Three Essays on Asset Pricing

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ABSTRACT
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The first essay examines the joint determination of the contract for a private equity (PE) fund manager and the equilibrium risk premium of the PE fund. My model relies on two realistic features of PE funds. First, I model agency frictions between PE fund’s investors and manager. Second, I model the illiquidity of PE fund investments. To alleviate agency frictions, compensation to the manager becomes sensitive to the PE fund performance, which makes investors excessively hold the PE fund to hedge the manager’s fees. This induces a negative effect on the risk premium in equilibrium. For the second feature, I add search frictions in the secondary market for PE fund’s shares. PE fund returns also contain a positive illiquidity premium since investors internalize the possibility of holding sub-optimal positions in the PE fund. Thus, my model delivers a plausible explanation for the inconclusive findings of the empirical literature regarding PE funds’ performance. Agency conflicts deliver a lower risk-adjusted performance of PE funds, while illiquidity risk can raise it.

In the second essay, coauthored with Andrew Ang and Pierre Collin-Dufresne, we investigate how often investors should adjust asset class allocation targets when returns are predictable and updating allocation targets is costly. We compute optimal tactical asset allocation (TAA) policies over equities and bonds. By varying how often the weights are reset, we estimate the utility costs of different frequencies of TAA decisions relative to the continuous optimal Merton (1971) policy. We find that the utility cost of infrequent switching is minimized when the investor updates the target portfolio weights annually. Tactical tilts taking advantage of predictable stock returns generate approximately twice as much value as those market-timing bond returns.

In the third essay, also coauthored with Andrew Ang and Pierre Collin-Dufresne, we revisit the question of a pension sponsor’s optimal asset allocation in the presence of a downside constraint
and the possibility for the pension sponsor to contribute money to the pension plan. We analyze the joint problem of optimal investing and contribution decisions, when there is disutility associated with contributions. Interestingly, we find that the optimal portfolio decision often looks like a “risky gambling” strategy where the pension sponsor increases the pension plan’s allocation to risky assets in bad states. This is very different from the traditional prediction, where in economy downturns the pension sponsor should fully switch to the risk-free portfolio. Our solution method involves a separation of the pension sponsor’s problem into a utility maximization problem and a disutility minimization one.
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Equilibrium Asset Prices and Manager’s Contracts: 
An Application to Private Equity Fund

1.1 Introduction

When investing in private equity (PE) funds, a contract with a manager (general partner) is particularly relevant for investors (limited partners). The manager plays an active role in portfolio companies, but investors cannot monitor the manager’s activities closely by the legal structure of limited partnerships. Moreover, investments in PE funds are illiquid, and this lack of liquidity is mainly due to the difficulty in finding a counterparty in secondary markets for fund interests. It takes time to find a counterparty with whom to trade, and the waiting time is uncertain.

Thus, understanding how contract terms are associated with the PE fund performance and how much premium investors require for illiquidity appears to be a critical task. It is also important for a more general understanding of delegated asset management. Surprisingly, there is little theoretical analysis of these institutional features of PE fund investments to help investors’ decisions, despite their large and increasing allocations to PE funds and usage of secondary markets.¹ In this paper, I build a dynamic asset-pricing model in which investors optimally choose the compensation to the fund’s manager and at the same time they can trade fund interests in a secondary market, where the price should be endogenous as well.

I find that higher performance fees are associated with lower net-of-fee performance. When the agency frictions between the manager and investors are more severe, investors make the manager’s compensation more sensitive to the PE fund’s performance since the manager’s diversion decreases the drift of the fund’s cash flow. This implies an increase in performance fees, which are a fraction of the PE fund’s performance. The increase in performance fees makes the manager’s compensation co-vary more with the fund returns: a positive shock to the fund’s cash flow raises the present value of the manager’s compensation, but also the secondary market price of fund interests. This covariance induces investors to tilt their portfolios towards the PE fund to hedge the manager’s compensation. The investors’ excessive demands for the PE fund thus lower risk premium in equilibrium. This finding is consistent with the data\(^2\) and can explain why in equilibrium PE funds earn lower risk-adjusted returns than public equity.\(^3\)

Additionally, I find that when the secondary market is illiquid, the fund return comprises also the illiquidity premium, which is proportional to a difference in investors’ reservation value for the fund. For example, suppose an investor is holding fund interests excessively, and she wants to transfer some of them. The best price at which she would transfer her shares is the buyer’s shadow valuation of the PE fund, whereas the minimum price which she would accept is her shadow valuations of the PE fund. Thus, the difference in investor’s shadow valuations is a maximum price discount that sellers would accept to rebalance their holdings of the PE fund, and the price is the weighted average of the buyer’s and seller’s shadow valuations. While in a competitive market without illiquidity the weight on the seller’s reservation value is the mass of sellers, in the presence of illiquidity, the weight also depends on how much bargaining power sellers have. Thus, the size of the illiquidity premium is higher when investors’ shadow valuations are more

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\(^2\)Robinson and Sensoy (2013) document that performance fees are negatively related to net-of-fee performance for venture capital funds when performance is measured as the levered public market equivalent (PME). They find that 1% increase in performance fees reduces the levered PME by 0.04 when a beta of 2.5 is used to calculate the levered PME.

\(^3\)For example, Kaplan and Schoar (2005) and Phalippou and Gottschalg (2009) find that the average PE fund slightly underperforms public markets.
heterogeneous. The illiquidity premium can raise the risk-premium on the PE fund and my model delivers a plausible explanation for the inconclusive findings of the empirical literature regarding PE funds’ performance.\(^4\)

My model, presented in Section 1.2, is as follows. I consider an economy with a continuum of risk averse investors and a risk averse manager with constant absolute risk aversion (CARA) utility. The manager has a portfolio of companies to invest in and expertise in overseeing them, but she does not have enough wealth. Investors have little ability to monitor portfolio companies, but they have enough wealth. Thus, at time zero, a PE fund is organized by raising a lump-sum capital from investors and hiring the manager. The manager may not exert effort choice, yielding insufficient monitoring of portfolio companies, and ultimately reducing the drift of a cash flow risk which is specific to the PE fund. The manager enjoys a private benefit from shirking, which is the source of agency frictions. Investors write a contract which specifies compensation to the manager in order to alleviate these frictions. Investors also have access to public equity, which is traded in liquid markets. Cash flows of two assets are correlated through a systematic risk factor.

Additionally, as investors may transfer their fund interests in the PE fund in a secondary market, to generate a trading motive of fund interests across investors, I assume that investors receive an endowment whose covariance with the PE fund’s cash flow is time-varying and heterogeneous.\(^5\) The key assumption is that there exists a background risk which can be hedged only by the PE fund investment, not by public equity. Investors are characterized by an intrinsic valuation for the PE fund that is “high” or “low.” A high-type investor values the PE fund higher than a low-type investor since high-type investors have relatively lower covariance between their endowments and the PE fund’s cash flow than low-type investors do.

Initially, I examine the frictionless economy in which there are no agency and search fric-

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\(^4\)Harris, Jenkinson, and Kaplan (2014) use the most comprehensive database and find that PE performance has consistently exceeded that of public markets.

\(^5\)I assume that existing contractual agreements with the manager are assumed by the new investor when the fund interests are transferred to the new investor, i.e. there’s no renegotiation of contract.
tions. Investors optimally share the risks with the manager by making the manager’s fees linear in the performance of aggregate risky assets. This coincides with the standard rule for the optimal risk-sharing under CARA utility. The optimal risk-sharing rule induces investors to increase the holdings of risky assets since the net exposure to risks is reduced. In equilibrium, to clear the market, these additional demands push up the prices of risky assets. Thus, in the economy with delegation of asset management, prices of risky assets are higher relative to those in an economy without delegation, and risks are priced as if there is a representative agent whose risk aversion is less than the investors’ risk aversion.

Heterogeneous valuations of the PE fund across investors affect the investors’ portfolio choices. High-type investors demand more limited partnerships in the PE fund and less public equity than low-type investors do. Increased holdings of the PE fund expose high-type investors to excessive systematic risks. Thus, high-type investors reduce their holdings of public equity to achieve the optimal exposure to the systematic risk. This finding is well supported with the recent asset allocation trend of U.S. university endowments: divesting from public equity and increasing holdings in alternative investments which include PE funds and hedge funds.\(^6\)

To capture the agency premium component in the risk premium of the PE fund, I introduce agency frictions. Relative to the optimal risk-sharing case, the compensation to the manager is more sensitive only to the PE fund’s performance. Since the manager’s shirking reduces the drift of the PE fund’s cash flow, it is natural to make the compensation more tied to the PE fund’s performance. The increase in performance fees has three effects on the PE fund returns. First, the risk averse manager requires a higher fixed component in her compensation, which lowers the price of the PE fund. Second, the PE fund returns co-vary more with the manager’s compensation, which makes investors unwilling to deviate from the PE fund and increases the investors’ valuation of fund interests. Lastly, risk averse investors impose a higher discount in the price of the PE fund

\(^6\)Ang, Ayala, and Goetzmann (2014) report that in 2006 the average U.S. university endowment allocated 46% of assets to U.S. stocks while at the end of 2012 the average allocation dropped to 32%. At the same time, the average allocation to PE funds increased from 5% in 2006 to 9% at the end of 2012.
for more volatile fees.

In the net-of-fee PE fund returns, the first effect is exactly canceled out by the increase in the manager’s fixed fees. Netting the second and third effects yields the negative agency premium. Higher performance fees raise the variance of fees in a second order, and they linearly increase the covariance between the PE fund’s performance and the manager’s compensation. Given that a fraction of the PE fund performance that goes to the manager is between zero and one, the second effect always dominates the third one.

I also consider the case where search frictions exist in the secondary market. The illiquid secondary market for limited partnerships is a variant of the over-the-counter market model of Duffie, Gârleanu, and Pedersen (2005, 2007). An investor seeks a buyer since she finds that her endowment is relatively more correlated with the PE cash flow than the buyer’s endowment. Similarly, an investor is willing to provide liquidity since her endowment is relatively less correlated with the PE cash flow than the seller’s endowment. Then, a candidate seller and buyer search for each other at some mean rate, a parameter that reflects the liquidity of the secondary market. When two investors meet, they bargain over the price of limited partnerships.

We see that search frictions in the secondary market do not alter the optimal contract. Although search frictions can cause some investors to hold sub-optimal positions in the PE fund, they are facing the same agency frictions as long as they are holding shares in the PE fund. Sub-optimal holders would choose the same contract as the optimal holders do, since that is the only way they can raise the value of their PE fund portfolio by alleviating agency frictions.

While the size of the illiquidity premium is higher when investors’ shadow valuations are more heterogeneous, its sign depends on market conditions. On the one hand, the positive illiquidity premium arises and more weight is put on the seller’s reservation value if the demand is scarce either because buyers no longer wish to purchase fund interests, or because buyers can meet other sellers more easily. On the other hand, the negative illiquidity premium can be obtained and more weight is put on the buyer’s shadow value when the supply is scarce either because investors need
more partnerships suddenly, or because when sellers can easily meet a candidate buyer. This finding can explain cross-sectional variations in empirical secondary market prices of PE funds.\footnote{Kleymenova, Talmor, and Vasvari (2012) find that excess demand is positively associated with the price of fund interests. The ratio of the final winning bid to the average bid as a percentage of net asset value (NAV) observed over the prior 6 months is regressed on the logarithm of the total monetary bid less the maximum bid divided by NAV. Nadauld, Vorkink, Sensoy, and Weisbach (2016) find higher price discounts for more thinly traded private equity fund shares.}

Finally, I relate secondary market returns to performance measures often used in empirical research. I claim that secondary market returns can serve as lower thresholds for empirical performance measures. The key intuition is the investors’ individual rationality constraint. Investors participate in the PE only when the benefits of investing in the PE fund are greater than the costs of doing so. The secondary market price represents the benefits of investing in the PE since investors would obtain this price if they sold their partnerships right after the inception of the PE fund. Thus, secondary market returns may Understate empirical performance measures which are based on the costs of the PE fund investment, i.e. the initial capital paid by investors to the manager.

This paper relates to several bodies of research. First, I complement the literature on delegated asset management. Cuoco and Kaniel (2011), Basak and Pavlova (2013), Buffa, Vayanos, and Woolley (2014), and Leung (2015) all study the implications of incentive contracts on manager’s portfolio choices and equilibrium prices of assets in which the manager invests. My model differs from theirs by allowing investors to trade fund interests in the secondary market and provides impacts of incentive contracts on the fund return itself. The most closely related paper is Ou-Yang (2005), who studies equilibrium asset prices in the presence of a moral hazard problem. However, in that paper, there is only one representative investor, and thus the illiquidity of PE fund investment would not have any impact on equilibrium prices.

Second, this paper relates to the literature on valuation and portfolio choice with illiquid assets. Duffie, Gârleanu, and Pedersen (2005, 2007) derive illiquidity discounts in equilibrium prices with search frictions and restrictions on illiquid asset holdings, and Gârleanu (2009) relaxes the latter assumption. However, in their papers investors have access to only illiquid asset. I complement the
literature by considering an additional risky asset (public equity) that can be traded without delay, which allows me to characterize the alpha and beta of the PE fund investment relative to the public equity. Ang, Papanikolaou, and Westerfield (2014) also consider an asset allocation problem of an investor who trades both liquid and illiquid assets, but by exogenously specifying return processes of both assets, not by endogenizing them.

Finally, my results complement the literature on returns of PE fund by endogenizing contracts and asset prices. Sørensen, Wang, and Yang (2014) and Bollen and Sensoy (2015) investigate whether the performance of a PE fund is sufficient to compensate investors for risk, illiquidity, and fees in a partial equilibrium setting. My model endogenizes the manager’s compensation and the secondary market price of fund interests via a contracting model in general equilibrium.

The remainder of the paper is organized as follows. Section 1.2 describes the model. Section 1.3 characterizes the equilibrium in the benchmark economy in which any number of fund interests can be traded without delay. Section 1.4 presents the equilibrium in the presence of search frictions, and section 1.5 relates secondary returns to empirical performance measures. Section 1.6 provides numerical exercises. Section 1.7 concludes.

1.2 Model

1.2.1 Security Market

I consider a continuous-time economy on a finite time horizon $[0, T]$. There are one riskless asset and two risky assets available. The riskless asset pays the interest rate which is normalized to zero. There is an unlimited supply of the riskless asset. Since the model does not have intermediate consumption, the assumption of zero interest rate is innocuous. The first risky asset is public equity. The cash flow process of a representative public firm is given by

$$d\delta_t = \mu\delta_t dt + \sigma dZ_{\delta t},$$
where the initial value $\delta_0$ is given, and $Z_{\delta t}$ is a standard Brownian motion. There are no intertemporal dividend payments, and the cash flow $\delta_T$ is paid at the terminal date $T$. A claim to the public firm’s cash flow, public equity, is available for continuous trading. I assume that the public firm has one share of the public equity.

There is another risky asset, a PE fund. The manager of the PE fund does not have enough wealth to invest in a portfolio of target companies, and thus he needs to raise a capital $I_0$ at time zero from investors by issuing one share of fund interest.\(^8\) If the manager receives the investment successfully at time zero, the total amount of raised capital is invested in the portfolio companies.\(^9\) I assume that there’s no intertemporal capital call and distribution, and at the terminal date $T$ the portfolio companies are liquidated for total proceeds of $y_T$:

$$y_T = b_\delta \delta_T + b_n n_T + s_T,$$

where $n_t$ is a background factor, and $s_t$ is an idiosyncratic factor. Since $\delta_t$ is a component common to both public equity and the PE fund’s cash flow, $\delta_t$ serves as a systematic factor in the model. The processes for the background and the idiosyncratic factors are given by

$$dn_t = \mu n_t dt + \sigma dZ_{nt}$$

$$ds_t = (\bar{a} - a_t + \mu s_t) dt + \sigma dZ_{st},$$

with the initial value $n_0$ and $s_0$. The $Z_{nt}$ and $Z_{st}$ are standard Brownian motions, and $Z_{\delta t}$, $Z_{nt}$, and $Z_{st}$ are mutually independent. The parameter $b_\delta$ and $b_n > 0$ represent the cash flow beta of the portfolio companies on the systematic and the background factor. I denote by $a_t \geq 0$ the

---

\(^8\)In reality, buyout funds leverage the raised capital with a debt to acquire portfolio companies, and venture capital funds do not use debt. Here, I assume that the only source of external financing is equity for expositional simplicity. If I endogenize a leverage choice by the manager, this serves as another moral hazard problem in addition to the unobservable diversion choice. If the leverage choice is observable, the compensation may depend on the actual risk exposure delivered to the LP. If the leverage choice is unobservable, the manager would achieve his optimal risk exposure regardless of shares he receives, and thus the contract may decrease the risk premium component.

\(^9\)The timing of fundraising is ahead of acquisition of portfolio companies, but for tractability I assume there’s no delay between fundraising and acquisition.
manager’s shirking. The manager’s shirking can be interpreted in two ways. First, \( a_t \) can represent insufficient monitoring of entrepreneurs. Less monitoring implies that the entrepreneurs of the portfolio companies make less effort, and ultimately decrease the idiosyncratic cash flow process in the drift term. Second, \( a_t \) can be interpreted as cash diversion by the manager. The parameter \( \mu \) captures the intrinsic growth rate of the public firm’s cash flow, and I assume \( \mu = 0 \) for expositional purposes. Since dividends are paid only at the terminal date, this assumption is innocuous and does not alter any of economic insights. The cash flow of the portfolio companies intrinsically grows faster than that of the public firm due to the manager’s value-adding activity \( \bar{a} > 0 \). This higher growth rate might be attenuated by the manager’s shirking \( a_t \).

I assume that fund interests in the PE fund are traded in a liquid secondary market after the inception of the PE fund. A seller may transfer her fund interests to a buyer who then assumes the existing contractual agreements with the manager.\(^\text{10}\) For a baseline model, investors can hold any number of PE fund shares, and can find counterparties immediately. In section 1.4, I extend the model to incorporate illiquidity of PE fund investment by assuming that the investors hold either large or small number of PE fund shares, and there exist search frictions.

I define a vector \( X_t^T = [\delta_t, n_t, s_t] \), which serves as state variables. Then, I have

\[
\frac{dX}{dt} = A_t dt + \Sigma_X dZ_t,
\]

where \( A_t \), \( \Sigma_X \), and \( Z_t^T \) can be obtained by stacking up appropriate variables. An undesirable feature of this cash flow process is that it can take negative values. As a result, the equilibrium price can take negative values. However, this specification allows for closed-form solutions, and thus I can analyze the impact of agency frictions and illiquidity of the PE investment on its expected return. The probability of the cash flow reaching negative can be made small by choosing appropriate parameter values.

\(^{10}\)There are other types of secondary transactions which involve renegotiation of contracts, such as GP restructuring. However, a transfer of a single fund is the most common transaction type, see â€” Prequin special report: private equity secondary market,” Prequin, March, 2015.
1.2.2 Agents

The manager enjoys a private benefit \( \int_0^T g(a_t) dt \) when he diverts \( a_t \). The benefit function is given by

\[
g(a) = \kappa_0 a - \frac{\kappa_1}{2} a^2,
\]

where the parameter \( \kappa_0 (0 \leq \kappa_0 \leq 1) \) is a proxy for a manager’s diversion skill, and the parameter \( \kappa_1 (\kappa_1 \geq 0) \) is a proxy for an inverse of severity of agency frictions. When \( \kappa_0 \) is equal one, a deadweight cost is zero. The more severe agency frictions (lower \( \kappa_1 \)), the higher marginal benefit from shirking. Total proceeds of \( y_T \) from the liquidation of the portfolio companies are divided among the manager and the investors according to the contract specifying the fees to the manager, \( F_T \). I assume that the manager’s personal wealth is zero, and that a securitization of his compensation is not allowed.\(^{11}\) The manager has a negative exponential utility function with a constant absolute risk aversion coefficient \( \bar{\gamma} \): \( \bar{u}(x) = -\frac{1}{\bar{\gamma}} \exp(-\bar{\gamma}x) \). Given that the investment is received, the manager’s objective is to maximize his expected utility

\[
\max_{\{a_t\}} \mathbb{E} \left[ \bar{u} \left( F_T + \int_0^T g(a_t) dt \right) \right].
\]

I denote by \( \{a_t(F_T)\} \) the manager’s optimal choice of diversion, and by \( V(X_t, t; F_T) \) the manager’s value function given the compensation scheme \( F_T \). Let \( \epsilon_0 \) be the manager’s reservation wage.

Then, given the compensation scheme \( F_T \), the manager’s individual rationality (IR) constraint is

\[
\text{(Manager’s IR)} \quad V(X_0, 0; F_T) \geq \bar{u}(\epsilon_0).
\]

The manager would earn the certainty equivalent wealth \( \epsilon_0 \) unless the investors invested in the PE fund, and thus IR constraint guarantees at least the utility evaluated at \( \epsilon_0 \).

\(^{11}\)This assumption implies that the only source of income for the manager is from the compensation paid by the investors for managing the portfolio companies. Suppose that the manager has personal wealth. If the manager’s initial wealth is observable, then the contract may ask the manager to commit his wealth to the PE fund. Even if the initial wealth is the manager’s private information, the absence of wealth effect implies that facing any contract, the manager will take the same diversion policy as another hypothetical manager with zero initial wealth.
There is a measure one of investors, who also have a negative exponential utility function with a constant absolute risk aversion coefficient $\gamma$: $u(x) = -\frac{1}{\gamma} \exp(-\gamma x)$. At time zero, the investors are endowed with the initial liquid wealth $\overline{W}$. The investors also receive an endowment $N_T = m_T n_T$ at the terminal date. This might be thought as cash flow from another investment. The key assumption is that there exists the background risk $n_t$ which can be hedged only by the PE fund investment, not by the public equity.

The investors are characterized by an intrinsic valuation for the PE fund that is “high” or “low.” A low-type investor has $m_t = m_l$ and values the PE fund investment lower than a high-type investor with $m_t = m_h$ ($m_h < m_l$) since the covariance between the PE fund’s cash flow and the terminal endowment is higher than that of a high-type investor. The investor’s exposure to the background factor $m_t$ is a Markov chain, switching from $m_l$ to $m_h$ with intensity $\lambda_l$, and back with intensity $\lambda_h$. A transition of $m_t$ represents an endowment shock. The exposure processes of any two investors are independent. This generates a trading motive of the fund interest across investors.

I denote by $i = \{l, h\}$ the investors’ types. The letter “$l$” and “$h$” represent the investor’s intrinsic valuation of the partnership in the PE fund as low or high, respectively. I assume that the investment horizon $T$ is sufficiently long (usually 10-year or longer for a typical PE fund) such that the distribution of the investors’ types is in the steady-state. I denote by $\pi^i$ the steady-state mass of type $i$ investors. Each investor decides the number of shares $\theta_t$ of the public equity, the number of shares $\psi_t$ of the PE fund, and designs the optimal contract for the manager to maximize the expected utility over the terminal wealth:

$$\max_{\{\theta_t, \psi_t, a_t\}, F_T} \mathbb{E}[u(W_T + m_T n_T - \psi_T F_T)],$$ (1.1)

subject to the manager’s IR constraint and incentive compatibility constraint:

$$(IC) \ a_t = a_t(F_T).$$

The initial liquid wealth is $W_0 = \overline{W} + \psi_0(Q_0 - I_0)$ and the dynamics of $W_t$ is $dW_t = \theta_t dP_t + \psi_t dQ_t$, where $P_t$ and $Q_t$ is the price of public equity and the PE fund per share. I denote by $J(W_t, X_t, i, t)$
the type \( i \) investor’s value function. If the investor does not invest in the PE fund, the entire initial wealth is allocated to the riskless asset and the public equity. The investor’s optimization problem is to choose \( \theta_t \) to maximize (1.1) with the initial liquid wealth \( W_0 = \bar{W} \) and \( \psi_t = 0 \). I denote by \( J_u(W, X, i, t) \) his value function when not investing in the PE fund. Then, the investor’s individual rationality constraint is

\[
(\text{Investor’s IR}) \quad J(\bar{W} + \psi_0(Q_0 - I_0), X_0, i, 0) \geq J_u(\bar{W}, X_0, i, 0).
\] (1.2)

### 1.2.3 Prices

I conjecture that the price of one share of the public equity is given by

\[ P_t = p_{0,t} + p_{1,t}X_t. \]

I have the following boundary conditions: \( p_{0,T} = 0 \) and \( p_{1,T} = [1 \ 0 \ 0] \) since the final price is equal to its payoff. This implies that I can write the excess return for holding one share of the public equity within time interval \( dt \) as:

\[
dP_t = \mu_{P,t}dt + p_{1,t}\Sigma_X dZ_t,
\]

where the expression for \( \mu_{P,t} \) can be found in the appendix. I also conjecture that the price of one share of the PE fund is given by

\[ Q_t = q_{0,t} + q_{1,t}X_t, \]

with the boundary conditions: \( q_{0,T} = 0 \), and \( q_{1,T} = [b_\delta \ b_n \ 1] \). This price is reflecting the investors’ costs of compensating the manager. Thus, the price can also be interpreted as an ex-dividend price. In this case, the compensation to the manager is a negative-dividend payment. This definition of price avoids a jump in the price right before the terminal date. When computing a net-of-fee return (or cum-dividend return), one must take the manager’s compensation into account. I can derive the dynamics of \( Q_t \):

\[
dQ_t = \mu_{Q,t}dt + q_{1,t}\Sigma_X dZ_t,
\]
where the expression for $\mu_{Q,t}$ can also be found in the appendix.

### 1.2.4 Contract Space

The compensation to the manager is given by

$$F_T = \int_0^T dF_t = \phi_T + \int_0^T h_t dt + f_t dX_t.$$  

The functions $\phi_t$, $h_t$, and $f_t$ might depend on $X_t$. This contract space is quite general, which can include a linear or nonlinear contract, and benchmarking to the return of public equity, etc. In actuality, the manager’s compensation consists of intertemporal management fees, typically $1.5\%-2\%$ of the raised capital and a performance based incentive fee (carried interest), typically $20\%$ of profits.\footnote{In practice, the final proceeds, $y_T$, are divided among the manager, investors, and creditors (if levered up) according to the “waterfall” schedule. Typically, the creditors are paid first, the investors receive the raised capital back plus a hurdle rate, the manager is paid then to catch up to the prescribed carried interest, and finally additional proceeds are split between the investors and the manager according the the carried interest. See Gompers and Lerner (1999), Metrick and Yasuda (2010), and Robinson and Sensoy (2013) for more details.}

Here, the accumulated term $\int_0^T h_t dt$ serves as the management fees, and the term $\phi_T + \int_0^T f_t dX_t$ serves as the carried interest for expositional simplicity. Since I assume that the manager is not allowed to participate in the securities markets using his own account, the manager finds no difference between intertemporal management fees and the accumulated fees paid at the terminal date. The terminal compensation $F_T$ also has the interpretation of the accumulated payment $dF_t$ from time zero to $T$. Thus, an investor who holds one share of partnership in the PE fund within time interval $[t, t+dt]$ has an obligation to compensate the manager by $dF_t$. Thus, the net-of-fee return is defined as $d\hat{Q}_t = dQ_t - dF_t$, and the expected net-of-fee return is defined as $\mu_{\hat{Q},t} = \mu_{Q,t} - \mathbb{E}_t[dF_t]$.

### 1.2.5 Equilibrium

The economy introduced so far is summarized in Figure 1.1. I look for equilibria in which the investors participate in the PE fund, i.e. their IR constraint is satisfied and the manager accepts the...
contract. These equilibria are described by price processes \( \{P_t, Q_t\} \), a compensation contract \( F_T \), the manager’s diversion choice \( \{a_t\} \), and the investment \( \{\theta^i_t\} \) in the public equity and \( \{\psi^i_t\} \) in the PE fund by the type \( i \) investors.

**Assumption 1.1.** The average investor’s exposure to the background factor is positive: \( 0 < \bar{m} = \frac{\lambda^l m^b + \lambda^h m^l}{\lambda^l + \lambda^h} \) and the cash flow beta on the background factor satisfies \( b_n \geq \frac{\gamma}{\gamma + \gamma} \bar{m} \).

This assumption ensures that in equilibrium the net-of-fee PE fund return is positively exposed to the background factor, and thus it is justified that limited partnerships in the PE fund are valued higher by the investors with low covariance between the terminal endowment and the PE fund’s cash flow than by the investors with high covariance.

**Assumption 1.2.** The manager’s diversion technology satisfies \( \kappa_0 > \frac{\gamma}{\gamma + \gamma} \).

This assumption makes sure that in the absence of incentive contracts there exists inefficiency due to agency frictions, i.e. if a substantial fraction of the benefits from shirking disappears as a deadweight cost, the manager would not shirk at all, and thus there’s no effect of agency frictions.

**Definition 1.3.** Price processes \( \{P_t, Q_t\} \), a compensation contract \( F_T \), the manager’s diversion \( \{a_t\} \), and the investment \( \{\theta^i_t, \psi^i_t\} \) form an equilibrium if:
(i) Given $P_t$, $Q_t$, and $F_T$, $a_t$ solves the manager’s optimization problem.

(ii) Given $P_t$ and $Q_t$, the investors choose to invest in the PE fund, hold the entire public equity and PE: $1 = \sum_i \pi^i \theta^i_t$ and $1 = \sum_i \pi^i \psi^i_t$ for $i = \{l, h\}$. The investors decide a contract $(\{a_t\}, F_T)$, which solves the investors’ optimization problem.

### 1.3 Equilibrium without Illiquidity

In this section, I consider a competitive equilibrium in which the investors can immediately trade any number of fund interests i) when there are no agency frictions, and ii) when there exist agency frictions. Before solving for equilibrium, I derive the property of optimal contract.

#### 1.3.1 Optimal Contract

In this subsection, I transform the original contract form using the manager’s IR constraint. I provide an expression for the optimal compensation to the manager in terms of the original contract coefficients and the manager’s value function. Formally, the manager’s value function is defined as

$$V(X_t, t; F_T) = \max_{a_t} E_t \left[ \bar{u} \left( \phi_T + \int_t^T (h_u + g(a_u)) \, du + f_u dX_u \right) \right].$$

Although the manager’s original problem is to maximize the expected utility over the entire period, I use a dynamic programming approach. The key point is that the value function at time zero corresponds to the manager’s expected utility, which is the original problem. I will use the similar approach to solve the investors’ problem.

**Lemma 1.4.** The optimal compensation to the manager is given by

$$F_T = \epsilon_0 - \int_0^T g(a_t) \, dt + \frac{\gamma}{2} \int_0^T \bar{f}_t \Sigma \Sigma^T \bar{f}_t^T \, dt + \bar{f}_t \Sigma \Sigma dZ_t,$$

where $\bar{f}_t = f_t - \frac{V(X_t, t; F_T)}{\tilde{V}(X_t, t; F_T)}$. Then, the manager’s optimal diversion is given by

$$\kappa_0 - \kappa_1 a_t = \bar{f}_t 1_{3},$$

(1.3)
where $1_u$ is the $3 \times 1$ zero vector with $u$-th element of one.

At the terminal date, the investors pay the manager the reservation wage, charge the manager’s private benefits, compensate the manager the risk premium in the third term, and share the risk with the manager. Note that $V(X_0, 0; F_T) = \bar{u}(e_0)$, that is, under the optimal contract the manager’s IR constraint is always binding. The sensitivity of the contract to the shocks, $\tilde{f}_t$, is adjusted to reflect the sensitivity to the state vector of the manager’s value function in monetary terms (scaled by $-\bar{\gamma}V(X_t, t; F_T)$). Thus, the investors’ problem becomes to choose $a_t$ and $\tilde{f}_t$ subject to IC constraint (1.3). The manager’s shirking marginally decreases the drift of the idiosyncratic factor by one unit, and thus the third element of $\tilde{f}_t$ measures the manager’s marginal cost of shirking. To be incentive compatible, the marginal cost must equal the manager’s marginal private benefit $\kappa_0 - \kappa_1 a_t$.

### 1.3.2 Equilibrium without Agency Frictions

I first solve for equilibrium in the absence of agency frictions. I eliminate agency frictions by setting the parameter $\kappa_1$ in the manager’s benefit function to infinity. This implies that the marginal private benefit from shirking is minus infinity for all values of $a_t > 0$, and thus the optimal decision is not to shirk, i.e. $a_t = 0$ for $t \in [0, T]$. The alpha of the PE fund in this case will guide us to find the component due to agency frictions of the PE fund investment. A type $i$ investor’s value function is now given by

$$J(W_t, X_t, i, t) = \max_{\theta_t, \psi_t, f_t} \mathbb{E}_t \left[ u \left( W_T + m_T n_T - \int_t^T \psi_u dF_u \right) \right],$$

with the boundary condition: $J(W_T, X_T, i, T) = u(W_T + m_T n_T)$. The interpretation of $\psi_t dF_t$ is the investor’s payment to the manager for holding $\psi_t$ shares of the partnership within time interval $[t, t + dt]$. I conjecture that the value function has the following form: $J(W_t, X_t, i, t) = u(W_t + k^0_{0,t} + k^1_{1,t} X_t)$. Thus, the term $(k^0_{0,t} + k^1_{1,t} X_t)$ captures a certainty equivalent wealth of the
Theorem 1.5. When any number of shares of the PE fund is traded in the liquid secondary market and there are no agency frictions, the optimal contract is characterized by

\[ \bar{f}_t^w = \frac{\gamma}{\gamma + \bar{y}} (p_{1,t} + q_{1,t} + \bar{k}_{1,t}) \]

where \( p_{1,t} = [1 0 0] \), \( q_{1,t} = [b_1 b_n 1] \), and \( \bar{k}_{1,t} = [0 \bar{m} 0] \). The risk premium of the public equity return and the net-of-fee PE fund return per share are given by

\[
\begin{align*}
\mu_{P,t} &= -\text{Cov}(dP_t, dM_t) \\
\mu_{\hat{Q},t} &= \alpha^w + \beta \mu_{P,t},
\end{align*}
\]

where \( M_t \) is a stochastic discount factor (SDF) and follows

\[ dM_t = -\gamma(p_{1,t} + q_{1,t} + \bar{k}_{1,t} - \bar{f}_t^w)\Sigma X^t dZ_t. \]

The alpha and beta of the net-of-fee PE return is given by

\[
\begin{align*}
\alpha^w &= -\text{Cov}\left(d\hat{Q}_t - \beta dP_t, dM_t\right), \quad \beta = \frac{\text{cov}(d\hat{Q}_t, dP_t)}{\text{var}(d\hat{Q}_t)} = \frac{s}{\gamma + \bar{y}} \left(b_\delta - \frac{2}{\gamma}\right).
\end{align*}
\]

The optimal holdings of the public equity and the PE fund of the type \( i \) investor are given by

\[
\begin{align*}
\theta^i_t &= \frac{\mu_{P,t}}{\gamma \text{Var}(dP_t)} - \beta \psi^i_t \\
\psi^i_t &= \frac{\alpha^w}{\gamma \text{Var}(d\hat{Q}_t - \beta dP_t)} - \frac{J_X(W, X, i, t)}{\gamma J(W, X, i, t)} \frac{\text{Cov}(d\hat{Q}_t - \beta dP_t, dX_t)}{\text{var}(d\hat{Q}_t - \beta dP_t)}. \end{align*}
\]

The optimal contract has an intuitive interpretation. The manager is compensated based on the total risk in the economy. The manager receives a fraction \( \frac{\gamma}{\gamma + \bar{y}} \) of the aggregate risk, and the investors receive the complementary fraction \( \frac{s}{\gamma + \bar{y}} \). This coincides with the standard rule for optimal risk-sharing under CARA utility. The fact that the contract is a standard form might be surprising given that a transfer of contractual agreements with the manager is allowed in the model. The intuition is that the processes for investor’s exposure to the background risk are identical in a sense that the transition intensities are homogeneous. Thus, the model essentially has a representative
investor *ex ante*. Of course, the investors become heterogeneous *ex post* since any two investors’ exposure processes are independent.

