ENLARGEMENT OF FILTRATION AND THE STRICT LOCAL MARTINGALE PROPERTY IN STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

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In this thesis, we study the strict local martingale property of solutions of various types of stochastic differential equations and the effect of an initial expansion of the filtration on this property. For the models we consider, we either use existing criteria or, in the case where the stochastic differential equation has jumps, develop new criteria that can can detect the presence of the strict local martingale property. We develop deterministic sufficient conditions on the drift and diffusion coefficient of the stochastic process such that an enlargement by initial expansion of the filtration can produce a strict local martingale from a true martingale. We also develop a way of characterizing the martingale property in stochastic volatility models where the local martingale has a general diffusion coefficient, of the form $\mu(S_t, v_t)$,where the function $\mu(x_1, x_2)$ is locally Lipschitz in x.

Contents

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Introduction

In this thesis, we devote ourselves to studying mechanisms by which strict local martingales can arise from martingales. A strict local martingale is a local martingale which is not a martingale. We will study how expanding the original filtration with respect to which a process is a martingale can lead to a strict local martingale. That is, if we begin with a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F}$ denotes $(\mathcal{F}_t)_{t\geq 0}$, and with an $\mathbb{F}$ martingale $M = (M_t)_{t\geq 0}$, and consider an expanded filtration $\mathbb{G}$ such that, for all t we have the inclusion $\mathcal{F}_t \subset \mathcal{G}_t$, when can we obtain a filtration $\mathbb{G}$ such that M becomes a strict local martingale, possibly under a different but equivalent probability measure Q?

Strict local martingales have recently been a popular subject of study. Some relatively recent papers concerning strict local martingales include Biagini et al [3], Bilina-Protter [4], Chybiryakov [7], Cox-Hobson [8], Delbaen-Schachermayer [9], Föllmer-Protter [13], Lions-Musiela [29], Hulley [17], Keller-Ressel [24], Klebaner-Liptser [26], Kreher-Nikeghbali [27], Larsson [28], Madan-Yor [31], Mijatovic-Urusov [32], Protter [37], Protter-Shimbo [38], and Sin [40], and from this list we can infer a certain interest. Our motivation comes from the analysis of financial bubbles, as explained in [37], for example. The theory tells us that on a compact time set, the (nonnegative) price process of a risky asset is in a bubble, i.e., undergoing speculative pricing, if and only if the price process is a strict local martingale under the risk neutral measure governing the situation. Therefore one can model the formation of bubbles by observing when the price process changes from being a martingale to being a strict local martingale. This is discussed in detail in [22], [3], and [37], for example.

The models on which we perform an initial expansion of the filtration are stochastic volatility models. We work with the settings examined in Lions and Musiela [29], Mijatovic-Urusov [32] and the case in which the martingale driving the diffusion has jumps. In these cases, we assume always that a component of the stochastic volatility process is an Itô diffusion, so that we can use Feller's test for explosions in our quest to characterize the stochastic processes in question.

An expansion of the filtration using initial expansion involves adding the information encoded in a random variable to the original σ algebra at time zero. This augmentation doesn't have to happen at time zero, however; it can happen at any stopping time. This random variable will be denoted L. We will see that this type of enlargement of filtration from $\mathbb{F}$ to $\mathbb{G}$ changes the risk-neutral measure from P to Q and our stochastic process, that we will call S and which is assumed to be a $(P, \mathbb{F})$

martingale, becomes a strict local martingale under $(Q, \mathbb{G})$ on a stochastic interval that depends on the model and the random variable that we add to $\mathbb{F}$.

The case of initial expansions is particularly tractable, since Jacod [20] has developed the theory that provides us with the dynamics of the process under the enlarged filtration. (See [36, Chapter VI] for a pedagogic exposition.) That is, he provides us with the semimartingale decomposition of the process in the enlarged filtration, which permits us to detect the presence of the strict local martingale property of the process, or lack thereof.

In Chapter 1 we begin with an introduction to the paper of Delbaen and Shirakawa, which treats the one-dimensional case of a local martingale. After this, we reintroduce the models of P.L. Lions and M. Musiela, Mijatovic and Urusov, and Andersen and Piterbarg on stochastic volatility (in the style of what are known as Heston-type models). For each of these models, we conduct a detailed analysis of the techniques used to characterize the martingale property. In Chapter 2 we show how the addition of more information via an “expansion of the filtration” can lead what was originally a martingale to become a strict local martingale, under a risk neutral measure chosen from the infinite selection available in an incomplete market. We do this first for the models of Lions and Musiela, Mijatovic and Urusov. Then we drop the hypothesis of continuous paths and extend our results to the case of discontinuous martingales replacing Brownian motions. This of course requires deriving necessary and sufficient conditions such that the martingale property is satisfied in these models with jumps. We display deterministic sufficient conditions such that we begin with a true martingale, perform an expansion of the filtration via initial expansions, and end up with a strict local martingale for each of these paradigms. In Chapter 3, we examine the multifaceted subject of stochastic stability of systems of stochastic differential equations. We also encounter sufficient conditions for explosion and for non-explosion of these systems of stochastic differential equations, and note well that Feller's test for explosion does not apply here, as such a test only works for one-dimensional diffusions. We see that Lyapunov functions play a crucial role in the study of stochastic stability. We proceed to study how to characterize the martingale property of solutions of stochastic differential equations where the diffusion coefficient of the local martingale is of a general form, namely, $\mu(S_t, v_t)$.

Chapter 1

0.1 The One-Dimensional Case

Let us begin a study of the martingale property of solutions of stochastic differential equations of various types. First, we deal with the simplest kind: we assume that we are on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0<t<T}$ and that B is an $\mathbb{F}$ Brownian motion. Let S be an adapted process that satisfies the following stochastic differential equation:

$$dS_t = \sigma(S_t)dB_t; \quad S(0) = S_0 \tag{1}$$

F. Delbaen and H. Shirakawa, in 2002, showed that we can verify whether or not the martingale property of the solution of (1) fails. They found deterministic necessary and sufficient conditions on the diffusion coefficient σ of S that can tell us whether or not S is a martingale. We note immediately that the solution S satisfies the Markov property as well as weak uniqueness.

Before we begin our discussion, let us state two very important theorems, namely, Feller's test for explosions, which gives us necessary and sufficient conditions such that the solution of a one-dimensional stochastic differential equation explodes, and the Comparison Theorem, which allows us to compare values of solutions of different stochastic differential equations.

Let X be a diffusion in $\mathbb{R}$ solving the SDE

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt; \quad X_0 = x.$$

Denote by X^x the solution starting from x. Let $D_n = (-n, n)$ for $n = 1, 2, 3, 4...$ and let $\tau_n = \inf\{t : X_t \in D_n\}$. Since X is a continuous process, we have that τ_n is a non-decreasing sequence of $\mathbb{F}$ stopping times that converge to τ_∞, which is called the *explosion time* of X. We say that the diffusion X *explodes* if $P_x(\tau_\infty<\infty)>0$. Feller's test for explosions is the following:

Theorem 1 (Feller's test for explosions)**.** *Suppose μ and σ are locally bounded and that $\sigma(x)$ is continuous and that $\sigma(x)>0$. Then, the diffusion X explodes if and only if there exists an x_0 such that one of the following two conditions is satisfied:*

1. $\int_{-\infty}^{x_0} e^{\int_{x_0}^{x} \frac{2\mu(s)}{\sigma^2(s)}ds}(\int_{x}^{x_0} \frac{e^{\int_{x_0}^{y} \frac{2\mu(s)}{\sigma^2(s)}ds}}{\sigma^2(y)}dy)dx<\infty$
2. $\int_{x_0}^{\infty} e^{\int_{x_0}^{x} \frac{2\mu(s)}{\sigma^2(s)}ds}(\int_{x}^{x_0} \frac{e^{\int_{x_0}^{y} \frac{2\mu(s)}{\sigma^2(s)}ds}}{\sigma^2(y)}dy)dx<\infty$

See [23] for a proof.

Below, we state the comparison theorem, as it appears in [18]:

Theorem 2 (Comparison Theorem)**.** *Assume the existence of:*

1. *A real-valued function* $\sigma : [0, \infty) \times \mathbb{R}$ *such that:*

$$|\sigma(t, x) - \sigma(t, y)| \leq \psi(|x - y|); x, y \in \mathbb{R}, t \geq 0$$

   Where ψ *is an increasing function that satisfies*

$$\int_0^\infty \frac{1}{\psi^2(u)} du = \infty \tag{2}$$

2. *Real valued functions* $b^1(t, x)$ *and* $b^2(t, x)$ *on* $[0, \infty) \times \mathbb{R}$ *such that* $b^1(t, x) < b^2(t, x)$ *for* $t \geq 0$, $x \in \mathbb{R}$.

Assume that we are on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. *Assume that we have two adapted, measurable processes* $\beta^1(t, \omega)$ *and* $\beta^2(t, \omega)$ *that satisfy* $\beta^1(t) \leq b^1(t, x)$ *for all* $t \geq 0$, *and* $\beta^2(t) \geq b^2(t, x)$ *as well as two adapted processes that satisfy:*

$$\begin{aligned} x_t^i - x^i(0) &= \int_0^t \sigma(s, x^i(s)) dB_s + \int_0^t \beta_s^i ds; i = 1, 2. \\ & x^1(0) \leq x^2(0) \end{aligned}$$

Then, almost surely, we have

$$x^1(t) \leq x^2(t); t \geq 0$$

Now we may begin the foray into the subject of the martingale property in a simple stochastic differential equation. What follows now is a summary of the work done in [10].

Let us assume that we have a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0<t<T}$. and an $\mathbb{F}$ Brownian motion B. We will make some assumptions about the function $\sigma : \forall\epsilon>0,\ M<\infty$,

$$\sup_{\epsilon\leq u\leq M} \sigma(u) \geq \inf_{\epsilon\leq u\leq M} \sigma(u)>0.$$

We also assume that $\sigma(x) = 0$ for all $x<0$. This implies that the process S remains at zero after reaching it.

Let us discuss the weak solution of equation (1). First, consider a Brownian Motion B such that $B_0 = 1$. Define $T_0 = \inf\{t : B_t = 0\}$, the hitting time of 0. The increasing process A is defined as $A = \int_0^t \frac{1}{\sigma^2(B_u)} du$. The assumptions we have made on the function σ as well as the continuity of the Brownian motion, it follows that for $t < T_0$ we have that $A_t<\infty$. Define the inverse function of A, $A_t^{-1} = C_t = \inf\{s : A_s \geq t\}$. Define the process, for $t \leq A_{T_0}$ $Y_t = B_{C_t}$. Y solves, for $t \leq T_0$,

$$dY_t = \sigma(Y_t)dB_t^{'}$$

for some Brownian motion $B^{'}$, on a possibly enlarged space.

We have that C solves

$$dC_t = \sigma(Y_t)dt$$

From [10], we have the following characterization of the process S : It reaches the point zero in finite time if and only if $A_{T_0}<\infty$. Our next step will be to find conditions under which $A_{T_0}<\infty$ holds. Before we begin displaying these conditions, we recall the definition of a Bessel-Squared process: Denote by $|B_t|$ the norm of n- dimensional Brownian motion B. More precisely, let $\mathcal{F}_t = \sigma(B_s \colon s \leq t)$ be its natural filtration, and define the n-dimensional Bessel-Squared process, to be: $X \equiv \|B\|^2$.

This will be a good time to state the Ray-Knight Theorem:

Theorem 3 (Ray-Knight)**.** *Let B be a standard Brownian motion and let a be its family of local times. Let T_1 be the first time B hits* 1*. Then, process $Z_a, 0 \leq a \leq 1$ defined as $Z_a = l_{T_1}^{1-a}$ is a two-dimensional Bessel-squared process.*

Remember that we have assumed that our Brownian motion starts at 1. We would like to apply the theorem of Ray-Knight theorem to the process $W = 1 - B$. W hits 1 only if B hits 0. We have that the local time of W at $1 - a$ is just the local time L_a of B at a. We may now state the following theorem:

Theorem 4 (Ray-Knight)**.** *The process $L_{T_0}^a$, $0 \geq a \leq 1$, has the same law as the two-dimensional Bessel-squared process.*

Recall that, for any non-negative Borel function f, the following condition holds:

$$\int_0^{T_0} f(B_u)du = \int_0^{\infty} f(x)L_{T_0}^x dx$$

Let us apply this to the increasing process A. This gives us

$$A_{T_0} = \int_0^{\infty} \frac{1}{\sigma^2(x)} L_{T_0}^x dx = \int_0^1 \frac{1}{\sigma^2(x)} L_{T_0}^x dx + \int_1^{\infty} \frac{1}{\sigma^2(x)} L_{T_0}^x dx$$

The second term in this equation is certainly finite, and we arrive at the following condition: $A_{T_0} < \infty$ if and only if $\int_0^1 \frac{1}{\sigma^2(x)} L_{T_0}^x dx < \infty$.

By the Ray-Knight theorem, we may replace the local time L_{T_0} by a two-dimensional Bessel-Squared process: Let B^1 and B^2 be two independent standard Brownian motions defined on a probability space. Thus, $A_{T_0} < \infty$ if and only if

$$\int_0^1 \frac{1}{\sigma^2(x)} ((B_x^1)^2 + (B_x^2)^2) dx < \infty$$

almost surely.

Since these two Brownian motions are independent, we have that $A_{T_0} < \infty$ if and only if

$$\int_0^1 \frac{1}{\sigma^2(x)} (B_x)^2 dx < \infty$$

We have arrived at the following theorem:

Theorem 5 (Theorem 1.4 in [10])**.** *Let the process S solve* (1)*. Assume that the* *function σ satisfies: $\forall\epsilon>0$, $M<\infty$. Then,*

- *Either $\int_0^t \frac{x}{\sigma^2(x)}dx = \infty$ and S doesn't reach 0 in finite time*
- *Or $\int_0^t \frac{x}{\sigma^2(x)}dx<\infty$ and S reaches the origin in finite time almost surely.*

Note that, by the Markov Property, we have that, for any $\tau>0$, that $P(S_\tau = 0)>0$. It is now time to answer the question as to when S is a martingale. If S is a true martingale then, for all t, we have that the measure defined by

$$dQ = S_t dP$$

is a true probability measure. If this is the case, we have that $Q[S_u = 0] = 0$. for all $u \leq t$. Under the measure Q, then, the process S does not hit 0. Let us use the formula of Itô and Girsanov-Maruyama to compute the dynamics of S and $\frac{1}{S}$ under the measure Q : Under Q, the process S satisfies

$$S_t = S_0 + \int_0^t \sigma(S_s)dB_s + \int_0^t \frac{1}{S}\sigma^2(S_s)ds$$

Under the measure Q, the dynamics of the process $X = \frac{1}{S}$ are

$$dX_u = X_u\sigma(\frac{1}{X_u})dB_u$$

for $u \leq t$.

The observation of the fact that the process $\frac{1}{S}$ does not hit zero under Q leads us to conclude that the following condition **must** hold:

$$\int_0^1 \frac{y}{y^4\sigma^2(\frac{1}{y})}dy = \infty \tag{3}$$

We now turn to characterizing the strict local martingale property in certain stochastic volatility models. We will see that the subject of explosions continues to play a crucial role in the characterization of the martingale property in these models.

But, of course, by a change of variables, this is equivalent to requiring that

$$\int_0^\infty \frac{x}{\sigma^2(x)}dx = \infty$$

It turns out that this necessary condition is also sufficient to ensure that S is a true martingale. We shall not include the entire proof of the sufficiency here, but its essence is the following: For all n and fixed t, define the sequence of stopping times $T_n := \inf\{u : S_u \geq n\} \wedge t$. The stopped process $S_t^{T_n}$ is a true martingale, and we have $E[S_t^{\tau_n}] = S_0$. One can then show that the family of random variables $\{S_t^{\tau_n}\}_n$ is uniformly integrable, which allows us to conclude that, for all t, $E[S_t] = \lim_{n\to\infty} E[S_t^{\tau_n}] = S_0$.

Of course, the interested reader can consult [10] for the full proof.

Before we continue, let us point out that we can relate the condition (3) to Feller's test for explosions. Note that the process $\frac{1}{S}$ hits zero under Q if and only if the process S hits ∞, or explodes under Q. We have worked out the dynamics of S under Q : Its diffusion coefficient is $\sigma(x)$ and its drift coefficient is $\frac{\sigma^2(x)}{x}$.

One very easily can check that the condition (3) is sufficient to ensure that, with $\mu(x) = \frac{\sigma^2(x)}{x}$, $\int_{-\infty}^{1} \mathrm{e}^{\int_{x_0}^{x} \frac{2\mu(s)}{\sigma^2(s)}ds} (\int_{x}^{x_0} \frac{\mathrm{e}^{\int_{x_0}^{y} \frac{2\mu(s)}{\sigma^2(s)}ds}}{\sigma^2(y)} dy) dx < \infty$, ensuring that S explodes under the measure Q.

0.2 Stochastic Volatility Models

Let us begin our discussion of more complicated models, namely stochastic volatility models, with the work of Carlos Sin as it appears in [40]. In stochastic volatility models, the volatility of the stochastic process S itself follows a stochastic differential equation, rather than being constant or deterministic. This is a more realistic assumption if S is a candidate for a model of, say, an asset price. Inspired by John Donne's poem "No Man is an Island," we note indeed that no stock is an island. Stochastic volatility accounts for the influence of various variables on the stock price. Stochastic volatility accounts better for the presence of heavy-tailed distribution of stock returns.

We define a probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, and a two-dimensional standard Brownian motion $W = (W^1, W^2)$. Let S and v solve:

$$dS_t = S_t v_t^{\alpha}(\sigma^1 dW_t^1 + \sigma^2 dW_t^2)$$

$$dv_t = v_t(a^1 dW_t^1 + a^2 dW_t^2) + \rho(L - v_t)dt$$

for constant vectors $a = \begin{bmatrix} a^1 \\ a^2 \end{bmatrix}$ and $\sigma = \begin{bmatrix} \sigma 1 \\ \sigma 2 \end{bmatrix}$. In the above, v is referred to as the *stochastic volatility.*

We shall write this in condensed form, where by adW_t we mean $a^1 dW_t^1 + a^2 dW_t^2$ and by σdW_t we mean $\sigma^1 dW_t^1 + \sigma^2 dW_t^2$:

$$dS_t = S_t v_t^{\alpha} \sigma dW_t; \quad S_0 = S_0 \tag{4}$$

$$dv_t = v_t a dW_t + \rho(L - v_t)dt; \quad v_0 = 1 \tag{5}$$

In the above, $\rho>0$.

Define the following sequence of $\mathbb{F}$ stopping times: $\tau_n = \inf\{t : |v_t| \geq n\}$. We have that $\tau_\infty = \lim_{n\to\infty} \tau_n$ is the explosion time of v.

Theorem 6. *S is a Q martingale if and only if $(a.\sigma) \leq 0$.*

In order to prove this, he first proves the following three lemmas:

Lemma 1 (Sin)**.** *Let v solve*

$$dv_t = v_t a dW_t + \rho(L - v_t)dt; v_0 = 1$$

Then, almost surely, $v_t > 0$ for $t \in \mathbb{R}^+$.

Lemma 2 (Sin)**.** *Let (S, v) solve* (4)*. Then, S is a supermartingale, and, for all $T \in \mathbb{R}$, we have*

$$E_Q[S_T] = S_0 Q(\hat{\tau}_\infty > T)$$

In the above, $\hat{\tau}_\infty$ is the explosion time of $\hat{v}$, which is the unique solution to:

$$d\hat{v}_t = \hat{v}_t a dW_t + \rho(L - \hat{v}_t)dt + (a \cdot \sigma)\hat{v}_t^{\alpha+1}dt; \quad \hat{v}_0 = 1 \tag{6}$$

Lemma 3 (Sin)**.** *The unique solution to* (6) *explodes (to $+\infty$) in finite time if and only if $(a \cdot \sigma) > 0$.*

The proofs of these lemmas rely on the comparison theorem and Feller's test for explosions. We now discuss other stochastic volatility models, where the drift and diffusion coefficients of the stochastic volatility assume more general forms. We will again see that the determination of whether an auxiliary diffusion process related to the stochastic volatility explodes or not is crucial to the categorization of the process S as a martingale or a strict local martingale.

