AAS 14-450

OKID AS A UNIFIED APPROACH TO SYSTEM IDENTIFICATION

Francesco Vicario*, Minh Q. Phan†, Raimondo Betti‡ and Richard W. Longman§

This paper presents a unified approach for the identification of linear state-space models from input-output measurements in the presence of noise. It is based on the established Observer/Kalman filter IDentification (OKID) method of which it proposes a new formulation capable of transforming a stochastic identification problem into a (simpler) deterministic problem, where the Kalman filter corresponding to the unknown system and the unknown noise covariances is identified. The system matrices are then recovered from the identified Kalman filter. The Kalman filter can be identified with any deterministic identification method for linear state-space models, giving rise to numerous new algorithms and establishing the Kalman filter as the unifying bridge from stochastic to deterministic problems in system identification.

INTRODUCTION

System identification as a research topic has attracted a lot of interest over the last decades with applications in many fields. The basic purpose of system identification is to develop a mathematical model of a system for analysis or controller design, and state-space models are particularly suitable since they lend themselves to linear algebra techniques, robust numerical integration, and modern control design methods. Many algorithms have been developed, some of them of deterministic nature, i.e. without considering noise in the measured data, and others stochastic, i.e. with formulations minimizing the noise uncertainty in the identification. Providing a comprehensive review is beyond the scope of the present paper, and the task is complicated by the large number of methods that researchers have devised. One of the most successful identification algorithms for linear state-space models is OKID/ERA (Reference 1), which relies on an observer equation to compress the dynamics of the system and efficiently estimate its Markov Parameters. The latter are then passed to the Eigensystem Realization Algorithm (ERA, Reference 2) or some improved variants of it, e.g. ERA with Data Correlation (ERA/DC, Reference 3), to complete the identification process. The observer at the core of the method was proven to be the steady-state Kalman filter corresponding to the system to be identified and to the covariance of the process and measurement noise. A remarkable result of OKID/ERA is that the method provides simultaneously both the system matrices and the Kalman gain, extracting all the possible information present in the data. Indeed, not only is desirable to identify the system matrices, but also to estimate the covariance of the process and measurement noises so that one can then design the corresponding Kalman filter to estimate the system state and implement a state-feedback control loop. Whereas the measurement noise covariance

*Doctoral Student, Department of Mechanical Engineering, Columbia University, New York, NY 10027, USA.
†Associate Professor, Thayer School of Engineering, Dartmouth College, Hanover, NH 03755, USA.
‡Professor, Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY 10027, USA.
§Professor, Department of Mechanical Engineering and Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY 10027, USA.

can usually be assessed via dedicated experiments, quantifying the process noise is harder, if at all possible. OKID/ERA overcomes the difficulty identifying simultaneously both the system and the Kalman filter from the measured data.

OKID/ERA has been successfully applied for over twenty years, especially in the aerospace community (it was originally distributed by NASA for the identification of lightly-damped structures), and it keeps receiving attention as researchers try to further improve it (e.g Reference 4), apply to linear-time-varying problems (Reference 5) or even to nonlinear systems (e.g. Reference 6). In this paper we show how ERA (or ERA/DC) is not the only method to complete the identification process. Thanks to a novel interpretation of the main OKID result, we prove that it is possible to use a Kalman filter to optimally transform a problem of identification from noisy data into a simpler, noise-free problem. As a result, we propose and demonstrate with numerical examples several new OKID-based identification algorithms optimal in the presence of noise, which can be as many as the number of deterministic identification algorithms that one can find. We establish then the Kalman filter as the bridge from stochastic to deterministic system identification and OKID as a unified optimal approach to handle noisy data in system identification, paralleling the central role that the Kalman filter has in signal estimation. Another, more practical interpretation of the main contribution of this paper is the following: many deterministic identification algorithms are available in the literature and some of them have also proven to be robust to noise, but their formulation does not specifically address the presence of noise in the data, leaving their robustness entirely in the hands of the numerical techniques used in the implementation. Instead we propose a general approach which optimally filters the noise and lets the deterministic algorithms operate on noise-free data, i.e. in the conditions for which they are designed. A well-known alternative to the OKID approach to explicitly and optimally handle noise is given by the family of subspace methods (see for example Reference 7). It is worth noting that several subspace algorithms of deterministic nature have also been formulated, and two of them are chosen not by chance in this paper to identify the Kalman filter within the proposed approach. The choice aims to show how OKID and subspace methods are not necessarily parallel paths but can be combined, with potential for synergy.

The paper is organized as follows. After rigorously formulating the stochastic system identification problem, the OKID core equation is derived, in a slightly different way with respect to Reference 8 in order to better highlight the central role of the Kalman filter in system identification. Then the novel interpretation of the main OKID result is presented and it is shown how to convert the original stochastic problem into an equivalent deterministic form. The resulting new algorithms are outlined and their features are illustrated via a simple numerical example, which provides the ground to present the conceptual contribution of the work. Finally, an example on a 4-degree-of-freedom structure is given to show the method in action on a more realistic system.

PROBLEM STATEMENT

Consider the following linear dynamical system in state-space form

$$x(k+1) = Ax(k) + Bu(k) + w_p(k) \tag{1a}$$

$$y(k) = Cx(k) + Du(k) + w_m(k) \tag{1b}$$

where $x \in \mathbb{R}^{n\times 1}$ is the state vector, $u \in \mathbb{R}^{m\times 1}$ is the input vector, $y \in \mathbb{R}^{q\times 1}$ is the output vector, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, $C \in \mathbb{R}^{q\times n}$ is the output matrix and $D \in \mathbb{R}^{q\times m}$ is the direct influence matrix. Additionally, the vectors $w_p(k) \in \mathbb{R}^{n\times 1}$ and $w_m(k) \in \mathbb{R}^{q\times 1}$ represent the zero-mean white process and measurement noise, with covariance matrices R and Q, respectively. They are uncorrelated with u and y and, for simplicity, mutually uncorrelated.

A single set of length l of input-output data, measured from the system starting at some unknown initial state $x(0)$, is given

$$\{u(k)\} = \{u(0),\ \ u(1),\ \ u(2),\ \ ...,\ \ u(l-1)\} \tag{2a}$$
$$\{y(k)\} = \{y(0),\ \ y(1),\ \ y(2),\ \ ...,\ \ y(l-1)\} \tag{2b}$$

The objective is to identify the system of Eq. (1) from the measured input-output data provided in Eq. (2), i.e. to find the matrices A, B, C, D given the sequences $\{u(k)\}$ and $\{y(k)\}$. The data is assumed to be of sufficient length and richness so that the system of Eq. (1) can be correctly identified. Neither the noise sequences $\{w_p(k)\}$ and $\{w_m(k)\}$ or their covariance matrices R and Q are known.

