{o}(\mathbf{x})$. We define an action on $X(\mathcal{H})$ of $\mathcal{I}(-\partial\mathcal{H})$ by

$$I(\mathbf{s}) \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } I(\mathbf{s}) = I_D(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widehat{CFD}(\mathcal{H})$ is defined as an $\mathcal{A}(-\partial\mathcal{H})$-module by

$$\widehat{CFD}(\mathcal{H}) = \mathcal{A}(-\partial\mathcal{H}) \otimes_{\mathcal{I}(-\partial\mathcal{H})} X(\mathcal{H}).$$

Since there are no explicit computations of $\widehat{CFD}$ in this thesis, we omit the definition of the map δ_1 and refer the reader to [LOT, Section 6.1].

Define $I_A(\mathbf{x}) = o(\mathbf{x})$. The module $\widehat{CFA}(\mathcal{H})$ is generated over $\mathbb{F}_2$ by $X(\mathcal{H})$, and the right action of $\mathcal{I}(\partial\mathcal{H})$ on $\widehat{CFA}(\mathcal{H})$ is defined by

$$\mathbf{x} \cdot I(\mathbf{s}) = \begin{cases} \mathbf{x} & \text{if } I(\mathbf{s}) = I_A(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathcal{A}_\infty$ multiplication maps count certain holomorphic representatives of the homology classes defined in this section [LOT, Definition 7.3].

2.4 Gradings

It turns out there is no $\mathbb{Z}$-grading on $\mathcal{A}(\mathcal{Z})$. Instead, the algebra is graded by a nonabelian group G, and the left or right modules over it are graded by left or right cosets of a subgroup of G. In the topological case, domains of a Heegaard diagram can be also graded by G, and the subgroup above is the image of periodic domains in G.

Given a group G and an element λ in the center of G, recall the definition of a differential algebra graded by (G, λ) (Definition 2.38 of [LOT]). Given such an algebra A, and a set S with a right G action, recall the definition of a right differential A-module graded by S (Definition 2.41 of [LOT]).

Specifically, recall the definitions of the gradings by $G'(4k)$, and the refined grading gr on $\mathcal{A}(\mathcal{Z}, i)$ in the smaller group $G(\mathcal{Z}) \subset G'(4k)$, as well as the grading of type D structures over $\mathcal{A}(\mathcal{Z})$ by cosets of a subgroup of $G'(4k)$ or $G(\mathcal{Z})$(Section 3.3 and Section 10 of [LOT]).

2.5 The torus boundary case

We now focus on the special case of torus boundary, in particular gluing a knot in the solid torus to a knot complement.

From here on, let $Y_{p,1}$ stand for the $(p, 1)$-cable in the 0-framed solid torus, and let $Y_{K,n}$ be the n-framed knot complement $S^3 \setminus K$, so that $Y_{p,1} \cup_\partial Y_{K,n}$ is the pair $(S^3, K_{p,pn+1})$. The separating surface $F = \partial Y_{p,1} = -\partial Y_{K,n}$ is a torus, parametrized by the circle $\mathcal{Z}$ in Figure 1. The four α-points divide the circle into the four upward-oriented arcs ρ_0, ρ_1, ρ_2, and ρ_3, where ρ_0 contains the basepoint z. The algebra $\mathcal{A}(\mathcal{Z})$ has two idempotents, one for each α-arc, and 6 Reeb elements, coming from the Reeb chords ρ_1, ρ_2 and ρ_3 (see [LOT, Section 11.1]). In this case, $\widehat{CFD}(Y_{K,n})$ can be derived explicitly from $CFK^-(K)$ (see Section 3.2.2), and is represented best using the coefficient maps $D_1, D_2, D_3, D_{12}, D_{23}$, or D_{123}, which describe the sequence of Reeb chords that a holomorphic curve passes through.

When at least one of *CFA* or *CFD* is bounded [LOT, Definititions 2.4 and 2.22], there is a particularly simple description for their $\mathcal{A}_\infty$ tensor product and the tensor differential. A product $a \boxtimes d$ is nonzero in $CFA \boxtimes CFD$ whenever a and d occupy complementary sets of α-arcs. The differential $\partial^{\boxtimes}(a_1 \boxtimes d_1)$ has $a_2 \boxtimes d_2$ in the image whenever there is a sequence of coefficient maps $D_{I_1}, \dots, D_{I_n}$ from d_1 to d_2 and a multiplication map $m_{n+1}(a_1, \rho_{I_1}, \dots, \rho_{I_n})$ with a_2 in the image, both indexed the same way. See [LOT, Definition 2.26 and Equation (2.29)].

In this case, we make use of a grading by a group G, which is somewhat different from

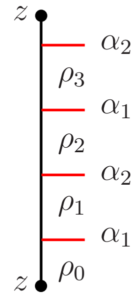


Figure 1: The circle $\mathcal{Z}$ in the case of torus boundary

$G(\mathcal{Z})$. The elements of G are quadruples of half-integers $(a; b, c; d)$ with $b+c \in \mathbb{Z}$ and $d \in \mathbb{Z}$, with multiplication given by

$$(a_1; b_1, c_1; d_1) \cdot (a_2; b_2, c_2; d_2) = (a_1 + a_2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}; b_1 + b_2, c_1 + c_2; d_1 + d_2).$$

The first number is called the Maslov component of the grading, and the pair (b, c) is the *spin*^{c} component. The fourth number is used in the case of knots to encode the U grading.

The grading on $\mathcal{A}(\mathcal{Z})$ is given by

$$gr(\rho_1) = (-\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}; 0)$$
$$gr(\rho_2) = (-\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 0)$$
$$gr(\rho_3) = (-\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}; 0).$$

For a homology solid torus, hence for both $Y_{p,1}$ and $Y_{K,n}$, the group of periodic domains is isomorphic to $\mathbb{Z}$, and so is its image in G.

For $CFD(Y_{K,n})$ we find a generator h for this image in Section 3.2.2. If D_I is a coefficient map from x to y then the gradings of x and y are related by

$$gr(y) = \lambda^{-1} gr(\rho_I)^{-1} gr(x) \in G/\langle h \rangle, \tag{2.5.1}$$

where $\lambda = (1; 0, 0; 0)$.

For $CFA(Y_{p,1})$, we find a generator g for the subgroup in Section 3.3. For a multiplication map $m_{l+1}(x, \rho_{I_1}, \ldots, \rho_{I_l}) = U^i y$ we have the formula

$$gr(y) = \lambda^{l-1} gr(x) gr(\rho_{I_1}) \cdots gr(\rho_{I_l})(0; 0, 0; i) \in \langle g \rangle \backslash G. \tag{2.5.2}$$

Let $\mathcal{H}$ be a provincially admissible Heegaard diagram for $Y_{p,1}$ with $\partial\mathcal{H} = \mathcal{Z}$. While $CFA^-(\mathcal{H})$ may not be an invariant of $Y_{p,1}$(Remark 11.20 of [LOT]), the pairing theorem [LOT, Theorem 11.21] says that there are homotopy equivalences

$$CFK^-(K_{p,pn+1}) \simeq CFA^-(\mathcal{H}) \boxtimes \widehat{CFD}(Y_{K,n})$$
$$\widehat{CFK}(K_{p,pn+1}) \simeq \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(Y_{K,n}),$$

which respect gradings in the following sense. The tensor product is graded by the double-coset space $\langle g\rangle\backslash G/\langle h\rangle$ via $gr(xy) = gr(x)gr(y)$ (we use the notation xy to mean $x \boxtimes y$). This double-coset space is in turn isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and for a homogeneous xy we can always choose a coset representative for $gr(xy)$ of the form $(a; 0, 0; d)$, where $a, d \in \mathbb{Z}$. We can achieve this by multiplying any other representative by appropriate powers of g to the left and h to the right. From there, we recover the absolute Maslov and Alexander grading by the formula

$$\begin{aligned} A &= d - p\tau(K) - \frac{np(p-1)}{2} \\ M &= a + 2A. \end{aligned}$$

We discuss this formula in Section 3.4.

Chapter 3

Cables of knots

In this chapter, we use bordered Floer homology to give a formula for $\widehat{HFK}(K_{p,pn+1})$ of any $(p, pn+1)$-cable of a thin knot K in terms of $\Delta_K(t)$, $\tau(K)$, p, and n. We also give a formula for the Ozsváth-Szabó concordance invariant $\tau(K_{p,q})$ in terms of $\tau(K)$, p, and q, for all relatively prime p and q.

3.1 Introduction

In [OS03b], Ozsváth and Szabó introduce a powerful knot invariant using Heegaard diagrams. Here we study its simplest version, the knot Floer homology $\widehat{HFK}(K)$, which has the structure of a bigraded vector space over $\mathbb{F}_2$, the field with two elements. Its Euler characteristic is the symmetrized Alexander polynomial $\Delta_K(T)$, in the sense that

$$\sum_{i,j}(-1)^i T^j \text{ rank } \widehat{HFK}_i(K, j) = \Delta_K(T).$$

The indices i and j in the summation stand for the *Maslov grading* M and the *Alexander grading* A, respectively. It is sometimes convenient to make use of a third grading, $\delta = A - M$.

Originally, knot Floer homology was defined by counting pseudo-holomorphic curves in the g-fold symmetric product of a genus g Heegaard surface. Later, combinatorial versions appeared, including a method using grid diagrams [MOS07]. The complex coming from a grid diagram has $n!$ generators, where n is the arc index of the knot, so this method only works well in practice for knots with few crossings or for special families of knots. In this chapter, we instead use the bordered Floer homology package that we described in Chapter 2, which generalizes Heegaard Floer homology to 3-manifolds with boundary, and to knots in 3-manifolds with boundary [LOT]. The beauty of this theory is that it allows us to compute invariants for a space by cutting the space into simpler pieces, and studying the pieces and their gluing instead. This approach is particularly well suited for studying knot satellites. It was used by Levine to study generalizations of Bing and Whitehead doubles [Lev09, Lev10]. Here, we apply the bordered method to cables of thin knots. Our Corollary 2 has since been generalized by Hom to cables of all knots [Hom11].

Let K be a knot in S^3. Recall that the (p, q)-cable of K, denoted $K_{p,q}$, is the satellite knot with pattern the torus knot $T_{p,q}$ and companion K. In other words, if $T_{p,q}$ is drawn on the surface of an unknotted solid torus, then we obtain $K_{p,q}$ by gluing the solid torus to the complement of K, identifying its meridian and preferred longitude with the meridian and preferred longitude of K. Thus, p and q refer to the winding of the cable in the longitudinal and meridional directions of K, respectively.

A knot K is called *Floer homologically thin* [MO08] if its knot Floer homology is supported in a single δ-grading. Throughout this chapter, we will say *thin* to mean Floer homologically thin. If the homology is supported on the diagonal $\delta = -\sigma/2$, where σ denotes the knot signature, then we say the knot is *σ-thin*, or *perfect* [Ras02]. The class of σ-thin knots contains as a proper subset all quasi-alternating knots [MO08], and in particular all alternating knots [OS03a].

Using the knot filtration on $\widehat{CF}(S^3)$, Ozsváth and Szabó define an integer knot invariant τ [OS03c], independently discovered by Rasmussen [Ras03], whose absolute value is a lower bound on the four-ball genus. The behavior of τ under various satellite operations, such as cabling, Bing, and Whitehead doubling, has been studied extensively in recent years [Hed08, LN06, Cot08, Rob09b, Rob09a, Lev09, Lev10]. In [Hed08], Hedden gives upper and lower bounds for $\tau(K_{p,pn+1})$ in terms of $\tau(K)$, p, and n, and, for sufficiently large $|n|$, describes the knot Floer homology of the cable in the topmost Alexander gradings. In the case where K is thin, we extend Hedden's results to a complete description of the knot

Floer homology of the cable. In particular, we derive a formula for $\widehat{HFK}(K_{p,pn+1})$ and for $\tau(K_{p,pn+1})$ in terms of $\tau(K)$, $\Delta_K(t)$, p, and n. Note that for a σ-thin knot, $\tau(K) = -\sigma(K)/2$ (Theorem 1). Our method can easily be adapted to compute $\widehat{HFK}(K_{p,q})$ for any relatively prime p and q, as we explain at the end of Section 3.3.

Note: A *Mathematica* [WR] program implementing this method is available online [Pet]. The program takes $\Delta_K(t)$, $\tau(K)$, p, and n as input, and outputs the generators of the homology $\widehat{HFK}(K_{p,pn+1})$ as a list of ordered pairs of Alexander and Maslov gradings. It then plots the result on the (A, M)-lattice. The program computes $\widehat{HFK}$ for cables with thousands of crossings in a matter of seconds, whereas the grid method would take billions of years. In Section 3.7 we give the result for the $(5, 16)$-cable of the knot $11n50$. This knot is interesting as it is the first known example of a homologically thin (with respect to $\widehat{HFK}$, $\overline{Kh}$ and $\overline{Kh}'$), non-quasi-alternating knot [Gre09].

3.2 $\widehat{CFD}$ of a thin knot

3.2.1 The complex CFK^- for thin knots

Recall that given a knot K, $CFK^-(K)$ is a free, finitely generated chain complex over $\mathbb{F}_2[U]$, endowed with an Alexander filtration A by the integers, and an integer grading, called the Maslov grading. The differential lowers the Maslov grading by one, respects the Alexander filtration, and does not decrease the U power. We can illustrate $CFK^-(K)$ graphically as follows. We choose a basis of generators $\mathcal{B}$ for $CFK^-(K)$ over $\mathbb{F}_2[U]$ which is homogeneous with respect to the Alexander filtration. Then $\mathcal{B} \otimes_{\mathbb{F}_2} \mathbb{F}_2[U]$ is a basis for $CFK^-(K)$ over $\mathbb{F}_2$. We plot $\mathcal{B} \otimes \mathbb{F}_2[U]$ on the (U, A)-lattice, and draw arrows for the differential ∂_-. To match preexisting conventions, a generator of the form $U^x\xi$ of Alexander depth y is at position $(-x, y)$, where $\xi \in \mathcal{B}$. If $\partial_-(x) = y_1 + \cdots + y_n$, where $x, y_1, \ldots, y_n$ are basis elements, then there is an arrow from x to each y_i. In this case we say that x *points* to each y_i. If y_i is below/to the left of x, we say that the arrow from x to y_i points *down/to the left*. Note that all arrows point non-strictly down and to the left. If the arrow is *vertical*, meaning that x

and y_i have the same U power, then the *length* of the arrow is $A(x) - A(y_i)$. If the arrow is *horizontal*, meaning that x and y_i are in the same Alexander filtration, then the *length* of the arrow is the difference between the U power of y and the U power of x.

From now on, K will be a thin knot. In this case, $\Delta_K(t)$ and $\tau(K)$ are sufficient to describe a model for the chain complex $CFK^-(K)$. Note that for a σ-thin knot, this means that the only information we need is the Alexander polynomial and the signature. This was stated without proof in [OS03a] with regard to alternating knots. We now state and prove the general claim.

Theorem 3.2.1. *If K is a thin knot, $CFK^-(K)$ is completely determined by $\tau(K)$ and $\Delta_K(t)$.*

The proof relies on two lemmas. First we perform a filtered chain homotopy to obtain a new complex with a simpler differential. Then we change basis to show that the complex is isomorphic to a direct sum of three special kinds of complexes.

Lemma 3.2.2. *There is a filtered chain homotopy equivalence*

$$(CFK^-(K), \partial_-) \cong (\widehat{HFK}(K) \otimes \mathbb{F}_2[U], \partial_z + U\partial_w),$$

where ∂_z counts holomorphic disks that pass once through the basepoint z, and ∂_w counts disks that pass once through w.

Proof. In each vertical column of the (U, A)-lattice, the arrows that go between elements in the same position count disks that do not pass through either basepoint, and hence form the differential $\hat{\partial}$. We take homology with respect to these arrows. In terms of basis elements, if $\partial_-(a) = b_1 + \cdots + b_n$, and a and b_1 have the same Alexander filtration and U power, and if $x_1, \ldots, x_k$ are all the other elements that point to b_1, then we replace the basis vectors $b_1, b_2, \ldots, b_n, x_1, \ldots, x_k$ with $b_1 + \cdots + b_n, b_2, \ldots, b_n, x_1 + a, \ldots, x_k + a$. In this way, we get an isolated arrow from a to $b_1 + \cdots + b_n$, so we can delete it. Repeating this until there are no more such arrows, we get a complex with generators $\widehat{HFK}(K) \otimes \mathbb{F}_2[U]$.

Since K is thin, the difference in the Maslov gradings of any two generators of $\widehat{HFK}$ is equal to the difference in their Alexander filtrations. Thus, if an arrow pointing from x

to $U^l y$ drops the Alexander filtration by k, then $M(x) - M(y) = A(x) - A(y) = k - l$, since multiplication by U drops the Alexander filtration by 1. On the other hand, since the differential always drops the Maslov grading by 1, and multiplication by U drops it by 2, then $1 = M(x) - M(U^l y) = k - l + 2l = k + l$. Then either $k = 0$ and $l = 1$, or $k = 1$ and $l = 0$. In the first case we have a horizontal arrow of length one pointing to the left and contributing to $U\partial_w$, and in the second case we have a vertical arrow of length one pointing down and contributing to ∂_z. □

Definition 3.2.3. *A free, finitely generated, chain complex $\mathcal{C}$ over $\mathbb{F}_2[U]$ is automatically endowed with a U-power filtration. An Alexander filtration A is a filtration such that*

- *multiplication by U lowers the A filtration by* 1*,*
- *the differential respects A.*

The complex $\mathcal{C}$ is said to be thin *if the differential lowers the sum of A and U-power filtration by exactly* 1*.*

This definition is equivalent to saying that in the graph of $\mathcal{C}$ all arrows are either vertical or horizontal and have length one.

Given a thin complex, call the map consisting of all vertical arrows ∂_z, and the map consisting of all horizontal arrows $U\partial_w$. We choose this notation in order to be consistent with the case of a knot Floer complex. For a homogeneous element x,

$$\partial^2 x = (\partial_z + U\partial_w)^2 x = \partial_z^2 x + U^2 \partial_w^2 x + (\partial_z(U\partial_w) + (U\partial_w)\partial_z)x,$$

where the three homogeneous summands have distinct positions on the lattice. Since $\partial^2 = 0$, all three summands must be identically zero, showing that ∂_z and $U\partial_w$ are differentials.

The *vertical complex* $\mathcal{C}^{\mathrm{v}} := \mathcal{C}/(U \cdot \mathcal{C})$ is a chain complex which inherits the Alexander filtration from $\mathcal{C}$. We call its homology the *vertical homology*, denoted $H^{\mathrm{v}}(\mathcal{C})$. We also define the *horizontal complex* $\mathcal{C}^{\mathrm{h}}$ as the degree zero part of the associated graded space to $\mathcal{C} \otimes_{\mathbb{F}_2[U]} \mathbb{F}_2[U, U^{-1}]$ with respect to the Alexander filtration. It is filtered by the U powers, and inherits a differential from $\mathcal{C}$. We call its homology the *horizontal homology*, denoted $H^{\mathrm{h}}(\mathcal{C})$.

When $\mathcal{C} \cong CFK^-(K)$, then ∂_z and ∂_w are the differentials for $\widehat{CF}(S^3)$ with respect to the two different basepoints. In that case $C^{\mathrm{v}}(CFK^-(K)) \cong C^{\mathrm{h}}(CFK^-(K)) \cong \widehat{CF}(S^3)$, and $H^{\mathrm{v}}(CFK^-(K)) \cong H^{\mathrm{h}}(CFK^-(K)) \cong \widehat{HF}(S^3) \cong \mathbb{F}_2$.