Let us examine the models analysed by Lions and Musiela in [29]: Assume that we are on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0<t<T}$. Lions and Musiela begin with an analysis of the model

$$\begin{aligned} dS_t &= S_t v_t dB_t; \quad S(0) = S_0 && (7) \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt; \quad v_0 = 1 && (8) \end{aligned}$$

In the above, B and W are two correlated Brownian motions with correlation ρ. We can obtain an explicit expression for the solutions S and v : introduce the sequence of $\mathbb{F}$ stopping times $\tau_n : \inf\{t \geq 0 : |v_t|>n\}$. We have that $\tau_n \to \infty$ as $n \to \infty$ and that solving (7) up to time t yields:

$$S_t^{\tau_n} = S_0 \int_0^t v_s 1_{s \leq \tau_n} dB_s + \frac{1}{2} \int_0^t v_s^2 1_{s \leq \tau_n} ds$$

We have, a.s. the convergence as $t \to \infty$ of $S_t^{\tau_n}$ to S_t that solves (7). This gives us the following fact about S, namely that it is a continuous, positive, integrable supermartingale.

Lions and Musiela prove the following:

Proposition 1. *If* $\rho = 0$ *then*

$$\sup_{t \in [0,T]} E[S_t |\log S_t|] < \infty$$

and S is a true martingale.

Proposition 2. *If $\rho>0$ then $\forall t>0$*

$$E[S_t]<S_0$$

The proof of this proposition uses the fact that the condition $\{\rho>0\}$ ensures that we can make the explosion time of the diffusion

$$dv_t = \alpha v_t 1_{t\leq\tau_n} dW_t + \alpha\rho v_t^2 1_{t\leq\tau_n} dt$$

as small as possible.

Lastly, they treat the case $\{\rho<0\}$ in the following proposition:

Proposition 3. *If $\rho<0$ then S is a martingale and for $m\geq 1$, we have $\sup_{t\in[0,T]} E[S_t^m]<\infty$ for all T if and only if $\rho\leq -\sqrt{\frac{m-1}{m}}$.*

The proof of this proposition obtains an upper bound for $E[S_{t\wedge\tau_n}^m]$. Specifically, they show that

$$E[S_{t\wedge\tau_n}^m] \leq S_0^m E_{\hat{P}}[\mathrm{e}^{\frac{m^2-m}{2}\int_0^{\tau_n} v_s^2 ds}]$$

where $\hat{P}$ is a measure under which v solves:

$$dv_t = \alpha v_t dW_t + \alpha\rho m v_t^2 dt$$

Proceeding to more general models for the stochastic volatility, as treated by Lions and Musiela, we examine the following case:

$$dS_t = S_t v_t dB_t; \quad S(0) = S_0 \tag{9}$$

$$dv_t = \mu(v_t)dW_t + b(v_t)dt; \quad v_0 = 1 \tag{10}$$

Here B and W are correlated Brownian motions, with correlation coefficient ρ. The time interval is assumed to be $[0, T]$. μ and b are assumed to be C^∞ functions on $[0, \infty)$ and μ is Lipschitz continous on $[0, \infty)$ such that:

$$\begin{aligned} \mu(0) &= 0 \\ b(0) &\geq 0 \\ \mu(x) &> 0 \quad \text{if} \quad x > 0 \\ b(x) &\leq C(1 + x) \end{aligned}$$

We have the following proposition which characterizes the martingale property of S :

Proposition 4. *If*

$$\limsup_{x \to +\infty} \frac{\rho \, x\mu(x) + b(x)}{x} < \infty \tag{11}$$

holds, then S is an integrable non negative martingale. For the same model, if the *condition*

$$\liminf_{x\to+\infty}(\rho\ x\mu(x)+b(x))\phi(x)^{-1}>0 \tag{12}$$

holds, then S is not a martingale but a supermartingale and a strict local martingale. Here, $\phi(x)$ is an increasing, positive, smooth function that satisfies

$$\int_a^{\infty}\frac{1}{\phi(x)}dx<\infty \tag{13}$$

with a being some positive constant.

In the proof, Lions and Musiela show that condition (11) is sufficient to obtain bounds on $E_{\hat{P}}[v_t^2]$ for all $t>0$, where $\hat{P}$ is a measure under which v solves up to τ_n:

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho\mu(v_t)v_t dt$$

Such a bound allows us to conclude that $\sup_n E[S_{t\wedge\tau_n}\log S_{t\wedge\tau_n}]<\infty$. This then implies that the family $\{S_{t\wedge\tau_n}\}_n$ is uniformly integrable, which allows us to conclude that $S_0 = \lim_{n\to\infty} E[S_{\tau_n\wedge t}] = E[\lim_{n\to\infty} S_{\tau_n\wedge t}]$, implying that S is a martingale.

It is also shown that (12) is sufficient to conclude that the blow-up time of the solution of the stochastic differential equation

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho\mu(v_t)v_t dt$$

can be made as small as one wishes with positive probability.

Lastly, Lions and Musiela treat the following model:

$$dS_t = S_t^\beta v_t^\delta dB_t \tag{14}$$

$$dv_t = \alpha v_t^\gamma dW_t + b(v_t)dt \tag{15}$$

They make the following assumptions and restrictions on the parameters and functions: α, γ, β, and δ are all positive, $b(0) \geq 0$, b is Lipschitz on $[0, \infty)$ and satisfies, for all x,

$$b(x) \leq C(1 + x)$$

If $\beta<1$, we have that the process S is a true martingale possessing moments of all orders. Therefore, they assume that $\beta \geq 1$, and assume no further restrictions on γ, since with the conditions specified on b, the above system of stochastic differential equations will not explode.

We have the following set of deterministic conditions on the correlation between the Brownian motions and the drift and diffusion coefficients of the volatility that allow us to characterize the martingale property of the solution S :

Proposition 5. *If the following conditions hold:* $\rho>0$, $\gamma + \delta>1$ *and*

$$\limsup_{x \to +\infty} \frac{\rho \alpha x^{\gamma+\delta} + b(x)}{x} \tag{16}$$

Then S is a true martingale.

If the following conditions hold:

$\rho>0$, $\gamma+\delta>1$ *and there exists* $\phi(x)$, *an increasing, positive, smooth function that satisfies*

$$\int_a^\infty \frac{1}{\phi(x)}dx<\infty, \tag{17}$$

where a is some positive constant, and

$$\liminf_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta}+b(x)}{\phi(x)}>0$$

Then S is a strict local martingale.

The proof of this proposition is very similar to that of the previous propositions involving the other models we have seen. In essence, by Itô's formula, we obtain $E[S_t \log S_t] = \frac{1}{2}E_{\hat{P}}[\int_0^t v_s^{2\delta} ds]$ where under $\hat{P}$, v solves

$$dv_t = \alpha v_t dW_t + b(v_t)dt + \rho\alpha v_t^{\gamma+\delta}dt$$

This leads us to conditions (16) and (17).

Let us now make mention of the work of Andersen and Piterbarg, as it appears in [2]. Their work is very similar to that of Lions and Musiela, and they treat the subject of the finiteness of moments of the local martingale S at hand in detail.

They analyse models of the following sort:

$$\begin{aligned} dS_t &= \lambda_t f(S_t)\sqrt{V_t}dB_t; \quad S(0) = S_0 \\ dV_t &= \kappa(\theta - V_t)dt + \epsilon V_t^p dW_t \quad V_0 = 1 \end{aligned}$$

They treat the case $f(x) = x$ as well as the case $f(x) = bx + h$ for $0<b \leq 1$. They also assume that λ_t is a constant, but note that their results can be extended to λ stochastic, as long as it is positive and bounded. In their model, p, κ, θ, λ are all strictly positive. In what follows, without loss of generality, we will take $\lambda = 1$.

For the model

$$dS_t = S_t\sqrt{V_t}dB_t; \quad S(0) = S_0 \tag{18}$$

$$dV_t = \kappa(\theta - V_t)dt + \epsilon V_t^p dW_t \quad V_0 = 1 \tag{19}$$

we have the following theorem, from [2]:

Theorem 7 (Andersen-Piterbarg)**.** *If either $p \leq \frac{1}{2}$ or $p>\frac{3}{2}$, then S is a true martingale. If $\frac{1}{2}<p<\frac{3}{2}$ then S is a martingale if $\rho \leq 0$ and is a supermartingale if $\rho>0$. If $p = \frac{3}{2}$ then S is a martingale if $\rho \leq \frac{1}{2}\epsilon^{-1}$ and is a strict local martingale if $\rho>\frac{1}{2}\epsilon$.*

Let us briefly relate some of the conditions displayed in the work of Lions and Musiela to those in the work of Andersen-Piterbarg. Let us call $\sqrt{V_t}$ in (18) v_t and use Itô's formula to compute its differential:

$$d(\sqrt{V_t}) = dv_t = \frac{\epsilon}{2}V_t^{\frac{2p-1}{2}}dW_t + (\frac{1}{2}\kappa\theta V_t^{-\frac{1}{2}} - \frac{1}{2}\kappa V_t^{\frac{1}{2}} - \frac{1}{8}\epsilon^2 V_t^{\frac{4p-3}{2}})dt \qquad (20)$$

. Thus, we obtain:

$$dv_t = \frac{\epsilon}{2}v_t^{2p-1}dW_t + (\frac{1}{2}\kappa\theta v_t^{-1} - \frac{1}{2}\kappa v_t - \frac{1}{8}\epsilon^2 v_t^{4p-3})dt$$

The model (18) is now tantamount to following the model, which we can interpret in the Lions-Musiela framework:

$$\begin{aligned} dS_t &= S_t v_t dB_t; \quad S(0) = S_0 \\ dv_t &= \frac{\epsilon}{2}v_t^{2p-1}dW_t + (\frac{1}{2}\kappa\theta v_t^{-1} - \frac{1}{2}\kappa v_t - \frac{1}{8}\epsilon^2 v_t^{4p-3})dt; \quad v_0 = 1 \end{aligned}$$

Let us examine what happens when $p \leq \frac{1}{2}$ or $p > \frac{3}{2}$: In the former case, we have:

$$\limsup_{x\to+\infty} \frac{\rho \ x\mu(x) + b(x)}{x} = \limsup_{x\to+\infty} \frac{\epsilon\rho x^{\alpha^{(1)}}}{2x} + \frac{\kappa\theta}{2x^2} - \frac{1}{2}\kappa - \frac{\epsilon^2 x^{\alpha^{(2)}}}{8x} < \infty$$

with $\alpha^{(1)} \leq 1$ and $\alpha^{(2)} \leq -1$. In the latter case, we have

$$\limsup_{x\to+\infty} \frac{\rho \ x\mu(x) + b(x)}{x} = \limsup_{x\to+\infty} \frac{\rho x^{\alpha^{(1)}}}{x} + \frac{\kappa\theta}{x^2} - \kappa - \frac{\epsilon^2 x^{\alpha^{(2)}}}{8x} < \infty$$

with $\alpha^{(1)}>3$ and $\alpha^{(2)}>3$ and $\alpha^{(2)}>\alpha^{(1)}$. Both of these imply that S is a true martingale.

In the case that $\frac{1}{2}<p<\frac{3}{2}$ and $\rho>0$, we have that, with $\phi(x) = x^{2p}$,

$$\frac{\rho x^{2p}}{\phi(x)} - \frac{\epsilon^2 x^{4p-3}}{8\phi(x)}>0$$

which implies that S is a strict local martingale.

Now if $\frac{1}{2}<p<\frac{3}{2}$ and $\rho \leq 0$, we have that $2p-1>4p-4$ and we have

$$\limsup_{x\to\infty} \frac{\rho x^{2p} - \frac{\epsilon^2}{8}x^{4p-3}}{x} = \limsup_{x\to\infty} \rho x^{2p-1} - \frac{\epsilon^2}{8}x^{4p-4}<\infty$$

which implies that S is a martingale.

In the case that $p = \frac{3}{2}$ and $\rho>\frac{1}{2}\epsilon$, we can take the function $\phi(x) = x^2$, which satisfies (2), to show that

$$\liminf_{x\to\infty} \frac{(\frac{\epsilon}{2}\rho - \frac{1}{8}\epsilon^2)x^3}{\phi(x)}>0.$$

Of course, we have seen that this implies that S is a strict local martingale. Thus, we can conclude that the results of Lions and Musiela contain those of Andersen and Piterbarg.

We next examine a slightly more general case, which has been studied in [32]. They consider the state space $J = (l, r)$ with $-\infty \leq l \leq r \leq \infty$ and a J-valued diffusion Y on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ that follows the stochastic differential equation

$$dY_t = \sigma(Y_t)dW_t + \mu(Y_t)dt; \quad Y(0) = Y_0 \tag{21}$$

Here, W is an $\mathcal{F}_t$ Brownian motion and μ and $\sigma : J \to \mathbb{R}$ are Borel functions satisfying the conditions

$$\sigma(x) \neq 0, \forall x \in J$$

$$\frac{1}{\sigma^2}, \frac{\mu}{\sigma^2} \in L^1_{loc}(J)$$

Here, $L^1_{loc}(J)$ refers to the set of functions that are integrable on compact subsets of J. The conditions on the coefficients of the diffusion Y ensure that the solution to (21) has a weak solution that is unique in law. This solution might exit its state space, and we denote the exit time of the state space by γ. If Y does exit its state space, $P(\gamma<\infty)>0$. We assume that l and r are absorbing boundaries. Lastly, we assume that

$$\frac{b^2}{\sigma^2} \in L^1_{loc}(J).$$

Next, they consider the stochastic exponential

$$S_t = \mathrm{e}^{\int_0^{t\wedge\gamma} b(Y_u)dW_u - \frac{1}{2}\int_0^{t\wedge\gamma} b^2(Y_u)du}; \quad t \in [0, \infty) \tag{22}$$

In this case, B and W are assumed to be independent Brownian motions. The solution S to (22) is a non-negative local martingale and by Fatou's lemma is hence a supermartingale. Therefore, verifying whether S is a martingale on the time interval $[0,T]$ is tantamount to verifying whether $E[S_T] = 1$.

Consider another diffusion process, called $\tilde{Y}$, on another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{P})$ which satisfies the stochastic differential equation

$$d\tilde{Y}_t = (\mu + b\sigma)(\tilde{Y}_t)dt + \sigma(\tilde{Y}_t)dW_t; \quad \tilde{Y}(0) = Y_0$$

Now let $\bar{J} = [l, r]$ and let $c \in J$ and define

$$\rho(x) = \mathrm{e}^{-\int_c^x \frac{2\mu}{\sigma^2}(y)dy}, x \in J \tag{23}$$

$$\tilde{\rho}(x) = \rho(x)\,\mathrm{e}^{-\int_c^x \frac{2b}{\sigma}(y)dy}, x \in \bar{J} \tag{24}$$

$$s(x) = \int_c^x \rho(y)dy, x \in \bar{J} \tag{25}$$

$$\tilde{s}(x) = \int_c^x \tilde{\rho}(y)dy, x \in \bar{J} \tag{26}$$

$$\tilde{v}(x) = \int_c^x \frac{\tilde{s}(x) - \tilde{s}(y)}{\tilde{\rho}(y)\sigma^2(y)}, x \in J \tag{27}$$

$$v(x) = \int_c^x \frac{s(x) - \tilde{s}(y)}{\rho(y)\sigma^2(y)}, x \in J \tag{28}$$

Feller's test for explosions tells us that the process $\tilde{Y}$ exits its state space at r if and only if

$$\tilde{v}(r)<\infty \tag{29}$$

The process Y exits its space at the point r if and only if

$$v(r)<\infty \tag{30}$$

$\tilde{Y}$ exits its state space at l if and only if

$$\tilde{v}(l)<\infty \tag{31}$$

The process Y exits its space at the point l if and only if

$$v(l)<\infty \tag{32}$$

The boundary point r is said to be *good* if the following set of conditions is satisfied:

$$s(r) \quad < \quad \infty \tag{33}$$

$$\frac{(s(r)-s)b^2}{\rho\sigma^2} \quad \in \quad L^1_{loc}(r-) \tag{34}$$

An equivalent condition for r to be good is the following:

$$\begin{aligned} \tilde{s}(r) &< \infty && (35) \\ \frac{(\tilde{s}(r)-\tilde{s})b^2}{\tilde{\rho}\sigma^2} &\in L^1_{loc}(r-) && (36) \end{aligned}$$

The endpoint l is said to be *good* if the following set of conditions is satisfied:

$$\begin{aligned} s(l) &> -\infty && (37) \\ \frac{(s-s(l))b^2}{\rho\sigma^2} &\in L^1_{loc}(l+) && (38) \end{aligned}$$

An equivalent condition for l to be good is the following:

$$\begin{aligned} \tilde{s}(l) &> -\infty && (39) \\ \frac{(\tilde{s}-\tilde{s}(l))b^2}{\tilde{\rho}\sigma^2} &\in L^1_{loc}(l+) && (40) \end{aligned}$$

We have the following theorem that characterizes the martingale property of (22):

Theorem 8. *[Theorem 2.1 in [32]] The process S given by* (22) *is a martingale on the interval* $[0, T]$ *if and only if at least one of the conditions* (1)$-$(2) *below is satisfied and at least one of the conditions* $(3)-(4)$ *below is also satisfied.*

1. *$\tilde{Y}$ does not exit J at r,*

2. *The endpoint r is good*

3. $\tilde{Y}$ does not exit J at l,

4. The endpoint l is good

Remark 9. *A particularly interesting corollary of this is the following: Assume that the diffusion Y does not exit its state space and let the assumptions of Theorem 8 be satisfied. Then S is a martingale if and only if $\tilde{Y}$ does not exit its state space.*

Let us also discuss a related paper, namely, that of Zhenyu, Bernard and McLeish([42]). They study nearly the same model as Mijatovic and Urusov do in [32], except for the fact that they allow for the Brownian motions driving the processes Y and S to be correlated, with correlation ρ :

That is, they assume that S and Y solve

$$dS_t = S_t b(Y_t) dB_t; \quad S(0) = S_0 \tag{41}$$

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t; \quad Y(0) = Y_0 \tag{42}$$

Denote by L_t the $\mathbb{F}$ martingale $\int_0^{t\wedge\gamma} b(Y_u)dB_u$ and by γ the explosion time of the diffusion Y. We have the following propositions from [42] which allow us to characterize the martingale property of S :

Proposition 6 (Proposition 2.1 in [42])**.** *Define, for all $i \in \mathbb{N}$, $R_i = \inf\{t \in [0,T] : S_t > i\}$ and $U_i = \inf\{t \in [0,T] : S_t \leq \frac{1}{i}\}$. Then, let $T_\infty := \lim_{i\to\infty} R_i$ and $T_0 := \lim_{i\to\infty} U_i$. T_0 and T_∞ denote the hitting times of 0 and ∞ of S respectively.*

Consider the probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, P)$ *with the process* S *defined as in* (41)*, with* $S_0 = 1$*. Then there exists a unique probability measure, call it* $\tilde{P}$*, on* $(\Omega, \mathcal{F}_{T_{\infty -}})$ *such that, for any stopping time* $0<\nu<\infty$*,*

1.

$$\tilde{P}(A \cap \{T_\infty>\nu \wedge T\}) = E_P[1_A S_{\nu\wedge T}] \tag{43}$$

for all $A \in \mathcal{F}_{\nu\wedge T}$*.*

2. *For all non-negative* $\mathcal{F}_{\nu\wedge T}$ *measurable random variables* U *taking values in* $[0,\infty]$*,*

$$E_{\tilde{P}}[U 1_{\{T_\infty>\nu\wedge T\}}] = E_P[U S_{\nu\wedge T} 1_{\{T_0>\nu\wedge T\}}] \tag{44}$$

and with $\tilde{S}_t = \frac{1}{S_t} 1_{\{T\infty>t\}}$*,*

3.

$$E_P[U 1_{\{T_0>\nu\wedge T\}}] = E_{\tilde{P}}[U \tilde{S}_{\nu\wedge T}] \tag{45}$$

4. *S is a uniformly integrable martingale if and only if*

$$\tilde{P}(T_\infty>T) = 1 \tag{46}$$

Proposition 7 (Proposition 2.3 in [42])**.** *Under the probability measure* $\tilde{P}$*, the diffusion* Y *solves the stochastic differential equation*

$$dY_t = (\mu(Y_t) + \rho b(Y_t)\sigma(Y_t))1_{t<\gamma}dt + \sigma(Y_t)1_{t<\gamma}d\tilde{W}_t; Y(0) = Y_0$$

Proposition 8 (Proposition 2.4 in [42]). *The process S given by* (41) *is a martingale on* $[0, T]$ *if and only if*

$$\tilde{P}(\int_0^{T\wedge\gamma} b^2(Y_u)du<\infty) = 1$$

Conditions such that this integral functional of the diffusion Y converges or diverges are given by Mijatovic and Urusov in their beautiful and elegant paper [33].

