Optimal Multiple Stopping Approach to Mean Reversion Trading

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ABSTRACT

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This thesis studies the optimal timing of trades under mean-reverting price dynamics subject to fixed transaction costs. We first formulate an optimal double stopping problem whereby a speculative investor can choose when to enter and subsequently exit the market. The investor's value functions and optimal timing strategies are derived when prices are driven by an Ornstein-Uhlenbeck (OU), exponential OU, or Cox-Ingersoll-Ross (CIR) process. Moreover, we analyze a related optimal switching problem that involves an infinite sequence of trades. In addition to solving for the value functions and optimal switching strategies, we identify the conditions under which the double stopping and switching problems admit the same optimal entry and/or exit timing strategies. A number of extensions are also considered, such as incorporating a stop-loss constraint, or a minimum holding period under the OU model.

A typical solution approach for optimal stopping problems is to study the associated free boundary problems or variational inequalities (VIs). For the double optimal stopping problem, we apply a probabilistic methodology and rigorously derive the optimal price intervals for market entry and exit. A key step of our approach involves a transformation, which in turn allows us to characterize the value function as the smallest concave majorant of the reward function in the transformed coordinate. In contrast to the variational inequality approach, this approach directly constructs the value function as well as

the optimal entry and exit regions, without *a priori* conjecturing a candidate value function or timing strategy. Having solved the optimal double stopping problem, we then apply our results to deduce a similar solution structure for the optimal switching problem. We also verify that our value functions solve the associated VIs.

Among our results, we find that under OU or CIR price dynamics, the optimal stopping problems admit the typical buy-low-sell-high strategies. However, when the prices are driven by an exponential OU process, the investor generally enters when the price is low, but may find it optimal to wait if the current price is sufficiently close to zero. In other words, the continuation (waiting) region for entry is *disconnected*. A similar phenomenon is observed in the OU model with stop-loss constraint. Indeed, the entry region is again characterized by a bounded price interval that lies strictly above the stop-loss level. As for the exit timing, a higher stop-loss level always implies a lower optimal take-profit level. In all three models, numerical results are provided to illustrate the dependence of timing strategies on model parameters.

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To my parents and my husband

Chapter 1

Introduction

One important problem commonly faced by individual and institutional investors is to determine when to open and close a position. While observing the prevailing market prices, a speculative investor can choose to enter the market immediately or wait for a future opportunity. After completing the first trade, the investor will need to decide when is the best to close the position. This motivates the investigation of the optimal sequential timing of trades.

Naturally, the optimal sequence of trading times should depend on the price dynamics of the risky asset. For instance, if the price process is a super/submartingale, then the investor, who seeks to maximize the expected liquidation value, will either sell immediately or wait forever. Such a trivial timing arises when the underlying price follows a geometric Brownian motion (see Example A below). Similar observations can also be found in, among others, Shiryaev et al. [2008].

On the other hand, it has been widely observed that many asset prices exhibit mean reversion, including commodities (see Schwartz [1997]), foreign exchange rates (see Engel and Hamilton [1989]; Anthony and MacDonald [1998]; Larsen and Sørensen [2007]), volatility indices (see Metcalf and Hassett [1995], Bessembinder et al. [1995], Casassus and Collin-Dufresne [2005], and references therein), as well as US and global equities (see Poterba and Summers [1988]; Malliaropulos and Priestley [1999]; Balvers et al. [2000]; Gropp [2004]). Mean-reverting processes are also used to model the dynamics of interest rate, and default risk. In industry, hedge fund managers and investors often attempt to construct mean-reverting prices by simultaneously taking positions in two highly correlated or co-moving assets. The advent of exchange-traded funds (ETFs) has further facilitated this *pairs trading* approach since some ETFs are designed to track identical or similar indexes and assets. For instance, Triantafyllopoulos and Montana [2011] investigate the mean-reverting spreads between commodity ETFs and design model for statistical arbitrage. Dunis et al. [2013] also examine the mean-reverting spread between physical gold and gold equity ETFs.

In this thesis, we study the optimal timing of trades under the Ornstein-Uhlenbeck (OU), exponential Ornstein-Uhlenbeck (XOU), or Cox-Ingersoll-Ross (CIR) model. In Chapter 2, we discuss a pairs trading example where we model the value of resulting position by an OU process. To incorporate mean-reversion for positive price processes, one popular choice for pricing and investment applications is the exponential OU model, as proposed by Schwartz [1997] for commodity prices, due to its analytical tractability. It also serves as the building block of more sophisticated mean-reverting models. The CIR process has been widely used as the model for short interest rates, volatility, and energy prices (see, for example, Cox et al. [1985], Heston [1993], Ribeiro and Hodges [2004], respectively). In the real option literature, mean reverting processes have been used to model the value of a project. For instance, Ewald and Wang [2010] studies the timing of an irreversible investment whose value is modeled by a CIR process, and they solve the associated optimal single stopping problem. Carmona and León [2007] examine the optimal investment timing where the interest rate evolves according to a CIR process.

In Chapter 2, we study the optimal timing of trades subject to transaction costs under the OU model. Specifically, our formulation leads to an *optimal double stopping* problem that gives the optimal entry and exit decision rules. We obtain analytic solutions for both the entry and exit problems. In addition, we incorporate a stop-loss constraint to our trading problem. We find that a higher stop-loss level induces the investor to voluntarily liquidate earlier at a lower take-profit level. Moreover, the entry region is characterized by a bounded price interval that lies strictly above stop-loss level. In other words, it is optimal to wait if the current price is too high or too close to the lower stop-loss level. This is intuitive since entering the market close to stop-loss implies a high chance of exiting at a loss afterwards. As a result, the delay region (complement of the entry region) is *disconnected*. Furthermore, we show that optimal liquidation level decreases with the stop-loss level until they coincide, in which case immediate liquidation is optimal at all price levels. Chapter 2 is adapted from Leung and Li [2015].

In Chapter 3, we study the optimal timing of trades under the XOU model. We consider the *optimal double stopping* problem, as well as a different but related formulation. In the second formulation, the investor is assumed to enter and exit the market infinitely many times with transaction costs. This gives rise to an *optimal switching* problem. We analytically derive the non-trivial entry and exit timing strategies. Under both approaches, it is optimal to sell when the asset price is sufficiently high, though at different levels. As for entry timing, we find that, under some conditions, it is optimal for the investor not to enter the market at all when facing the optimal switching problem. In this case for the investor who has a long position, the optimal switching problem reduces into an optimal stopping problem, where the optimal liquidation level is identical to that of the optimal double stopping problem. Otherwise, the optimal entry timing strategies for the double stopping and switching problem are described by the underlying's first passage time to an interval that lies above level zero. In other words, the continuation region for entry is *disconnected* of the form $(0, A) \cup (B, +\infty)$, with critical price levels A and B (see Theorems 3.2.4 and 3.2.7 below). This means that the investor generally enters when the price is low, but may find it optimal to wait if the current price is too close to zero. We find that this phenomenon is a distinct consequence due to fixed transaction costs under the XOU model. Indeed, when there is no fixed costs, even if there are proportional transaction costs (see Zhang and Zhang [2008]), the entry timing is simply characterized by a single price level. Chapter 3 is based on Leung et al. [2014a].