Lemma 3.2.4. *Suppose $\mathcal{C}$ is a thin complex with horizontal and vertical homologies of rank at most 1. Then $\mathcal{C}$ is isomorphic to a direct sum of complexes, each modeled by one of the complexes in Figure 2. In particular, $(\widehat{HFK}(K) \otimes \mathbb{F}_2[U], \partial_z + U\partial_w)$ has a model complex isomorphic to a direct sum of these model complexes.*

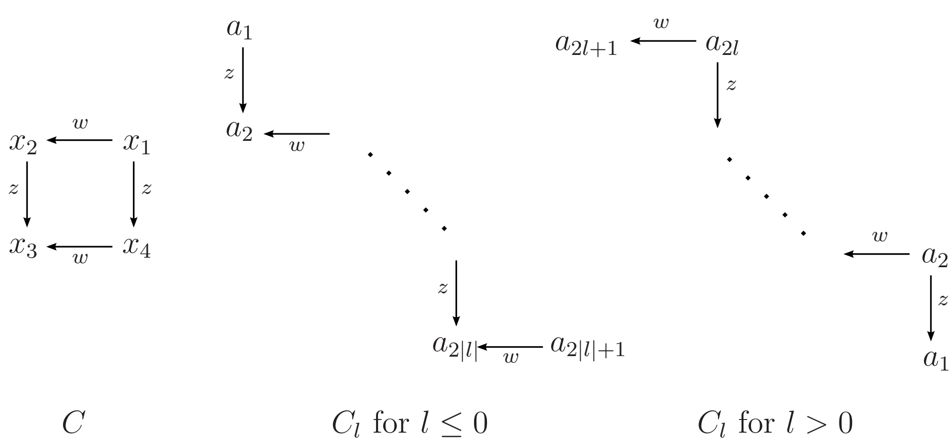


Figure 2: Model complexes for CFK^- of a thin knot K, where $l = \tau(K)$. A w-arrow from x to y means that y has coefficient U in ∂x.

Proof. For the sake of simplicity, we slightly abuse notation in this proof. We will say there is a w-arrow from x to y, to mean that there is a w-arrow from x to Uy. We will also denote $U\partial_w$ simply by ∂_w, and thus say $\partial_w x = y$, instead of $U\partial_w x = Uy$.

We prove the lemma by induction on $\mathrm{rk}_{\mathbb{F}_2[U]}(\mathcal{C})$, since $\mathcal{C}$ has finite rank over $\mathbb{F}_2[U]$. We change basis in $\mathcal{C}$ over $\mathbb{F}_2[U]$ homogeneously to split off a C or a C_i summand. Then $\mathcal{C} \cong C \oplus \mathcal{C}'$ or $\mathcal{C} \cong C_i \oplus \mathcal{C}'$, where $\mathcal{C}'$ has lower rank than $\mathcal{C}$, and vertical and horizontal homologies of rank at most 1, hence $\mathcal{C}'$ must split in the desired way too. Thus, $\mathcal{C}$ splits into a direct sum of the model complexes by induction.

On the (U, A)-lattice, the complex $\mathcal{C}$ is supported in a strip of finite width and slope 1. Choose a nonzero basis element b_1 over $\mathbb{F}_2[U]$ of smallest Alexander filtration possible (so b_1 is on the lower boundary edge of the strip).

Case 1: There is a vertical arrow pointing to b_1.

Let a be a generator that has a z-arrow to b_1. If $\partial_z a = b_1$, let $b = b_1$, and if $\partial_z a = b_1 + \cdots + b_n$ and $n > 1$, change basis by replacing $b_1, \ldots, b_n$ with $b = b_1 + \cdots + b_n, b_2, \ldots, b_n$. Now $\partial_z a = b$. By our choice of b_1, and since $\partial^2 = 0$, we know that there is no w-arrow pointing to a or b, and no z-arrow originating at b. If there are other generators with a z-arrow to b, add a to each of them, so that in the new basis only a has a z-arrow to b.

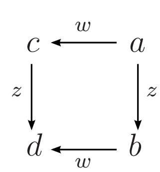
Case 1.1: $\partial_w b \neq 0$.

We will split off a C summand. Since $(\partial_z\partial_w + \partial_w\partial_z)a = 0$, we have $\partial_w a = c \neq 0$, $\partial_z c = \partial_w b$. By changing basis if necessary, we may assume c is a basis element. Since $\partial_z^2 c = 0$, and a is the only generator with b in the image of its ∂_z differential, it follows that a does not appear in $\partial_z c = \partial_w b$. Thus, we may change basis if necessary so that $\partial_w b = d$ with d a basis element, without affecting the choices made so far. Now $\partial_w^2 = 0$ implies that $\partial_w d = 0$, and $\partial_z^2 = 0$ implies that $\partial_z d = 0$.

For any other b' that has a w-arrow to d, replace it by $b + b'$, so that b remains the only generator with a w-arrow to d. In the same way we arrange that c is the only generator with a z-arrow to d. Now $\partial_z^2 = 0$ implies that no z-arrow points to c. After the last two changes, a may no longer be the only generator with a z-arrow to b.

Suppose there is some $a' \neq a$ with a z-arrow to b. Since b is the only generator with a w-arrow to d, and c is the only one with a z-arrow to d, then $\partial_z\partial_w + \partial_w\partial_z = 0$ implies that a' also points to c. Similarly, if a' points to c, it must also point to b. Add a to all such a', so that a is the only generator with a z-arrow to b, and the only one with a w-arrow to c. From $\partial^2 = 0$ it follows that nothing points to a.

Thus, we have changed basis to split off a C, modeled by the square

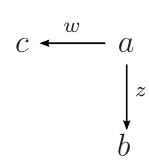


Case 1.2: $\partial_w b = 0$.

We will split off a C_l summand. Add a to any other generator that has a z-arrow to b, so that now only a does. Now no z-arrow points to a, since $\partial_z^2 = 0$. Note that $\partial_w b = 0$ implies that b survives in horizontal homology. By the rank assumption in this Lemma, $H^{\mathrm{h}}(\mathcal{C}) \cong \mathbb{F}_2$, represented by b, and so no other generator survives in horizontal homology. In particular a does not survive, so $\partial_w a \neq 0$. As before, we may assume that $\partial_w a = c$, where c is a basis element. Note that $\partial_w c = \partial_z c = 0$.

Suppose that some $a_1 \neq a$ has a w-edge to c, and add a_1 to all other such generators except a. Now only a and a_1 have a w-edge to c. If a_1 also has w-edges to generators other than c, change basis as before to arrange that $\partial_w a_1 = c + c_1$, where c_1 is a basis element. We can continue until we get a zig-zag, i.e. basis elements $a, a_1, \dots, a_n$ with $\partial_w a = c$, $\partial_w a_1 = c + c_1$, $\partial_w a_2 = c_1 + c_2$, ..., $\partial_w a_{n-1} = c_{n-2} + c_{n-1}$, and either $\partial_w a_n = c_{n-1} + c_n$, or $\partial_w a_n = c_{n-1}$, so that no other w-edge points to any c_i. In the first case, we replace the basis vectors $c, c_1, \dots, c_n$ with $c, c + c_1, c_1 + c_2, \dots, c_{n-1} + c_n$, and in the second, we get a contradiction to the fact that the horizontal homology has rank one. Now only a has a w-edge to c.

Case 1.2.1: If no z-arrow points to c, then we split off the C_1 staircase



Case 1.2.2: If there is a z-arrow pointing to c, we may assume that in fact only one basis element d has a z-arrow to c.

If $\partial_z d \neq c$, then we may arrange that $\partial_z d = c + c^1$, where c^1 is another basis element. As before, we can get a zig-zag $\partial_z d = c + c^1$, $\partial_z d_1 = c^1 + c^2, \dots, \partial_z d_{k-1} = c^{k-1} + c^k$ and either $\partial_z d_k = c^k + c^{k+1}$, or $\partial_z d_k = c^k$, so that d_i are in the basis and no other z-arrow points to any of the c^i. In the first case, we replace the basis vectors $c, c^1, \dots, c^{k+1}$ by $c, c + c^1, c^1 + c^2, \dots, c^k + c^{k+1}$, and we split off the abc staircase, i.e., a C_{-1}. In the second, we change basis by adding all d_i to d, so that only $d' = d + d_1 + \cdots + d_k$ has a z-arrow to

c. Then there is no w-arrow to d', so we can repeat the steps of Case 1.2, beginning at d' instead of a.

If $\partial_z d = c$, we can repeat the steps of Case 1.2, beginning at d instead of a.

Since the complex is supported in a diagonal strip of finite width, eventually we have to stop, and we split off a staircase C_l for some $l > 0$.

Case 2: There is no vertical arrow pointing to $b = b_1$.

We will split off a C_l summand. If $\partial_w b = 0$, then we split off a single b. Otherwise, we may assume that $\partial_w b = c$, where c is a basis element. Add b to any other b' with a w-arrow to c, so that now only b has a w-arrow to c. Since $\partial^2 b = 0$, then $\partial_w c = \partial_z c = 0$. Since there is no z arrow to b, then the vertical homology is $\mathbb{F}_2$, represented by b, so there is some d with a z-arrow to c, and we may assume that d is a basis vector. Add d to all other d' that have a z-arrow to c, so that now only d does. Since $\partial^2 = 0$, there is no w-arrow to d. We can proceed as in Case 1.2.2. Eventually we split off a C_l for some $l \leq 0$.

In each of the cases we managed to split off a model complex, so by induction on the rank of $\mathcal{C}$ over $\mathbb{F}_2[U]$, we are done.

In the special case of $(\widehat{HFK}(K) \otimes \mathbb{F}_2[U], \partial_z + U\partial_w)$, both the vertical and horizontal homologies have rank 1. Hence, the complex splits into exactly one C_l summand, and possibly multiple C summands. □

Proof of Theorem 3.2.1. We showed there is an isomorphism

$$\oplus_{i=1}^{k}(C'_i) \cong (\widehat{HFK}(K) \otimes \mathbb{F}_2[U], \partial_z + U\partial_w)$$

for some k, where each C'_i is one of the model complexes in Figure 2. If we restrict to the vertical column of the (U, A)-lattice where the U-power is zero, we see exactly one representative over $\mathbb{F}_2[U]$ of each generator of $\oplus_{i=1}^{k} C'_i$. For each square C, its representatives in this column appear in three adjacent Alexander gradings, with two representatives in the middle grading. For the staircase C_l, its representatives in the column appear one in each of $2|l| + 1$ adjacent gradings. Also note that this column is isomorphic to $\widehat{HFK}(K)$, so its rank in any Alexander grading a equals the rank of $\widehat{HFK}(K)$ in the same Alexander grading a (which also equals the absolute value of the coefficient of the symmetrized Alexander polynomial in degree a). Figure 3 illustrates these observations for the knot 5_2.

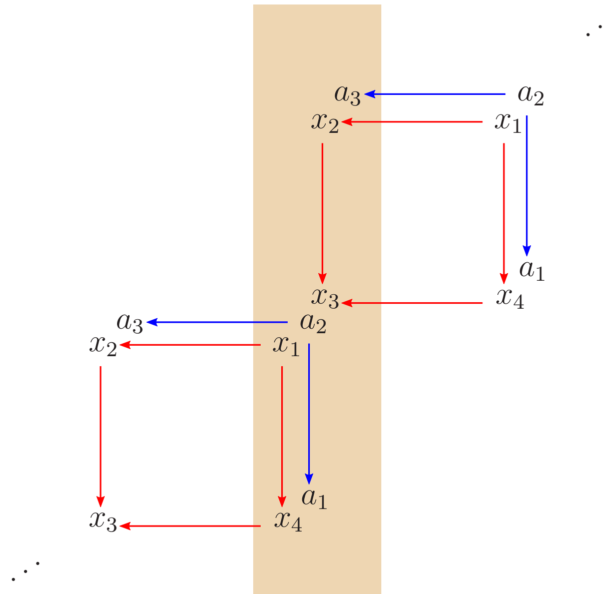


Figure 3: CFK^- of the 5_2 knot. The U^0 column is highlighted. To keep the figure simple, we omit writing the U-power of the generators when translating them by the U-action.

The two ends of the staircase are generators for $\widehat{HF}(S^3)$, a_1 with respect to the basepoint z, and $a_{2|l|+1}$ with respect to the basepoint w. Thus, the representative for a_1 has Alexander grading $-\tau(K)$, and the one for $a_{2|l|+1}$ has Alexander grading $\tau(K)$ (see Section 3.5 for the definition of τ). Then the staircase looks like $C_{\tau(K)}$. It contains $2|\tau(K)|+1$ elements, one in each Alexander grading i, where $-|\tau(K)| \leq i \leq |\tau(K)|$.

Let a'_i be the rank of the column in Alexander grading i after removing all the staircase generators. In other words,

$$a'_i = \begin{cases} |a_i| & \text{if } |i| > |\tau(K)| \\ |a_i| - 1 & \text{otherwise,} \end{cases}$$

where a_i is the coefficient of t^i in the symmetrized Alexander polynomial $\Delta_K(t)$. Let c_i be the number of squares with an upper right corner representative in Alexander grading i.

We see that $c_{g-1} = a'_g$, $c_{g-2} = a'_{g-1} - 2c_{g-1}$, and in general we get the recursive formula $c_i = a'_{i+1} - 2c_{i+1} - c_{i+2}$. Note, in particular, that $c_i = c_{-i}$. ☐

3.2.2 $\widehat{CFD}$ from CFK^-

Theorems 11.27 and A.11 of [LOT] together provide an algorithm for computing $\widehat{CFD}$ of any bordered knot complement with framing n from CFK^-. In particular, if K is thin, we take the simplified basis described in Section 3.2.1 and modify each square and the one staircase as in Figure 4. To simplify our notation when working with indices and gradings, we will often write τ for $\tau(K)$ when it is clear from the context what we mean.

The dashed diagonal arrow $\dashrightarrow$ in Figure 4 for

$$\xrightarrow{D_{12}} \qquad \text{if } n = 2\tau(K),$$

$$\xrightarrow{D_1} \mu_1 \xleftarrow{D_{23}} \mu_2 \xleftarrow{D_{23}} \cdots \xleftarrow{D_{23}} \mu_m \xleftarrow{D_3} \qquad \text{if } n = 2\tau(K) - m, \quad m > 0,$$

$$\xrightarrow{D_{123}} \mu_1 \xrightarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_{|m|} \xrightarrow{D_2} \qquad \text{if } n = 2\tau(K) - m, \quad m < 0.$$

Note that when $\tau(K) = 0$ and $n \geq 2\tau(K)$, the type D structure is not bounded. In that case, to obtain a bounded $\widehat{CFD}(K, n)$ we modify the dashed arrow to

$$\xrightarrow{D_1} \epsilon_1 \xleftarrow{1} \epsilon_2 \xrightarrow{D_2} \qquad \text{if } n = 2\tau(K),$$

$$\xrightarrow{D_1} \epsilon_1 \xleftarrow{1} \epsilon_2 \xrightarrow{D_{23}} \mu_1 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_{|m|} \xrightarrow{D_2} \qquad \text{if } n = 2\tau(K) - m, \ m < 0.$$

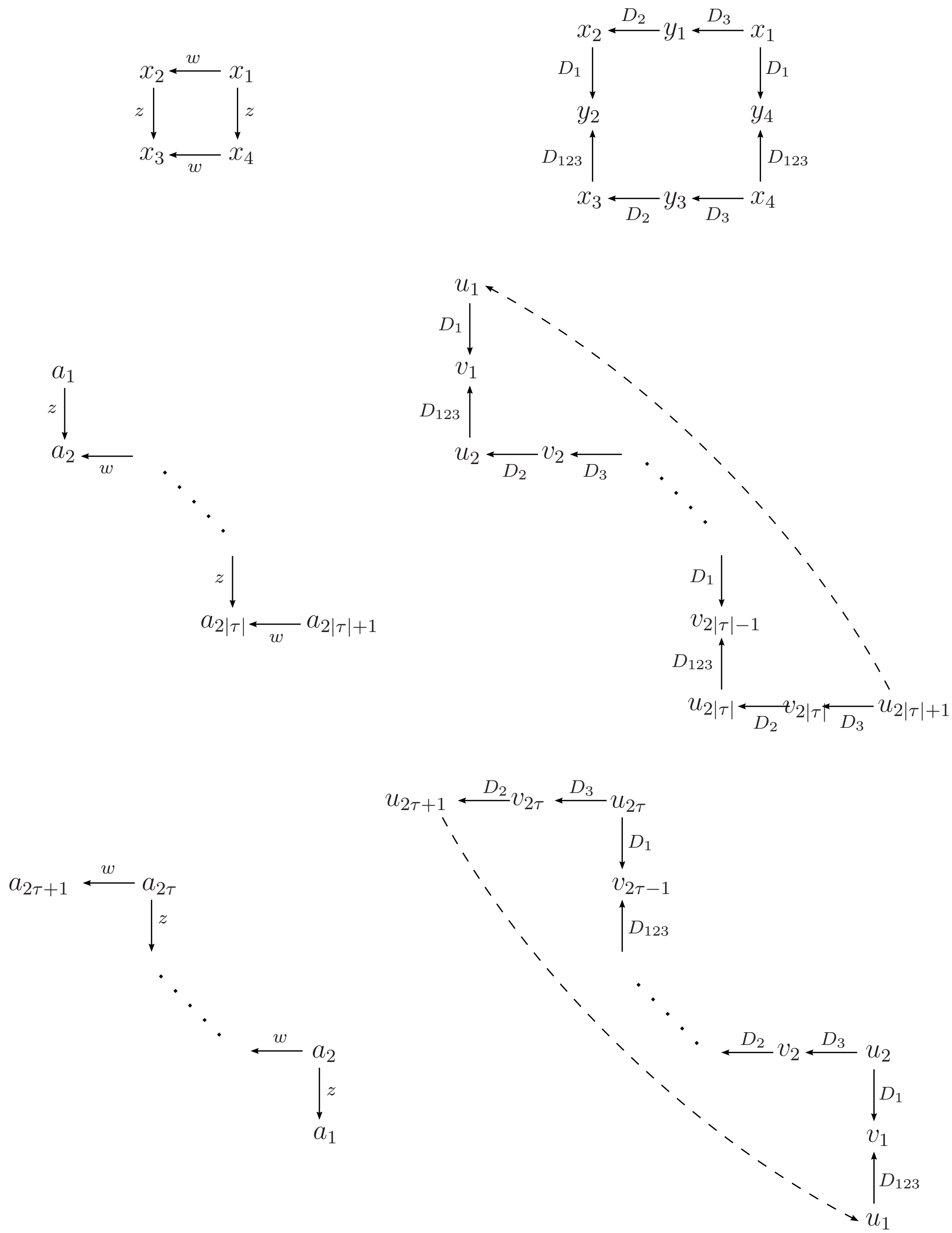


Figure 4: From CFK^- to $\widehat{CFD}$. The table shows each model knot Floer complex on the left, and the corresponding type D module on the right.