The contract can be expressed as

$$ dF_t = \text{constant} \, dt + \frac{\gamma \bar{m}}{\gamma + \gamma} \sigma dZ_{nt} + \frac{\gamma}{\gamma + \gamma} (dP_t + dQ_t), $$

(1.4)

where some constant can be derived. The compensation is path independent and linear in the aggregate portfolio return. Since there are three shocks and two risky assets, the economy is incomplete and the compensation (the second term) cannot be replicated only using the prices of two risky assets. However, what is interesting is the third component in the compensation, a fraction that goes to the manager. This corresponds to carried interests or performance fees of contracts observed in reality. I will show that in the presence of agency frictions, there will be additional sensitivity to the gross PE fund return bechmarked to the public equity return.

The volatility of the equilibrium stochastic discount factor (with minus sign) is the price of the risk:

$$ \text{vol}(-dM_t) = \frac{\gamma \gamma}{\gamma + \gamma} (p_{1,t} + q_{1,t} + \bar{k}_{1,t}) \Sigma X $$

Thus, the risks are priced as if there is a representative agent whose risk tolerance is the sum of the manager’s and the investor’s risk tolerance, and the total risks in the economy is $\sigma (p_{1,t} + q_{1,t} + \bar{k}_{1,t})$. Note that the total background risk is the PE fund’s cash flow beta $b_n$ plus the average investor’s exposure to the background factor $\bar{m}$. This result is surprising since the public equity and the PE fund are held by the investors whose risk aversion is $\gamma$. The intuition for this result is the investor’s hedging motives for the fees to the manager. Because of a risk-sharing motive, the investors make the manager’s fees sensitive to the aggregate risky assets performance. A positive covariance between the fees and risky asset returns implies positive hedging demands for the risky asset since the manager’s fees are negative dividends for the investors. The positive hedging demands increase the price of the risky asset in equilibrium by the market clearing condition, and decrease the investor’s effective risk aversion.
The risk premium of the public equity is a compensation for taking the systematic risk by the amount of $\sigma$. The gross PE fund return $dQ_t$ is overstating the true return to the investors since the price $Q_t$ is discounted by the amount of present value of the manager’s fees that the investor would pay if she held fund interests from time $t$ to $T$. This discount is decreasing over time since the remaining fees to the manager is also decreasing. To measure the true return to the investors, I adjust the gross PE fund return to reflect this decrease in the discount,

$$d\hat{Q}_t = dQ_t - dF_t = \mu_{\hat{Q}} dt + \Sigma_{\hat{Q}} \Sigma_X dZ_t,$$

where $\Sigma_{\hat{Q}} = q_{1,t} - \bar{f}^w_t$. The risk exposure of the net-of-fee PE fund return is reduced by the amount of $\bar{f}^w_t \Sigma_X$. The risk premium of the net-of-fee PE fund return is then the sum of compensation for being exposed to the systematic, background, and idiosyncratic risk by the amount of $\Sigma_{\hat{Q}} \Sigma_X$.

The beta can be obtained by regressing the net-of-fee PE fund return on the public equity return. If the PE fund’s cash flow is sufficiently exposed to the systematic risk, $b_\delta > -1$, the beta of the net-of-fee PE fund return is less than the systematic beta of the PE fund’s cash flow $b_\delta$. A positive covariance between the fees and the public equity return reduces the beta of the net-of-fee PE fund return. The alpha of the net-of-fee PE return is the background and idiosyncratic risk premium. This is intuitive since the systematic part of the PE return is completely spanned by the public equity return.

The first component of the optimal holdings of limited partnerships is a standard mean-variance efficient portfolio, which is the alpha divided by the investor’s risk aversion and the residual variance of the net-of-fee PE fund return. The second component is a hedging demand. An investor hedges against a high realization of the background factor by investing in the PE fund. The hedging demands are the products of the sensitivity of the investor’s value function with respect to the state vector (in monetary term) and the covariance between the residual net-of-fee PE return and the state vector. The sensitivity of the investor’s value function is

$$\frac{J_X(W, X, i, t)}{-\gamma J(W, X, i, t)} = k^i_{1,t} = [0 \hat{m}^i_t 0],$$
where
\[
\hat{m}^i_t = \bar{m} - \frac{\lambda^i}{\lambda^l + \lambda^h}(\tilde{m}^i - m^i)e^{-(\lambda^l + \lambda^h)(T-t)}.
\] (1.5)

The \(\hat{m}^i_t\) is the sensitivity to the background risk of the certainty equivalent wealth of the terminal endowment. The type \(i\) investor’s current exposure to the background factor is \(m^i\), but she takes into account a possibility of future endowment shocks. Thus, the sensitivity is the weighted average of \(m^i\) and \(\tilde{m}^i\), where the more weight is given to the current type. It can be immediately seen that \(\hat{m}^h_t < \hat{m}^l_t\) and \(\lim_{t \to T} \hat{m}^i_t = m^i\). This implies that the hedging demands are determined by the current type, and the hedging demands are time-varying as it is less likely to be affected by an endowment shock close to the terminal date.

The resulting hedging demands are
\[
\frac{J_X(W, X, i, t)}{-\gamma J(W, X, i, t)} \text{Cov} \left( d\hat{Q}_t - \beta dP_t, dX_t \right) = \frac{\tilde{\gamma}}{\gamma + \tilde{\gamma} \hat{m}^i_t \sigma^2} \left( b_n - \frac{\gamma}{\tilde{\gamma}} \bar{m} \right).
\]

A high-type investor demands more fund interests than a low-type investor since \(b_n - \frac{\gamma}{\tilde{\gamma}} \bar{m} > 0\) by the assumption 1.1. Thus, if the beta is positive, the high-type investor demands less public equity than the low-type investor. The intuition is that higher holdings of fund interests induce too much exposure to the systematic risk. Thus, the investor reduces the holdings of the public equity to achieve the optimal exposure to the systematic risk. This is consistent with the finding of Ang, Ayala, and Goetzmann (2014) who find that university endowments decrease allocation to public equity while increasing the investment in PE funds. They also find that the effective allocation to public equity is approximately 60% by taking into account the systematic risk embedded in PE funds, which confirms the intuition that the investors target the amount of the systematic risk.

**Lemma 1.6.** The risk premium of the public equity is increasing in \(\tilde{\gamma}, \gamma, b_n, \) and \(\sigma\). The alpha of the PE is increasing in \(\tilde{\gamma}, b_n, \) and \(\sigma\). If \(\tilde{\gamma} < \gamma\), the alpha is decreasing in \(\gamma, m^l, m^h, \) and \(\lambda^h, \) and is increasing in \(\lambda^l\).

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13This is consistent with empirical findings that net-of-fee PE returns have positive beta. See Ljungqvist and Richardson (2003) and Driessen, Lin, and Phalippou (2012) among others.
The PE fund’s cash flow beta and the volatility parameter increase the price of risks directly by increasing the aggregate risks in the economy. The manager’s risk aversion increases the risk premium of the public equity indirectly through the compensation. The covariance between the fees and the public equity returns decreases when the manager is more risk averse since the fraction that goes to the manager decreases. This implies a decrease in positive hedging demands for the public equity.

The manager’s risk aversion has two effects on the alpha: i) the risk-sharing through the contract is limited, and thus the price of risks increases, and ii) the limited risk-sharing also leaves more risk in the net-of-fee return. Both effects increase the alpha. The investor’s risk aversion and the average investor’s exposure to the background factor, \( \bar{m} \), also increases the prices of risks. However, the effect on the risk-sharing is opposite: the investors would unload the risks more through the contract when they are more risk-averse, or when their average exposure to the background risk is larger. The overall effect depends on the manager’s and investor’s risk aversion: the higher investor’s risk aversion implies the latter effect is dominating, and thus the alpha is decreasing in the investor’s risk aversion and the average exposure. Finally, the investor’s average exposure to the background risk is clearly increasing in \( m^l, m^h, \) and \( \lambda^h \) (more low-type investors, i.e. high average exposure in the steady-state), and is decreasing in \( \lambda^l \) (more high-type investors, i.e. low average exposure in the steady-state).

### 1.3.3 Equilibrium with Agency Frictions

I next solve for equilibrium in the presence of agency frictions. I introduce agency frictions by setting the parameter \( \kappa_1 \) in the manager’s benefit function to a finite value. I show that the following proposition holds:

**Proposition 1.7.** When there are agency frictions, and the secondary market for fund interests is liquid, an equilibrium is characterized by the public equity’s risk premium in Theorem 1.5. The
optimal contract is given by \( \bar{f}_t = \bar{f}_t^w + \frac{\gamma x}{\gamma + \gamma} \bar{1}_3 \), where \( x = \frac{\bar{\gamma}}{\gamma + \gamma(1 + \gamma)\kappa_1 \sigma^2} \). The optimal diversion level is given by (1.3). The risk premium of the net-of-fee PE return is given by \( \mu_{Q,t} = \alpha + \beta \mu_{P,t} \), where \( \alpha = \alpha^w + A \mathcal{P} \). The agency premium \( A \mathcal{P} \) is given by

\[
\mathcal{A} \mathcal{P} = \frac{\gamma \sigma^2 \bar{f}_t - \bar{f}_t^w (\bar{f}_t + \bar{f}_t^w) \bar{1}_3 - 2\gamma \sigma^2 (\bar{f}_t - \bar{f}_t^w) \bar{1}_3}{\text{Increase in variance of fees}} - \frac{\gamma \sigma^2 (\bar{f}_t - \bar{f}_t\\bar{1}_3)}{\text{Increase in covariance with fees}}
\]

\[
= \frac{\gamma^2 \sigma^2}{(\bar{\gamma} + \gamma)^2} x(\gamma - 2\gamma) < 0.
\]

The optimal contract is same as a standard principal-agent model (Holmstrom and Milgrom, 1987). Relative to the optimal risk-sharing case, the compensation to the manager is more sensitive to the idiosyncratic factor. This increase in the sensitivity makes the manager’s compensation more tied to the PE fund performance, and thus the manager’s marginal cost of diversion becomes larger. Then, the manager balances the marginal benefit and cost of diversion, which results in the optimal diversion \( a_t = (\kappa_0 - \bar{f}_t \bar{1}_3) / \kappa_1 \).

The compensation to the manager can be expressed as

\[
dF_t = \text{constant } dt + \frac{\gamma (\bar{m} - b_n x)}{\gamma + \gamma} \sigma dz_{nt} + \frac{\gamma}{\gamma + \gamma} (dP_t + dQ_t) + \frac{\gamma x}{\gamma + \gamma} (dQ_t - b_\delta dP_t).
\]

Comparing with (1.4), the second term is again the non-hedgeable component in the compensation, the third term is the fraction of the aggregate risky assets returns, and the last term is an incentive for the manager. Since the manager’s diversion decreases the drift of the idiosyncratic factor, it is natural to make the compensation more sensitive to the gross PE fund return benchmarked to the public equity return, \( dQ_t - b_\delta dP_t \). The degree of benchmarking is the beta of the gross PE fund return.

The risk premium of the public equity is unchanged. The public equity return is exposed only to the systematic risk, and the price of systematic risk remains the same since changing the exposure of the manager’s fees to the systematic risk does not resolve agency frictions.

The increase in the sensitivity to the idiosyncratic factor has three effects on the gross PE fund return. First, since the manager is risk averse, the increased exposure of the manager’s fee to the
idiosyncratic factor also makes the manager require more risk premium component (management fees) in the manager’s compensation such that the manager’s IR constraint is binding. The price $Q_t$ takes into account this increase in the management fees such that the price level is reduced and the risk premium of the gross return is increased.

Second, the compensation to the manager becomes more volatile, and this is costly for the investors since they are risk averse. The investor’s subjective valuation of fund interests decreases since the more volatile compensation imposes a higher quadratic variation in her value function. The first term in (1.6) is the increase in the risk premium due to this effect.

Third, the gross PE fund return co-varies more with the manager’s fee. Thus, a positive shock to the idiosyncratic factor implies that the compensation is more likely to be high, but at the same time the investors can transfer their fund interests at a higher price. This effect increases the investor’s shadow valuation of fund interests, and the second term in (1.6) is a decrease in the risk premium.

In the net-of-fee return, the first effect is exactly canceled out, and there’s no effect on the agency premium since the management fees are also increased by the same amount. It is immediately seen that netting the second and third effect yields the agency premium. The agency premium is negative since the increase in variance of fees is the second order, and thus it is always dominated by the increase in covariance with fees. The negative agency premium implies that the alpha of net-of-fee PE fund return is less than the one in an economy without agency frictions. This is surprising since the manager’s non-zero diversion decreases the expected cash flow of the PE fund. The intuition is that the investors facing agency frictions would tighten the initial fundraising condition, but after the capital is raised the investors only care about the risk exposure and hedging power of the PE.

**Lemma 1.8.** The sensitivity of the compensation to the idiosyncratic factor is decreasing in $\kappa_1$. The manager’s optimal diversion is decreasing in $\kappa_1$ if $\kappa_0 = 1$. The agency premium is increasing in $\kappa_1$. 

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Suppose that agency frictions become more severe, i.e. $\kappa_1$ decreases. Then, the manager can obtain the more marginal benefit from the same level of diversion. Thus, the investors increase the manager’s shares in the PE fund such that the additional benefit from diversion is canceled out by the additional cost of diversion. However, the increase in the manager’s shares in the PE fund does not offset the additional benefit from diversion completely, which raises the manager’s optimal diversion. This relation holds only when the manager’s diversion technology is efficient $\kappa_0 = 1$. Inefficient diversion skill can result in a decrease in the optimal diversion since the incentive contract can outweigh the increase in the diversion benefit. In the secondary market, the investors value the PE higher since the effect of more volatile fees is always dominated by the effect of higher covariance between the fees and the secondary price. Higher valuation of the PE fund reduces the agency premium, and thus the alpha.

### 1.4 Equilibrium with Illiquidity

In a competitive equilibrium, the investors trade their partnerships in the PE fund for two reasons. First, the hedging demands are time-varying since as time passes it is less likely to be hit by an endowment shock. Thus, the initial holdings look similar across the high- and low-type investors, but as time approaches to the terminal date the high-type investors will purchase more and the low-type investors will reduce partnerships in the PE fund. Second, when the investors are hit by an endowment shock, they switch the holdings of partnerships. In reality, the secondary market for fund interests is illiquid in a sense that only block size of fund interests is traded and search frictions exist. Thus, the investors can not gradually adjust the holdings of partnerships in the secondary market. They might hold either large or small number of shares of the PE fund. Also, when the investors are affected by an endowment shock, it takes time to find a counterparty.

To incorporate these institutional features, in this section, I assume that the secondary market is illiquid and the investors hold either large or small number of PE shares ($0 < \psi^n < \psi^o$). An
investor who wants to sell or buy fund interests finds a counterparty with an intensity \( v \). When a seller and a buyer meet each other, they bargain over the price of fund interests through bilateral Nash bargaining. A potential buyer pays at most his reservation value for obtaining more fund interests, whereas a potential seller requires a price of at least her reservation value. I denote by \( \eta \) the seller’s bargaining power.

I denote by \( ij = \{lo, ln, ho, hn\} \) the investors’ types. The first letter represents the investor’s intrinsic valuation of the PE fund as before. The letter “o” and “n” indicate whether the investor currently owns large or small number of shares of the PE fund, respectively. I again assume that the investment horizon \( T \) is sufficiently long such that the distribution of the investors’ types is in the steady-state. I denote by \( \pi^{ij} \) the steady-state mass of type \( ij \) investors. Given a total supply of one share of the PE fund, the market clearing condition requires that

\[
(\pi^{lo} + \pi^{ho})\psi^o + (\pi^{ln} + \pi^{hn})\psi^n = 1,
\]

which implies that the fraction of large holders is

\[
\pi \equiv \pi^{lo} + \pi^{ho} = \frac{1 - \psi^n}{\psi^o - \psi^n}.
\]

The investor’s optimization problem becomes

\[
\max_{\{\theta_t, a_t\}, F_T} \mathbb{E} \left[ u \left( W_T + \psi_T (y_T - F_T) + m_T n_T \right) \right].
\]

Note that holdings of the PE fund is not liquid wealth since an investor can not liquidate the holdings immediately. Also, allocation to the PE fund \( \psi_t \) is not a choice variable, but is associated with the investor’s type, i.e. \( \psi_t = \psi^j \) if the investor’s type is \( ij \). I conjecture that the type \( ij \) investor’s value function is given by

\[
J(W_t, X_t, ij, t) = u \left( W_t + \psi^j (\omega^i_{0,t} + \omega_{1,t} X_t) + k^i_{0,t} + k^i_{1,t} X_t \right),
\]

where the boundary conditions are \( \omega^i_{0,T} = 0, \omega_{1,T} = [b_s, b_n, 1], k^i_{0,T} = 0, \) and \( k^i_{1,T} = [0, m_T, 0] \). The term \( (\omega^i_{0,t} + \omega_{1,t} X_t) \) is the type \( i \) investor’s shadow valuation or reservation value of the PE fund.
per share. The shadow valuation is also linear in the state vector $X_t$ as the price $Q_t$. The investor’s heterogeneous valuation is reflected in $\omega_{0,t}$. The terms $(k^i_{0,t} + k^i_{1,t}X_t)$ is the certainty equivalent wealth of the terminal endowment, and the risk premium of the public equity and fund interests in the PE.

**Proposition 1.9.** When there exist agency frictions $\kappa_1 < \infty$ and the secondary market for the fund interest is illiquid $v < \infty$, the equilibrium is characterized by the optimal contract and the public equity’s risk premium in Theorem 1.7. The risk premium of the net-of-fee PE return is given by $\mu_{Q,t} = \alpha_t + \beta\mu_P$, where $\alpha_t = \alpha_{bc} + \frac{\psi^o + \psi^n}{2} A P + \mathcal{I}P_t$. The illiquidity premium $\mathcal{I}P_t$ is given by

$$\mathcal{I}P_t = \left[ \eta (\lambda^h + 2v \pi^lo (1 - \eta)) - (1 - \eta)(\lambda^l + 2v \pi^hn \eta) \right] \Delta \omega_{t} \Phi_{2d,T-t}$$

(1.7)

where the difference in the shadow valuation is given by

$$\Delta \omega_{t} \Phi_{2d,T-t} = -Cov(d\hat{Q}_t - \beta dP_t, d(M^l_t - M^h_t))\Phi_{2d,T-t},$$

the subjective SDF is

$$dM^i_t = -\gamma (p_{1,t} + q_{1,t} + k^i_{1,t} - \bar{f}_t) \Sigma X dZ_t,$$

and $\Phi_{2d,T-t} = \frac{1 - e^{-2d(T-t)}}{2d}$ with $d = v(\pi^lo (1 - \eta) + \pi^hn \eta)$. The illiquidity premium disappears when $t \to T$. As the secondary market becomes liquid $v \to \infty$,

$$\mathcal{I}P_t \to \begin{cases} -\Delta \omega_{t} (1 - \eta) & \text{if } \pi < \pi^h \\ \Delta \omega_{t} \eta & \text{if } \pi > \pi^h. \end{cases}$$

The illiquidity of the secondary market does not alter the optimal contract. This implies that the price of the systematic risk remains the same, and thus the risk premium of the public equity is not changed. The same optimal contract also implies the same agency premium in the alpha (The only difference is a factor $\frac{\psi^o + \psi^n}{2}$, which is due to the assumption that either large or small units are allowed to hold). The result of the unchanged contract is natural. As long as the investors are
holding either large or small number of shares in the PE fund, they are facing the same agency frictions. Although the illiquidity can cause some investors to hold sub-optimal positions in the PE fund, those investors would choose the same contract as the investors with the optimal holdings, since that is the only way they can raise their shadow valuation of the PE to alleviate agency frictions.

The unchanged contract does not imply the same secondary market price of the PE fund. The illiquidity premium arises since the investors internalize the possibility of holding sub-optimal shares when valuing the PE fund. Due to search frictions, there is a non-zero mass of the sub-optimal holders who should rebalance the holdings of the PE fund. If an endowment shock arrives close to the terminal date or time to search a counterparty is too long, then they may end up with higher (or lower) holdings of the PE fund relative to the optimal one. For example, if there were a large negative shock to the background factor $n_T$, this would make the low-type investors with large number of shares in the PE fund worse off since their terminal endowment and the cash flow from the PE fund are low at the same time.

The illiquidity premium is intuitive. It is proportional to the difference between the high- and low-type investor’s reservation value of fund interests, $\omega_{0,t}^h - \omega_{0,t}^l = \Delta_{\omega,t} \Phi_{2d,T-t}$. The higher difference in the reservation value, the more costly it is to hold sub-optimal holdings of fund interests. The difference in the reservation value is the annuity of the risk premium that is due to search frictions. The sensitivity of the investor’s value function to the state vector is $k_{1,t} = [0 \hat{m}_{t}^i 0]$ (after scaled by $-\gamma J(W_{t}, X_{t}, i, j, t)$), and thus the type $i$ investor’s subjective price of risk is $\gamma(p_{1,t} + q_{1,t} + k_{t}^i - \bar{f}_t)\Sigma X$. Note that $\hat{m}_{t}^l > \hat{m}_{t}^h$, and thus the low-type investor’s value function is more sensitive to the background factor, which implies that their subjective price of the background risk is higher than that of the high-type investors. Thus, the product of the difference in the subjective price of risk and the risk exposure of the net-of-fee returns captures the price discount (premium) that the sellers (buyers) would accept in terms of a risk premium to rebalance their PE holdings. When sellers and buyers can find each other more easily, the risk premium due
to search frictions is discounted with the higher rate \( g \) is increasing \( v \).

The final illiquidity premium depends on whether the secondary market is a buyer’s market or a seller’s market. On the one hand, a buyer bids a discounted price to a seller. This induces the positive illiquidity premium (Price Discount) in (1.7). The price discount is higher and thus illiquidity premium is higher if the high-type investors are frequently hit by an endowment shock, which removes the demand for fund interests (higher \( \lambda^h \)) quickly, and if it is easier for the buyer to find other sellers (higher \( \pi^{lo} \) and \( v \)). The effect of the seller’s bargaining power is mixed. The more bargaining power of the seller implies that the buyer may demand lower price discount, but at the same time the price is more weighted to the buyer’s shadow valuation, i.e. the more price discount.

On the other hand, the seller may ask a price premium to the buyer if supply is scarce. This induces the negative illiquidity premium (Price Premium) in (1.7). This price premium is higher when the low-type investors need more partnerships suddenly (higher \( \lambda^l \)), and when the seller can easily meet a candidate buyer (higher \( \pi^{hn} \) and \( v \)). Similarly, the higher bargaining power of the seller increases the price premium, but also the price becomes closer to the buyer’s shadow valuation, i.e. the more price discount. The net effect depends on the parameter values. This finding is consistent with the empirical evidence on pricing trends in the secondary market. Kleymenova, Talmor, and Vasvari (2012) report time series of secondary pricing as a percent of net asset value (NAV). They find that excess demands for fund interests are positively associated with the price of limited partnerships.

When the time-to-maturity becomes very short, the illiquidity premium is gone. The probability that the investors are affected by an endowment shock becomes low as \( t \to T \). Then, buyers or sellers would not command any price discount or premium to rebalance their holdings of the PE fund as soon as possible. As the secondary market becomes liquid \( v \to \infty \), the illiquidity premium does not vanish. The reason is the assumption of portfolio constraints. Only large or small units of the PE fund are allowed to hold, which implies that the fraction of investors who
hold large number of shares might be greater or less than the mass of high-type investors. For example, suppose that the fraction of investors with large number of shares exceeds the mass of high-type investors, \( \pi > \pi^h \). This implies that even when the secondary market is perfectly liquid, some low-type investors are holding large number of shares. These investors would sell at the discounted price, which causes a positive illiquidity premium even when the investors can find a counterparty immediately.

It should be noted that \( \alpha^{bc} \neq \alpha^w \), i.e. even when there’s no illiquidity premium, the risk premium of the PE is different with that of the economy in the absence of illiquidity. The reason is the portfolio constraint and bargaining process. The following lemma provides the conditions under which \( \alpha^{bc} = \alpha^w \).

**Lemma 1.10.** If the seller’s bargaining power is \( \eta = \frac{\lambda^l}{\lambda^l + \lambda^h} \) and \( \psi^o + \psi^m = 2 \), then \( \alpha^{bc} = \alpha^w \). The illiquidity premium becomes

\[
\mathcal{IP}_t = \frac{\psi \lambda^l \lambda^h}{(\lambda^l + \lambda^h)^2} (1 - 2\pi^h) \Delta \Phi_{2d, T-t}.
\]

(1.8)

The illiquidity premium is increasing \( \bar{\gamma}, \sigma, b_n, \) and \( m^l \), and is decreasing in \( m^h \).

The first condition is easy to understand. In the economy without illiquidity, the price of the PE fund is decided in the competitive market, and thus is the weighted average of the high- and low-type investor’s valuation, where the weight is the mass of each type investors. Whereas if the price is determined through bilateral Nash bargaining, the outcome of bargaining is

\[
(1 - \eta)(-\gamma)(1 - e^{\gamma \Delta \psi (Q_t - (\omega_{0,t}^h + \omega_{1,t} X_t))}) = \eta (-\gamma)(1 - e^{\gamma \Delta \psi (\omega_{0,t}^h + \omega_{1,t} X_t - Q_t)}).
\]

(1.9)

where \( \Delta \psi = \psi^o - \psi^m \). The seller’s surplus is intuitive: her liquid wealth increases by selling \( \Delta \psi \) shares of fund interests at the price \( Q_t \), but the certainty equivalent wealth also decreases by the shadow valuation of the sold shares in the PE fund. As long as the price is set to be higher than the low-type investor’s shadow valuation or reservation value, there is a positive surplus for a seller.
(since it is multiplied by $-\gamma$). The similar story can be applied to the buyer’s surplus. Then, the above equation tells that the price of the PE fund is decided such that the weighted seller’s and buyer’s surplus are identical. Solving (1.9) yields that the price is the weighted average of the low- and high-type investor’s shadow valuation:

$$Q_t = (1 - \eta)(\omega^l_{0,t} + \omega^l_1 X_t) + \eta(\omega^h_{0,t} + \omega^h_1 X_t),$$

where the weights are $1 - \eta$ for a seller, and $\eta$ for a buyer. The higher seller’s bargaining power, the more weight is given to the high-type’s shadow valuation, i.e. the higher price. When $\eta = \frac{\lambda_l}{\lambda_l + \lambda_h}$, the outcome of Nash bargaining exactly coincides with the competitive price. It is natural that the seller has more bargaining power when there are more buyers in the market.

The second condition is technical. The investor’s certainty equivalent wealth of the holdings of the PE fund contains the term, $\frac{(\psi^f)^2}{2}$, since the holdings generate a quadratic variation. For investors with the same intrinsic valuation of the PE fund, the difference between the large and small holder’s certainty equivalent wealth is $\Delta \psi(\omega^l_{0,t} + \omega^l_1 X_t)$, which contains the term of $\frac{1}{2}((\psi^o)^2 - (\psi^n)^2)$. The term $\Delta \psi$ is canceled out, then the term $\frac{1}{2}(\psi^o + \psi^n)$ remains. It turns out that if $\psi^o + \psi^n = 2$, the weighted average of $\omega^l_{0,t}$ coincides with $q^l_{0,t}$ in the economy without illiquidity (except the agency and illiquidity premium).

The sign of the illiquidity premium is determined by the steady-state mass of high-type investors. The mass of sellers net of buyers can be expressed as $\pi^{lo} - \pi^{hn} = (\pi - \pi^{ho}) - (\pi^h - \pi^{ho}) = \pi - \pi^h$. The fraction of large holders is one half because of the second condition: $\pi = \frac{1 - \psi^n}{\psi^o - \psi^n} = \frac{\frac{1}{2}(\psi^o + \psi^n) - \psi^n}{\psi^o - \psi^n} = \frac{1}{2}$. Thus, if $\pi^h < \frac{1}{2} = \pi$, the mass of sellers net of buyers is positive and the secondary market becomes the buyer’s market, i.e. the PE fund is traded at the discounted price, which yields the positive illiquidity premium. The illiquidity premium is increasing in $\bar{\gamma}$, $\sigma$, $b_n$, and $m^l$. The intuition is clear. The fraction that goes to the manager decreases in $\bar{\gamma}$, and thus the investor’s terminal endowment can be hedged more with the net-of-fee return, which implies an increase in the risk premium that is lost due to the illiquidity. Similarly, higher $\sigma$ and $b_n$ implies
more hedging power of the PE. An increase in $m^l$ and a decrease in $m^h$ implies the wider difference in the subjective price of the background risk. Thus, the cost of sub-optimal holdings increases. To compare with the case without the illiquidity, I focus on the special case of Lemma 1.10 in the remainder of the paper.

### 1.5 Empirical performance measures

So far, I compute the net-of-fee risk premium of the PE fund investment per share of fund interest assuming that an investor can sell her limited partnerships either in the liquid or in the illiquid secondary market. It is difficult to compare the model-implied risk premium to empirically reported performance measures for two reasons. First, I compute the risk premium per share, not per dollar invested. This is because I assume the normality of cash flow and CARA utility for tractability. Second, I compute the risk premium of the secondary market returns, not the hold-to-maturity return, which is commonly reported in PE databases and empirical research. Given that the annual turnover rate of the secondary market is relatively low to those of other liquid markets, the secondary market return may not be informative for the hold-to-maturity return.

However, the model can provide the lower threshold for empirical performance measures. If empirical performance measures exceed this threshold, the PE fund is compensating the investors sufficiently for bearing the risk, illiquidity, and agency frictions. The key intuition is the investor’s IR constraint. An individual investor participates in the PE fund only when it makes her better off than she invests only in the riskless asset and the public equity. In the model, the monetary cost of investing in the PE fund is the initial lump sum capital. The monetary benefits of investing in the PE fund are the investor’s shadow valuation of limited partnerships and the certainty equivalent wealth of the risk premium that she can earn by trading partnerships in the secondary market. Note that these benefits already take into account of agency and search frictions of the PE fund investment. Then, the investor’s IR constraint implies that these benefits are greater than the initial investment.
In other words, the secondary market return is based on the investor’s shadow valuation, and thus it may underestimate the empirical performance measures which are computed using the actual cash flow paid by the investor to the manager, i.e. the initial capital investment \( I_0 \) in the model. The following corollary formalizes this argument.

**Corollary 1.11.** Suppose that there exist agency frictions \( \kappa_1 < \infty \) and the secondary market for fund interests is illiquid \( v < \infty \). The expected hold-to-maturity return or the internal rate of return is defined as \( \varphi = \frac{1}{T} \log \mathbb{E} \left[ 1 + \frac{r_T - F_T - I_0}{I_0} \right] \). This is always greater than

\[
\varphi \geq \varphi = \frac{1}{I_0 T} \left[ \int_0^T (\alpha + \beta \mu) dt - \frac{\eta}{\psi^o} \Delta_k^i - \frac{1 - \eta}{\psi^n} \Delta_k^i \right],
\]

where \( \Delta_k^i = k_{0,0}^i - \hat{k}_0^i \) and \( k_{0,0}^i \) are the investor’s value function coefficient, and can be found in the appendix.

The expected hold-to-maturity return or the internal rate of return (IRR) is defined as the return that equates the present value of distribution to the investors and the cash flow paid by the investors. The drawback of the IRR is that it assumes that the net-of-fee cash flow to the investors is reinvested in the PE fund. However, in the model there’s no intermediate distribution, and thus this assumption is innocuous. The advantage of the IRR is that the alpha and beta of the PE fund investment can be identified by assuming a structural form of the IRR, such as CAPM. Thus, it has been used extensively in empirical research recently.\(^{14}\)

The term \( k_{0,0}^i \) is the time zero certainty equivalent wealth of the risk premium when an investor participates in the PE fund, and when she does not, respectively. Thus, the term \( \Delta_k^i = k_{0,0}^i - \hat{k}_0^i \) represents the change in the certainty equivalent wealth of the risk premium due to the secondary trading of the PE fund. The initial investment \( I_0 \) is always less than the sum of two:

i) \( Q_0 \), the time zero price that an investor would obtain if she sold her limited partnerships in the

\(^{14}\)Researchers assume that the CAPM can be applied to the IRR such that \( \varphi = \alpha + \beta \mu_H \). Other factors are also included, like Fama-French three factors. See Franzoni, Nowak, and Phalippou (2012), Driessen, Lin, and Phalippou (2012), and Ang, Chen, Goetzmann, and Phalippou (2014) among others.
secondary market right after she invested in the PE fund, ii) the weighted average of the increase in the certainty equivalent wealth due to the risk premium that she would capture by trading fund interests in the secondary market. Then, I can substitute $I_0$ in the numerator of the hold-to-maturity return (IRR) with the upper boundary of $I_0$, which gives the lower threshold for the expected hold-to-maturity return (IRR).

1.5.1 Public Market Equivalent

Another performance measure, a public market equivalent (PME), is also often used in empirical research. The PME measure is first suggested by Kaplan and Schoar (2005) for determining whether the PE fund generates sufficient returns relative to the public equity. The PME measure is defined as a ratio of present value of returns to the investors to present value of cash flow paid by the investors. The key assumption is that both terms are discounted using the historical market returns. The PME measure which is greater than one is usually interpreted as an outperformance of the PE fund over the public market. This criteria can be justified when investors have logarithm preferences and their wealth portfolio coincides with public market (Sørensen and Jagannathan, 2015). There are several issues with the PME measure. First, investors may not have logarithm preferences, or wealth portfolio may not be public market. Second, the PME measure does not take into account the cost of illiquidity and agency frictions.