As mentioned in the introduction, it would be ideal to extract from the measured input-output data also the optimal linear observer of the system state, i.e. the Kalman gain K. It is demonstrated later in the paper how not only is desirable to estimate K, but the identification of the system via the identification of the optimal observer is superior to the direct identification of the system. In other words, the estimation of K can be seen as a valuable by-product of the proposed system identification approach.

NEW APPROACH TO SYSTEM IDENTIFICATION

The new identification strategy consists in two main parts. As in OKID/ERA, we start from an observer equation to derive an expression between the input and the output without the state appearing explicitly. This results in the OKID core equation, whose least-squares (LS) solution establishes that the observer used in the derivation is the optimal observer in the presence of noise (Kalman filter). In contrast to OKID/ERA, the OKID core equation is used to estimate the Kalman filter output residuals. In the second part, we use the estimated residuals to construct a new identification problem with nominally no noise in its formulation. The only source of noise is the estimation error in the observer residuals. The dynamic system to be identified in the new problem is the Kalman filter. From the matrices of the Kalman filter, those of the system can be easily recovered.

Estimation of Observer/Kalman Filter Residuals

Consider the following observer for the system of Eq. (1)

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K\big(y(k) - \hat{y}(k)\big) \tag{3a}$$
$$\hat{y}(k) = C\hat{x}(k) + Du(k) \tag{3b}$$

where $\hat{x}(k) \in \mathbb{R}^{n\times 1}$ and $\hat{y}(k) \in \mathbb{R}^{q\times 1}$ are the observer state and output and K is the observer gain. The observer's role is to estimate the actual system state $x(k)$ from the past input-output measurements. It is then a state estimator or, with the control engineering terminology, an observer.

Define the observer output residuals as

$$\epsilon(k) = y(k) - \hat{y}(k) \tag{4}$$

Plugging Eq. (3b) into Eq. (3a) and Eq. (4) into Eq. (3b), the observer in Eq. (3) can be written in the equivalent form

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + \bar{B}v_x(k) \tag{5a}$$

$$y(k) = C\hat{x}(k) + Du(k) + \epsilon(k) \tag{5b}$$

where

$$\bar{A} = A - KC \tag{6a}$$

$$\bar{B} = \begin{bmatrix} B - KD & K \end{bmatrix} \tag{6b}$$

$$v_x(k) = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \tag{6c}$$

Propagating Eq. (5) forward in time by p time steps and then shifting the time index backward by $p+1$, we obtain

$$\hat{x}(k) = \bar{A}^p\hat{x}(k-p) + Tz(k) \tag{7}$$

where

$$z(k) = \begin{bmatrix} v_x(k-1) \\ v_x(k-2) \\ \vdots \\ v_x(k-p) \end{bmatrix} \tag{8}$$

$$T = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \ldots & \bar{A}^{p-2}\bar{B} & \bar{A}^{p-1}\bar{B} \end{bmatrix} \tag{9}$$

The stability of the observer guarantees that $\bar{A}^p$ becomes negligible for sufficiently large values of p $(p >> n)$. Equation (7) yields then the following relation expressing the current state as a function of the sole past input and output values

$$\hat{x}(k) = Tz(k) \tag{10}$$

Plugging Eq. (10) into Eq. (5b), we obtain the classic OKID equation (Reference 1)

$$y(k) = \bar{Y}v(k) + \epsilon(k) \tag{11}$$

where $\bar{Y}$ and $v(k)$ are augmented versions of T and $z(k)$ to take into account the direct influence of the input on the output through the D matrix, i.e.

$$v(k) = \begin{bmatrix} u(k) \\ v_x(k-1) \\ v_x(k-2) \\ \vdots \\ v_x(k-p) \end{bmatrix} \tag{12}$$

$$\bar{Y} = \begin{bmatrix} D & C\bar{B} & C\bar{A}\bar{B} & \ldots & C\bar{A}^{p-2}\bar{B} & C\bar{A}^{p-1}\bar{B} \end{bmatrix} \tag{13}$$

Equation (11) relates the input and output, without the state appearing explicitly. In the time-series literature it is known as AutoRegressive model with eXogenous input (ARX). Also, note that $\bar{Y}$ contains the sequence of Markov parameters (or unit pulse response) of the observer in the form of Eq. (5). Equation (11) can be written for each time step $k \geq p$ of the measured data record, to obtain the following system of equations in matrix form

$$Y = \bar{Y}V + E \tag{14}$$

where

$$Y = \begin{bmatrix} y(p) & y(p+1) & \dots & y(l-1) \end{bmatrix} \tag{15a}$$
$$V = \begin{bmatrix} v(p) & v(p+1) & \dots & v(l-1) \end{bmatrix} \tag{15b}$$
$$E = \begin{bmatrix} \epsilon(p) & \epsilon(p+1) & \dots & \epsilon(l-1) \end{bmatrix} \tag{15c}$$

Equation (14) is at the core of the OKID approach. Y and V are known (from measurements), $\bar{Y}$ and E are not. By having $l > (m+q)p + m$ and considering E as an error term, it is possible to find the least-squares (LS) solution to Eq. (14)

$$\tilde{Y} = YV^T \left(VV^T\right)^{-1} = YV^\dagger \tag{16}$$

where † denotes the Moore-Penrose pseudoinverse of a matrix. Right-multiplying Eq. (14) by V^T and replacing $\bar{Y}$ with its LS estimate $\tilde{Y}$, we obtain

$$YV^T = YV^T \left(VV^T\right)^{-1} VV^T + EV^T = YV^T + EV^T \tag{17}$$

which implies that $EV^T = 0$. From the definition of $v(k)$, we conclude that

$$\sum_{k=p}^{l-1} \epsilon(k) u^T(k-j) = 0 \qquad j = 0, 1, ..., p \tag{18a}$$
$$\sum_{k=p}^{l-1} \epsilon(k) y^T(k-j) = 0 \qquad j = 1, 2, ..., p \tag{18b}$$

Since the stated assumptions make the process of Eq. (1) stationary, then by the ergodic property we can estimate the ensemble average of each entry of the products between the current residual and the current input or past input and output by their time average over a sufficiently long record. Assuming l is large and dividing Eq. (18) by $l - p$, we recognize the left-hand side as the time average of each entry of $\epsilon(k)u^T(k-j)$ and $\epsilon(k)y^T(k-j)$. The ergodic property brings us to conclude that, for all $k \geq p$,

$$\mathbb{E}\left[\epsilon(k) u(k-j)^T\right] = 0 \qquad j = 0, 1, ..., p \tag{19a}$$
$$\mathbb{E}\left[\epsilon(k) y(k-j)^T\right] = 0 \qquad j = 1, 2, ..., p \tag{19b}$$

The residuals ϵ of the LS problem of Eq. (14) are then orthogonal to the current and past input values and to the past output values. This is the same property that uniquely characterizes the Kalman filter output residuals, which proves that the solution to the LS problem of Eq. (14) provides us with the estimate of the output residuals of the Kalman filter corresponding to the unknown system matrices A, B, C, D and noise statistics R, Q that generated the input-output data $\{u(k)\}, \{y(k)\}$. Among all the possible linear observers, the Kalman filter is optimal in the sense that it minimizes the expected value of the square of the state estimation error $\mathbb{E}\big[\left(x(k) - \hat{x}(k)\right)^T \left(x(k) - \hat{x}(k)\right)\big]$ at each time step k. Under the assumption of stationary noise (constant R and Q) and after a certain number of steps p after which the filter transient has vanished, the optimal gain becomes constant in time and is referred to as steady-state Kalman gain. Given the system matrices A, B, C, D and the noise covariance matrices R, Q, the steady-state Kalman gain K can be computed from the well-known algebraic Riccati equation. The choice of the letter K, usually reserved to the Kalman gain, for the gain of the observer in Eq. (3) is now justified.