Next, we find the gradings of the elements of $\widehat{CFD}(K, n)$. Recall that for a homology solid torus, such as a knot complement, the group of periodic domains is isomorphic to $\mathbb{Z}$

(see, for example, the discussion above [LOT, Lemma 11.40]), so working with base generator u_1, the image of this group $\pi_2(u_1, u_1)$ in G has a single generator h. Thus, $\widehat{CFD}(K, n)$ is graded by $G/\langle h \rangle$. We normalize the grading by setting $gr(u_1) = (0; 0, 0; 0)/\langle h \rangle$. Starting at u_1 and using (2.5.1) and the grading on the algebra, we go along the staircase, then along the dashed arrow:

<u>Case 1:</u> If $\tau(K) \leq 0$, the staircase C_τ is graded as follows

$$\begin{aligned} gr(u_{2k+1}) &= (k; 0, 2k; 0)/\langle h \rangle \\ gr(v_{2k+1}) &= (-\tfrac{1}{2}; -\tfrac{1}{2}, 2k + \tfrac{1}{2}; 0)/\langle h \rangle \\ gr(u_{2k}) &= (k - \tfrac{1}{2}; 0, 2k - 1; 0)/\langle h \rangle \\ gr(v_{2k}) &= (2k - \tfrac{1}{2}; \tfrac{1}{2}, 2k - \tfrac{1}{2}; 0)/\langle h \rangle. \end{aligned}$$

If $m > 0$, we have the extra elements $\mu_1, \ldots, \mu_m$, with gradings

$$gr(\mu_{i+1}) = (i - \tfrac{1}{2}; -\tfrac{1}{2}, i + \tfrac{1}{2} - 2\tau; 0)/\langle h \rangle.$$

If $m \leq 0$ and $\tau(K) = 0$, we have ϵ_1 and ϵ_2 graded as

$$\begin{aligned} gr(\epsilon_1) &= (-\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}; 0)/\langle h \rangle \\ gr(\epsilon_2) &= (\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}; 0)/\langle h \rangle, \end{aligned}$$

and for any $\tau(K) \leq 0$ the additional elements $\mu_1, \ldots, \mu_{|m|}$ when $m < 0$, with gradings

$$gr(\mu_{i+1}) = (-i - \tfrac{1}{2}; -\tfrac{1}{2}, -i - \tfrac{1}{2} - 2\tau; 0)/\langle h \rangle.$$

In each case, by closing the loop back at u_1 along the dashed arrow, we see that the grading of u_1 is also given by

$$gr(u_1) = (\tfrac{m}{2} - \tfrac{1}{2} + \tau; -1, m - 2\tau; 0)/\langle h \rangle.$$

The difference $(\frac{m}{2} - \frac{1}{2} + \tau; -1, m - 2\tau; 0)$ of the two grading representatives then lies in $\langle h \rangle$. Since this difference is primitive, it equals h or its inverse, so we choose

$$h = (\tfrac{m}{2} - \tfrac{1}{2} + \tau; -1, m - 2\tau; 0).$$

Case 2: If $\tau(K) > 0$, the staircase C_τ is graded as follows

$$\begin{aligned} gr(u_{2k+1}) &= (-k; 0, -2k; 0)/\langle h\rangle \\ gr(v_{2k+1}) &= (-\tfrac{1}{2}; -\tfrac{1}{2}, -2k - \tfrac{1}{2}; 0)/\langle h\rangle \\ gr(u_{2k}) &= (-k + \tfrac{1}{2}; 0, -2k + 1; 0)/\langle h\rangle \\ gr(v_{2k}) &= (-2k + \tfrac{1}{2}; \tfrac{1}{2}, -2k + \tfrac{1}{2}; 0)/\langle h\rangle, \end{aligned}$$

and h and the gradings of the extra elements for each framing are given by the same formula as in the $\tau(K) \leq 0$ case.

To compute the gradings of all squares, we rely on the following lemma.

Lemma 3.2.5. *All elements of* CFK^- *on a fixed line of slope* 1 *on the* (U, A)*-lattice are converted to elements of the same grading in* $\widehat{CFD}$*. In fact, if* x *and* y *are the generators of* $\widehat{CFD}$ *in idempotent* ι_0 *corresponding to* x' *and* y' *in* CFK^-*, and if*

$$M(x') - M(y') = n = A(x') - A(y'),$$

then the relative $G/\langle h\rangle$ *grading of* x *and* y *is given by*

$$gr(y) = (\tfrac{n}{2}; 0, n; 0)gr(x).$$

Proof. The Lemma follows directly from [LOT]. The changes of bases in the proofs of [LOT, Theorems 11.27, 11.35, and 11.37] all respect gradings, so it suffices to verify the statement for generators in a pair of Heegaard diagrams $\mathcal{H}_K$ and $\mathcal{H}(n)$, as in [LOT, Figure 11.8]. There is only one $Spin^c$ structure for S^3, so $\pi_2(x', y')$ is nonempty. Take any domain D from x' to y', and add to it $-n_z(D)$ copies of the Heegaard surface for $\mathcal{H}_K$, to obtain a domain B' from x' to y' that misses the basepoint z. From

$$\begin{aligned} M(x') - M(y') &= \text{ind}(B') - 2n_w(B') = n \\ A(x') - A(y') &= n_z(B') - n_w(B') = n \end{aligned}$$

it follows that $n_w(B') = -n$ and $\text{ind}(B') = -n$. In the bordered diagram $\mathcal{H}(n)$, there is a corresponding domain B from x to y which crosses the boundary regions labeled by 2 and 3 with multiplicity $-n$ each. We make use of the more general grading theory for a moment, and work with the grading group $G'(4)$ [LOT, Section 10]. From [LOT, Equations 10.2, 10.19, and 10.27],

$$\begin{aligned} R(g'(B)) &= (-e(B) - n_x(B) - n_y(B); r_*(\partial^\partial(B))) \\ gr'(y) &= R(g'(B))gr'(x). \end{aligned}$$

Observe that

$$e(B) + n_x(B) + n_y(B) = e(B') + \frac{n}{2} + n_{x'}(B') + n_{y'}(B') = \mathrm{ind}(B') + \frac{n}{2} = -\frac{n}{2},$$

so $gr'(y) = (\frac{n}{2}; 0, n, n)gr'(x)$. Switching back to the grading group G, we get that the refined grading is $gr(y) = (\frac{n}{2}; 0, n; 0)gr(x)$. □

We say that a square of $\widehat{CFD}$ lies in level t if the upper right corner of the corresponding small square of $CFK^-(K)$ is on a line of slope 1 that is t units below the line through a_1, i.e., the upper right corner element has Maslov grading $-2\tau(K) - t$ as an element of $\widehat{HFK}$. Note that t can be negative, meaning that the square is above the a_1-line. By Theorem 3.2.1, there are $c_{t+\tau(K)}$ squares in level t. By Lemma 3.2.5, each square in level t has upper right corner x_1 in grading $(\frac{t}{2}; 0, t; 0)/\langle h\rangle$, and using (2.5.1) again, the grading of the whole square is given by

$$\begin{aligned} gr(x_1) &= (\tfrac{t}{2}; 0, t; 0)/\langle h\rangle \\ gr(x_2) &= (\tfrac{t}{2} - \tfrac{1}{2}; 0, t-1; 0)/\langle h\rangle \\ gr(x_3) &= (\tfrac{t}{2}; 0, t; 0)/\langle h\rangle \\ gr(x_4) &= (\tfrac{t}{2} + \tfrac{1}{2}; 0, t+1; 0)/\langle h\rangle \\ gr(y_1) &= (t - \tfrac{1}{2}; \tfrac{1}{2}, t - \tfrac{1}{2}; 0)/\langle h\rangle \\ gr(y_2) &= (-\tfrac{1}{2}; -\tfrac{1}{2}, t - \tfrac{1}{2}; 0)/\langle h\rangle \\ gr(y_3) &= (t + \tfrac{1}{2}; \tfrac{1}{2}, t + \tfrac{1}{2}; 0)/\langle h\rangle \\ gr(y_4) &= (-\tfrac{1}{2}; -\tfrac{1}{2}, t + \tfrac{1}{2}; 0)/\langle h\rangle. \end{aligned}$$

3.3 CFA^- of the $(p,1)$-cable in the solid torus

Figure 5 shows a bordered Heegaard diagram for the $(p,1)$-cable in the solid torus.

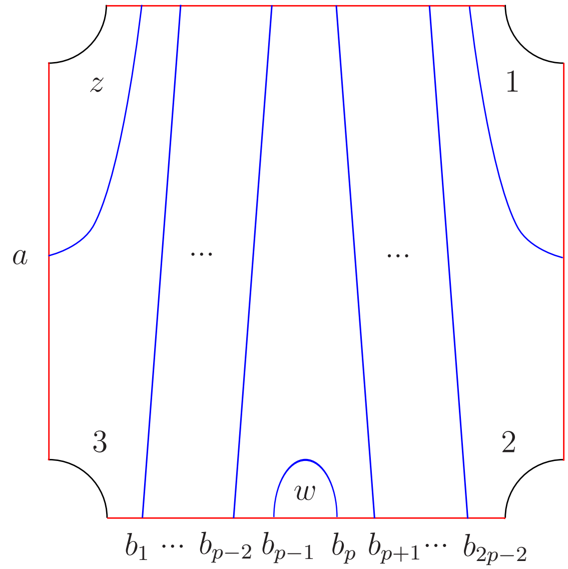


Figure 5: The $(p,1)$-cable in the solid torus

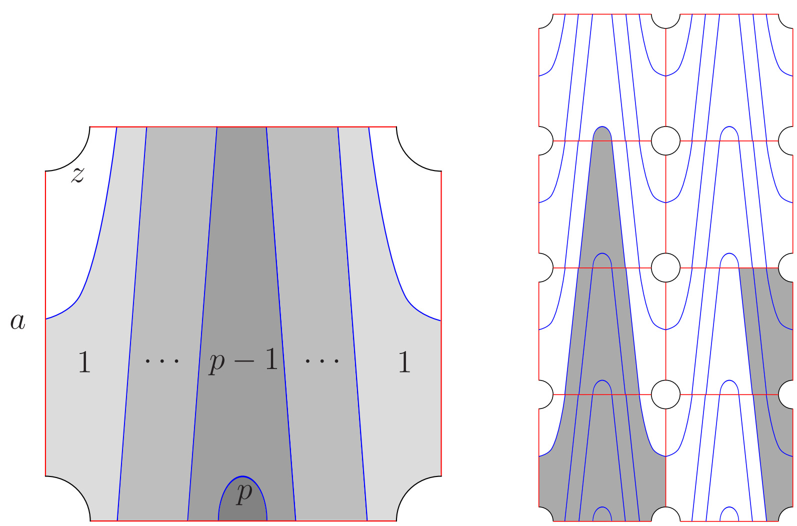


Figure 6: Left: The generator B_a for positive periodic domains. Right: The lift of the Heegaard diagram to the universal cover of the punctured torus in the case of the $(4,1)$-cable, along with B_a and the domain for $m_4(b_6, \rho_2, \rho_{12}, \rho_1)$.

The module $CFA^-(Y_{p,1})$ is generated over $\mathbb{F}_2[U]$ by $a, b_1, \ldots, b_{2p-2}$. The multiplication maps count certain J-holomorphic curves in $\Sigma\times[0,1]\times\mathbb{R}$, whose relative homology class has index 1. For more detail on the moduli spaces, indices, and expected dimension for bordered diagrams, see [LOT, Section 5]. Since the Heegaard surface Σ is a punctured torus, we can instead count embedded disks of index 1 in its universal cover $\mathbb{C}\setminus(\mathbb{Z}\times\mathbb{Z})$ connecting lifts of generators and missing the preimage of z, modulo vertical and horizontal translations of the lattice. The positive periodic domains are generated by the domain B_a in Figure 6. It is straightforward to enumerate the finitely many embedded disks whose boundary does not project to all of β. Any other disk that contributes to the multiplication maps is a sum of one of these and a positive number of copies of B_a. Figure 6 shows the periodic domain, and the only domain that contributes to $m_4(b_6,\rho_2,\rho_{12},\rho_1)$ on a portion of the universal cover for the $(4,1)$-cable.

We conclude that the only multiplication maps are

$$m_1(b_k)=U^{p-k}\cdot b_{2p-k-1}\qquad\text{for } 1\le k\le p-1$$

$$m_{3+i}(b_k,\rho_2,\underbrace{\rho_{12},\dots,\rho_{12}}_{i},\rho_1)=U^{i+1}\cdot b_{k+i+1}\qquad\text{for }\begin{array}{l}1\le k\le p-2,\\ 0\le i\le p-k-2\end{array}$$

$$m_{3+i}(b_k,\rho_2,\underbrace{\rho_{12},\dots,\rho_{12}}_{i},\rho_1)=b_{k-i-1}\qquad\text{for }\begin{array}{l}p+1\le k\le 2p-2,\\ 0\le i\le k-1-p\end{array}$$

$$m_{2+i}(a,\underbrace{\rho_{12},\dots,\rho_{12}}_{i},\rho_1)=b_{2p-i-2}\qquad\text{for } 0\le i\le p-2$$

$$m_{4+i+j}(a,\rho_3,\underbrace{\rho_{23},\dots,\rho_{23}}_{j},\rho_2,\underbrace{\rho_{12},\dots,\rho_{12}}_{i},\rho_1)=U^{pj+i+1}\cdot b_{i+1}\qquad\text{for }\begin{array}{l}0\le i\le p-2,\\ 0\le j\end{array}$$

$$m_{3+j}(a,\rho_3,\underbrace{\rho_{23},\dots,\rho_{23}}_{j},\rho_2)=U^{p(j+1)}\cdot a\qquad\text{for } 0\le j$$

Setting $U=0$ yields $\widehat{CFA}(Y_{p,1})$ - the generators are $a,b_1,\dots,b_{2p-2}$, and the multiplication maps are given by the third and fourth map above.

Applying (2.5.2) to the last multiplication map for $j=0$, we find an indeterminacy of $(-\frac{1}{2};0,1;p)$ for the grading of a, i.e. the element $g=(-\frac{1}{2};0,1;p)$ is in the image of $\pi_2(a,a)$

in G. Since g is primitive in G, it is a generator for that image.

Next, we normalize the $\langle g\rangle\backslash G$ grading on the generators by setting

$$gr(a) := \langle g\rangle\backslash(0;0,0;0)$$

From the fourth multiplication map we see that

$$gr(b_{2p-i-2}) = \langle g\rangle\backslash(-\tfrac{1}{2}; i+\tfrac{1}{2}, -\tfrac{1}{2}; 0) \quad \text{for } 0 \le i \le p-2,$$

and from the first map get

$$gr(b_i) = \langle g\rangle\backslash(\tfrac{1}{2}; i-\tfrac{1}{2}, -\tfrac{1}{2}; i-p) \quad \text{for } 1 \le i \le p-1.$$

We remind the reader that an alternate choice of multiplication maps for this computation may at first seem to provide different gradings, but one can verify that those grading representatives lie in the same coset as the ones provided above.

Note: We can construct a Heegaard diagram similar to the one in Figure 5 to study any cable $K_{p,q}$. Given any relatively prime integers p and q, with $p > 1$, let i be the unique integer $1 \le i < p$, $i = q \bmod p$. One can construct a genus 1 bordered diagram for the (p,i)-cable as follows. Theorem 3.5 of [Ord06] provides an algorithm for constructing genus 1 Heegaard diagrams for $(1,1)$ knots. In particular, the construction for the diagram of the torus knot $T_{p,i}$ can be modified to provide a bordered diagram for the (p,i)-cable in the solid torus. We first find the Heegaard normal form of the attaching curve β (Figure 3.11 of [Ord06] demonstrates this process for $T_{5,3}$) so that the corners of the fundamental domain for the flat torus coincide with the basepoint w. Then we draw two α-circles through the basepoint w, one identified with the two vertical segments of the boundary of the domain, hence isotopic to the meridian of the torus, the other identified with the two horizontal segments of the boundary, hence isotopic to the standard longitude and to β, and remove a neighborhood of w. To be consistent with our conventions, we rename z to w, and place a new basepoint z in the bottom rightmost region of the diagram. Since the diagram has genus 1, it is straightforward to compute the multiplication maps for $CFA^-(p,i)$. By tensoring with $\widehat{CFD}(K,(q-i)/p)$ as in Section 3.4, we can compute $\widehat{HFK}(K_{p,q})$ and $\tau(K_{p,q})$.

3.4 The tensor product $CFA^- \boxtimes \widehat{CFD}$

We now compute the gradings of the tensor product in (N, A') notation, and list the differentials. Then we find the shifting constant c.

Since $\widehat{CFD}(K, n)$ splits as a direct sum of squares and a staircase, its tensor product with $CFA^-(Y_{p,1})$ splits as a direct sum of the tensor products of $CFA^-(Y_{p,1})$ with each square and the staircase too. For this reason, we refer to the corresponding direct summands of the tensor product as squares and a staircase too.

Each of the $c_{t+\tau(K)}$ squares in level t is graded by

$$
\begin{aligned}
gr(ax_1) &= (t, -pt) \\
gr(ax_2) &= (-1+t, p-pt) \\
gr(ax_3) &= (t, -pt) \\
gr(ax_4) &= (1+t, -p-pt) \\
gr(b_k y_1) &= (-2k + 2k\tau - kn - k^2 n + t + 2kt, k + knp - pt) \\
gr(b_{2p-i-2} y_1) &= (-3 - 2i + 2\tau + 2i\tau - 2n - 3in - i^2 n + 3t + 2it, p + np + inp - pt) \\
gr(b_k y_2) &= (1 - 2k - 2\tau + 2k\tau + kn - k^2 n - t + 2kt, k - np + knp - pt) \\
gr(b_{2p-i-2} y_2) &= (-2 - 2i + 2i\tau - in - i^2 n + t + 2it, p + inp - pt) \\
gr(b_k y_3) &= (1 + 2k\tau - kn - k^2 n + t + 2kt, k - p + knp - pt) \\
gr(b_{2p-i-2} y_3) &= (-2l + 2i\tau - 2n - 3in - i^2 n + 3t + 2it, np + inp - pt) \\
gr(b_k y_4) &= (2l + 2k\tau + kn - k^2 n - t + 2kt, k - p - np + knp - pt) \\
gr(b_{2p-i-2} y_4) &= (-1 + 2i\tau - in - i^2 n + t + 2it, inp - pt),
\end{aligned}
$$

where $1 \leq k \leq p-1$ and $0 \leq i \leq p-2$.

Matching up the coefficient maps of $\widehat{CFD}(K, n)$ with the multiplication maps in the beginning of this section, we see that the non-trivial differentials on each square are

$$\partial(ax_1) = b_{2p-2}y_4 + Ub_1y_2 + U^p ax_2 \qquad \partial(b_iy_1) = U^{p-i}b_{2p-i-1}y_1 + Ub_{i+1}y_2$$
$$\partial(ax_2) = b_{2p-2}y_2 \qquad \partial(b_{p-1}y_1) = Ub_py_1$$
$$\partial(ax_4) = U^p ax_3 \qquad \partial(b_{2p-i-1}y_1) = b_{2p-i-2}y_2$$
$$\partial(b_ky_j) = U^{p-k}b_{2p-k-1}y_j$$

where $1 \leq i \leq p-2$, $j = 2,3,4$ and $1 \leq k \leq p-1$.