Our main contribution in Chapter 4 is the analytical derivation of the non-trivial optimal entry and exit timing strategies and the associated value functions. Under both double stopping and switching approaches, it is optimal to exit when the process value is sufficiently high, though at different levels. As for entry timing, we find the necessary and sufficient conditions whereby it is optimal not to enter at all when facing the optimal switching problem. In this case, the optimal switching problem in fact reduces to an optimal single stopping problem, where the optimal stopping level is identical to that of the optimal double stopping problem. Chapter 4 is built on Leung et al. [2014b].

A typical solution approach for optimal stopping problems driven by diffusion involves the analytical and numerical studies of the associated free boundary problems or variational inequalities (VIs); see e.g. Bensoussan and Lions [1982], Øksendal [2003], and Sun [1992]. For our double optimal stopping problem, this method would determine the value functions from a pair of VIs and require regularity conditions to guarantee that the solutions to the VIs indeed correspond to the optimal stopping problems. As noted by Dayanik [2008], "the variational methods become challenging when the form of the reward function and/or the dynamics of the diffusion obscure the shape of the optimal continuation region." In our optimal entry timing problem, the reward function involves the value function from the exit timing problem, which is not monotone and can be positive and negative.

In contrast to the variational inequality approach, our proposed methodology starts with a characterization of the value functions as the smallest concave majorant of any given reward function. A key feature of this approach is that it allows us to directly construct the value function, without *a priori* finding a candidate value function or imposing conditions on the stopping and delay (continuation) regions, such as whether they are connected or not. In other words, our method will derive the structure of the stopping and delay regions as an output.

Having solved the optimal double stopping problem, we determine the optimal structures of the buy/sell/wait regions. We then apply this to infer a similar solution structure for the optimal switching problem and verify using the variational inequalities.

In earlier studies, Dynkin and Yushkevich [1969] analyze the concave characterization of excessive functions for a standard Brownian motion, and Dayanik and Karatzas [2003] and Dayanik [2008] apply this idea to study the optimal single stopping of a one-dimensional diffusion. Alvarez [2003] discusses the conditions for the convexity of an r-excessive mapping under a linear, time-homogeneous and regular diffusion process. In this regard, we contribute to this line of work by solving a number of optimal double stopping problems under the OU, XOU, or CIR model, and incorporating a stop-loss exit under the OU model.

Among other related studies, Ekstrom et al. [2011] analyze the optimal single liquidation timing under the OU model with zero long-run mean and no transaction cost. Chapter 2 extends their model in a number of ways.

First, we analyze the optimal entry timing as well as the optimal liquidation timing. Our model allows for a non-zero long-run mean and transaction costs, along with a stop-loss level. Song et al. [2009] propose a numerical stochastic approximation scheme to solve for the optimal buy-low-sell-high strategies over a finite horizon. Song and Zhang [2013] study the optimal switching problem with stop-loss under the OU price dynamics. Under a similar setting, Zhang and Zhang [2008] and Kong and Zhang [2010] also investigate the infinite sequential buying and selling/shorting problem under exponential OU price dynamics with slippage cost. In contrast to these studies, we study both optimal double stopping and switching problems specifically under exponential OU with fixed transaction costs. In particular, the optimal entry timing with fixed transaction costs is characteristically different from that with slippage.

Zervos et al. [2013] consider an optimal switching problem with fixed transaction costs under a class of time-homogeneous diffusions, including the GBM, mean-reverting CEV underlying, and other models. However, their results are not applicable to the exponential OU model as it violates Assumption 4 of their paper (see also Remark 3.3.4 below). Indeed, their model assumptions restrict the optimal entry region to be represented by a single critical threshold, whereas we show that in the XOU model the optimal entry region is characterized by two positive price levels.

As for related applications of optimal stopping, Karpowicz and Szajowski [2007] analyze the double stopping times for a risk process from the insurance company's perspective. The problem of timing to buy/sell derivatives has also been studied in Leung and Ludkovski [2011] (European and American options). Leung and Liu [2012] study the optimal timing to liquidate credit derivatives where the default intensity is modeled by an OU or CIR process. They focus on the finite-horizon trading problem, and identify the conditions under which immediate stopping or perpetual holding is optimal. Menaldi et al. [1996]

study an optimal starting-stopping problem for general Markov processes, and provide the mathematical characterization of the value functions. Czichowsky et al. [2015] investigate the optimal trading problem under exponential OU dynamics with proportional transaction costs and log utility.

In the context of pairs trading, a number of studies have also considered market timing strategy with two price levels. For example, Gatev et al. [2006] study the historical returns from the buy-low-sell-high strategy where the entry/exit levels are set as ± 1 standard deviation from the long-run mean. Similarly, Avellaneda and Lee [2010] consider starting and ending a pairs trade based on the spread's distance from its mean. In Elliott et al. [2005], the market entry timing is modeled by the first passage time of an OU process, followed by an exit at a fixed finite horizon. In comparison, rather than assigning *ad hoc* price levels or fixed trading times, our approach will generate the entry and exit thresholds as solutions of an optimal double stopping problem. Considering an exponential OU asset price with zero log mean, Bertram [2010] numerically computes the optimal enter and exit levels that maximize the expected return per unit time. Gregory et al. [2010] also apply this approach to log-spread following the CIR and GARCH diffusion models. Other timing strategies adopted by practitioners have been discussed in Vidyamurthy [2004].

On the other hand, the related problem of constructing portfolios and hedging with mean reverting asset prices has been studied. For example, Benth and Karlsen [2005] study the utility maximization problem that involves dynamically trading an exponential OU underlying asset. Jurek and Yang [2007] analyze a finite-horizon portfolio optimization problem with an OU asset subject to the power utility and Epstein-Zin recursive utility. Chiu and Wong [2012] consider the dynamic trading of co-integrated assets with a mean-variance criterion. Tourin and Yan [2013] derive the dynamic trading strategy for two co-integrated stocks in order to maximize the expected terminal utility of wealth over a fixed horizon. They simplify the associated Hamilton-Jacobi-Bellman equation and obtain a closed-form solution. In the stochastic control approach, incorporating transaction costs and stop-loss exit can potentially limit model tractability and is not implemented in these studies.