As a corollary, $\tilde{Y}$ contains the estimates of the Markov parameters of the Kalman filter. The original OKID/ERA algorithm would compute the Markov parameters of the system from the ones of the Kalman filter and, feeding them to the ERA (or ERA/DC), would find a realization of the matrices A, B, C, D and K as desired. Instead, in the new approach presented in this paper, we focus on the output residuals of the Kalman filter. Thanks to the above proof, their sequence can be estimated from Eq. (14)

$$\tilde{E} = Y - \tilde{Y}V \tag{20}$$

For simplicity of notation, the estimated residuals will be denoted in the rest of the paper simply as $\epsilon(k)$, with no tilde.

Before jumping into the second part of the new identification method, it is worth noting the following remarkable fact. Despite not knowing the system or the noise covariance (both necessary to find the corresponding Kalman filter), we managed to use the equation of the (unknown) Kalman filter to derive a relationship between the measured input and output. Finally, the LS solution to the resulting system of equations confirms that the equation we started from was not just the equation of an observer, but that of the Kalman filter. This is the essence of the OKID approach and allows one to go beyond the identification of the above input-output relationship (large-order ARX model) and find a state-space model of the system.

Identification of the Observer/Kalman Filter

Recalling Eq. (4), Eq. (3) can be written as follows

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K\epsilon(k) \tag{21a}$$
$$\hat{y}(k) = C\hat{x}(k) + Du(k) \tag{21b}$$

which is usually known in the literature as the *innovation form* of the Kalman filter. Equation (21) can also be looked at as the state space model of a dynamic system with u and ϵ as input and $\hat{y}$ as output. Such interpretation is at the basis of the work presented in this paper. Indeed, once an estimate for the time history of the Kalman filter residual ϵ is available, that can be used to obtain via Eq. (4) an estimate for the time history of the Kalman filter output $\hat{y}$ as well. Both the input and the output sequences of the dynamic system in Eq. (21) are then known. Additionally, in Eq. (21) no (unknown) noise term is present. We have just constructed a new noise-free identification problem: given the time histories of $u, \epsilon, \hat{y}$, find the matrices A, B, C, D and K. Thanks to the absence of noise, *any* deterministic identification method can be used to solve the new problem. Note that the solution to the new problem is also the solution to the original problem.

This gives rise to many OKID-based identification algorithms, as many as the deterministic identification methods one can think of. In this paper, to illustrate the effectiveness of the new approach, we demonstrate via examples two possible choices, namely the Deterministic Intersection (DI) and the Deterministic Projection (DP) method. In the literature of subspace methods, several intersection and projection algorithms have been developed (Reference 7). In the examples of this paper, we refer to the DI algorithm of Reference 9 and the DP algorithm of Reference 10. The Matlab®

codes of both algorithms are provided in Reference 7. The DI and DP methods are considered deterministic because their formulation is based on purely deterministic state-space models (with no process or measurement noise). It is however worth noting that, although they do not qualify as stochastic identification algorithms, the numerical techniques used in the implementation of DI and DP (essentially Singular Value Decomposition, SVD) make them very robust to noise and in some specific cases even unbiased (Reference 7). The resulting new OKID-based algorithms are referred to as OKID/DIi and OKID/DPi to remark that the underlying Kalman filter is identified in its innovation form, distinguishing them from the following variant.

An alternative way to complete the identification is given by an equivalent state-space model to describe the dynamics of the Kalman filter. Recalling Eq. (4), Eq. (5) can be rewritten as

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + \bar{B}v_x(k) \tag{22a}$$

$$\hat{y}(k) = C\hat{x}(k) + Du(k) \tag{22b}$$

which in this paper is referred to as the *bar form* of the Kalman filter. Similarly to the innovation form, Eq. (22) represents a dynamic system whose input u and y and output $\hat{y}$ are known. *Any* deterministic identification method can be applied to find a realization of the matrices $\bar{A}$, $\bar{B}$, C, D. Relabeling the matrix blocks in the definition of $\bar{B}$ in Eq. (6) as

$$\bar{B} = \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \end{bmatrix} \tag{23}$$

we can complete the identification recovering K, B and A from Eq. (6) as follows

$$K = \bar{B}_2 \tag{24a}$$

$$B = \bar{B}_1 + KD \tag{24b}$$

$$A = \bar{A} + KC \tag{24c}$$

The new OKID-based identification algorithms proposed in this paper can then be formulated either via the innovation form of the Kalman filter to give OKID/DIi and OKID/DPi, or via the bar form to give OKID/DIb and OKID/DPb. Both alternatives are demonstrated in the examples. It is worth adding that other algorithms based on the new identification strategy can be devised simply by replacing DI and DP by other deterministic methods. For instance, one could use the subspace Algorithm 1 and Algorithm 2 in Reference 7 or the algorithms from the superspace family (References 11, 12, 13).

As a last comment, note that the input to the observer to be identified (in either form) is different in the state and observation equations. More precisely, the state equation has an additional input (ϵ or y), which makes the form of the deterministic identification problem slightly different from the standard form usually considered in the literature, including in the DI and DP algorithms. Two ways to address the issue are possible. One consists in feeding the deterministic identification algorithms with the same additional input in the observation equation as well, relying on the associated coefficients in the corresponding extended D matrix being identified as 0. The other approach is to tailor the deterministic identification algorithms so that they identify the observer taking into account its peculiar form. The required modification is very simple to apply, for example, in the case of the deterministic intersection methods or the superspace algorithms. Numerical experiments show negligible difference between the two approaches when the innovation form is used for the Kalman filter. In the case of bar form, the tailored algorithms tend to provide better results. In the examples given in this paper, for simplicity no modification is adopted.

ALGORITHM

The detailed steps to implement the new method are given below, in a comprehensive algorithm along which the user can choose which form of the Kalman filter to use and whether to implement DI, DP or other deterministic methods for the identification of the observer. The input to the algorithm are the sequences $\{u(k)\}$ and $\{y(k)\}$ of Eq. (2). The output is the set of matrices A, B, C, D and K.