The gradings and differentials on the staircase, including the diagonal string connecting a_1 and $a_{2|\tau|+1}$, depend on $\tau(K)$:

<u>Case 1:</u> If $\tau(K) \leq 0$, the gradings on the staircase are

$$gr(au_{2t+1}) = (2t, -2pt)$$
$$gr(au_{2t}) = (-1+2t, p-2pt)$$
$$gr(b_kv_{2t+1}) = (-2\tau + 2k\tau + kn - k^2n - 2t + 4kt, k + knp - np - p - 2pt)$$
$$gr(b_kv_{2t}) = (-2k + 2k\tau - kn - k^2n + 2t + 4kt, k + knp - 2pt)$$
$$gr(b_{2p-i-2}v_{2t+1}) = (-1 + 2i\tau - in - i^2n + 2t + 4it, inp - 2pt)$$
$$gr(b_{2p-i-2}v_{2t}) = (-3 - 2i + 2\tau + 2i\tau - 2n - 3in - i^2n + 6t + 4it, p + np + inp - 2pt)$$

For $m = 2\tau(K) - n > 0$,

$$gr(b_{2p-i-2}\mu_{j+1}) = (-1 + 2j + 2ij - 2\tau - 2i\tau - in - i^2n, -jp + 2p\tau + inp)$$
$$gr(b_k\mu_{j+1}) = (2jk - 2k\tau + kn - k^2n, k - p - jp + 2p\tau - np + knp),$$

and for $m < 0$,

$$gr(b_{2p-i-2}\mu_{j+1}) = (-2 - 2i - 2j - 2ij - 2\tau - 2i\tau - in - i^2n, p + jp + 2p\tau + inp)$$
$$gr(b_k\mu_{j+1}) = (1 - 2k - 2jk - 2k\tau + kn - k^2n, k + jp + 2p\tau - np + knp).$$

When $\tau(K) = 0$ and $m \leq 0$, we also have

$$\begin{aligned}
gr(b_k\epsilon_1) &= (kn - k^2n, k - p - np + knp)\\
gr(b_k\epsilon_2) &= (1 + kn - k^2n, k - p - np + knp)\\
gr(b_{2p-i-2}\epsilon_1) &= (-1 - in - i^2n, inp)\\
gr(b_{2p-i-2}\epsilon_2) &= (-in - i^2n, inp).
\end{aligned}$$

<u>Case 2:</u> If $\tau(K) > 0$, the gradings on the staircase are

$$\begin{aligned}
gr(au_{2t+1}) &= (-2t, 2pt)\\
gr(au_{2t}) &= (1 - 2t, -p + 2pt)\\
gr(b_kv_{2t+1}) &= (1 - 2k - 2\tau + 2k\tau + kn - k^2n + 2t - 4kt, k - np + knp + 2pt)\\
gr(b_kv_{2t}) &= (1 + 2k\tau - kn - k^2n - 2t - 4kt, k - p + knp + 2pt)\\
gr(b_{2p-i-2}v_{2t+1}) &= (-2 - 2i + 2i\tau - in - i^2n - 2t - 4it, p + inp + 2pt)\\
gr(b_{2p-i-2}v_{2t}) &= (2\tau + 2i\tau - 2n - 3in - i^2n - 6t - 4it, np + inp + 2pt)
\end{aligned}$$

The gradings on all $b_i\mu_j$ are given by the same formula as in Case 1.

Next, we list the non-trivial differentials. When $\tau < 0$ and $m > 0$, the non-trivial differentials are

$$\begin{aligned}
\partial(au_1) &= b_{2p-2}v_1 & \partial(b_kv_s) &= U^{p-k}b_{2p-k-1}v_s\\
\partial(au_{2t+1}) &= b_{2p-2}v_{2t+1} + U^pau_{2t} & \partial(b_k\mu_j) &= U^{p-k}b_{2p-k-1}\mu_j\\
\partial(au_{2|\tau|+1}) &= b_{2p-2}\mu_1 + U^pau_{2|\tau|}
\end{aligned}$$

When $\tau < 0$ and $m = 0$ the non-trivial differentials are

$$\begin{aligned}
\partial(au_1) &= b_{2p-2}v_1 & \partial(au_{2|\tau|+1}) &= b_{2p-3}v_1 + U^pau_{2|\tau|}\\
\partial(au_{2t+1}) &= b_{2p-2}v_{2t+1} + U^pau_{2t} & \partial(b_kv_s) &= U^{p-k}b_{2p-k-1}v_s
\end{aligned}$$

When $\tau < 0$ and $m < 0$ the non-trivial differentials are

$$\partial(au_1) = b_{2p-2}v_1 \qquad \partial(b_k\mu_j) = U^{p-k}b_{2p-k-1}\mu_j$$
$$\partial(au_{2t+1}) = b_{2p-2}v_{2t+1} + U^p au_{2t} \qquad \partial(b_i\mu_{|m|}) = Ub_{i+1}\mu_{|m|} + U^{p-i}b_{2p-i-1}\mu_{|m|}$$
$$\partial(au_{2|\tau|+1}) = U^p au_{2|\tau|} \qquad \partial(b_{p-1}\mu_{|m|}) = Ub_p\mu_{|m|}$$
$$\partial(b_kv_s) = U^{p-k}b_{2p-k-1}v_s \qquad \partial(b_{2p-i-1}\mu_{|m|}) = b_{2p-i-2}\mu_{|m|}$$

When $\tau = 0$ and $m > 0$, the non-trivial differentials are

$$\partial(au_1) = b_{2p-2}\mu_1 \qquad \partial(b_k\mu_j) = U^{p-k}b_{2p-k-1}\mu_j$$

When $\tau = 0$ and $m = 0$ the non-trivial differentials are

$$\partial(au_1) = b_{2p-2}\epsilon_1 \qquad \partial(b_{p-1}\epsilon_2) = b_{p-1}\epsilon_1 + Ub_p\epsilon_2$$
$$\partial(b_k\epsilon_1) = U^{p-k}b_{2p-k-1}\epsilon_1 \qquad \partial(b_p\epsilon_2) = b_p\epsilon_1$$
$$\partial(b_i\epsilon_2) = b_i\epsilon_1 + Ub_{i+1}\epsilon_1 + U^{p-i}b_{2p-i-1}\epsilon_2 \qquad \partial(b_{2p-i-1}\epsilon_2) = b_{2p-i-1}\epsilon_1 + b_{2p-i-2}\epsilon_1$$

When $\tau = 0$ and $m < 0$ the non-trivial differentials are

$$\partial(au_1) = b_{2p-2}\epsilon_1 \qquad \partial(b_k\mu_j) = U^{p-k}b_{2p-k-1}\mu_j$$
$$\partial(b_k\epsilon_1) = U^{p-k}b_{2p-k-1}\epsilon_1 \qquad \partial(b_i\mu_{|m|}) = Ub_{i+1}\epsilon_1 + U^{p-i}b_{2p-i-1}\mu_{|m|}$$
$$\partial(b_k\epsilon_2) = b_k\epsilon_1 + U^{p-k}b_{2p-k-1}\epsilon_2 \qquad \partial(b_{p-1}\mu_{|m|}) = Ub_p\mu_{|m|}$$
$$\partial(b_{2p-k-1}\epsilon_2) = b_{2p-k-1}\epsilon_1 \qquad \partial(b_{2p-i-1}\mu_{|m|}) = b_{2p-i-2}\epsilon_1$$

When $\tau > 0$ and $m > 0$, the non-trivial differentials are

$$\partial(au_{2t}) = b_{2p-2}v_{2t-1} + U^p au_{2t+1} \qquad \partial(b_{2p-i-1}v_{2\tau}) = b_{2p-i-2}\mu_1$$
$$\partial(au_{2\tau}) = b_{2p-2}v_{2\tau-1} + Ub_1\mu_1 + U^p au_{2\tau+1} \qquad \partial(b_{p-1}v_{2\tau}) = Ub_pv_{2\tau}$$
$$\partial(au_{2\tau+1}) = b_{2p-2}\mu_1 \qquad \partial(b_kv_s) = U^{p-k}b_{2p-k-1}v_s$$
$$\partial(b_iv_{2\tau}) = Ub_{i+1}\mu_1 + U^{p-i}b_{2p-i-1}v_{2\tau} \qquad \partial(b_k\mu_j) = U^{p-k}b_{2p-k-1}\mu_j$$

When $\tau > 0$ and $m = 0$ the non-trivial differentials are

$$\partial(au_{2t}) = b_{2p-2}v_{2t-1} + U^p au_{2t+1} \qquad \partial(b_kv_s) = U^{p-k}b_{2p-k-1}v_s$$

When $\tau > 0$ and $m < 0$ the non-trivial differentials are

$$\partial(au_{2t}) = b_{2p-2}v_{2t-1} + U^p au_{2t+1} \qquad \partial(b_k\mu_j) = U^{p-k}b_{2p-k-1}\mu_j$$
$$\partial(b_kv_s) = U^{p-k}b_{2p-k-1}v_s$$

The indices vary as follows

$$1 \le s \le \begin{cases} 2|\tau|-1 & \text{if } m>0, \tau>0 \\ 2|\tau| & \text{otherwise} \end{cases} \qquad 1 \le j \le \begin{cases} |m|-1 & \text{if } m<0, \tau\le 0 \\ |m| & \text{otherwise} \end{cases}$$
$$1 \le t \le \begin{cases} |\tau| & \text{if } m<0, \tau>0 \\ |\tau|-1 & \text{otherwise} \end{cases} \qquad \begin{array}{l} 1\le i \le p-2 \\ 1 \le k \le p-1 \end{array}$$

Recall that the absolute N grading is obtained by requiring that the homology

$$H_*\left(CFK^-(K_{p,pn+1})/U=1\right) \cong \mathbb{F}_2$$

lives in N grading 0, see the discussion in [LOT, Section 11.3], specifically Equation 11.17 and the paragraph preceding it. Set $U = 1$ above.

When $\tau > 0$, au_1 splits as a direct summand of the chain complex, so it represents $H_*\left(CFK^-(K_{p,pn+1})/U=1\right) \cong \mathbb{F}_2$, implying that $N(au_1) = 0$.

When $\tau < 0$, the subcomplex D generated by au_1, b_1v_1, and $b_{2p-2}v_1$ splits, and

$$H_*(D) = \frac{\ker(D)}{\operatorname{im}(D)} = \langle au_1 + b_1v_1\rangle,$$

so $N(au_1 + b_1v_1) = 0$.

When $\tau = 0$ and $m > 0$, the subcomplex D generated by $au_1, b_1\mu_1$, and $b_{2p-2}\mu_1$ splits, and

$$H_*(D) = \frac{\ker(D)}{\operatorname{im}(D)} = \langle au_1 + b_1\mu_1\rangle,$$

so $N(au_1 + b_1\mu_1) = 0$.

When $\tau = 0$ and $m = 0$, The image and kernel of the differential are

$$\begin{aligned} \operatorname{im} =& \langle b_1\epsilon_1 + b_2\epsilon_1 + b_{2p-2}\epsilon_2, b_2\epsilon_1 + b_3\epsilon_1 + b_{2p-3}\epsilon_2, \ldots, b_{p-2}\epsilon_1 + b_{p-1}\epsilon_1 + b_{p+1}\epsilon_2, \\ & b_{p-1}\epsilon_1 + b_p\epsilon_2, b_p\epsilon_1, b_{p+1}\epsilon_1, \ldots, b_{2p-2}\epsilon_1\rangle \\ \ker =& \operatorname{im} \oplus \langle au_1 + b_{2p-2}\epsilon_1\rangle, \end{aligned}$$

so $au_1 + b_{2p-2}\epsilon_1$ survives in homology, implying that $N(au_1 + b_{2p-2}\epsilon_1) = 0$.

When $\tau = 0$ and $m < 0$, the subcomplex D generated by au_1, $b_1\epsilon_1$, $b_1\epsilon_2$, $b_{2p-2}\epsilon_1$, and $b_{2p-2}\epsilon_2$ splits, and

$$H_*(D) = \frac{\ker(D)}{\operatorname{im}(D)} = \frac{\langle au_1 + b_1\epsilon_1, b_1\epsilon_1 + b_{2p-2}\epsilon_2\rangle}{\langle b_1\epsilon_1 + b_{2p-2}\epsilon_2\rangle},$$

so $N(au_1 + b_1\epsilon_1) = 0$.

In each case $N(au_1) = 0$, so the first component of the grading provided earlier in this section is, in fact, the absolute N grading.

Next, we find the absolute Alexander grading, by requiring that the Euler characteristic of $\widehat{CFK}$ is the symmetrized Alexander polynomial. The formula

$$\Delta_{K_{p,pn+1}}(t) = \Delta_K(t^p)\Delta_{T_{p,pn+1}}(t)$$

implies that if the degree of the symmetrized $\Delta_K(t)$ is d, then the degree of the symmetrized $\Delta_{K_{p,pn+1}}(t)$ is

$$pd + \frac{(|p|-1)(|pn+1|-1)}{2} = \begin{cases} pd + \frac{np(p-1)}{2} & \text{if } n \geq 0 \\ pd - \frac{np(p-1)}{2} - p + 1 & \text{otherwise,} \end{cases}$$

so we look for the highest relative Alexander grading in which generators survive when taking Euler characteristic, and shift to make it equal this degree.

If $d > |\tau|$, then this highest grading is realized by the following generators coming from all squares in level $d - \tau - 1$:

If $n > 0$, $\quad b_p y_1$

If $n = 0$, $\quad ax_2, b_p y_1, b_{p+1}y_1, \ldots, b_{2p-2}y_1, b_p y_2, b_{p+1}y_2, \ldots, b_{2p-2}y_2$

If $n < 0$, $\quad b_1 y_2$.

In each case, the contribution of each square to $\chi(\widehat{CFK})$ is rank 1 and in Maslov grading $d - \tau \bmod 2$.

If $d < |\tau|$ or if K is the unknot, then the highest grading is realized by staircase generators:

If $\tau > 0, n > 0,$ $\quad b_p v_{2\tau}$

If $\tau > 0, n = 0,$ $\quad au_{2\tau+1}, b_p v_{2\tau}, b_{p+1} v_{2\tau}, \ldots, b_{2p-2} v_{2\tau}, b_p \mu_1, b_{p+1}\mu_1, \ldots, b_{2p-2}\mu_1$

If $\tau \geq 0, n < 0,$ $\quad b_1 \mu_1$

If $\tau \leq 0, n > 0,$ $\quad b_p \mu_{|m|}$

If $\tau = 0, n = 0,$ $\quad au_1$

If $\tau < 0, n = 0,$ $\quad au_1, b_p \mu_{|m|}, b_{p+1}\mu_{|m|}, \ldots, b_{2p-2}\mu_{|m|}, b_p v_1, b_{p+1} v_1, \ldots, b_{2p-2} v_1$

If $\tau < 0, n < 0,$ $\quad b_1 v_1.$

In each case, the contribution of the staircase to $\chi(\widehat{CFK})$ is rank 1 and in Maslov grading 0 mod 2.

If $d = |\tau|$, then the highest grading is realized by the listed generators from squares in level $d - \tau - 1$ combined with the listed staircase generators. The rank 1 contribution to $\chi(\widehat{CFK})$ has Maslov grading 0 mod 2 both for the staircase, and for each square, so there are no further cancelations.

In each of these cases, we need to shift by the constant $c = -p\tau - np(p-1)/2$. Together with the fact that $N = M - 2A$, we now have a complete description of $CFK^-(K_{p,pn+1})$. Setting $U = 0$ in the above differentials gives $\widehat{CFK}(K_{p,pn+1})$.

For completeness, we include a list of the generators of $\widehat{HFK}(K_{p,pn+1})$.

For each square direct summand, all generators survive in homology except ax_1, ax_2, $b_{2p-2}y_2$, $b_{2p-2}y_4$, $b_i y_1$, $b_{i-1}y_2$, for all $p+1 \leq i \leq 2p-2$. The staircase summand depends on τ and the framing:

If $\tau > 0$, all generators survive except au_{2t} and $b_{2p-2}v_{2t-1}$ for any framing, as well as $au_{2\tau+1}$, $b_i v_{2\tau}$, and $b_{i-1}\mu_1$ if $m > 0$. Here $p+1 \leq i \leq 2p-2$ and $1 \leq t \leq \tau$.

If $\tau < 0$, all generators survive except au_{2t+1}, $b_{2p-2}v_{2t+1}$, and $au_{2|\tau|+1}$ for any framing, as well as $b_{2p-2}\mu_1$ if $m > 0$, $b_{2p-2}v_{2|\tau|+1}$ if $m < 0$, and $b_{2p-3}v_1$ if $m = 0$. Here $1 \leq t \leq |\tau| - 1$.

If $\tau = 0$ and $m > 0$, all generators survive except au_1 and $b_{2p-2}\mu_1$.

If $\tau = 0$ and $m = 0$, the homology has rank one and is represented by $au_1 + \sum_{k=p}^{2p-2} b_k \epsilon_2$.

If $\tau = 0$ and $m < 0$, the homology is generated by $au_1 + b_{2p-2}\epsilon_2$, , $b_i\mu_{|m|}$, $b_{2p-2-i}\epsilon_2 + b_{2p-1-i}\mu_{|m|}$, $b_k\mu_j$, for $1 \leq i \leq p$, $1 \leq k \leq 2p-2$, $1 \leq j \leq |m|-1$.

This completes the description of $\widehat{HFK}(K_{p,pn+1})$.

3.5 τ of the cable

In [OS03c], Ozsváth and Szabó define a concordance invariant $\tau(K)$ arising from the Alexander filtration on $\widehat{CF}(S^3)$. Alternatively, τ can be defined in terms of the associated graded object $HFK^-(K)$ by

$$\tau(K) = -\max\{s | \forall d \geq 0, U^d HFK^-(K,s) \neq 0\}.$$

(see Lemma A.2 of [OST08]).

We do not need to fully compute the homology of $CFK^-(K_{p,pn+1})$. It is enough to observe that $U^p HFK^-$ vanishes for each direct summand of the tensor product coming from a square in the D module. Thus, τ only depends on the staircase summand, which agrees with CFK^- of the $(2, -2\tau+1)$-torus knot, and hence

$$\tau(K_{p,pn+1}) = \tau((T_{2,-2\tau+1})_{p,pn+1}).$$

Then one can work out the computation fully using the complex $CFK^-(K_{p,pn+1})$ provided in Section 3.4. For example, when $\tau(K) < 0$ and $m > 0$, the staircase summand splits further, and all direct summands vanish in $U^{2p}HFK^-$, except the one generated by au_1, b_1v_1, and $b_{2p-2}v_1$. Its homology is generated by $U^{p-1}au_1 + b_1v_1$, and survives in all powers of U. The Alexander gradings are

$$A(U^{p-1}au_1) = A(au_1) - 2(p-1) = -p\tau - \frac{np(p-1)}{2} - 2p + 2,$$
$$A(b_1v_1) = -p\tau - \frac{np(p-1)}{2} - p + 1,$$

so the Alexander filtration level of $U^{p-1}au_1 + b_1v_1$ is $-p\tau - \frac{np(p-1)}{2} - p + 1$, hence

$$\tau(K_{p,pn+1}) = -(-p\tau - \frac{np(p-1)}{2} - p + 1).$$

The computation in the remaining cases goes the same way. The only generator in HFK^- that survives in all U-powers is $U^{p-1}au_1 + b_1v_1$ if $\tau < 0$, $U^{p-1}au_1 + b_1\mu_1$ if $\tau = 0$ and $n < 0$, in which cases

$$\tau(K_{p,pn+1}) = p\tau(K) + \frac{np(p-1)}{2} + p - 1,$$

and au_1 if $\tau = 0$ and $n \geq 0$, or if $\tau > 0$, in which cases

$$\tau(K_{p,pn+1}) = p\tau(K) + \frac{np(p-1)}{2}.$$

This completes the proof of Theorem 1. Observe that our work agrees with the results in [Hed08], where Hedden computes τ of $(p, pn+1)$-cables for sufficiently large $|n|$.

3.6 Proof of corollaries

We prove Corollaries 2 and 3.

Proof of Corollary 2. Since $K_{1,q} = K$ for all q, the result for $p = 1$ is a tautology.

Fixing K and $p > 1$, Van Cott [Cot08] defines the function

$$h(q) = \tau(K_{p,q}) - \frac{p-1}{2}q$$

with domain all integers relatively prime to p, and proves that h is non-increasing.