Thus, the threshold for the outperformance may be greater (or less) than one, and it may vary with the degree of agency frictions, the liquidity of the secondary market, and the risk aversion of manager and investors. Using the model, I can provide the lower threshold for the PME measure at which the investors are sufficiently compensated for the risk, illiquidity, and agency frictions of the PE fund investment. The model implied PME is

$$\text{PME} = \frac{1}{I_0} \mathbb{E} \left[ \left( 1 + \frac{\mu P_T}{P_0} \right)^{-1} (y_T - F_T) \right].$$
Note that the only cash flow paid by the investors is the initial lump sum investment (thus no discounting), and that the net-of-fee distribution is discounted with the expected public market return per dollar, \( \mathbb{E} \left[ \frac{P_T}{P_0} \right] = 1 + \frac{\mathbb{E}[P_T - P_0]}{P_0} = 1 + \frac{\mu_p T}{P_0} \). Using the lower threshold for the IRR, I can also find the lower threshold for the PME:

\[
PME \geq 1 + \varphi T - \frac{\mu_p T}{P_0}.
\]

Here, I use an approximation that 

\[
\frac{1+e}{1+e} \approx 1 + d - e.
\]

Note that when \( \int_0^T \alpha dt = \frac{\eta}{\psi_o I_0} \Delta_k^h + \frac{1-\eta}{\psi_o I_0} \Delta_k^l \) and \( \frac{I_0}{\psi_o I_0} \beta = 1 \), the PME is greater than one. The first condition states that the accumulated alpha is equal to the weighted average of the increase in the certainty equivalent wealth of the risk premium. The second condition implies that the beta of the net-of-fee PE fund return per dollar is one.\(^{15}\) The both conditions do not hold in general, and thus the fact that the PME measure exceeds one does not imply that the investors are sufficiently compensated. In the next section, I provide the threshold using the calibrated parameter values.

### 1.6 Numerical Example

#### 1.6.1 Calibration

I select parameters for a numerical example to match the annual asset turnover rate of the secondary market for fund interests. I also focus on the special case of Lemma 1.10 by assuming \( \psi^o = 0.67 \) and \( \psi^o = 1.33 \). I use the search intensity of \( v = 10 \), which implies that investors expect to be in contact with \( 2v = 20 \) other investors each year, that is, 1.7 investors per month. The switching intensities \( \lambda^l = 0.4 \) and \( \lambda^h = 0.6 \) mean that a high-type (low-type) investor remains a high (low) type for an average of 1.7 (2.5) years, and the seller’s bargaining power is \( \eta = \frac{\lambda^l}{\lambda^l + \lambda^h} = 0.4 \). Given

\[^{15}\text{The beta of the net-of-fee PE fund return per dollar is } \text{Cov} \left( \frac{d\hat{Q}_t}{Q_t}, \frac{dP_t}{P_t} \right) / \text{Var} \left( \frac{dP_t}{P_t} \right) = \frac{P_t}{Q_t} \beta. \text{ Since the PME measure takes the initial lump sum capital as the cost of acquiring partnerships in the PE fund, I can substitute } \hat{Q}_t \text{ with } I_0, \text{ and then the beta of the return per dollar evaluated at time zero is } \frac{I_0}{Q_t} \beta.\]

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these transition intensities and the search intensity, the steady-state masses of each type investors are $\pi^{ln} = 45\%$, $\pi^{lo} = 15\%$, $\pi^{hn} = 5\%$, and $\pi^{ho} = 35\%$. Given the equilibrium mass of potential buyers, the average time needed to sell is $\frac{12}{2\psi_{hn}} = 12$ months while the average time needed to buy is $\frac{12}{2\psi_{lo}} = 4$ months. These intensities and the investor positions imply an annual turnover rate of $2\psi_{lo}\pi^{hn}(\psi^o - \psi^n) = 10\%$, which is close to empirical estimates.\footnote{See Ang, Papanikolaou, and Westerfield (2014) for turnover rates of various illiquid asset classes.}

I assume the volatility parameter of 15\% to match the volatility of the public equity return. I also use the cash flow beta on the systematic factor of two, and the cash flow beta on the background factor of 1.5. These cash flow betas and the volatility parameter imply that the beta and the volatility of the portfolio companies are 2 and 40\%, respectively, which are close to empirical estimates. Cochrane (2005) estimate the systematic beta of two and the volatility of 90\% for individual companies in venture capital portfolios. Metrick and Yasuda (2010) assume a pairwise correlation of 20\% between any two investments, and 24 companies in venture capital portfolio, which yields the volatility of portfolio companies’ cash flow of 43\%.

I set the investor’s risk aversion parameter $\gamma$ to 1, the manager’s risk aversion parameter $\bar{\gamma}$ to 40, the investor’s initial liquid wealth $\bar{W}$ to 20, the manager’s reservation wage $\epsilon_0$ to 0.5 to make the manager and investors have the same relative risk aversion evaluated at time zero: $\gamma_R = \gamma \bar{W} = \gamma \epsilon_0 = 20$. I also normalize the initial capital investment $I_0$ to one. This implies that investors allocate $\frac{I_0}{W} = 5\%$ of the initial liquid wealth to the PE fund. These risk aversion parameters imply that without agency frictions the optimal risk-sharing rule is to specify the manager’s shares to be $\frac{\gamma}{\gamma + \gamma} = 2.44\%$. I set the initial capital investment to one.\footnote{Ang, Ayala, and Goetzmann (2014) find that university endowments allocate 7.31\% of the total assets to PE funds. This number is based on net asset value of PE fund investment, and thus the ratio of the initial capital investment to the initial liquid wealth should be higher. I assume the ratio of 5\%. The manager’s IR constraint is always binding, and thus the ratio of her reservation wage over the initial capital investment is 50\%, which matches the estimate of Sørensen, Wang, and Yang (2014).} High-type investors have the exposure of $m^h = -4$ and low-type investors have the exposure of $m^l = 3$ to the background risk. I set the manager’s diversion skill to 40\%, the growth rate of PE fund cash flow to $\bar{a} = 5\%$, and the
severity of agency frictions to $1/\kappa_1 = 0.2$. The PE fund has 12-year of life. Table 1.1 summarizes the definitions of the variables and the parameter values used in the baseline example.

The assumption of the investor’s large or small number of shares in the PE fund is realistic. Figure 1.2 plots the optimal positions in the PE fund employed by the high- and low-type investors in the associated frictionless market. A high (low)-type investor gradually invests more (less) in the PE fund. The initial holdings are almost identical since a longer time-to-maturity makes the investors reluctant to take extreme positions. As time passes, it is less likely to be hit by an endowment shock since the PE fund has a finite life. Thus, the difference between the high-and low-type investor’s holdings becomes larger. Note that if a high-type investor experiences an endowment shock and becomes a low-type investor, she rebalances her holdings of the PE fund (and also public equity accordingly), and thus switches from the high-type allocation to the low-type allocation. The two horizontal lines are the large or small number of shares in the PE fund in the illiquid secondary market. The large (small) number of shares are close to the average holdings of the high (low)-type investor’s holdings in the associated frictionless market.
Table 1.1: Summary of key variables and parameters

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Systematic factor</td>
<td>$\delta$</td>
<td>Growth rate of cash flow</td>
<td>$\mu$</td>
<td>0%</td>
</tr>
<tr>
<td>Systematic Brownian motion</td>
<td>$Z_\delta$</td>
<td>Volatility of the public equity’s cash flow</td>
<td>$\sigma$</td>
<td>15%</td>
</tr>
<tr>
<td>Background factor</td>
<td>$n$</td>
<td>Cash flow beta on the systematic factor</td>
<td>$b_\delta$</td>
<td>2</td>
</tr>
<tr>
<td>Background Brownian motion</td>
<td>$Z_n$</td>
<td>Cash flow beta on the background factor</td>
<td>$b_n$</td>
<td>1.5</td>
</tr>
<tr>
<td>Idiosyncratic factor</td>
<td>$s$</td>
<td>Initial capital investment</td>
<td>$I_0$</td>
<td>1</td>
</tr>
<tr>
<td>Idiosyncratic Brownian motion</td>
<td>$Z_s$</td>
<td>Search intensity</td>
<td>$v$</td>
<td>10</td>
</tr>
<tr>
<td>Mass of type $ij$ investors</td>
<td>$\pi^{ij}$</td>
<td>Seller’s bargaining power</td>
<td>$\eta$</td>
<td>0.4</td>
</tr>
<tr>
<td>Manager’s diversion</td>
<td>$a$</td>
<td>Manager’s diversion skill</td>
<td>$\kappa_0$</td>
<td>0.4</td>
</tr>
<tr>
<td>Compensation to the manager</td>
<td>$F_T$</td>
<td>Inverse of severity of agency frictions</td>
<td>$\kappa_1$</td>
<td>5</td>
</tr>
<tr>
<td>Manager’s value function</td>
<td>$V$</td>
<td>Life of PE investment</td>
<td>$T$</td>
<td>12</td>
</tr>
<tr>
<td>Investor’s value function</td>
<td>$J$</td>
<td>Manager’s risk aversion</td>
<td>$\bar{\gamma}$</td>
<td>40</td>
</tr>
<tr>
<td>Price of the public equity</td>
<td>$P$</td>
<td>Investor’s risk aversion</td>
<td>$\gamma$</td>
<td>1</td>
</tr>
<tr>
<td>Price of the PE fund</td>
<td>$Q$</td>
<td>Manager’s reservation wage</td>
<td>$\epsilon_0$</td>
<td>0.5</td>
</tr>
<tr>
<td>Expected public equity return</td>
<td>$\mu_P$</td>
<td>Low-type investor’s exposure to the background factor</td>
<td>$m^l$</td>
<td>3</td>
</tr>
<tr>
<td>Expected gross PE fund return</td>
<td>$\mu_Q$</td>
<td>High-type investor’s exposure to the background factor</td>
<td>$m^h$</td>
<td>-4</td>
</tr>
<tr>
<td>Expected net-of-fee PE fund return</td>
<td>$\mu_{\hat{Q}}$</td>
<td>Intensity to become high type from low type</td>
<td>$\lambda^l$</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Intensity to become low type from high type</td>
<td>$\lambda^h$</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Manager’s value-adding activity without diversion</td>
<td>$\bar{a}$</td>
<td>5%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Large number of shares in the PE fund</td>
<td>$\psi^o$</td>
<td>1.33</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Small number of shares in the PE fund</td>
<td>$\psi^n$</td>
<td>0.67</td>
</tr>
</tbody>
</table>
1.6.2 Contract and Risk Premium

Using baseline parameter values, I compute the optimal compensation to the manager:

\[ F_T = 2.25\% \times T - 25.59\% \times \sigma \triangle Z_n - 32.33\% \times \triangle P + 19.82\% \times \triangle Q, \]

where \( \triangle u = u_T - u_0 \). The constant term (management fee) is close to 2% per year which is the standard in the PE fund industry.\(^{18}\) The sensitivity to the PE fund performance (carried interest) is around 20%, which is also commonly observed number. The fourth term is an incentive for the manager since the gross return can be improved by the manager’s action.

The risk premium of public equity is \( \mu_P = 6.59\% \) and the volatility is \( \text{vol}(dP_t) = 15\% \). The beta of net-of-fee PE fund return is \( \beta = 1.93 \). The systematic component in the net-of-fee risk premium is \( 1.93 \times 6.59\% = 12.72\% \). Since the gross PE fund return is shared between the manager and the investors, the net-of-fee beta is less than the cash flow beta.

Then, I compute the risk premium and volatility of the PE fund along the time. Figure 1.3 plots the risk premium of the gross and net-of-fee PE fund returns along the time. The risk premium of the gross return is around 24.86\% and reaches the highest value when the PE is 11.58-year old. The risk premium of the net-of-fee return is smaller than that of the gross return by 5.05\%, which is the cost of compensating the manager in terms of the risk premium. Compared to the risk premium of the public equity, both the gross and net-of-fee return generate the substantial premium, but the risk-adjusted return is a more proper measure of risk-return trade-off. The volatility of the gross PE fund return is \( \text{vol}(dQ_t) = 40.39\% \), and that of the net-of-fee return is \( \text{vol}(d\hat{Q}_t) = 38.19\% \). This reflects that some of risks in the PE fund’s cash flow are shared between the manager and the investors through the contract, and the ultimate risk that the investors bear when investing in the PE fund is reduced.

---

\(^{18}\)Since I assume that the initial capital investment is one, I can interpret the numbers in the compensation as the percentages of the committed capital.
To examine the risk-adjusted return of the PE fund, Figure 1.4 plots the alpha of the net-of-fee PE fund return. The frictionless alpha is constant at 7.58%. The alpha accounting the illiquidity and the agency friction is 6.89% initially, increases to 8.56%, and then decreases back to 6.89%.

In the frictionless market, the investors require the alpha in excess of the systematic premium for bearing non-systematic risks. It can be immediately seen that the illiquidity premium is the source of time-varying risk premium. Initially, the net supply for fund interests $\pi^{lo} - \pi^{hn}$ is zero, i.e. every investors are holding the optimal amount of the PE fund. As some of the investors experience endowment shocks, a transaction occurs in the secondary market, but the speed of matching between a seller and a buyer is not fast enough to clear the net supply in the market. Thus, the net supply gradually increases to the steady-state value, which is 10% of the whole investors. This positive net supply makes the sellers transfer their fund interests at more discounted price, which yields higher illiquidity premium. However, as time passes, the buyers would bid a price close to the frictionless price since they know that their intrinsic valuation is more likely to be their final valuation. If they kept bidding a discounted price, they may not obtain the fund interests.
This force drives down the illiquidity premium to zero at the maturity. The maximum illiquidity premium is 1.67%, which accounts for 21.99% of the frictionless alpha.

Agency frictions between the manager and the investors drives down the alpha through the agency premium, $AP$ of $-0.70\%$. In absolute term, this accounts for $9.17\%$ of the frictionless alpha. When the PE fund is young, the alpha can be lower than the frictionless alpha. The intuition is that the investors facing agency frictions would design the contract to give an incentive to the manager by increasing the sensitivity of the compensation to the idiosyncratic factor. Thus, this incentive ultimately changes the distribution of the net cash flow to the investors even if the underlying cash flow process, $y$, is identical. Given that the illiquidity premium is close to zero when the PE is young, this negative effect of the agency frictions can result in the lower alpha than the frictionless alpha.
1.6.3 Empirical Performance Measure

Next, I compute the threshold for the IRR and the PME measure at which the investors break-even. The average net-of-fee PE fund return per dollar is \( \frac{1}{T_0} \int_{T_0}^{T} (\alpha + \beta \mu_P) dt = 19.81\% \). The investors also gain the benefit from secondary trading, and thus to obtain the hold-to-maturity return this benefit should be subtracted, \( \frac{1}{T_0} \left( \frac{n}{\nu_w} \Delta h_k + \frac{1-n}{\nu_u} \Delta l_k \right) = 3.81\% \). Then, I have the threshold for the IRR of \( \varphi = 19.81\% - 3.81\% = 16.00\% \). Similarly, I can proceed to obtain the threshold for the PME measure, and the result is \( \text{PME} \geq 1 + \varphi T - \frac{\mu_P T}{P_0} = 2.13 \). Here, I assume that the initial price of public equity is equal to the initial lump sum investment in the PE fund.

The IRR of 16.00\% implies that the PE fund should return at least 1.16 dollars to the investors after compensating the manager for the investors to break-even. Similarly, the PME measure of 2.13 implies that to obtain the expected distribution of the PE fund by investing in the public equity, the investors are required to allocate the amount which is 2.13 times of the initial lump sum capital to the public equity.

1.6.4 Effects of Agency Frictions

In Figure 1.5, I explore the effects of the severity of agency frictions by varying \( \kappa_1 \) from one to 100. As agency frictions become more severe, performance fees increase to make sure the manager not to divert too much. Under the optimal contract, there is still inefficiency in manager’s value-adding activity \( (\bar{a} - a) \). The manager increases the level of diversion as their marginal benefit from diversion becomes flatter. However, when agency frictions are extremely severe, performance fees give excessive incentives to the manager so that she would decrease the diversion level to zero.

The volatility of the net-of-fee PE fund return is decreasing in the severity of agency frictions. Since the more fraction of the PE performance goes to the manager, the less risks are remained in the net-of-fee PE fund return. I also plot the risk premium of gross PE fund return (blue solid), net-of-fee PE fund (red dash), and public equity (green dash-dot). Obviously, the risk premiun
Figure 1.5: Effects of agency frictions

- Performance fees (% of PE performance)
- Value adding activity
- Volatility of net−of−fee PE fund return (%)
- Risk Premium (%)
- Alpha (%)
- CAPM Beta
of public equity is constant over the severity of agency frictions. We see that the gross PE fund return increases in performance fees. This result implies that, relative to PE funds with lower performance fees, more expensive PE funds typically earn higher gross returns to offset their higher fees. However, in terms of the net-of-fee return, investors’ additional demands for the PE fund push up the price of the PE fund and reduce the risk premium. These additional demands are induced by the increase in performance fees.

To measure the relative performance of the PE fund, I plot the alpha and beta of the PE fund. Since the sensitivity of manager’s compensation to the systematic risk remains the same relative to the case without agency frictions, the beta also remains the same. We can see that the alpha is decreasing in the severity of agency frictions. Without agency frictions, investors would give a fraction \( \frac{\gamma}{\gamma + \gamma} = 2.44\% \) of the PE fund performance to the manager, and the PE fund would deliver the alpha of 8.41%. Because of agency frictions, performance fees increase and the net exposure to risks of the PE fund is reduced. Investors demand more PE fund, and in equilibrium the PE fund earns lower alpha.

### 1.7 Conclusion

In this paper I study how agency frictions between the PE fund managers and investors, and search frictions in the secondary market, affect the manager’s contract and equilibrium asset prices. Unlike previous models, I include public equity in investment opportunity sets as a liquid alternative asset, and simultaneously determine both the optimal compensation to managers and the equilibrium asset prices. This allows me to derive the alpha and beta of the PE fund returns and to decompose the alpha into the frictionless, agency, and illiquidity premium components. I show that, because of agency frictions, the manager’s compensation becomes more volatile, and co-vary more with PE cash flow. These two forces move the equilibrium price of the PE fund in opposite directions, but the latter effect dominates. As a consequence, a negative agency premium arises.
also show that, because of search frictions, the PE fund can be overpriced or underpriced relative to the one in an economy without search frictions. The size of distortion depends on the heterogeneity in investors’ valuations of the PE fund and the mean rate of contacting a counterparty. The sign of the distortion depends on the sign of net selling pressure.

My model combines a standard principal-agent model with an asset pricing framework. After investors design the optimal contract for the manager, investors can transfer contractual agreements with the manager. Also, I allow secondary markets to exhibit search frictions. Both assumptions are in line with institutional features of PE fund investment, and are, to the best of my knowledge, new to the literature. This also enables me to derive the risk premium of both the gross and net-of-fee PE fund returns.

A natural extension of my analysis is to endogenize the risk exposure of portfolio companies’ cash flows. This can be achieved by allowing the manager to choose either the leverage ratio of the PE fund or portfolio companies to invest in among a set of portfolio companies with different risk-return profiles. In that case, to align incentives of the manager and investors, an option-like contract would be optimal, and the impact of non-linear contracts on equilibrium asset prices can be studied.
Chapter 2

How Often Should You Take Tactical Asset Allocation Decisions?

2.1 Introduction

Tactical asset allocation (TAA) policies aim to generate value by periodically adjusting asset class allocation targets to take advantage of time-varying expected returns. We investigate how often institutions should take such tactical decisions.¹

We consider a long-term investor with constant relative risk aversion (CRRA) utility defined over final wealth allocating between stocks and bonds.² Both asset classes exhibit predictability, which we calibrate to data: bond returns are time-varying and depend on the risk-free rate and yield spread, and equities can be forecasted by the same two variables as well as the dividend yield. These predictors have a long history in finance. Many researchers have used short rates and term spreads to forecast excess bond returns (see Fama and Bliss (1987) and Campbell and Shiller (1991) among others). The dividend yield has been used to forecast equity returns at least since Dow (1920) and is intuitively appealing because high valuation ratios embed low future discount rates.

We distinguish the frequency with which an institution might change tactical targets, such as moving from a 60/40 bond/equity target to a 50/50 target, from the actual trading frequency, since the two are often different in practice. Indeed, institutional investors, such as pension and

¹For the purposes of this article, we consider dynamic asset allocation (DAA) and TAA to be equivalent.

²The simplified setting of only two assets, equities and bonds, is a reasonable first-order approximation for most institutional portfolios given that equity risk, and exposure to equity risk of many alternatives, dominates (see Leibowitz, Bova, and Hammond (2010) and Ang (2014) among others).
sovereign-wealth funds, often have an investment board (the principal) who chooses tactical investment targets and then delegates the actual portfolio construction to in-house or external portfolio managers (the agent).

We allow the principal to optimally switch portfolio target weights at discrete points in time. Between these times of tactical shifts, the agent continuously rebalances back to constant portfolio weights. We refer to these switching strategies as TAA because the portfolio weights are updated only infrequently. The TAA policies are optimally set for a given calendar-time updating frequency, and respond to time-varying changes in investment opportunities.

We assume that information processing is costly and exogenously given. The principal should pay a fraction of the asset under management to research changes in investment opportunities or to decide the optimal switching target portfolio weight. These costs are the source of infrequent switching. As the updating intervals become more frequent, the principal can respond to time-varying investment opportunities better. However, this also induces higher information processing costs. The principal will decide the optimal switching frequency by trading off the benefit of frequent switching against high information processing costs while taking predictability of returns as given. We consider cases of optimal tactical switching for the predictable bond returns only, predictable stock returns only, and for both time-varying bond and stock returns.

We estimate the utility costs and certainty equivalent return (CER) losses of periodically updating TAA strategies at different horizons with the information processing cost compared to the optimal first-best policy of taking continuous TAA decisions without any cost (the Merton (1971) case). We show that for the annual frequency, as long as TAA strategies are implemented optimally, the utility losses are minimized at 1.42 percent of initial wealth, and the CER losses are minimized at 14 basis points of annualized return. TAA programs are approximately twice as valuable for exploiting variation in the equity risk premium compared to the bond premium.

We analyze the effects of the cost of switching on the performance of TAA strategies by varying the information processing costs. We find that performance of TAA strategies updating at different
intervals become worse as the information processing cost increases. The benefit of marginal decrease in the switching interval is fixed, and the cost of marginal increase in the number of switching is proportional to the cost of switching. Thus, the investor finds it optimal to change the target weights less often when the switching cost is high. We also find that the benefit of frequent switching of target weight is asymmetric such that when the market conditions indicate that future returns are likely bad, the utility costs and CER losses are greater. The optimal switching frequency, however, is independent of the initial market conditions.

As a robustness check, we consider a case that the investor can hold long or short positions in cash. Given the same predictability of returns, the enhanced investment opportunity sets make the benefit of marginal decrease in the switching interval more valuable to the investor. This induces the shorter optimal switching frequency and the lower minimum utility costs and CER losses. We also consider more predictors. Our model can be easily extended to incorporate multiple predictors. We create “predictive index” for each asset, and use them as a proxy for the state variables in the baseline model. The increased predictability by including additional predictors implies that the higher utility costs and CER losses.

Despite a large literature on asset allocation, little is known about the impact of making TAA decisions at lower frequencies than the rebalancing frequency, because the literature typically assumes both to be the same, i.e., that principal and agent investors are identical.\(^3\) Two recent practitioner studies examining the optimal frequency of TAA decisions are Leibowitz and Bova (2011) and Almadi, Rapach, and Suri (2014).\(^4\) Neither derive the optimal portfolio strategies for predictable asset returns or for different frequencies of TAA decisions. Thus, they do not compute investors’ utility costs for suboptimal rebalancing behavior. An advantage of our framework is that the optimal time-varying TAA policies are derived for different rebalancing frequencies over

\(^3\)See Brandt (2010) and Wachter (2010) for recent summaries on asset allocation literature.

\(^4\)There are also academic studies that investigate optimal discrete rebalancing intervals in the presence of inattention costs, e.g., Abel, Eberly, and Panageas (2007) but they assume a constant opportunity set (i.e., no predictability).
predictable equity returns, bond returns, or both.

2.2 Model

We write the dynamics of bonds and stocks such that the expected returns of bonds and stocks vary over time. Bond returns are predictable by short rates, \( r(t) \), and the risk-premium factor, \( y(t) \). Stock returns are also predictable by short rates and risk-premium factor, but in addition dividend yields, \( z(t) \), also have forecasting ability.

2.2.1 Bond Returns

We employ a two-factor version of the Vasicek (1977) term structure model with the short rate and the risk-premium factor. Stating the factor dynamics, we have

\[
\begin{align*}
    dr(t) &= [\kappa_r \{ \bar{r} - r(t) \} + \kappa_{ry} \{ \bar{y} - y(t) \}] dt + \sigma_r dZ_r(t) \\
    dy(t) &= \kappa_y (\bar{y} - y(t)) dt + \sigma_{ry} dZ_r(t) + \sigma_y dZ_y(t),
\end{align*}
\]

where \( Z_r \) and \( Z_y \) are independent Brownian motions. The instantaneous correlation between the short rate and the risk-premium factor is \( \rho_{ry} = \sigma_{ry}/\sqrt{\sigma_{rr}^2 + \sigma_{yy}^2} \). In equation (2.1), the risk premium factor influences, and is correlated with, the short rate.

We take the short rate as the three-month T-bill rate and proxy the risk-premium factor with the term spread measured as the difference between the 10-year and two-year Treasury bond yields. Since we use the short rate and term spread as state variables, bond returns reflect predictable deviations from the Expectations Hypothesis (see Dai and Singleton (2002) and Duffee (2002)). We specify a price of risk-free rate risk as an affine function of state variables:

\[
\Lambda_r(t) = \lambda_r + \phi_r r(t) + \phi_y y(t).
\]

Further, we assume that

\[
\kappa_{ry} = -\sigma_r \phi_y.
\]
This restriction effectively makes the risk-free rate follow a one-factor version of the Vasicek (1977) under the risk-neutral measure (but is driven by two factors under the empirical measure). In other words, the yield spread shock is not spanned in bond markets.\(^5\)

In the utility maximization problem we consider later, an investor who takes bond and stock prices as given and allocates his wealth only in constant time-to-maturity bond index and stock, not cash. Thus, we need to specify the return process of the constant time-to-maturity bond index. We assume without loss of generality that the time-to-maturity of bond index the investor can trade is \(T\), which is same as the investment horizon.

Denote \(B(t)\) as the index value of \(T\) time-to-maturity bond index. Then, the following holds

\[
\frac{dB(t)}{B(t)} = (\alpha_b + \beta_b, r(t) + \beta_b, y(t))dt + \sigma_b dZ_r(t), \tag{2.3}
\]

where the constant and coefficients on the short rate, \(r(t)\), and risk-premium factor, \(y(t)\), are determined by no-arbitrage relations and are functions of the data-generating process in equation (2.1) and the price of risk in equation (2.2) (see Appendix B.1.1). In our empirical work, we choose total returns of 10-year Treasury constant maturity bonds to represent returns of bond index.\(^6\)

### 2.2.2 Stock Returns

We build on the models of Campbell and Viceira (1999) and Stambaugh (1999) who forecast equity returns using dividend yields. In addition, we also allow short rates and term spreads to predict equity premiums, whose predictive power has been studied by Campbell (1986), Hodrick (1992), Ang and Bekaert (2007), and others. Specifically, equity returns follow

\[
\frac{dS(t)}{S(t)} = (\alpha_s + \beta_s, r(t) + \beta_s, y(t) + \beta_s, z(t))dt + \sigma_s \left( \rho_{rs} dZ_r(t) + \sqrt{1 - \rho_{rs}^2} dZ_s(t) \right), \tag{2.4}
\]

\(^5\)One bond is sufficient to complete bond markets in our framework, even though the unspanned risk-premium factor drives bond excess returns and thus induces non-trivial hedging demands for investors (see Duffee (2002) and Collin-Dufresne and Goldstein (2002)).

\(^6\)Since the investor has an horizon of \(T\), the risk-free asset is a zero-coupon bond of \(T\) years. The 10-year maturity is a standard benchmark, tradeable, and also enables the model to be calibrated using standard Vector Autoregression (VAR) techniques as we detail below.
where \( Z_s(t) \) is a Brownian motion, which is independent of \( Z_r(t) \) and \( Z_y(t) \). We assume that stock markets are incomplete in the model by specifying time-varying prices of equity specific risk which depend on the dividend yield:

\[
\Lambda_s(t) = \lambda_s + \phi_s z(t),
\]

(2.5)

where \( z(t) \) is the dividend yield and follows

\[
dz(t) = \kappa_z (\overline{z} - z(t)) dt + \sigma_z \left( \rho_{zs} dZ_s(t) + \sqrt{1 - \rho_{zs}^2} dZ_z(t) \right).
\]

(2.6)

Note that \( Z_z \) is a Brownian motion, which is independent of \( Z_r, Z_y, \) and \( Z_s \). A negative value of \( \rho_{zs} \) allows the dividend yield to be strongly negatively correlated with innovations to equity returns, as found by Stambaugh (1999). The conditional mean parameters \((\alpha_s, \beta_{sr}, \beta_{sy}, \beta_z)\) can be found in Appendix B.1.2 and can be calibrated using standard predictability regressions. We take total returns on the S&P 500 index as stock returns, and dividend yields, \( z(t) \), are constructed using the sum of the previous 12 months of dividends.

### 2.2.3 Asset Allocation Problem

Following Merton (1969, 1971), Brennan, Schwartz, and Lagnado (1997) and others, we consider an investor with horizon \( T \) who maximizes constant relative risk aversion (CRRA) utility over terminal wealth:

\[
\max_{\{w(t)\}_{t=0}^T} \mathbb{E} \left[ \frac{W(T)^{1-\gamma}}{1-\gamma} \right],
\]

(2.7)

where \( \gamma \) is the investor’s degree of risk aversion, and \( w(t) \) is the weight in the investor’s portfolio held in stocks at time \( t \). We assume the remainder, \( 1 - w(t) \) is held in bonds.\(^7\) Note that while

\(^7\)This case is the most relevant for investors with leverage constraints, like pension funds. The case where investors can hold short or long positions in cash leads to similar results to our analysis. A disadvantage of allowing cash holdings is that the risk aversion coefficient has to be carefully calibrated, otherwise the equity premium puzzle often leads to aggressively levered positions in equities (see, for example, Brennan and Xia (2000)). We report a case in which cash holdings is allowed in Section 2.5.1.
the investor does not hold cash, time-varying short rates influence risk premiums of bonds and equities. The wealth process follows
\[
dW(t) = (1 - w(t)) \frac{dB(t)}{B(t)} + w(t) \frac{dS(t)}{S(t)}.
\]

### 2.3 TAA Policies

We define a TAA investment policy as follows. An investor can switch his portfolio weights \(n\) times at evenly spaced points. During the period between two adjacent switching dates, the investor maintains a constant portfolio weight.\(^8\) The investor changes the target portfolio weight by paying information processing costs at switching dates. Information processing costs are \(c\) fraction of the contemporaneous value of the investment portfolio and they are utilized to research changes in the investment opportunities every time she switches the target portfolio weight.

We solve for the optimal TAA policy, which is a function of the number of rebalancing intervals, \(n\), the horizon of the investor, \(T\), and the state of the economy summarized by the variables that predict returns, \(X(t) \equiv (r(t), y(t), z(t))\). The optimal TAA policy is time-consistent in a sense that an investor derives the optimal policy once at time zero using dynamic programming techniques (see Appendix B.2.3 for further details) and the policy, which is a function of the state, remains optimal as time passes. Given the optimal TAA policy, we calculate the expected utility for a TAA policy with \(n\) number of switching.

To decide the optimal number of switching, we take the continuous Merton (1971) policy as a benchmark. The continuous Merton (1971) policy is the first-best policy in the absence of information processing costs. However, in the presence of information processing costs, it should be avoided since it is associated with infinite number of switching of the target portfolio weight.

\(^8\) As we work in continuous time, the investor maintains constant weights in between the switching TAA decision dates by trading continuously. If the investor were to rebalance discretely, then she would buy-and-hold between two rebalancing dates. Pure discrete switching strategies do not have closed-form solutions. For simple systems with one state variable, the discrete switching strategies can be solved numerically and are close to our analytical TAA strategies. See Appendix B.4 for further details.
Given the continuous Merton (1971) policy, we calculate the expected utility of this policy while assuming there are no information processing costs. See Appendix B.2.1 for further details.

Then, we measure the utility costs in terms of the percentage increase of initial wealth required for an investor to be indifferent between a TAA strategy switching at a given frequency with the information processing cost and the continuous Merton (1971) strategy without any costs. We also measure the certainty equivalent return losses in terms of the annualized certainty equivalent return that achieves the same expected utility as the continuous Merton (1971) for TAA investor. Let $J(X(0))$ denote the expected utility when the continuous Merton (1971) policy is employed, and $\hat{J}(X(0); n)$ denote the expected utility when the TAA with $n$-switching is used and there are information procession costs. Then, the utility cost and CER loss of TAA switching $n$ times is calculated as

\[
\text{utility cost } (X(0); n) = \left( \frac{J(X(0))}{\hat{J}(X(0); n)} \right)^{\frac{1}{1-\gamma}} - 1
\]

\[
\text{CER } (X(0); n) = \frac{1}{T(1-\gamma)} \log \left( \frac{J(X(0))}{\hat{J}(X(0); n)} \right)
\]

Note that the utility cost or CER depends on the initial value of state variables. To find the utility cost or CER which is independent of the initial state variables, we numerically integrate over the initial state variables using the stationary distribution.

The investor would increase the number of switching while the marginal benefit of shortening the switching interval is greater than the marginal cost of doing so. The marginal benefit would be the ability to react to changes in investment opportunities and the marginal cost would be the marginal increase in information processing costs. At the optimal number of switching, the investor would get the lowest utility cost or CER. We take the information processing costs, $c$ as given, and derive the optimal number of switching.

Figure 2.1 illustrates two TAA strategies for one path of simulated return predictors. For a 10-year horizon, we plot the equity weight of TAA strategies rebalanced every year ($n = 10$),
The figure plots the portfolio weights in equity of three different trading strategies: the five-year TAA, the one-year TAA, and the optimal continuous-time Merton (1971) weights (“continuous”) for an investor with a $T = 10$ year horizon over one simulated path of state variables.

every five years ($n = 2$), and the optimal Merton (1971) continuous strategy (which we label “continuous”). The TAA strategies change only at rebalancing dates, so they are step functions. As expected, the one-year TAA strategy follows the continuous strategy more closely than the TAA strategy switching every five years. It is important to note that these optimal strategies are solved at time zero and change over time as the state variables change and the horizon decreases.
Table 2.1: Summary Statistics of State Variables and Asset Returns

<table>
<thead>
<tr>
<th></th>
<th>Mean (%)</th>
<th>Stdev (%)</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short Rate</td>
<td>3.86</td>
<td>2.94</td>
<td>1.00</td>
</tr>
<tr>
<td>Term Spread</td>
<td>0.70</td>
<td>0.77</td>
<td>-0.61 1.00</td>
</tr>
<tr>
<td>Dividend Yield</td>
<td>3.62</td>
<td>1.54</td>
<td>0.00 -0.17 1.00</td>
</tr>
<tr>
<td>Excess Bond Return</td>
<td>1.35</td>
<td>6.83</td>
<td>-0.07 0.13 -0.02 1.00</td>
</tr>
<tr>
<td>Excess Stock Return</td>
<td>6.72</td>
<td>14.55</td>
<td>-0.10 0.09 -0.02 0.10 1.00</td>
</tr>
</tbody>
</table>

The table reports means, standard deviations, and correlations of the short rate, term spread, dividend yield, and excess returns of bonds and stocks. We take monthly frequency data from January 1941 to December 2013. The short rate is the three-month T-bill rate and the term spread is the difference between the 10-year and two-year Treasury bond yields. The stock data is the monthly return of S&P 500 index. The bond data is the monthly return of 10-year Treasury constant maturity bond index. All data are continuously compounded and means and standard deviations are annualized.