1. Construct the matrices Y and V from Eqs. (15a) and (15b)

2. Compute

$$\begin{aligned}\tilde{Y} &= YV^{\dagger} \\ \begin{bmatrix}\hat{y}(p) & \hat{y}(p+1) & \dots & \hat{y}(l-1)\end{bmatrix} &= \tilde{Y}V \\ \begin{bmatrix}\epsilon(p) & \epsilon(p+1) & \dots & \epsilon(l-1)\end{bmatrix} &= Y - \tilde{Y}V \qquad \text{(for innovation form only)}\end{aligned}$$

<u>Algorithms with Kalman filter in innovation form</u>

3. Define the following input and output sequences

$$\begin{aligned}\{u_i\} &= \left\{ \begin{bmatrix}u(p)\\ \epsilon(p)\end{bmatrix}, \begin{bmatrix}u(p+1)\\ \epsilon(p+1)\end{bmatrix}, \dots, \begin{bmatrix}u(l-1)\\ \epsilon(l-1)\end{bmatrix} \right\} \\ \{y_i\} &= \{\hat{y}(p), \hat{y}(p+1), \dots, \hat{y}(l-1)\}\end{aligned}$$

4. Execute, with input $\{u_i\}$ and output $\{y_i\}$,
   - the DI algorithm for OKID/DIi
   - the DP algorithm
   - any other algorithm for deterministic state-space model identification

   and read the output matrices A_i, B_i, C_i, D_i

5. Extract the desired matrices

$$A = A_i, B = B_i(:, 1:m), K = B_i(:, m+1:m+q), C = C_i, D = D_i(:, 1:m)$$

<u>Algorithms with Kalman filter in bar form</u>

3. Define the following input and output sequences

$$\begin{aligned}\{u_b\} &= \left\{ \begin{bmatrix}u(p)\\ y(p)\end{bmatrix}, \begin{bmatrix}u(p+1)\\ y(p+1)\end{bmatrix}, \dots, \begin{bmatrix}u(l-1)\\ y(l-1)\end{bmatrix} \right\} \\ \{y_b\} &= \{\hat{y}(p), \hat{y}(p+1), \dots, \hat{y}(l-1)\}\end{aligned}$$

4. Execute, with input $\{u_b\}$ and output $\{y_b\}$,
   - the DI algorithm for OKID/DIb

- the DP algorithm for OKID/DPb
- any other algorithm for deterministic state-space model identification

and read the output matrices A_b, B_b, C_b, D_b

5. Extract the desired matrices

$$C = C_b, D = D_b(:, 1:m), K = B_b(:, m{+}1 : m{+}l), B = B_b(:, 1:m){+}KD, A = A_b{+}KC$$

Matlab® notation has been used in step 5 to indicate how to extract A, B, C, D, K from the matrices of the identified Kalman filter.

DEMONSTRATION AND INTERPRETATION

In this section we introduce a small example to demonstrate the above algorithms, discuss their main features and provide an interpretation of the new identification strategy.

Example

Consider the state-space model of Eq. (1) with the following matrices

$$A = \begin{bmatrix} 0 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D = 0 \tag{28}$$

The measured input-output data in Eq. (2) are simulated as follows. First we generate a white input sequence $\{u(k)\}$ of length $l = 10,000$ (from a normal distribution with zero mean and standard deviation of 1) and two zero-mean gaussian noise sequences $\{w_p(k)\}$ and $\{w_m(k)\}$ with covariance

$$Q = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \times 10^{-2} \qquad\qquad R = 4 \times 10^{-2} \tag{29}$$

Said sequences are used to generate $\{y(k)\}$ via Eq. (1). Such $\{u(k)\}$ and $\{y(k)\}$ can also be interpreted as input-output measurements with mutually uncorrelated zero-mean gaussian noise affecting the input-output channels with standard deviation of 0.1 and 0.2, respectively. The resulting signal-to-noise ratio is about 20 dB in both channels.

Estimation of Kalman Residuals

The algorithm starts with the choice of the parameter p. Let us assume that the system order is unknown but we have reason to believe it is small, say less than 5. Let us then choose $p = 20$ and run the first part of the identification method (steps 1 and 2), which is common to all the proposed OKID-based algorithms and leads to the estimation of the Kalman filter output residuals. Figure 1 compares the obtained estimates with the theoretical residuals coming from the Kalman filter of Eq. (3) with the gain computed from the true system and covariance matrices of Eqs. (28) and (29) via the Riccati equation. As part of the properties of the Kalman filter, the residuals are known to be a white process. It is then remarkable how it is possible to accurately estimate the time history of such a random process, as shown in Figure 1.

For the purpose of illustration, steps 3, 4 and 5 are executed for all the four algorithms described above to get the corresponding identified matrices A, B, C, D and K. The parameter i to be set in the DI and DP methods is chosen equal to 5, consistently with the above mentioned a priori belief on

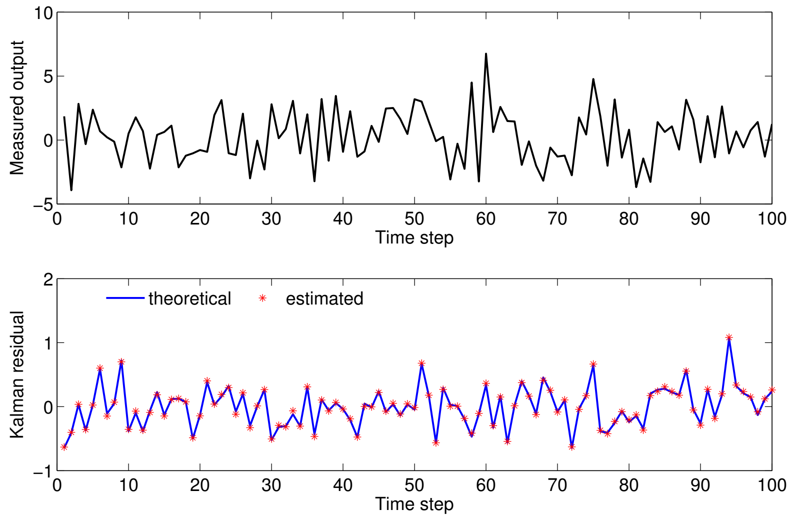


Figure 1: **Estimation of Kalman filter output residuals.**

the system order. All the new algorithms are able to identify the right order ($n = 2$). It is here worth remarking that when noise corrupts the data, it is impossible to get exact identification. This fact generally makes the comparison of different methods and algorithms a difficult task, often addressed via lenghty numerical simulations whose generality is difficult to claim. The task is beyond the scope of this paper. Table 1 reports the eigenvalues of the true A matrix and of the same matrix identified via the new algorithms. The identified values are shown in terms of mean and standard deviation of the results of a Monte Carlo simulation with 100 replications of the same example varying the noise sequences. All of the proposed algorithms provide good identification. None of them outperforms the others and neither form of the Kalman filter seems to provide significantly better results, suggesting the proposed algorithms are all equivalent, at least in the example.