By Theorem 1, $h(pn+1)$ is constant as a function of n for $\tau = 0$ and $n \geq 0$, or $\tau > 0$, and is given by

$$h(pn+1) = p\tau(K) - \frac{p-1}{2},$$

implying that for $\tau = 0$ and any $q > 0$, or $\tau > 0$ and any q,

$$h(q) = p\tau(K) - \frac{p-1}{2}.$$

We see that

$$\begin{aligned}\tau(K_{p,q}) &= h(q) + \frac{p-1}{2}q \\ &= p\tau(K) + \frac{(p-1)(q-1)}{2}.\end{aligned}$$

It is shown that τ changes sign under reflection[OS03c]. Thus, since $\overline{K_{p,q}} = \overline{K}_{p,-q}$, it follows that for $\tau(K) = 0$ and $q < 0$, or $\tau(K) < 0$ we have

$$\begin{aligned}\tau(K_{p,q}) &= -\tau(\overline{K}_{p,-q})\\ &= -\left(p\tau(\overline{K}) + \frac{(p-1)(-q-1)}{2}\right)\\ &= p\tau(K) + \frac{(p-1)(q+1)}{2}.\end{aligned}$$

□

Proof of Corollary 3. Since $\tau(K) = g_4(K) \geq 0$ and $q > 0$, Corollary 2 implies that

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}.$$

Substituting $g_4(K)$ for $\tau(K)$ we see that

$$\tau(K_{p,q}) = pg_4(K) + \frac{(p-1)(q-1)}{2}.$$

On the one hand, we know that

$$g_4(K_{p,q}) \leq pg_4(K) + \frac{(p-1)(q-1)}{2},$$

since we can construct a surface for $K_{p,q}$ in the four-ball by connecting p parallel copies of the surface for K via $(p-1)q$ twisted bands. Thus $g_4(K_{p,q}) \leq \tau(K_{p,q})$. On the other hand, $g_4(K_{p,q}) \geq \tau(K_{p,q})$ for any knot, implying the desired result. □

3.7 An example

We programmed the results of this chapter into *Mathematica* [WR] to be able to compute specific examples. The program takes $\Delta_K(t), \tau(K), p$, and n as input, and outputs $\widehat{HFK}(K_{p,pn+1})$. We use the program to compute $\widehat{HFK}$ of the $(5,16)$-cable of the knot $11n50$. We include the relevant data for $11n50$ for the reader's convenience:

$$\begin{aligned}&\Delta_{11n50}(t) = 2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2,\\ &\tau(11n50) = 0.\end{aligned}$$

Note that here $p = 5$ and $n = 3$.

We describe $\widehat{HFK}$ of the cable as a polynomial, where the coefficient of $x^A y^M$ is the rank of $\widehat{HFK}$ in Alexander grading A and Maslov grading M:

$$\begin{aligned}
&2x^{-40}y^{-78} + 2x^{40}y^{2} + 2x^{-39}y^{-77} + 2x^{39}y + 4x^{-35}y^{-69} + 4x^{35}y + 4x^{-34}y^{-68} + 4x^{34} \\
&+ 5x^{-30}y^{-60} + 5x^{30} + 5x^{-29}y^{-59} + 5x^{29}y^{-1} + x^{-25}y^{-52} + 2x^{-25}y^{-51} + x^{25}y^{-2} \\
&+ 2x^{25}y^{-1} + x^{-24}y^{-51} + 4x^{-24}y^{-50} + x^{24}y^{-3} + 4x^{24}y^{-2} + 2x^{-23}y^{-49} + 2x^{23}y^{-3} \\
&+ 3x^{-20}y^{-44} + 2x^{-20}y^{-43} + 3x^{20}y^{-4} + 2x^{20}y^{-3} + 5x^{-19}y^{-43} + 5x^{19}y^{-5} + 4x^{-18}y^{-42} \\
&+ 4x^{18}y^{-6} + 2x^{-15}y^{-37} + 3x^{-15}y^{-36} + 2x^{15}y^{-7} + 3x^{15}y^{-6} + 4x^{-14}y^{-36} + 4x^{14}y^{-8} \\
&+ 5x^{-13}y^{-35} + 5x^{13}y^{-9} + 3x^{-10}y^{-30} + 2x^{-10}y^{-29} + 3x^{10}y^{-10} + 2x^{10}y^{-9} + 2x^{-9}y^{-29} \\
&+ 2x^{9}y^{-11} + x^{-8}y^{-29} + 4x^{-8}y^{-28} + x^{8}y^{-13} + 4x^{8}y^{-12} + 2x^{-7}y^{-27} + 2x^{7}y^{-13} \\
&+ 3x^{-5}y^{-24} + 2x^{-5}y^{-23} + 3x^{5}y^{-14} + 2x^{5}y^{-13} + 5x^{-3}y^{-23} + 5x^{3}y^{-17} + 4x^{-2}y^{-22} \\
&+ 4x^{2}y^{-18} + 2y^{-19} + 3y^{-18}
\end{aligned}$$

We also plot the result on the (A, M)-axis (without marking the rank at each coordinate):

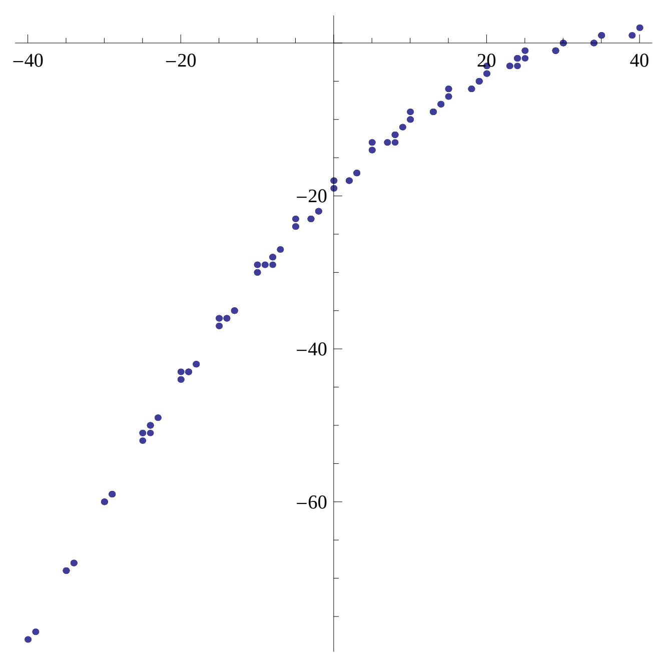


Figure 7: $\widehat{HFK}$ of the $(5, 16)$-cable of the knot $11n50$

Chapter 4

The decategorification of bordered Floer homology

In this chapter, we describe the Grothendieck group $K_0(\mathcal{A}(\mathcal{Z}))$ of $\mathbb{Z}/2$-graded differential modules over the algebra $\mathcal{A}(\mathcal{Z})$. When there is an additional grading, we study the image of graded dg modules in the larger group $K_0(\mathcal{A}(\mathcal{Z})_{gr})$ as a generalization of the Alexander polynomial of a knot in S^3 to knots in 3-manifolds with boundary. As one application, we show that the pairing theorem for bordered Floer homology categorifies the classical Alexander polynomial formula for satellites. We also show that the Euler characteristic of bordered Floer homology agrees with that of sutured Floer homology.

4.1 A $\mathbb{Z}/2$ grading

Instead of working with the grading groups $G'(4k)$ and $G(\mathcal{Z})$, we will define a homological $\mathbb{Z}/2$ grading m on the algebra $\mathcal{A}(\mathcal{Z})$, and study the Grothendieck group of $\mathbb{Z}/2$-graded modules over $\mathcal{A}(\mathcal{Z})$. The reason we prefer this is because we would like to easily relate the Euler characteristic of *CFA* or *CFD* of bordered 3-manifolds in $K_0(\mathcal{A}(\mathcal{Z}))$ to other familiar 3-manifold invariants.

Let $\mathcal{Z} = (Z, \mathbf{a}, M, z)$ be a a pointed matched circle with associated surface $F = F(\mathcal{Z})$ of genus k. For each $i \in [2k]$, let ρ_i be the Reeb chord connecting the two points in $M^{-1}(i)$. Now choose grading refinement data ψ, i.e. a base idempotent $I(\mathbf{s})$ and an element $\psi(\mathbf{t}) \in G'(4k)$ for each $\mathbf{t}$, as in [LOT, Section 10.5], and get a grading gr on $\mathcal{A}(\mathcal{Z})$ with values in $G(\mathcal{Z})$. Let $g_i = \mathrm{gr}(a(\rho_i))$ for $i \in [2k]$. The set $\{[a(\rho_i)] | i \in [2k]\}$ generates $H_1(F;\mathbb{Z})$, and λ generates the center of $G(\mathcal{Z})$, so $\{\lambda, g_1, \ldots, g_{2k}\}$ is a generating set for $G(\mathcal{Z})$. Since the commutators of $G(\mathcal{Z})$ are precisely the even powers of λ, we see that the abelianization of $G(\mathcal{Z})$ is

$$G(\mathcal{Z})^{Ab} = \mathbb{Z}/2 \times \mathbb{Z}^{2k}.$$

Thus, given any abelian group H, any map from $\{\lambda, g_1, \ldots, g_{2k}\}$ to $\mathbb{Z}/2 \times H$ sending λ into $\mathbb{Z}/2 \times \{0\}$ determines a homomorphism h from $G(\mathcal{Z})$ to $\mathbb{Z}/2 \times H$. Moreover, if $h(\lambda) = (1, 0)$, the composition $h \circ \mathrm{gr} : \mathcal{A}(\mathcal{Z}) \to \mathbb{Z}/2 \times H$ is a grading, with the differential lowering the grading by $(1, 0)$.

Here we choose H to be the trivial group, and define a homomorphism

$$f : G(\mathcal{Z}) \to \mathbb{Z}/2$$
$$\lambda \mapsto 1$$
$$g_i \mapsto 1$$

Definition 4.1.1. *The $\mathbb{Z}/2$ grading of an element $a \in \mathcal{A}(\mathcal{Z})$ is defined as $m(a) := f \circ gr(a)$.*

The discussion above shows that m is a differential grading on $\mathcal{A}(\mathcal{Z})$ (with ∂ lowering the grading by 1 mod 2).

Note that with this $\mathbb{Z}/2$ grading on $\mathcal{A}(\mathcal{Z})$, we are ready to prove Theorem 4 and 5, which are statements for modules that come equipped with a $\mathbb{Z}/2$ grading compatible with m. However, in order to define an invariant in $K_0(\mathcal{A}(\mathcal{Z}))$ of bordered manifolds, we need to define a $\mathbb{Z}/2$ grading on $\mathit{CFA}(\mathcal{H}, \mathfrak{s})$ and $\mathit{CFD}(\mathcal{H}, \mathfrak{s})$.

First we define a $\mathbb{Z}/2$ grading on domains from the refined grading g by $G(\mathcal{Z})$.

Definition 4.1.2. *For any $x, y \in \mathfrak{S}(\mathcal{H})$ and $B \in \pi_2(x, y)$, define $m(B) \in \mathbb{Z}/2$ by*

$$m(B) := f \circ g(B).$$

Theorem 4.1.3. *For periodic domains, the map m is independent of the grading refinement data. In fact, if B is a periodic domain, $m(B) = 0$.*

Before we prove the theorem, we make an observation about the refined grading on the generators $a(\rho_i)$, and then give an alternate definition of m.

Lemma 4.1.4. *Reducing the Maslov component modulo 2, the refined grading on $a(\rho_i)$ is given by*

$$g_i = \begin{cases} (-\frac{1}{2}; [a(\rho_i)]) & \text{if } i \in \mathbf{s} \\ (\frac{1}{2}; [a(\rho_i)]) & \text{if } i \notin \mathbf{s}. \end{cases}$$

Proof. If $i \in \mathbf{s}$, then $g_i = gr'(a(\rho_i)) = (-\frac{1}{2}; [a(\rho_i)])$. If $i \notin \mathbf{s}$, then choose some $j \in \mathbf{s}$ and define $\mathbf{t} := (\mathbf{s} \setminus j) \cup i$. Then

$$\begin{aligned} g_i &= \psi(\mathbf{t}) gr'(a(\rho_i)) \psi(\mathbf{t})^{-1} \\ &= (-\frac{1}{2} + L([\psi(\mathbf{t})], [a(\rho_i)]) + L([\psi(\mathbf{t})], [\psi(\mathbf{t})^{-1}]) + L([a(\rho_i)], [\psi(\mathbf{t})^{-1}]); [a(\rho_i)]) \\ &= (-\frac{1}{2} + L([\psi(\mathbf{t})], [a(\rho_i)]) + L([a(\rho_i)], [\psi(\mathbf{t})^{-1}]); [a(\rho_i)]) \\ &= (-\frac{1}{2} + L([\psi(\mathbf{t})], [a(\rho_i)]) - L([a(\rho_i)], [\psi(\mathbf{t})]); [a(\rho_i)]) \\ &= (-\frac{1}{2} + 2L([\psi(\mathbf{t})], [a(\rho_i)]); [a(\rho_i)]) \end{aligned}$$

Now, write $\partial [\psi(\mathbf{t})] = \sum_{p=1}^{4k} n_p a_p$. Since $M_* \partial [\psi(\mathbf{t})] = \mathbf{t} - \mathbf{s} = j - i$, and $M^{-1}(i) = \{\rho_i^+, \rho_i^-\}$, it must be that $n_{\rho_i^+} + n_{\rho_i^-} = 1$. Then

$$\begin{aligned} L([\psi(\mathbf{t})], [a(\rho_i)]) &= m([a(\rho_i)], [\psi(\mathbf{t})]) \\ &= m([a(\rho_i)], \sum_{p=1}^{4k} n_p a_p) \\ &= \sum_{p=1}^{4k} n_p m([a(\rho_i)], a_p) \\ &\equiv n_{\rho_i^+} m([a(\rho_i)], \rho_i^+) + n_{\rho_i^-} m([a(\rho_i)], \rho_i^-) \\ &= \frac{1}{2} n_{\rho_i^+} + \frac{1}{2} n_{\rho_i^-} \\ &= \frac{1}{2} \mod 1. \end{aligned}$$

It follows that $2L([\psi(\mathbf{t})], [a(\rho_i)]) \equiv 1 \mod 2$, so $g_i = (\frac{1}{2}; [a(\rho_i)])$. ∎

Using $[a(\rho_i)]$ as a standard basis for $H_1(F;\mathbb{Z})$, if $\alpha = \sum_{i=1}^{2k} h_i\,[a(\rho_i)]$, we will write $(j;\alpha)$ as $(j;h_1,\ldots,h_{2k})$.

Proposition 4.1.5. *Let* $\delta_{ij} := [a(\rho_i)] \cap [a(\rho_j)] = L(\rho_i,\rho_j)$, *and define a map* $f_{\mathbf{s}}$ *from* $G(\mathcal{Z})$ *to* $\mathbb{Z}/2$ *by*

$$f_{\mathbf{s}}(j;h_1,\ldots,h_{2k}) = j - \frac{1}{2}\sum_{i\in\mathbf{s}} h_i + \frac{1}{2}\sum_{i\notin\mathbf{s}} h_i + \sum_{i_1<i_2} h_{i_1}h_{i_2}\delta_{i_1i_2}.$$

Then $f_{\mathbf{s}}$ *agrees with the homomorphism* f.

Proof. Note that $[a(\rho_{i_1})] \cap [a(\rho_{i_2})] = -\,[a(\rho_{i_2})] \cap [a(\rho_{i_1})] \in \mathbb{Z}$ implies $\delta_{i_1i_2} \equiv \delta_{i_2i_1} \bmod 2$.

Given $a = (j_a;h_1^a,\ldots,h_{2k}^a)$ and $b = (j_b;h_1^b,\ldots,h_{2k}^b)$,

$$\begin{aligned}
f_{\mathbf{s}}(ab) &= f_{\mathbf{s}}(j_a + j_b + L([a],[b]);h_1^a + h_1^b,\ldots,h_{2k}^a + h_{2k}^b)\\
&= j_a + j_b + L([a],[b]) - \frac{1}{2}\sum_{i\in\mathbf{s}}(h_i^a + h_i^b) + \frac{1}{2}\sum_{i\notin\mathbf{s}}(h_i^a + h_i^b)\\
&\qquad + \sum_{i<j}(h_i^a + h_i^b)(h_j^a + h_j^b)\delta_{ij}\\
&= j_a + j_b + L([a],[b]) - \frac{1}{2}\sum_{i\in\mathbf{s}}(h_i^a + h_i^b) + \frac{1}{2}\sum_{i\notin\mathbf{s}}(h_i^a + h_i^b)\\
&\qquad + \sum_{i<j} h_i^a h_j^a\delta_{ij} + \sum_{i<j} h_i^a h_j^b\delta_{ij} + \sum_{i<j} h_j^a h_i^b\delta_{ij} + \sum_{i<j} h_i^b h_j^b\delta_{ij}\\
&= f_{\mathbf{s}}(a) + f_{\mathbf{s}}(b) + L([a],[b]) + \sum_{i<j} h_i^a h_j^b\delta_{ij} + \sum_{i>j} h_i^a h_j^b\delta_{ij}\\
&= f_{\mathbf{s}}(a) + f_{\mathbf{s}}(b) + L([a],[b]) + \sum_{i\neq j} h_i^a h_j^b\delta_{ij}\\
&= f_{\mathbf{s}}(a) + f_{\mathbf{s}}(b)
\end{aligned}$$

The last equality is true since $L([a],[b])$ is an integer, and $\sum_{i\neq j} h_i^a h_j^b\delta_{ij} \equiv L([a],[b]) \bmod 2$. Thus, $f_{\mathbf{s}}$ is a homomorphism.

We evaluate

$$f_{\mathbf{s}}(g_i) = \begin{cases} -\frac{1}{2} - \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 + 0 = 1 \bmod 2 & \text{if } i \in \mathbf{s}\\ \frac{1}{2} - \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 + 0 = 1 \bmod 2 & \text{if } i \notin \mathbf{s}\end{cases}$$

and $f_{\mathbf{s}}(\lambda) = 1$, so $f_{\mathbf{s}}$ agrees with f on the generators g_i and λ. Thus, $f_{\mathbf{s}} \equiv f$. □

Proof of Theorem 4.1.3. We may assume that $B \in \pi_2(\mathbf{x},\mathbf{x})$ for some $\mathbf{x}$ with $I_A(\mathbf{x}) = I(\mathbf{s})$, since if $B_1 \in \pi_2(\mathbf{x},\mathbf{x})$ and $I_A(\mathbf{x}) \neq I(\mathbf{s})$, we can choose a generator $\mathbf{y}$ with $I_A(\mathbf{y}) = I(\mathbf{s})$, and a domain $B_2 \in \pi_2(\mathbf{y},\mathbf{x})$, so that $B := B_2 * B_1 * (-B_2) \in \pi_2(\mathbf{y},\mathbf{y})$, and we see that

$$\begin{aligned} f_{\mathbf{s}}(g(B)) &= f_{\mathbf{s}}(g(B_2 * B_1 * (-B_2))) \\ &= f_{\mathbf{s}}(g(B_2)g(B_1)g(-B_2)) \\ &= f_{\mathbf{s}}(g(B_2)g(B_1)g(B_2)^{-1}) \\ &= f_{\mathbf{s}}(g(B_2)) + f_{\mathbf{s}}(g(B_1)) - f_{\mathbf{s}}(g(B_2)) \\ &= f_{\mathbf{s}}(g(B_1)). \end{aligned}$$

We construct a surface F as in the proof of [LOT, Lemma 10.3]. We follow the notation of that proof without explaining it for the next few paragraphs, so we advise the reader to get familiar with it before proceeding.