2.4 Empirical Results

2.4.1 Parameter Estimates

We take monthly frequency data from January 1941 to December 2013. In our analysis, we consider systems with no predictability, predictability of bond returns only, predictability of stock returns only, and predictability in both asset classes. The continuous-time parameters are estimated by deriving the discrete-time version of the model and recovering the parameters from VAR and predictive regression coefficients (see Appendix B.3).

Table 2.1 provides summary statistics of the state variables and excess returns. All numbers are annualized. Note that the short rate and yield spread are negatively correlated. The historical excess bond return is 1.35% and the excess equity return is 6.72%. The volatilities of excess bond and stock returns are 6.83% and 14.55%, respectively. The mean and volatilities translate into Sharpe ratios of 0.20 for bonds and 0.46 for equities.

Table 2.2 reports the parameter estimates for a restricted VAR implied by the model (Panel A)
Table 2.2: Vector Autoregression and Predictive Regressions

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>Short Rate</th>
<th>Yield Spread</th>
<th>Dividend Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: VAR for the Short Rate, Yield Spread, and Dividend Yield</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Short Rate</td>
<td>0.0006</td>
<td>0.9894</td>
<td>-0.0239</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0001)**</td>
<td>(0.0240)**</td>
<td>(0.0017)**</td>
<td></td>
</tr>
<tr>
<td>Yield Spread</td>
<td>0.0002</td>
<td>-</td>
<td>0.9766</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0001)**</td>
<td></td>
<td>(0.0087)**</td>
<td></td>
</tr>
<tr>
<td>Dividend Yield</td>
<td>0.0007</td>
<td>-</td>
<td>-</td>
<td>0.9826</td>
</tr>
<tr>
<td></td>
<td>(0.0002)**</td>
<td></td>
<td></td>
<td>(0.0059)**</td>
</tr>
<tr>
<td><strong>Panel B: Predictability Regressions</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Excess Bond Returns</td>
<td>-0.0020</td>
<td>0.0340</td>
<td>0.2653</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0007)**</td>
<td>(0.1223)</td>
<td>(0.3457)</td>
<td></td>
</tr>
<tr>
<td>Excess Stock Returns</td>
<td>-0.0028</td>
<td>0.0055</td>
<td>0.0108</td>
<td>0.2235</td>
</tr>
<tr>
<td></td>
<td>(0.0036)</td>
<td>(0.0026)*</td>
<td>(0.0007)**</td>
<td>(0.0910)*</td>
</tr>
</tbody>
</table>

Panel A reports coefficients of a discrete-time restricted Vector Autoregression (VAR) implied by the model in equations (2.1) and (2.4). Panel B reports coefficients of predictive regressions for excess stock and bond returns. All parameters are estimated at the monthly frequency with data from January 1941 to December 2013. Standard errors are in parentheses. We denote 5% and 1% levels of significance by * and **, respectively.

and regressions predicting excess stock and bond returns (Panel B). Not surprisingly in Panel A, the VAR displays high persistence of the short rates, spreads, and yields. In the first line, yield spreads predict T-bill short rates echoing Campbell and Shiller (1991). This result indicates the importance of allowing for multiple factors in short rate dynamics (see Longstaff and Schwartz (1992) and Duffee (2002)).

In Panel B, the point estimates in the predictive regression coefficients for excess bond returns show that when short rates and spreads are high, bond risk premiums are high. The coefficients, however, are not statistically significant. After the short rate process is determined, the relatively large standard errors partly reflect the difficulty in estimating time-varying prices of risk in term
structure models (see Dai and Singleton (2002)). In the predictive regression for stock returns, all three variables—short rates, term spreads, and dividend yields—are jointly statistically significant. For our analysis with no predictability, and predictability only in one of stock or bond returns, we re-estimate the predictive regressions imposing these constraints. The model parameters correspondingly change for these various cases.

Figure 2.2 graphs the instantaneous total returns of bonds and equities implied by the model. We mark NBER recessions in the shaded areas. The implied average returns are 5.0% and 10.6% for bonds and equities which are very close to the empirical values of 5.3% and 10.7% over the sample. Expected bond returns are relatively volatile during the 1970s and early 1980s, coinciding with high inflation and monetary policy efforts to counter the high inflation during this time. Bond risk premiums are high after the 1990 and 2000 recessions, and are also very high during and after the 2008 financial crisis. Since the 1980s, expected stock returns decrease from 26.2% during 1981 to 2.1% in 2003. Recently from 2004 to 2008, equity returns average 4.9% and then increase to 2009 to 7.2% during the financial crisis. In our model, high expected returns are associated with low stock prices, which is reflected in the large, positive coefficient on the dividend yield in the equity predictive regression (equation (2.4) and Table 2.2).

2.4.2 Characterizing TAA Policies

Figure 2.3 examines how quickly the optimal TAA portfolio weights converge to continuous-time weights. We plot bond and equity holdings as a function of the rebalancing frequency. The weights are computed at time zero for an investor with a horizon of $T = 10$ years with the predictive variables set at their long-term means. We choose the risk aversion coefficient so that at time zero the optimal weights are 60% equities and 40% bonds for a TAA strategy which selects the portfolio weights once only at time zero. This level of risk aversion corresponds to Merton (1971) continuously rebalanced weights of approximately 50%, which are plotted in the dashed horizontal lines.
As the switching intervals become more frequent, the TAA weights converge to the continuously rebalanced weights. In particular, TAA rebalancing at a frequency of one year or less produces TAA weights similar to the continuous weights.

In Panel B of Figure 2.3, the TAA equity weights are larger than the continuous strategy weights. As we will see in Figure 2.4, the hedging demands are negative, i.e. the continuous strategy holds less equity weight than the myopic strategy, which refers to an instantaneous mean-variance strategy. Intuitively, the TAA strategy’s hedging are less aggressive than the continuous strategy (with the TAA producing larger hedging demands in absolute value) since under TAA, the investor does not have the ability to react to future changes in the investment opportunity set, except at discrete TAA rebalancing dates.

To examine how the optimal TAA weights change as a function of the predictive variables,
Figure 2.3: Optimal TAA Portfolio Weights

Panel A: Bond Weight

The figure plots portfolio weights of TAA and the Merton (1971) continuously rebalanced strategy at time 0 for an investor with a horizon of $T = 10$ years, risk aversion of 7.9, with steady-state values of the predictive variables. The optimal TAA weights in the solid line are a function of the rebalancing frequency. The optimal Merton (1971) continuous-time weights are drawn in the horizontal dashed line.
Figure 2.4 plots the TAA equity portfolios weights together with the Merton (1971) strategy and the myopic strategy.\footnote{The myopic policy times the market over the next (instantaneous) period, but ignores the conditional covariance structure between asset returns and state variables. Thus, there are no hedging demands in the myopic policy. See Appendix B.2.2 for further details.} The weights are optimal at time zero for an investor with a 10-year horizon. We consider annual TAA decisions. We vary the short rate, term spread, and dividend yield in each panel. In all panels, we change only the state variable on the $x$-axis and hold constant all other parameters and state variables.

The first notable fact in Figure 2.4 is that the annual TAA and continuous weights are similar, but both are significantly different from the myopic weights. This means that mean-variance strategies, which do not take into account the hedging demands induced by predictable asset returns, can result in significantly sub-optimal holdings.

Second, Figure 2.4 shows that equity weights are decreasing in the short rate and term spread. As can be inferred from the coefficients of the predictive regressions in Table 2.2, the returns of equities in excess of bonds decrease as both short rates and term spreads increase. Thus, as these predictive variables decrease, the investor favors equities over bonds. Similarly, the equity weights increase as dividend yields increase. Higher dividend yields imply higher equity risk premiums (the dividend yield does not influence the dynamics of short rates and spreads in the model in equation (2.1)), and the investor takes advantage of higher equity premiums by larger equity weights.

Third, the myopic weights are larger than the TAA and continuous weights. The difference between the myopic equity weight and TAA and continuous equity weight is the hedging demand. Since the investor is risk-averse, she would hedge future risks of negative shocks to investment opportunity sets. This can be done by increasing the holdings of an asset which exhibits negative covariance between the risk premiums. The negative hedging demands result from the investor optimally lowering the equity weight to protect her against possible negative shocks to the equity risk premium. While Campbell and Viceira (1999) found large, positive hedging demands for
We plot equity weights at time zero for the optimal TAA strategy switching allocations every year, the strategy with continuous rebalancing, and the myopic strategy. The last is valid for an instantaneous mean-variance portfolio. Panel A, B, C correspond to short rate, term spreads, and dividend yield predictors, respectively. Vertical lines indicate the steady-state mean of each state variable. In all panels, we vary only the state variable on the $x$-axis and hold constant all other parameters and state variables. We use an horizon of $T = 10$ years and a risk aversion of 7.9.
equity, our investor holds a 100% risky asset portfolio. Campbell and Viceira’s investor shorts the risk-free asset to fund her equity hedging demands. In our system, this role is taken by bonds: long-term equity positions are smaller than myopic weights because bond returns hedge negative shocks to expected returns better than equities. It also turns out that with our estimated parameter values the variations in the conditional covariance between asset returns and the risk premium is small, and thus the hedging demands are mainly determined by unconditional covariance between asset returns and the risk premium. Thus, most of times the TAA and continuous equity weights are lower than the myopic equity weights. We confirm that if we turn off the bond predictability entirely, and retain only stock predictability, then the optimal TAA position in stocks is indeed larger than the myopic position (while the bond position is lower than its myopic counterpart).

Finally, while the TAA portfolio weights are close to the continuous Merton (1971) positions, they react to changes in the state variables more conservatively. This can be seen in the flatter slopes of the TAA line compared to the continuously rebalanced Merton (1971) line in Panels B and C. (The effect is also there, but harder to discern, in Panel A.) This is due to TAA investors having only limited opportunities to switch their allocations. Since they cannot react instantaneously, TAA investors’ portfolios exhibit less sensitivity to changes in predictors because they recognize these opportunities are mean-reverting.

2.4.3 TAA Decisions at Different Frequencies

Figure 2.5 and Table 2.3 report the utility costs of TAA as a function of switching frequencies. The utility costs are stated in terms of the percentage increase of initial wealth required for an investor to be indifferent between a TAA strategy switching at a given frequency and the continuous Merton (1971) strategy. Put another way, how much does an investor need to be compensated for making TAA decisions more infrequently? We use a horizon of 10 years and set the risk aversion so that a TAA strategy choosing the portfolio weights only once at the beginning of the ten-year period
We plot the utility costs of taking TAA decisions at various intervals with switching costs of 5 basis points relative to the continuous Merton (1971) strategy which switches instantaneously without any costs. The utility costs are integrated over the steady distribution of state variables. The utility costs are computed for a 10-year horizon and a risk aversion of 7.9, which corresponds to a 60% equity and 40% bond portfolio at time zero for the TAA strategy which chooses the portfolio weights only once at the beginning of the ten-year period (the 10-year TAA strategy). We consider the full case of predictability in both stocks and bonds.

yields a 60% equity and 40% bond portfolio (as in Figures 2.3 and 2.4).

Figure 2.5 considers the case of joint stock and bond predictability. The baseline case is the optimal Merton (1971) policy, or the continuous switching TAA policy without any information processing costs. Since there is no cost of switching and returns are predictable, the continuous switching TAA is the first-best scenario. Then, we consider the investor with switching costs of 5 basis points of the contemporaneous value of portfolio. As the switching TAA intervals become more frequent, the utility costs decrease. Investors require an additional 3.1% of wealth when switching allocations twice over a 10-year horizon to have the same utility as employing the continuous Merton (1971) strategy. The utility costs keep declining until the switching interval
1-year. In this region, the benefit of shortening the switching interval outweighs the cost of paying the cost of more frequent switching. For the six-month, one-quarter and one-month TAA strategies, the utility costs are increasing, at 1.50%, 2.28%, and 6.27%. This indicates that the investor does not gain from the return predictability by switching the target weight more often as much as paying a fraction of the value of portfolio. Thus, we can conclude that the optimal switching frequency is around every year. It should be emphasized that this conclusion of course depends on how much returns are predictable.

In Table 2.3, we record the case of no predictability or IID returns, and predictability in only one of stock or bond returns. The last column of Table 2.3 lists the joint stock and bond predictability case and is the same as Figure 2.5. For the IID case, the utility costs are very small for longer switching intervals; there is little benefit of taking TAA decisions more frequently because there is little value in market timing. The investor prefers switching the target weight less often since returns are IID, i.e. the benefit of frequent switching is relatively small to the switching costs. The smallest utility cost can be achieved when the investor switch twice over the horizon.

Table 2.3 also reports performance statistics of various TAA frequencies relative to the first-best case, continuous switching without the costs. Panel A reports in terms of utility costs, and Panel B provides in terms of certainty equivalent return (CER) loss. It shows that the utility costs and CER losses are approximately twice as large for equity predictability compared to bond predictability for switching intervals longer than one-year. For example, changing portfolio weights every five-year results in utility costs of 1.45% of initial wealth in the predictable bond case compared to 3.26% for the predictable stock case. When equities are predictable, there is significantly more benefit in taking TAA decisions at longer intervals because the investor can capitalize on times when equity returns are potentially much higher than bonds. The smaller predictable variation in bond returns makes it more difficult for an investor to capitalize on bond predictability (see insignificant slopes

10 Even when expected returns of bond and stock are constant, unexpected shocks to bond and stock returns are still correlated with state variables so hedging demands matter. The IID TAA weights are optimally computed to take into account hedging demands.
Table 2.3: Performance Statistics of Various TAA Frequencies

Panel A: Utility Costs (% of Initial Wealth)

<table>
<thead>
<tr>
<th>Switching Interval</th>
<th>IID Returns</th>
<th>Predictable Bond Only</th>
<th>Predictable Stock Only</th>
<th>Predictable Stocks and Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 Years</td>
<td>0.47</td>
<td>2.43</td>
<td>4.82</td>
<td>4.75</td>
</tr>
<tr>
<td>5 Years</td>
<td>0.27</td>
<td>1.45</td>
<td>3.26</td>
<td>3.14</td>
</tr>
<tr>
<td>2 Years</td>
<td>0.30</td>
<td>1.06</td>
<td>1.87</td>
<td>1.88</td>
</tr>
<tr>
<td>1 Year</td>
<td>0.52</td>
<td><strong>0.99</strong></td>
<td><strong>1.40</strong></td>
<td><strong>1.42</strong></td>
</tr>
<tr>
<td>6 Months</td>
<td>1.02</td>
<td>1.27</td>
<td>1.49</td>
<td>1.50</td>
</tr>
<tr>
<td>1 Quarter</td>
<td>2.03</td>
<td>2.16</td>
<td>2.27</td>
<td>2.28</td>
</tr>
<tr>
<td>1 Month</td>
<td>6.19</td>
<td>6.23</td>
<td>6.27</td>
<td>6.27</td>
</tr>
</tbody>
</table>

Panel B: Certainty Equivalent Return Loss (annualized return, %)

<table>
<thead>
<tr>
<th>Switching Interval</th>
<th>IID Returns</th>
<th>Predictable Bond Only</th>
<th>Predictable Stock Only</th>
<th>Predictable Stocks and Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 Years</td>
<td>0.05</td>
<td>0.24</td>
<td>0.47</td>
<td>0.46</td>
</tr>
<tr>
<td>5 Years</td>
<td>0.03</td>
<td>0.14</td>
<td>0.32</td>
<td>0.31</td>
</tr>
<tr>
<td>2 Years</td>
<td>0.03</td>
<td>0.11</td>
<td>0.18</td>
<td>0.19</td>
</tr>
<tr>
<td>1 Year</td>
<td>0.05</td>
<td><strong>0.10</strong></td>
<td><strong>0.14</strong></td>
<td><strong>0.14</strong></td>
</tr>
<tr>
<td>6 Months</td>
<td>0.10</td>
<td>0.13</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>1 Quarter</td>
<td>0.20</td>
<td>0.21</td>
<td>0.22</td>
<td>0.23</td>
</tr>
<tr>
<td>1 Month</td>
<td>0.60</td>
<td>0.60</td>
<td>0.61</td>
<td>0.61</td>
</tr>
</tbody>
</table>

The table reports utility costs and certainty equivalent return (CER) loss of TAA strategies switching at set periodic calendar intervals with switching costs versus the continuous Merton (1971) strategy which switches instantaneously without any costs. Utility costs are reported in percentage terms of initial wealth and represent the increase in initial wealth required to make the investor with a given TAA strategy have the same utility if she had the ability to implement the optimal Merton (1971) strategy with predictable returns. CER losses are reported in annualized percentage return and represent the decrease in CER that the investor would experience if she took costly TAA decision. We take an investor with a 10-year horizon, switching costs 5 basis points of the value of portfolio, and risk aversion of 7.9, which corresponds to a 60% equity and 40% bond portfolio at time zero for the TAA strategy which fixes the portfolio weights only once at time zero (the 10-year switching TAA strategy). We compute the utility costs and CER losses integrating across the steady-state distribution of the state variables. The various cases correspond to non-predictable asset returns, only bond returns are predictable, only stock returns are predictable, or both bond and stock returns are predictable. The minimum utility costs and CER losses for each case are marked with bold numbers.
in the predictive regression in Table 2.2). However, the optimal frequency, every year, is the same for the bond and equity predictability. The value-added benefit of TAA bond or equity decisions is outweighed by the higher switching costs. The minimum level of utility costs are decreasing as the asset returns become more predictable. For example, in terms of utility costs the minimum utility costs are increasing from 0.27% to 1.42% by adding marginal predictability to asset returns. In terms of CER losses, the investor would give up only CER of 0.39% when returns are IID, but she would lose CER of 3.22% when both returns are predictable.

### 2.4.4 Effect of Switching Costs

In the baseline case, we take the switching costs of 5 basis points as given, and then obtain the optimal switching frequency. To analyze the effects of switching costs on the utility costs and the optimal switching frequency, we vary the switching costs from 5 to 50 basis points, and find the number of switching that achieves the minimum utility costs relative to the first-best case.

The results are reported in Figure 2.6. The minimum utility costs are increasing as the cost of switching increases. For example, when the investor should pay 50 basis points of the value of portfolio for updating the target weight, the minimum utility cost relative to the first-best case is around 4% of the initial wealth, and can be achieved by updating the target weight every 3.3-year. For every costs of switching, we use the same parameter values in which both stocks and bonds are predictable. Thus, the marginal benefit of shortening the switching interval is fixed. The switching cost only increases the marginal cost of increasing the number of switching, which makes the minimum utility cost and the optimal number of switching increase as the switching cost increases.
We plot the minimum utility costs of taking TAA decisions relative to the continuous Merton (1971) strategy which switches instantaneously without any costs for various switching costs from 5 to 50 basis points (left axis). We also plot the optimal switching frequency that achieves the minimum utility costs on the right axis. The utility costs are integrated over the steady distribution of state variables. The utility costs are computed for a 10-year horizon and a risk aversion of 7.9, which corresponds to a 60% equity and 40% bond portfolio at time zero for the TAA strategy which chooses the portfolio weights only once at the beginning of the ten-year period (the 10-year TAA strategy). We consider the full case of predictability in both stocks and bonds.

### 2.4.5 Effect of Business Cycles

So far, we report the unconditional utility costs and CER losses to provide performance statistics of TAA strategies by integrating across the steady-state distribution of the return predictors. This enables us to compare TAA strategies at various switching frequencies with the first-best case systemically, and thus the optimal number of switching is independent of the current condition of investment opportunities. A natural question is then how performance statistics of TAA strategies respond to the market conditions. To obtain the conditional utility costs and CER losses, we proxy the business cycles by one of state variables, dividend yields. Low (high) valuation at troughs
(peaks) of business cycles is naturally embedded in high (low) dividend yields. We keep the other state variables at their long-run mean and estimate performance statistics of TAA conditional on dividend yields regime. High (trough) and low (peak) regimes correspond to the dividend yield which deviates from the long-run mean by three standard deviations.

The results are reported in the first two columns of Table 2.4. We use the same parameter values used in the last column of Table 2.3. The most notable thing is that the utility costs and CER losses are lower across all frequencies when the initial market conditions indicate that future returns are likely high. Thus, we can conclude that the benefit of frequent switching is asymmetric such that a freedom to respond to negative shocks to the risk premium is more valued by the investor. Another thing we should note is that the optimal frequency is independent of the market conditions. This is intuitive since the initial investment opportunity sets affect TAA at various switching frequencies systemically, and thus the optimal switching interval is same across the initial market conditions.

2.5 Extension

2.5.1 Cash

We exclude cash from an available asset class in the baseline model. In this section, we show that the case where investors can hold short or long positions in cash leads to similar results to our analysis, but much higher utility costs. We believe that utility costs in this scenario are overstated since most of investors, especially institutions, do not take a leverage. Our approach to exclude cash from an available asset class can be viewed as an extreme version of borrowing constraint.

The same solution method can be used for deriving the optimal allocation rule and the value function when cash is available with a slight modification. Table 2.4 reports the utility costs and CER losses when the investor is allowed to hold long or short positions in cash. We use the same parameter values used in the last column in Table 2.3, i.e. both bond and stock returns
Table 2.4: Robustness Check - Performance Statistics of Various TAA Frequencies

Panel A: Utility Costs (% of Initial Wealth)

<table>
<thead>
<tr>
<th>Switching Interval</th>
<th>Business Cycle Peak</th>
<th>Business Cycle Trough</th>
<th>Including Cash</th>
<th>More Predictors</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 Years</td>
<td>5.73</td>
<td>2.91</td>
<td>24.02</td>
<td>21.23</td>
</tr>
<tr>
<td>5 Years</td>
<td>3.37</td>
<td>2.56</td>
<td>12.71</td>
<td>20.32</td>
</tr>
<tr>
<td>2 Years</td>
<td>2.02</td>
<td>1.67</td>
<td>6.57</td>
<td>17.53</td>
</tr>
<tr>
<td>1 Year</td>
<td><strong>1.51</strong></td>
<td><strong>1.31</strong></td>
<td>3.72</td>
<td>14.25</td>
</tr>
<tr>
<td>6 Months</td>
<td>1.55</td>
<td>1.44</td>
<td>2.06</td>
<td>10.88</td>
</tr>
<tr>
<td>1 Quarter</td>
<td>2.30</td>
<td>2.25</td>
<td>1.24</td>
<td><strong>8.42</strong></td>
</tr>
<tr>
<td>1 Month</td>
<td>6.28</td>
<td>6.27</td>
<td><strong>1.08</strong></td>
<td>8.93</td>
</tr>
</tbody>
</table>

Panel B: Certainty Equivalent Return Loss (annualized return, %)

<table>
<thead>
<tr>
<th>Switching Interval</th>
<th>Business Cycle Peak</th>
<th>Business Cycle Trough</th>
<th>Including Cash</th>
<th>More Predictors</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 Years</td>
<td>0.56</td>
<td>0.29</td>
<td>2.11</td>
<td>1.92</td>
</tr>
<tr>
<td>5 Years</td>
<td>0.33</td>
<td>0.25</td>
<td>1.19</td>
<td>1.85</td>
</tr>
<tr>
<td>2 Years</td>
<td>0.20</td>
<td>0.17</td>
<td>0.63</td>
<td>1.61</td>
</tr>
<tr>
<td>1 Year</td>
<td><strong>0.15</strong></td>
<td><strong>0.13</strong></td>
<td>0.36</td>
<td>1.33</td>
</tr>
<tr>
<td>6 Months</td>
<td>0.15</td>
<td>0.14</td>
<td>0.20</td>
<td>1.03</td>
</tr>
<tr>
<td>1 Quarter</td>
<td>0.23</td>
<td>0.22</td>
<td>0.12</td>
<td><strong>0.81</strong></td>
</tr>
<tr>
<td>1 Month</td>
<td>0.61</td>
<td>0.61</td>
<td><strong>0.11</strong></td>
<td>0.85</td>
</tr>
</tbody>
</table>

The table reports utility costs and certainty equivalent return (CER) loss of TAA strategies switching at set periodic calendar intervals with switching costs versus the continuous Merton (1971) strategy which switches instantaneously without any costs. Utility costs are reported in percentage terms of initial wealth and represent the increase in initial wealth required to make the investor with a given TAA strategy have the same utility if she had the ability to implement the optimal Merton (1971) strategy with predictable returns. CER losses are reported in annualized percentage return and represent the decrease in CER that the investor would experience if she took costly TAA decision. We take an investor with a 10-year horizon, switching costs 5 basis points of the value of portfolio, and risk aversion of 7.9, which corresponds to a 60% equity and 40% bond portfolio at time zero for the TAA strategy which fixes the portfolio weights only once at time zero (the 10-year switching TAA strategy). The first two columns correspond to low and high dividend yields regime. In the third column, we allow the investor to hold cash. Finally, in the last column, we use parameter values obtained by adding more predictors.
are predictable. Thus, we allow bigger investment opportunity sets while fixing the degree of assets’ predictability and the cost of switching. We obtain that the utility costs and CER losses are significantly increased for longer switching interval. This is due to the fact that the investor with the continuous switching can capitalize the benefit of return predictability more easily by taking a leverage. This also implies that the investor using TAA policy could obtain higher marginal benefit if she decreased the switching interval, and thus the optimal interval becomes shorter, around one month. Note that the minimum utility cost and CER loss are lower than those in the case without cash. The bigger investment opportunity sets make the investor take more frequent TAA decisions relative to when she is constrained not to hold cash, which generate lower utility or return loss.

2.5.2 More Predictors

To extend the model to have more than three predictors, we create a “predictive index” for bond and stock returns, respectively. In the predictive regression of returns, we can put more predictors. Consider simple discretized version of equation (2.3) and (2.4). However, take a vector of predictors instead of taking just the yield spread and dividend yield:

\[
R_b(t + \Delta) = \Delta(\alpha_b + \beta_{b,r} r(t) + \beta_{b,x} x(t)) + \sigma_b \sqrt{\Delta} \epsilon_b(t + \Delta)
\]

\[
R_s(t + \Delta) = \Delta(\alpha_s + \beta_{s,r} r(t) + \beta_{s,x} x(t)) + \sigma_s \sqrt{\Delta} (\rho_{rs} \epsilon_r(t + \Delta) + \sqrt{1 - \rho_{rs}^2} \epsilon_s(t + \Delta)),
\]

where \(\Delta = 1\)-month, \(R_i(t + \Delta)\) is bond \((i = b)\) or equity \((i = s)\) return over \([t, t + \Delta]\), \(r(t)\) is the short rate as usual, \(x(t)\) is a vector of predictors, including yield spread, dividend yield, inflation, default spread, and output gap, and finally \(\epsilon_i(t + \Delta)\) is a standard normal variable. In the baseline model, we restrict that the slope \(\beta_{b,x}\) has a non-zero element only for the yield spread, and that the slope \(\beta_{s,x}\) has non-zero elements only for the yield spread and dividend yield. We can relax these restrictions and can take \(y(t) = \beta_{b,x} x(t)\) as a predictive index for bond returns, and take \(z(t) = (\beta_{s,x} - \beta_{b,x}) x(t)\) as a predictive index for stock returns. That is, bond and stock returns can
be expressed as

\[ R_b(t + \Delta) = \Delta(\alpha_b + \beta_b r(t) + y(t)) + \sigma_b \sqrt{\Delta \epsilon_r(t + \Delta)} \]

\[ R_s(t + \Delta) = \Delta(\alpha_s + \beta_s r(t) + y(t) + z(t)) + \sigma_s \sqrt{\Delta(\rho_{rs} \epsilon_r(t + \Delta) + \sqrt{1 - \rho_{rs}^2 \epsilon_s(t + \Delta)})} \]

In this way, we can easily incorporate multiple predictors in our framework and use our methodology to solve for the optimal TAA policy.

The effect of introducing more predictors is obvious. We obtain higher \( R^2 \) in predictive regressions, which implies the more predictability. We recover parameter values under this specification, and estimate the utility costs and CER losses. We keep restricting the investor from holding positions in cash. The results are reported in the last column of Table 2.4. The utility costs and CER losses are significantly increased across all switching intervals. Strong predictability in returns indicates that the investor can be better off by switching her target portfolio whenever the expected returns implied by predictors are changed. Thus, the minimum level of utility cost and CER loss that the investor using the TAA policy can achieve is increasing, and the optimal switching interval becomes shorter.

### 2.6 Conclusion

We solve for optimal TAA policies which switch target portfolio weights at periodic calendar intervals. The TAA policies are optimally computed for a long-horizon CRRA investor with time-varying expected returns. Under predictability, the optimal TAA weights are very different from the myopic, or instantaneous mean-variance, weights. Given 5 basis point of information processing costs, we find that the utility cost of infrequent switching is minimized when the investor updates the target portfolio weight annually. At this frequency, the marginal benefit and cost of shortening the switching interval are balanced and the utility cost is 1.42% of initial wealth compared to the Merton (1971) case.
There are at least three useful extensions to our approach. First, we do not consider variation in conditional volatilities. Given the relatively high mean reversion of volatilities, it is conceivable that TAA utility costs would still be small at intervals shorter than one quarter. Second, we ignore transaction costs in assuming that the agent can rebalance continuously in between TAA decision dates. While the literature started by Constantinides (1983) shows that closed-form solutions for portfolio weights in a non-IID environment are rarely available, we expect that the presence of transaction costs would only serve to lengthen the optimal TAA decision interval. On the other hand, the small transaction costs for implementing overlay TAA strategies with future contracts would likely not change our results. Lastly, we have focused only on the TAA decision between two asset classes: equity and bonds. It would be natural to extend this analysis to more asset classes, such as commodities, inflation protected bonds, and real estate.
Chapter 3

Asset Allocation and Pension Liabilities

in the Presence of a Downside Constraint

3.1 Introduction

A large decline in pension plans’ funding ratio motivated the creation of mandatory contribution rules and public insurance on defined benefit pension plans. For example, in the U.S. Employee Retirement Income Security Act (ERISA) in 1974 created the minimum funding contribution (MFC) and Pension Benefit Guaranty Corporation (PBGC).\(^1\) Despite of these government’s interventions to save underfunded pension plans, unfortunately large number of defined benefit pension plans are still underfunded.\(^2\) Thus, we believe that it is important to understand how underfunded pension plans can end up with funded status through the optimal asset allocation and contribution policy in the first place.

We revisit the question of a pension sponsor’s optimal asset allocation in the presence of a downside constraint. It is well-known (Grossman and Vila (1989)) that when markets are complete a put-based strategy is optimal by combining the unconstrained optimal portfolio and a put option on that unconstrained portfolio to hedge the downside. This analysis ignores however, the

\(^1\)MFC requirements specify that sponsors of underfunded pension plans must contribute an amount equal to any unfunded liabilities. PBGC has insurance obligations to pay defined benefits to employees when pension sponsors fail to fulfill due to firms’ bankruptcy.

\(^2\)In 2013 the largest 100 corporate defined benefits pension plans in the U.S. reported 1.78 trillion USD of liabilities guaranteed with only 1.48 trillion USD of asset, which represents underfunding of more than 15%. See Milliman 2014 Corporate Pension Funding Study, www.milliman.com.
possibility for the pension sponsor to contribute money to the pension plan over time. We analyze
the joint problem of optimal investing and contribution decisions, when there is disutility asso-
ciated with contributions.\footnote{Rauh (2006) finds that mandatory contributions leads to a reduction in corporate investment. Thus, the disutility from contributions is a reduced form of costs of foregone investment opportunities due to a use of internal cash for contributions.} Interestingly, we find that with the possibility of costly contributions to the pension plan in bad states to satisfy the downside constraint, the optimal portfolio deci-
sion often looks like a “risky gambling” strategy where the pension sponsor increases the pension plan’s allocation to risky assets in bad states. This is very different from the traditional prediction, where in economy downturns the pension sponsor should fully switch to the risk-free portfolio that replicates the downside constraint.

There are two effects of bad states of the economy in the optimal portfolio weight. First, the pension sponsor starts to contribute contemporaneously and keeps doing so as long as the economy is in bad states. Thus, the pension sponsor can take the present value of contributions and invest more aggressively by increasing the equity weight, which is hedged by contemporaneous and future contributions. Second, the pension sponsor decreases the equity weight to hedge the downside risk. If the former effect dominates the latter one, then a risky gambling behavior can be observed. However, note that this risk taking incentive is induced not by a moral hazard problem, but by a commitment to contributions.

We propose a separation approach to solve the optimal contribution and portfolio policy. The pension sponsor’s problem is cast in two separate shadow price problems. The first problem solves for the shadow price of maximizing the utility over the terminal pension plan’s asset. The second problem solves for the shadow price of minimizing the intermediate disutility from contributions. We interpret the shadow price of the utility maximization problem as the marginal benefit of increasing contributions. Similarly, the shadow price of the disutility minimization problem is the marginal cost of doing so. We show that the shadow prices for two problems are identical such that the marginal benefit and cost of increasing contributions are equal at the optimal solution.
Our approach allows us to characterize the optimal contribution, portfolio policy, and the value of put option in a simple way. Especially, the optimal contribution and the value of put option shed light on the level of minimum mandatory contributions and the premium of PBGC should charge to the pension sponsor. Also, by comparing with a case without a downside constraint, we can predict morally hazardous reactions of the pension sponsor in the presence of government insurance.