More interestingly, Table 1 shows how the OKID-based algorithms give better identification than the straight application of the corresponding deterministic methods. Running the DI and DP algo-

Table 1: Eigenvalue comparison between true A and corresponding identified matrices (Monte Carlo simulation with 100 replications).

Method	Eigenvalue 1		Eigenvalue 2	
	mean	std. dev.	mean	std. dev.
True	-0.80902	-	0.30902	-
OKID/DIi	-0.80894	0.00097	0.30938	0.00265
OKID/DPi	-0.80896	0.00101	0.30935	0.00267
OKID/DIb	-0.80881	0.00098	0.30933	0.00266
OKID/DPb	-0.80826	0.00227	0.30915	0.00269
DI	-0.80864	0.00098	0.30760	0.00262
DP	-0.81375	0.00102	0.29896	0.00283

rithms directly on the measured $\{u(k)\}$ and $\{y(k)\}$ sequences give in general less accurate results than running them after the estimation of the Kalman filter residuals. This leads to the interpretation of the first part of the OKID approach as a pre-filtering stage. The Kalman filter embedded in the OKID core equation, Eq. (14), provides new input-output signals which are then passed to the second part of the new OKID approach. Such pre-filtering lets the chosen deterministic identification method operate in the conditions for which it was formulated, i.e. with no noise or at least significantly attenuated noise. To emphasize the role played by the pre-filtering stage, Figures 2 and 3 show the plots of the normalized singular values arising from the Singular Value Decomposition (SVD) at the core of the DI and DP methods. Such SVDs are meant to split the zero and non-zero singular values, the number of the latter being the order of the system. If the input-output data are corrupted by noise, no singular value is exactly zero and the user might experience difficulties in deciding which singular values can be considered negligible and be discarded. Figure 2 shows the singular values of the DI and DP algorithms when applied directly, without pre-filtering. The separation line between zero and non-zero singular values is somewhat arguable. The pre-filtering gives rise to clearer plots, where two singular values stand out as being the ones to be considered different from zero, as shown for example for the algorithms based on the bar-form Kalman filter (Figure 3).

Note that the singular values considered to be negligible in the plots of Figure 3 are not exactly 0. Even though pre-filtering makes the order of the system clearly equal to 2, some noise is still present in the data fed to the the second part of the OKID-based algorithms. The OKID core equation relies on the assumption that $\bar{A}^p$ in Eq. (7) is negligible, which is true for sufficiently large p. The approximation resulting from truncating p to a finite value gives then rise to some noise in the estimation of the residuals. Theoretically, increasing p asymptotically leads to no truncation error. In practice, p cannot grow indefinitely for numerical issues (condition number of the matrix to be pseudo-inverted in Eq. (16)) and because that would increase the number of parameters to be estimated and at the same time decrease the number of equations available in the LS problem of Eq. (14), reducing its overdeterminacy.

Residual Whitening

An alternative to the classic OKID equation was proposed in Reference 14. The technique is called *residual whitening* and relies on the following ARMAX (AutoRegressive model with Moving Average and eXogenous input) equation in place of the ARX model of Eq. (11)

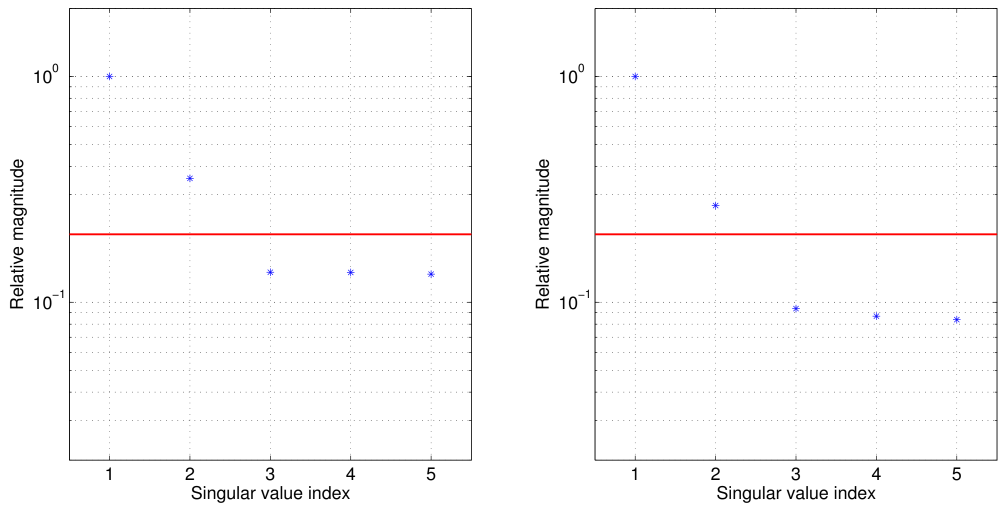
$$y(k) = \Phi v(k) + \Psi\gamma(k) + \epsilon(k) \tag{30}$$

where Φ and Ψ contain products of the system matrices, the deadbeat observer gain and the Kalman gain (Reference 14), $v(k)$ is defined as in Eq. (12) and γ is given by

$$\gamma(k) = \begin{bmatrix} \epsilon(k-1) \\ \epsilon(k-2) \\ \vdots \\ \epsilon(k-p) \end{bmatrix} \tag{31}$$

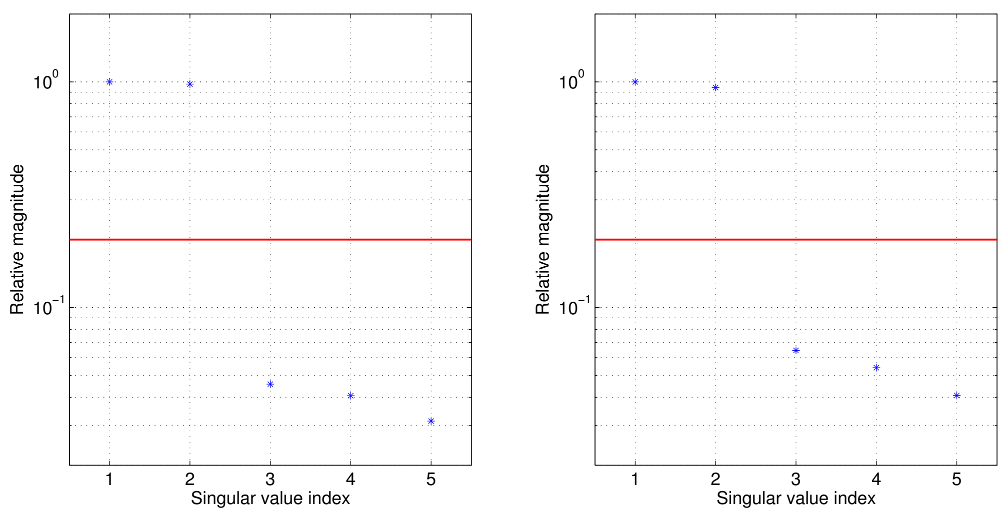
The resulting set of equations is, in matrix form,