If necessary, perform a β-curve isotopy as in Figure 8 to arrange that distinct segments of $\partial\Sigma$ lie in distinct regions of the Heegaard diagram, and label the regions $R_0, R_1, \ldots, R_{4k-1}$, beginning at the basepoint z, and following the orientation of $\partial\Sigma$. Let $\delta_i = m(R_i) - m(R_{i-1})$.

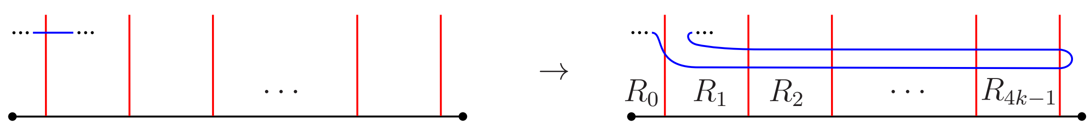


Figure 8: An isotopy ensuring that there are $4k$ distinct regions at $\partial\Sigma$. On the left, a collar neighborhood of $\partial\Sigma$ and the closest intersection point to a_1 along α_1. On the right, a finger move along $\partial\Sigma$ of the β-curve near that intersection point.

If $x \in \mathbf{x}$, then $C(x) = 0$, so the lift $\Phi^{-1}D(x)$ of a neighborhood of x is a union of disks and $h(x)$ half-disks. Observe that $e([\Sigma]) + 2n_{\mathbf{x}}([\Sigma]) = 1$, so

$$\begin{aligned} e(B) + 2n_{\mathbf{x}}(B) &\equiv e(\widetilde{B}) + 2n_{\mathbf{x}}(\widetilde{B}) - l(e([\Sigma]) + 2n_{\mathbf{x}}([\Sigma])) \\ &\equiv e(F) + 2\sum_{x\in\mathbf{x}} n_x(F) - l \end{aligned}$$

$$\equiv \chi(F) - \frac{1}{4}\sum |\delta_i| + \sum_{x\in\mathbf{x}} h(x) - l$$
$$\equiv \#(\partial F) - \frac{1}{4}\sum |\delta_i| + \sum_{x\in\mathbf{x}} h(x) - l$$

The boundary of any half-disk lies above an α- or a β-circle, or an α-arc. Since each α or β-circle is occupied exactly once by an $x \in \mathbf{x}$, we can cancel in pairs all half disks with boundaries lying above α- or β-circles with the components of ∂F that are lifts of the same α- or β-circles. We are left only with half disks with boundary projecting to an α-arc, and with components of ∂F whose projection intersects $\partial\Sigma$. Let the number of these connected components of ∂F be t, and let $h^a(x)$ be the number of half disks at x with boundary projecting to an α-arc. Then

$$e(B) + 2n_{\mathbf{x}}(B) \equiv t - \frac{1}{4}\sum |\delta_i| + \sum_{x\in\mathbf{x}} h^a(x) - l.$$

Now,

$$\begin{aligned} |\delta_i| &= \#\text{ corners of } F \text{ above } a_i \\ &= |\#\text{ preimages of } \alpha_{M(i)} \text{ in } \partial F| \\ &= |\text{ multiplicity of } \alpha_{M(i)} \text{ in } \partial B| \\ &= |h_{M(i)}|, \end{aligned}$$

and

$$\sum_{x\in\mathbf{x}} h^a(x) = \sum_{x\in\mathbf{x}} |\text{ multiplicity of the } \alpha\text{-arc occupied by } x \text{ in } \partial B| = \sum_{i\in\mathbf{s}} |h_i|,$$

so

$$\begin{aligned} e(B) + 2n_{\mathbf{x}}(B) &\equiv t - l - \frac{1}{4}\sum_{i=1}^{4k} |h_{M(i)}| + \sum_{i\in\mathbf{s}} |h_i| \\ &\equiv t - l - \frac{1}{2}\sum_{i=1}^{2k} |h_i| + \sum_{i\in\mathbf{s}} |h_i| \\ &\equiv t - l - \frac{1}{2}\sum_{i\notin\mathbf{s}} |h_i| + \frac{1}{2}\sum_{i\in\mathbf{s}} |h_i|. \end{aligned}$$

Thus,

$$\begin{aligned} f_{\mathbf{s}}(g(B)) &= -e(B) - 2n_{\mathbf{x}}(B) - \frac{1}{2}\sum_{i\in\mathbf{s}} h_i + \frac{1}{2}\sum_{i\notin\mathbf{s}} h_i + \sum_{i_1<i_2} h_{i_1}h_{i_2}\delta_{i_1i_2} \\ &= -t + l + \frac{1}{2}\sum_{i\notin\mathbf{s}}(|h_i| + h_i) - \frac{1}{2}\sum_{i\in\mathbf{s}}(|h_i| + h_i) + \sum_{i_1<i_2} h_{i_1}h_{i_2}\delta_{i_1i_2} \\ &= -t + l + \sum_{i\notin\mathbf{s}, h_i>0} h_i - \sum_{i\in\mathbf{s}, h_i>0} h_i + \sum_{i_1<i_2} h_{i_1}h_{i_2}\delta_{i_1i_2} \\ &= -t + l + \sum_{h_i>0} h_i + \sum_{i_1<i_2} h_{i_1}h_{i_2}\delta_{i_1i_2}. \end{aligned}$$

Since lifts of adjacent regions are identified along α-arcs in pairs, starting at the highest index and going down, the lift of $\partial^{\partial}\widetilde{B}$ consists of positively oriented arcs of $\partial\Sigma$ such that any two are either disjoint or nested, but never interleaved or abutting. In other words, we can represent the result of this identification geometrically on an annulus lying above $\mathcal{Z}$ by layers of horizontal arcs, so that the lowest layer consists of all $\partial^{\partial}R_i^{(j)}$ of highest index (j), after gluing, the second layer from the bottom contains the second highest indices (j), and so on. In this representation, each arc not in the lowest layer projects to a subset of the projection of the arc that it lies over. Since the identification along β-arcs is from lowest to highest index, following the regions along the higher multiplicity side of an α-arc at $\partial\Sigma$ shows that the boundaries of the set of arcs are matched along α-arcs in order from the highest level (i.e. lowest index) to the lowest possible, to form the t circles of ∂F that contain α-arcs. See, for example, Figure 9.

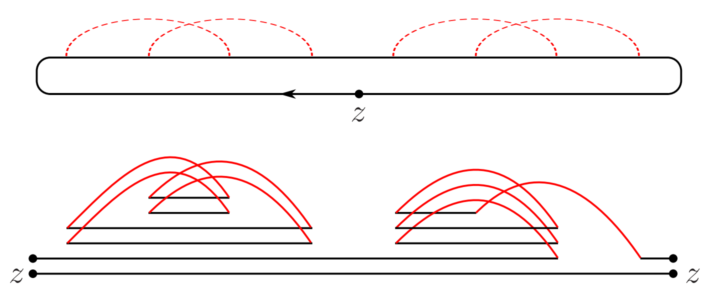


Figure 9: Top: The genus 2 split circle. Bottom: The layers representation of the boundary components of F which contain parts of $\partial\Sigma$, in the case when B is a domain with boundary $(2, 2, 3, -1)$ and after two copies of Σ have been added to B. Here $l = 2$, $l_0 = 1$.

Rotating 90° counterclockwise, we can draw $\mathcal{Z}$ in the plane as a vertical line segment oriented upwards with ends identified with the basepoint z, and we can draw the annulus as a rectangle, so that the top and bottom edges are identified and project to z, and so that when we endow all $\partial\Sigma$-arcs and α-arcs with an orientation arising from the orientation of F, all $\partial\Sigma$-arcs are oriented upwards.

Note that for a matched pair (i, j) with $i < j$, $h_{M(i)} = \delta_i = -\delta_j$, i.e. for $i \in [2k]$, $h_i = \delta_{\rho_i^-} = -\delta_{\rho_i^+}$. If $h_i < 0$, then $\delta_{\rho_i^-} < 0$, so there are $|h_i|$ $\partial\Sigma$-arcs ending at ρ_i^-, hence $|h_i|$ α-arcs starting at ρ_i^- and ending at ρ_i^+. In other words, all $|h_i|$ copies of α_i are oriented upwards when $h_i < 0$. Similarly, when $h_i > 0$, there are h_i $\partial\Sigma$-arcs starting at ρ_i^-, hence h_i α-arcs ending at ρ_i^- and starting at ρ_i^+, i.e. all h_i copies of α_i are oriented downwards when $h_i < 0$. (see Figure 10).

For $h_i < 0$, we can draw the copies of α_i as parallel arcs open to the right, curved so that they do not intersect any $\partial\Sigma$-arcs. For $h_i > 0$, we draw the copies of α_i similarly as arcs starting at ρ_i^+ moving upwards, passing through the top horizontal line, continuing to move up from the bottom to the copies of ρ_i^-, also in a way as to not intersect $\partial\Sigma$-arcs. All arcs now move strictly upwards, and in this way we represent the relevant boundary components of F as a closed braid (where we don't care about the sign of crossings). Note that a copy of α_i and a copy of α_j cross an even number of times if $\delta_{ij} = 0$, and an odd number of times if $\delta_{ij} = 1$. It follows that if the braid is given by a permutation σ,

$$\begin{aligned} t &= \#\text{ boundary components of } F \text{ intersecting } \partial\Sigma \\ &= \#\text{ components in the closure of the braid} \\ &= \#\text{cycles in } \sigma\text{, including cycles of length 1} \\ &\equiv \#\text{ involutions of } \sigma \ + \ \text{length of } \sigma \\ &\equiv \#\text{ crossings in the braid } + \#\text{ strands in the braid} \\ &\equiv \sum_{i_1 < i_2} h_{i_1} h_{i_2} \delta_{i_1 i_2} + \Big(\sum_{h_i > 0} h_i + l\Big). \end{aligned}$$

Hence, $f_{\mathbf{s}}(gr(B)) = 0$.

□

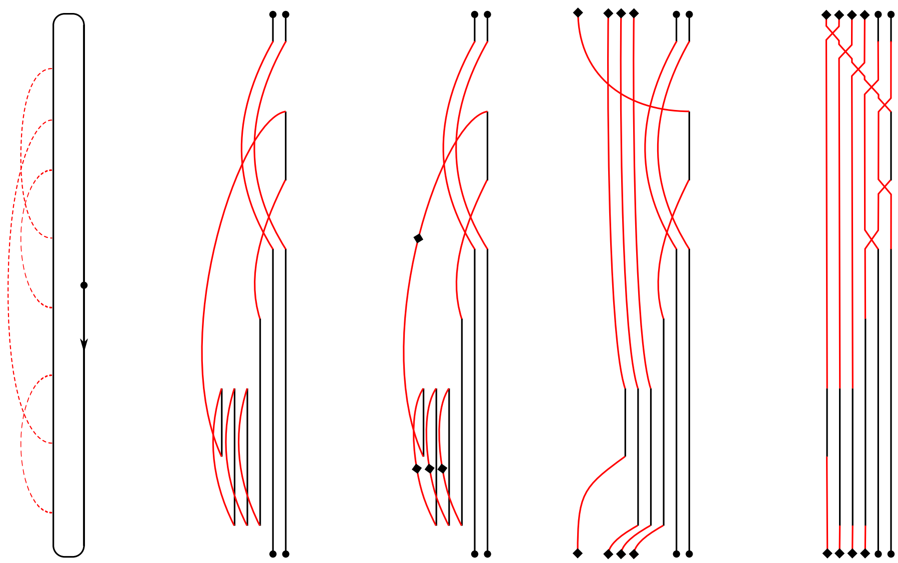


Figure 10: Obtaining a braid representation of the boundary components of F that intersect $\partial\Sigma$. In this example $(h_1, h_2, h_3, h_4) = (3, 1, -1, -2)$.

Note that if B is periodic, then $f_{\mathbf{s}}(R(B)) = 0$ too.

Now we define m as a relative grading on the modules.

Definition 4.1.6. *Suppose $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \neq 0$. Pick $x \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and define $m : \widehat{CFA}(\mathcal{H}, \mathfrak{s}) \to \mathbb{Z}/2$ by $m(\mathbf{x}) = 0$ and*

$$m(\mathbf{y}) = m(\mathbf{x}) + m(B),$$

where $\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$. Similarly, define $m : \widehat{CFD}(\mathcal{H}, \mathfrak{s}) \to \mathbb{Z}/2$ by $m(\mathbf{x}) = 0$ and

$$m(\mathbf{y}) = m(\mathbf{x}) + m(R(B)),$$

where $\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$.

This is well defined since it factors through the $G(\partial\mathcal{H})$-set grading defined in [LOT, Chapter 10].

In some cases there is a natural choice of a base generator for each $Spin^c$ structure so that m agrees with the absolute Maslov grading after gluing. We do not discuss these choices here.

4.2 The Grothendieck group of $\mathbb{Z}/2$-graded $\mathcal{A}(\mathcal{Z})$-modules

We proceed to define our main object of interest, the Grothendieck group of a differential graded algebra. As a warm-up example, recall the definition of the Grothendieck group of a ring:

Definition 4.2.1. *Given a $\mathbb{Z}$-graded associative ring A, the Grothendieck group $K_0(A)$ is defined as the $\mathbb{Z}[q, q^{-1}]$-module with*

- *generators: $[P]$, P - finitely generated graded projective A-module*
- *relations:*
  1. *$[P] = [P'] + [P'']$ whenever there is a short exact sequence $0 \to P' \to P \to P'' \to 0$*
  2. *$[P\{n\}] = q^n[P]$, where $\{n\}$ denotes the grading shift up by n*

Example: The most familiar and basic example is $\mathbb{Z}$-graded modules/chain complexes over $\mathbb{Z}$. The ring $\mathbb{Z}$ does not have an internal grading, i.e. sits completely in grading 0, and the reader can verify that $K_0(\mathbb{Z}) \cong \mathbb{Z}$ via $[C] = \chi(C)$.

As mentioned in Section 1, specific examples have been studied in low dimensional topology, namely, the image in the corresponding Grothendieck groups of the homology theories with $\mathbb{Z}$ or $\mathbb{Z}/2$ coefficients *SFH*, *Kh*, *HF*, *HFK* [FJR11, Kho00, OS04, OS03b].

Given a differential graded algebra A, recall the definition of $K_0(A)$ from [Kho10]. Let $\mathcal{K}(A)$ be the homotopy category of modules over A, $\mathcal{KP}(A)$ - the full subcategory of projective modules, and $\mathcal{P}(A) \subset \mathcal{KP}(A)$ - the full subcategory of compact projective modules, which is a triangulated category.

Definition 4.2.2. *Given an algebra A with differential grading by $\mathbb{Z}$ or $\mathbb{Z}/2$, we define $K_0(A)$ to be the Grothendieck group of the category $\mathcal{P}(A)$. It has*

- *generators: $[P]$ over all compact projective differential graded A-modules P*
- *relations:*

1. $[P_2] = [P_1] + [P_3]$ *whenever* $P_1 \to P_2 \to P_3$ *is a distinguished triangle (cite a reference for triangulated categories)*

2. $[P[1]] = -[P]$*, where* $[1]$ *is the grading shift by* 1

3. *If we also introduce an additional* $\frac{1}{2}\mathbb{Z}$*-grading along with the relation* $[P\{k\}] = t^k[P]$*, where* $\{k\}$ *denotes the grading shift up by* $k \in \frac{1}{2}\mathbb{Z}$*, we get a* $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$*-module for* K_0*.*

Note that relation (3) is usually seen in reference to a $\mathbb{Z}$-grading, but we find it more convenient in the bordered Floer homological context to work with a $\frac{1}{2}\mathbb{Z}$-grading, which we define and study in Section 4.3.

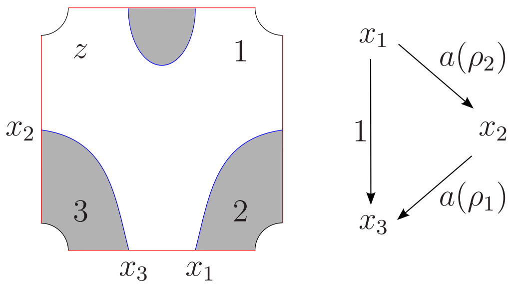
By [LOT10], there's an equivalence of categories of bounded left type D structures and right $\mathcal{A}_\infty$-modules, and along with [LOT10, Corollary 2.3.25], we see that we can work either in the homotopy category of finitely generated projective modules or bounded type D structures. We will prove Theorem 4 for left type D structures. The same result for right $\mathcal{A}_\infty$-modules follows by the equivalence of categories.

For the rest of this Section, we will write $\mathcal{A}$ for $\mathcal{A}(\mathcal{Z})$.

Recall that a type D structure N over $\mathcal{A}$ is called *bounded* if for all $x \in N$ there is some integer n, so that for all $i \geq n$, $\delta_i(x) = 0$. Note that for a finitely generated N we can find a universal n, so that for all x and all $i \geq n$, $\delta_i(x) = 0$. Thinking of N as a dg module, if we fix an ordering of the basis $x_1, \ldots, x_n$, we can represent the differential by the matrix formed by the coefficients of $\partial x_i = \sum_j a_{ij} x_j$. Observe that N being finitely generated and bounded is equivalent to the existence of an ordered basis for N over $\mathcal{A}$ with an upper triangular differential matrix with zeros on the diagonal (see Fig. 11 for example). By Proposition 2.3.10 of [LOT10], every type D structure over $\mathcal{A}$ is homotopy equivalent to a bounded one, by tensoring it with a left and right bounded AD identity bimodule.

Next, note that in the homotopy category, homotopy equivalent type D structures map to the same symbol in K_0, so it is enough to study bounded type D structures.

Note: In the special case of $\widehat{CFD}$ of a 3-manifold, the bounded type D structures are the ones coming from admissible Heegaard diagrams.



$$\begin{bmatrix} 0 & a(\rho_2) & 1 \\ 0 & 0 & a(\rho_1) \\ 0 & 0 & 0 \end{bmatrix}$$

Figure 11: An example of a bounded type D structure. Left: A Heegaard diagram. The domains that contribute to the differential are shaded. Center: The type D structure for the diagram. Right: A matrix representation of the "differential" with respect to the given ordering of the basis. Note that the notation ρ_i here comes from the diagram and carries different meaning than in Section 4.1 and the rest of the paper.

Given a pointed matched circle $\mathcal{Z}$ for a surface F, recall that in Section 4.1 we defined ρ_i as the Reeb chord connecting the two points in $M^{-1}(i)$. The elements $[a(\rho_i)]$ generate $H_1(F;\mathbb{Z})$. Denote the initial endpoint of ρ_i by ρ_i^- and the final endpoint by ρ_i^+, and order the basis $\{[a(\rho_i)]|i \in [2k]\}$ so that $[a(\rho_i)] < [a(\rho_j)]$ if and only if ρ_i^- comes before ρ_j^- when we follow the orientation of the circle starting at z. In other words, we order the basis of $H_1(F;\mathbb{Z})$ according to the order of the initial endpoints of the corresponding Reeb chords, where this order is induced by the orientation of the circle. Call the ordered basis $a_1 < a_2 < \cdots < a_{2k}$.