The investment behavior of pension plans has been studied by Sharpe (1976), Sundaresan and Zapatero (1997), Boulier, Trussant, and Florens (1995), and Van Binsbergen and Brandt (2007). Sharpe (1976) first recognized the value of implicit put option in pension plan’s asset to insure shortfall at the maturity. Sundaresan and Zapatero (1997) consider the interaction of pension sponsors and their employees. Given the marginal productivity of workers, the retirement date is endogenously determined. Then, pension sponsors solve the investment problem of maximizing the utility over excess assets in liabilities. Our focus is to derive the optimal contribution and portfolio policy, we model the exogeneous and deterministic benefits of the pension plan.4

Our paper is closely related to Boulier, Trussant, and Florens (1995). In their problem, the investment manager chooses his portfolio weights and contribution rates to minimize the quadratic disutility from contributions with the downside constraint. However, from the perspective of the pension sponsor the surplus at the end of the pension plan also matters since it is usually refunded to the pension sponsor and can be used to fund profitable projects. We model this motive as the utility over the terminal pension plan’s asset. Van Binsbergen and Brandt (2007) solve for the optimal asset allocation of the pension sponsor under regulatory constraints. They assume time-varying investment opportunity sets, and explore the impact of regulatory constraints on asset allocations. However, a contribution is not a control variable and a downside constraint is not explicitly specified. Instead, we assume an absence of any government regulations and derive the optimal contribution and portfolio policy. By doing this, we can have policy implications on how

4As long as the market is complete, our model can be extended to incorporate a stochastic feature of liabilities, and the solution technique goes through.
minimum contribution rules and premium paid to PBGC should be decided.

Our methodology is based on Karatzas, Lehoczky, and Shreve (1987) and El Karoui, Jeanblanc, and Lacoste (2005). Karatzas, Lehoczky, and Shreve (1987) solve a consumption and portfolio choice problem. They find that the initial wealth can be allocated in two problems, maximizing the utility over intermediate consumption and maximizing the utility over the terminal wealth. The optimal allocation leads to the optimal solution to the original problem. In our model, a contribution is a counterpart of consumption, but it generates the disutility and the pension sponsor’s objective is to minimize this disutility. Thus, the problem can be cast in a problem to decide how much to contribute to satisfy the downside constraint while minimizing the disutility. El Karoui, Jeanblanc, and Lacoste (2005) find a put option based solution to maximize the utility over the terminal wealth with the downside constraint. However, their solution can be applied to only initially overfunded pension plans. We allow initially underfunded pension plans to contribute in order to guarantee the terminal benefits.

There are at least three important aspects that we do not address explicitly. First, we do not incorporate time-varying investment opportunities. The expected returns of bonds and equities are predicted by macro variables, such as short rates, yield slopes, and dividend yields. This induces non-trivial hedging demands and liability risks, which drive a wedge between myopic and dynamic investment. Second, we do not consider the taxation issues. Drawing contributions from firm’s internal resources is costly for sure, however there is also a benefit from tax deductions. Third, our model do not include inflation. Depending on whether the pension sponsor’s preference is in real or nominal term, the allocation to real assets such as TIPS should be considered.

The paper is organized as follows. Section 3.2 describes the pension plan’s benefits and asset return dynamics. Section 3.3 considers a constrained case in which there is the downside constraint, and the separation method for the optimal investment and contribution policy. Section 3.4 presents the pension sponsor’s problem without the downside constraint as a benchmark case. Section 3.5 presents our results and Section 3.6 concludes.
3.2 Model

3.2.1 Liability

A defined benefit pension plan pays pre-defined benefits to employees on their retirement date. Usually, the benefits depend on the last 5-year average of salary and the number of years of employment. Let $L_t$ be an index of the pension benefits, i.e. if employees retire right now, they receive $L_t$. It follows:

$$dL_t = g L_t dt.$$ 

The pension benefits grow with the rate of $g$. This reflects an increase in years of employment and growth of salary. The terminal date $T$ is exogenously given. This can be thought as the average duration of employment. We define the downside constraint as

$$K = L_T = L_0 e^{gT}.$$ 

The pension sponsor should optimally manage the pension plan’s asset and contribute to the pension plan’s asset such that the terminal value of the pension plan’s asset is greater than the amount of the benefits promised to retiring employees.

3.2.2 Investment Sets

The pension sponsor has two available assets, a risky stock and a risk-free money market account. Let $r$ be the risk-free rate. We assume that $r$ is constant. The stock price follows

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$

where $\mu$ is the expected return of the stock, $\sigma$ is the volatility parameter, and $Z$ is a standard Brownian motion. Hence, in our model there is only one shock and one risky asset, and the market is complete. This implies that there exists a unique pricing kernel or stochastic discount factor. We
have following dynamics of the pricing kernel:

\[
\frac{dM_t}{M_t} = -rdt - \eta dZ_t,
\]

where \( \eta = \frac{\mu - r}{\sigma} \) is the market price of risk. Without loss of generality, we assume that the initial value of the pricing kernel is normalized to one, \( M_0 = 1 \). Now, the pension plan’s asset value follows

\[
dW_t = [(r + \pi_t(\mu - r)) W_t + Y_t] dt + \pi_t \sigma W_t dZ_t,
\]

where \( \pi \) is a fraction of the asset invested in the risky stock, and \( Y_t \) is the pension sponsor’s contribution to the pension plan’s asset.

### 3.2.3 Pension Sponsor’s Problem

The pension sponsor’s problem is

\[
\max_{\pi,Y} \mathbb{E} \left[ e^{-\beta T} u(W_T) - \int_0^T e^{-\beta t} \phi(Y_t) dt \right]
\]

subject to \( W_T \geq K \),

where \( u(x) = \frac{x^{1-\gamma}}{1-\gamma} \) and \( \phi(x) = k \frac{x^{\theta}}{\theta} \). The first term in equation (3.1) is a standard power utility with a relative risk aversion of \( \gamma \) over the final pension plan’s asset. The utility over the final pension plan’s asset can be justified since at the maturity the pension sponsor receives any pension plan’s surplus, which is valuable when internal financing is scarce or external financing is too costly.\(^5\)

The second term in equation (3.1) represents the pension sponsor’s disutility from contributing to the pension plan. The pension sponsor has limited internal resources for profitable projects which might be foregone if the pension sponsor uses the internal cash to contribute to the pension plan. We capture the cost of foregone projects due to contributions as the separable disutility function. A parameter \( \theta \) will capture a desire to smooth contributions over time. To have convex

\(^5\)Petersen (1992) uses plan-level data to find evidence in support of the financing motives.
disutility, we assume that $\theta > 1$. A parameter $k$ captures the importance of the disutility from contributing relative to the utility over the final pension plan’s asset. For example, if the pension sponsor is financially healthy (sufficiently high internal resources), the impact of contributing to the pension plan is relatively small and thus the disutility function have low $k$. Finally, $\beta$ is the subjective discount rate of the pension sponsor.

The disutility from contributions is a counterpart of adjustment costs in the investment literature.\(^6\) The key difference is that the disutility shows up as a separable objective while adjustment costs decrease firm’s liquidity. This motivates us to decompose the pension sponsor’s problem into two separate ones:

- **Utility Maximization Problem**

  Without a possibility of contributions, the pension sponsor maximizes the expected utility over the final pension plan’s asset given the upper bound for the present value of the final pension plan’s asset and the downside constraint:

  $$\max_{\pi^u} \mathbb{E} \left[ e^{\beta T} u (W_T^u) \right]$$
  s.t. $W_0^u \geq \mathbb{E}^Q [e^{-r T} W_T^u]$ 
  $W_T^u \geq K,$

  where $\pi^u$ is a fraction of $W^u$ invested in the risky stock, $\mathbb{E}^Q[\cdot]$ is an expectation under the risk-neutral measure $Q$.

- **Disutility Minimization Problem**

  The pension sponsor minimizes the expected disutility from contributions given the lower

\(^6\)An investment can increase firm’s capital, but also incur adjustment costs. See Caballero (1999) for summaries on this literature.
bound for the present value of contributions:

\[
\min_Y \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi(Y_t) \, dt \right] \quad (3.3)
\]

s.t \( X_0 \leq \mathbb{E}^Q \left[ \int_0^T e^{-rt} Y_t \, dt \right] \).

- **Budget Constraint**

The sum of the pension plan’s original initial asset value and the lower bound for the present value of contributions should be equal to the upper bound for the present value of the final pension plan’s asset value:

\[
W_0 + X_0 = W^u_0. \quad (3.4)
\]

Whenever \( K > 0 \), we consider the problem as a constrained case. When \( K = 0 \), there is no downside constraint and it serves as a benchmark case. We will show that solving two problems separately and satisfying the budget constraint (3.4) will lead us to the solution to the original problem (3.1).

## 3.3 Constrained Case

### 3.3.1 Utility Maximization Problem

First, we solve the utility maximization problem. The budget constraint (3.4) implies that the initial endowment for the first problem \( W^u_0 \) is greater than the original endowment, \( W_0 \), and that the difference \( W^u_0 - W_0 \) is the present value of the contribution stream. That is, the pension sponsor expects the contribution stream in the future and thus, at time zero the pension sponsor can behave as if the pension sponsor borrows the present value of the contribution stream. The optimal amount of borrowing will be determined later by taking into account both the utility over the final asset and the disutility from the contribution. If the initial endowment for the first problem
is less than the present value of the downside constraint, \( W_0^u \leq Ke^{-rT} \), there is no solution that guarantees the benefits for sure at the maturity. This implies that the present value of the contribution \( X_0 = W_0^u - W_0 \) should be greater than \( \max(Ke^{-rT} - W_0, 0) \). For example, if the pension plan is initially underfunded, the present value of the contribution should be greater than the initial shortfall, \( Ke^{-rT} - W_0 \). The dynamic budget constraint for the first problem is

\[
dW_t^u = (r + \pi_t^u (\mu - r)) W_t^u dt + \pi_t^u W_t^u \sigma dZ_t. \tag{3.5}
\]

Note that there’s no contribution process since it’s already reflected in the increased initial endowment \( W_0^u \).

**Put-based Strategy**

It is well-known (Grossman and Vila (1989)) that when the market is complete the optimal strategy of the first problem consists in investing a fraction of asset in the unconstrained optimal portfolio and using the remaining fraction of asset to purchase a put option on that unconstrained portfolio to hedge the downside. We call this strategy a put-based strategy. To decide the optimal fraction in the unconstrained optimal portfolio, we define the following functions for any \( 0 < y < \infty \):

\[
\mathcal{W}_u(y) = \mathbb{E}^Q \left[ e^{-rT} I_u (y\xi_T) \right] + \mathbb{E}^Q \left[ e^{-rT} (K - I_u (y\xi_T))^+ \right],
\]

where \( I_u (\cdot) \) is the inverse function of \( u' (\cdot) \), \( \xi_t = M_t e^{\beta t} \), and \( (x)^+ = \max(x, 0) \). This function calculates the cost of constructing the put-based strategy when the terminal asset value is random variable \( I_u (y\xi_T) \) and the put option’s strike price is \( K \). The terminal asset value is chosen such that the marginal utility is proportional to the marginal rate of substitution of the economy at the terminal date. It will be shown that the parameter \( y \) is a shadow price, i.e. a marginal increase in the utility when the initial endowment \( W_0^u \) for the first problem is marginally increased or the present value of the contribution stream is marginally increased. Proposition 3.1 explicitly computes this function.
Proposition 3.1. The function $W_u(y)$ is given by

$$W_u(y) = y^{-1/\gamma}e^{-\alpha_u T} N(\delta_1(y,T)) + Ke^{-rT} N(-\delta_2(y,T)),$$

where $\alpha_u = \frac{\beta}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\eta^2}{2\gamma}\right)$. $\delta_1$ and $\delta_2$ can be found in Appendix C.1. Also, the first derivative of $W_u(y)$ is given by

$$W'_u(y) = -\frac{1}{\gamma}y^{-1/\gamma-1}e^{-\alpha_u T} N(\delta_1(y,T)) < 0.$$ (3.7)

Since the market is complete and the put-based strategy consists in the underlying asset and the put option, the expression for $W_u(y)$ looks like Black and Scholes (1973) option pricing formula. The first part is the present value of the terminal unconstrained optimal portfolio value multiplied by the probability that the downside constraint is met at the maturity under the forward measure. Note that the final unconstrained optimal portfolio value is discounted with a rate of $\alpha_u$ which is a weighted average of the pension sponsor’s subjective discount rate and subjective risk-adjusted expected return. Suppose that the pension sponsor is extremely risk averse. Then, the pension sponsor will allocate all pension plan’s asset in the risk-free asset, and thus the terminal unconstrained optimal portfolio value can be discounted with the risk-free rate: $\lim_{\gamma \to \infty} \alpha_u = r$. The second part is the present value of the benefits multiplied by the probability that the put option is in-the-money under the risk-neutral measure.

Since we have the concave utility function, a higher shadow price implies a lower cost of constructing the put-based strategy. Thus, we can see that $W_u(y)$ is decreasing, which implies that $W_u(y)$ is invertible. Let $Y_u$ denote the inverse of this function. For a fixed initial endowment for the first problem, $W_0^u \geq Ke^{-rT}$, we introduce the following random variable

$$W_T^u = I_u (Y_u (W_0^u) \xi_T) + (K - I_u (Y_u (W_0^u) \xi_T))^+. $$

The following Theorem 3.2 states that the constructed terminal asset value is optimal for the problem (3.2).
Theorem 3.2. For any $W_0^u \geq Ke^{-rT}$, $W_T^u$ is optimal for the problem (3.2), and the optimal portfolio weight is given by
\[ \pi_t^u = \frac{\eta}{\gamma \sigma} (1 - \varphi_t), \] (3.8)
where $\varphi_t = \frac{Ke^{-r\tau}}{W_t^u} N (-\delta_2 (yt, \tau)) < 1$, $\tau = T - t$, and $yt = \mathcal{Y}_u(W_0^u) \xi_t$.

Since the put-based strategy is constructed by combining the underlying unconstrained optimal portfolio and its put option, the downside constraint is always satisfied not only at the terminal date, but also along the horizon. Now, the question is how much the pension sponsor should hold the underlying unconstrained optimal portfolio to achieve the maximum utility. Theorem 3.2 states that the optimal shadow price should be $\mathcal{Y}_u(W_0^u)$ such that the cost of constructing the put-based strategy is exactly same as the initial endowment for the first problem, $W_0^u$. Then, the optimal portfolio weight is a weighted average of the mean-variance efficient portfolio and zero investment in the equity. The weight on the mean-variance efficient portfolio is $1 - \varphi_t$. The parameter $\varphi_t$ measures how far away the current asset value is from the present value of the benefits. The closer the asset is to the present value of the benefits, the less fraction of the asset is invested in the equity.

Now, we compute the value function of the first problem and relate its first derivative to the shadow price. Let $J(W_0^u)$ be the value function of the first problem and define the following function $G(y)$ for $0 < y < \infty$:
\[ G(y) = \mathbb{E} \left[ e^{-\beta T} u \left( I_u (y\xi_T) + (K - I_u (y\xi_T))^+ \right) \right]. \] (3.9)
This function computes the expected utility when the put-based strategy is used with the shadow price of $y$. At the optimal solution, we choose the shadow price satisfying the budget constraint with equality, $y = \mathcal{Y}_u(W_0^u)$, so that we can obtain the value function $J(W_0^u)$ by substituting $y$ in $G(y)$ with $\mathcal{Y}_u(W_0^u)$. Proposition 3.3 states that the first derivative of the value function, i.e. the shadow price is indeed $\mathcal{Y}_u(W_0^u)$. 

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Proposition 3.3. The function $G(y)$ is given by

$$G(y) = \frac{y^{1-\frac{1}{\alpha}}}{1-\gamma} - e^{-\alpha_t T} N\left(\delta_1(y, T)\right) + e^{-\beta T} \frac{y^{1-\gamma}}{1-\gamma} N\left(-\delta_3(y, T)\right),$$

(3.10)

where $\delta_3$ can be found in Appendix C.1. Also, $G(y)$ satisfies

$$J(W_0^u) = G\left(\chi_u(W_0^u)\right)$$

(3.11)

$$J'(W_0^u) = \chi_u(W_0^u).$$

3.3.2 Disutility Minimization Problem

The second problem is to decide how to contribute along the horizon to minimize the expected disutility while satisfying the minimum present value of the contribution. Alternatively, the problem can also be stated that the pension sponsor has the initial endowment $X_0$ in its internal cash account to fund the future contribution stream and decides how to manage this internal resource. The assumption is that the pension sponsor considers only self-financing strategies. Let $X_t$ be the time $t$ value of this account. Then, the dynamic budget constraint of the second problem is

$$dX_t = \left[\left(r + \pi^\phi_t (\mu - r)\right) X_t - Y_t\right] dt + \pi^\phi_t \sigma X_t dZ_t,$$

(3.12)

where $\pi^\phi_t$ is a fraction of this account invested in the equity and the rest of it is invested in the risk-free asset. The contribution to the pension plan decreases the account balance.

Now, the problem becomes a standard portfolio choice problem with intermediate outflow (contribution) and no bequest objective. However, there are two important differences. First, contribution does not increase the pension sponsor’s utility, but increase the disutility. Second, the static budget constraint states that the present value of the contribution should be greater than the initial endowment. At the optimal solution, the static budget constraint is binding and thus the terminal value of the internal cash account is zero, $X_T = 0$.

We define the following function for any $0 < y < \infty$:

$$W_\phi(y) = E^\mathbb{Q}\left[\int_0^T e^{-rt} I_\phi(y\xi_t) dt\right],$$
where $I_{\phi}(\cdot)$ be the inverse function of $\phi'(\cdot)$. The function $W_\phi(y)$ computes the present value of the contribution stream from time zero to the terminal date when an intermediate contribution is set to be $I_{\phi}(y\xi_t)$, i.e. the marginal disutility is proportional to the marginal rate of substitution of the economy at each time. As the first problem, it will be shown that the parameter $y$ is a shadow price, i.e. a marginal increase in the disutility when the minimum present value of the contribution $X_0$ is marginally increased. Proposition 3.4 explicitly computes this function.

**Proposition 3.4.** The function $W_\phi(y)$ is given by

$$W_\phi(y) = \left(\frac{y}{\theta - 1}\right)^{\alpha_{\phi}} \frac{1 - e^{-\alpha_{\phi}T}}{\alpha_{\phi}},$$

where $\alpha_{\phi} = \frac{\theta - 1}{\theta - 1} \left( r - \frac{\eta^2}{2(\theta - 1)} \right) - \frac{\beta}{\theta - 1}$. Also, the first derivative of $W_\phi(y)$ is given by

$$W_\phi'(y) = \frac{1}{y(\theta - 1)} W_\phi(y) > 0.$$

The present value of the contribution stream has a form of annuity with a rate of return $\alpha_{\phi}$, which is a weighted average of the pension sponsor’s subjective discount rate and the subjective risk adjusted expected return. An incentive to smooth contributions over time (high $\theta$) implies that the contribution stream can be discounted with a rate $r$: $\lim_{\theta \to \infty} \alpha_{\phi} = r$. Since we have the convex disutility function, the present value of the contribution would be higher if a marginal disutility (shadow price) is higher. Thus, we can see that $W_\phi(y)$ is increasing, which implies that $W_\phi(y)$ is invertible. Let us denote $Y_\phi$ be the inverse of the function $W_\phi$. For the minimum contribution requirement $X_0 > 0$, we introduce the contribution process

$$Y_t = I_{\phi}(Y_\phi(X_0) \xi_t).$$

Theorem 3.5 states that the above contribution policy is optimal for the problem (3.3).

**Theorem 3.5.** For any $X_0 > 0$, $Y_t$ constructed above is optimal for the problem (3.3), and the optimal hedging policy is

$$\pi^\phi = -\frac{\eta}{(\theta - 1)\sigma}.$$
By setting the marginal disutility of the contribution to be proportional to the marginal rate of substitution of the economy at each time, the minimum disutility can be achieved. The shadow price is determined such that the present value of the contribution stream is identical with the minimum contribution requirement \( X_0 \). The optimal hedging policy is to short the equity, since the contribution is increasing in the marginal rate of substitution or decreasing in the stock return. Whenever the stock price decreases, the pension sponsor should increase the contribution which can be funded with profits from short positions in the equity. If the pension sponsor has a strong desire to smooth the contribution (higher \( \theta \)), the pension sponsor would decrease short positions in the equity since the contribution stream is stable.

Finally, we compute the value function of the second problem. Let \( L(X_0) \) be the value function of the second problem and define the following function \( C(y) \) for \( 0 < y < \infty \):

\[
C(y) = \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi (I_\phi (y \xi_t)) \, dt \right].
\]

(3.15)

This function computes the expected disutility when the contribution is set to be \( I_\phi (y \xi_t) \) as a function of \( y \). At the optimal solution, we choose \( y = \mathcal{Y}_\phi (X_0) \) so that we can obtain the value function \( L(X_0) \) by substituting \( y \) in \( C(y) \) with \( \mathcal{Y}_\phi (X_0) \). Proposition 3.6 states that the first derivative of \( L(X_0) \) (shadow price) is indeed \( \mathcal{Y}_\phi (X_0) \).

**Proposition 3.6.** The function \( C(y) \) is given by

\[
C(y) = \frac{k}{\theta} \left( \frac{y}{k} \right)^{\theta-1} \frac{1 - e^{-\alpha \phi T}}{\alpha \phi},
\]

and satisfies

\[
L(X_0) = C(\mathcal{Y}_\phi (X_0))
\]

(3.16)

\[
L'(X_0) = \mathcal{Y}_\phi (X_0).
\]
3.3.3 Optimality of Separation

So far, we derive the solutions for the utility maximization problem and the disutility minimization problem while taking the present value of the contribution as given. We now show that the optimal choice of the present value of the contribution $X_0$ leads that separately solved solutions are indeed the solution for the original problem. The pension sponsor behaves as if taking a leverage at time zero, $W_0^u = W_0 + X_0$. With $W_0^u$, the pension sponsor solves the utility maximization problem with the downside constraint. Then, the pension sponsor solves the disutility minimization problem to pay back the borrowing $X_0$ through the contributions. The next theorem shows that how the initial borrowing amount $X_0$ is decided to achieve the optimality of the original problem.

**Theorem 3.7.** Consider an arbitrary portfolio and contribution policy pair $(\tilde{\pi}, \tilde{Y})$ satisfying the downside constraint. Then, there exists a pair $(\pi, Y)$ dominating $(\tilde{\pi}, \tilde{Y})$. In particular, the value function of the original problem $V(W_0)$ is

$$V(W_0) = \max_{X_0} J(W_0 + X_0) - L(X_0) = \max_{W_0(y_u) - W_0(y_\phi) = W_0} G(y_u) - C(y_\phi). \quad (3.17)$$

For an arbitrary portfolio and contribution policy pair, we can take the present value of that contribution stream, $X_0 = E_Q \left[ \int_0^T e^{-rt} \tilde{Y}_t dt \right]$. Then, for $X_0$, $\tilde{\pi}$ is a feasible strategy to the utility maximization problem (3.2), and $\tilde{Y}$ is a feasible strategy to the disutility minimization problem (3.3). We can find the optimal solutions to each problem and they will (weakly) dominate $(\tilde{\pi}, \tilde{Y})$. Thus, finding the optimal solution to the original problem (3.1) can be translated into the problem to find the optimal present value of the contribution $X_0$ to maximize the difference between two value functions of (3.2) and (3.3), $J(W_0 + X_0) - L(X_0)$.

Suppose that (3.17) has an interior solution. This implies that the FOC with respect to $X_0$ equals zero:

$$J'(W_0 + X_0) = L'(X_0).$$
This condition states that at the optimal solution, the marginal increase in the value function of the utility maximization problem should be identical with the marginal increase in the value function of the disutility minimization problem. Thus, we can interpret LHS as the marginal benefit of increasing the present value of the contribution, and RHS as the marginal cost of increasing the present value of the contribution. Recall that the shadow prices of both problems are satisfying the static budget constraints with equality. Hence, we have $y = \mathcal{V}_u(W_0 + X_0) = \mathcal{V}_\phi(X_0)$, which is determined by the budget constraint:

$$W_u(y) - W_\phi(y) = W_0. \quad (3.18)$$

Define the following function for $0 < y < \infty$:

$$W(y) = W_u(y) - W_\phi(y).$$

This function computes the initial pension fund’s asset required to have the shadow price of $y$ for both problems. Proposition 3.8 shows that there exists a unique $y$ solving $W(y) = W_0$, and thus we obtain the optimal solution to the original problem.

**Proposition 3.8.** The function $W(y)$ is decreasing in $y$ and $\lim_{y \to 0} W(y) = \infty$ and $\lim_{y \to \infty} W(y) = -\infty$. Hence, there exists a unique $y$ satisfying $W(y) = W_0$.

Suppose that we find $y$ solving (3.18). Then, the time $t$ pension plan’s asset can be expressed as $W_t = W_t^u - X_t$. This implies that the present value of the terminal pension plan’s asset ($W_T = W_T^u$ since $X_T = 0$) is the sum of the current pension plan’s asset and the pension sponsor’s internal fund for hedging the contributions. Proposition 3.9 describes the optimal portfolio weight and contribution policy to the original problem.

**Proposition 3.9.** The optimal portfolio weight is given by

$$\pi_t = \pi_t^u \rho_t + \pi_\phi (1 - \rho_t),$$
and the optimal contribution rate is given by

\[ Y_t = \frac{(\rho_t - 1) \frac{\alpha_\phi}{1 - e^{-\alpha_\phi (T-t)}}}{W_t}, \]

where \( \rho_t = \frac{W_t^c}{W_t} = 1 + \frac{X_t}{W_t}. \)

The optimal portfolio weight is a weighted average of two weights, \( \pi_t^u \) and \( \pi_t^\phi. \) The weight is the ratio of the present value of the terminal pension plan’s asset over the current pension plan’s asset. We call \( \rho_t \) the pension plan’s leverage ratio. Note that because a possibility of future contributions, this ratio is generally not equal to one. When the state of the economy is good and the expected contribution is small, then the weight \( \rho \) is close to one. Also, \( \pi_t^u \) becomes the mean-variance efficient portfolio \( \left( \frac{\mu}{\sigma} \right) \) since it is more likely that the downside constraint is not binding. Thus, the optimal portfolio weight, \( \pi_t \) is close to the mean-variance efficient portfolio.

As the economy gets worse (the equity price drops), the pension plan’s asset gets close to the downside constraint. There are two effects of bad states of the economy in the optimal portfolio weight. First, the pension sponsor will hold large internal resources \( X_t \) to hedge large contemporaneous and future contributions, which indicates an increase in \( \rho_t. \) Thus, the pension sponsor will increases the equity weight, which is hedged by contemporaneous and future contributions. Second, the optimal equity weight for the utility maximization problem \( \pi_t^u \) will decrease, since the present value of the terminal pension plan’s asset, \( W_t^u \) approaches to the present value of the benefits. If the latter effect dominates the former one, then a risk management behavior can be observed, i.e. a decrease in the equity weight as the economy gets worse. On the other hand, if the former effect dominates, we can see a risk taking behavior. However, note that this risk taking incentive is induced not by a moral hazard problem, but by a commitment to contributions in the future.

The optimal contribution policy as a fraction of the current pension plan’s asset also depends on the pension plan’s leverage ratio \( \rho_t \) and time-to-maturity \( T - t. \) The pension sponsor contributes more when the pension plan’s asset return is low so that the leverage ratio is high. For the same
pension plan’s leverage ratio $\rho_t$, the ratio of the contribution to the pension plan’s asset is higher when time-to-maturity is short. Since the pension plan’s objective is to minimize the expected disutility, the pension plan would defer a contribution as much as it can.

### 3.4 Benchmark Case

Now, we consider the benchmark case. There’s no downside constraint and the pension sponsor contributes purely for maximizing the terminal pension plan’s asset while taking into account the disutility from contributions. The pension sponsor’s problem becomes

$$\max_{\pi,Y} \mathbb{E} \left[ e^{-\beta T} u(W_T) - \int_0^T e^{-\beta t} \phi (Y_t) \, dt \right].$$

Everything we derive for the constrained case goes through, except for the first problem. Now, let $V^M_u(y)$ be the counterpart of $V_u(y)$ in the constrained case:

$$V^M_u(y) = \mathbb{E}^Q \left[ e^{-r T} I_u (y \xi_T) \right].$$

Note that the pension plan holds just the unconstrained optimal portfolio since there’s no downside constraint. Similarly, we can consider $V^M_u(W^*_0), J^M(W^*_0), G^M(y)$ as the benchmark version of $V_u(W^*_0), J(W^*_0), G(y)$. Proposition 3.10 summarizes the results for the first problem in the benchmark case.

**Proposition 3.10.** The function $V^M_u(y)$ is given by

$$V^M_u(y) = y^{\frac{1}{\gamma}} e^{-\alpha u T}. \quad (3.19)$$

Also, the first derivative is given by

$$V^M_u (y)' = y^{\frac{1}{\gamma} - \frac{1}{\gamma} - 1} e^{-\alpha u T} < 0. \quad (3.20)$$

For a given $y$, we have $V^M_u(y) < V_u(y)$. For any $W^*_0$, $W^*_T = I_u \left( V^M_u (W^*_0) \xi_T \right)$ is optimal for the utility maximization problem, and the optimal portfolio weight is given by $\pi^*_BC = \frac{n}{\gamma \sigma}$. The
function $G^{BC}(y)$ is given by

$$G^{BC}(y) = \frac{y^{1-\frac{1}{\gamma}}}{1 - \gamma} e^{-\alpha u T},$$

and satisfies

$$J^{BC}(W^u_0) = G^{BC}(Y^{BC}_u(W^u_0))$$

(3.21)

$$J^{BC}(W^u_0)' = Y^{BC}_u(W^u_0).$$

Without the downside constraint, the present value of the terminal pension plan’s asset is smaller than the constrained case. To achieve the same level of marginal utility, the benchmark case requires smaller initial asset since the put option doesn’t have to be purchased. As we expect, the optimal portfolio weight is the mean-variance efficient portfolio. Now, Theorem 3.7, Proposition 3.8 and 3.9 can be stated for the benchmark case by substituting corresponding counterparts with $J^{BC}(W^u_0), G^{BC}(y), W^{BC}_u(y), Y^{BC}_u(W^u_0),$ and $\pi^{u}_{BC}.$

### 3.5 Quantitative Analysis

We now turn to quantitative analysis of the model. For a baseline case, we use 10-year for the pension plan’s maturity $T$. According to Bureau of Labor Statistics, as of 2014 the median years of tenure with current employer for workers with age over 65 years is 10.3-year. Also, we use $\eta = 0.4$ for the market price of risk, $\sigma = 20\%$ for the volatility of the equity, $r = 2\%$ for the risk-free rate, and $\beta = 1\%$ for the pension sponsor’s subjective discount rate. These numbers are standard assumptions in the literature. The expected excess return of the equity is $\mu - r = \sigma \eta = 8\%$. We use $\gamma = 5$, which implies the equity weight of the mean-variance efficient portfolio is $\frac{\eta}{\gamma \sigma} = 40\%$. For the disutility function, we use $k = 100$ and $\theta = 2$. The quadratic disutility function is common in the investment literature, in which a firm is assumed to be risk-neutral and faces quadratic costs of investment adjustment.\(^7\) Finally, we use two values for the initial funding ratio,

\(^7\)See Gould (1968); more recently Bolton, Chen, and Wang (2011); among others.
Table 3.1: Summary of key variables and parameters

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terminal benefits</td>
<td>$K$</td>
<td>Pension plan’s investment horizon</td>
<td>$T$</td>
<td>10-year</td>
</tr>
<tr>
<td>Pension’s asset</td>
<td>$W$</td>
<td>Price of Risk</td>
<td>$\eta$</td>
<td>0.4</td>
</tr>
<tr>
<td>Present value of the terminal asset</td>
<td>$W^u$</td>
<td>Risk-free rate</td>
<td>$r$</td>
<td>2%</td>
</tr>
<tr>
<td>Pension sponsor’s internal resources</td>
<td>$X$</td>
<td>Pension sponsor’s subjective discount rate</td>
<td>$\beta$</td>
<td>1%</td>
</tr>
<tr>
<td>for hedging contribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shadow price</td>
<td>$y$</td>
<td>Pension sponsor’s risk aversion</td>
<td>$\gamma$</td>
<td>5</td>
</tr>
<tr>
<td>Pension sponsor’s marginal rate of substitution</td>
<td>$\xi$</td>
<td>Pension sponsor’s elasticity of disutility</td>
<td>$\theta$</td>
<td>2</td>
</tr>
<tr>
<td>Portfolio weight of equity</td>
<td>$\pi$</td>
<td>Relative importance of disutility</td>
<td>$k$</td>
<td>100</td>
</tr>
<tr>
<td>Contribution flow</td>
<td>$Y$</td>
<td>Initial funding ratio</td>
<td>$\lambda_0$</td>
<td>80%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Initial funding ratio</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(underfunded)</td>
<td>$\lambda_0$</td>
<td>120%</td>
</tr>
</tbody>
</table>

This table summarizes the symbols for the key variables used in the model and the parameter values in the baseline case.

\[
\lambda_0 = \frac{W_0}{Ke^{-rT}} = 80\% \text{ or } 120\%. \]

We will vary preference parameters, $(\gamma, k, \theta)$, and the price of risk to see the impacts on the optimal present value of the contribution, portfolio and contribution policy. Table 3.1 summarizes all the key variables and parameters in the model.

### 3.5.1 Present Value of Contribution

Figure 3.1 plots the determination of $X_0$ by equating the shadow prices of the first and second problem: $\mathcal{Y}_u(W_0 + X_0) = \mathcal{Y}_\phi(X_0)$. The initial pension plan’s asset is normalized to one $W_0 = 1$ and thus the present value of the contribution can be interpreted as a fraction of the initial pension plan’s asset. Panel A is a case when the pension plan is initially underfunded, $\lambda_0 = 80\%$, and Panel B is a case when overfunded, $\lambda_0 = 120\%$. We also plot the benchmark case. Since we assume the quadratic disutility function, the shadow price of the disutility minimization problem is linear in the present value of the contribution. The shadow price of the utility maximization problem is decreasing in the present value of the contribution, since the utility function is concave. Also, the shadow price of the first problem for the constrained case is always above that of the benchmark.
We can see that the optimal present value of the contribution is $X_0 = 3.68\%$ and the shadow price is $y = 0.18$ for the benchmark case. This indicates that along the horizon the pension sponsor contributes 3.68\% even though there is no downside constraint. This is because the marginal benefit of contributing is greater than the marginal cost of doing so when $X_0 < 3.68\%$ as we can see in Figure 3.1.

For the underfunded pension plan, the marginal benefit curve $\mathcal{Y}_u(W_0 + X_0)$ has the left asymptote line at $X_0 = Ke^{-rT} - W_0 = 25\%$, at which the solution for the utility maximization problem is zero investment in the equity. The optimal present value of the contribution is $X_0 = 25.10\%$ and the shadow price is $y = 1.22$. Compared to the benchmark case, the pension sponsor contributes more to make the pension plan overfunded at the maturity. For the overfunded pension plan, the present value of the contribution is $X_0 = 4.15\%$ and the shadow price is $y = 0.20$. This implies that relative to the benchmark case the additional contributions of 0.47\% are required to guarantee the benefits for the initially overfunded pension plan.