$$Y = \Phi V + \Psi W + E \tag{32}$$



(a) **DI algorithm.** (b) **DP algorithm.**

Figure 2: **SVD of deterministic algorithms.**



(c) **OKID/DIb algorithm.** (d) **OKID/DPb algorithm.**

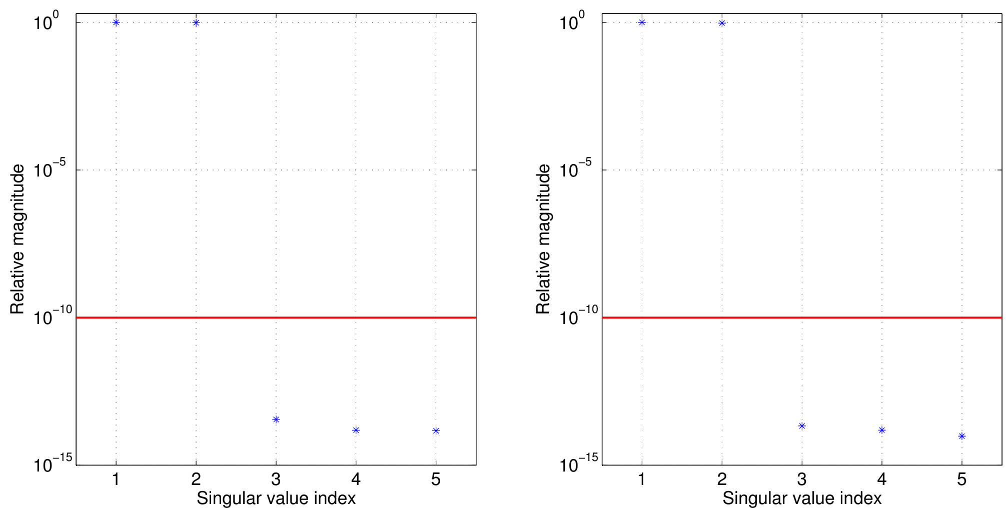
Figure 3: **SVD of OKID algorithms based on Kalman filter bar form.**

where, similarly to Y, V and E in Eq. (15), the matrix W is defined as

$$W = \begin{bmatrix} \gamma(p) & \gamma(p+1) & \dots & \gamma(l-1) \end{bmatrix} \tag{33}$$

The LS solution to Eq. (32) yields residuals that are orthogonal not only to the current input and past input and output, but also to the past and future residuals. The latter implies that the estimated residuals are explicitly forced by the LS solution to be white, from which the name of the technique.

The goal in Reference 14 was to limit the value of p in problems where the data record was



(a) **OKID/DIb algorithm.** (b) **OKID/DPb algorithm.**

Figure 4: **SVD of OKID algorithms based on Kalman filter bar form with residual whitening.**

Table 2: Eigenvalue of A matrix identified by different algorithms with residual whitening.

Method	Eigenvalue 1	Eigenvalue 2
OKID/DIi	-0.8109258705999016	0.3077103118249301
OKID/DPi	-0.8109258705999016	0.3077103118249320
OKID/DIb	-0.8109258705999021	0.3077103118249284
OKID/DPb	-0.8109258705999010	0.3077103118249288

relatively short, in order to maintain a high ratio between the number of equations ($l - p$) and the number of parameters (proportional to p) in the LS problem to be solved for the estimation of the Markov parameters to be fed to ERA. The value of p to make Eq. (32) hold without any approximation just needs to equal to n or larger. On the other side, since W is initially unknown, Eq. (32) must be solved iteratively, updating W with the residuals E estimated at each iteration by LS. The procedure is known as Generalized Least Squares (GLS, References 14, 15). As a conclusion, the residual whitening technique can also be interpreted as trading the truncation error of the classic OKID equation with the iteration error of the GLS procedure. As opposed to the former, the latter can be made as small as desired just by running more iterations. We can then think of residual whitening as a technique to improve the estimation of the residuals and make the pre-filtering exact, i.e. yielding a noise-free set of data to be fed to the DI or DP method. To show the concept, we estimate the Kalman residuals by residual whitening with $p = 2$ and execute the steps 3 to 5 for all the proposed algorithms getting the SVD plots of Figure 4 and the eigenvalues of the identified A matrix summarized in Table 2. The zero singular values are now really such, since they are close to the working precision of Matlab®. Thanks to residual whitening, 15 order of magnitude separate the zero and non-zero singular values, making the selection of the right order of the system crystal clear. In conclusion, residual whitening with $p = n$ makes the pre-filtering exact and the SVD shows no trace of noise. As a consequence, the DI and DP methods are run on noise-free data and all the proposed algorithms provide the same identified matrices, as shown by the eigenvalues of the identified A. The numerical values in Table 2 differ only after the 15$^{\text{th}}$ significant digit.

The first part of the OKID-based algorithms can now be interpreted as a conversion of the original stochastic identification problem of Eq. (1) into the deterministic problem of Eq. (21) or Eq. (22). The new approach consists then in converting the original problem, whose data are corrupted by noise, into a simpler noise-free problem which can be solved by any deterministic identification method. When the first part is solved approximately (e.g. due to truncation error in the classic OKID equation, limited number of iterations in the residual whitening technique, violation of the initial assumptions on the process and measurement noise), the error in the residual estimates makes the conversion not exact and the new identification problem is not completely noise-free, yet the noise is significantly reduced (pre-filtering).

For the sake of clarity, we used residual whitening in the example above to highlight the fact that the exact LS solution to the OKID equation would lead to completely noise-free identification of the observer. Residual whitening can generally be used to improve the estimates of the observer residuals, but it must be kept in mind that the LS problem of Eq. (32) is non-linear (both Ψ and W are unknowns) and the convergence of the GLS procedure to the (global) minimum is not always guaranteed. When the procedure converges, then it is a powerful technique to refine the identification.

As a last note, it is worth clarifying that it is impossible to estimate exactly the theoretical Kalman residuals from a finite-length record. This is due to the stochastic nature of the noise in the identification problem of Eq. (1). Even in numerical simulations, since the process and measurement noises are random, an infinitely-long record would be necessary to make them really satisfy the problem assumptions (in particular their whiteness). The consideration is mainly of academic interest, since in real applications the noises can be of diverse nature and to some extent always violate the problem assumptions. It is however important to realize that the residuals given by residual whitening are exact in the sense that they correspond to the linear observer minimizing exactly the OKID equation with no truncation error. However, the finiteness of the record prevents the minimizing observer from being exactly the theoretical Kalman filter.