0.3 Expansion of the Filtration by Initial Expansions

Now that we have seen how to characterize the strict local martingale property in various stochastic volatility models, let us turn to the subject of expansions of filtrations. We will eventually see how certain expansions of the filtration at hand can alter the martingale property of solutions of stochastic differential equations.

When we perform an expansion of the filtration $\mathcal{F}_{t(t\geq 0)}$ we mean an enlargement of the filtration by which we obtain a bigger filtration $\mathcal{G}_{t(t\geq 0)}$ such that for all $t \geq 0$, we have $\mathcal{F}_t \subseteq \mathcal{G}_t$.

It was Kiyosi Itô who first worked on this notion in 1976. He began with $\mathcal{F}_{t(t\geq 0)}$, the natural filtration of a Brownian motion B, and added to it the sigma algebra generated by the random variable B_1. He demonstrated the important and interesting fact that B is still a semimartingale for the expanded filtration and calculated its canonical decomposition in this bigger filtration. Indeed, once we have established the hypotheses under which every semimartingale in the smaller filtration remains a semimartingale in the bigger filtration, we shall be very interested in computing the decomposition of a semimartingale in the expanded filtration. The expansion that Itô performed is of a type that we call *intial expansion*. By this we mean that we add the sigma algebra generated by a random variable, say L, to the filtration $\mathcal{F}_{t(t\geq 0)}$, at time 0. We can also add it at a stopping time τ. The new, enlarged filtration, which we will call $\mathbb{G}$ can be denoted as

$$\mathcal{G}_t = \bigcap_{\epsilon>0}(\mathcal{F}_{t+\epsilon} \vee \sigma(L)).$$

Let us state some important theorems that give us insight into the nature of initial expansions of filtrations. We are especially interested in conditions on the random variable L that ensure that every semimartingale in $\mathbb{F}$ remains a semimartingale in $\mathbb{G}$:

Theorem 10. *[Jacod's Criterion] Let L be a random variable with values in a standard Borel space $(E, \mathcal{E})$ and let $Q_t(\omega, dx)$ denote the regular conditional distribution of L given $\mathcal{F}_t$ for each $t \geq 0$. Suppose that for each t there exists a positive σ-finite measure η_t on $(E, \mathcal{E})$ such that $Q_t(\omega, dx) \ll \eta_t(dx)$ a.s. Then every $\mathbb{F}$ semimartin-*

gale is also an $\mathbb{G}$ *semimartingale.*

We have a useful amelioration of Theorem 10 which allows us to replace the family of measures η_t with one measure η :

Theorem 11. *Let* L *be a random variable with values in a standard Borel space* $(E, \mathcal{E})$ *and let* $Q_t(\omega, dx)$ *denote the regular conditional distribution of* L *given* $\mathcal{F}_t$ *each* $t \geq 0$.*Then there exists for each* t *a positive* σ*-finite measure* η_t *on* $(E, \mathcal{E})$ *such that* $Q_t(\omega, dx) \ll \eta_t(dx)$ *a.s. if and only if there exists one positive* σ*-finite measure* $\eta(dx)$ *such that* $Q_t(\omega, dx) \ll \eta(dx)$ *for all* ω*, each* $t>0$. *In this case* η *can be taken to be the distribution of* L*.*

We have the following corollary:

Corollary. *Let* L *be independent of the filtration* $\mathbb{F}$*. Then every* $\mathbb{F}$ *semimartingale is also a* $\mathbb{G}$ *semimartingale.*

The independence of L of $\mathbb{F}$ implies that $Q_t(\omega, dx) = \eta(dx)$, and specifically that $Q_t(\omega, dx) \ll \eta(dx)$. The conclusion of the corollary now follows from 11.

Before we conclude this chapter, we perfunctorily discuss another way in which one can expand the filtration, namely, via what is known as *progressive* expansions. When we progressively expand a filtration, we add a random variable gradually to this filtration in order to create a minimal expanded filtration allowing the random variable to be a stopping time. In this case, the expression for the enlarged filtration, which we call $\mathcal{G}_{t(t\geq 0)}$, takes the form

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma(\{L \wedge u\})$$

We do not address the subject of progressive expansions, as it has been shown in [**?**], for example, that such an expansion can create arbitrage opportunities.

Now that we have discussed ways in which to detect the martingale property, or lack thereof, of solutions of certain stochastic differential equations, and have introduced the topic of initial expansion of filtrations, we would like to see what such an expansion can do to the solution of a stochastic differential equation. The question we pose is: given a certain model in which the solution to the stochastic differential equation at hand is a martingale, when and how does an initial expansion of the filtration produce a strict local martingale? Answering this question shall be the subject of the next chapter.

Chapter 2

0.4 The Model of Lions and Musiela

Let us begin with the framework established by P.L. Lions and M. Musiela [29], that treats the case of stochastic volatility. We will begin working on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0<t<T}$. We assume that the stochastic process $S = (S_t)_{0\leq t\leq T}$, which we can think of as a stock price, and the stochastic volatility satisfy SDEs of the following system of two equations:

$$dS_t = S_t v_t dB_t; \quad S_0 = 1 \tag{47}$$

$$dv_t = \mu(v_t)dW_t + b(v_t)dt; \quad v_0 = 1 \tag{48}$$

Here B and W are correlated Brownian motions, with correlation coefficient ρ. Our time interval is assumed to be $[0, T]$. We will assume that μ and b are C^∞ functions on $[0, \infty)$ and that μ is Lipschitz continous on $[0, \infty)$ such that:

$$\begin{aligned} \mu(0) &= 0 \\ b(0) &\geq 0 \\ \mu(x) > 0 &\quad \text{if} \quad x > 0 \\ b(x) &\leq C(1+x) \end{aligned}$$

We recall the conditions of Lions and Musiela, which allow us to determine whether the solution to (47) is a strict local martingale or an integrable, non-negative martingale: If

$$\limsup_{x \to +\infty} \frac{\rho\, x\mu(x) + b(x)}{x} < \infty$$

holds, then S is an integrable non negative martingale.

For the same model, recall that if the condition

$$\liminf_{x \to +\infty} (\rho\, x\mu(x) + b(x))\phi(x)^{-1} > 0$$

holds, then S is not a martingale but a supermartingale and a strict local martingale.

Here, $\phi(x)$ is an increasing, positive, smooth function that satisfies

$$\int_a^\infty \frac{1}{\phi(x)} dx < \infty \tag{49}$$

with a being some positive constant.

We would like to determine whether or not an enlargement of the filtration can give rise to a strict local martingale in the bigger filtration, when one begins with a true martingale in the smaller one. More specifically, we would like to answer the following question: beginning with a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and a price process S that is an $\mathbb{F}$ martingale, if we perform a countable expansion of $\mathbb{F}$, resulting in an enlarged filtration $\mathbb{G}$, can we obtain a $\mathbb{G}$ strict local martingale under an equivalent measure?

We perform an initial expansion of the filtration $\mathbb{F}$ by adding a random variable $L \in \mathcal{F}$ to $\mathcal{F}_0$. We assume that this random variable L takes values in a Polish space $(E, \mathcal{E})$. The new, enlarged filtration, which we will call $\mathbb{G}$ will be denoted as

$$\mathcal{G}_t = \bigcap_{\epsilon>0} (\mathcal{F}_{t+\epsilon} \vee \sigma(L))$$

We use the results of Jean Jacod [20], on the initial expansion of filtrations: If S is a continuous $\mathbb{F}$ martingale, there exists a process $(x, \omega, t) \to k^x(t, \omega)$, measurable with respect to the sigma algebra $\mathcal{E} \otimes \mathcal{P}(\mathbb{F})$, where $P(\mathbb{F})$ denotes the predictable sigma algebra on $\Omega \otimes \mathbb{R}_+$, such that $\langle q^x, S \rangle = (k^x q^x) \cdot \langle S, S \rangle$.

The function q^x is given by the following: Let η be the distribution of L, and let $Q_t(\omega, dx)$ be the regular conditional distribution of L, given $\mathcal{F}_t$. $q^L(t,\omega)\eta(dx)$ is an $\mathbb{F}$ martingale, and a version of $Q_t(\omega, dx)$.

We have

$$Q_t(\omega, \cdot) = q_t(\omega, \cdot)\eta(\cdot)$$

Jacod proves the existence of an $\mathbb{F}$ predictable function k_t^L such that in our case,

$$[q^L, S] = k^L q^L \cdot [S, S]$$

The function k_t^L satisfies $k_t^L = \frac{h_t^L}{q_t^L}$ if $q_t^L > 0$ and $k_t^L = 0$ otherwise. In the above, h_t^L is the density process such that we have

$$d[q^L, S]_t = h_t^L d[S, S]_t$$

Jacod's theorem tells us next that the following process is a $\mathbb{G}$ local martingale:

$$\widetilde{S}_t = S_t - \int_0^t k_s^L d[S, S]_s$$

Let us illustrate this concept with an example, the case where the random variable L takes on only a finite number of values:

Let $A_1, A_2....A_n$ be a sequence of events such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{n} A_k = \Omega$. The enlarged filtration, $\mathbb{G}$, is the filtration generated by $\mathbb{F}$ and the random variable $L = \sum_{i=1}^{n} a_i 1_{A_i}$.

In this case, we have

$$k_t^L q_t^L = \sum_{i=1}^{n} \frac{\xi_t^i}{P(A_i)} 1_{\{L=a_i\}},$$

and

$$q_t^L = \sum_{i=1}^{n} \frac{P(L = a_i | \mathcal{F}_t)}{P(A_i)} 1_{\{L=a_i\}}$$

Here, ξ_t^i are processes arising from the Kunita-Watanabe inequality: If we let N_t^i be the $\mathcal{F}$-martingale $P(L = a_i | \mathcal{F}_t)$ we have that $d[N^i, S]_t = \xi_t^i d[S, S]_t$.

We will henceforth work with the general case of initial expansions, wherein we don't necessarily have a countable partition of the sample space.

Returning to the Lions-Musiela framework, we have that, under $(P, \mathbb{G})$, the price process and stochastic volatility satisfy:

$$S_t = \int_0^t (S_s v_s) dB_s - \int_0^t k_s^L d[S, S]_s + \int_0^t k_s^L d[S, S]_s,$$

where

$$\int_0^t (S_s v_s) dB_s - \int_0^t k_s^L d[S, S]_s$$

is a $(P, \mathbb{G})$ martingale, and

$$\int_0^t k_s^L d[S,S]_s$$

is a finite variation process. The stochastic volatility in turn satisfies:

$$v_t = \int_0^t \mu(v_s)dW_s - \int_0^t k_s^L \mu(v_s)^2 ds + \int_0^t k_s^L \mu(v_s)^2 ds + \int_0^t b(v_s)ds.$$

Here,

$$v_t = \int_0^t \mu(v_s)dW_s - \int_0^t k_s^L \mu(v_s)^2 ds$$

is a $(P, \mathbb{G})$ martingale, and

$$\int_0^t k_s^L \mu(v_s)^2 ds + \int_0^t b(v_s)ds$$

is a finite variation process.

We perform a Girsanov transform, to switch to a probability measure under which S is a local martingale: under $(Q, \mathbb{G})$, where the measure Q is equivalent to P, S possesses the following decomposition:

$$S_t = \int_0^t (S_s v_s)dB_s - \int_0^t k_s^L (S_s v_s)^2 ds - \int_0^t \frac{1}{Z} d[Z, \sigma \cdot B]_s + \int_0^t k_s^L (S_s v_s)^2 ds + \int_0^t \frac{1}{Z} d[Z, \sigma \cdot B]_s.$$

Here, $Z_t = E[\frac{dQ}{dP}|\mathcal{G}_t]$. Writing

$$Z_t = 1 + ZH \cdot B_t + ZJ \cdot W_t,$$

where $\cdot$ represents stochastic integration, for predictable processes J and H, we have the above equal to (recall that $d[B,W]_t = \rho dt$):

$$\int_0^t (S_s v_s) dB_s - \int_0^t k_s^L (S_s v_s)^2 ds - \int_0^t ((S_s v_s)H_s + \rho J_s) ds + \int_0^t k_s^L (S_s v_s)^2 ds + \int_0^t ((S_s v_s)H_s + \rho J_s) ds.$$

In order to get rid of the finite variation term in this decomposition, we set

$$\int_0^t k_s^L (S_s v_s)^2 ds + \int_0^t ((S_s v_s)H_s + \rho J_s) ds = 0.$$

The volatility, in turn, has the following decomposition:

$$\begin{aligned} v_t &= \int_0^t \mu(v_s) dW_s - \int_0^t k_s^L \mu(v_s)^2 ds - \\ &\int_0^t \frac{1}{Z} d[Z, \mu \cdot W]_s + \int_0^t k_s^L \mu(v_s)^2 + \int_0^t b(v_s) ds + \int_0^t \frac{1}{Z} d[Z, \mu \cdot W]_s \end{aligned}$$

This equals:

$$v_t = \int_0^t \mu(v_s)dW_s - \int_0^t k_s^L \mu(v_s)^2 ds - \int_0^t (\rho\mu(v_s)H_s + \mu(v_s)J_s)ds$$
$$+ \int_0^t b(v_s)ds + \int_0^t k_s^L \mu(v_s)^2 ds + \int_0^t (\rho\mu(v_s)H_s + \mu(v_s)J_s)ds$$

Here

$$\int_0^t \mu(v_s)dW_s - \int_0^t k_s^L \mu(v_s)^2 ds - \int_0^t (\rho\mu(v_s)H_s + \mu(v_s)J_s)ds$$

is a $(Q, \mathbb{G})$ local martingale, and

$$\int_0^t b(v_s)ds + \int_0^t k_s^L \mu(v_s)^2 ds + \int_0^t (\rho\mu(v_s)H_s + \mu(v_s)J_s)ds$$

is a finite variation process.

Recall that under $(Q, \mathbb{G})$, we would like S to be a local martingale. This entails the finite variation part of the decomposition of S under $(Q, \mathbb{G})$ being zero. Given that there are two Brownian motions, there are infinitely many combinations of H and J that will work. What we need is

$$k_t^L (S_t v_t)^2 = -(S_t v_t)H_t - \rho J_t. \tag{50}$$

Note that k_t^L in our framework is defined by the relation, for a right-continuous martingale $(q_t^L)_{t\geq 0}$, by

$$[q^L, S]_t = \int_0^t k_s^L q_s^L d[S,S]_s = \int_0^t k_s^L q_s^L (S_s v_s)^2 ds$$

In the rest of this chapter we will make the following assumptions on the processes k, H and J :

Hypothesis 1 (Standing Assumptions)**.**

(1) *We assume that each of* $k, H,$ *and* J *have right continuous paths a.s.*

(2) $Q(\omega : k_0^L > 0) > 0$

We will give examples and also a framework where the all important process k^L has right continuous paths, a.s., which shows that Hypothesis (1) is not unreasonable. Continuing, our new drift, which we will call $\hat{b}(v_t)$, is given by

$$\hat{b}(v_t) = b(v_t) + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t). \tag{51}$$

Notice that we can no longer represent the drift in deterministic terms as simply functions of the real variable x, so we cannot immediately invoke the results of Lions & Musiela. To address this, let us fix $0 < \varepsilon^{(1)} < k_0^L$ and $|\rho H_0 + J_0| < \varepsilon^{(2)}$ and define the following random times:

$$\tau^k = \inf\{t : |k_t^L| < \varepsilon^{(1)}\}$$

$$\tau^{H,J} = \inf\{t : |\rho H_t + J_t| > \varepsilon^{(2)}\}$$

Note that since the processes k_t^L, H_t and J_t are assumed to be $\mathbb{G}$ predictable, right continuous processes, we can indeed claim that these random times are $\mathbb{G}$ stopping times, by the theory of débuts, as originally developed by Dellacherie [12].

Now define the stopping time τ to be

$$\tau = (\tau^k \wedge \tau^{H,J}). \tag{52}$$

By the assumptions in 1 we have that $Q(\tau > 0) > 0$.

On the stochastic interval $[0, \tau]$, we have the following lower bound on our drift coefficients:

$$\hat{b}(v_t) = b(v_t) + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t) \geq b(v_t) + \min(\varepsilon^{(1)}, \varepsilon^{(2)})\mu^2(v_t) - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(v_t)$$

The above discussion gives us the following result.

Theorem 12. *Assume Hypothesis* 1 *holds, as well as the following conditions:*

$$\limsup_{x\to+\infty} \quad \frac{\rho\ x\mu(x)+b(x)}{x} < \infty$$
$$\liminf_{x\to+\infty} \quad (\rho\ x\mu(x)+b(x)+\min(\varepsilon^{(1)},\varepsilon^{(2)})\mu^2(x)-\max(\varepsilon^{(1)},\varepsilon^{(2)})\mu(x))\phi(x)^{-1} > 0$$

on the functions μ,b *are satisfied, and assume that* B *and* W *are correlated Brownian motions with correlation* $\rho > 0$*. Let the process* S *be the unique strong solution of the SDE*

$$\begin{aligned} dS_t &= S_t v_t dB_t \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt \end{aligned}$$

on $(P, \mathbb{F})$*, and assume that* S *is strictly positive. The solution* S *is also the solution of*

$$\begin{aligned} dS_t &= S_t v_t dB_t \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt + k_t^L \mu^2(v_t)dt + (\rho H_t + J_t)\mu(v_t)dt \end{aligned}$$

on $(Q, \mathbb{G})$*. Then* S *is a* $(P, \mathbb{F})$ *martingale and a* $(Q, \mathbb{G})$ *strict local martingale on the stochastic interval* $[0, \tau]$*, where* τ *is given in* (78)*. More specifically, we have*

$E[S_t^\tau]<S_0$.

In the above, $\phi(x)$ *is an increasing, positive, smooth function that satisfies*

$$\int_a^\infty \frac{1}{\phi(x)}dx<\infty$$

Before we continue, let us recall a result (proved for example in [36]) that allows us to compare the values of solutions of stochastic differential equations. It is well known, but we include it here for the reader's convenience. Let us denote by $\mathcal{D}^n$ the set of $\mathbb{R}^n-$ valued càdlàg processes. We will denote by $\mathcal{D}$ the set of $\mathbb{R}$ valued càdlàg processes. An operator $\mathbf{F}$ from $\mathcal{D}^n$ to $\mathcal{D}$ is said to be *Process Lipschitz* if for all X, $Y \in \mathcal{D}^n$ and for all stopping times T :

1. $X^{T_-} = Y^{T_-} \to F(X)^{T_-} = F(Y)^{T_-}$

2. There exists an adapted process K_t such that $||F(X_t) - F(Y_t)|| \leq K_t||X_t - Y_t||$

Theorem 13 (Comparison Theorem)**.** *[36, p. 324] Let Z be a continuous semimartingale, let F be process Lipschitz, and let A_t be adapted, increasing, and continuous. Assume that G and H are process Lipschitz functionals such that $G(X)_{t_-}>H(X)_{t_-}$ for all semimartingales X. Let $x_0 \geq y_0$, and X and Y be the unique solutions of*

$$X_t = x_0 + \int_0^t G(X)_{s\_} dA_s + \int_0^t F(X)_{s-} dZ_s$$
$$Y_t = y_0 + \int_0^t H(Y)_{s\_} dA_s + \int_0^t F(Y)_{s-} dZ_s$$

Then, $P\{\exists t \geq 0 : X_t \leq Y_t\} = 0$.