A summary of the chapters is presented in Table 1.1.

	OU	XOU	CIR
Double Stopping	Chapter 2	Chapter 3	Chapter 4
Switching	–	Chapter 3	Chapter 4

Table 1.1: Summary of our results on the optimal double stopping and swiching problems under the OU, XOU, and CIR processes.

Chapter 2

Trading under OU Dynamics

Motivated by the industry practice of pairs trading, we study the optimal timing strategies for trading a mean-reverting price spread. An optimal double stopping problem is formulated to analyze the timing to start and subsequently liquidate the position subject to transaction costs. Modeling the price spread by an Ornstein-Uhlenbeck process, we apply a probabilistic methodology and rigorously derive the optimal price intervals for market entry and exit. A number of extensions are also considered, such as incorporating a stop-loss constraint, or a minimum holding period. We show that the entry region is characterized by a bounded price interval that lies strictly above the stop-loss level. As for the exit timing, a higher stop-loss level always implies a lower optimal take-profit level. Both analytical and numerical results are provided to illustrate the dependence of timing strategies on model parameters such as transaction costs and stop-loss level.

In Section 2.1, we discuss a pairs trading example with OU price spreads, and formulate the optimal trading problem. Our method of solution is presented in Section 2.2. In Section 2.3, we analytically solve the optimal double stopping problem and examine the optimal entry and exit strategies. In Section 2.4, we study the trading problem with a stop-loss constraint. In Section

2.5, we present a number of extensions. The proofs of all lemmas are provided in Appendix B.

2.1 Problem Overview

In the background, we fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the historical probability measure $\mathbb{P}$. We consider an Ornstein-Uhlenbeck (OU) process driven by the SDE:

$$dX_t = \mu(\theta - X_t)\, dt + \sigma\, dB_t, \tag{2.1.1}$$

with constants $\mu, \sigma > 0$, $\theta \in \mathbb{R}$, and state space $\mathbb{R}$. Here, B is a standard Brownian motion under $\mathbb{P}$. Denote by $\mathbb{F} \equiv (\mathcal{F}_t)_{t \geq 0}$ the filtration generated by X.

2.1.1 A Pairs Trading Example

Let us discuss a pairs trading example where we model the value of the resulting position by an OU process. The primary objective is to motivate our trading problem, rather than proposing new estimation methodologies or empirical studies on pairs trading. For related studies and more details, we refer to the seminal paper by Engle and Granger [1987], the books Hamilton [1994], Tsay [2005], and references therein.

We construct a portfolio by holding α shares of a risky asset $S^{(1)}$ and shorting β shares of another risky asset $S^{(2)}$, yielding a portfolio value $X_t^{\alpha,\beta} = \alpha S_t^{(1)} - \beta S_t^{(2)}$ at time $t \geq 0$. The pair of assets are selected to form a mean-reverting portfolio value. In addition, one can adjust the strategy (α, β) to enhance the level of mean reversion. For the purpose of testing mean reversion, only the ratio between α and β matters, so we can keep α constant while varying β without loss of generality. For every strategy (α, β), we observe the resulting portfolio values $(x_i^{\alpha,\beta})_{i=0,1,\dots,n}$ realized over an n-day period. We then apply the method of maximum likelihood estimation (MLE) to fit the observed portfolio values to an OU process and determine the model parameters. Under the OU model, the conditional probability density of X_{t_i} at time t_i given $X_{t_{i-1}} = x_{i-1}$ with time increment $\Delta t = t_i - t_{i-1}$ is given by

$$f^{OU}(x_i|x_{i-1};\theta,\mu,\sigma) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left(-\frac{(x_i - x_{i-1}e^{-\mu\Delta t} - \theta(1-e^{-\mu\Delta t}))^2}{2\tilde{\sigma}^2} \right),$$

with the constant

$$\tilde{\sigma}^2 = \sigma^2 \frac{1-e^{-2\mu\Delta t}}{2\mu}.$$

Using the observed values $(x_i^{\alpha,\beta})_{i=0,1,\dots,n}$, we maximize the average log-likelihood defined by

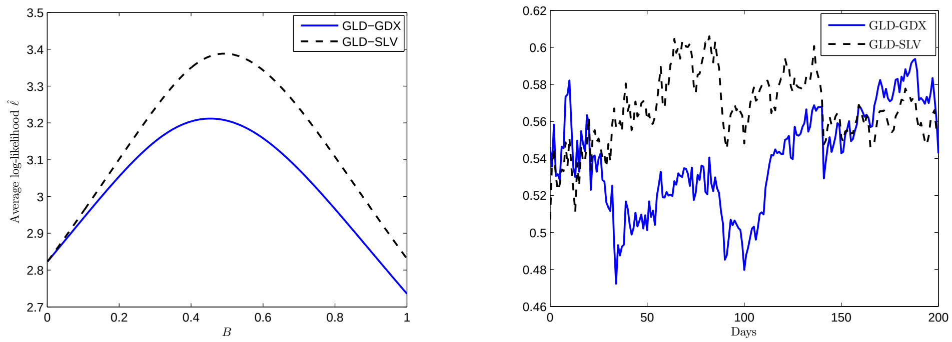
$$\begin{aligned}
&\ell(\theta,\mu,\sigma|x_0^{\alpha,\beta},x_1^{\alpha,\beta},\dots,x_n^{\alpha,\beta}) \\
&:= \frac{1}{n}\sum_{i=1}^{n} \ln f^{OU}\left(x_i^{\alpha,\beta}|x_{i-1}^{\alpha,\beta};\theta,\mu,\sigma\right) \\
&= -\frac{1}{2}\ln(2\pi) - \ln(\tilde{\sigma}) - \frac{1}{2n\tilde{\sigma}^2}\sum_{i=1}^{n}[x_i^{\alpha,\beta} - x_{i-1}^{\alpha,\beta}e^{-\mu\Delta t} - \theta(1-e^{-\mu\Delta t})]^2,
\end{aligned}$$

and denote by $\hat{\ell}(\theta^*,\mu^*,\sigma^*)$ the maximized average log-likelihood over θ, μ, and σ for a given strategy (α,β). For any α, we choose the strategy (α,β^*), where

$$\beta^* = \arg\max_{\beta} \hat{\ell}(\theta^*,\mu^*,\sigma^*|x_0^{\alpha,\beta},x_1^{\alpha,\beta},\dots,x_n^{\alpha,\beta}).$$

For example, suppose we invest A dollar(s) in asset $S^{(1)}$, so $\alpha = A/S_0^{(1)}$ shares is held. At the same time, we short $\beta = B/S_0^{(2)}$ shares in $S^{(2)}$, for $B/A = 0.001, 0.002, \dots, 1$. This way, the sign of the initial portfolio value depends on the sign of the difference $A - B$, which is non-negative. Without loss of generality, we set $A = 1$.