Given a set $\mathbf{s} \subset [2k]$, let $J(\mathbf{s})$ be the multi-index $(j_1, \ldots, j_n)$, so that $1 \leq j_1 < \ldots < j_n \leq 2k$ and $\{j_1, \ldots, j_n\} = \mathbf{s}$. Given a multi-index $J = (j_1, \ldots, j_n)$ of increasing numbers as above, define $a_J = a_{j_1} \wedge \ldots \wedge a_{j_n}$. We will use the shortcut notation $a_{\mathbf{s}} := a_{J(\mathbf{s})}$. Note that $a_{\mathbf{s}}$ form a basis for $\Lambda^*(H_1(F;\mathbb{Z}))$.

Suppose M is a right $\mathcal{A}_\infty$-module over $\mathcal{A}(\mathcal{Z})$ with a set of "homogeneous" generators $\mathfrak{S}(M)$, i.e. for each generator there is a unique indecomposable idempotent that acts non-trivially (by the identity) on that generator. For $x \in \mathfrak{S}(M)$, let $o(x)$ be the subset of $[2k]$ for which $I_A(x) := I(o(x))$ is the unique primitive idempotent acting non-trivially on x. Define a function $h : \mathfrak{S}(M) \to \Lambda^* H_1(F;\mathbb{Z})$ by $h(x) = a_{o(x)}$. Similarly, if N is a left type D structure with a set of generators $\mathfrak{S}(N)$, let $\overline{o}(x)$ be the subset of $[2k]$ for which $I_D(x) := I(\overline{o}(x))$ is the unique primitive idempotent acting non-trivially on x. Again define

$h : \mathfrak{S}(N) \to \Lambda^* H_1(F;\mathbb{Z})$ by $h(x) = a_{\overline{o}(x)}$. Recall that for *CFA* of a Heegaard diagram, $I_A(x)$ is the idempotent corresponding to the α arcs occupied by x, and for *CFD*, $I_D(x)$ is the idempotent corresponding to the unoccupied α arcs, and in either case $h(x) \in \Lambda^k(H_1(F;\mathbb{Z}))$.

Below is the full version of Theorem 4 and its proof.

Theorem 4.2.3. *Let $\mathcal{Z}$ be a pointed matched circle with associated surface F of genus k. The Grothendieck group of the category of $\mathbb{Z}/2$-graded left type D structures over $\mathcal{A}(\mathcal{Z})$ is equivalent to that of right $\mathcal{A}_\infty$-modules over $\mathcal{A}(\mathcal{Z})$ and is given by*

$$K_0(\mathcal{A}(\mathcal{Z})) = \Lambda^*(H_1(F;\mathbb{Z})).$$

Moreover, if M is a left type D structure or a right $\mathcal{A}_\infty$-module over $\mathcal{A}(\mathcal{Z})$, its image in this group is given by

$$[M] = \sum_{x \in \mathfrak{S}(M)} (-1)^{m(x)} h(x),$$

where $h(x)$, as defined above, is the wedge in $\Lambda^*(H_1(F;\mathbb{Z}))$ of generators of $H_1(F;\mathbb{Z})$ given by the set of matched points in $\mathcal{Z}$ corresponding to $I_A(x)$ for $\mathcal{A}_\infty$-modules, or to $I_D(x)$ for type D structures (with order induced by the orientation of the circle). In other words, $[M]$ counts generators of M in each primitive idempotent.

Proof. We show that if M is a finitely generated bounded type D structure, then $[M]$ is a linear combination of symbols of elementary projectives, i.e., type D structures of form $\mathcal{A}I(s)$. We use induction on the rank of M over $\mathcal{A}$. Clearly, if M has only one generator, it is an elementary projective, since, because of the boundedness condition, the differential must be zero. Otherwise, chose an upper triangular differential matrix for M and let x be the generator corresponding to the bottommost row of the matrix. Then x is a cycle, and we have a distinguished triangle

$$\mathcal{A}I_D(x)[m(x)] \to M \to M/\mathcal{A}x,$$

so $[M] = (-1)^{m(x)}[\mathcal{A}I_D(x)] + [M/\mathcal{A}x]$. The matrix for $M/\mathcal{A}x$ is obtained from the matrix for M by removing the last row and column, hence $M/\mathcal{A}x$ is bounded and of rank one lower than M, so $[M/\mathcal{A}x]$ is the sum of symbols of elementary projectives. Thus, $[M]$ is of that form too. In fact, applying this process repeatedly, we see that

$$[M] = \sum_{x \in \mathfrak{S}(M)} (-1)^{m(x)} [\mathcal{A}I_D(x)].$$

So far we have shown that $K_0(\mathcal{A})$ is generated (over $\mathbb{Z}$) by the symbols of elementary projectives.

Given $s \subset [2k]$, let a be the generator of $\mathcal{A}I(s)$, and let M be the module generated by x and y with $I_D(x) = I(s) = I_D(y)$, and with differential given by $\partial(x) = y$, $\partial y = 0$. In other words, M is the mapping cone of the identity map on $\mathcal{A}I(s)$. Then we have a distinguished triangle

$$\mathcal{A}I(s) \to \mathcal{A}I(s) \to M,$$

which rotates to

$$\mathcal{A}I(s) \to M \to \mathcal{A}I(s)[1],$$

where the first map is $a \to y$, and the second is $x \to a$, $y \to 0$. Since M is homotopic to zero, the second distinguished triangle implies $[\mathcal{A}I(s)] + [\mathcal{A}I(s)[1]] = [M] = 0$, so

$$[\mathcal{A}I(s)[1]] = -[\mathcal{A}I(s)].$$

Note that there is a correspondence between elementary projectives and elements of $\Lambda^*(H_1(F;\mathbb{Z}))$ given by

$$I(\mathbf{s}) \to a_{\mathbf{s}}.$$

Thus, $K_0(\mathcal{A})$ is a quotient of $\Lambda^*(H_1(F;\mathbb{Z}))$. In fact we see that $K_0(\mathcal{I}) = K_0(\mathbb{Z}/2^{2^{2k}}) = \Lambda^*(H_1(F;\mathbb{Z}))$, and we have an equivalence $K_0(\mathcal{A}) \cong K_0(\mathcal{I})$ for the following reason.

The algebra $\mathcal{A}(\mathcal{Z})$ has a $\mathbb{Z}$-grading given by the total support of an element in $H_1(Z, \mathbf{a})$, i.e. the sum of the coefficients of the projection from $gr'(a) \in G'(\mathcal{Z})$ onto $H_1(Z, \mathbf{a})$. Since we only work with upward going and horizontal strands, this grading is in fact by non-negative numbers, and the degree 0 part is precisely the ground ring $\mathcal{I}$. The part of positive degree, call it $\mathcal{A}_+$, is an augmentation ideal - dividing by it yields $\mathcal{I}$. This augmentation map $\mathcal{A}(\mathcal{Z}) \to \mathcal{I}$ induces a map of the categories of modules over $\mathcal{A}(\mathcal{Z})$ and $\mathcal{I}$ by $M \mapsto$

$M/\mathcal{A}_+M$, which induces an equivalence of Grothendieck groups. The last fact is well known to experts. Alternatively, the reader can verify that the map sends distinguished triangles to distinguished triangles. Thus,

$$K_0(\mathcal{A}) = \Lambda^*(H_1(F;\mathbb{Z})).$$

□

4.3 K_0 of Alexander-graded dg modules over $\mathcal{A}(\mathcal{Z})$

In this Section, we define a relative $\frac{1}{2}\mathbb{Z}$-grading a on the dg algebra $\mathcal{A}(\mathcal{Z})$, as well as on the left type D structures and right $\mathcal{A}_\infty$-modules over $\mathcal{A}(\mathcal{Z})$. We refer to it as Alexander grading, and show that it has properties similar to the Alexander grading on *CFK* for links in closed 3-manifolds. For now we restrict to the case of torus boundary, i.e. $\mathcal{Z}(T^2)$, and define an Alexander grading for *CFD* of knot complements and for *CFA* of knots in the solid torus.

Recall that the unrefined grading on the torus algebra takes values in the group G' which consists of quadruples $(j;a,b,c)$, with $j \in \frac{1}{2}\mathbb{Z}$, $a,b,c \in \mathbb{Z}$, and j is an integer if b and $a-c$ are even. There is also a refined grading by a subgroup G, different from $G(\mathcal{Z})$, which consists of triples $(j;p,q)$ with $j,p,q \in \frac{1}{2}\mathbb{Z}$ and $p+q \in \mathbb{Z}$. For the group law on G' and G and further details, see Section 11.1 of [LOT].

We have a surjection

$$G' \twoheadrightarrow H_1(Z,\mathbf{a})$$
$$(j;a,b,c) \mapsto (a,b,c)$$

As with the refined grading group $G(\mathcal{Z})$, we can think of G as a $\mathbb{Z}$ central extension of $H_1(T^2;\frac{1}{2}\mathbb{Z})$. In other words, we have a map

$$G \to H_1(T^2;\mathbb{Q})$$
$$(j;p,q) \mapsto (p,q)$$

with image inside $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ and kernel generated by $\lambda = (1; 0, 0)$.

Let K be a knot in S^3 and let $CFD(K, n)$ be a type D structure for the n-framed complement $S^3 \setminus K$. Fix a generator x and recall that we can choose the sign of the generator B of $\pi_2(x, x)$ so that B does not cover the basepoint z, and has multiplicities $1, n+1, n$ at the regions corresponding to ρ_1, ρ_2, ρ_3, respectively.

We can define the Alexander grading in two ways: by composing gr' with map from G' to $\frac{1}{2}\mathbb{Z}$, or by composing gr with a map from the subgroup G to $\frac{1}{2}\mathbb{Z}$. The two maps to $\frac{1}{2}\mathbb{Z}$ are obtained by composing the two maps above with

$$\begin{aligned} H_1(Z, \mathbf{a}) &\to \tfrac{1}{2}\mathbb{Z} \\ (a, b, c) &\mapsto \tfrac{n+1}{2}a + \tfrac{n-1}{2}b + \tfrac{-n-1}{2}c \end{aligned}$$

and

$$\begin{aligned} H_1(T^2; \mathbb{Q}) &\to \tfrac{1}{2}\mathbb{Z} \\ (p, q) &\mapsto np - q. \end{aligned}$$

The gradings $g'(B)$ and $g(B)$ are in the kernel of the respective maps, so we get maps from the quotients $G'/P(x)$ and $G/P(x)$ to $\frac{1}{2}\mathbb{Z}$, and also note that the two maps commute with the grading maps (i.e. transitioning between G and G'), hence the two maps define the same grading on $CFD(K, n)$. Note that this grading agrees with the function S from 11.39 of [LOT].

Next, we define the Alexander grading for a knot in the solid torus. One might like to just add the number of times we pass through the second basepoint of the Heegaard diagram to the Alexander grading above, but we also have to keep track of the homological class of the knot in the solid torus.

Given a 0-framed solid torus and a knot K in it with homology class $[p]$, fix a generator x of $\widehat{CFA}(K)$, and note that we can choose the generator B of $\pi_2(x, x)$ to have multiplicities $0, 1, 1$ at ρ_1, ρ_2, ρ_3 respectively, and to avoid the basepoint z. This B covers w with multiplicity p: B has boundary a set of complete α and β circles, and the complete arc α_1. By capping off all circles with the disks they bound, we get an immersed surface in the solid torus with boundary the meridian of the solid torus. In other words, the surface we obtain from B is homologically equivalent to the disk D^2 (with positive orientation) that the meridian bounds, hence intersects the knot homologically p any times. Hence, in the Heegaard diagram, B covers the basepoint w a total of p many times.

This time we have the algebra graded by $G' \times \{0\}$ or $G \times \{0\}$ as subgroups of $G' \times \mathbb{Z}$ or $G' \times \mathbb{Z}$, and corresponding maps to $H_1(Z, \mathbf{a}) \times \mathbb{Z}$ or $H_1(T^2; \mathbb{Q}) \times \mathbb{Z}$ defined as above on the first factor, and as the identity on the second. To define the Alexander grading, we compose these maps with the maps to $\frac{1}{2}\mathbb{Z}$ given by $(a, b, c; d) \mapsto d - p(b + c)$ and $(q_1, q_2; d) \mapsto d - pq_2$ respectively.

The domain B has grading of the form $g'(B) = (\_; 0, 1, 1; p)$ in G', or $g(B) = (\_; 0, 1; p)$ in G, and the grading on $\widehat{CFA}(K)$ takes values in $G' \times \mathbb{Z}/\langle g'(B)\rangle$ or $G \times \mathbb{Z}/\langle g(B)\rangle$, respectively. The gradings of B are in the kernels of the maps, so we get well defined maps from the quotients $G' \times \mathbb{Z}/\langle g'(B)\rangle$ and $G \times \mathbb{Z}/\langle g(B)\rangle$ to $\frac{1}{2}\mathbb{Z}$, and also note that the two maps commute with the grading maps, hence we have a well defined grading on $\widehat{CFA}(K)$.

Note that a $\frac{1}{2}\mathbb{Z}$-grading introduces relation (3) in Definition 4.2.2, and the Grothendieck group for left type D structures and right $\mathcal{A}_\infty$-modules over $\mathcal{A}(\mathcal{Z})$ becomes

$$K_0(\mathcal{A}(\mathcal{Z})_{gr}) \cong \Lambda^*(H_1(F; \mathbb{Z})) \otimes \mathbb{Z}[t^{1/2}, t^{-1/2}].$$

4.4 Tensor products

We first show that as a relative grading, the grading m defined in Section 2.4 agrees with the relative Maslov grading for closed manifolds.

Let Y_1 and Y_2 be bordered 3-manifolds which agree along their boundary, with Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ which can be glued along their boundary $\mathcal{Z} = \partial\mathcal{H}_1 = -\partial\mathcal{H}_2$ to form a closed Heegaard diagram $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$ representing the closed 3-manifold $Y = Y_1 \cup_\partial Y_2$. Let $\mathfrak{s} \in \text{spin}^c(Y)$, and let $\mathfrak{s}_i = \mathfrak{s}|_{Y_i}$.

Let $u, v \in \widehat{CFA}(\mathcal{H}_1, \mathfrak{s}_1)$ and $x, y \in \widehat{CFD}(\mathcal{H}_2, \mathfrak{s}_2)$ be homogeneous elements with respect to the grading gr, and suppose u and x occupy complementary α-arcs, and so do v and y.

Proposition 4.4.1. *Let t be the relative Maslov grading mod 2 between $u \boxtimes x$ and $v \boxtimes y \in \widehat{CF}(\mathcal{H}, \mathfrak{s})$ (see Theorem 10.41 of [LOT]). Then*

$$[m(u) + m(x)] - [m(v) + m(y)] = t \bmod 2$$

Proof. Recall that $\widehat{CFA}(\mathcal{H}_1, \mathfrak{s}_1)$ is graded by $P_1(x_1)\backslash G(\mathcal{Z})$, where $x_1 \in \mathfrak{S}(\mathcal{H}_1, \mathfrak{s}_1)$ is a chosen base generator, and $P_1(x_1)$ is the image of $\pi_2(x_1, x_1)$ in $G(\mathcal{Z})$. Similarly, $\widehat{CFD}(\mathcal{H}_2, \mathfrak{s}_2)$ is graded by $G(\mathcal{Z})/R(P_2(x_2))$, where $x_2 \in \mathfrak{S}(\mathcal{H}_2, \mathfrak{s}_2)$ is a chosen base generator and $P_2(x_2)$ is the image of $\pi_2(x_2, x_2)$ in $G(\mathcal{Z})$. Recall that the map f from Section 2.4 maps any element of $P_1(x_1)$ or $R(P_2(x_2))$ to zero.

By Theorem 10.41 of [LOT],

$$gr^{\boxtimes}(u \boxtimes x) = \lambda^t gr^{\boxtimes}(v \boxtimes y) \in P_1(x_1)\backslash G(\mathcal{Z})/R(P_2(x_2)).$$

This means that if we fix representatives $g_u, g_v, g_x, g_y \in G(\mathcal{Z})$ so that $[g_u] = gr(u) \in P_1(x_1)\backslash G(\mathcal{Z})$, etc., then

$$[g_u g_x] = [\lambda^t g_v g_y] \in P_1(x_1)\backslash G(\mathcal{Z})/R(P_2(x_2)).$$

Then there exist $h_1 \in P_1(x_1)$ and $h_2 \in R(P_2(x_2))$, such that

$$h_1 g_u g_x h_2 = \lambda^t g_v g_y \in G(\mathcal{Z}).$$

Applying f to both sides, we see that

$$f(h_1 g_u g_x h_2) = f(\lambda^t g_v g_y),$$

so

$$\begin{aligned} 0 &= f(h_1 g_u g_x h_2) - f(\lambda^t g_v g_y) \\ &= f(h_1) + f(g_u) + f(g_x) + f(h_2) - t f(\lambda) - f(g_v) - f(g_y) \\ &= f(g_u) + f(g_x) - t - f(g_v) - f(g_y) \\ &= \overline{f}(gr(u)) + \overline{f}(gr(x)) - t - \overline{f}(gr(v)) - \overline{f}(gr(y)) \\ &= m(u) + m(x) - t - m(v) - m(y). \end{aligned}$$

□

A similar statement can be made about the Alexander grading on the manifolds studied in Section 4.3. Let $\mathcal{H}_1$ be a Heegaard diagram for a knot in a 0-framed solid torus $(S^1 \times D^2, C)$, and let $\mathcal{H}_2$ be a Heegaard diagram for a 0-framed knot complement $(S^3 \setminus K, 0)$, so that $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$ is a Heegaard diagram for (S^3, K_C). Recall that there is only one spin^c structure for each of the bordered manifolds and for S^3.

Proposition 4.4.2. *Let $u, v \in \widehat{CFA}(\mathcal{H}_1)$ and $x, y \in \widehat{CFD}(\mathcal{H}_2)$ be homogeneous elements with respect to the grading gr, and suppose u and x occupy complementary α-arcs, and so do v and y. Let A be the relative Alexander grading between $u \boxtimes x$ and $v \boxtimes y \in \widehat{CF}(\mathcal{H})$ (see the discussion precedingTheorem 11.21 of [LOT]). Then*

$$(a(u) + p \cdot a(x)) - (a(v) + p \cdot a(y)) = A$$

Proof. The Maslov component does not affect the computation, so we leave it blank. Recall that in this case $\widehat{CFA}(\mathcal{H}_1)$ is graded by the coset $\langle h_A \rangle \backslash G(\mathcal{Z})$, and $\widehat{CFD}(\mathcal{H}_2)$ is graded by the coset $G(\mathcal{Z})/\langle h_D \rangle$, where $h_A = (\_; 0, 1; p)$ and $h_D = (\_; 1, 0; 0)$.

Fix representatives

$$\begin{aligned} g_u &= (\_; p_u, q_u; d_u), \\ g_v &= (\_; p_v, q_v; d_v), \\ g_x &= (\_; p_x, q_x; d_x), \\ g_y &= (\_; p_y, q_y; d_y) \end{aligned}$$

$\in G(\mathcal{Z})$ so that $[g_u] = gr(u) \in \langle h_A \rangle \backslash G(\mathcal{Z})$, etc. Then

$$[g_u g_x] = [h_A^{-q_u - q_x} g_u g_x h_D^{-p_u - p_x}] = [(\_; 0, 0; d_u - pq_u - pq_x)],$$

$$[g_v g_y] = [h_A^{-q_v - q_y} g_v g_y h_D^{-p_v - p_y}] = [(\_; 0, 0; d_v - pq_v - pq_y)],$$

and so $A = A(u \boxtimes x) - A(v \boxtimes y) = (d_u - pq_u - pq_x) - (d_v - pq_v - pq_y)$, which equals precisely $(a(u) + p \cdot a(x)) - (a(v) + p \cdot a(y))$. ∎

Now we define a product operation on $\Lambda^* H_1(F; \mathbb{Z})$. Define an inner product on the vector space $H_1(F; \mathbb{Z})$ by

$$\langle a_i, a_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

i.e. so that $\{a_1, \ldots, a_{2k}\}$ is an orthonormal basis. This extends to an inner product on $\Lambda^n H_1(F;\mathbb{Z})$ by

$$\langle v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n \rangle = \det(\langle v_i, w_j \rangle)$$

and $\{a_J \,||J| = n\}$ is an orthonormal basis for $\Lambda^n H_1(F;\mathbb{Z})$. We define an operation $\cdot$ on $\Lambda^* H_1(F;\mathbb{Z})$ by

$$x \cdot y := \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle + \cdots + \langle x_{2k}, y_{2k} \rangle,$$

where x_i and y_i are the Λ^i components of x and y, respectively.