Now we may begin the proof of Theorem 12.

Proof of Theorem 12. Notice that the condition

$$\limsup_{x \to +\infty} \frac{\rho \; x\mu(x) + b(x)}{x} < \infty$$

is sufficient to show that the solution to the SDE

$$dS_t = S_t v_t dB_t \tag{53}$$

$$dv_t = \mu(v_t) dW_t + b(v_t) dt \tag{54}$$

is a true martingale (Lions and Musiela [29, Theorem 2.4(i)]) and that the condition

$$\liminf_{x \to +\infty} (\rho \; x\mu(x) + b(x) + \epsilon\mu^2(x) + \epsilon\mu(x))\phi(x)^{-1} > 0$$

is enough to show that the solution to the SDE

$$dS_t = S_t v_t dB_t \tag{55}$$

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \epsilon\mu^2(v_t)dt + \epsilon\mu(v_t)dt \tag{56}$$

is a strict local martingale, by Lions and Musiela [29, Theorem 2.4 , (ii)].

Define a sequence of stopping times $T_n := \inf\{t : v_t \geq n\}$. We have that the stopped process $S_{t\wedge\tau\wedge T_n}$ is a martingale. The stopping time T_∞ is the explosion time of v. Therefore, we may write

$$S_0 = E[S_{t\wedge\tau\wedge T_n}] = E[S_{t\wedge\tau}1_{\{t\wedge\tau<T_n\}}] + E[S_{T_n}1_{\{T_n\leq t\wedge\tau\}}].$$

Since $E[S_{t\wedge\tau}1_{\{t\wedge\tau<T_n\}}]$ increases to $E[S_{t\wedge\tau}]$, we would have that $E[S_{t\wedge\tau}]<S_0$ for all t if we can show that $\liminf_{n\to+\infty} E[S_{T_n}1_{\{T_n\leq t\wedge\tau\}}]>0$.

We have: $E[S_{T_n}1_{\{T_n\leq t\wedge\tau\}}] = \hat{P}(T_n \leq t\wedge\tau)$ where under $\hat{P}$, v solves

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^L\mu^2(v_t) + (\rho H_t + J_t)\mu(v_t)dt$$

Now, the condition

$$\liminf_{x\to+\infty}(\rho\, x\mu(x) + b(x) + \min(\varepsilon^{(1)},\varepsilon^{(2)})\mu^2(x) - \max(\varepsilon^{(1)},\varepsilon^{(2)})\mu(x))\phi(x)^{-1}>0$$

is sufficient to guarantee that the explosion time of the stochastic differential equation

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho v_t\mu(v_t) + \min(\varepsilon^{(1)}, \varepsilon^{(2)})\mu^2(v_t)dt - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(v_t)dt$$

can be made as small as we wish.

It is easy to see that the comparison theorem stated above implies that the solution to the SDE

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^L\mu^2(v_t) + (\rho H_t + J_t)\mu(v_t)dt$$

is Q almost surely greater than or equal to that of the SDE

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \min(\varepsilon^{(1)}, \varepsilon^{(2)})\mu^2(v_t)dt - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(v_t)dt$$

for all $t \in [0, \tau]$. Thus, since the explosion time of the SDE

$$\begin{aligned} dS_t &= S_t v_t dB_t \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt + \min(\varepsilon^{(1)}, \varepsilon^{(2)})\mu^2(v_t)dt - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(v_t)dt \end{aligned}$$

can be made as small as possible, the explosion time T_∞ of

$$\begin{aligned} dS_t &= S_t v_t dB_t \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t)dt \end{aligned}$$

can be made as small as possible as well.

This means that, for all t, we have $\hat{P}(T_\infty \leq t \wedge \tau)>0$. This implies that for all t we have

$$E[S_t^\tau] = <S_0,$$

implying that S_t is a local martingale that is not a martingale, and hence a strict local martingale. □

Remark 14. *It can be checked that the functions $\mu(x) = x$ and $b(x) = x - \rho x^2$ satisfy the criteria*

$$\limsup_{x\to+\infty} \frac{\rho\ x\mu(x) + b(x)}{x} < \infty$$

$$\liminf_{x\to+\infty} (\rho\ x\mu(x) + b(x) + \min(\varepsilon^{(1)}, \varepsilon^{(2)})\mu^2(x) - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(x))\phi(x)^{-1} > 0$$

In fact, for $k \geq 1$, the functions $\mu(x) = x^k$ and $b(x) = x - \rho x^{k+1}$ work as well, if ρ is *positive. The reason we need ρ to be positive is that we need the following condition on the drift, in order for it to have a non-exploding, positive solution:*

$$b(0) \geq 0$$

$$b(x) \leq C(1 + x)$$

for some $C \geq 0$.

Thus, if we work with an SDE with such diffusions and drift coefficients, we begin with a true martingale and end up with a strict local martingale, due to initial expansions.

Indeed, one can check using Feller's test for explosions (see, for example, [23]), that the SDE

$$dv_t = v_t^k dW_t + (v_t - \rho v_t^{k+1})dt$$

does not explode (in other words, that the time of explosion is infinite, almost surely): If we assume our state space for v to be $(-\infty, +\infty)$, we need only to show that the scale function,

$$p(x) = \int_c^x e^{\{-2 \int_c^\psi \frac{b(y)}{\mu^2(y)} dy\}} d\psi$$

satisfies the following:

$$p(-\infty) = -\infty$$

$$p(\infty) = \infty$$

In the above expression for the scale function, $\mu(x)$ is the diffusion coefficient, and $b(x)$ is the drift. If we take $\mu(x) = x$, and $b(x) = x - \rho x^2$, a quick computation shows that indeed

$$p(-\infty) = -\infty$$

and

$$p(\infty) = \infty.$$

Remark 15. *One should note that, in the case that the random variable L is independent of the sigma algebra generated by by the process $S = (S_t)_{0 \leq t \leq T}$, we have the process k^L identically equal to zero, which means that the decomposition of the process S does not change under an expansion of filtrations, and the martingale nature of solutions doesn't change.*

Before we continue, we must ensure that the subprobability measure Q defined above is a *true* probability measure. Let us begin by defining the sequence of probability measures Q_m by

$$dQ_m = Z_{T \wedge T_m} dP$$

where $T_m = \inf\{t : \int_0^t (H_s^2 + J_s^2 + 2\rho J_s H_s) ds \geq h(m)\}$, for some function h. We then have

$$E[e^{\frac{1}{2}\int_0^{t\wedge T_m}(H_s^2+J_s^2+2\rho J_s H_s)ds}] \leq e^{\frac{1}{2}h(m)}<\infty.$$

Recall that the relation

$$k_t^L(S_t v_t)^2 = -(S_t v_t)H_t - \rho J_t$$

holds true for all $t \geq 0$.

And so, we have $Q_m \ll P$ on $[0, T_m]$ for each m, as well as that the Q_m are true probability measures, since $Z_t^{T_m}$ is a true $\mathbb{G}$ martingale.

Note that if $\{Z_{T\wedge T_m}\}_m$ is a uniformly integrable martingale, then Q is equivalent to P on $[0, T]$. This is because the uniform integrability of $(Z^{T_m})_m$ ensures the L^1 convergence of Z^{T_m}, i.e.

$$\lim_{m\to\infty} E[Z_{t\wedge T_m}] = E[Z_{t\wedge \widetilde{T}}] = 1,$$

where $\widetilde{T} = \lim_{m\to\infty} T_m$. It is assumed that $\widetilde{T} \geq T$. So we obtain that, for all t in the interval $[0, T]$: $E[Z_{t\wedge \widetilde{T}}] = E[Z_t] = 1$. Thus, Q is equivalent to P on $[0, T]$.

We next perform a similar analysis for the following case:

$$dS_t = S_t^\beta v_t^\delta dB_t \quad (57)$$

$$dv_t = \alpha v_t^\gamma dW_t + b(v_t)dt \quad (58)$$

Here, we make the following assumptions and restrictions on the parameters and functions: α, γ, β, and δ are all positive, $b(0) \geq 0$, b is Lipschitz on $[0, \infty)$ and satisfies, for all x,

$$b(x) \leq C(1 + x)$$

In addition, we assume that

$$\mu(0) = 0,$$

$$\mu(x) > 0, x > 0,$$

and lastly that μ is locally Lipschitz.

Next we note that if $\beta < 1$, we have that the process S is a true martingale possessing moments of all orders. Therefore, we assume that $\beta \geq 1$, and assume no further restrictions on γ, since with the conditions specified on b, the above system of stochastic differential equations will not explode. The details of this case are almost identical to that of the previous case, and we omit most them here. We work with a new probability measure and enlarged filtration $(Q, \mathbb{G})$. The measure Q is defined by $E[\frac{dQ}{dP}]\mathcal{G}_t] = Z_t$. We choose Q such that it is a local martingale measure for S. Writing

$$Z_t = 1 + ZH \cdot B_t + ZJ \cdot W_t,$$

where $\cdot$ represents stochastic integration, for predictable processes J and H, and recalling that $d[B, W]_t = \rho dt$, we arrive at the $(Q, \mathbb{G})$ decomposition for the volatility after doing a calculation very similar to that done for the previous model:

$$\begin{aligned} v_t &= \int_0^t \alpha v_s^\gamma dW_s - \int_0^t \alpha^2 k_s^L v_s^{2\gamma} ds - \int_0^t (\alpha v_s^\gamma H_s \rho + \alpha J_s v_s^\gamma) ds + \int_0^t b(v_s) ds \\ &+ \int_0^t \alpha^2 k_s^L v_s^{2\gamma} ds + \int_0^t (\alpha v_s^\gamma H_s \rho + \alpha J_s v_s^\gamma) ds. \end{aligned}$$

Our new drift, then, call it $\hat{b}(v_t)$ satisfies

$$\hat{b}(v_t) = b(v_t) + \alpha^2 k_t^L v_t^{2\gamma} + v_t^\gamma(\alpha H_t \rho + \alpha J_t).$$

Let us fix $0<\varepsilon^{(1)}<\alpha^2 k_0^L$ and $|\alpha\rho H_0 + \alpha J_0|<\varepsilon^{(2)}$

Define the random times

$$\tau^k = \inf\{t : |\alpha^2 k_t^L|<\varepsilon^1\}$$

$$\tau^{J,H} = \inf\{t : |\alpha H_t \rho + \alpha J_t|>\varepsilon^2\}.$$

Define the stopping time τ to be

$$\tau = (\tau^k \wedge \tau^{J,H}) \tag{59}$$

Proceeding, we have, on the stochastic interval $[0, \tau]$, the following lower bound on our drift:

$$\hat{b}(v_t) \geq b(v_t) + \min(\varepsilon^{(1)}, \varepsilon^{(2)}) v_t^{2\gamma} - \max(\varepsilon^{(1)}, \varepsilon^{(2)}) v_t^\gamma$$

Let us recall the conditions on the coefficients and parameters of this system of stochastic differential equations

$$\begin{aligned} dS_t &= S_t^\beta v_t^\delta dB_t \\ dv_t &= \alpha v_t^\gamma dW_t + b(v_t)dt \end{aligned}$$

such that S is a martingale: $\rho>0$, $\gamma+\delta>1$ and

$$\limsup_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta}+b(x)}{x}<\infty$$

Let us also recall the conditions on the coefficients and parameters of this system such that the process S is a *strict* local martingale:

$\rho>0$, $\gamma+\delta>1$ and there exists $\phi(x)$, an increasing, positive, smooth function that satisfies

$$\int_a^\infty \frac{1}{\phi(x)}dx<\infty,$$

where a is some positive constant, and

$$\liminf_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta}+b(x)}{\phi(x)}>0$$

Our discussion has given rise to the following theorem:

Theorem 16. *Assume the Standing Assumptions given in Hypothesis* 1. *Assume* *in addition that the following conditions are satisfied:*

$$\limsup_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta}+b(x)}{x}<\infty \tag{60}$$

$$\liminf_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta}+b(x)+\min(\varepsilon^{(1)},\varepsilon^{(2)})x^{2\gamma}-\max(\varepsilon^{(1)},\varepsilon^{(2)})x^{\gamma}}{\phi(x)}>0 \tag{61}$$

Let W and B be correlated Brownian motions with correlation ρ. Assume that $\rho>0$ and that $\gamma+\delta>1$. Let the process S be the unique strong solution of the SDE

$$dS_t = S_t^{\beta} v_t^{\delta} dB_t \tag{62}$$

$$dv_t = \alpha v_t^{\gamma} dW_t + b(v_t)dt \tag{63}$$

on $(P,\mathbb{F})$, and assume that S is strictly positive. The solution S is also the solution of

$$dS_t = S_t^{\beta} v_t^{\delta} dB_t \tag{64}$$

$$dv_t = \alpha v_t^{\gamma} dW_t + b(v_t)dt + \alpha^2 k_t^L v_t^{2\gamma} dt + (\alpha v_t^{\gamma} H_t \rho + \alpha J_t v_t^{\gamma})dt \tag{65}$$

on $(Q,\mathbb{G})$.

Then S is a $(P,\mathbb{F})$ martingale and a $(Q,\mathbb{G})$ strict local martingale on the stochastic interval $[0,\tau]$, where τ is given in (59). More specifically, we have $E[S_t^{\tau}]<S_0$. In the

above, ϕ is an increasing, positive, smooth function that satisfies

$$\int_a^\infty \frac{1}{\phi(x)}dx<\infty$$

where a is some positive constant.

Proof. Defining the sequence of stopping times $T_n := \inf\{t : v_t \geq n\}$. We have that the stopped process $S_{t\wedge\tau\wedge T_n}$ is a martingale. The stopping time T_∞ is the explosion time of v. Therefore, we may write

$$S_0 = E[S_{t\wedge\tau\wedge T_n}] = E[S_{t\wedge\tau}1_{\{t<T_n\}}] + E[S_{T_n}1_{\{T_n\leq t\wedge\tau\}}].$$

Since $E[S_{t\wedge\tau}1_{\{t<T_n\}}]$ increases to $E[S_{t\wedge\tau}]$ as $n\to\infty$, we would have that $E[S_{t\wedge\tau}]<S_0$ for all t if we can show that $\liminf_{n\to+\infty} E[S_{T_n}1_{\{T_n\leq t\wedge\tau\}}]>0$.

We have: $E[S_{T_n}1_{\{T_n\leq t\wedge\tau\}}] = \hat{P}(T_n \leq t\wedge\tau)$ where under $\hat{P}$, v solves

$$dv_t = \alpha v_t^\gamma dW_t + b(v_t)dt + \alpha^2 k_t^L v_t^{2\gamma}dt + v_t^\gamma(\alpha H_t\rho + \alpha J_t)dt + \rho v_t^{\gamma+\delta}dt$$

Now the condition

$$\liminf_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta} + b(x) + \min(\varepsilon^{(1)},\varepsilon^{(2)})x^{2\gamma} - \max(\varepsilon^{(1)},\varepsilon^{(2)})x^\gamma}{\phi(x)}>0$$

is sufficient to guarantee that the explosion time of the SDE

$$dv_t = \alpha v_t^{\gamma} dW_t + b(v_t)dt + \min(\varepsilon^{(1)}, \varepsilon^{(2)}) v_t^{2\gamma} dt - \max(\varepsilon^{(1)}, \varepsilon^{(2)}) v_t^{\gamma} dt + \rho v_t^{\gamma+\delta}$$

can be made as small as we wish.

Thus we can invoke the comparison lemma and conclude that that the explosion time of the solution of the SDE

$$dv_t = b(v_t)dt + \alpha^2 k_t^L v_t^{2\gamma} dt + v_t^{\gamma}(\alpha H_t \rho + \alpha J_t)dt + \rho v_t^{\gamma+\delta} dt$$

can be made as small as possible. This means that, for any $t>0$, we have

$$\liminf_{n \to +\infty} \hat{P}(T_n \leq t \wedge \tau) > 0.$$

This implies that for all t we have

$$E[S_t^{\tau}] = < S_0,$$

implying that S_t is a local martingale that is not a martingale, and hence a strict local martingale.

□

Remark 17. *If we assume that there exists an $\epsilon>0$ such that $\gamma \geq \frac{1+\epsilon}{2}$, we can use*

$\phi(x) = x^{1+\epsilon}$, *and one can easily check that the following forms of* $b(x)$ *satisfy*

$$\limsup_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta} + b(x)}{x} < \infty$$

and

$$\liminf_{x\to+\infty} \frac{\rho\alpha x^{\gamma+\delta} + b(x) + \min(\varepsilon^{(1)}, \varepsilon^{(2)})x^{2\gamma} - \max(\varepsilon^{(1)}, \varepsilon^{(2)})x^{\gamma}}{\phi(x)} > 0 :$$

$$\begin{aligned} b(x) &= K\ln(x) - \rho\alpha x^{\gamma+\delta} \\ b(x) &= K\sin(x) - \rho\alpha x^{\gamma+\delta} \\ b(x) &= Ke^{-ax} - \rho\alpha x^{\gamma+\delta} \\ b(x) &= Kx^{m} - \rho\alpha x^{\gamma+\delta} \end{aligned}$$

In the above, a*,* K *and* α *are positive constants, and* m *is a constant satisfying* $m \leq 1$.

Before we continue, we must ensure that the sub-probability measure Q defined above is a *true* probability measure. Let us begin by defining the sequence of probability measures Q_m by

$$dQ_m = Z_{T\wedge T_m} dP$$

where $T_m = \inf\{t : \int_0^t (H_s^2 + J_s^2 + 2\rho J_s H_s)ds \geq h(m)\}$, for some function h. We then have

$$E[e^{\frac{1}{2}\int_0^{t\wedge T_m}(H_s^2+J_s^2+2\rho J_s H_s)ds}] \leq e^{\frac{1}{2}h(m)} < \infty.$$

In this case, H and J must satisfy

$$k_t^L S_t^2 v_t^{2\delta} + H_t S_t v_t^{2\delta} + \rho J_t S_t v^{\delta} = 0$$

for all $t \geq 0$ since we have assumed Q to be a local martingale measure for S.

And so, we have $Q_m \ll P$ on $[0, T_m]$ for each m, as well as that the Q_m are true probability measures, since $Z_t^{T_m}$ is a true $\mathbb{G}$ martingale.

Note that if $\{Z_{T\wedge T_m}\}_m$ is a uniformly integrable martingale, then Q is equivalent to P on $[0, T]$. This is because the uniform integrability of $(Z^{T_m})_m$ ensures the L^1 convergence of Z^{T_m}, i.e.

$$\lim_{m\to\infty} E[Z_{t\wedge T_m}] = E[Z_{t\wedge \widetilde{T}}] = 1,$$

where $\widetilde{T} = \lim_{m\to\infty} T_m$. It is assumed that $\widetilde{T} \geq T$. So we obtain that, for all t in the interval $[0, T] : E[Z_{t\wedge\widetilde{T}}] = E[Z_t] = 1$. Thus, Q is equivalent to P on $[0, T]$.

0.5 The Case of Jump Discontinuities

Let us now turn to the discontinuous case. That is, we assume that S and v follow SDEs of the form:

$$dS_t = S_{t-} v_t^\alpha dM_t \tag{66}$$

$$dv_t = \mu(v_t) dB_t + b(v_t) dt \tag{67}$$

We will assume that μ and b are C^∞ functions on $[0, \infty)$ and that μ is Lipschitz continuous on $[0, \infty)$ such that:

$$\begin{aligned} \mu(0) &= 0 \\ b(0) &\geq 0 \\ \mu(x) > 0 &\quad \text{if} \quad x > 0 \text{ and } \mu(x) = x\tilde{\mu}(x) \\ b(x) &\leq C(1 + x) \text{ and } b(x) = x\tilde{b}(x) \end{aligned}$$

Note that the assumptions that μ and b factor as $\mu(x) = x\tilde{\mu}(x)$ and $b(x) = x\tilde{b}(x)$ ensures a positive solution of the equation for v in (66), even though it seems always true; but it is not, since we also require μ to be Lipschitz, and even if $\tilde{\mu}$ is Lipschitz, the function $x\mu(x)$ need be only locally Lipschitz.