Figure 3.2 plots the the cost of constructing the put-based strategy for the utility maximization problem with the downside constraint. Again, Panel A is a case when the pension plan is initially underfunded, and Panel B is a case when overfunded. We put a fraction of the initial endowment $W^{u}_0 = W_0 + X_0$ invested in the unconstrained optimal portfolio or the mean-variance efficient portfolio on $x$-axis. The dashed line is a 45-degree line, i.e. the cost of the unconstrained optimal portfolio in the put-based strategy and the solid line represents the total cost of the put-based strategy. Thus, the difference between two lines represents the cost of purchasing the put option.

We can see that for the initially underfunded pension plan, without contributions there’s no feasible put-based strategy. That is, the cost of the put-based strategy is greater than $W^{u}_0 = W_0 = 100\%$ when $X_0 = 0$. However, with the optimal present value of the contribution $X_0 = 25.10\%$, the initial endowment of the first problem is increased to $W^{u}_0 = W_0 + X_0 = 125.10\%$ and thus there is the optimal put-based strategy which costs exactly $W^{u}_0 = 125.10\%$. At the optimal put-
Figure 3.1: Determination of Present Value of Contribution

Panel A: Initially Underfunded Pension

This figure plots shadow prices of first and second problem as a function of present value of contribution. Panel A is for an initially underfunded pension with 80% funding ratio, and Panel B is for an initially overfunded pension with 120% funding ratio.
based strategy, the effective allocation to the mean-variance efficient portfolio is 70.60%, and the
rest 54.50% is used to replicate the put option on 70.70% of the mean-variance efficient portfolio.

On the other hand, for the initially overfunded pension plan, even with zero contribution there’s
a feasible put-based strategy, which costs exactly \( W_0^u = W_0 = 100\% \) and consists in the mean-
variance efficient portfolio of 95.92% and the put option of 4.08% on that portfolio. However, with
contribution, the pension sponsor can be better off by allocating 101.21% to the mean-variance
efficient portfolio and 2.94% to the put option on that portfolio. The total cost of this put-based
strategy is \( W_0^u = 104.15\% \) and the shortfall \( X_0 = W_0^u - W_0 = 4.15\% \) will be funded through
contributions.

### 3.5.2 Portfolio Weight and Contribution Policy

Figure 3.3 plots the equity weight at time \( t = 5\)-year as a function of an annualized equity return
over the last five years. We fix the initial pension plan’s asset and vary the terminal benefits, \( K \).
We set \( K = 153\% \) for Panel A, and \( K = 102\% \) for Panel B such that the initial funding ratios
are 80% and 120%, respectively. First, we can see that the equity weight of the benchmark case
is decreasing in the past equity return. The low equity return over time zero to 5-year indicates
that the state of economy is bad, i.e. the marginal rate of substitution is high. As we will see in
Figure 3.4, the optimal contribution rule is to increase contributions in a such state. The pension
sponsor expects that future contributions will be made, and thus can take more risks by increasing
the equity weight. Put differently, when the state of economy is bad, future contributions can hedge
positions in the equity, and thus the pension sponsor can take more risks. When the past equity
return is higher, the equity weight of the benchmark case is approaching to the mean-variance
efficient portfolio, which is \( \frac{\mu}{\sigma} = 40\% \). We can see that the equity weight of the benchmark case
is identical for the initially underfunded and overfunded pension plans. This is obvious since how
far away from the present value of the benefits doesn’t matter for the pension sponsor without the
This figure plots costs of put-based strategy as a function of fraction of allocation to the mean-variance efficient portfolio. Panel A is for an initially underfunded pension with 80% funding ratio, and Panel B is for an initially overfunded pension with 120% funding ratio.
Next, the equity weight of the constrained case exhibits an U-shaped pattern. When the state of economy gets worse, the pension sponsor defers contributions and employs the risk management policy, i.e. decreases the equity weight, since the pension sponsor wants to avoid costly contributions as much as it can. On the other hand, when the economy downturn is significant, the pension sponsor starts to contribute. Since contemporaneous and future contributions can hedge high equity positions, the pension sponsor takes more risks. When the pension sponsor switches from risk management to risk taking depends on the initial funding status. For the initially underfunded pension plan, risk taking incentives dominate risk management incentives. The intuition is that for same negative shocks to the economy, the impact is greater for the initially underfunded pension plan so that the pension sponsor starts to contribute earlier, which yields risk taking incentives.

By comparing the benchmark case and the constrained case, we can predict a situation in which a government insurance exists. In the benchmark case, the pension sponsor has only risk taking incentives, which are hedged by contributions. Even if the pension plan ends up with underfunded, the government agency, such as PBGC will guarantee the benefits. Thus, as the economy gets worse the pension sponsor would take more risks. On the other hand, the pension sponsor without the government insurance would avoid large contributions as much as it can by managing risks, i.e. decreasing the equity weight. However, when the pension plan’s asset is severely deteriorated, the pension sponsor will take more risks than the benchmark case since to save the pension plan the pension sponsor will contribute large amount, which can hedge high exposure to the equity shocks.

Figure 3.4 plots the contribution rate, $Y_t/W_t$ as a function of an annualized equity return over time zero to 5-year. We can see that contribution rates of the benchmark case are decreasing in the state of economy and identical across initial funding status. The pension sponsor with the downside constraint behaves differently depending on the initial funding status. The initially underfunded pension sponsor contributes more than the benchmark case for the same state of the economy. The effect of negative shocks to the economy is greater to the initially underfunded pension plan,
This figure plots equity weights at time $t = 5$-year as a function of annualized equity return over the last five years. Panel A is for the initially underfunded pension with 80% funding ratio, and Panel B is for the initially overfunded pension with 120% funding ratio. We fix the initial asset value at one and vary the terminal downside constraint, $K$. 

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and thus to satisfy the downside constraint higher contributions should be made. Next, consider the initially overfunded pension plan in Panel B. Since the disutility from contributions is more important \((k = 100)\) relative to the utility, the contributions are slightly higher than the benchmark case, which can explain why risk management incentives dominate risk taking incentives for the initially overfunded pension plan to guarantee the downside constraint.

### 3.5.3 Effects of Relative Importance of Disutility

Figure 3.5 plots the optimal present value of the contribution and the put option value at time zero for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as we vary the relative importance of the disutility, \(k\). We also plot the optimal present value of the contribution for the benchmark case. Since drawing contributions from the pension sponsor’s internal resources is more costly when \(k\) is large, we can see that the optimal present value of the contribution decreases as \(k\) increases for both the benchmark and constrained cases. However, there is a key difference between the initially underfunded and overfunded pension plans. As we see in Figure 3.2, the initially overfunded pension plan has a put-based strategy even without a contribution. Hence, when \(k\) is sufficiently large, the pension sponsor won’t contribute at all and just use the put-based strategy without any contribution. However, the initially underfunded pension plan can not construct a put-based strategy without a contribution. Thus, we can see that even if \(k\) is sufficiently large, the underfunded pension plan takes the present value of the contribution, which is equal to the time zero shortfall \(K e^{-rT} - W_0 = 25\%\).

The put option value at time zero increases as \(k\) increases for both underfunded and overfunded pension plans. When contributing is more costly, the pension sponsor decreases the present value of the contribution and allocations in the unconstrained optimal portfolio, which makes the overall pension plan’s asset less risky and increases the put option value. For low \(k\), contributing more than the put option value is optimal since contributing is less costly and the pension sponsor can
This figure plots contribution rates at time $t = 5$-year as a function of annualized equity return over the last five years. Panel A is for the initially underfunded pension with 80% funding ratio, and Panel B is for the initially overfunded pension with 120% funding ratio. We fix the initial asset value at one and vary the terminal downside constraint, $K$. 

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This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the relative importance of the disutility ($k$). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.
hold more unconstrained optimal portfolio. However, when $k$ is high, the opposite happens. A fraction of the put option is funded by the initial pension plan’s asset.

3.5.4 Effects of Elasticity of Disutility

The elasticity of the disutility, $\theta$ has impacts on the determination of the optimal present value of the contribution. In Figure 3.6, we vary $\theta$ from 1.2 to 2 and see the optimal $X_0$ for the initially underfunded (Panel A), and overfunded (Panel B) pension plans. To focus on the shape of the optimal present value of the contribution, we omit the benchmark case and the put option value here. We can see that the present value of the contribution is U-shaped in $\theta$ for both cases, but it is clearer for the underfunded pension plan. Since the elasticity of the disutility only moves the marginal cost curve $W_\phi(y)$ in Figure 3.1, given a contribution policy whether an increase in $\theta$ raises the present value of the contribution is our interest. If the present value of the contribution increases, the marginal cost curve moves downward and the optimal $X_0$ increases, and vice versa.

The optimal contribution policy, $Y_t = I_\phi(y\xi_t) = \left(\frac{y\xi_t}{k}\right)^{\theta-1}$ is convex in $\xi_t$ when $1 < \theta < 2$ and is concave in $\xi_t$ when $\theta > 2$. Given that $y < k$ (this is the case for our parameter values), an increase in $\theta$ always increases the present value of the contribution since contributions become less convex when $1 < \theta < 2$ and more concave when $\theta > 2$. This can be seen clearly through the first derivative of $W_\phi(y)$ with respect to $\theta$:

$$\frac{\partial W_\phi(y)}{\partial \theta} = -W_\phi(y) \left[ \frac{\log \frac{y}{k}}{(\theta - 1)^2} + \frac{\partial \alpha_\phi}{\partial \theta} \frac{1 - (1 + \alpha_\phi T)e^{-\alpha_\phi T}}{\alpha_\phi} \right].$$

If the insides of the bracket are negative, the marginal cost curve moves downward and the optimal contribution increases, and vice versa. The first term in the bracket is negative since with our parameter values $y < k$.

The second part is the effect of $\theta$ on the annuity term. We find that for low $\theta$ the second term in the bracket is positive and dominates the first term, and thus the marginal cost curve moves upward and the optimal contribution decreases. On the other hand, for high $\theta$, the opposite happens.
The intuition is that to smooth contributions, the pension sponsor increases contributions in more likely states of the economy and decreases contributions in less likely states of the economy when $1 < \theta < 2$. The former action increases the present value of the contribution, while the latter decreases it. When $\theta$ is small, the latter effect is dominating.

### 3.5.5 Effects of Risk Aversion and Price of Risk

Now, we investigate effects of the risk aversion and the price of risk. First, we report the optimal present value of the contribution and the put option value at time zero for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as we vary the relative risk aversion, $\gamma$ in Figure 3.6. We also plot the optimal present value of the contribution for the benchmark case. When the risk aversion is high, the mean-variance efficient portfolio holds less equity. The risk of the underlying asset is reduced, and thus the put option value decreases, which implies that less contributions are required. For the initially overfunded pension plan, when the risk aversion is very high, the put option value becomes worthless since the pension sponsor holds zero equity position and the downside constraint is always satisfied (note that it is initially overfunded). Some contributions are still optimal since the pension sponsor can achieve higher utility even taking into account the disutility from contributions.

Next, we vary the price of risk while keeping the volatility of the equity returns at $\sigma = 20\%$. An increase in the price of risk moves both the marginal benefit curve $W_u(y)$ and the marginal cost curve $W_\phi(y)$ in Figure 3.1. If the marginal benefit curve moves upward and the marginal cost curve moves downward as the price of risk increases, we will see that the optimal present value of the contribution also increases. This is the case for the initially overfunded pension plan (Panel B of Figure 3.8). Higher volatility of the mean-variance efficient portfolio ($\Sigma$) increases the put option value, and thus higher initial endowment is required to have the same level of the marginal benefit, i.e. the marginal benefit curve moves upward. On the other hand, when the price of risk
This figure plots the optimal present value of the contribution for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the elasticity of the disutility \( (\theta) \). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.
Figure 3.7: Effects of Relative Risk Aversion

Panel A: Initially Underfunded

Panel B: Initially Overfunded Pension

This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the relative risk aversion ($\gamma$). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.
Figure 3.8: Effects of Price of Risk

Panel A: Initially Underfunded

Panel B: Initially Overfunded Pension

This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the price of risk ($\eta$). We fix the volatility of the equity returns at $\sigma = 20\%$. The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.
is higher, for a given contribution policy, the present value of the contribution is higher since a
distribution of states of the economy are more positive skewed and it is more likely to contribute,
i.e. the marginal cost curve moves downward. The net effect is an increase in the optimal present
value of the contribution and the put option value.

For the initially underfunded pension plan, the movement of the marginal benefit curve is
opposite. The same level of the marginal benefit can be financed with lower initial asset using
higher expected equity return. However, if this downward movement of the marginal benefit curve
is dominated by the downward movement of the marginal cost curve, we will still see an increase
in the optimal present value of the contribution, but see a decrease in the put option value, which
is due to a decrease in allocations to the unconstrained optimal portfolio. This is the case for the
initially underfunded pension plan (Panel A of Figure 3.8).

3.6 Conclusion

We develop the separation approach to analyze the pension sponsor’s contribution and portfolio
policy in the presence of the downside constraint at the terminal date. The problem can be cast
in two separate shadow price problems, the utility maximization problem and the disutility mini-
mization problem. At the optimal solution, two shadow prices are identical. We show that while
guaranteeing the benefits, both risk management and risk taking behaviors can emerge. When the
pension plan’s asset decreases, the pension sponsor first decreases the equity weight and defers
contributions as much as it can to avoid costly contributions. Then, only when the pension plan’s
asset is significantly deteriorated, the pension sponsor starts to contribute and increases the equity
weight, which is hedged by large contemporaneous and future contributions. In our model, the
pension sponsor’s risk taking behavior is induced not by a moral hazard problem, but by commit-
ment to contributions. We hope to extend our analysis to include time-varying expected returns,
and stochastic benefits in future research.
Bibliography


Campbell, John Y., and Luis M. Viceira, 1999, Consumption and portfolio decisions when expected returns are time varying., *Quarterly Journal of Economics* 114.


A.1 State Variables and Return Dynamics

The vector of state variable $X_t = [\delta_t, n_t, s_t]$ follows the multivariate arithmetic Brownian motion:

$$dX_t = \begin{bmatrix} 0 \\ 0 \\ \bar{a} - a_t \end{bmatrix} dt + \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} dZ_{\delta t} \\ dZ_{nt} \\ dZ_{st} \end{bmatrix}.$$

Then, the definitions of $A_t$ and $\Sigma_X$ are straightforward. I define a vector $S_t = [P_t, Q_t]$. Then, I have

$$dS_t = \begin{bmatrix} p_{0,t}' + p_{1,t} A + p_{1,t}' s_t \\ q_{0,t}' + q_{1,t} A + q_{1,t}' s_t \end{bmatrix} dt + \begin{bmatrix} p_{1,t} \\ q_{1,t} \end{bmatrix} \Sigma_X dZ_t.$$

Then, the definitions of $\mu_{P,t}$, $\mu_{Q,t}$, $\mu_{S,t}$, and $\Sigma_{S,t}$ are straightforward.

A.2 Proofs

I omit time subscripts for simplicity unless there is a confusion.

**Proof of Lemma 1.4.** Consider the following variable

$$\exp\left(-\tilde{\gamma} \left( \int_0^t (h_u + g(a_u)) \, du + f_u \, dX_u \right) \right) V(X, t; F_T).$$

This variable is martingale, and thus the expected instantaneous variation should be zero, which leads to the Hamilton-Jacobi-Bellman (HJB) equation for $V(X, t; F_T)$:

$$0 = \max_{\alpha} -\tilde{\gamma} V \left( h + f A - \frac{\tilde{\gamma}}{2} f\Sigma_X \Sigma_X^T f^T + g \right) + V_t + V_X (A - \tilde{\gamma} \Sigma_X \Sigma_X^T f^T) + \frac{1}{2} \text{tr}(V_{XX} \Sigma_X \Sigma_X^T). \tag{A.1}$$
Thus, I can express $dV$ as

$$
dV = V \left[ \tilde{\gamma} (h + fA + g) - \frac{1}{2} \tilde{\gamma}^2 f \Sigma_X \Sigma_X^T f^T \right] dt + V_X \tilde{\gamma} \Sigma_X \Sigma_X^T f^T dt + V_X \Sigma_X dZ.
$$

Define a value $\varepsilon_t = -\frac{1}{\tilde{\gamma}} \log(-\tilde{\gamma}V_t)$. Apply Ito’s lemma to $\varepsilon$,

$$
\tilde{\gamma} d\varepsilon = -\tilde{\gamma} \left[ h + fA + g - \frac{1}{2} \tilde{\gamma} f \Sigma_X \Sigma_X^T f^T \right] dt + \tilde{\gamma} \left( \bar{f} - f \right) \Sigma_X dZ,
$$

where $\bar{f} = f - \frac{V_X}{\tilde{\gamma}^2}$. Therefore, I have

$$
d\varepsilon + h dt + f dX = -g dt + \frac{\tilde{\gamma}}{2} \bar{f} \Sigma_X \Sigma_X^T \bar{f}^T dt + \bar{f} \Sigma_X dZ.
$$

Integrating the above equation between 0 and $T$ yields

$$
F_T = \varepsilon_0 - \int_0^T g dt + \frac{\tilde{\gamma}}{2} \int_0^T \bar{f} \Sigma_X \Sigma_X^T \bar{f}^T dt + \bar{f} \Sigma_X dZ.
$$

Note that I use $\varepsilon_T = \phi_T$. The manager’s value function at time zero is

$$
V(X_0, 0; F_T) = -\frac{1}{\tilde{\gamma}} \exp \left( -\tilde{\gamma} \varepsilon_0 \right)
$$

Thus, the manager’s IR constraint reduces to $\varepsilon_0 \geq \varepsilon_0$. Then, the investors would choose $\varepsilon_0$ that meets this constraint with equality: $\varepsilon_0 = \varepsilon_0$. The FOC with respect to $a$ in the manager’s HJB equation (A.1) yields

$$
\kappa_0 - \kappa_1 a = \bar{f} 1_3.
$$

Lemma A.1. When the investors do not participate in the PE, the value function is given by $J_u(W, X, i, t) = u(W + \hat{k}_i^1 + X^1) X$ where $k_1^1 = [\hat{m}^i 0]$, where $\hat{m}^i$ is defined in (1.5), and $\hat{k}_0$ solves

$$
(\hat{k}_0^1)' = -\frac{\mu^2}{2\gamma^2} + \frac{\hat{\gamma}}{2} (\hat{m}^i)^2 \sigma^2 - \lambda^i (\hat{k}_0^1 - \hat{k}_0),
$$

with the boundary condition $\hat{k}_0^1, T = 0$.

Proof. There is a continuum of investors and I assume that the manager raises the initial capital from the rest of the investors even if one investor decides not to invest in the PE. An investor’s HJB equation when she doesn’t participate in the PE is

$$
0 = \max_{\theta} J_{ut} + J_{uw} \theta \mu + \frac{1}{2} J_{ww} \theta^2 p_1 \Sigma_X \Sigma_X^T p_1 + \theta p_1 \Sigma_X \Sigma_X^T \Sigma_X \Sigma_X^T J_{wX} + J_{ux} (A + \mu_X X) + \frac{1}{2} \text{tr} \left( J_{uxX} \Sigma_X \Sigma_X^T \right) + \lambda^i (J_u(W, X, i, t) - J_u(W, X, i, t)).
$$

\]
Using the conjectured value function, the HJB becomes

\[
0 = \max_{\theta}(\dot{k}_0) + (\dot{k}_1) X + \theta \mu_P - \gamma \theta^2 p_1 \Sigma_X \Sigma_X^T p_1 - \gamma \theta p_1 \Sigma_X \Sigma_X^T (k_1^i)^T \\
+ \dot{k}_1 (A + \mu_X X) - \frac{\gamma}{2} \text{tr} ((k_1^i) (k_1^i) \Sigma_X \Sigma_X^T) + \frac{\lambda^i}{\gamma} \left(1 - e^{-\gamma(k_0 - k_0 + (k_1 - k_1) X)}\right).
\]

In the HJB equation, I approximate the last term by

\[
\frac{1}{\gamma} \left(1 - e^{-\gamma(k_0 - k_0 + (k_1 - k_1) X)}\right) \approx \dot{k}_0 - \dot{k}_1 + (k_1 - k_1) X.
\]

Then, I can collect coefficients on \(X\):

\[
0 = (k_1^i) + \lambda^i (k_1 - k_1^i).
\]

The solution is \(k_1^i = [0 \hat{m}^i 0]\), where \(\hat{m}^i\) is given in (1.5). The FOCs with respect to \(\theta\) is \(\mu_P = \gamma p_1 \Sigma_X \Sigma_X^T (\theta p_1 + k_1^i)^T\).

The risk premium of the public equity is decided by the rest of investors who participate in the PE, and I show \(\mu_P = \frac{\hat{\gamma}}{\gamma + \hat{\gamma}} (1 + b_0) \sigma^2\) in Theorem 1.5. Substitute the equilibrium risk premium, then I have \(\theta = \frac{\mu_P}{\gamma \sigma^2}\). Then, collecting constant terms yields (A.2).

**Proof of Theorem 1.5.** This is a special case of Proposition 1.7 by setting \(\kappa_1 = \infty\) and thus \(x = 0\). Please refer to the proof of Proposition 1.7.

**Proof of Lemma 1.6.** The risk premium of the public equity is clearly increasing in \(b_0\), and \(\sigma\). Also, it is increasing in \(\hat{\gamma}\) and \(\gamma\) since

\[
\frac{\partial}{\partial \hat{\gamma}} \left(\frac{\hat{\gamma} \gamma}{\hat{\gamma} + \gamma}\right) = \frac{\gamma^2}{(\hat{\gamma} + \gamma)^2} > 0.
\]

The alpha of PE is clearly increasing in \(\sigma\). Also, it is increasing in \(b_n\) since

\[
\frac{\partial}{\partial b_n} \left((b_n + \hat{m}) \left(b_n - \frac{\gamma}{\hat{\gamma}} \hat{m}\right)\right) = b_n - \frac{\gamma}{\hat{\gamma}} \hat{m} + b_n + \hat{m} > 0.
\]

The alpha is increasing in \(\hat{\gamma}\) since

\[
\frac{\partial}{\partial \hat{\gamma}} \left(\frac{\hat{\gamma}^2 \gamma}{(\hat{\gamma} + \gamma)^2} (1 + b_n(b_n + \hat{m})) - \frac{\hat{\gamma} \gamma^2}{(\hat{\gamma} + \gamma)^2} (b_n + \hat{m})\right)
\]

\[
= \frac{\gamma}{(\hat{\gamma} + \gamma)^3} \left[2 \hat{\gamma} + 2 \hat{\gamma} (b_n + \hat{m}) \left(b_n - \frac{\gamma}{\hat{\gamma}} \hat{m} + \frac{\gamma + \hat{\gamma}}{2 \hat{\gamma}} \hat{m}\right)\right] > 0.
\]

Suppose \(\hat{\gamma} < \gamma\). Then, the alpha is decreasing in \(\gamma\) and \(\hat{m}\) since

\[
\frac{\partial}{\partial \gamma} \left(\frac{\hat{\gamma}^2 \gamma}{(\hat{\gamma} + \gamma)^2} (1 + b_n + \hat{m}) - \frac{\hat{\gamma} \gamma^2}{(\hat{\gamma} + \gamma)^2} (b_n + \hat{m})\right)
\]

\[
= -\frac{\gamma}{(\hat{\gamma} + \gamma)^3} \left[\hat{\gamma} (\gamma - \hat{\gamma}) + \gamma (\gamma - \hat{\gamma}) (b_n + \hat{m}) \left(b_n - \frac{\gamma}{\hat{\gamma}} \hat{m} + \frac{\gamma + \hat{\gamma}}{\hat{\gamma} (\gamma - \hat{\gamma})} \hat{m}\right)\right] < 0,
\]

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and
\[
\frac{\partial}{\partial \bar{m}} \left( (b_n + \bar{m}) \left( b_n - \frac{\gamma}{\bar{m}} \right) \right) = \left( 1 - \frac{\gamma}{\bar{m}} \right) b_n - \frac{2\gamma}{\bar{m}} \bar{m} < 0.
\]
Note that \( \bar{m} = \frac{\lambda^l m^h + \lambda^h m^l}{\lambda^l + \lambda^h} \) is increasing in \( m^l, m^h \), and \( \lambda^h \) and decreasing in \( \lambda^l \).

**Proof of Proposition 1.7.** Consider a variable
\[
\exp \left( \gamma \int_0^t \psi_u dF_u \right) J(W, X, i, t).
\]
Then, clearly it is a martingale. Thus, the investor’s HJB equation is
\[
0 = \max_{\theta, f, a} J_i + J_W(\theta \mu_S + \gamma \psi \Sigma_S \Sigma_X \Sigma_X^\top f^\top) + \frac{1}{2} J_{WW} \theta \Sigma_S \Sigma_X \Sigma_X^\top \Sigma_S^\top \theta^\top + \gamma \Sigma_S \Sigma_X \Sigma_X^\top J_W^\top
\]
\[
+ \frac{1}{2} \text{tr} \left( J_{XX} \Sigma_X \Sigma_X^\top \right) + J_X \left( A + \gamma \psi \Sigma_X \Sigma_X^\top f^\top \right) + J \gamma \psi \left( \dot{\epsilon} - g + \frac{\gamma + \gamma \psi}{2} \Sigma_X \Sigma_X^\top \dot{f} \right).
\]
subject to IC constraint (1.3). The first term is the expected time variation in the value function, the second term is the variation due to the change in the liquid wealth, the third, fourth, and fifth terms are quadratic variations, the sixth term is the variation due to the change in the state variables, the seventh term is the variation due to the change in the compensation, and the last term captures the change in the investor’s type. Conjecture that the investor’s value function is given by \( J(W, X, i, t) = u(W + k_0^i + k_1^i X) \) with the boundary condition \( k_0^{i, T} = 0 \), and \( k_1^{i, T} = [0 m^l 0] \).

The HJB becomes
\[
0 = \max_{\theta, f, a} (k_0^i)' + (k_1^i)' X + \theta \mu_S + \gamma \psi \Sigma_S \Sigma_X \Sigma_X^\top f^\top - \frac{\gamma}{2} \theta \Sigma_S \Sigma_X \Sigma_X^\top \Sigma_S^\top \theta^\top - \gamma \theta \Sigma_S \Sigma_X \Sigma_X^\top (k_1^i) \Sigma_X + k_1^i \left( A + \gamma \psi \Sigma_X \Sigma_X^\top f^\top \right) - \frac{\gamma}{2} \text{tr} \left( (k_1^i) \Sigma_X \Sigma_X^\top \right) - \psi \left( \dot{\epsilon} - g + \frac{\gamma + \gamma \psi}{2} \Sigma_X \Sigma_X^\top \dot{f} \right)
\]
\[
+ \lambda^l (k_0^i - k_0^l + (k_1^i - k_1^l) X).
\]

The FOC with respect to \( \theta \) is
\[
\mu_P + \gamma \psi p_1 \Sigma_X \Sigma_X^\top f^\top = \gamma p_1 \Sigma_X \Sigma_X^\top (\theta p_1 + \psi q_1 + k_1^i) \Sigma_X \Sigma_X^\top f^\top.
\]
Multiply \( \pi^i \) and sum over, then I have
\[
\mu_P = \gamma p_1 \Sigma_X \Sigma_X^\top (p_1 + q_1 + \bar{k}_1 - \bar{f}) \Sigma_X \Sigma_X^\top,
\]
where \( \bar{k}_1 = \sum_i \pi^i k_1^i \). This implies
\[
p_0' = \gamma p_1 \Sigma_X \Sigma_X^\top (p_1 + q_1 + \bar{k}_1 - \bar{f}) \Sigma_X \Sigma_X^\top
\]
\[
p_1' = 0_{1 \times 3}.
\]
Thus, I have \( p_1 = [1 \ 0 \ 0] \). The FOCs with respect to \( \psi \) is

\[
    \mu_Q = \dot{c} - g - \gamma (\theta p_1 + 2\psi q_1 + \bar{k}_1) \Sigma X \Sigma X f^\top + \gamma q_1 \Sigma X \Sigma X (\theta p_1 + \psi q_1 + \bar{k}_1)^\top + \frac{\bar{\gamma} + 2\gamma \psi}{2} f \Sigma X \Sigma X f^\top.
\]

Multiplying \( \pi^i \) and summing over yields

\[
    \mu_Q = \dot{c} - g - \gamma (p_1 + 2q_1 + \bar{k}_1) \Sigma X \Sigma X f^\top + \gamma q_1 \Sigma X \Sigma X (p_1 + q_1 + \bar{k}_1)^\top + \frac{\bar{\gamma} + 2\gamma}{2} f \Sigma X \Sigma X f^\top. \tag{A.4}
\]

Since this should hold for any \( X \), I have

\[
    q_0' = \dot{c} - (\bar{a} - a)q_1 3 - \gamma (p_1 + 2q_1 + \bar{k}_1) \Sigma X \Sigma X f^\top + \gamma q_1 \Sigma X \Sigma X (p_1 + q_1 + \bar{k}_1)^\top + \frac{\bar{\gamma} + 2\gamma}{2} f \Sigma X \Sigma X f^\top \\
    q_1' = 0_{1 \times 3},
\]

This implies that \( q_1 = [b\delta \ b\eta \ 0] \). Then, I can collect coefficients on \( X \):

\[
    (k_1^i)' = -\lambda^i (k_1^i - k_1^i).
\]

The solution is \( k_1^i = [0 \ n \ 0] \). This implies that \( \bar{k} = [0 \ n \ 0] \). The FOC with respect to the first and second element of \( \vec{f} \) are

\[
    (\bar{\gamma} + \gamma \psi) \vec{f}_{11} = \gamma (\theta + \psi b_\delta) \\
    (\bar{\gamma} + \gamma \psi) \vec{f}_{12} = \gamma (\bar{m} + \psi b_\eta)
\]

Multiply \( \pi^i \) respectively and sum over, then I have \( \vec{f}_{11} = \vec{f}_{w1} = \frac{\bar{\gamma}}{\gamma + \bar{\gamma}} (1 + b_\delta) \) and \( \vec{f}_{12} = \vec{f}_{w2} = \frac{\bar{\gamma}}{\gamma + \bar{\gamma}} (b_\eta + \bar{m}) \).

The FOC with respect to \( a \) is

\[
    (\kappa_0 - \kappa_1 a) (1 + (\bar{\gamma} + \gamma \psi) \kappa_1 \sigma^2) = 1 + \gamma \kappa_1 \psi \sigma^2.
\]

Note that I use the fact that only the first element of \( p_1 \) is nonzero and the only second element of \( k_1^i \) is nonzero.

Multiply \( \pi^i \) respectively and sum over, then I have

\[
    \kappa_0 - \kappa_1 a = \frac{1 + \gamma \kappa_1 \sigma^2}{1 + (\bar{\gamma} + \gamma) \kappa_1 \sigma^2} = (1 + x) \frac{\gamma}{\bar{\gamma} + \gamma},
\]

where \( x = \frac{\bar{\gamma}}{\gamma + (\gamma + \bar{\gamma}) \kappa_1 \sigma^2} > 0 \). Notice that when \( \kappa_1 = \infty \), I have \( x = 0 \) and this implies that \( \vec{f}_{13} = \vec{f}_{w13} = \frac{\bar{\gamma}}{\gamma + \bar{\gamma}} > 0 \).