EXAMPLE

As a more realistic example, consider the lumped model of a 4-story building, shown in Figure 5, with each mass equal to $m = 0.259$ and each lateral spring of stiffness $k = 122.889$. The building is also supposed to have viscous damping, quantified by a damping factor of $\zeta = 0.01$ for each of the 4 vibration modes. The force is applied in correspondence of the third floor via a zero-order-hold (ZOH) system with sampling time of 0.01s. The excitation used in the example is a white signal normally distributed with zero mean and standard deviation of 1 and duration of 100s ($l = 10,000$). The input channel is affected by gaussian noise of standard deviation equal to 0.15, for a signal-to-noise ratio of about 16 dB. The lateral acceleration at each floor is measured, for a total of 4 outputs. Gaussian noise of standard deviation of 1 is present in each output channel, resulting in signal-to-noise ratios of about 13 db, 26 dB, 52 dB and 50 dB (from the ground up). The discrete-time state-space model of the structure of Figure 5 is therefore in the form of Eq. (1), with $n = 8$, $m = 1$ and $q = 4$. The process noise w_p is due to the noise in the input channel. Its covariance matrix is then $Q = BB^T$, whereas the covariance of the measurement noise w_m is the identity matrix.

All the four variants of the algorithm described above are executed, with $p = 40$ in the OKID equation and $i = 20$ for the DI and DP algorithms. The identification results are reported in Table 3, in the form of natural frequencies and damping factors, together with the true values as well as those obtained via the direct application of the DI and DP methods and via traditional OKID/ERA algorithm.

The algorithms based on the innovation form of the Kalman filter perform sensibly better then their bar-form counterparts, in particular for OKID/DP. The accuracy of OKID/DIi and OKID/DPi is in line with if not better than OKID/ERA. Very significant is the fact that the OKID pre-filtering makes OKID/DIi and OKID/DPi generally perform better than DI and DP, as expected from the theoretical framework previously presented. This confirms the benefit of pre-filtering the data via the OKID equation, making the DI and DP algorithms work in conditions closer to the ones for which they are formulated. For completeness, the SVD plots of some of the algorithms in Table 3 are reported in Figures 6, 7 and 8. The advantage of OKID is evident with the DI method, leading to the correct identification of 4 vibration modes (Figure 7a), whereas without OKID pre-filtering

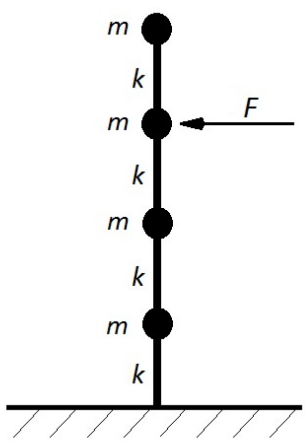
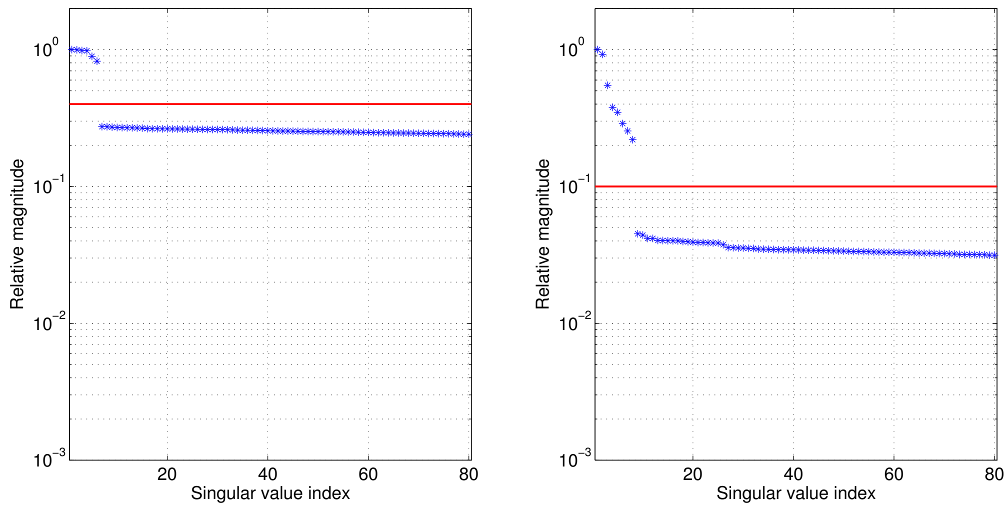


Figure 5: **Lumped model of 4-story building.**

Table 3: Identified natural frequencies (Hz) and damping factors of the structure of Fig. 5 (Monte Carlo simulation, average over 100 replications).

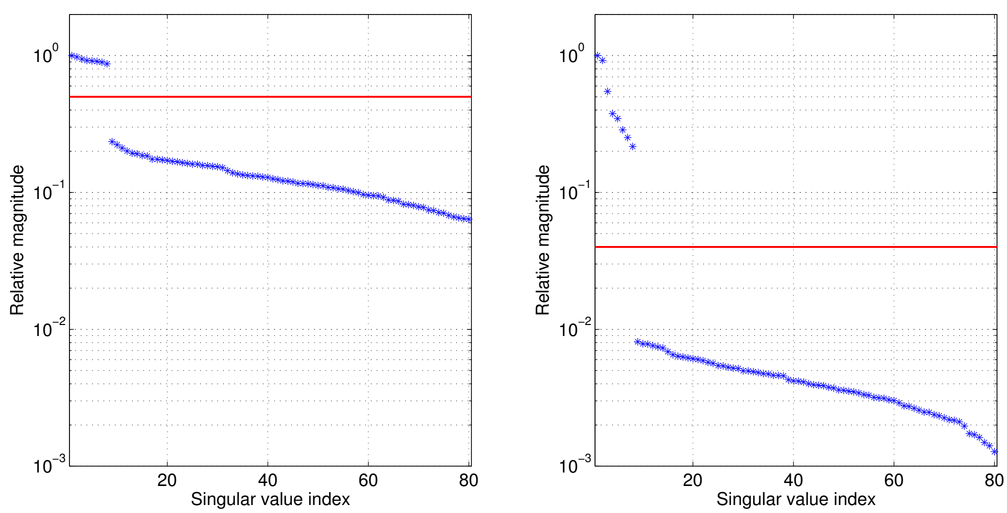
Method	f_1	ζ_1	f_2	ζ_2	f_3	ζ_3	f_4	ζ_4
True	1.3948	0.0100	3.9721	0.0100	5.9447	0.0100	7.0122	0.0100
OKID/DIi	1.3955	0.0109	3.9725	0.0109	5.9450	0.0104	7.0122	0.0100
OKID/DPi	1.3958	0.0105	3.9730	0.0106	5.9451	0.0103	7.0121	0.0100
OKID/DIb	1.3979	0.0154	3.9718	0.0129	5.9458	0.0121	7.0126	0.0120
OKID/DPb	1.3653	0.0013	3.9991	0.0131	5.9499	0.0140	7.0197	0.0120
DI	1.3892	0.0120	3.9677	0.0106	5.9429	0.0105	7.0111	0.0104
DP	1.3951	0.0098	3.9754	0.0097	5.9453	0.0098	7.0129	0.0098
OKID/ERA	1.3956	0.0120	3.9732	0.0120	5.9458	0.0111	7.0125	0.0100



(a) **DI algorithm.**

(b) **DP algorithm.**

Figure 6: **SVD of deterministic algorithms.**



(a) **OKID/DIi algorithm.**

(b) **OKID/DPi algorithm.**

Figure 7: **SVD of OKID algorithms based on Kalman filter innovation form.**

the first mode is missed (Figure 6a). Its value in Table 3 is reported by forcing the selection of 8 non-zero singular values in SVD plots like Figure 6a. In the DP case, the advantage of OKID shows up in pushing the negligible singular values down towards 0 (Figures 6b and 7b). A similar gap between the zero and non-zero singular values characterizes the SVD plot of OKID/ERA, where the pre-filtering action is performed by the same OKID core equation.