We assume α to be positive. In the above, B is a standard Brownian motion and

M is a discontinuous martingale such that $[M,M]$ is locally in L^1 and such that $d\langle M,M\rangle_t = \lambda_t dt$. Let us note that the conditions imposed on the coefficients b and μ of the volatility are sufficient to ensure the existence and uniqueness of a nonnegative solution v_t such that $E[\sup_{t\in[0,T]}|v_t^p|]<\infty$ for $1 \leq p \leq \infty$. Lastly, we assume that the processes v and M satisfy:

$$\Delta(\int_0^t v_s^\alpha dM_s) > -1, \tag{68}$$

i.e., for all t, ${v_t}_-^\alpha \Delta(M_t) > -1$. (We are using the standard notation that for a càdlàg process X that $\Delta X_t = X_t - X_{t-}$, the jump of X at time t.) The above condition (68) ensures that S remains positive for all $t \geq 0$.

Let us proceed to expand the filtration $\mathbb{F}$ to obtain $\mathbb{G}$ by an initial expansion, and compute the canonical expansion of S under $(P,\mathbb{G})$. We obtain the canonical decomposition of the process S under $\mathbb{G}$ via the theory of Jacod [20]. (The reader can consult [36, Chapter VI] for a pedagogic treatment of the subject.) Jacod proves the existence of an $\mathbb{F}$ predictable process k_t^L such that

$$\langle q^L, S\rangle = k^L q_-^L \cdot < S, S >$$

The function k_t^L satisfies $k_t^L = \frac{h_t^L}{q_{t-}^L}$ if $q_{t-}^L > 0$ and $k_t^L = 0$ otherwise. In the above, h_t^L

is the density process such that we have

$$d\langle q_t^L, S\rangle_t = h_t^L d\langle S, S\rangle_t$$

Jacod's theorem also tells us that the following process is a $\mathbb{G}$ local martingale:

$$\widetilde{S}_t = S_t - \int_0^t k_t^L d\langle S, S\rangle_t$$

We obtain:

$$\begin{aligned} S_t &= \int_0^t S_{s-} v_s^\alpha dM_s - \int_0^t k_s^L S_{s-}^2 v_s^{2\alpha} \lambda_s ds + \int_0^t k_s^L S_{s-}^2 v_s^{2\alpha} \lambda_s ds \\ v_t &= \int_0^t \mu(v_s) dB_s - \int_0^t k_s^L \mu(v_s)^2 ds + \int_0^t b(v_s) ds + \int_0^t k_s^L \mu(v_s)^2 ds \end{aligned}$$

Here $\int_0^t S_{s-} v_s^\alpha dM_s - \int_0^t k_s^L S_{s-}^2 v_s^{2\alpha} \lambda_s ds$ and $\int_0^t \mu(v_s) dB_s - \int_0^t k_s^L \mu(v_s)^2 ds$ are $(P, \mathbb{G})$ local martingales, and $\int_0^t k_s^L S_{s-}^2 v_s^{2\alpha} \lambda_s ds$ and $\int_0^t b(v_s) ds + \int_0^t k_s^L \mu(v_s)^2 ds$ are finite variation processes.

We perform a Girsanov transform, to switch to a probability measure Q which is equivalent to P, under which S is a local martingale. We can do this as long as we assume the condition (74), given in Theorem 18 (below). As in the previous cases, let $Z_t = E[\frac{dQ}{dP}|\mathcal{G}_t]$. Writing

$$Z_t = 1 + ZH \cdot B_t + ZJ \cdot M_t,$$

where $\cdot$ represents stochastic integration, for $\mathbb{G}$ predictable processes J_t and H_t, we have the following decompositions for S and v under $(Q, \mathbb{G})$:

$$\begin{aligned} S_t &= \int_0^t S_{s-} v_s^{\alpha} dM_s - \int_0^t k_s^L S_{s-}^2 v_s^{2\alpha} \lambda_s ds - \int_0^t (\lambda_s H_s S_{s-} v_s^{\alpha} + \rho J_s S_{s-} v_s^{\alpha}) ds \\ &\quad + \int_0^t k_s^L S_{s-}^2 v_s^{2\alpha} \lambda_s ds + \int_0^t (\lambda_s H_s S_{s-} v_s^{\alpha} + \rho J_s S_{s-} v_s^{\alpha}) ds \end{aligned}$$

$$\begin{aligned} v_t &= \int_0^t \mu(v_s) dB_s - \int_0^t k_s^L \mu(v_s)^2 ds - \int_0^t (H_s \rho \mu(v_s) + J_s \mu(v_s)) ds \\ &\quad + \int_0^t b(v_s) ds + \int_0^t (H_s \rho \mu(v_s) + J_s \mu(v_s)) ds + \int_0^t k_s^L \mu(v_s)^2 ds \end{aligned}$$

Since we have assumed that under $(Q, \mathbb{G})$, S is a local martingale, we set the finite variation term in its decomposition to zero:

$$\lambda_t k_t^L S_{t-}^2 v_t^{2\alpha} + \rho J_t S_{t-} v_t^{\alpha} + \lambda_t H_t S_{t-} v_t^{\alpha} = 0 \tag{69}$$

Our new drift, which we will call $\hat{b}(v_t)$, under $(Q, \mathbb{G})$ is given by the following:

$$\hat{b}(v_t) = b(v_t) + H_t \rho \mu(v_t) + J_t \mu(v_t) + k_t^L \mu(v_t)^2. \tag{70}$$

S and v now solve, under $(Q, \mathbb{G})$

$$\begin{aligned} dS_t &= S_{t-} v_t^{\alpha} dM_t + k_t^L S_{t-}^2 v_t^{2\alpha} \lambda_t dt + (\lambda_t H_t S_{t-} v_t^{\alpha} + \rho J_t S_{t-} v_t^{\alpha}) dt \\ dv_t &= \mu(v_t) dB_t + b(v_t) dt + k_t^L \mu(v_t)^2 dt + (H_t \rho \mu(v_t) + J_t \mu(v_t)) dt \end{aligned}$$

Let us remark that in this case, the following relation holds:

$$\langle q^L, S \rangle = \int_0^t k_s^L q_s^L S_{s-}^2 v_s^{2\alpha} \lambda_s ds$$

holds. Returning to the decomposition of the volatility we just arrived at, we again note that we can no longer represent the drift in deterministic terms as simply functions of the real variable x, so we cannot immediately invoke the results of Lions & Musiela. To address this, let us fix $0 < \varepsilon^{(1)} < k_0^L$ and $|\rho H_0 + J_0| < \varepsilon^{(2)}$ and define the following random times:

$$\tau^k = \inf\{t : |k_t^L| < \varepsilon^{(1)}\}$$

$$\tau^{H,J} = \inf\{t : |\rho H_t + J_t| > \varepsilon^{(2)}\}$$

Now define the stopping time τ to be

$$\tau = (\tau^k \wedge \tau^{H,J}). \tag{71}$$

On the stochastic interval $[0, \tau]$, we have the following lower bound on our drift coefficients:

$$\hat{b}(v_t) = b(v_t) + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t) \geq b(v_t) + \min(\varepsilon^{(1)}, \varepsilon^{(2)})\mu^2(v_t) - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(v_t)$$

The above discussion gives us the following result.

From this discussion, we have arrived at the following theorem:

Theorem 18. *Let S_t be the strong solution under $(P, \mathbb{F})$ of*

$$dS_t = S_{t-} v_t^\alpha dM_t \tag{72}$$

$$dv_t = \mu(v_t) dB_t + b(v_t) dt \tag{73}$$

S is also the solution, under $(Q,\mathbb{G})$ of

$$\begin{aligned} dS_t &= S_{t-}v_t^{\alpha}dM_t \\ dv_t &= \mu(v_t)dB_t + b(v_t)dt + k_t^L\mu(v_t)^2dt + (H_t\rho\mu(v_t) + J_t\mu(v_t))dt \end{aligned}$$

Assume:

$$E[\mathrm{e}^{\frac{1}{2}\int_0^T v_s^{2\alpha}d\langle M^d,M^d\rangle_s+\int_0^T v_s^{2\alpha}d\langle M^c,M^c\rangle_s}]<\infty. \tag{74}$$

Assume also that

$$\liminf_{x\to+\infty}(\rho\ x\mu(x) + b(x) + \min(\varepsilon^{(1)},\varepsilon^{(2)})\mu^2(x) - \max(\varepsilon^{(1)},\varepsilon^{(2)})\mu(x))\phi(x)^{-1}>0$$

Then, the process S is a true $(P,\mathbb{F})$ martingale and a $(Q,\mathbb{G})$ strict local martingale. Specifically, we have $E[S_t^{\tau}]<S_0$ where τ is given by (71).

Proof of Theorem 18. From [38], a sufficient condition for the solution S of $dS_t = S_{t-}v_t^{\alpha}dM_t$ to be a martingale on $[0,T]$ is that $E[\mathrm{e}^{\frac{1}{2}\int_0^T v_s^{2\alpha}d\langle M^d,M^d\rangle_s+\int_0^T v_s^{2\alpha}d\langle M^c,M^c\rangle_s}]<\infty$. (In Remark 19 following this proof we present an alternative condition.)

Let us now display sufficient conditions for the solution S of (72) under $(Q,\mathbb{G})$ to be a strict local martingale.

Define a sequence of stopping times $T_n := \inf\{t : v_t \geq n\}$. We have that the stopped process $S_{t\wedge\tau\wedge T_n}$ is a martingale. The stopping time T_∞ is the explosion time of v.

Therefore, we may write

$$S_0 = E[S_{t\wedge\tau\wedge T_n}] = E[S_t 1_{\{t\wedge\tau<T_n\}}] + E[S_{T_n} 1_{\{T_n\le t\wedge\tau\}}].$$

As we saw in the continuous case, since $E[S_{t\wedge\tau} 1_{\{t\wedge\tau<T_n\}}]$ increases to $E[S_{t\wedge\tau}]$, we would have that $E[S_{t\wedge\tau}]<S_0$ for all t if we can show that $\liminf_{n\to+\infty} E[S_{T_n} 1_{\{T_n\le t\wedge\tau\}}]>0$. Now $E[S_{T_n} 1_{\{T_n\le t\wedge\tau\}}]>0 = \hat{P}(T_n \le t\wedge\tau)$, where under $\hat{P}$, v solves

$$dv_t = \mu(v_t)dB_t + b(v_t)dt + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t) + \rho v_t \mu(v_t)dt$$

The condition

$$\liminf_{x\to+\infty}(\rho\ x\mu(x) + b(x) + \min(\varepsilon^{(1)},\varepsilon^{(2)})\mu^2(x) - \max(\varepsilon^{(1)},\varepsilon^{(2)})\mu(x))\phi(x)^{-1}>0$$

is sufficient to guarantee that the explosion time of the stochastic differential equation

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho v_t\mu(v_t) + \min(\varepsilon^{(1)},\varepsilon^{(2)})\mu^2(v_t)dt - \max(\varepsilon^{(1)},\varepsilon^{(2)})\mu(v_t)dt$$

can be made as small (in an appropriate sense) as we wish.

The comparison theorem implies that the solution to the SDE

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t)dt$$

is Q almost surely greater than or equal to that of the SDE

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \min(\varepsilon^{(1)}, \varepsilon^{(2)})\mu^2(v_t)dt - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(v_t)dt$$

for all $t \in [0, \tau]$. Thus, since the explosion time of the SDE

$$dv_t \ = \ \mu(v_t)dW_t + b(v_t)dt + \min(\varepsilon^{(1)}, \varepsilon^{(2)})(v_t)dt - \max(\varepsilon^{(1)}, \varepsilon^{(2)})\mu(v_t)dt$$

can be made as small as possible, the explosion time T_∞ of

$$dv_t = \mu(v_t)dW_t + b(v_t)dt + \rho\mu(v_t)v_t dt + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t)dt$$

can be made as small as possible as well.

This means that, for all t, we have $\hat{P}(T_\infty \leq t \wedge \tau)>0$. This implies that for all t we have

$$E[S_t^\tau] = <S_0,$$

implying that S_t is a local martingale that is not a martingale, and hence a strict local martingale. □

Corollary. *Let M be a Lévy martingale. Then, by the Lévy-Itô decomposition,*

$$M_t = W_t + \int_{|x|<1} x(N(;[0,t],dx) - t\nu(dx) + \sum_{0<s<t} \Delta M_s 1_{\{|\Delta M_s|\geq 1\}} - \alpha t$$

In the above, $N_t(\Lambda)$ is a Poisson random measure, $\alpha t = E[\sum_{0<s<t} \Delta M_s 1_{|\Delta M_s|\geq 1}]$ and $\nu(dx)$ is the Lévy measure of the process M_t :

$$\nu(\Lambda) = E[N^1(\Lambda)].$$

M satisfies: $d\langle M, M\rangle_t = (1 + \int_{\mathbb{R}} x^2\nu(dx))dt = cdt$.
Assume that $E[\mathrm{e}^{\int_0^T(\frac{1}{2}+\int_{\mathbb{R}} x^2\nu(dx))v_s^{2\alpha}ds}]<\infty$. This is satisfied if $\int v_s^\alpha dM_s$ is locally square integrable. Assume also that

$$\liminf_{x\to+\infty}(\rho\ x\mu(x) + b(x) + \min(\varepsilon^{(1)},\varepsilon^{(2)})\mu^2(x) - \max(\varepsilon^{(1)},\varepsilon^{(2)})\mu(x))\phi(x)^{-1}>0$$

Then, the process S of (66) is a true $(P,\mathbb{F})$ martingale and a $(Q,\mathbb{G})$ strict local martingale.

Remark 19. *We now give an alternative way to ensure that when we change probabilities from P to Q after a filtration enlargement, that Q is indeed a true probability measure and not a sub probability measure. This is an alternative to assuming that the continuous paths equivalent of (74) holds, although it is related. Let us ensure that, in the discontinuous case we have just encountered, the subprobability measure*

Q we defined is a true probability measure. We will begin by defining the sequence of probability measures Q_m by

$$dQ_m = Z_{T \wedge T_m} dP$$

where Z is the Doléans exponential of $(\int_0^t H_s dB_s + \int_0^t J_s dM_s)$ *and* $T_m = \inf\{t : \int_0^t (H_s^2 + J_s^2 + 2\rho J_s H_s + J_s^2 \int_{\mathbb{R}} x^2 \nu(dx))ds \geq h(m)\}$*, for some function h. We then have*

$$E[e^{\frac{1}{2}\int_0^{t \wedge T_m}(H_s^2 + J_s^2 + 2\rho J_s H_s + H_s^2(\int_{\mathbb{R}} x^2 \nu(dx)))ds}] \leq e^{\frac{1}{2}h(m)} < \infty.$$

Recall that the relation

$$k_t^L S_t^2 v_t^{2\alpha} c + \rho J_t S_t v_t^{\alpha} + c H_t S_t v_t^{\alpha} = 0$$

holds true for all $t \geq 0$.

Continuing, we have $Q_m \ll P$ *on* $[0, T_m]$ *for each m, as well as that the* Q_m *are true probability measures, since* $Z_t^{T_m}$ *is a true* $\mathbb{G}$ *martingale.*

Note that if $\{Z_{T \wedge T_m}\}_m$ *is a uniformly integrable martingale, then Q is equivalent to P on* $[0, T]$*. This is because the uniform integrability of* $(Z^{T_m})_m$ *ensures the* L^1 *convergence of* Z^{T_m}*, i.e.*

$$\lim_{m \to \infty} E[Z_{t \wedge T_m}] = E[Z_{t \wedge \widetilde{T}}] = 1,$$

where $\widetilde{T} = \lim_{m\to\infty} T_m$. *It is assumed that* $\widetilde{T} \geq T$. *So we obtain that, for all* t *in the interval* $[0, T]$: $E[Z_{t\wedge\widetilde{T}}] = E[Z_t] = 1$. *Thus,* Q *is equivalent to* P *on* $[0, T]$.

We take this opportunity to mention that this idea (discovered independently by the first author) is developed in a beautiful (and more general) way in the recent paper of J. Blanchet and J. Ruf [5].

0.6 The Model of Mijatovic and Urusov

Let us now perform an analysis of when expansions by initial expansions of filtrations can take us from a martingale to a strict local martingale for the model studied by Mijatovic and Urusov in [32].

Let us assume that we begin with the probability space probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ and assume that v follows the stochastic differential equation

$$dv_t = \sigma(v_t)dW_t + \mu(v_t)dt; \quad v(0) = v_0 \tag{75}$$

Here, B and W are independent Brownian motions.

Denote by γ the exit time of v from its state space $J = (0, \infty)$.

Assume that the process S solves

$$S_t = \mathrm{e}^{\int_0^{t\wedge\gamma} b(v_u)dW_u - \frac{1}{2}\int_0^{t\wedge\gamma} b^2(v_u)du}; \quad t \in [0,\infty) \tag{76}$$

Let us perform an initial expansion of the filtration $\mathbb{F}$ to obtain a bigger filtration $\mathbb{G}$ under which S and v solve respectively:

$$\begin{aligned} dS_t &= S_t b(v_t)dB_t - k_t^L d[S,S]_t \\ dv_t &= \sigma(v_t)dW_t + \mu(v_t)dt + k_t^L \sigma^2(v_t)dt \end{aligned}$$

Now we switch to an equivalent probability measure, which we call Q. Here, $Z_t = \frac{dQ}{dP}|\mathcal{G}_t$. Writing

$$Z_t = 1 + ZH \cdot B_t + ZJ \cdot W_t,$$

S and v solve, under $(Q, \mathcal{G})$:

$$\begin{aligned} dS_t &= S_t b(v_t)dB_t \\ dv_t &= \sigma(v_t)dW_t + \mu(v_t)dt + k_t^L \sigma^2(v_t)dt + (H_t + J_t)\sigma(v_t)dt \end{aligned}$$

Here, B and W are $\mathcal{G}$ Brownian motions.

The new drift of v_t, which we will call $\hat{\mu}(v_t)$, under $(Q,\mathcal{G})$ is given by the following:

$$\hat{\mu}(v_t) = \mu(v_t) + k_t^L\sigma^2(v_t) + (H_t + J_t)\sigma(v_t). \tag{77}$$

Once again, we can no longer represent the drift in deterministic terms as simply functions of the real variable x, so we are not immediately sure . To address this, let us fix $0<\varepsilon^{(1)}<k_0^L$ and $|H_0 + J_0|<\varepsilon^{(2)}$ and define the following random times:

$$\begin{aligned} \tau^k &= \inf\{t : |k_t^L|<\varepsilon^{(1)}\} \\ \tau^{H,J} &= \inf\{t : |H_t + J_t|>\varepsilon^{(2)}\} \end{aligned}$$

Note that since the processes k_t^L, H_t and J_t are assumed to be $\mathbb{G}$ predictable, right continuous processes, we can indeed claim that these random times are $\mathbb{G}$ stopping times, by the theory of débuts, as originally developed by Dellacherie [12].

Now define the stopping time τ to be

$$\tau = (\tau^k \wedge \tau^{H,J}). \tag{78}$$

By 1 we have that $Q(\tau>0)>0$.