Plug the equilibrium contract in (A.3), then I have

\[
    \mu_P = \frac{\bar{\gamma} \gamma}{\bar{\gamma} + \gamma} \sigma^2 (1 + b_\delta).
\]

Plug the equilibrium contract in (A.4), then I have

\[
    \mu_Q = \dot{c} - g + \gamma q_1 \Sigma X \Sigma X f^\top - \frac{\gamma}{2} \vec{f}_{w} \Sigma X \Sigma X (\vec{f}_{w})^\top + \frac{\gamma^2 \sigma^2}{2(\bar{\gamma} + \gamma)^2} x (\gamma (x - 2) + 2\gamma x).
\]

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Consider the net-of-fee return of the PE, \( d\hat{Q} = dQ - dF = \mu_{\hat{Q}} dt + \Sigma_{\hat{Q}} dZ \), where the adjusted risk premium is given by

\[
\mu_{\hat{Q}} = \mu_Q - \hat{\epsilon} + g - \frac{\gamma}{2} f \Sigma_X \Sigma_X^T f^T \\
= \frac{\gamma^2}{(\gamma + \gamma)^2} \sigma^2 \left( (1 + b_\delta) \left( b_\delta - \frac{\gamma}{\gamma} \right) + (b_\mu + \bar{m}) \left( b_\mu - \frac{\gamma}{\gamma} \bar{m} \right) + 1 \right) + AP
\]

and the adjusted volatility is given by \( \Sigma_{\hat{Q}} = (q_1 - \hat{f}) \Sigma_X \). The beta of the PE return on the public equity return is

\[
\beta = \frac{\text{Cov}(dP, d\hat{Q})}{\text{Var}(dP)} = \frac{\gamma}{\gamma + \gamma} \left( b_\delta - \frac{\gamma}{\gamma} \right).
\]

Thus, the risk premium of the PE can be expressed as \( \mu_{\hat{Q}} = \alpha + \beta \mu_P \), where

\[
\alpha = \alpha^w + AP
\]

and

\[
\alpha^w = \frac{\gamma^2}{(\gamma + \gamma)^2} \sigma^2 \left( (b_\mu + \bar{m}) \left( b_\mu - \frac{\gamma}{\gamma} \bar{m} \right) + 1 \right)
\]

\[
AP = \frac{\gamma^2}{(\gamma + \gamma)^2} \sigma^2 x(\gamma x - 2\gamma).
\]

Substitute the equilibrium risk premium in the FOC with respect to \( \theta \) and \( \psi \), then I have the optimal holdings of the public equity and the PE. Finally, the investor’s IR constraint is

\[
J(W + \psi_{i,0}(Q_0 - I_0), X_0, i, 0) \geq J_u(W, X, i, 0).
\]

This is identical with

\[
\psi_{i,0}(q_{0,0} + q_{1,0} X_0) + k_{i,0}^0 \geq \psi_{i,0} I_0 + \hat{k}_{i,0}^0.
\]

The initial value of \( X \) decides whether this inequality is met or not. I only consider cases in which this constraint is satisfied. \( \square \)

**Proof of Lemma 1.8.** Differentiate \( \hat{f}_{1,3} \) with respect to \( \kappa_1 \), then I have

\[
\frac{\partial}{\partial \kappa_1} \left( \frac{\gamma}{\gamma + \gamma} (1 + x) \right) = \frac{\gamma}{\gamma + \gamma} \frac{\partial x}{\partial \kappa_1} < 0,
\]

since \( \frac{\partial x}{\partial \kappa_1} < 0 \). Similarly, differentiating \( AP \) with respect to \( \kappa_1 \) gives

\[
\frac{\partial AP}{\partial \kappa_1} = \frac{2 \gamma^2 \sigma^2}{(\gamma + \gamma)^2} (\gamma x - \gamma) \frac{\partial x}{\partial \kappa_1} > 0
\]

since \( x < \frac{\gamma}{\gamma} \). Finally, assume \( \kappa_0 = 1 \). Then, the optimal level of diversion is \( a_{\ell} = \frac{\gamma \sigma^2}{1 + (\gamma + \gamma) \kappa_0 \sigma^2} \). Thus, it can be immediately seen that \( a_{\ell} \) is decreasing in \( \kappa_1 \). \( \square \)
Figure A.1: The Flow of Masses of Investor’s Types

\[ \begin{align*}
\lambda^h & \quad \text{High type Large holder} \\
\lambda^l & \quad \text{Low type Large Holder} \\
2v\pi^{lo} & \quad \text{Trade } \psi^o - \psi^n \text{ shares at the price } Q \\
2v\pi^{hn} & \\
\lambda^h & \quad \text{High type Small holder} \\
\lambda^l & \quad \text{Low type Small holder}
\end{align*} \]

**Proof of Proposition 1.9.** The rates of change of the mass of the investor’s types are

\[ \begin{align*}
\left( \pi^{lo}_t \right)' &= -2v\pi^{hn}_t \pi^{lo}_t - \lambda^l \pi^{lo}_t + \lambda^h \pi^{ho}_t \\
\left( \pi^{hn}_t \right)' &= 2v\pi^{hn}_t \pi^{lo}_t - \lambda^l \pi^{hn}_t + \lambda^h \pi^{hn}_t \\
\left( \pi^{ho}_t \right)' &= 2v\pi^{hn}_t \pi^{lo}_t - \lambda^l \pi^{ho}_t + \lambda^h \pi^{lo}_t \\
\left( \pi^{ln}_t \right)' &= -2v\pi^{hn}_t \pi^{lo}_t - \lambda^h \pi^{hn}_t + \lambda^l \pi^{ln}_t.
\end{align*} \]

The first term in the right hand side is due to a transfer of fund interests from the low-type large holders to the high-type small holders in the secondary market. The rate of searching a counterparty is \( 2v\pi^{hn}_t \pi^{lo}_t \) since a buyer and a seller search each other. The second and third term are the outflow and inflow due to an endowment shock, respectively. Figure A.1 plots the flow between the investor’s types. Note that \((\pi^{ho}_t)' + (\pi^{lo}_t)' = 0\) and \((\pi^{hn}_t)' + (\pi^{ln}_t)' = 0\). This implies that the mass of large (small) holders is constant over time: \( \pi^{ho}_t + \pi^{lo}_t = \pi \) and \( \pi^{hn}_t + \pi^{ln}_t = 1 - \pi \). This guarantees that the market clearing condition is satisfied at any time \( t \). I can solve for the steady-state mass of each type by setting \((\pi^{lo}_t)' = 0\)

\[ 0 = 2v(\pi^{lo}_t)^2 + (2v(\pi^h - \pi) + \lambda^l + \lambda^h)\pi^{lo}_t - \lambda^h \pi, \]

where \( \pi^h = \frac{\lambda^l}{\lambda^l + \lambda^h} \) is the steady-state mass of the high-type investors. Note that for the steady masses, I drop the time subscript. I take the positive root which is in the interval \((0, 1)\).
Now, consider a variable
\[ \exp \left( \gamma \int_0^t \psi_u dF_u \right) J(W_t, X_t, ij, t), \]
for \( ij = \ln, hn, lo, \) and \( ho. \) Clearly it is a martingale. The type \( ij \)'s HJB equation is
\[
0 = \max_{\theta, a, f} \left[ J_x + J_W(\theta \mu_P + \gamma \theta \psi^j p_1 \Sigma X J X^T f^T) + \frac{1}{2} J_{WW} \theta^2 p_1 \Sigma X \Sigma X p_1^T + \theta p_1 \Sigma X \Sigma X J W X \right.
\]
\[
+ \frac{1}{2} \text{tr} \left( J X X \Sigma X X^T f^T + J X (A + \mu_X X + \gamma \psi^j \Sigma X X^T f^T) \right) + J \gamma \psi^j \left( \dot{\epsilon} - g + \frac{\epsilon + \gamma \psi^j}{2} f \Sigma X X^T f^T \right)
\]
\[+ \lambda^t (J(W, X, i, j, t) - J(W, X, i, j, t)) + 2 \nu \pi^t 1(ij = lo, hn)(J(W - Q(\psi^j - \psi^o), X, i, j, t) - J(W, X, i, j, t)), \]
subject to IC constraint: \( \kappa_0 - \kappa_{1a} = f_{13}. \) The last term is new and is a jump in the value function due to a trade of the private equity. A trading occurs only when the investor's type is \( lo \) or \( hn \) with the speed of \( 2 \nu_{hn} \) or \( 2 \nu_{lo}, \) respectively. If the type \( lo \) and \( hn \) investors meet, they immediately trade \( \psi^o - \psi^o \) shares of the PE at the price \( Q, \) which is determined using Nash bargaining. Then, the type \( lo \) and \( hn \) investors become the type \( ln \) and \( ho \) investors, respectively. Plug in the conjectured value function in the HJB equation:
\[
0 = \max_{\theta, a, f} \psi^j \left( (\omega^0_0) + (k^0_i X) + (k^0_i X + \theta \mu_P + \gamma \theta \psi^j p_1 \Sigma X X^T f^T \right.
\]
\[
- \frac{1}{2} \gamma p_1 \Sigma X X^T p_1 - \gamma \theta \Sigma X X^T (\psi^j \omega_1 + k^1_i) \left( \psi^j \omega_1 + k^1_i \right)
\]
\[+ (\psi^j \omega_1 + k^1_i)(A + \mu_X X + \gamma \psi^j \Sigma X X^T f^T) - \frac{1}{2} \text{tr} \left( (\psi^j \omega_1 + k^1_i) \Sigma X X^T \right)
\]
\[\left. - \psi^j \left( \dot{\epsilon} - g + \frac{\epsilon + \gamma \psi^j}{2} f \Sigma X X^T f^T \right) + \lambda^t (\psi^j (\omega^0_0 - \omega^0_0) + k^0_i - k^0_i + (k^1_i - k^1_i) X) \right)
\]
\[+ 2 \nu \pi_{hn} 1(ij = lo)(\Delta_\psi Q - \Delta_\psi (\omega^0_0 + \omega_1 X))
\]
\[+ 2 \nu \pi_{lo} 1(ij = hn)(-\Delta_\psi Q + \Delta_\psi (\omega^0_0 + \omega_1 X)), \]
where \( \Delta_\psi = \psi^o - \psi^o. \) The outcome of Nash bargaining is
\[
Q = \arg \max_{Q} \left( J(W + Q \Delta_\psi, ln) - J(W, lo) \right)^\eta \left( J(W - Q \Delta_\psi, ho) - J(W, hn) \right)^{1-\eta}.
\]
The FOC with respect to \( Q \) is
\[
\eta \frac{J_W(W + Q \Delta_\psi, ln)}{J(W + Q \Delta_\psi, ln) - J(W, lo)} = (1 - \eta) \frac{J_W(W - Q \Delta_\psi, ho)}{J(W - Q \Delta_\psi, ho) - J(W, hn)}.
\]
Plug the conjectured value function in, then I have
\[
(1 - \eta) \left( 1 - e^\gamma (\Delta_\psi Q - \Delta_\psi (\omega^0_0 + \omega_1 X)) \right) = \eta \left( 1 - e^\gamma (-\Delta_\psi Q + \Delta_\psi (\omega^0_0 + \omega_1 X)) \right).
\]
I use approximation \( e^x \approx 1 + x. \) Then,
\[
Q = (1 - \eta)(\omega^0_0 + \omega_1 X) + \eta(\omega^0_0 + \omega_1 X).
\]
This should hold for any $X$. This implies that

$$
q_0 = (1 - \eta)\omega^l_0 + \eta\omega^h_0
$$

$$
q_1 = \omega_1.
$$

This is intuitive since the price of fund interest is the weighted average of the type $l$ and $h$ investor’s shadow values. The weight is the seller’s (type $l$ investors) bargaining power. The more the low type sellers have the bargaining power, the price is closer to the high type investor’s shadow value. Now, deriving the equilibrium risk premium of the PE return is straightforward. From the HJB equation, find the derivatives $(\omega^l_0)'$. Notice that thanks to the assumption of normality of cash flow, the optimal choice of $\theta$, $a$, and $\bar{f}$ do not involve $X$. Thus, I can collect the terms involving $X$ before computing the optimal choices:

$$
\psi^j\omega_1' + (k^i_1)' = -\lambda^i(k^i_1 - k^i_1).
$$

Note that for type $io$ and $in$ investors, $k^i_1$ is common and thus I have

$$
\omega_1' = 0_{1 \times 3}
$$

$$
(k^i_1)' = -\lambda^i(k^i_1 - k^i_1).
$$

The solution to these ODEs are $\omega_1 = q_1$ and $k^i_1 = [0 \, m^i \, 0]$. The FOC with respect to $\theta$ is

$$
\mu_P + \gamma\psi^j p_1 \Sigma_X \Sigma_X^T \bar{f}^T = \gamma p_1 \Sigma_X \Sigma_X^T (\theta p_1 + \psi^j \omega_1 + k^i_1)^T.
$$

Multiply $\pi^{ij}$ and sum over, then I have

$$
\mu_P = \gamma p_1 \Sigma_X \Sigma_X^T (p_1 + \omega_1 + \bar{k}_1 - \bar{f})^T.
$$

(A.5)

This implies

$$
p_0' = \gamma p_1 \Sigma_X \Sigma_X^T (p_1 + \omega_1 + \bar{k}_1 - \bar{f})^T
$$

$$
p_1' = 0_{1 \times 3}.
$$

Thus, I have $p_{1,t} = [1 \, 0 \, 0]$. The FOCs with respect to the first and second element of $\bar{f}$ and with respect to $a$ gives the same optimal contract as Proposition 1.7. This also implies that the risk premium of the public equity remains same.

Plug the equilibrium contract and the risk premium of the public equity in the FOC with respect to $\theta$, then the optimal holding of the public equity can be expressed as $\theta^{ij} = \frac{\gamma}{\nu + \gamma}(1 + b_6) - \beta \psi^j$. Define $B = \frac{\gamma}{\nu + \gamma}(1 + b_6)$. Thus, I can express the terms including $\theta$ in the HJB equation as

$$
\theta \mu_P + \gamma\theta^j p_1 \Sigma_X \Sigma_X^T \bar{f}^T - \frac{\gamma}{2}\theta^2 p_1 \Sigma_X \Sigma_X^T p_1^T - \gamma \theta p_1 \Sigma_X \Sigma_X^T (\psi^j \omega_1 + k^i_1)^T = \frac{\gamma}{2}\sigma^2 (B^2 - 2B\beta \psi^j + \beta^2 (\psi^j)^2).
$$
Now, I can collect constant terms in the HJB equation:

\[ 0 = \psi^j(\omega^0_0') + (k^0_0) + \frac{\gamma^2}{2}(B^2 - 2B\beta + \beta^2(\psi^j)^2) \]

\[ + \psi^j(\bar{\omega}_0 + k^1_1 (A + \gamma\psi^j \Sigma_X \Sigma_X^T f^T) - \frac{\gamma}{2} \text{tr} \left( (\psi^j \bar{\omega}_0 + k^1_1) \Sigma_X \Sigma_X^T \right) \]

\[ -\psi^j \left( -\frac{\gamma}{2} + \gamma \psi^j \frac{f \Sigma_X \Sigma_X^T f^T}{2} \right) + \lambda^i(\psi^j - \omega^0_0) + k^i - k^0_0 \]

\[ + 2\nu\pi^h (i = m) \psi_1(i = m) \psi_1(\bar{\omega}_0 - \omega^0_0) + 2\nu\pi^o(i = h) \psi_1(i = o) \psi_1(\bar{\omega}_0 - \omega^0_0). \]

I can subtract the equation for the type in investors from that for the type io investors, then I have

\[ (\omega^0_0') = -\frac{\gamma}{2} \sigma^2 (-2B\beta + \beta^2(\psi^o + \psi^n)) - \bar{a} + a \]

\[ -\gamma(\psi^o + \psi^n) \bar{\omega}_0 + k^1_1 \Sigma_X \Sigma_X^T f^T \]

\[ -\frac{\gamma}{2} \sum X \Sigma_X^T f^T \]

\[ -\lambda^i(\bar{\omega}_0 - \omega^0_0) \]

\[ -2\nu\pi^h (i = m) \eta(\bar{\omega}_0 - \omega^0_0) + 2\nu\pi^o(i = h) (1 - \eta)(\bar{\omega}_0 - \omega^0_0). \]

Now, the \((k^0_0)\) can be obtained by substituting \((\omega^0_0')\) in (A.6):

\[ (k^0_0)' = \frac{\gamma}{2} \sigma^2 \psi^o \psi^n + \gamma \psi^o \psi^n \bar{\omega}_0 + k^1_1 \Sigma_X \Sigma_X^T f^T \]

\[ -\frac{\gamma}{2} \psi^o \psi^n \bar{\omega}_0 \Sigma_X \Sigma_X^T \omega^0_0 \]

\[ -\frac{\gamma}{2} \sigma^2 B^2 + \frac{\gamma}{2} k^1_1 \Sigma_X \Sigma_X^T (k^1_1)^T - \lambda^i(k^i - k^0_0) \]

\[ + 2\nu\pi^h (i = m) \eta(\bar{\omega}_0 - \omega^0_0) - 2\nu\pi^o(1 - \eta)(\bar{\omega}_0 - \omega^0_0). \]

The risk premium of the PE return is then

\[ \mu_Q = -\bar{a} + (1 - \eta)(\omega^0_0') + \eta(\omega^0_0') \]

\[ = \mu_{Q} + \mathcal{IP} + \frac{x\gamma^2}{(\gamma + 2\gamma^2)^2} \sigma^2 \left[ x(\gamma + \gamma(\psi^o + \psi^n)) + \gamma(1 - (\psi^o + \psi^n)) \right], \]

where \(\mu_{Q} = \mu_{Q} + \sigma^2 B, \mu_{Q} = \alpha + \beta \mu_P, \) and

\[ B = \frac{\gamma}{\gamma + \gamma} b \eta (1 - \eta) \bar{m}^l + \eta \bar{m}^h + b \cdot H \left( \frac{\psi^o + \psi^n}{2}, 1 \right) - b - \bar{m} \]

\[ -\frac{\gamma}{\gamma + \gamma} \bar{m} (1 - \eta) \bar{m}^l + \eta \bar{m}^h - \bar{m} H \left( 1, \frac{\psi^o + \psi^n}{2} \right) - \frac{\gamma}{\gamma + \gamma} \left( \frac{\gamma}{\gamma + \gamma} - 2 \bar{m} \right) \left( -2 \psi^o - \psi^n \right), \]

and \(H(x, y) = \frac{\gamma}{\gamma + \gamma} x + \frac{\gamma}{\gamma + \gamma} y.\) The illiquidity premium can be found:

\[ \mathcal{IP} = (\omega^0_0 - \omega^0_0)[\eta(\lambda^h + 2\nu\pi^o(1 - \eta)) - (1 - \eta)(\lambda^i + 2\nu\pi^h \eta)]. \]

where the difference between \(\omega^0_0\) can be computed using (A.7):

\[ (\omega^0_0 - \omega^0_0)' = -\frac{\gamma}{\gamma + \gamma} \sigma^2 (m^l - m^h) \left( b - \frac{\gamma}{\gamma} \bar{m} \right) e^{-(\lambda^i + \lambda^h)(T - \epsilon)} + (\lambda^i + \lambda^h + 2\nu\pi^o(1 - \eta) + 2\nu\pi^h \eta)(\omega^0_0 - \omega^0_0). \]
The solution is
\[
\omega^h_0 - \omega^l_0 = \frac{\gamma}{\gamma + \gamma} \sigma^2 (\hat{m}^l - \hat{m}^h) \left( b \eta - \frac{\gamma}{\gamma} \hat{m}^l \right) \Phi_{2d, \tau},
\]
where \(d = v(\pi^{lo}(1 - \eta) + \pi^{hn} \eta)\). Then, the net-of-fee return is
\[
\mu_{\hat{Q}} = \alpha + \beta \mu_P
\]
\[
\Sigma_{\hat{Q}} = \frac{\gamma}{\gamma + \gamma} \left[ \left( b \eta - \frac{\gamma}{\gamma} \right) \left( b \eta - \frac{\gamma}{\gamma} \hat{m}^l \right) \left( 1 - \frac{\gamma}{\gamma} \right) \right],
\]
where the alpha is given by
\[
\alpha = \alpha_{bc} + \frac{\psi^o + \psi^n}{2} A_P + T_P,
\]
the beta is given by \(\beta = \frac{\gamma}{\gamma + \gamma} \left( b \eta - \frac{\gamma}{\gamma} \right)\), and \(\alpha_{bc} = \alpha_w + \sigma^2 B\). The investor’s IR constraint is
\[
J(w + \psi^j (\omega^l_{0,i} + \omega^h_{1,0} X_0 - I_0), X_0, i, j, 0) \geq J_u(w, X_0, i, 0).
\]
for \(ij = ho\) and \(ln\) since at time zero the high-type investors would hold \(\psi^o\) shares of the PE and the low-type investors would hold \(\psi^n\) shares. This is identical with
\[
\psi^j (\omega^l_{0,0} + \omega^h_{1,0} X_0) + k^i_{0,0} \geq \psi^j I_0 + \hat{k}^i_{0,0}.
\]
The initial value \(X_0\) decides whether this inequality is met or not. I only consider cases in which this constraint is satisfied. The illiquidity premium goes to zero as \(t \to T\) since \(\Phi_{2d, \tau} \to 0\). Finally, it can be shown that when the secondary market becomes liquid \(v \to \infty\), \(\pi^{lo} \to 0\) and \(\pi^{hn} \to \pi^h - \pi\) if \(\pi < \pi^h\), or \(\pi^{lo} \to \pi - \pi^h\) and \(\pi^{hn} \to 0\) if \(\pi > \pi^h\). Thus, I have
\[
\lim_{v \to \infty} T_P = \frac{\triangle_\omega}{\pi^{lo}(1 - \eta) + \pi^{hn} \eta} \eta(1 - \eta)
\]
\[
\begin{cases} 
-\triangle_\omega (1 - \eta) & \text{if } \pi < \pi^h \\
\triangle_\omega \eta & \text{if } \pi > \pi^h.
\end{cases}
\]

Proof of Lemma 1.10. First, \(\psi^o + \psi^n = 2\) implies that the fraction of investors who hold large number of PE shares is exactly one half: \(\pi = \frac{1 - \psi^n}{\psi^o + \psi^n} = \frac{1}{2} \frac{\lambda_0 \hat{m}^l}{\lambda_0 \hat{m}^l + \lambda_0 \hat{m}^h} = \frac{1}{2}\). Substitute \(\eta\) with \(\frac{\lambda_0 \hat{m}^l}{\lambda_0 \hat{m}^l + \lambda_0 \hat{m}^h}\) and \(\psi^o + \psi^n\) with two in \(B\), then I have \(B = 0\) since \(H(1, 1) = 1\) and \(\frac{\lambda_0 \hat{m}^l + \lambda_0 \hat{m}^h}{\lambda_0 \hat{m}^l + \lambda_0 \hat{m}^h} = \frac{\hat{m}}{\hat{m}}\). Also, substitute \(\eta\) in \(T_P\) then I have (1.8). Note that I use the mass of sellers net of buyers is given by \(\pi^{lo} - \pi^{hn} = \pi - \pi^h = \frac{1}{2} - \pi^h\) since the mass of type \(hn\) is
\[ \pi_{hn} = \pi^h - \pi^{lo} = \pi^h - (\pi - \pi^{lo}) \]. The illiquidity premium is clearly increasing in \( \sigma \) and \( b_n \). The illiquidity premium is also increasing in \( \gamma \) and \( m^l \), and is decreasing in \( m^h \) since

\[
\begin{align*}
\frac{\partial (\omega^h_0 - \omega^l_0)}{\partial \gamma} &= \frac{\gamma^2}{(\gamma + \gamma)^2} \sigma^2 (\hat{m}^l - \hat{m}^h) (b_n + \hat{m}) \Phi_{2d,\tau} > 0 \\
\frac{\partial (\omega^h_0 - \omega^l_0)}{\partial m^l} &= \frac{\bar{\gamma} m^l}{\gamma + \gamma} \sigma^2 \left( b_n - \frac{\gamma}{\gamma} m^h \right) e^{-\lambda_l m^h} \Phi_{2d,\tau} > 0 \\
\frac{\partial (\omega^h_0 - \omega^l_0)}{\partial m^h} &= - \frac{\bar{\gamma} m^h}{\gamma + \gamma} \sigma^2 \left( b_n - \frac{\gamma}{\gamma} m^l + \frac{\gamma \lambda}{\gamma (\lambda^l + \lambda^h)} \right) e^{-\lambda_l m^h} \Phi_{2d,\tau} < 0
\end{align*}
\]

\[ \Box \]

**Proof of Corollary 1.11.** Suppose that the secondary market is illiquid. Then, the investor’s IR constraint becomes

\[ J(\bar{W} - \psi^i I_0, X_0, i, 0) \geq J_u(\bar{W}, X_0, i, 0), \]

for the type \( ij = ho \) and \( ln \) investors, i.e. initially the high- (low-) type investors choose the large (small) number of PE shares. The \( J_u(\bar{W}, X_0, 0) \) is the value function that the investor would obtain if she decided not to participate in the PE, and is derived in the Lemma A.1. Using the conjectured value functions, the investor’s IR constraint can be expressed as

\[ \omega^0_{l,0} + \omega_{1,0} X_0 + \frac{1}{\psi^0} (k^h_{0,0} - \hat{k}^h_{0,0}) \geq I_0, \]

Multiply the bargaining power of seller and buyer, respectively and sum over, then I have

\[ Q_0 + \frac{\eta}{\psi^0} (k^h_{0,0} - \hat{k}^h_{0,0}) + \frac{1 - \eta}{\psi^0} (k^l_{0,0} - \hat{k}^l_{0,0}) \geq I_0, \quad (A.8) \]

Thus, the expected hold-to-maturity return (IRR) is

\[
\begin{align*}
\varphi &= \frac{1}{T} \log \mathbb{E} \left[ 1 + \frac{y_T - F_T - I_0}{I_0} \right] \\
&\geq \frac{1}{T} \log \mathbb{E} \left[ 1 + \frac{1}{I_0} \left( y_T - F_T - Q_0 - \frac{\eta}{\psi^0} \Delta^h_k - \frac{1 - \eta}{\psi^0} \Delta^l_k \right) \right] \\
&= \frac{1}{I_0 T} \log \left( 1 + \int_0^T (\alpha + \beta \mu_F) dt - \frac{\eta}{\psi^0} \Delta^h_k - \frac{1 - \eta}{\psi^0} \Delta^l_k \right) \\
&\approx \frac{1}{I_0 T} \left( \int_0^T (\alpha + \beta \mu_F) dt - \frac{\eta}{\psi^0} \Delta^h_k - \frac{1 - \eta}{\psi^0} \Delta^l_k \right)
\end{align*}
\]

where the first inequality is due to (A.8), and the second equality is because \( \mathbb{E}[y_T - F_T - Q_0] = \mathbb{E}[Q_T - Q_0 - F_T] = \mathbb{E}[\int_0^T dQ_t - dF_t] = \int_0^T \mu_d dt \). \[ \Box \]
Appendix B

Appendix to Chapter 2

B.1 Asset Returns

B.1.1 Constant Time-to-maturity Bond Index

The dynamics of the risk-free rate under the risk-neutral measure is

\[ dr(t) = \left[ \kappa_r \{ \bar{r} - r(t) \} + \kappa_{ry} \{ \bar{y} - y(t) \} \right] dt + \sigma_r \left( dZ_r^Q(t) - \Lambda_r(t) \right) \]

\[ = \kappa_r^Q \left( r^Q - r(t) \right) dt + \sigma_r dZ_r^Q(t), \]

where \( r^Q = \frac{\kappa_r^r \bar{r} + \kappa_{ry} \bar{y} - \sigma_r \lambda_r}{\kappa_r^r} \), and \( \kappa_r^Q = \kappa_r + \sigma_r \phi_r \). Also, \( Z_r^Q \) represents a Brownian motion under the risk-neutral measure. Note that the mean-reverting speed and mean level differ in two measures. Now, the time \( t \) price of zero coupon bond maturing at time \( T \geq t \) can be derived:

\[ P(t, T) = \exp \left( A_1(T - t) + A_2(T - t) r(t) \right) \]

\[ A_2(\tau) = -\frac{1 - e^{-\kappa_r^Q \tau}}{\kappa_r^Q}, \]

\[ A_1(\tau) = -\left( r^Q - \frac{\sigma_r^2}{2\kappa_r^r} \right) (A_2(\tau) + \tau) - \frac{\sigma_r^2 A_2(\tau)^2}{4k_r^Q}. \]

Then, returns of \( T \) time-to-maturity bond index are

\[ dB_t = \frac{dP(t, s)}{P(t, s)} \bigg|_{s=t+T} + \frac{1}{P(t, s)} \frac{\partial P(t, s)}{\partial s} \bigg|_{s=t+T} dt + \sigma_b dZ_r(t), \]

\[ = (\alpha_b + \beta_{b, r} r(t) + \beta_{b, y} y(t)) dt + \sigma_b dZ_r(t), \]

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where

\[ \alpha_b = a_1(T) + A_2(T)\sigma_r\lambda_r \]
\[ \beta_{b,r} = 1 + a_2(T) + A_2(T)\sigma_r\phi_r \]
\[ \beta_{b,y} = A_2(T)\sigma_y\phi_y \]
\[ \beta_z = \sigma_s\sqrt{1 - \rho_{rs}^2}\sigma_s \]
\[ a_i(T) = \left. \frac{\partial A_i(s - t)}{\partial s} \right|_{s=t+T} (i = 1, 2). \]

The return of the \( T \) time-to-maturity bond index consists of two parts: the return from holding \( P(t, t + T) \) between time \( t \) and \( t + dt \) and the rollover return from selling \( P(t + dt, t + T) \) and buying \( P(t + dt, t + dt + T) \).

### B.1.2 Stock

Under the risk-neutral measure, equity returns can be written as

\[
\frac{dS(t)}{S(t)} = r(t)dt + \sigma_s \left( \rho_{rs} dZ^Q_r(t) + \sqrt{1 - \rho_{rs}^2} dZ^Q_s(t) \right),
\]

where \( Z^Q_r \) and \( Z^Q_s \) are Brownian motions under the risk-neutral measure. Under the physical measure, equity returns follow

\[
\frac{dS(t)}{S(t)} = r(t)dt + \sigma_s \rho_{rs} (dZ_r(t) + \Lambda_r(t)) + \sigma_s \sqrt{1 - \rho_{rs}^2} (dZ_s(t) + \Lambda_s(t))
\]

\[ = (\alpha_s + \beta_{s,r} r(t) + \beta_{s,y} \phi_r(t) + \beta_z \phi_s(t))dt + \sigma_s \left( \rho_{rs} dZ_r(t) + \sqrt{1 - \rho_{rs}^2} dZ_s(t) \right), \]

where

\[ \alpha_s = \sigma_s \left( \rho_{rs}\lambda_r + \sqrt{1 - \rho_{rs}^2}\lambda_s \right) \]
\[ \beta_{s,r} = 1 + \sigma_s \rho_{rs}\phi_r \]
\[ \beta_{s,y} = \sigma_s \rho_{rs}\phi_y \]
\[ \beta_z = \sigma_s \sqrt{1 - \rho_{rs}^2}\phi_s. \]
B.2 Trading Strategies

B.2.1 Continuous Merton (1971) Policy

We use the stochastic control approach to solve the problem. Let $J(W, X, t)$ denote the indirect utility function. The principle of optimality leads to the following Hamilton-Jacobi-Bellman equation for $J$:

$$\max_{w(t)} J_t + \mathcal{L}J = 0,$$

(B.1)

where

$$\mathcal{L}J = J_W W \mu_W + J_X K (\theta - X) + J_{WX} W \Sigma_X \Sigma_W^T + \frac{1}{2} J_{WW} W^2 \Sigma_W \Sigma_W^T + \frac{1}{2} \text{tr} (J_{XX} \Sigma_X \Sigma_X^T),$$

with boundary condition

$$J(W(T), X(T), T) = \frac{W(T)^{1-\gamma}}{1 - \gamma}.$$

The expected return and volatility of the wealth process are defined as $\mu_W = (1 - w(t)) \mu_b + w(t) \mu_s$ and $\Sigma_W = (1 - w(t)) \Sigma_b + w(t) \Sigma_s$, where $\mu_b$ and $\mu_s$ are the conditional expected returns of bond and stock, and $\Sigma_b$ and $\Sigma_s$ are the volatility matrices of bond and stock. The coefficients $K$, $\theta$, and $\Sigma_X$ can be obtained by stacking all three state variables. The indirect utility function $J$ is conjectured to have the form:

$$J(W(t), X(t), t) = \frac{W(t)^{1-\gamma}}{1 - \gamma} F(X(t), t)^\gamma.$$

Under this conjecture, the optimal portfolio weight of stock is given by

$$w^*(t) = \frac{\mu_s - \mu_b}{\gamma \Sigma_s \Sigma_s^T \Sigma_s^T} - \frac{\Sigma_b \Sigma_s^T \Sigma_s^T}{\Sigma_s \Sigma_s^T} + \frac{F X \Sigma_s X \Sigma_s^T}{F \Sigma_s \Sigma_s^T},$$

(B.2)

where $\Sigma_s = \Sigma_s - \Sigma_b$. We can interpret the optimal portfolio weight as two parts: the myopic demand and the hedging demand. The first two terms in equation (B.2) represent the myopic demand. The term $(\mu_s - \mu_b)/(\gamma \Sigma_s \Sigma_s^T)$ is the standard formula for an IID environment with a constant risk premium. In our setting, the risk-free rate changes over time, so the investor also cares about the covariance of stock and bond returns represented in the second term, $(\Sigma_b \Sigma_s^T)/(\Sigma_s \Sigma_s^T)$. The last term in equation (B.2) is the hedging demand, which allows the investor to hedge possible future variation of the state variables by holding an off-setting position in assets whose return is correlated with those state variables.

To solve for $F(X(t), t)$, we conjecture its form and then verify. Our conjecture is that

$$F(X(t), t) = \exp \left( B_1(\tau) + B_2(\tau) X(t) + \frac{1}{2} X(t)^\top B_3(\tau) X(t) \right),$$

(B.3)
where \( \tau = T - t \) and the matrix \( B_3 \), the vector \( B_2 \), and the scalar \( B_1 \) satisfy a system of ordinary differential equations (ODEs). Substituting the optimal portfolio weight into equation (B.1) gives us the following partial differential equation (PDE):

\[
F_t + F \left( \frac{1 - \gamma}{2\gamma^2} (\Lambda^* - \gamma \Sigma_b^T) \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* (\Lambda^* - \gamma \Sigma_b^T) + \frac{1 - \gamma}{\gamma} r^* \right) + F_X \left[ K (\theta - X) + \frac{1 - \gamma}{\gamma} \Sigma_X \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* (\Lambda^* - \gamma \Sigma_b^T) + (1 - \gamma) \Sigma_X \Sigma_b^T \right] + \frac{\gamma - 1}{2F} F_X \left[ \Sigma_X \Sigma_X^T - \Sigma_X \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* \Sigma_X^T \right] F_X + \frac{1}{2} \text{tr} (F_X \Sigma_X \Sigma_X^T) = 0,
\]

where \( \Lambda^* = \lambda^* + \phi^* X \) such that \( \Sigma_s \Lambda^* = \mu_s - \mu_b, r^* = \delta_0^* + \delta_1^* X \), and \( r^* = \mu_b - \frac{\gamma}{2} \Sigma_b \Sigma_b^T \). Plugging equation (B.3) into the PDE and matching coefficients on \( X(t)^T [\cdot] X(t), X(t) \), and the constant term leads us to a system of ODEs:

\[
\begin{align*}
\hat{B}_3(\tau) &= 2B_3 P + B_3 Q B_3^T + \frac{1 - \gamma}{\gamma^2} \phi^* \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* \phi^* \\
\hat{B}_2(\tau) &= B_2 P + B_2 Q B_2^T + R B_3 \\
&\quad + \frac{1 - \gamma}{\gamma^2} (\lambda^* - \gamma \Sigma_b^T) \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* \phi^* + \frac{1 - \gamma}{\gamma} \delta_1^* \\
\hat{B}_1(\tau) &= B_2 R + \frac{1}{2} B_2 Q B_2^T + \frac{1}{2} \text{tr} (B_3 \Sigma_X \Sigma_X^T) \\
&\quad + \frac{1 - \gamma}{2\gamma^2} (\lambda^* - \gamma \Sigma_b^T) \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* (\lambda^* - \gamma \Sigma_b^T) + \frac{1 - \gamma}{\gamma} \delta_0^*,
\end{align*}
\]

where

\[
\begin{align*}
P &= -K + \frac{1 - \gamma}{\gamma} \Sigma_X \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* \phi^* \\
Q &= \gamma \Sigma_X \Sigma_s^T + (1 - \gamma) \Sigma_X \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* \Sigma_X \\
R &= K \theta + \frac{1 - \gamma}{\gamma} \Sigma_X \Sigma_s^T (\Sigma_s \Sigma_s^T)^{-1} \Sigma_s^* (\lambda^* - \gamma \Sigma_b^T) + (1 - \gamma) \Sigma_X \Sigma_b^T.
\end{align*}
\]

The boundary conditions are

\[
B_1(T) = 0, \quad B_2(T) = 0_{1 \times 3}, \quad B_3(T) = 0_{3 \times 3}.
\]

**B.2.2 Myopic Policy**

We introduce a myopic policy which ignores the hedging demands present in the continuous Merton (1971) policy. A myopic investor times the market over the next (instantaneous) period and has a portfolio weight represented by the first two terms in equation (B.2). The portfolio weights can be expressed as

\[
w(t) = \alpha_0 + \alpha_1 X(t),
\]

(B.5)
where $\alpha_0 = \frac{\Sigma_\gamma (\lambda - \gamma \Sigma_\gamma)}{\gamma \Sigma_\gamma}$ and $\alpha_1 = \frac{\Sigma_\alpha}{\gamma \Sigma_\alpha}$. We solve for the indirect utility when the investor follows the above strategy. Denote $\hat{J}(W, X, t)$ as the indirect utility corresponding to $(\alpha_0, \alpha_1)$. Then $\hat{J}(W, X, t)$ should also satisfy equation (B.1). Since $w(t)$ is linear in $X(t)$, $\hat{J}(W, X, t)$ takes the same form as the continuous Merton (1971) policy:

$$
\hat{J}(W(t), X(t), t) = \frac{W(t)^{1-\gamma}}{1-\gamma} \hat{F}(X(t), t)^{\gamma}.
$$

Similarly, we conjecture that $\hat{F}(X, t)$ is exponential quadratic:

$$
\hat{F}(X(t), t) = \exp \left(\hat{B}_1(\tau) + \hat{B}_2(\tau)X(t) + \frac{1}{2}X(t)^\top \hat{B}_3(\tau)X(t)\right),
$$

(B.6)

where $\hat{B}_1$, $\hat{B}_2$, and $\hat{B}_3$ satisfy a system of ODEs. With a similar procedure as solving the continuous Merton (1971) policy in the previous section, we obtain the system of ODEs:

$$
\dot{\hat{B}}_3(\tau) = 2\hat{B}_3 \hat{P} + \hat{B}_3 \hat{Q} \hat{B}_3^\top + 2\frac{(1-\gamma)}{\gamma} \left\{ \alpha_1^{\top} \Sigma_* \phi^* - \frac{\gamma}{2} \alpha_1^{\top} \Sigma_* \Sigma_\alpha \alpha_1 \right\}, \tag{B.7}
$$

$$
\dot{\hat{B}}_2(\tau) = \hat{B}_2 \hat{P} + \hat{B}_2 \hat{Q} \hat{B}_3^\top + \hat{R} \hat{B}_3^\top + \frac{1}{\gamma} \left\{ \alpha_0^{\top} \Sigma_* \phi^* + \lambda^* \Sigma_\alpha \alpha_1 - \gamma (\Sigma_0 + \alpha_0 \Sigma_*^\top)(\Sigma_\alpha \alpha_1) + \delta_1^* \right\}, \tag{B.8}
$$

$$
\dot{\hat{B}}_1(\tau) = \hat{B}_2 \hat{R} + \frac{1}{2} \hat{B}_2 \hat{Q} \hat{B}_2^\top + \frac{1}{2} \text{tr} \left( \hat{B}_3 \Sigma_X \Sigma_X^\top \right) + \frac{1}{\gamma} \left\{ \alpha_0^{\top} \Sigma_*^\top - \frac{\gamma}{2} (\Sigma_0 + \alpha_0 \Sigma_*) (\Sigma_0 + \alpha_0 \Sigma_*)^\top + \delta_0^* \right\}, \tag{B.9}
$$

where

$$
\begin{align*}
\hat{P} &= -K + (1-\gamma) \Sigma_X \Sigma_*^\top \alpha_1, \\
\hat{Q} &= \gamma \Sigma_X \Sigma_X^\top, \\
\hat{R} &= K \theta + (1-\gamma) \Sigma_X (\Sigma_0 + \alpha_0 \Sigma_*)^\top,
\end{align*}
$$

and $\delta_0^*$, $\delta_1^*$ are such that $\mu_b = \delta_0^* + \delta_1^* X$. The boundary conditions are

$$
\hat{B}_1(T) = 0, \hat{B}_2(T) = 0_{1 \times 3}, \hat{B}_3(T) = 0_{3 \times 3}.
$$

### B.2.3 Tactical Asset Allocation Policy

Let $\hat{J}_k(W_k, X_k; n)$ be the value function at $k$-th switching date of TAA switching $n$ times, and take the following form:

$$
\hat{J}_k(W_k, X_k; n) = \frac{W_k^{1-\gamma}}{1-\gamma} \hat{F}_k(X_k; n)^{\gamma},
$$

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where $\hat{F}_k(X_k; n) = \exp\left(\hat{B}_{1,k} + \hat{B}_{2,k}X_k + \frac{1}{2}X_k^\top\hat{B}_{3,k}X_k\right)$. We refer to time 0 as the zero-th switching date. Then, the recursion formulas for $\hat{B}_1$, $\hat{B}_2$, and $\hat{B}_3$ of the TAA policy are equations (B.7), (B.8), and (B.9), but $\alpha_0$ is undetermined and $\alpha_1 = 0_{1 \times 3}$. Also, in the recursive equation for $\hat{B}_1$, the cost of switching $\frac{1-c}{\gamma}\log(1-c)$ should be incorporated. Now, the question is what is the optimal $\alpha_0$? The agent chooses $\alpha_0$ to maximize the value function at the $k$-th rebalancing date. The FOC with respect to $\alpha_0$ is

$$\frac{\partial \hat{B}_{1,k}}{\partial \alpha_0} + \frac{\partial \hat{B}_{2,k}}{\partial \alpha_0} X_t = 0.$$  

Note that $\hat{B}_{3,k}$ does not depend on $\alpha_0$. The above equation tells us that the optimal target portfolio weight at $k$-th rebalancing date is linear in the state variables:

$$\alpha_0^* = c_{0,k} + c_{1,k}X_k.$$  

We substitute $\alpha_0^*$ in $\hat{B}_{1,k}$ and $\hat{B}_{2,k}$, and re-collect coefficients on constant, $X_k$, and $X_k \cdot X_k$. Then, we have a new $\hat{B}_{1,k}$ and $\hat{B}_{2,k}$. We repeat this procedure until time 0 to obtain the optimal target portfolio weight at each TAA decision point and the value function at time 0.