In Figure 2.1, we illustrate an example based on two pairs of exchange-traded funds (ETFs), namely, the Market Vectors Gold Miners (GDX) and



(a) (b)

Figure 2.1: (a) Average log-likelihood plotted against B. (b) Historical price paths with maximum average log-likelihood. The solid line plots the portfolio price with longing \$1 GLD and shorting \$0.454 GDX, and the dashed line plots the portfolio price with longing \$1 GLD and shorting \$0.493 SLV.

iShares Silver Trust (SLV) against the SPDR Gold Trust (GLD) respectively. These liquidly traded funds aim to track the price movements of the NYSE Arca Gold Miners Index (GDX), silver (SLV), and gold bullion (GLD) respectively. These ETF pairs are also used in Triantafyllopoulos and Montana [2011] and Dunis et al. [2013] for their statistical and empirical studies on ETF pairs trading.

Using price data from August 2011 to May 2012 ($n = 200$, $\Delta t = 1/252$), we compute and plot in Figure 2.1(a) the average log-likelihood against the cash amount B, and find that $\hat{\ell}$ is maximized at $B^* = 0.454$ (resp. 0.493) for the GLD-GDX pair (resp. GLD-SLV pair). From this MLE-optimal B^*, we obtain the strategy (α, β^*), where $\alpha = 1/S_0^{(1)}$ and $\beta^* = B^*/S_0^{(2)}$. In this example, the average log-likelihood for the GLD-SLV pair happens to dominate that for GLD-GDX, suggesting a higher degree of fit to the OU model. Figure 2.1(b) depicts the historical price paths with the strategy (α, β^*).

We summarize the estimation results in Table 2.1. For each pair, we first estimate the parameters for the OU model from empirical price data. Then, we use the estimated parameters to simulate price paths according the corresponding OU process. Based on these simulated OU paths, we perform another MLE and obtain another set of OU parameters as well as the maximum average log-likelihood $\hat{\ell}$. As we can see, the two sets of estimation outputs (the rows names "empirical" and "simulated") are very close, suggesting the empirical price process fits well to the OU model.

	Price	$\hat{\theta}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\ell}$
GLD-GDX	empirical	0.5388	16.6677	0.1599	3.2117
	simulated	0.5425	14.3893	0.1727	3.1304
GLD-SLV	empirical	0.5680	33.4593	0.1384	3.3882
	simulated	0.5629	28.8548	0.1370	3.3898

Table 2.1: MLE estimates of OU process parameters using historical prices of GLD, GDX, and SLV from August 2011 to May 2012. The portfolio consists of \$1 in GLD and -\$0.454 in GDX (resp. -\$0.493 in SLV). For each pair, the second row (simulated) shows the MLE parameter estimates based on a simulated price path corresponding to the estimated parameters from the first row (empirical).

2.1.2 Optimal Stopping Problem

Given that a price process or portfolio value evolves according to an OU process, our main objective is to study the optimal timing to open and subsequently close the position subject to transaction costs. This leads to the analysis of an optimal double stopping problem.

First, suppose that the investor already has an existing position whose value process $(X_t)_{t\geq 0}$ follows (2.1.1). If the position is closed at some time τ, then the investor will receive the value X_τ and pay a constant transaction cost $c_s \in \mathbb{R}$. To maximize the expected discounted value, the investor solves the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau}(X_\tau - c_s) \right\}, \tag{2.1.2}$$

where $\mathcal{T}$ denotes the set of all $\mathbb{F}$-stopping times, and $r > 0$ is the investor's subjective constant discount rate. We have also used the shorthand notation: $\mathbb{E}_x\{\cdot\} \equiv \mathbb{E}\{\cdot | X_0 = x\}$.

From the investor's viewpoint, $V(x)$ represents the expected liquidation value associated with X. On the other hand, the current price plus the transaction cost constitute the total cost to enter the trade. The investor can always choose the optimal timing to start the trade, or not to enter at all. This leads us to analyze the entry timing inherent in the trading problem. Precisely, we solve

$$J(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-\hat{r}\nu}(V(X_\nu) - X_\nu - c_b) \right\}, \tag{2.1.3}$$

with $\hat{r} > 0$, $c_b \in \mathbb{R}$. In other words, the investor seeks to maximize the expected difference between the value function $V(X_\nu)$ and the current X_ν, minus transaction cost c_b. The value function $J(x)$ represents the maximum expected value of the investment opportunity in the price process X, with transaction costs c_b and c_s incurred, respectively, at entry and exit. For our analysis, the pre-entry and post-entry discount rates, $\hat{r}$ and r, can be different, as long as $0 < \hat{r} \leq r$. Moreover, the transaction costs c_b and c_s can also differ, as long as $c_s + c_b > 0$. Furthermore, since $\tau = +\infty$ and $\nu = +\infty$ are candidate stopping times for (2.1.2) and (2.1.3) respectively, the two value functions $V(x)$ and $J(x)$ are non-negative.

As extension, we can incorporate a stop-loss level of the pairs trade, that caps the maximum loss. In practice, the stop-loss level may be exogenously imposed by the manager of a trading desk. In effect, if the price X ever reaches level L prior to the investor's voluntary liquidation time, then the position will be closed immediately. The stop-loss signal is given by the first passage time

$$\tau_L := \inf\{t \geq 0 \,:\, X_t \leq L\}.$$

Therefore, we determine the entry and liquidation timing from the constrained optimal stopping problem:

$$J_L(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-\hat{r}\nu}(V_L(X_\nu) - X_\nu - c_b) \right\}, \tag{2.1.4}$$

$$V_L(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r(\tau \wedge \tau_L)}(X_{\tau \wedge \tau_L} - c_s) \right\}. \tag{2.1.5}$$

Due to the additional timing constraint, the investor may be forced to exit early at the stop-loss level for any given liquidation level. Hence, the stop-loss constraint reduces the value functions, and precisely we deduce that $x - c_s \leq V_L(x) \leq V(x)$ and $0 \leq J_L(x) \leq J(x)$. As we will show in Sections 2.3 and 2.4, the optimal timing strategies with and without stop-loss are quite different.