On the stochastic interval $[0, \tau]$, we have the following lower bound on our drift coefficients:

$$\hat{\mu}(v_t) = \mu(v_t)+k_t^L\sigma^2(v_t)+(H_t+J_t)\sigma(v_t) \geq \mu(v_t)+\min(\varepsilon^{(1)}, \varepsilon^{(2)})\sigma^2(v_t)-\max(\varepsilon^{(1)}, \varepsilon^{(2)})\sigma(v_t)$$

Let $M^1 = \min(\varepsilon^{(1)}, \varepsilon^{(2)})$ and $M^2 = \max(\varepsilon^{(1)}, \varepsilon^{(2)})$

Now would be a good time for us recall a condition, from [32], that is sufficient to ensure that the solution to the stochastic differential equation

$$\begin{aligned} dS_t &= S_t b(v_t) dB_t \\ dv_t &= \mu(v_t)dt + \sigma(v_t)dW_t \end{aligned}$$

is a *strict* local martingale:

The diffusion v does not explode to ∞ but the diffusion $\tilde{v}$ given by

$$d\tilde{v}_t = \mu(\tilde{v}_t)dt + b(\tilde{v}_t)dt + \sigma(\tilde{v}_t)dW_t$$

does explode to ∞.

The above discussion gives us the following result:

Theorem 20. *Assume Hypothesis* 1 *holds. Define*

$$\rho^p(x) = \mathrm{e}^{-\int_c^x \frac{2(\mu+b\sigma)}{\sigma^2}(y)dy}, x \in J \tag{79}$$

$$\tilde{\rho}^p(x) = \rho^p(x)\,\mathrm{e}^{-\int_c^x \frac{2b}{\sigma}(y)dy}, x \in \bar{J} \tag{80}$$

$$s^p(x) = \int_c^x \rho^p(y)dy, x \in \bar{J} \tag{81}$$

$$\tilde{s}^p(x) = \int_c^x \tilde{\rho}^p(y)dy, x \in \bar{J} \tag{82}$$

$$v^p(x) = \int_c^x \frac{s^p(x) - s^p(y)}{\rho^p(y)\sigma^2(y)}, x \in J \tag{83}$$

$$\tilde{v}^p(x) = \int_c^x \frac{\tilde{s}^p(x) - \tilde{s}^p(y)}{\tilde{\rho}^p(y)\sigma^2(y)}, x \in J \tag{84}$$

$$\rho^q(x) = \mathrm{e}^{-\int_c^x \frac{2(\mu + b\sigma + M^1\sigma^2 - M^2\sigma)}{\sigma^2}(y)dy}, x \in J \tag{85}$$

$$\tilde{\rho}^q(x) = \rho^q(x)\,\mathrm{e}^{-\int_c^x \frac{2b}{\sigma}(y)dy}, x \in \bar{J} \tag{86}$$

$$s^q(x) = \int_c^x \rho^q(y)dy, x \in \bar{J} \tag{87}$$

$$\tilde{s}^q(x) = \int_c^x \tilde{\rho}^q(y)dy, x \in \bar{J} \tag{88}$$

$$v^q(x) = \int_c^x \frac{s(x) - s(y)}{\rho^q(y)\sigma^2(y)}, x \in J \tag{89}$$

$$\tilde{v}^q(x) = \int_c^x \frac{\tilde{s}^q(x) - \tilde{s}^q(y)}{\tilde{\rho}^q(y)\sigma^2(y)}, x \in J \tag{90}$$

Assume that

$$v^p(\infty) = \infty \tag{91}$$

$$\tilde{v}^p(\infty) = \infty \tag{92}$$

$$v^q(\infty) = \infty \tag{93}$$

$$\tilde{v}^q(\infty) < \infty \tag{94}$$

Let the process S be the unique strong solution of the SDE

$$dS_t = S_t b(v_t) dB_t \tag{95}$$

$$dv_t = \sigma(v_t) dW_t + \mu(v_t) dt \tag{96}$$

on $(P, \mathbb{F})$, and assume that S is strictly positive. The solution S is also the solution of

$$dS_t = S_t b(v_t) dB_t \tag{97}$$

$$dv_t = \sigma(v_t) dW_t + \mu(v_t) dt + k_t^L \sigma^2(v_t) dt + (H_t + J_t)\sigma(v_t) dt \tag{98}$$

on $(Q, \mathbb{G})$. Then S is a $(P, \mathbb{F})$ martingale and a $(Q, \mathbb{G})$ strict local martingale on the stochastic interval $[0, \tau]$, where τ is given in (78). More specifically, we have $E[S_t^\tau] < S_0$.

Proof. The conditions (91) are sufficient to ensure that the following holds:

1. The solution to the stochastic differential equation

$$dv_t = \mu(v_t)dt + \sigma(v_t)dW_t$$

   does not explode to ∞.

2. The solution to the stochastic differential equation

$$d\tilde{v}_t = \mu(\tilde{v}_t)dt + b(\tilde{v}_t)\sigma(\tilde{v}_t)dt + \sigma(\tilde{v}_t)dW_t$$

   does not explode to ∞.

3. The solution to the stochastic differential equation

$$dv_t = \mu(v_t)dt + M^1\sigma^2(v_t)dt - M^2\sigma(v_t)dt + \sigma(v_t)dW_t$$

   does not explode to ∞.

4. The solution to the stochastic differential equation

$$d\tilde{v}_t = \mu(\tilde{v}_t)dt + b(\tilde{v}_t)\sigma(v_t)dt + M^1\sigma^2(\tilde{v}_t)dt - M^2\sigma(\tilde{v}_t)dt + \sigma(\tilde{v}_t)dW_t \qquad (99)$$

   does explode to ∞.

Now we invoke the comparison lemma, 13: we have that the solution to the stochastic differential equation

$$dv_t = \sigma(v_t)dW_t + \mu(v_t)dt + k_t^L\sigma^2(v_t)dt + (H_t + J_t)\sigma(v_t)dt + b(v_t)\sigma(v_t)dt \qquad (100)$$

is Q almost surely greater than the solution to (99). This implies that, since the solution to (99) explodes to ∞, so does that of (100).

From Remark 9, we have that this is sufficient to ensure that the solution to (95) is a true martingale and that the solution to (97) is a strict local martingale.

□

Remark 21. *Take*

1. $\sigma(x) = x^q$, $q > \frac{1}{2}$
2. $b(x) = x^r$, $r + q < 1$
3. $\mu(x) = x^p$, $p < 1$

One can check that such a choice of b, μ *and* σ *is such that* (91) *is satisfied, and that an enlargement of the filtration will produce a* $(Q, \mathbb{G})$ *strict local martingale from a* $(P, \mathbb{F})$ *martingale.*

0.7 Examples

Let us now consider some examples of the random variable L that we can add to a filtration:

Example 1 (Mansuy & Yor)**.** *S and v solve*

$$\begin{aligned} dS_t &= S_t v_t dB_t; \qquad S_0 = 1 \\ dv_t &= \mu(v_t) dW_t + b(v_t) dt; \quad v_0 = 1 \end{aligned}$$

and $L = B_T$. In this case, we have

$$k_t^L = S_t v_t \frac{B_T - B_t}{T - t}$$

We have $k_0^L = S_0 v_0 \frac{B_T}{T}$, and indeed, $Q(\omega : k_0^L){>}0$. Here, it is immediately apparent that the process k has continous paths.

Example 2 (Mansuy & Yor)**.** *S and v solve*

$$\begin{aligned} dS_t &= S_t v_t dB_t; \qquad S_0 = 1 \\ dv_t &= \mu(v_t) dW_t + b(v_t) dt; \quad v_0 = 1 \end{aligned}$$

and $L = T_a$, the first hitting time of a of the Brownian motion B_t. In this case, we have $k_t^L = -\frac{1}{a - B_t} + \frac{a - B_t}{T_a - t}$. Again, it is immediately apparent that the process k has *continous paths and that* $Q(\omega : k_0^L) > 0$.

We conclude this section with a more intuitive example of the random variable L :

Example 3. *Let* S *and* v *solve*

$$\begin{aligned} dS_t &= S_t v_t dB_t; \qquad S_0 = 1 \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt; \quad v_0 = 1 \end{aligned}$$

Assume the filtration $\mathbb{F}$ *is separable. As was the case previously,* B *and* W *are two correlated Brownian motions with correlation coefficient* ρ*. Note that the vector process* $Y := \begin{bmatrix} S \\ v \end{bmatrix}$ *is a strong Markov process. Fix a time* $T > 0$ *and let* $\mathcal{A}_i = (\alpha_i, \beta_i)$ *where* (α_i, β_i) *are open sets such that* $\bigcup_{i=1}^{n}(\alpha_i, \beta_i) = \mathbb{R}$ *and* $(\alpha_i, \beta_i) \cap (\alpha_j, \beta_j) = \phi$, $i \neq j$*. Let* $f^i(x)$ *be a sequence of strictly positive, bounded, smooth functions whose support is contained in the set* (α_i, β_i)*. Assume that the information encoded in the random variable* L *that we add to the filtration at time* 0 *can be modeled as* $L = f(S_T, v_T) = \sum_{i=1}^{n} a_i f^i(S_T)$*. Intuitively, we can think of this manner of adding information to the filtration as knowing approximately where the stock price might be at time* T*. Let us make the perfectly reasonable assumption that the random variable* L *has a continuous density, which we will call* $\gamma(x)$.

Recall that the relation

$$q_t^L \eta(dx) = Q_t(dx)$$

holds. Here, Q_t *denotes the regular conditional distribution of* $f(S_T, v_t)$ *given* S_t *and* η *denotes the law of* $f(S_T, v_T)$*. We have*

$$\langle q^L, S\rangle_t = k^L q_t^L \cdot \langle S, S\rangle_t = \int_0^t k_s^L q_s^L (S_s v_s)^2 ds. \tag{101}$$

Denote by N_t *the* $\mathbb{F}$ *martingale* $E[f(S_T, v_t)|\mathcal{F}_t]$*.*

By the martingale representation theorem, we may write: $N_t = \int_0^t \xi_s^1 dB_s + \int_0^t \xi_s^2 dW_s$ *for* $\mathbb{F}$ *predictable processes* ξ^1 *and* ξ^2*.*

Denote by M_t *the martingale part of the process* $\begin{bmatrix} S \\ v \end{bmatrix}$: $M_t = \begin{bmatrix} M_t^1 \\ M_t^2 \end{bmatrix} = \begin{bmatrix} S_t v_t dB_t \\ \sigma(v_t) dW_t \end{bmatrix}$.

Now we have that the vector process $\overrightarrow{X} = \begin{bmatrix} B \\ W \end{bmatrix}$ *solves* $d\overrightarrow{X}_t = \gamma_t^1 dM_t^1 + \gamma_t^2 dM_t^2$ *where* $\gamma_t^1 = \frac{1}{S_t v_t}$ *and* $\gamma_t^2 = \frac{1}{\sigma(v_t)}$*. Lastly, denote by* $\mathcal{D}_T$ *the set of all Borel functions* g *such that* $g(Y_T) \in \mathbb{L}^2(P)$ *and that the function* $(t, y) \to P_t g(y)$ *on* $(0, \infty) \times \mathbb{R}^2$ *is once differentiable in* t *and twice-differentiable in* y*, with all partial derivatives being continuous.* P_t *denotes the transition semigroup of the process* Y*. In our case, we have assumed that the function* $f(Y_T)$ *is bounded and smooth, as well as that the functions* μ *and* b *are* C^∞*, and this is sufficient for* P_t *to be differentiable in* t *and twice differentiable in* x*, since our function* f *is indeed in* $\mathcal{D}_T$*. By Theorem* 2.4

in [21], we have an explicit representation for the process $\overrightarrow{\xi} = \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$:

$$\xi_t = S_t v_t \nabla(P_{T-t} f)(Y_t)$$

Since the function f *is assumed to be smooth and bounded, and is in* $\mathcal{D}_T$*, the process* ξ *possesses continuous paths, almost surely.*

Let us now show when, in this situation, the process k_t^L *possesses càdlàg paths. We have the relation, for all* x*,*

$$[q^x, S]_t = \int_0^t k_s^x q_s^x (Sv)_s^2 ds.$$

Using the definition of Q_t*, we have that* $N_t = E[f(S_T, v_t)|\mathcal{F}_t] = \int_{\mathbb{R}} x\eta(dx) q_t^x = \int_{\mathbb{R}} x\eta(dx)(\int_0^t h_s^{1,x} dB_s + \int_0^t h_s^{2,x} dW_s)$ *for some functions* $h^{i,x} : i \in \{1, 2\}$ *that are jointly measurable for each fixed* x*. We now have* $(\int_{\mathbb{R}} x\eta(dx))(\int_0^t h_s^{1,x} dB_s + \int_0^t h_s^{2,x} dW_s)) = \int_0^t (\int_{\mathbb{R}} x\eta(dx)(h_s^{1,x} dB_s + h_s^{2,x} dW_s)$.

This means that $\xi_t = \int_{\mathbb{R}} \eta(dx) h_t^x$ *in* $\mathbb{L}^2(dP \times dt)$ *where* $h^x = \begin{bmatrix} h^{1,x} \\ h^{2,x} \end{bmatrix}$*. Recall that we have determined already that* ξ *has continuous paths. Thus, we can, without loss of generality, take* $t \to \int_{\mathbb{R}} x\eta(dx) h_t^x$ *to be continuous.*

It remains to answer the following question: does $t \to \int_{\mathbb{R}} x\eta(dx) h_t^x$ *being continuous* *imply that $t \to h_t^x$ is continuous for almost all x?*

Note that we may write $\eta(dx) = \gamma(x)dx$ for γ continuous in x. We are assuming that $t \to \int_{\mathbb{R}} x\eta(dx)h_t^x$ is continuous.

Suppose that $t \to h_t^x$ is not continuous for all $x \in \mathcal{A}$, where $\mathcal{A}$ is a set of positive Lebesgue measure. In particular, let Λ^+ be a set with positive Lebesgue measure and assume there exists a t_0 such that $\lim_{t\to t_0} h_t^x = h_{t_0}^x + \epsilon(x)$ with $\epsilon(x){>}0$ for all $x \in \Lambda^+$.

Then, for a sequence t_n converging to t_0, and if h^x is such that $\int_{\Lambda^+}(h_{t_n}^x)^2\gamma(x)dx{<}\infty$ we obtain $\lim_{t_n\to t_0}\int_{\Lambda^+} h_t^x\gamma(x)dx = \int_{\Lambda^+}(h_{t_0}^x + \epsilon(x))\gamma(x)dx \neq \int_{\Lambda^+} h_{t_0}^x\gamma(x)dx$.

Note that the condition $\int_{\Lambda^+}(h_{t_n}^x)^2\gamma(x)dx{<}\infty$ ensures the uniform integrability, for each n, of $h_{t_n}^x$ in x, which allows us to exchange limits and integrals in the above calculation.

The same argument works if we take Λ^- to be the set of positive Lebesgue measure such that there exists a t_0 such that $\lim_{t_n\to t_0} h_t^x = h_{t_0}^x + \epsilon(x)$ with $\epsilon(x){>}0$. We have arrived at a contradiction in assuming that $t \to h_t^x$ is not continuous for all $x \in \mathcal{A}$.

Recall that (101)*holds for almost all x, almost surely, $d\lambda$, where λ represents the Lebesgue measure. This holds for all $\omega \notin N^x$, for a given x where $P(N^x) = 0$. Note that we cannot take $N := \bigcup_{x\in\mathbb{R}} N^x$, infer that $P(N) = 0$, and then conclude that for the particular form of L we have chosen, k^L has càdlàg paths, since we have an uncountable number of x. Let us now address this problem. Since we have assumed that the random variable $L = f(S_T, v_T)$ has a density that is continuous, we have* *the continuity in* x *of the application* $x \to q_t^x \eta(dx)$. *Thus, we have the continuity in* x *of* $Q_t = q_t^x \gamma(x)dx$. *If we assume* $\gamma(x)$ *to be continuous, we have that* $x \to q_t^x$ *is continuous for almost all* (t, ω), $d(P \times \lambda)$. *We can now find* Λ *with* $P(\Lambda) = 0$, *outside of which*

$$(x, t, \cdot) \to q_t^x(\cdot) \tag{102}$$

is jointly continuous in (t, x).

This gives us the following: the left side of (101) *has a version which is continuous in* (t, x), *almost surely. For all* ω *outside of* N, *such that* $P(N) = 0$, *we have a version of* $(x, t) \to \langle q^x, S \rangle_t$ *jointly continuous in* (x, t) *almost surely. The right side of* (101) *is also jointly continuous in* (x, t).

Equation (101) *gives us*

$$\langle q^x, S \rangle_t = \int_0^t h_s^x (S_s v_s) ds = \int_0^t k_s^x q_s^x (S_s v_s)^2 ds$$

holds. We then have

$$k_t^x = \frac{h_t^x}{q_t^x (S_t v_t)} \tag{103}$$

Equation (103) *holds for all* (t, x, ω) *such that* $\omega \notin N$, *with* $P(N) = 0$. *The right side of* (103) *is jointly càdlàg in* (x, t), *almost surely. Therefore, so is the left side of this same equation, and we can conclude that for the particular form of* L *we have* *chosen, k_t^L has càdlàg paths in t.*

In general, we have the following theorem:

Theorem 22. *Let S and v solve*

$$\begin{aligned} dS_t &= S_t v_t dB_t; \qquad S_0 = 1 \\ dv_t &= \mu(v_t) dW_t + b(v_t) dt; \quad v_0 = 1 \end{aligned}$$

Assume that (101) holds. Then we have that $(x,t) \to k_t^x$ is jointly càdlàg in t and x.

Chapter 3

0.8 Stochastic Stability

We now turn to the study of the stability of stochastic differential equations. We shall see some similarities and parellels between techniques that are used in the deterministic and stochastic cases. Specifically, we shall see that Lyapunov functions play an important role in the determination of the stability of certain solutions. What follows in this chapter is a summary of some part of the theory of stochastic stability as it appears in [25].

Throughout this chapter, we will assume that we are working on the probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_t$ such that $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{F}_t$ is a filtration of sub-σ fields of $\mathcal{F}$ satisfying the usual conditions. By this, we mean that it is complete and right-continuous.

Define $E = \{t>0\} \times E_l$ where E_l is an open set in $\mathbb{R}^2$. we will consider the following system:

$$dX_t = b(X_t)dt + \sum_{i=1}^{k} \sigma^i(X_t)dB_t^i; \quad X(0) = X_0 \tag{104}$$

where X, b and σ are vectors in E and the B_t^i are independent Brownian motions. We also assume that on domains that are bounded in $x \in \mathbb{R}^2$, the functions b and σ satisfy the local Lipschitz condition.

We will denote by X^x the solution of (104) started at x. Note that (104) defines a system of time-homogeneous stochastic differential equations whose solution is a time-homogeneous Markov process. We consider the conditions for stability of the solution $X_t = 0$. In order for the zero solution to be a solution to the system of equations (104), we require that

$$b(0) = 0; \quad \sigma(0) = 0$$

Now let U be a domain whose closure is denoted $\tilde{U}$ in E. Let $U^\epsilon(0) = \{(t, x) : |x| < \epsilon\}$. A function $V(t, x)$ is said to be in $C_0^2(U)$ if it is twice continuously differentiable with respect to x and continuously differentiable with respect to t in U except possibly at the point $\{x = 0\}$ and continuous in the set $\tilde{U} \backslash U^\epsilon(0)$ for all $\epsilon > 0$. Such a function will be called a Lyapunov function.

Recalling the definitions of martingales, supermartingales and submartingales, we note the following:

Example 4 (Example 2 in Chapter 2 of [25]). *Let $V(t,x)$ be a twice continuously differentiable function with respect to x and continuously differentiable with respect to t in $I \times U$ where U is a bounded closed domain in E_l. Suppose that in U, the following holds:*

$$\mathcal{L}V(t,x) = \frac{dV}{dt} + \frac{1}{2}\sum_{i,j=1}^{2} a_{i,j}(x)\frac{d^2V}{dx_i dx_j} + \sum_{i,j=1}^{2} b^i(x)\frac{dV}{dx_i}$$
$$= \frac{dV}{dt} + \frac{1}{2}\sum_{i=1}^{2}[\sigma^i(x), \frac{d}{dx}]^2 V + [b(x), \frac{d}{dx}]V \leq 0. \quad (105)$$

Let $\tau = \inf\{s : X_s \notin U\}$.