### B.3 Estimation

Let $\hat{X}$ be the augmented state variables vector: $\hat{X}(t) = [r(t)\ y(t)\ z(t)\ \log B(t)\ \log S(t)]^\top$. Then, our model can be written as

$$d\hat{X}(t) = \left(\hat{\mu} + \hat{K}\hat{X}(t)\right)dt + \hat{\Sigma}dZ(t),$$

where $\hat{\mu}$, $\hat{K}$, and $\hat{\Sigma}$ are vector representations of the parameters of each variable in $\hat{X}$. The discrete-time process implied by the above continuous-time mean-reverting process is

$$\hat{X}(t + \Delta) = \left(\int_t^{t+\Delta} e^{\hat{K}(t+\Delta-s)}ds\right)\hat{\mu} + e^{\hat{K}\Delta} \hat{X}(t) + \int_t^{t+\Delta} e^{\hat{K}(t+\Delta-s)} \hat{\Sigma}dZ(t),$$  \hspace{1cm} (B.10)

where $e^{\hat{K}\Delta}$ is a matrix exponential. The variance-covariance matrix $\Sigma$ is

$$\Sigma = \int_t^{t+\Delta} e^{\hat{K}(t+\Delta-s)}\hat{\Sigma}\hat{\Sigma}^\top e^{\hat{K}(t+\Delta-s)}ds.$$  \hspace{1cm} (B.11)

This is a restricted Vector Autoregression (VAR).

We estimate equation (B.10) and report the coefficients in Table 2.2. We recover the continuous-time parameters from the discrete-time VAR estimates, which we report in Table B.1. Panel D corresponds to the case that both bond and stock returns are predictable (recovered from Table 2.2). The other cases are:
• For IID returns, we set $\phi_r = \phi_y = \phi_s = 0$, i.e. the prices of risks are constant.

• For the case of predictable bond returns only, we set $\phi_s = \nu = 0$, i.e. stock returns are not predictable and uncorrelated with the short rate risk.

• For the case of only predictable stock returns, we set $\phi_r = \phi_y = 0$, i.e. bond returns are not predictable, but still correlated with stock returns.

For these three special cases, we derive the discrete-time VAR, re-estimate the discrete-time coefficients, and back out the corresponding continuous-time parameters.

**B.4 Discussion on TAA vs Discrete Rebalancing**

We consider a trading strategy closely related with TAA, namely discrete rebalancing. The investor using a discrete rebalancing strategy employs a buy-and-hold strategy during the period between two rebalancing dates. On the other hand, TAA keeps the target portfolio weights by trading continuously. However, TAA is very similar to discrete rebalancing and TAA allows us to have closed-form solutions. This feature is especially useful when an investment opportunity set is a function of more than one state variable, as in our model. Solving optimal weights of discrete rebalancing requires computationally burdensome numerical techniques. Thus, in this section we consider a simpler model to compare TAA and discrete rebalancing with one state variable, the divided yield, which governs the expected return of equities. This implies that the short rate is constant and the term structure is flat. Stock returns now follow

$$\frac{dS(t)}{S(t)} = (r + \sigma_s \Lambda_s(t))dt + \sigma_s dZ_s(t).$$

Dividend yields follow the same mean-reverting process as equation (2.6), and the price of risk $\Lambda_s$ takes the same form as equation (2.5). An investor allocates her wealth in the stock and the risk-free asset, which pays a constant return $r$.

We can derive the optimal portfolio weights and value functions of continuous Merton (1971) and TAA policy as we do for the full multivariate model. We now provide a solution method to derive the optimal portfolio weights at each point and to compute the discrete rebalancing value function. Suppose that the agent is allowed to trade only $n$ times at evenly spaced dates over the investment horizon $T$. We treat time zero as the zero-the rebalancing date. Define the value function at the $k$-th rebalancing date as

$$\tilde{J}_k(W_k, z_k) = \max_{\{w_i\}_{i=k}^{n-1}} \mathbb{E}_k \left[ \frac{W_T^1}{1 - \gamma} \right],$$
Table B.1: Continuous-Time Parameters

Panel A: IID Returns

<table>
<thead>
<tr>
<th>VAR Parameters</th>
<th>( \theta )</th>
<th>( r )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
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<td>0.1119</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
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<td></td>
<td>0.4169</td>
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</tr>
<tr>
<td>( z )</td>
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Volatility Parameters

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<th>( Z_r )</th>
<th>( Z_y )</th>
<th>( Z_z )</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r )</td>
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<td></td>
</tr>
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<td>( y )</td>
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<td>0.0054</td>
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<td>( z )</td>
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<td>0.0081</td>
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</table>

Prices of Risk

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<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
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<tr>
<td>( \Lambda_r )</td>
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<td></td>
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</table>

Panel B: Predictable Bond Returns Only

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<thead>
<tr>
<th>VAR Parameters</th>
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<th>( z )</th>
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</thead>
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<tr>
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Volatility Parameters

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<th>( Z_y )</th>
<th>( Z_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( dS/S )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r )</td>
<td>-</td>
<td>0.0134</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>-</td>
<td>-0.0050</td>
<td>0.0054</td>
<td></td>
</tr>
<tr>
<td>( z )</td>
<td>-0.0052</td>
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<td></td>
<td>0.0081</td>
</tr>
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</table>

Prices of Risk

<table>
<thead>
<tr>
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<th>( \lambda )</th>
<th>( r )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
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<td>( \Lambda_z )</td>
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</tbody>
</table>
We report continuous-time parameters corresponding to systems with no predictability (Panel A), predictability of bond returns only (Panel B), only predictable stock returns (Panel C), and when expected returns of both assets vary over time (Panel D). Panel D corresponds to the results in Table 1 of the main paper. For the other systems, we re-estimate the model with restricting some coefficients to be zero and recover the corresponding continuous-time parameters.

### Table B.1: Continuous-Time Parameters (cont.)

<table>
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<tr>
<th>Panel C: Predictable Stock Returns Only</th>
<th>Panel D: Predictable Stock and Bond Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>VAR Parameters</strong></td>
<td><strong>VAR Parameters</strong></td>
</tr>
<tr>
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</tr>
<tr>
<td>$r$</td>
<td>$r$</td>
</tr>
<tr>
<td>0.0387</td>
<td>0.0358</td>
</tr>
<tr>
<td>$y$</td>
<td>$y$</td>
</tr>
<tr>
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<tr>
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<tr>
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<td>0.0375</td>
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<td><strong>Volatility Parameters</strong></td>
<td><strong>Volatility Parameters</strong></td>
</tr>
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<td>$dS/S$</td>
<td>$dS/S$</td>
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<tr>
<td><strong>Prices of Risk</strong></td>
<td><strong>Prices of Risk</strong></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
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<td>$\Lambda_r$</td>
<td>$\Lambda_r$</td>
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<td>$\Lambda_s$</td>
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<td>-0.1496</td>
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</table>

We report continuous-time parameters corresponding to systems with no predictability (Panel A), predictability of bond returns only (Panel B), only predictable stock returns (Panel C), and when expected returns of both assets vary over time (Panel D). Panel D corresponds to the results in Table 1 of the main paper. For the other systems, we re-estimate the model with restricting some coefficients to be zero and recover the corresponding continuous-time parameters.
where $\mathbb{E}_k$ is the conditional expectation on the information upto $k$-th rebalancing date, and $w_i$ is a portfolio weight in stock at $i$-th rebalancing date. Then, the following holds

$$\tilde{J}_k (W_k, z_k) = \max_{w_k} \mathbb{E}_k \left[ \tilde{J}_{k+1} (W_{k+1}, z_{k+1}) \right].$$

We conjecture that $\tilde{J}_k (W_k, z_k) = \frac{(W_k)^{1-\gamma}}{1-\gamma} \tilde{F}_k (z_k)^\gamma$ with a boundary condition $\tilde{F}_n (z_n) = 1$. Plugging this into the above equation, we get

$$\tilde{F}_k (z_k)^\gamma = \min_{w_k} \mathbb{E}_k \left[ \tilde{F}_{k+1} (z_{k+1})^\gamma (e^{\gamma \Delta} + w_k g)^{1-\gamma} \right],$$

where $\Delta$ is a trading interval and

$$g = \exp \left( \int_{t_k}^{t_{k+1}} \log S(u) du \right) - e^{r \Delta}.$$

The first order condition is

$$\mathbb{E}_k \left[ \tilde{F}_{k+1} (z_{k+1})^\gamma (e^{\gamma \Delta} + w_k g)^{-\gamma} g \right] = 0,$$

which we solve by Gaussian Quadrature. By plugging the optimal portfolio weight policy into equation (B.12), we obtain $\tilde{F}_k (z_k)$. Doing this recursively, we derive the portfolio weights rule at each rebalancing date, and solve for $\tilde{F}_0 (z_0)$.

To capture an effect of inability to trade in more detail, we compute utility costs of a buy-and-hold strategy versus a TAA strategy over $T$ periods. The TAA strategy rebalances back to a constant portfolio weight, and thus is a single-switching TAA strategy. The utility costs are reported in percentage terms of initial wealth and represent the increase in initial wealth required to make the buy-and-hold investor have the same utility if she had the ability to undertake a TAA strategy with a single switch. We take the same parameters as Panel C of Table B.1, except we set $r$ at a constant level $r = \bar{r}$. We set the risk aversion to be 7.9, which is in line with the results in the paper. We compute the utility costs integrating across the steady-state distribution of the single state variable, the dividend yield. Figure B.1 plots the results. As we expect, utility costs of buy-and-hold policy are positive. As the horizon increases, the utility costs increase. However, even in 10-year of horizon the utility cost is less than 0.9%, which indicates that we can take TAA as good approximation for discrete rebalancing.
The figure plots utility costs of a buy-and-hold strategy versus a TAA strategy over $T$ periods. The TAA strategy rebalances back to a constant portfolio weight, and thus is a single-switching TAA strategy. The utility costs are reported in percentage terms of initial wealth and represent the increase in initial wealth required to make the buy-and-hold investor have the same utility if she had the ability to undertake a TAA strategy with a single switch. We take a risk aversion of 7.9 to be in line with the results in the paper. We compute the utility costs integrating across the steady-state distribution of the single state variable, the dividend yield.
Appendix C

Appendix to Chapter 3

C.1 Proofs

Proof of Proposition 3.1. By Girsanov’s Theorem, there exists a unique equivalent measure $Q$ in which all traded assets earn the risk-free rate, and under $Q$ measure the following stochastic process is a standard Brownian motion.

$$dZ_t^Q = dZ_t + \eta dt.$$ 

To compute $W_u(y)$, we can derive the dynamics of $\xi_t = M_t e^{\beta t}$ under $Q$ measure.

$$\frac{d\xi_t}{\xi_t} = (\beta - r)dt - \eta dZ_t$$

The random variable $I_u(y) = \xi_T$ can be expressed as

$$I_u(y) = y^\gamma \exp \left( -\frac{T}{\gamma} (\beta - r + \frac{1}{2} \eta^2) + \frac{\eta}{\gamma} (Z_T^Q - Z_0^Q) \right),$$

given that $\xi_0 = 1$. Let $A$ be the event in which $K > I_u(y)$. The event $A$ is equivalent to $x < -\delta_2(y, T)$, where $x$ is a standard normal random variable, and $\delta_2(y, T)$ is given by

$$\delta_2(y, T) = \frac{\log \left( \frac{y}{K} \right) + \frac{T}{\gamma} (r - \beta - \frac{\eta^2}{2})}{\frac{T}{\gamma}}.$$ 

since $Z_T^Q - Z_0^Q$ is normally distributed with zero mean and variance of $T$. Then, $W_u(y)$ can be expressed as

$$W_u(y) = E^Q \left[ e^{-rT} I_u(y) (1 - 1(A)) \right] + K e^{-rT} N(-\delta_2(y, T)),$$
where \(1(A)\) is an indicator function of event \(A\) and \(N(\cdot)\) is a cumulative distribution function of standard normal random variable. The first part can be easily computed:

\[
E^Q \left[ e^{-rT} I_u(y \xi_T) (1 - 1(A)) \right] = \exp \left( - \left( r + \frac{1}{\gamma} (\beta - r + \frac{1}{2} \eta^2) \right) T \right) y^{-\frac{1}{\gamma}} \int_{-\delta_2(y,T)}^\infty \exp \left( \frac{\eta \sqrt{T}}{\gamma} x \right) n(x) dx \\
= y^{-\frac{1}{\gamma}} e^{-\alpha_u T} \int_{-\delta_2(y,T)}^\infty n \left( x - \frac{\eta \sqrt{T}}{\gamma} \right) dx \\
= y^{-\frac{1}{\gamma}} e^{-\alpha_u T} N(\delta_1(y,T)),
\]

where \(n(\cdot)\) is a probability distribution function of a standard normal random variable, \(\alpha_u\) and \(\delta_1(y,T)\) are given by

\[
\alpha_u = \frac{\beta}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) \left( r + \frac{\eta^2}{2\gamma} \right) \\
\delta_1(y,T) = \delta_2(y,T) + \frac{\eta \sqrt{T}}{\gamma}.
\]

Now, the first derivative of \(W_u(y)\) can be computed as

\[
W_u'(y) = -\frac{1}{\gamma} y^{-\frac{1}{\gamma}} \exp \left( -e^{-\alpha_u T} N(\delta_1(y,T)) \right) \frac{\partial \delta_1(y,T)}{\partial y} \\
- K e^{-r T} n(\delta_2(y,T)) \frac{\partial \delta_2(y,T)}{\partial y}.
\]

Note that \(\frac{\partial \delta_1(y,T)}{\partial y} = \frac{\partial \delta_2(y,T)}{\partial y}\) and

\[
y^{-\frac{1}{\gamma}} e^{-\alpha_u T} n(\delta_1(y,T)) = y^{-\frac{1}{\gamma}} \exp \left( -\alpha_u T + \frac{\eta \sqrt{T}}{\gamma} \right) \\
= y^{-\frac{1}{\gamma}} \exp \left( -\alpha_u T - \frac{\eta \sqrt{T}}{\gamma} \delta_2(y,T) - \frac{\eta^2 T}{2\gamma^2} \right) n(\delta_2(y,T)) \\
= K e^{-r T} n(-\delta_2(y,T)).
\]

Hence, last two terms cancel out.

\[\square\]

**Proof of Theorem 3.2.** Consider any random variable \(\hat{W}_T^u \geq K\), which is feasible by a self financing trading strategy and the initial endowment \(W_0^u\). This implies that

\[
W_0^u \geq E^Q \left[ e^{-rT} \hat{W}_T^u \right].
\]

Then, we want to show that

\[
E \left[ e^{-\beta T} u(W_T^u) \right] \geq E \left[ e^{-\beta T} u(\hat{W}_T^u) \right].
\]

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Since $u$ is a concave utility, we have
\[ u(W_T^u) - u(W_T^{\tilde{u}}) \geq u'(W_T^u) \left( W_T^u - \tilde{W}_T^u \right). \] (C.1)

We can compute $u'(W_T)$:
\[
  u'(W_T^u) = u' \left( \max \left( I_u \left( \mathcal{Y}_u \left( W_0^u \right) \xi_T \right), K \right) \right) \\
  = \min \left( \mathcal{Y}_u \left( W_0^u \right) \xi_T, u'(K) \right) \\
  = \mathcal{Y}_u \left( W_0^u \right) \xi_T - \left( \mathcal{Y}_u \left( W_0^u \right) \xi_T - u'(K) \right)^+.
\]

Substitute in (C.1), we have
\[
  u(W_T^u) - u(W_T^{\tilde{u}}) \geq \mathcal{Y}_u \left( W_0^u \right) \xi_T \left( W_T^u - \tilde{W}_T^u \right) + \left( \mathcal{Y}_u \left( W_0^u \right) \xi_T - u'(K) \right)^+ \left( \tilde{W}_T^u - K \right).
\]

The second term is due to that $W_T^u = K$ corresponds to $\mathcal{Y}_u \left( W_0^u \right) \xi_T > u'(K)$. The second term is always greater or equal to zero since $\tilde{W}_T^u \geq K$. Multiplying $e^{-\beta T}$ and taking expectation under the physical measure of the first term of RHS yields
\[
  \mathcal{Y}_u \left( W_0^u \right) \mathbb{E} \left[ e^{-\beta T} \xi_T \left( W_T^u - \tilde{W}_T^u \right) \right] = \mathcal{Y}_u \left( W_0^u \right) \mathbb{E}^Q \left[ e^{-r T} \left( W_T^u - \tilde{W}_T^u \right) \right] \\
  \geq \mathcal{Y}_u \left( W_0^u \right) \left( W_0^u - \mathbb{E}^Q \left[ e^{-r T} \tilde{W}_T^u \right] \right) \\
  \geq 0.
\]

Hence, we obtain the desired inequality. Now, the optimal portfolio weight can be obtained by matching volatility of (3.5) and $W_T^u = \mathbb{E}^Q_t \left[ e^{-r(T-t)} W_T^u \right]$. By Proposition 3.1, we can easily compute the latter:
\[
  W_T^u = y_t e^{-\frac{1}{2} \sigma^2 \alpha_u(T-t)} \mathbb{N} \left( \delta_1(y_t, T-t) \right) + K e^{-r(T-t)} \mathbb{N} \left( -\delta_2(y_t, T-t) \right),
\]
where $y_t = \mathcal{Y}_u \left( W_0^u \right) \xi_t$. The diffusion part of the above is
\[
  \text{diff} \left( dW_T^u \right) = \frac{\eta}{\gamma} y_t e^{-\frac{1}{2} \sigma^2 \alpha_u(T-t)} \mathbb{N} \left( \delta_1(y_t, T-t) \right).
\]

This should be equal to the diffusion part of (3.5), $\pi_t^u W_t^u \sigma$. Hence, we have
\[
  \pi_t^u = \frac{\eta}{\gamma \sigma} \left( 1 - \varphi_t \right),
\]
where $\varphi_t = \frac{K e^{-r(T-t)} \mathbb{N} \left( -\delta_2(y_t, T-t) \right)}{W_t} < 1$. \hfill \Box
Proof of Proposition 3.3. We first compute \( G(y) \):

\[
G(y) = \mathbb{E}\left[ e^{-\beta T \left( \frac{(y \xi_T)^{1-\frac{1}{\gamma}}}{1-\gamma} (1 - 1(A)) + e^{-\beta T \frac{K^{1-\gamma}}{1-\gamma} 1(A)} \right)} \right]
\]

Note that the expectation is under the physical measure. The random variable \( I_u(y \xi_T) \) can be expressed as

\[
I_u(y \xi_T) = y^{-\frac{1}{\gamma}} \exp\left( -\frac{T}{\gamma} (\beta - r - \frac{1}{2} \eta^2) + \frac{\eta}{\gamma} (Z_T - Z_0) \right).
\]

The event \( A \) is equivalent to \( x < -\delta_3(y, T) \), where \( x \) is a standard normal random variable, and \( \delta_3(y, T) \) is given by \( \delta_3(y, T) = \delta_2(y, T) + \eta \sqrt{T} \), since \( Z_T - Z_0 \) is normally distributed with zero mean and variance of \( T \). If we follow similar steps as Proposition 3.1, we can obtain (3.10). (3.11) is the direct result of Theorem 3.2. Take the first derivative of (3.9), then we have

\[
G'(y) = \mathbb{E}\left[ e^{-\beta T u' (I_u(y \xi_T))} I_u'(y \xi_T) (1 - 1(A)) \right]
\]

From (3.11), we have

\[
J'(W^u_0) = G'(Y_u(W^u_0)) Y_u'(W^u_0)
\]

\[
= Y_u(W^u_0) W'_u(Y_u(W^u_0)) Y'_u(W^u_0)
\]

\[
= Y_u(W^u_0).
\]

\[\square\]

Proof of Proposition 3.4. We can interchange the integral and expectation:

\[
\mathcal{W}_\phi(y) = \int_0^T e^{-rt} \mathbb{E}^Q[ I_\phi(y \xi_t) ] dt,
\]

where

\[
I_\phi(y \xi_t) = \left( \frac{y}{K} \right)^{\frac{1}{1+\theta}} \exp\left( \frac{t}{\theta - 1} \left( \beta - r + \frac{1}{2} \eta^2 \right) - \frac{\eta}{\theta - 1} (Z^Q_t - Z^0_Q) \right).
\]

The inner expectation is

\[
\mathbb{E}^Q[I_\phi(y \xi_t)] = \left( \frac{y}{K} \right)^{\frac{1}{1+\theta}} \exp\left( \frac{t}{\theta - 1} \left( \beta - r + \frac{\theta \eta^2}{2(\theta - 1)} \right) \right).
\]

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Now, we can express $\mathcal{W}_\phi(y)$ as

$$\mathcal{W}_\phi(y) = \left(\frac{y}{k}\right) \frac{1}{\alpha_\phi} \int_0^T e^{-\alpha_\phi t} dt$$

$$= \left(\frac{y}{k}\right) \frac{1 - e^{-\alpha_\phi T}}{\alpha_\phi},$$

where $\alpha_\phi = \frac{\theta}{\theta - 1} \left( r - \frac{\sigma^2}{2(\theta - 1)} \right) - \frac{\beta}{\theta - 1}$. The first derivative is straightforward.

\[\square\]

**Proof of Theorem 3.5.** Consider any random variable $\tilde{Y}$, whose present value is greater than $X_0$. This implies that

$$\mathbb{E}_Q \left[ \tilde{Y}_t \right] \geq X_0.$$

Then, we want to show that

$$\mathbb{E} \left[ \int_0^T e^{-\beta t} \phi (Y_t) dt \right] \leq \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi \left( \tilde{Y}_t \right) dt \right].$$

Since $\phi$ is a convex disutility, we have

$$\phi (Y_t) \leq \phi \left( \tilde{Y}_t \right) + \phi' (Y_t) \left( Y_t - \tilde{Y}_t \right)$$

$$\leq \phi \left( \tilde{Y}_t \right) + \mathcal{Y}_\phi (X_0) \xi_t \left( Y_t - \tilde{Y}_t \right).$$

Multiplying $e^{-\beta t}$ and taking integral and expectation under the physical measure of the second term of RHS yields

$$\mathcal{Y}_\phi (X_0) \mathbb{E} \left[ \int_0^T e^{-\beta t} \xi_t \left( Y_t - \tilde{Y}_t \right) dt \right] = \mathcal{Y}_\phi (X_0) \left( \mathbb{E}_Q \left[ \int_0^T e^{-rt} Y_t dt \right] - \mathbb{E}_Q \left[ \int_0^T e^{-rt} \tilde{Y}_t dt \right] \right)$$

$$= \mathcal{Y}_\phi (X_0) \left( X_0 - \mathbb{E}_Q \left[ \int_0^T e^{-rt} \tilde{Y}_t dt \right] \right)$$

$$\leq 0.$$

Hence, we obtain the desired inequality. Now, the optimal hedging of contributions can be obtained by matching volatility of (3.12) and $X_t = \mathbb{E}_t^Q \left[ \int_t^T e^{-r(t-s)} Y_s ds \right]$. By Proposition 3.4, we can easily compute the latter:

$$X_t = \left(\frac{y_t}{k}\right) \frac{1 - e^{-\alpha_\phi (T-t)}}{\alpha_\phi}, \quad \text{(C.2)}$$

where $y_t = \mathcal{Y}_\phi (X_0) \xi_t$. The diffusion part of $X_t$ is

$$\text{diff} (dX_t) = -\frac{\eta}{\theta - 1} X_t.$$

This should be equal to the diffusion part of (3.12), $\pi^\phi_t X_t \sigma$. Hence, we have

$$\pi^\phi_t = -\frac{\eta}{(\theta - 1)\sigma}.$$
Proof of Proposition 3.6. We first compute $C(y)$:

$$
C(y) = E \left[ \int_0^T e^{-\beta t} \frac{k}{\theta} \left( \frac{y \xi_t}{k} \right)^{\frac{\alpha}{\beta}} dt \right]
$$

$$
= \frac{k}{\theta} \left( \frac{y}{k} \right)^{\frac{\alpha}{\beta}} \int_0^T e^{-\beta t} E \left[ \frac{\theta}{e^{\beta t}} \right] dt
$$

$$
= \frac{k}{\theta} \left( \frac{y}{k} \right)^{\frac{\alpha}{\beta}} \int_0^T e^{-\alpha \phi t} dt
$$

$$
= \frac{k}{\theta} \left( \frac{y}{k} \right)^{\frac{\alpha}{\beta}} \frac{1}{\alpha \phi} - e^{-\alpha \phi T}.
$$

(3.16) is the direct result of Theorem 3.5. Take the first derivative of (3.15) is

$$
C'(y) = E \left[ \int_0^T e^{-\beta t} \phi' (I_{\phi}(y \xi_t)) I_{\phi}'(y \xi_t) \xi_t dt \right]
$$

$$
= y E \left[ \int_0^T e^{-r t} \xi_t I_{\phi}'(y \xi_t) dt \right]
$$

$$
= y W_{\phi}'(y).
$$

From (3.16), we have

$$
L'(X_0) = C'(Y_{\phi}(X_0)) Y_{\phi}'(X_0)
$$

$$
= Y_{\phi}(X_0).
$$

Proof of Theorem 3.7. We can compute the present value of arbitrary contribution policy. Let

$$
X_0 = E^Q \left[ \int_0^T e^{-r t} Y_t dt \right]
$$

$$
W_0^u = W_0 + X_0.
$$

Then, $(\tilde{\pi}, \tilde{Y})$ satisfies the following static budget constraint:

$$
W_0^u \geq E^Q \left[ e^{-r T \tilde{W}_T} \right],
$$

where $\tilde{W}$ is a corresponding wealth process to $(\tilde{\pi}, \tilde{Y})$. Hence, $\tilde{\pi}$ is a feasible trading strategy to the first problem with the initial wealth $W_0^u$, and $\tilde{Y}$ is a feasible contribution policy to the second problem with the present value of contribution $X_0$. Let $\pi^u_t$ and $\pi^\phi$ be the optimal trading strategy to the first and the optimal hedging strategy to the second problem, respectively. Also, let $W^u_t$ and $X_t$ be the optimal path of asset value to the first problem, and the
optimal path of internal resources for hedging contributions to the second problem, respectively. Finally, let $Y$ denote the optimal contribution policy to the second problem. Then, we can construct the following portfolio and contribution policy, and path of the pension plan’s asset:

\[
\pi_t = \frac{\pi_t^u - \pi_t^\phi X_t}{W_t^u - X_t} \quad (C.3)
\]

\[
Y_t = Y_t \quad (C.4)
\]

\[
W_t = W_t^u - X_t. \quad (C.5)
\]

We need to prove that these policies are feasible for the original problem. Consider the discounted pension plan’s asset:

\[
e^{-rt}W_t = e^{-rt}W_t^u - e^{-rt}X_t
\]

\[
= W_0 + X_0 + \int_0^t e^{-rs}\pi_s^u \sigma W_s^u dZ_s^Q - X_0 + \int_0^t e^{-rs}Y_s d\bar{s} - \int_0^t e^{-rs}\pi_s^\phi X_s dZ_s^Q
\]

\[
= W_0 + \int_0^t e^{-rs}Y_s d\bar{s} + \int_0^t e^{-rs}\pi_s^u \sigma W_s^u dZ_s^Q
\]

\[
= \mathbb{E}^Q_0 \left[ e^{-rT}W_T - \mathbb{E}^Q_0 \left[ \int_0^T e^{-rs}Y_s d\bar{s} \right] \right],
\]

since $W_T = W_T^u - X_T = W_T^u$. Hence, $(\pi, Y)$ is a admissible portfolio and contribution policy to the original problem. Then, we have

\[
\mathbb{E} \left[ e^{\beta T} u (W_T) \right] \geq \mathbb{E} \left[ e^{\beta T} u (\tilde{W}_T) \right]
\]

\[
\mathbb{E} \left[ \int_0^T e^{-\beta \phi (Y_t)} dt \right] \leq \mathbb{E} \left[ \int_0^T e^{-\beta \phi (\tilde{Y}_t)} dt \right].
\]

Hence, we have a desired inequality. Then, (3.17) is straightforward. \[\square\]

**Proof of Proposition 3.8.** From (3.6), we can easily see that $\mathcal{W}_u(y)$ is decreasing, $\lim_{y \to 0} \mathcal{W}_u(y) = \infty$, and $\lim_{y \to \infty} \mathcal{W}_u(y) = Ke^{-rT}$. Also, from (3.13) we can see that $\mathcal{W}_\phi(y)$ is increasing, $\lim_{y \to 0} \mathcal{W}_\phi(y) = 0$, and $\lim_{y \to \infty} \mathcal{W}_\phi(y) = \infty$. \[\square\]

**Proof of Proposition 3.9.** Suppose that we find $y$ solving (3.18), i.e. the optimal present value of the contribution, $X_0$. Then, we can set the optimal portfolio and contribution policy, and the optimal path of the pension plan’s asset to the
original problem as (C.3), (C.4), (C.5) using solutions to the first and second problems. Then, the optimal portfolio weight is straightforward. Note that by (C.2) the optimal path of the internal resources for hedging contributions is

\[ X_t = \left( \frac{y_t e^r}{k} \right)^{1/\alpha} \frac{1 - e^{-\alpha(T-t)}}{\alpha \phi} = Y_t \frac{1 - e^{-\alpha(T-t)}}{\alpha \phi}. \]

Hence, the optimal contribution rate is

\[ \frac{Y_t}{W_t} = \frac{X_t}{W_t} \frac{\alpha \phi}{1 - e^{-\alpha(T-t)}}. \]

Proof of Proposition 3.10. The first part of (3.6) is the present value of the terminal pension plan’s asset, \( I_u(y_T) \) if \( I_u(y_T) > K \), otherwise zero. Hence, \( W_u(y) \) can be easily computed from that. The first derivative is straightforward. Now, we can express \( W_u(y) \) as

\[ W_u(y) = y^{-1} e^{-\alpha u T} + K e^{-r T} N(-\delta_2(y, T)) - y^{-1} e^{-\alpha u T} N(-\delta_1(y, T)) > W_u^{BC}(y). \]

The last two terms are the present value of \((K - I_u(y_T))^+\), and thus positive. The remaining part can be proved following similar procedures as in Theorem 3.2 and Proposition 3.3.