2.2 Method of Solution

In this section, we disucss our method of solution. First, we denote the infinitesimal generator of the OU process X by

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{d^2}{dx^2} + \mu(\theta - x)\frac{d}{dx}, \tag{2.2.1}$$

and recall the classical solutions of the differential equation

$$\mathcal{L}u(x) = ru(x), \tag{2.2.2}$$

for $x \in \mathbb{R}$, are (see e.g. p.542 of Borodin and Salminen [2002] and Prop. 2.1 of Alili et al. [2005]):

$$F(x) \equiv F(x;r) := \int_0^\infty u^{\frac{r}{\mu}-1} e^{\sqrt{\frac{2\mu}{\sigma^2}}(x-\theta)u - \frac{u^2}{2}}\, du, \tag{2.2.3}$$

$$G(x) \equiv G(x;r) := \int_0^\infty u^{\frac{r}{\mu}-1} e^{\sqrt{\frac{2\mu}{\sigma^2}}(\theta-x)u - \frac{u^2}{2}}\, du. \tag{2.2.4}$$

Direct differentiation yields that $F'(x) > 0$, $F''(x) > 0$, $G'(x) < 0$ and $G''(x) > 0$. Hence, we observe that both $F(x)$ and $G(x)$ are strictly positive and convex, and they are, respectively, strictly increasing and decreasing.

Define the first passage time of X to some level κ by $\tau_\kappa = \inf\{t \geq 0 : X_t = \kappa\}$. As is well known, F and G admit the probabilistic expressions (see Itō and McKean [1965] and Rogers and Williams [2000]):

$$\mathbb{E}_x\{e^{-r\tau_\kappa}\} = \begin{cases} \frac{F(x)}{F(\kappa)} & \text{if } x \leq \kappa, \\ \frac{G(x)}{G(\kappa)} & \text{if } x \geq \kappa. \end{cases}$$

A key step of our solution method involves the transformation

$$\psi(x) := \frac{F}{G}(x). \tag{2.2.5}$$

Starting at any $x \in \mathbb{R}$, we denote by $\tau_a \wedge \tau_b$ the exit time from an interval $[a, b]$ with $-\infty \leq a \leq x \leq b \leq +\infty$. With the reward function $h(x) = x - c_s$, we compute the corresponding expected discounted reward:

$$\begin{aligned} &\mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)} h(X_{\tau_a \wedge \tau_b})\} \\ &= h(a)\mathbb{E}_x\{e^{-r\tau_a} 1_{\{\tau_a < \tau_b\}}\} + h(b)\mathbb{E}_x\{e^{-r\tau_b} 1_{\{\tau_a > \tau_b\}}\} && (2.2.6) \\ &= h(a)\frac{F(x)G(b) - F(b)G(x)}{F(a)G(b) - F(b)G(a)} + h(b)\frac{F(a)G(x) - F(x)G(a)}{F(a)G(b) - F(b)G(a)} && (2.2.7) \\ &= G(x)\left[\frac{h(a)}{G(a)}\frac{\psi(b) - \psi(x)}{\psi(b) - \psi(a)} + \frac{h(b)}{G(b)}\frac{\psi(x) - \psi(a)}{\psi(b) - \psi(a)}\right] \\ &= G(\psi^{-1}(z))\left[H(z_a)\frac{z_b - z}{z_b - z_a} + H(z_b)\frac{z - z_a}{z_b - z_a}\right], && (2.2.8) \end{aligned}$$

where $z_a = \psi(a)$, $z_b = \psi(b)$, and

$$H(z) := \begin{cases} \frac{h}{G} \circ \psi^{-1}(z) & \text{if } z > 0, \\ \lim\limits_{x \to -\infty} \frac{(h(x))^+}{G(x)} & \text{if } z = 0. \end{cases} \tag{2.2.9}$$

The second equality (2.2.7) follows from the fact that $f(x) := \mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)} 1_{\{\tau_a < \tau_b\}}\}$ is the unique solution to (2.2.2) with boundary conditions $f(a) = 1$ and $f(b) =$

0. Similar reasoning applies to the function $g(x) := \mathbb{E}_x\{e^{-r(\tau_a\wedge\tau_b)}1_{\{\tau_a>\tau_b\}}\}$ with $g(a) = 0$ and $g(b) = 1$. The last equality (2.2.8) transforms the problem from x coordinate to $z = \psi(x)$ coordinate (see (2.2.5)).

The candidate optimal exit interval $[a^*, b^*]$ is determined by maximizing the expectation in (2.2.6). This is equivalent to maximizing (2.2.8) over z_a and z_b in the transformed problem. This leads to

$$W(z) := \sup_{\{z_a,z_b:z_a\leq z\leq z_b\}} \left[H(z_a)\frac{z_b - z}{z_b - z_a} + H(z_b)\frac{z - z_a}{z_b - z_a} \right]. \tag{2.2.10}$$

This is the smallest concave majorant of H. Applying the definition of W to (2.2.8), we can express the maximal expected discounted reward as

$$G(x)W(\psi(x)) = \sup_{\{a,b:a\leq x\leq b\}} \mathbb{E}_x\{e^{-r(\tau_a\wedge\tau_b)}h(X_{\tau_a\wedge\tau_b})\}.$$

Remark 2.2.1. *If $a = -\infty$, then we have $\tau_a = +\infty$ and $1_{\{\tau_a<\tau_b\}} = 0$ a.s. In effect, this removes the lower exit level, and the corresponding expected discounted reward is*

$$\begin{aligned}\mathbb{E}_x\{e^{-r(\tau_a\wedge\tau_b)}h(X_{\tau_a\wedge\tau_b})\} &= \mathbb{E}_x\{e^{-r\tau_a}h(X_{\tau_a})1_{\{\tau_a<\tau_b\}}\} + \mathbb{E}_x\{e^{-r\tau_b}h(X_{\tau_b})1_{\{\tau_a>\tau_b\}}\}\\ &= \mathbb{E}_x\{e^{-r\tau_b}h(X_{\tau_b})\}.\end{aligned}$$

Consequently, by considering interval-type strategies, we also include the class of stopping strategies of reaching a single upper level b (see Theorem 2.3.3 below).

Next, we prove the optimality of the proposed stopping strategy and provide an expression for the value function.

Theorem 2.2.2. *The value function $V(x)$ defined in (2.1.2) is given by*

$$V(x) = G(x)W(\psi(x)), \tag{2.2.11}$$

where G, ψ and W are defined in (2.2.4), (2.2.5) and (2.2.10), respectively.