Proof of Theorem 5. For the first part, $M \boxtimes N$ inherits a $\mathbb{Z}/2$ grading from M and N by $m(x \boxtimes y) = m(x) + m(y)$. We have

$$\begin{aligned}
[M] \cdot [N] &= \sum_{x \in \mathfrak{S}(M)} (-1)^{m(x)} h(x) \cdot \sum_{y \in \mathfrak{S}(N)} (-1)^{m(y)} h(y) \\
&= \left(\sum_{\mathbf{s} \subset [2k]} a_{\mathbf{s}} \sum_{x \in \mathfrak{S}(M), h(x) = a_{\mathbf{s}}} (-1)^{m(x)} \right) \cdot \left(\sum_{\mathbf{t} \subset [2k]} a_{\mathbf{t}} \sum_{y \in \mathfrak{S}(N), h(y) = a_{\mathbf{t}}} (-1)^{m(y)} \right) \\
&= \sum_{\mathbf{s}, \mathbf{t} \in [2k]} \langle a_{\mathbf{s}}, a_{\mathbf{t}} \rangle \sum_{\substack{x \in \mathfrak{S}(M), h(x) = a_{\mathbf{s}} \\ y \in \mathfrak{S}(N), h(y) = a_{\mathbf{t}}}} (-1)^{m(x)} (-1)^{m(y)} \\
&= \sum_{\substack{x \in \mathfrak{S}(M), y \in \mathfrak{S}(N) \\ h(x) = a_{\mathbf{s}} = h(y)}} (-1)^{m(x) + m(y)} \\
&= \sum_{x \boxtimes y} (-1)^{m(x) + m(y)} \\
&= \chi(M \boxtimes N).
\end{aligned}$$

The second part is the specialization of the first part to $\widehat{CFA}$ and $\widehat{CFD}$ with the grading m defined in Section 4.1 and $\widehat{CF}$ graded by the Maslov grading. We remark that $[\widehat{CFD}]$ and $[\widehat{CFA}]$ are elements of $\Lambda^k H_1(F;\mathbb{Z})$. The last step in the above equation follows, up to an overall sign, from Proposition 4.4.1. $\square$

Proof of Theorem 6. This is the Alexander-graded version of Theorem 5. For $\mathcal{H}_1$ and $\mathcal{H}_2$ as in Proposition 4.4.2,

$$[\widehat{CFA}(\mathcal{H}_1)] = \sum_{\substack{x\in\mathfrak{S}(\mathcal{H}_1)\\ I_A(x)=\iota_0}} (-1)^{m(x)}t^{a(x)}a_1 + \sum_{\substack{x\in\mathfrak{S}(\mathcal{H}_1)\\ I_A(x)=\iota_1}} (-1)^{m(x)}t^{a(x)}a_2$$

$$[\widehat{CFD}(\mathcal{H}_2)] = \sum_{\substack{y\in\mathfrak{S}(\mathcal{H}_2)\\ I_D(y)=\iota_0}} (-1)^{m(y)}t^{a(y)}a_1 + \sum_{\substack{y\in\mathfrak{S}(\mathcal{H}_2)\\ I_D(y)=\iota_1}} (-1)^{m(y)}t^{a(y)}a_2.$$

We multiply out and see that $[\widehat{CFA}(\mathcal{H}_1)]\cdot[\widehat{CFD}(\mathcal{H}_2)]$ equals

$$\sum_{\substack{x\in\mathfrak{S}(\mathcal{H}_1),y\in\mathfrak{S}(\mathcal{H}_2)\\ I_A(x)=\iota_0=I_D(y)}} (-1)^{m(x)+m(y)}t^{a(x)+pa(y)} + \sum_{\substack{x\in\mathfrak{S}(\mathcal{H}_1),y\in\mathfrak{S}(\mathcal{H}_2)\\ I_A(x)=\iota_1=I_D(y)}} (-1)^{m(x)+m(y)}t^{a(x)+pa(y)}$$

$$= \sum_{\substack{x\in\mathfrak{S}(\mathcal{H}_1),y\in\mathfrak{S}(\mathcal{H}_2)\\ x\boxtimes y\neq 0}} (-1)^{m(x)+m(y)}t^{a(x)+pa(y)}$$

$$= \sum_{\substack{x\in\mathfrak{S}(\mathcal{H}_1),y\in\mathfrak{S}(\mathcal{H}_2)\\ x\boxtimes y\neq 0}} (-1)^{M(xy)}t^{A(xy)}$$

$$= \pm t^k\chi(\widehat{CFK}(K_C))$$

Gluing a Heegaard diagram for the unknot in the 0-framed solid torus with one generator (see Section 3.3) to a diagram for $(S^3\setminus K, 0)$ shows that the a_1 component of $[\widehat{CFD}(S^3\setminus K, 0)]$ is $\chi(\widehat{CFK}(K)) = \Delta_K(t)$. Similarly, gluing a diagram with one genertor for the 0-framed complement of the unknot to a diagram for $C \hookrightarrow S^1\times D^2$ in a 0-framed $S^1\times D^2$ shows that the a_1 component of $[\widehat{CFA}(S^1\times D^2, C)]$ is $\chi(\widehat{CFK}(C)) = \Delta_C(t)$.

Proving that the a_2 component of $[\widehat{CFD}(S^3\setminus K, 0)]$ vanishes requires a different argument. Say rank $\widehat{HFK}(S^3, K) = n$ and assume we have a basis $\{x_0, \dots, x_{2n}\}$ for $CFK^-(K)$ which is both horizontally and vertically simplified. We illustrate $CFK^-(K)$ on the (U, A) lattice as in Section 3.2.1. Recall this means we have $2n+1$ points $\{\xi_0, \dots, \xi_{2n}\}$ representing the basis, and a vertical (respectively horizontal) arrow from x_i to x_j of length l is represented by a vertical (respectively horizontal) arrow of length l pointing down (respectively to the left) staring at ξ_i and ending at ξ_j. Note that there is at most one vertical (respectively horizontal) arrow starting or ending at each basis element, and there is a unique element

ξ^v (respectively ξ^h) with no in-coming or out-going vertical (respectively horizontal) arrow. See Figure 12.

Recall that we can obtain $\widehat{CFD}(S^3 \setminus K, 0)$ by replacing arrows with chains of coefficient maps, and adding one more chain from ξ^v to ξ^h, called the unstable chain.

Choose ι_0 to be the base idempotent, and define $\psi(\iota_1) = (\frac{1}{2}; 1, 0, 0) \in G'(4)$. This refinement give rise to a $G(\mathcal{Z})$ grading, and hence to a $\mathbb{Z}/2$ grading m. With this choice, we list the relative (m, a) gradings of a horizontal chain of length l, a vertical chain of length l, and the unstable chain when $\tau(K) = 0$, $\tau(K) > 0$, and $\tau(K) < 0$, in this order.

$$(0,0) \xrightarrow{D_1} (0,-\tfrac{1}{2}) \xleftarrow{D_{23}} (0,-\tfrac{3}{2}) \xleftarrow{D_{23}} \cdots \xleftarrow{D_{23}} (0,\tfrac{1}{2}-l) \xleftarrow{D_{123}} (1,-l)$$

$$(0,0) \xrightarrow{D_3} (0,\tfrac{1}{2}) \xrightarrow{D_{23}} (0,\tfrac{3}{2}) \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} (0,l-\tfrac{1}{2}) \xrightarrow{D_2} (1,l)$$

$$(0,0) \xrightarrow{D_{12}} (0,0)$$

$$(0,0) \xrightarrow{D_1} (0,-\tfrac{1}{2}) \xleftarrow{D_{23}} (0,-\tfrac{3}{2}) \xleftarrow{D_{23}} \cdots \xleftarrow{D_{23}} (0,\tfrac{1}{2}-2\tau) \xleftarrow{D_3} (0,-2\tau)$$

$$(0,0) \xrightarrow{D_{123}} (1,\tfrac{1}{2}) \xrightarrow{D_{23}} (1,\tfrac{3}{2}) \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} (1,-2\tau-\tfrac{1}{2}) \xrightarrow{D_2} (0,-2\tau)$$

We plot the chains on the (U, A) coordinate system so that the grading a of a generator with coordinates (x, y) is given by $x - y$. Draw each chain corresponding to a vertical (respectively horizontal) arrow also as a vertical (respectively horizontal) chain, and represent coefficient maps between a generator in ι_0 and a generator in ι_1 by arrows of length $\frac{1}{2}$, and coefficient maps between two generators in ι_1 by arrows of length one. The choice for the unstable chain depends on $\tau(K)$.

Case 1: $\tau(K) = 0$. Ignore the D_{12} map, and identify ξ^v and ξ^h if they are not the same basis element. Note this may not result in the correct model for $\widehat{CFD}(S^3 \setminus K, 0)$, but the information about the ι_1 elements is intact, which is all we are interested in.

<u>Case 2:</u> $\tau(K) > 0$.

Note that in this case ξ^v is $\tau(K)$ units above and $\tau(K)$ units to the left of ξ^h. Draw the unstable chain in an L-shape, as follows. Starting at ξ^v, represent the first coefficient map by a vertical arrow of length $\frac{1}{2}$, so that the first ι_1 element is half a unit below ξ^v. Proceed downwards, drawing the D_{12} maps to have length one, until half the ι_1 elements have been plotted. Repeat the process for the other half, starting at ξ^h and going to the left. Connect the middle two elements by a straight arrow to represent the coefficient map between them.

<u>Case 3:</u> $\tau(K) < 0$.

Here ξ^v is $|\tau(K)|$ units below and $|\tau(K)|$ units to the right of ξ^h. Rotate the construction for Case 2 by 180°.

Figure 12 illustrates the above description with a couple of examples.

Note that each element lies on a line L_t of slope 1 passing through $(t, 0)$, for some $t \in \frac{1}{2}\mathbb{Z}$. For ι_0 elements $t \in \mathbb{Z}$, and for ι_1 elements $t \in \mathbb{Z} + \frac{1}{2}$. If $a(x) - a(y) = t \in \frac{1}{2}\mathbb{Z}$, then the line through x is t units above the line through y.

Let G be the graph on vertices $\xi_0, \ldots, \xi_{2n}$ and edges the $2n + 1$ chains (note that if $\tau(K) = 0$ we may have only $2n$ vertices and $2n$ edges), embedded as above. Every vertex has degree 2, so G is a union of cycles. Endow edges with the orientation induced by the horizontal and vertical arrows for CFK^-, and orient the edge corresponding to the unstable chain to start at ξ^v and end at ξ^h.

Rotate the plane clockwise by 45°, so that the lines L_t are now horizontal, and smoothen G locally at the vertices, so we can think of it as an immersion of a union of circles. The vertices now comprise the local minima, local maxima, and the points with vertical tangents of the immersion. A vertex $v \neq \xi^v, \xi^h$ is a local maximum if it has an incoming horizontal edge and an outgoing vertical edge, and a local minimum if it has an incoming vertical edge and an outgoing horizontal edge. The vertex ξ^v is a local maximum if it has an incoming horizontal edge and $\tau(K) \geq 0$, and a local minimum if it has an outgoing horizontal edge and $\tau(K) \leq 0$. Otherwise it has a vertical tangent. Similarly, ξ^h is a local maximum if it has an outgoing vertical edge and $\tau(K) \leq 0$, and a local minimum if it has an incoming vertical edge and $\tau(K) \geq 0$. Note that this covers the case when $\xi^v = \xi^h$.

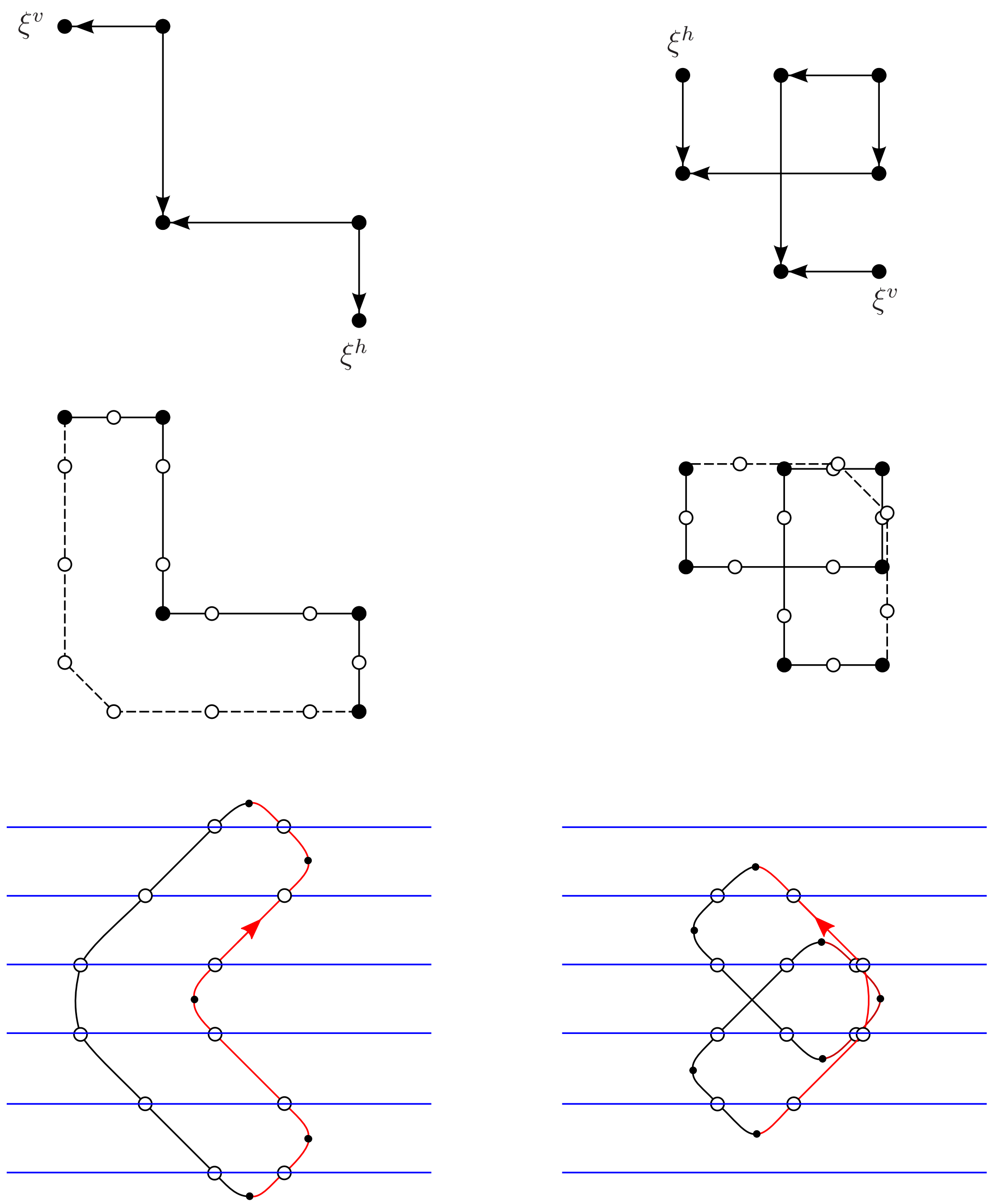


Figure 12: Two examples of the graphical interpretation of CFK^- and $\widehat{CFD}$. Left: The $(3, 4)$ torus knot ($\tau = 3$). Right: the $(2, -1)$ cable of the left-handed trefoil ($\tau = -2$). On top are the models for CFK^-, below are the models for $\widehat{CFD}$, with dashed unstable chain, and on the bottom are the rotated, smoothened graphs. Red and black represent opposite m gradings.

Tracing the ι_1 elements along a connected component of G, observe that the m grading changes exactly when passing through a local maximum or minimum. On the other hand, think of a connected component as an immersed circle, ignore the orientations of the edges, and fix an orientation for the circle. The derivative of the height function changes sign exactly at the local maxima or minima, so two ι_1 elements have the same m grading exactly when the derivatives of the height function at their coordinates have the same sign. For any $t \in \mathbb{Z} + \frac{1}{2}$, the line L_t crosses G away from any local minima and maxima, and the oriented intersection number of L_t and G is zero, since G consists of immersed circles. This means that half the intersection points have positive derivative, and the other half have negative derivative (see Figure 12). In other words, half of the ι_1 elements of a given a grading have m grading 0, and the other half have m grading 1, and so they cancel each other out in the summation for $[\widehat{CFD}(S^3, K), 0]$, i.e.

$$\sum_{x \in \mathfrak{S}(\widehat{CFD}(S^3 \setminus K, 0)), I_D(x) = \iota_1} (-1)^{m(x)} t^{a(x)} = 0.$$

In other words, the a_2 component of $[\widehat{CFD}(S^3 \setminus K, 0)]$ is zero.

Last, we show that we do not in fact need the assumption that we have a basis which is both vertically and horizontally simplified.

Let $\Xi = \{\xi_0, \ldots, \xi_{2n}\}$ be the plot of a vertically simplified basis with ξ^v in position $(0, \tau)$, and let $H = \{\eta_0, \ldots, \eta_{2n}\}$ be the plot of a horizontally simplified basis with η^h in position $(\tau, 0)$. Also plot the vertical and horizontal chains.

The symmetries of CFK^∞ discussed in [OS03b, Section 3.5] imply that when we have a reduced complex, the vertical and horizontal complexes are isomorphic as graded, filtered complexes. The change of basis for this isomorphism may not be a bijection, and in fact may not even map generators to homogeneous linear combinations of generators, but since the isomorphism preserves gradings and filtrations, we may deduce that the number of elements of Ξ in position (x, y) with given Maslov grading is the same as the number of elements of H with the same grading in position (y, x).

In addition, the symmetry

$$\widehat{HFK}_i(K, j) \cong \widehat{HFK}_{i-2j}(K, -j)$$

implies that there is the same number of elements of Ξ with given parity of the Maslov grading in position (x, y), as in position (y, x).

Since the grading m agrees with the Maslov grading, by combining the two symmetries, we see that for a given m grading (0 or 1) there the number of elements of Ξ of that grading in a given position (x, y) equals the number of elements H of the same grading in the same position. In other words, we can find a bijection $b : H \to \Xi$ that preserves coordinates and also preserves the m grading. Identify the two bases under this bijection, i.e. think of a horizontal chain from η_i to η_j, as a horizontal chain from $b(\eta_i)$ to $b(\eta_j)$, and think of the unstable chain as going from ξ^v to $b(\eta^h)$. While the result of this identification may not represent $\widehat{CFD}(S^3 \setminus K, 0)$, it has the same graphical structure that we already analyzed in the case of a basis which is simultaneously horizontally and vertically simplified. The bigradings on the chains when moving along a connected component of the graph obey the same rules as before, since b respects the m grading, and by Lemma 3.2.5 the a grading of any element is specified by its coordinates. This allows is to make the same cancelation argument as before. $\square$

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