Then the process $V(\tau \wedge t, X_{\tau\wedge t})$ is an $\mathbb{F}$ supermartingale. We have

$$E[V(\tau \wedge t, X_{\tau\wedge t})|\mathcal{F}_s] \leq V(s, x_s)$$

We also have the following lemma:

Lemma 4 (Lemma 2.4 in [25]). *Define $\tau_U(t) = \inf\{s : X_s \notin U\} \wedge t$.*

Let $V(t,x)$ be a $C_0^2(t{>}0 \times U)$ that is bounded in $(t{>}0) \times U$ and such that $LV \leq 0$ in this domain. Then the process $V(\tau_U(t), X_{\tau_U(t)})$ is a supermartingale.

Specifically, we have

$$E[V(\tau_U(t), X_{\tau_U(t)}|\mathcal{F}_s] \leq V(x, X_s)$$

Now we are ready to discuss the subject of stochastic stability in these stochastic differential equations. Our first notion of stability is *stability in probability*:

Definition 1. *A solution of* (104) *is said to be stable in probability if for all* $t \geq 0$ *and for any* $s \geq 0$ *and* $\epsilon{>}0$ *we have*

$$\lim_{x \to 0} P(\sup_{t>s} |X_t^{s,x}| > \epsilon) = 0$$

We have the following theorem that gives us a deterministic condition that, if verified, ensures the presence of stability in probability:

Theorem 23 (Theorem 3.1 in [25])**.** *Let* $\{t{>}0 \times U\} = U_1$ *be a domain that contains the line* $x = 0$ *and assume the existence of a function* $V(t,x) \in C_0^2(U_1)$ *that satisfies* $V(t,0) = 0$ *and in the neighborhood of the point* $x = 0$*, satisfies* $V(t,x){>}W(x)$ *where the function* $W(x){>}0$ *for all* $x \neq 0$*. Assume that* V *also satisfies:*

$$\mathcal{L}V \leq 0$$

for $x \neq 0$*. Where the expression for* $\mathcal{L}V$ *is given by* (105)*. Then the solution* $X = 0$ *of* (104) *is stable in probability.*

We have a stronger kind of stability in probability, namely, *asymptotic stability in probability*:

Definition 2. *The solution* $X = 0$ *of [25] is said to be asymptotically stable in* *probability if it is stable and probability and, in addition,*

$$\lim_{x\to 0} P(\lim_{t\to\infty} X_t^{s,x} = 0) = 1$$

We have the following theorem, which gives us conditions under which the zero solution of (104) is asymptotically stable in probability:

Theorem 24. *Assume that any solution of* (104) *that begins in* $\epsilon<|x|<r$ *almost surely reaches the boundary of this domain in finite time for all sufficiently small and positive* r *and* ϵ*.*

Suppose we have the existence of a function $V(t,x) \in C_0^2(t>0 \times U)$ *that satisfies* $V(t,0) = 0$ *and in the neighborhood of the point* $x = 0$, *satisfies* $V(t,x)>W(x)$ *where the function* $W(x)>0$ *for all* $x \neq 0$. *Assume that this function has an infinitesimal upper limit and that* $LV \leq 0$. *Then the solution* $X = 0$ *is asymptotically stable in probability.*

Here is another theorem which gives us sufficient conditions under which the property of asymptotic stability is satisfied:

Theorem 25. *Assume the existence of a function* $V(t,x)$ *that satisfies* $V(t,x) \in C_0^2(t>0 \times U)$ *that satisfies* $V(t,0) = 0$ *and in the neighborhood of the point* $x = 0$, *satisfies* $V(t,x)>W(x)$ *where the function* $W(x)>0$ *for all* $x \neq 0$. *Assume this function* V *has an infinitesimal upper limit. Assume lastly that* LV *is negative definite in the domain* $(t>0 \times U)$. *Then the solution* $X = 0$ *is asymptotically stable* *in probability.*

Let us present a theorem that gives us sufficient conditions such that the solution to (104) is *not* asymptotically stable:

Theorem 26 (Theorem 4.2 in [25])**.** *Let U_r denote the subset $\{|x|<r\}$ of E_l. Assume that there exists a function $V(t,x) \in C_0^2(t>0 \times U_r)$ such that*

$$LV<0; \quad x \in U_r; \quad x \neq 0$$

$$\lim_{x\to 0} \inf_{t>0} V(t,x) = \infty$$

Assume also that any solution of (104) *that begins in $\epsilon<|x|<r$ almost surely reaches the boundary of this domain in finite time for all sufficiently small and positive r and ϵ. Then the solution $X = 0$ of* (104) *is not stable in probability. The event $\{\sup_{t>0} |X_t^x|<r\}$ has zero probability for all $x \in U_r$.*

Let us now turn to some different notions of stability, namely, *p-stability* and *Exponential p-stability*:

Definition 3. *The solution $X = 0$ of* (104) *in E_l is said to be p-stable if:*

For $t \geq 0$,

$$\sup_{|x|\leq\delta, t\geq 0} E[|X_t^x|^p| \to 0$$

as $\delta \to 0$.

Definition 4. *The solution $X = 0$ of* (104) *in E_l is said to be exponentially p-stable* *if for positive constants A and α*

$$E[|X_t^x|^p] \leq A|x|^p \, \mathrm{e}^{-\alpha t}$$

We have the following theorem which will allow us to verify whether exponential p-stability is satisfied for (104):

Theorem 27. *The solution $X = 0$ of (104) is exponentially p-stable if there exists a function $V(t,x) \in C_0^2(E)$ that satisfies, for positive constants k_1, k_2, k_3 :*

$$k_1|x|^p \leq V(t,x) \leq k_2|x|^p$$

$$LV(t,x) \leq -k_3|x|^p$$

We end this chapter with one last characterization of the system of stochastic differential equation (104): whether or not it has a stationary solution. By this we mean the following: suppose that the initial condition X_0 is distributed according to a probability measure μ. In other words, $P(X_0 \in B) = \mu(B)$ for all Borel sets B. If for all $t \geq 0$ we have that X_t is distributed according to μ then μ is called an invariant measure for (104) and the system has a stationary solution.

We have the following theorem from [25]:

Theorem 28. *Let X solve* (104). *Let V be a non-negative twice continuously differentiable function such that $LV(x) \to -\infty$ as $|x| \to \infty$. Assume that X does not explode almost surely. Then there exists a stationary solution to* (104).

0.9 The Martingale Property in the Case of a General Diffusion Coefficient

In this chapter, we attempt to arrive at conditions such that the solution to more general stochastic differential equations is a martingale or not. In the models we study here, the concept of explosions play a crucial role in our analysis, but we will see that we cannot any longer use Feller's test for explosions. This is because Feller's test for explosions can only tell us whether or not the solution to a **one** dimensional stochastic differential explodes.

Let us begin by presenting the weak solution of the vector $\overrightarrow{X} = \begin{bmatrix} S \\ v \end{bmatrix}$ of SDEs

$$dS_t = S_t \mu(S_t, v_t) dB_t; \quad S_0 = 1 \tag{106}$$

$$dv_t = \sigma(S_t, v_t) dW_t + b(S_t, v_t) dt; \quad v_0 = 1 : \tag{107}$$

By this we mean a triple (v, S, B, W), $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_t$ such that $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{F}_t$ is a filtration of sub-σ fields of $\mathcal{F}$ satisfying the usual conditions and B and W are correlated $\mathcal{F}_t$ Brownian motions. Our time interval is compact, $[0,T]$. In the above, μ is a Borel function on $\mathbb{R}^2$. S and v solve

$$dS_t = S_t \mu(S_t, v_t) 1_{t<\xi} dB_t; \quad S_0 = 1 \tag{108}$$

$$dv_t = \sigma(S_t, v_t) 1_{t<\xi} dW_t + b(S_t, v_t) 1_{t<\xi} dt; \quad v_0 = 1 \tag{109}$$

In the above, the stopping time ξ, called the explosion time of the process v is defined as follows: Let $e_n = \inf\{t : |v_t| \geq n\} \wedge n$. ξ is then defined as $\xi = \lim_{n\to\infty} e_n$ Note that we have an explicit expression for the solution S :

$$S_t = e^{\int_0^{\xi\wedge t} \mu(S_s,v_s)dB_s - \frac{1}{2}\int_0^{\xi\wedge t} \mu^2(S_s,v_s)ds} \tag{110}$$

It is immediately apparent that S is a non-negative local martingale, and it will be a true martingale on the time interval $[0,T]$ if and only if $E[S_T] = S_0 = 1$.

Now, let Ω^1 denote the space of continuous functions $\omega^1 = \omega^1 : (0,\infty) \to [l,r]$ such that l and r are absorbing boundaries and $\omega^1(0) = 1$.

Let Ω^2 denote the space of continuous functions $\omega^2 = (0,\infty) \to [0,\infty)$ such that $\omega^2(0) = 1$ and $\omega^2_t = \omega^2_{(t\wedge T_0 \wedge T_\infty)}$. Denote by Ω^3 the space of continuous functions $\omega^3 : [0,\infty) \to (-\infty,\infty)$ with $\omega^3(0) = 0$.

Denote by Ω^4 the space of continuous functions $\omega^4 : [0,\infty) \to (-\infty,\infty)$ with $\omega^4(0) =$

0.

Define the canonical process by $(v, S, B, W) = (\omega^1(t), \omega^2(t), \omega^3(t), \omega^4(t))$, for all $t \geq 0$. Let the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the right-continuous filtration generated by the canonical process.

Note that T_0 and T_∞ are $\mathbb{F}$ stopping times.

Let $\Omega = \prod_{i=1}^4 \Omega_i$ and $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$. Henceforth, processes will be defined on the filtered space $(\Omega, \mathcal{F}_T, \mathcal{F}_t)$, $t \in [0, \infty)$. Let P be the probability measure induced by the canonical process on the space $(\Omega, \mathbb{F})$.

Define, as in [6], for all $i \in \mathbb{N}$, $R_i = \inf\{t \in [0,T] : \omega_t^2 > i\}$ and $S_i = \inf\{t \in [0,T] : \omega^2(t) \leq \frac{1}{i}\}$. Then, let $T_\infty := \lim_{i\to\infty} R_i$ and $T_0 := \lim_{i\to\infty} S_i$. T_0 and T_∞ denote the hitting times of 0 and ∞ of ω^2 respectively.

Given the canonical space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]})$, the processes (v, S, B, W) correspond, respectively, to the four components of ω. We assume that the processes v, S are adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ as well as B and W, which are Brownian motions with respect to this very filtration.

Define by L_t the continuous $(P, \mathbb{F})$ local martingale $L_t = \int_0^{t\wedge\xi} \mu(S_s, v_s) dB_s$. We have that the process S is given by: $S = \mathcal{E}(L)$.

We have, from [39], the following lemma:

Lemma 5. *Under P, the local martingale L_t has quadratic variation $\langle L\rangle_t = \int_0^{t\wedge\xi} \mu^2(S_s, v_s) ds$.*

For a positive, predictable stopping time τ, define $S_t = \mathcal{E}(L_t)$, $t \in [0, \tau)$. Then the random variable $S_\tau = \lim_{t\uparrow\tau}$ exists and is non-negative. It satisfies, P a.s, for a sequence of announcing stopping times τ_n, $\{\lim_{n\uparrow\infty} \int_0^{\tau_n\wedge\xi} \mu^2(S_s, v_s)<\infty\} = \{S_\tau>0\}$.

Proof. By Doob's downcrossing inequality, we have that $S_\tau(\omega)$ exists for almost all $\omega \in \Omega$. As was mentioned in [39], one should consult the proof of Theorem 1.3.15 in Karatzas and Shreve(1991) with ∞ replaced by τ and n replaced by τ_n for all $n \in \mathbb{N}$. Note that $S_\tau = 0$ if and only if $\log(S_\tau) = -\infty$. Next, note that $\log(S_t) = \langle L\rangle_t(\frac{L_t}{\langle L\rangle_t} - \frac{1}{2})$ for $t \in (0, \tau)$ with $\langle L\rangle_t>0$. So, in order to prove the statement in the lemma, one must show that $\lim_{t\uparrow\tau} \frac{L_t}{\langle L\rangle_t}$ exists. But this follows from the Dambis-Dubins-Schwarz theorem, since L_t is a time-changed one-dimensional Brownian motion, possibly on an extended probability space.

□

Theorem 29. *Consider the probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, P)$ with the process S defined as in (114), with $S_0 = 1$. Then there exists a unique probability measure, call it $\tilde{P}$, on $(\Omega, \mathcal{F}_{T_{\infty-}})$ such that, for any stopping time $0<\nu<\infty$,*

1.
$$\tilde{P}(A \cap \{T_\infty>\nu \wedge T\}) = E_P[1_A S_{\nu\wedge T}] \tag{111}$$

   for all $A \in \mathcal{F}_{\nu\wedge T}$.

2. *For all non-negative $\mathcal{F}_{\nu\wedge T}$ measurable random variables U taking values in*

$[0, \infty]$,

$$E_{\tilde{P}}[U 1_{\{T_\infty > \nu \wedge T\}}] = E_P[U S_{\nu \wedge T} 1_{\{T_0 > \nu \wedge T\}}] \tag{112}$$

and with $\tilde{S}_t = \frac{1}{S_t} 1_{\{T\infty > t\}}$,

3.

$$E_P[U 1_{\{T_0 > \nu \wedge T\}}] = E_{\tilde{P}}[U \tilde{S}_{\nu \wedge T}] \tag{113}$$

4. S *is a martingale if and only if*

$$\tilde{P}(T_\infty > T) = 1 \tag{114}$$

Before we begin the proof of this theorem, let us, for the convenience of the reader, recall the definition of a *standard system* as defined in [35] and its implications: Let $\mathcal{T}$ be a partially ordered, non-void indexing set and let $(\mathcal{F}_t^0)_{t \in \mathcal{T}}$ be an increasing family of σ fields on Ω. We say that $(\mathcal{F}_t^0)_{t \in \mathcal{T}}$ is a *standard system* if:

1. Each measurable space $(\Omega, \mathcal{F}_t^0)$ is a standard Borel space. In other words, $\mathcal{F}_t^0$ is σ isomorphic to the σ field of Borel sets on some complete separable metric space.

2. For any increasing sequence $t_i \in \mathcal{T}$, and decreasing sequence $\mathcal{A}_i \in \Omega$ such that $\mathcal{A}_i$ is an atom of $\mathcal{F}_{t_i}$, we have $\cap_i \mathcal{A}_i \neq \phi$.

Let us also state the implications of the property of being a standard system: Let

$\mathcal{F}_i$ be a sequence of σ fields on Ω satisfying (1) and let μ_i be a consistent sequence of probability measures on $\mathcal{F}_{i(i \geq 1)}$. Then, from [35], we have the following theorem:

Theorem 30. *[Parthasarathy] If condition* (2) *holds, then* $\mu_{i(i\geq 1)}$ *admits an extension to* $\bigvee_{i\geq 1} \mathcal{F}_i$.

Proof. Recall our assumption that $S_0 = 1$. Observe that the stopped process S^{R_i} is a nonnegative martingale. Therefore, it generates a measure $\tilde{P}_i$ on $(\Omega, \mathcal{F}_{R_i-})$ by $d\tilde{P}_i := S_T^{R_i} dP$ for all $i \in \mathbb{N}$. Note that the family of probability measures $\{\tilde{P}_i\}$ is consistent for all i, in that $\tilde{P}_{i+j}|\mathcal{F}_{R_i-} = \tilde{P}_i \; \forall i, j \in \mathbb{N}$ and $\mathcal{F}_{R-} = \bigvee_{i\in\mathbb{N}} \mathcal{F}_{R_i-}$ The extension theorem V.4.1 of [35], also, stated above as 30 gives us the existence of a probability measure $\tilde{P}$ on $(\mathcal{F}_{R-})$ such that $\tilde{P}|\mathcal{F}_{R_i-} = \tilde{P}_i$. Let us now check that the conditions of this theorem are indeed satisfied in our case.

We need to check that $\{\mathcal{F}_{R_i}\}_{i\in\mathbb{N}}$ is a standard system. If this is true, we may apply the aforementioned extension theorem of Parthasarathy and also conclude that every probability measure on $\mathcal{F}_{R-}$ has an extension to a probability measure on $\mathcal{F}_T$.

We have, from [6], that a sufficient condition for $\{\mathcal{F}_{R_i}\}_{i\in\mathbb{N}}$ to be a standard system is the following: $\{\hat{\mathcal{F}}_t\}_{t\in[0,T]} := \{\mathcal{F}_t \cap \mathcal{F}_{R-}\}_{t\in[0,T]}$ is the right-continuous modification of a *standard system.* In [6], an example of an Ω and a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ such that $\{\mathcal{F}_t \cap \mathcal{F}_{R-}\}_{t\in[0,T]}$ is a right-continuous modification of a standard system is given:

Let E denote a locally compact space with a countable base (for example, $E = \mathbb{R}^n$ for some $n \in \mathbb{N}$) and let (Ω) be the space of right-continuous paths $\omega : [0,T] \to [0,\infty] \times E$ whose first component ω^1 of ω is such that $\omega^1(R(\omega)+t) = \infty$ for all $t \geq 0$ and that have left limits on $(0, R(\omega))$ where $R(\omega)$ denotes the first time that $\omega^1 = \infty$. Let $\{\mathcal{F}_t^0\}_{t\in[0,T]}$ denote the filtration generated by the paths and $\{\mathcal{F}_t\}_{t\in[0,T]}$ its right-continuous modification. Then, it follows from from Dellacherie, Meyer and Follmer, that $\{\mathcal{F}_t \cap \mathcal{F}_{R-}\}_{t\in[0,T]}$ is a right-continuous modification of a standard system.

In the example we are studying, we can equate the process S_t with ω^1, the first component of ω. Thus, we have that, in our case, $\{\mathcal{F}_{R_i-}\}_{i\in\mathbb{N}}$ is a standard system.

From [6], we also have that any probability measure P on $(\Omega, \mathcal{F}_{R_-})$ can be extended to a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}_T)$.

□

Let us now turn to the task of displaying deterministic criteria that, if satisfied, tell us whether the solution S to the SDE (108) is a martingale or strict local martingale. Recall that from (114), S will be a martingale if $\tilde{P}(T_\infty \leq T) = 0$ and S will be a strict local martingale if $\tilde{P}(T_\infty \leq T)>0$. Let us display the stochastic differential satisfied by S and v under $\tilde{P}$:

$$\begin{aligned} dS_t &= S_t\mu(S_t,v_t)dB_t 1_{t<\xi} + S_t\mu^2(S_t,v_t)1_{t<\xi}dt; \quad S_0 = 1 \\ dv_t &= \sigma(S_t,v_t)1_{t<\xi}dW_t + b(S_t,v_t)1_{t<\xi}dt + \rho\sigma(S_t,v_t)\mu(S_t,v_t)1_{t<\xi}dt; \quad v_0 = 1 \end{aligned}$$

0.10 Explosion of Multidimensional Diffusions

We state a few theorems regarding explosion from [41] and [34]. In each of these theorems, denote by x the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. In addition, denote by $\mathcal{A}$ the generator of the diffusion $X_t = \begin{bmatrix} S_t \\ v_t \end{bmatrix}$ under $\tilde{P}$:

$$\begin{aligned} \mathcal{A} &= x_1\mu(x_1,x_2)\frac{d}{dx^1} + b(x_1,x_2)\frac{d}{dx_2} + \\ &\rho\sigma(x_1,x_2)\mu(x_1,x_2)\frac{d}{dx^1} + \frac{1}{2}{x_1}^2\mu^2(x_1,x_2)\frac{d^2}{d{x_1}^2} + \frac{1}{2}\sigma^2(x_1,x_2)\frac{d^2}{d{x_2}^2}. \end{aligned}$$

Denote by $\mathcal{L}$ the operator $\frac{d}{dt} + \mathcal{A}$.