Proof. Since $\tau_a \wedge \tau_b \in \mathcal{T}$, we have $V(x) \geq \sup_{\{a,b:a\leq x\leq b\}} \mathbb{E}_x\{e^{-r(\tau_a\wedge\tau_b)}h(X_{\tau_a\wedge\tau_b})\} = G(x)W(\psi(x))$.

To show the reverse inequality, we first show that

$$G(x)W(\psi(x)) \geq \mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})W(\psi(X_{t\wedge\tau}))\},$$

for $\tau \in \mathcal{T}$ and $t \geq 0$. The concavity of W implies that, for any fixed z, there exists an affine function $L_z(\alpha) := m_z\alpha + c_z$ such that $L_z(\alpha) \geq W(\alpha)$ and $L_z(z) = W(z)$ at $\alpha = z$, where m_z and c_z are both constants depending on z. This leads to the inequality

$$\begin{aligned}
&\mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})W(\psi(X_{t\wedge\tau}))\}\\
&\leq \mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})L_{\psi(x)}(\psi(X_{t\wedge\tau}))\}\\
&= m_{\psi(x)}\mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})\psi(X_{t\wedge\tau})\} + c_{\psi(x)}\mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})\}\\
&= m_{\psi(x)}\mathbb{E}_x\{e^{-r(t\wedge\tau)}F(X_{t\wedge\tau})\} + c_{\psi(x)}\mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})\}\\
&= m_{\psi(x)}F(x) + c_{\psi(x)}G(x) && (2.2.12)\\
&= G(x)L_{\psi(x)}(\psi(x))\\
&= G(x)W(\psi(x)), && (2.2.13)
\end{aligned}$$

where (2.2.12) follows from the martingale property of $(e^{-rt}F(X_t))_{t\geq 0}$ and $(e^{-rt}G(X_t))_{t\geq 0}$.

By (2.2.13) and the fact that W majorizes H, it follows that

$$\begin{aligned}
G(x)W(\psi(x)) &\geq \mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})W(\psi(X_{t\wedge\tau}))\}\\
&\geq \mathbb{E}_x\{e^{-r(t\wedge\tau)}G(X_{t\wedge\tau})H(\psi(X_{t\wedge\tau}))\}\\
&= \mathbb{E}_x\{e^{-r(t\wedge\tau)}h(X_{t\wedge\tau})\}. && (2.2.14)
\end{aligned}$$

Maximizing (2.2.14) over all $\tau \in \mathcal{T}$ and $t \geq 0$ yields that $G(x)W(\psi(x)) \geq V(x)$. ◻

Let us emphasize that the optimal levels (a^*, b^*) may depend on the initial value x, and can potentially coincide, or take values $-\infty$ and $+\infty$. As such, the structure of the stopping and delay regions can potentially be characterized by multiple intervals, leading to *disconnected* delay regions (see Theorem 2.4.5 below).

We follow the procedure for Theorem 2.2.2 to derive the expression for the value function J in (2.1.3). First, we denote $\hat{F}(x) = F(x;\hat{r})$ and $\hat{G}(x) = G(x;\hat{r})$ (see (2.2.3)–(2.2.4)), with discount rate $\hat{r}$. In addition, we define the transformation

$$\hat{\psi}(x) := \frac{\hat{F}}{\hat{G}}(x) \quad \text{and} \quad \hat{h}(x) = V(x) - x - c_b. \tag{2.2.15}$$

Using these functions, we consider the function analogous to H:

$$\hat{H}(z) := \begin{cases} \frac{\hat{h}}{\hat{G}} \circ \hat{\psi}^{-1}(z) & \text{if } z > 0, \\ \lim\limits_{x \to -\infty} \frac{(\hat{h}(x))^+}{\hat{G}(x)} & \text{if } z = 0. \end{cases} \tag{2.2.16}$$

Following the steps (2.2.6)–(2.2.10) with F, G, ψ, and H replaced by $\hat{F}$, $\hat{G}$, $\hat{\psi}$, and $\hat{H}$, respectively, we write down the smallest concave majorant $\hat{W}$ of $\hat{H}$, namely,

$$\hat{W}(z) := \sup_{\{z_{\hat{a}}, z_{\hat{b}} : z_{\hat{a}} \le z \le z_{\hat{b}}\}} \left[\hat{H}(z_{\hat{a}}) \frac{z_{\hat{b}} - z}{z_{\hat{b}} - z_{\hat{a}}} + \hat{H}(z_{\hat{b}}) \frac{z - z_{\hat{a}}}{z_{\hat{b}} - z_{\hat{a}}} \right].$$

From this, we seek to determine the candidate optimal entry interval $(z_{\hat{a}^*}, z_{\hat{b}^*})$ in the $z = \hat{\psi}(x)$ coordinate. Following the proof of Theorem 2.2.2 with the new functions $\hat{F}$, $\hat{G}$, $\hat{\psi}$, $\hat{H}$, and $\hat{W}$, the value function of the optimal entry timing problem admits the expression

$$J(x) = \hat{G}(x)\hat{W}(\hat{\psi}(x)). \tag{2.2.17}$$

An alternative way to solve for $V(x)$ and $J(x)$ is to look for the solutions to the pair of variational inequalities

$$\min\{rV(x) - \mathcal{L}V(x), V(x) - (x - c_s)\} = 0, \tag{2.2.18}$$

$$\min\{\hat{r}J(x) - \mathcal{L}J(x), J(x) - (V(x) - x - c_b)\} = 0, \tag{2.2.19}$$

for $x \in \mathbb{R}$. With sufficient regularity conditions, this approach can verify that the solutions to the VIs, $V(x)$ and $J(x)$, indeed correspond to the optimal stopping problems (see, for example, Theorem 10.4.1 of Øksendal [2003]). Nevertheless, this approach does not immediately suggest candidate optimal timing strategies or value functions, and typically begins with a conjecture on the structure of the optimal stopping times, followed by verification. In contrast, our approach allows us to directly construct the value functions, at the cost of analyzing the properties of H, W, $\hat{H}$, and $\hat{W}$.

2.3 Analytical Results

We will first study the optimal exit timing in Section 2.3.1, followed by the optimal entry timing problem in Section 2.3.2.

2.3.1 Optimal Exit Timing

We now analyze the optimal exit timing problem (2.1.2) under the OU model. First, we obtain a bound for the value function V in terms of F.

Lemma 2.3.1. *There exists a positive constant K such that, for all $x \in \mathbb{R}$, $0 \leq V(x) \leq KF(x)$.*

In preparation for the next result, we summarize the crucial properties of H.