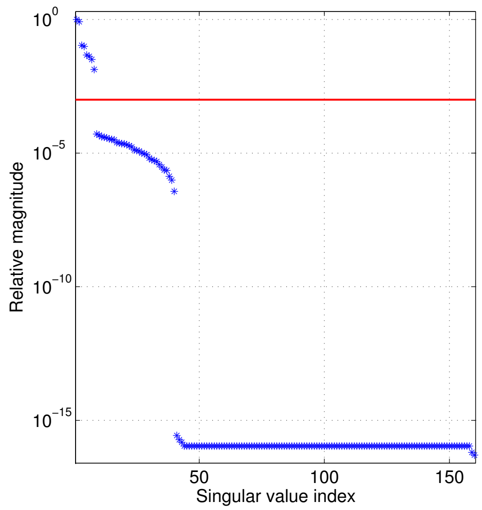


Figure 8: **SVD of ERA/DC in OKID/ERA.**

CONCLUSIONS

This paper presented a new identification strategy for state-space model identification of linear dynamic systems from data corrupted by noise. The approach is based on two main steps: first the estimation of the output residuals of the optimal observer (Kalman filter) for the system and the (unknown) noise statistics, and then the identification of the Kalman filter by solving a new, simpler problem. The key feature of the latter is that of being noise-free, which makes any deterministic identification algorithm suitable for its solution. The first part of the method is similar to the first part of the well-known OKID/ERA algorithm, leading to a generalization of the OKID approach where ERA (or ERA/DC) is not the only algorithm that can complete the identification process. The fact that any deterministic method can be applied establishes OKID as a unified approach to system identification. Via numerical examples, four algorithms resulting from the new approach have been demonstrated and shown to significantly improve the identification with respect to the direct application of the corresponding deterministic methods. The intuitive interpretation of OKID as optimal pre-filtering or as conversion from stochastic to deterministic identification is given and supported by the examples.

The new approach does not only explicitly give a central role to the Kalman filter in system identification, paralleling the one it has in signal estimation, or generate a large number of new algorithms, paving the way to potential improvement over the existing linear stochastic identification methods. It also provides a new general framework that can be applied to address non-linear identification problems, a first example of which is presented in Reference 16 on bilinear systems. More interesting results in the field of system identification are expected.

REFERENCES

[1] J.-N. Juang, M. Q. Phan, L. G. Horta, and R. W. Longman, "Identification of Observer/Kalman Filter Markov Parameters: Theory and Experiments," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 2, 1993, pp. 320–329.

[2] J.-N. Juang and R. S. Pappa, "An Eigensystem Realization Algorithm for Modal Parameter Identification and Model Reduction," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 5, 1985, pp. 620–627.

[3] J.-N. Juang, J. E. Cooper, and J. R. Wright, "An Eigensystem Realization Algorithm Using Data Correlations (ERA/DC) for Modal Parameter Identification," *Journal of Control Theory and Advanced Technology*, Vol. 4, No. 1, 1988, pp. 5–14.

[4] J. Aitken and T. Clarke, "Observer/Kalman Filter Identification With Wavelets," *IEEE Transactions on Signal Processing*, Vol. 60, No. 7, 2012, pp. 3476–3485.

[5] M. Majji, J.-N. Juang, and J. L. Junkins, "Observer/Kalman-Filter Time-Varying System Identification," *Journal of Guidance, Control, and Dynamics*, Vol. 33, No. 3, 2010, pp. 887–900.

[6] T. Clarke and X.-D. Sun, "Minimal State-Space Model Realisation of a Nonlinear Helicopter," *Control Theory and Applications, IEE Proceedings*, Vol. 145, No. 4, 1998, pp. 415–422.

[7] P. Van Overschee and B. Moor, *Subspace Identification for Linear Systems: Theory, Implementation, Applications*. Dordrecht: Kluwer Academic Publishers, 1996.

[8] M. Q. Phan, L. G. Horta, J.-N. Juang, and R. W. Longman, "Linear System Identification via an Asymptotically Stable Observer," *Journal of Optimization Theory and Applications*, Vol. 79, No. 1, 1993, pp. 59–86.

[9] M. Moonen, B. De Moor, L. Vandenberghe, and J. Vanderwalle, "On- and Off-Line Identification of Linear State Space Models," *International Journal of Control*, Vol. 49, No. 1, 1989, pp. 219–232.

[10] B. De Moor and J. Vanderwalle, "A Geometrical Strategy for the Identification of State Space Models of Linear Multivariable Systems with Singular Value Decomposition," *Proceedings of the 3rd International Symposium on Applications of Multivariable System Techniques*, Plymouth, UK, 1987, pp. 59–69.

[11] M. Q. Phan, J. A. Solbeck, and L. R. Ray, "A Direct Method For State-Space Model and Observer/Kalman Filter Gain Identification," *Proceedings of the AIAA Guidance, Navigation, and Control Conference and Exhibit*, Providence, Rhode Island, 2004.

[12] P. Lin, M. Q. Phan, and S. A. Ketcham, "State-Space Model and Kalman Filter Gain Identification by a Superspace Method," *Proceedings of the 5th International Conference on Modeling, Simulation and Optimization of Complex Processes*, Hanoi, Vietnam, 2012.

[13] M. Q. Phan and H. Celik, "A Superspace Method for Discrete-Time Bilinear Model Identification by Interaction Matrices," *Advances in the Astronautical Sciences*, Vol. 139, 2011, pp. 445–464.

[14] M. Q. Phan, L. G. Horta, J.-N. Juang, and R. W. Longman, "Improvement of Observer/Kalman Filter Identification (OKID) by Residual Whitening," *Journal of Vibrations and Acoustics*, Vol. 117, 1995, pp. 232–238.

[15] T. C. Hsia, "On Least Squares Algorithms for System Parameter Identification," *IEEE Transactions on Automatic Control*, Vol. 21, 1976, pp. 104–108.

[16] F. Vicario, M. Q. Phan, R. Betti, and R. W. Longman, "Bilinear Observer/Kalman Filter Identification," *24th AAS/AIAA Space Flight Mechanics Meeting*, Santa Fe, NM, 2013, AAS14-307.