Theorem 31. *[Theorem 10.2.1 in [41]]*

Let the process S and v solve the system of stochastic differential equations (115). *Assume the existence of a non-negative function $V \in C^{1,2}([0,T]\times\mathbb{R}^2)$ as well as the existence of a λ>0 such that*

$$\lim_{|x|\to\infty} \inf_{0\le t\le T} V(t,x) = \infty \tag{115}$$

$$\mathcal{L}V - \lambda V \le 0 \tag{116}$$

Then, with probability 1, *the process S does not explode before T.*

Proof. Define the sequence of stopping times $\tau_n = \inf_t : |X_t| \geq n$. Since we have that $\mathcal{L} - \lambda V \leq 0$, on $[0,T] \times \mathbb{R}^2$, we obtain that

$$V(0, S_0, \upsilon_0) \geq E[\mathrm{e}^{-\lambda(T\wedge\tau_n)} V(T\wedge\tau_n, S_{T\wedge\tau_n}, \upsilon_{T\wedge\tau_n})] \geq \mathrm{e}^{-\lambda T} E_{\tilde{P}}[V(\tau_n, S_{\tau_n}, \upsilon_{T\wedge\tau_n})1_{\tau_n<T}]$$

Since it is true that $|X_{\tau_n}| = n$ if $\tau_n<T$ and that we have assumed that

$$\lim_{|x|\to\infty} \inf_{0\leq t\leq T} V(t,x) = \infty,$$

we must have that

$$\lim_{n\to\infty} \tilde{P}(\tau_n<T) = 0$$

Since the vector process X does not explode before T, the process S does not explode before T, and we have $\tilde{P}(T_\infty>T) = 1$.

□

Lemma 6. *[Lemma in [34]] Let the functions μ, σ, b be continuous in x and let $x_1\mu^2(x_1,x_2)$ and let the drift and diffusion coefficients of the process X be Lipschitz in x for $t \leq T$, and suppose there exist positive constants c and r and a function $V \in C^{1,2}([0,T] \times \mathbb{R}^2)$ such that*

$$\mathcal{L}V \leq c$$

for all $t \in T$ *and* $|x| \geq r$ *and*

$$\lim_{|x|\to\infty} \inf_{0\leq t\leq T} V(t,x) = \infty$$

Then, $\tilde{P}(T_\infty < T) = 0$*. Again, since the vector process* X *does not explode before* T*, the process* S *does not explode before* T*.*

Proof. Assume that there exists some $T_0 < T$ such that $\tilde{P}(\tau_\infty < T_0) > 0$. Define τ_∞ to be the explosion time of the vector process X. Take a sample path on which $\tau_\infty \leq T_0$. Set $\rho = \sup\{t > 0 : |X_t| = r\}$.

Then, for V, c, and r as in the hypothesis of this lemma, using Itô's formula, we may write:

$$V(t, S_t, v_t) = V(\rho, S_\rho, v_\rho) + \int_\rho^t \mathcal{L}V\,ds + M_t - M_\rho$$

where

$$M_t = \int_0^t \frac{dV}{dS}_s \mu(S_s, v_s) dB_s$$

By the Dambis-Dubins-Schwartz theorem, we have that $M_t = Z_{\psi_t}$ is a time-changed Brownian motion run according to the clock $\psi_t = \int_0^t (\frac{dV}{dS_s}\mu(S_s, v_s))^2$. From the assumptions we have made on the function V, we obtain:

$$\infty = V(t, S_t, v_t) \leq V(\rho, S_\rho, v_\rho) + c(t-\rho) + \liminf_{t\uparrow\tau_\infty} Z_{\psi_t} - Z_{\psi_\rho} < \infty.$$

We have arrived at a contradiction in assuming that $\tilde{P}(\tau_\infty{<}T){>}0$. Of course, since the vector process X does not explode before T, the process S does not explode before T.

□

We have another theorem which ensures that the explosion time of the process S is infinite, $\tilde{P}$ almost surely:

Theorem 32 (Theorem 2.1 in [34])**.** *Assume the existence of positive numbers c and r, as well as the existence of a non-negative function $V \in C^{1,2}([0,T] \times \mathbb{R}^2)$ and a non-decreasing, differentiable function $\beta : [0,\infty) \to [0,\infty)$ such that, for all $t \in [0,T]$ and $|x| \geq r$,*

$$\mathcal{L}V \leq c\beta(V(t,x))$$

and

$$\lim_{|x|\to\infty} \inf_{0\leq t\leq T} V(t,x) = \infty$$

Assume that the function β satisfies

$$\int_0^\infty \frac{du}{1+\beta(u)} = \infty$$

Then, $\tilde{P}(T_\infty{<}T) = 0$.

Proof. Let V, β, c and r be as in the hypotheses of this lemma. Set $f(v) = \int_0^v \frac{du}{1+\beta(u)}$

and $W(t,x) = f(V(t,x))$.

We have that $W \in C^{1,2}([0,T] \times \mathbb{R}^2)$ and

$$\begin{aligned}\mathcal{L}W &= (\mathcal{L}V)f'(V(t,x)) + \frac{1}{2}(x_1\mu(x_1,x_2)\frac{dV}{dx_1})^2 f''(V(t,x)) + \\ &\quad \frac{1}{2}(\sigma(x_1,x_2)\frac{dV}{dx_2})^2 f''(V(t,x)) = \frac{\mathcal{L}V}{1+\beta(V(t,x))} \\ &\quad +(\frac{1}{2}(x_1\mu(x_1,x_2)\frac{dV}{dx_1})^2 + \frac{1}{2}(\sigma(x_1,x_2)\frac{dV}{dx_2})^2 \frac{-\beta'(V(t,x))}{1+\beta(V(t,x))} \le c\end{aligned}$$

for $t \le T$ and $|x| \ge r$. This is because of our assumptions on the functions V and β.

Then, since f is non-decreasing, we obtain: $W(t,x) \ge f(\inf_{t\in[0,T]} V(t,x))$.

W then satisfies: $\inf_{t\in[0,T]} W(t,x) \to \infty$ as $|x| \to \infty$. Therefore, W satisfies the conditions of (115) and we can use Lemma 6 to conclude that $\tilde{P}(T_\infty < T) = 0$.

□

We now state a theorem that gives us sufficient conditions that the explosion time of the vector process X_t is less than T *does* have positive probability:

Theorem 33 (Theorem 10.2.1 in [41])**.** *Assume the existence of a number $\lambda>0$ and a bounded function $V \in C^{1,2}([0,T] \times \mathbb{R}^2)$ such that*

$$V(0,x_0) > e^{-\lambda T} \sup_{x\in\mathbb{R}^2} V(T,x) \tag{117}$$

and

$$\mathcal{L}V \geq \lambda V$$

Then, we have $\lim_{n\to\infty} \tilde{P}(\tau_n \leq T){>}0$.

Proof. Define $\tau_n = \inf\{t : |X_t| \geq n\}$. Now, we are supposing that $\mathcal{L}V \geq \lambda V$ for $t \in [0, \infty)$, we have:

$$V(0, S_0, v_0) \leq \mathrm{e}^{-\lambda T}(\sup_{x\in\mathbb{R}^2} V(T, x))\tilde{P}(\tau_n{>}T) + (\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^2} V(t, x))\tilde{P}(\tau_n \leq T)$$

If $\lim_{n\to\infty} \tilde{P}(\tau_n \leq T)$ were zero then we would arrive at

$$V(0, S_0, v_0) \leq \mathrm{e}^{-\lambda T} \sup_{x\in\mathbb{R}^2} V(T, x)$$

which is a contradiction because we assumed that the function V satisfies (117). We are done, and we have established that we must have $\tilde{P}(\tau_\infty \leq T){>}0$.

□

We now have deterministic criteria that can check to determine whether S is a martingale. Let us proceed to discuss some examples.

Theorem 34. *Assume that there exist positive numbers c and r as well as a non-*

decreasing, non-negative function β that satisfies

$$\int_0^\infty \frac{du}{1+\beta(u)} = \infty \tag{118}$$

and such that

$$2x_1^2\mu^2(x_1,x_2) + 2x_2(b(x_1,x_2) + \rho\sigma(x_1,x_2)\mu(x_1,x_2)) + x_1^2\mu^2(x_1,x_2) \leq c\beta(|x|^2)$$

for all $t \leq T$ and $|x| \geq r$. Then, the solution S to (108) *is a martingale on* $[0,T]$. *Note that in this case, we can take the function $V(t,x) = |x|^2$ and we obtain that $\frac{dV}{dt} + \frac{1}{2}x_1^2\mu^2(x_1,x_2)\frac{d^2V}{dx_1^2} + x_1\mu^2\frac{dV}{dx_1} + (b(x_1,x_2) + \rho\sigma(x_1,x_2)\mu(x_1,x_2))\frac{dV}{dx_2} + \frac{1}{2}\sigma^2(x_1,x_2)\frac{d^2V}{dx_2^2} < \beta(V(t,x))$ and we can conclude that $\tilde{P}(T_\infty < T) = 0$, since the vector process X does not explode, and that, from* (114)*, S is a martingale.*

Example 5. *Suppose the following condition holds: There exists some positive C such that, for all $t \geq 0$ and $x \in \mathbb{R}^2$,*

$$\begin{aligned} &x_1^2\mu^4(x_1,x_2) + x_1^2\mu^2(x_1,x_2) + \\ &\quad \sigma^2(x_1,x_2) + (b(x_1,x_2) + \rho\sigma(x_1,x_2)\mu(x_1,x,2))^2 \leq C(1+|x|^2)\log(1+|x|) \end{aligned} \tag{119}$$

Then, we have

$$2x_1^2\mu^2(x_1,x_2) + 2x_2(b(x_1,x_2)\rho\sigma(x_1,x_2)\mu(x_1,x_2)) \\ + x_1^2\mu^2(x_1,x_2) \leq C^{'}(|x|^2 + (1+|x|^2)(\log(1+|x|)) \quad (120)$$

for a constant $C^{'}$ *and for all* $t \geq 0$ *and* $x \in \mathbb{R}^2$*. If* (119) *holds, we have* (118) *holds, and then the conditions of* 34 *will hold, allowing us to conclude that* S *is a martingale.*

Example 6. *Let* $\mu(x_1,x_2) = \mathrm{e}^{f(x_1,x_2)}$*, where* $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *is a strictly negative function. In this case, if we take* $V(t,x) = |x|^2$*, we indeed have that* V *is non-negative and* $\lim_{|x|\to\infty} \inf_{t\in[0,T]} V(t,x) = \infty$*. On* $[0,T] \times [0,\infty) \times [0,\infty)$*, if we take* $\lambda = 4$*, we have:*

$$\frac{dV}{dt} + \frac{1}{2}x_1^2\mu^2(x_1,x_2)\frac{d^2V}{dx_1^2} + \\ x_1\mu^2\frac{dV}{dx_1} + (b(x_1,x_2) + \rho\sigma(x_1,x_2)\mu(x_1,x_2))\frac{dV}{dx_2} + \frac{1}{2}\sigma^2(x_1,x_2)\frac{d^2V}{dx_2^2} - \lambda V \leq 0$$

From this of course, it follows that (114) *holds, and that we may conclude that* S *is a martingale.*

0.11 An Application of Exponential p-Stability

We end this chapter by stating a theorem, inspired by the concept of exponential 2-stability, that ensures that the solution S of

$$\begin{aligned} dS_t &= S_t\mu(S_t, v_t)dB_t 1_{t<\xi} + S_t\mu^2(S_t, v_t)1_{t<\xi}dt; \quad S_0 = 1 \\ dv_t &= \sigma(S_t, v_t)1_{t<\xi}dW_t + b(S_t, v_t)1_{t<\xi}dt + \rho\sigma(S_t, v_t)\mu(S_t, v_t)1_{t<\xi}dt; \quad v_0 = 1 \end{aligned}$$

is a strict local martingle. As before, ξ is the explosion time of the diffusion v. Our time interval is, as always, assumed to be $[0, T]$. Here, B and W are correlated Brownian motions with correlation coefficient ρ. We assume that the functions $b(x)$ and $\sigma(x)$ are continuous functions on $[0, \infty) \times [0, \infty)$ and that σ is locally Lipschitz continous on $[0, \infty) \times [0, \infty)$. In addition, we assume that

$$\begin{aligned} \sigma(0) &= 0 \\ b(0) &= 0 \\ \sigma(x) > 0 \quad &\text{if} \quad |x| > 0 \end{aligned}$$

In the above, μ is a Borel function of (x_1, x_2). To show that S is a strict local martingale, of course, we need to show that $E[S_T] < S_0$. Alternatively, we can show that there exists a time interval on which $t \to E[S_t]$ is decreasing. In this case, S would be a local martingale that is not a martingale, and hence a strict local martingale.

Let $E = I \times E_l$ be a domain in $\mathbb{R}_+ \times \mathbb{R}^2$. Denote by $\mathcal{C}_2^0(E)$ the set of functions that are twice continuously differentiable with respect to x and continuously differentiable with respect to t throughout E, except possibly for the set $\{x = 0\}$, and continuous in the closed ensemble $\tilde{E} \setminus E^{\epsilon}(0)$. Here $\tilde{E}$ denotes the closure of E.

Recall that the generator, $\mathcal{A}$ of the diffusion $X_t = \begin{bmatrix} S_t \\ v_t \end{bmatrix}$ takes the form

$$\begin{aligned} \mathcal{A} \ &= \ x_1\mu^2(x_1,x_2)\frac{d}{dx_1} + b(x_1,x_2)\frac{d}{dx_2} + \rho\sigma(x_1,x_2)\mu(x_1,x_2)\frac{d}{dx_2} + \\ &\quad \frac{1}{2}{x_1}^2\mu^2(x_1,x_2)\frac{d^2}{d{x_1}^2} + \frac{1}{2}\sigma^2(x_1,x_2)\frac{d^2}{d{x_2}^2}. \end{aligned}$$

As always, denote by $\mathcal{L}$ the operator $\frac{d}{dt} + \mathcal{A}$.

Finally, denote by $X_t^{s,x}$ the solution $\begin{bmatrix} S \\ v \end{bmatrix}$ started at the point x at time $0 \leq s \leq t$.

We have the following theorem:

Theorem 35. *Assume the existence of a function* $V(t,x)$ *of class* $\mathcal{C}_2^0(E)$ *such that*

$$k_1|x|^2 \leq V(t,x) \leq k_2|x|^2 \tag{121}$$

$$\mathcal{L}V(t,x) \leq -k_3|x|^2 \tag{122}$$

For some positive constants k_1, k_2 *and* k_3.

Assume also that the function $b(x) + \rho\sigma(x)\mu(x)$ *satisfies*

$$|x|>0 \implies b(x) + \rho\sigma(x)\mu(x)>0$$

Then, S *is a* $(P, \mathbb{F})$ *strict local martingale.*

Proof. Using Itô's lemma, we can write:

$$E[V(t, X_t^{0,x}) - V(0,x)] = \int_0^t E[\mathcal{L}V(u, X_u^{0,x})]du$$

If we differentiate this equation with respect to t, and keeping in mind that $\mathcal{L}V(t,x) \leq -k_3|x|^2$ and $k_1|x|^2 \leq V(t,x) \leq k_2|x|^2$, we obtain

$$\frac{d}{dt}E[V(t, X_t^{0,x})] \leq -\frac{k_3}{k_2}E[V(t, X_t^{0,x})]$$

From this we obtain $E[V(t, X_t^{0,x})] \leq V(0,x)\,\mathrm{e}^{-\frac{k_3}{k_2}}(t)$. Given the conditions on the function on V, we finally obtain: $E[|X_t^{0,x}|^2] = E^{0,x}[(S_t)^2 + (v_t)^2] \leq \frac{k_2}{k_1}((S_0)^2 + (v_0)^2)\,\mathrm{e}^{-\frac{k_3}{k_2}(t)}$. Now fix $T > \frac{k_2}{k_3}\ln(k_2 k_1)$. Then, we have

$$E[{S_T}^2 + {v_T}^2] < {S_0}^2 + {v_0}^2 \tag{123}$$

Now the condition on the function $b(x)$ ensures for all t, the function $t \to E[v_t]$ is increasing: we have$E[v_t] - E[v_s] = E[\int_s^t b(S_s, v_s) + \rho\sigma(S_s, v_s)\mu(S_s, v_s)ds] + E[\int_s^t \sigma(S_s, v_s)dW_s] = E[\int_s^t b(S_s, v_s) + \rho\sigma(S_s, v_s)\mu(S_s, v_s)ds] > 0$. Using Jensen's inequality, we may write, $E[{v_T}^2] > {v_0}^2$. Thus, in order for (123) to be satisfied, we must have $E[{S_T}^2] < {S_0}^2$. Using Jensen's inequality again, we obtain that $E[S_T] < S_0$, implying that the process S is a local martingale that is not a martingale, and hence a strict local martingale.

$\square$

Conclusion

We study in detail the characterization of the strict local martingale property in a wide variety of models. Such models include the one-dimensional case studied, for example, by [10], stochastic volatility models as studied by [29] and those that involve a general form of the diffusion coefficient, namely, $\mu(S_t, v_t)$ for the local martingale at hand. We faced a significant challenge in exhibiting Lyapunov functions whose existence establishes the strict local martingale property in these models.

The motivation for this work is to relate possible economic causes of financial bubbles to mathematical models of how they might arise, from within the martingale oriented absence of arbitrage framework. We use the economic cause of speculative pricing that comes from overexcitement of the market due to the disclosure of new information. Examples might be the announcement of a new medicine with major financial consequences (such as a "cure" for the common cold, to exaggerate a bit), a technological breakthrough (this is the thesis of John Kenneth Galbraith, for example [16]), a resolution of some sort of political instability, a weather event (such as an early frost for the Florida orange crop), etc. The obvious and intuitive manner to model such an event is by the addition of new observable events to the underlying filtration, and an established way to do that is via the theory of the "expansion of filtrations." This theory was developed in the 1980s, and a recent presentation can be found, for example, in [36, Chapter VI].

The theory of the expansion of filtrations and the martingale theory of an absence of arbitrage do not mesh well, as papers of Imkeller [19], Fontana et al [15], and the PhD thesis of Anna Aksamit [1] have detailed. Many more references are provided in those papers. Therefore one has to be careful both as to how one expands the filtration as well as what one means by an absence of arbitrage. Here we use the approach of an "initial expansion," although we interpret it as occurring at a random (stopping) time. We work in an incomplete market setting where there are an infinite number of risk neutral measures; in particular we take a stochastic volatility framework. We show how the expansion of filtrations creates a drift even in a drift free model (this is well known) and then we need to change the risk neutral measure to remove the drift created by the addition of new information. The insight is that under this new risk neutral measure with the new enlarged filtration, the price process changes from a martingale to a strict local martingale. This has financial significance: It has been shown over the last decade that on compact time sets, a price process models a financial bubble if and only if it is strict local martingale under the risk neutral measure; thus we have shown how a non bubble price process can become a bubble price process after the arrival of new information (via an expansion of the filtration). Our ideas were inspired by the previous work of Carlos Sin [40] and Biagini-Föllmer-Nedelcu [3] who were interested in bubble formation, but did not relate it to the expansion of filtrations.

We remarking that this is different from the modeling of insider information, another popular use of the expansion of filtrations; see for example [4]. We can think of expansions of filtrations in the following manner: filtrations can be viewed as a collection of events that are observable. This collection evolves with time. In finance, we can think of this as an augmentation of the information available at any given time. Thus, filtration enlargement is a natural candidate to model insider trading, and this had been investigated by Protter, Bilina-Falafala, Jeanblanc, Fontana and Song. The subject of bubble detection is salient too. In a financial bubble, the market price of the asset exceeds its fundamental price. If a price process solving a stochastic differential equation is a strict local martingale, we have a bubble. Thus, our work shows us that we can model a way in which extra information might lead to excessive speculation, which could lead to the birth of a bubble.

We reiterate that we have confined our study, for now, to the case of initial expansions. We did not consider, for example, the case of progressive expansions, wherein the enlarged filtration consists of the smallest filtration that makes the random variable L a stopping time. This is because such an enlargement can lead to the existence of arbitrage opportunities. Natural next steps would be to perform this analysis for other types of filtration enlargements and improve upon the techniques used involving stochastic stability to display sufficient conditions such that the local martingale is a strict local martingale.

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