Lemma 2.3.2. *The function H is continuous on $[0, +\infty)$, twice differentiable on $(0, +\infty)$ and possesses the following properties:*

(i) $H(0) = 0$, *and*

$$H(z) \begin{cases} < 0 & \text{if } z \in (0, \psi(c_s)), \\ > 0 & \text{if } z \in (\psi(c_s), +\infty). \end{cases}$$

(ii) Let x^* be the unique solution to $G(x) - (x - c_s)G'(x) = 0$. Then, we have

$$H(z) \text{ is strictly} \begin{cases} \text{decreasing} & \text{if } z \in (0, \psi(x^*)), \\ \text{increasing} & \text{if } z \in (\psi(x^*), +\infty), \end{cases}$$

and $x^* < c_s \wedge L^*$ with

$$L^* = \frac{\mu\theta + rc_s}{\mu + r}. \tag{2.3.1}$$

(iii)

$$H(z) \text{ is } \begin{cases} \text{convex} & \text{if } z \in (0, \psi(L^*)], \\ \text{concave} & \text{if } z \in [\psi(L^*), +\infty). \end{cases}$$

Based on Lemma 2.3.2, we sketch H in Figure 2.2. The properties of H are essential in deriving the value function and optimal liquidation level, as we show next.

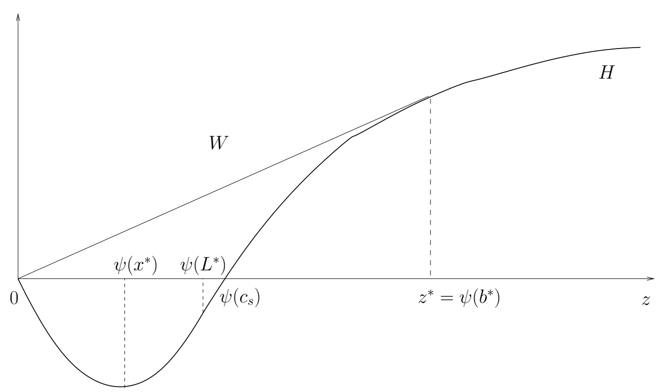


Figure 2.2: Sketches of H and W. By Lemma 2.3.2, H is convex on the left of $\psi(L^*)$ and concave on the right. The smallest concave majorant W is a straight line tangent to H at z^* on $[0, z^*)$, and coincides with H on $[z^*, +\infty)$.

Theorem 2.3.3. *The optimal liquidation problem* (2.1.2) *admits the solution*

$$V(x) = \begin{cases} (b^* - c_s)\frac{F(x)}{F(b^*)} & \text{if } x \in (-\infty, b^*), \\ x - c_s & \text{otherwise,} \end{cases} \tag{2.3.2}$$

where the optimal liquidation level b^* *is found from the equation*

$$F(b) = (b - c_s)F'(b), \tag{2.3.3}$$

and is bounded below by $L^* \vee c_s$*. The corresponding optimal liquidation time is given by*

$$\tau^* = \inf\{t \geq 0 \,:\, X_t \geq b^*\}. \tag{2.3.4}$$

Proof. From Lemma 2.3.2 and the fact that $H'(z) \to 0$ as $z \to +\infty$ (see also Figure 2.2), we infer that there exists a unique number $z^* > \psi(L^*) \vee \psi(c_s)$ such that

$$\frac{H(z^*)}{z^*} = H'(z^*). \tag{2.3.5}$$

In turn, the smallest concave majorant is given by

$$W(z) = \begin{cases} z\frac{H(z^*)}{z^*} & \text{if } \ z < z^*, \\ H(z) & \text{if } \ z \geq z^*. \end{cases} \tag{2.3.6}$$

Substituting $b^* = \psi^{-1}(z^*)$ into (2.3.5), we have the LHS

$$\frac{H(z^*)}{z^*} = \frac{H(\psi(b^*))}{\psi(b^*)} = \frac{b^* - c_s}{F(b^*)}, \tag{2.3.7}$$

and the RHS

$$\begin{aligned} H'(z^*) &= \frac{G(\psi^{-1}(z^*)) - (\psi^{-1}(z^*) - c_s)G'(\psi^{-1}(z^*))}{F'(\psi^{-1}(z^*))G(\psi^{-1}(z^*)) - F(\psi^{-1}(z^*))G'(\psi^{-1}(z^*))} \\ &= \frac{G(b^*) - (b^* - c_s)G'(b^*)}{F'(b^*)G(b^*) - F(b^*)G'(b^*)}. \end{aligned}$$

Equivalently, we can express condition (2.3.5) in terms of b^*:

$$\frac{b^* - c_s}{F(b^*)} = \frac{G(b^*) - (b^* - c_s)G'(b^*)}{F'(b^*)G(b^*) - F(b^*)G'(b^*)},$$

which can be further simplified to

$$F(b^*) = (b^* - c_s)F'(b^*).$$

Applying (2.3.7) to (2.3.6), we get

$$W(\psi(x)) = \begin{cases} \psi(x)\frac{H(z^*)}{z^*} = \frac{F(x)}{G(x)}\frac{b^*-c_s}{F(b^*)} & \text{if } x < b^*, \\ H(\psi(x)) = \frac{x-c_s}{G(x)} & \text{if } x \geq b^*. \end{cases} \tag{2.3.8}$$

In turn, we obtain the value function $V(x)$ by substituting (2.3.8) into (2.2.11).

□

Next, we examine the dependence of the investor's optimal timing strategy on the transaction cost c_s.

Proposition 2.3.4. *The value function $V(x)$ of (2.1.2) is decreasing in the transaction cost c_s for every $x \in \mathbb{R}$, and the optimal liquidation level b^* is increasing in c_s.*

Proof. For any $x \in \mathbb{R}$ and $\tau \in \mathcal{T}$, the corresponding expected discounted reward, $\mathbb{E}_x\{e^{-r\tau}(X_\tau - c_s)\} = \mathbb{E}_x\{e^{-r\tau}X_\tau\} - c_s\,\mathbb{E}_x\{e^{-r\tau}\}$, is decreasing in c_s. This implies that $V(x)$ is also decreasing in c_s. Next, we treat the optimal threshold $b^*(c_s)$ as a function of c_s, and differentiate (2.3.3) w.r.t. c_s to get

$$b^{*\prime}(c_s) = \frac{F'(b^*)}{(b^* - c_s)F''(b^*)} > 0.$$

Since $F'(x) > 0$, $F''(x) > 0$ (see (2.2.3)), and $b^* > c_s$ according to Theorem 2.3.3, we conclude that b^* is increasing in c_s. □

In other words, if the transaction cost is high, the investor would tend to liquidate at a higher level, in order to compensate the loss on transaction cost. For other parameters, such as μ and σ, the dependence of b^* is generally not monotone.

2.3.2 Optimal Entry Timing

Having solved for the optimal exit timing, we now turn to the optimal entry timing problem. In this case, the value function is

$$J(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x\{e^{-\hat{r}\nu}(V(X_\nu) - X_\nu - c_b)\}, \quad x \in \mathbb{R},$$

where $V(x)$ is given by Theorem 2.3.3.

Lemma 2.3.5. *There exists a positive constant $\hat{K}$ such that, for all $x \in \mathbb{R}$, $0 \leq J(x) \leq \hat{K}\hat{G}(x)$.*

To solve for the optimal entry threshold(s), we will need several properties of $\hat{H}$, as we summarize below.

Lemma 2.3.6. *The function $\hat{H}$ is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, and twice differentiable on $(0, \hat{\psi}(b^*)) \cup (\hat{\psi}(b^*), +\infty)$, and possesses the following properties:*

(i) $\hat{H}(0) = 0$. Let $\bar{d}$ denote the unique solution to $\hat{h}(x) = 0$, then $\bar{d} < b^*$ and

$$\hat{H}(z) \begin{cases} > 0 & \text{if } z \in (0, \hat{\psi}(\bar{d})), \\ < 0 & \text{if } z \in (\hat{\psi}(\bar{d}), +\infty). \end{cases}$$

(ii) $\hat{H}(z)$ is strictly decreasing if $z \in (\hat{\psi}(b^*), +\infty)$.

(iii) Let $\underline{b}$ denote the unique solution to $(\mathcal{L} - \hat{r})\hat{h}(x) = 0$, then $\underline{b} < L^*$ and

$$\hat{H}(z) \text{ is } \begin{cases} \text{concave} & \text{if } z \in (0, \hat{\psi}(\underline{b})), \\ \text{convex} & \text{if } z \in (\hat{\psi}(\underline{b}), +\infty). \end{cases}$$

In Figure 2.3, we give a sketch of $\hat{H}$ according to Lemma 2.3.6. This will be useful for deriving the optimal entry level.

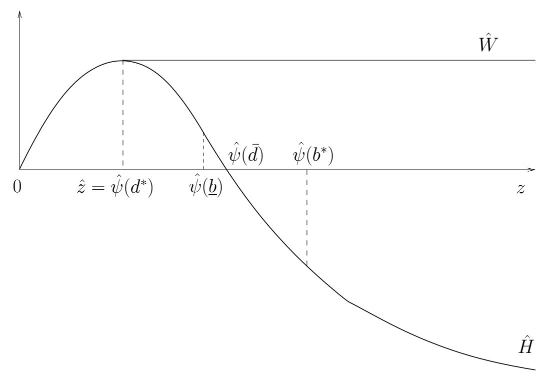


Figure 2.3: Sketches of $\hat{H}$ and $\hat{W}$. The function $\hat{W}$ coincides with $\hat{H}$ on $[0, \hat{z}]$ and is equal to the constant $\hat{H}(\hat{z})$ on $(\hat{z}, +\infty)$.

Theorem 2.3.7. *The optimal entry timing problem* (2.1.3) *admits the solution*

$$J(x) = \begin{cases} V(x) - x - c_b & \text{if } x \in (-\infty, d^*], \\ \frac{V(d^*) - d^* - c_b}{\hat{G}(d^*)}\hat{G}(x) & \text{if } x \in (d^*, +\infty), \end{cases} \tag{2.3.9}$$

where the optimal entry level d^* *is found from the equation*

$$\hat{G}(d)(V'(d) - 1) = \hat{G}'(d)(V(d) - d - c_b). \tag{2.3.10}$$

Proof. We look for the value function of the form: $J(x) = \hat{G}(x)\hat{W}(\hat{\psi}(x))$, where $\hat{W}$ is the the smallest concave majorant of $\hat{H}$. From Lemma 2.3.6 and Figure 2.3, we infer that there exists a unique number $\hat{z} < \hat{\psi}(b^*)$ such that

$$\hat{H}'(\hat{z}) = 0. \tag{2.3.11}$$

This implies that

$$\hat{W}(z) = \begin{cases} \hat{H}(z) & \text{if } z \le \hat{z}, \\ \hat{H}(\hat{z}) & \text{if } z > \hat{z}. \end{cases} \tag{2.3.12}$$

Substituting $d^* = \hat{\psi}^{-1}(\hat{z})$ into (2.3.11), we have

$$\hat{H}'(\hat{z}) = \frac{\hat{G}(d^*)(V'(d^*) - 1) - \hat{G}'(d^*)(V(d^*) - d^* - c_b)}{\hat{F}'(d^*)\hat{G}(d^*) - \hat{F}(d^*)\hat{G}'(d^*)} = 0,$$

which is equivalent to condition (2.3.10). Furthermore, using (2.2.15) and (2.2.16), we get

$$\hat{H}(\hat{z}) = \frac{V(d^*) - d^* - c_b}{\hat{G}(d^*)}. \tag{2.3.13}$$

To conclude, we substitute $\hat{H}(\hat{z})$ of (2.3.13) and $\hat{H}(z)$ of (2.2.16) into $\hat{W}$ of (2.3.12), which by (2.2.17) yields the value function $J(x)$ in (2.3.9). $\square$

With the analytic solutions for V and J, we can verify by direct substitution that $V(x)$ in (2.3.2) and $J(x)$ in (2.3.9) satisfy both (2.2.18) and (2.2.19).

Since the optimal entry timing problem is nested with another optimal stopping problem, the parameter dependence of the optimal entry level is complicated. Below, we illustrate the impact of transaction cost.

Proposition 2.3.8. *The optimal entry level d^* of* (2.1.3) *is decreasing in the transaction cost c_b.*

Proof. Considering the optimal entry level d^* as a function of c_b, we differentiate (2.3.10) w.r.t. c_b to get

$$d^{*\prime}(c_b) = \frac{-\hat{G}'(d^*)}{\hat{G}(d^*)}[V''(d^*) - \frac{V(d^*) - d^* - c_b}{\hat{G}(d^*)}\hat{G}''(d^*)]^{-1}. \tag{2.3.14}$$

Since $\hat{G}(d^*) > 0$ and $\hat{G}'(d^*) < 0$, the sign of $d^{*\prime}(c_b)$ is determined by $V''(d^*) - \frac{V(d^*)-d^*-c_b}{\hat{G}(d^*)}\hat{G}''(d^*)$. Denote $\hat{f}(x) = \frac{V(d^*)-d^*-c_b}{\hat{G}(d^*)}\hat{G}(x)$. Recall that $\hat{h}(x) = V(x) - x - c_b$,

$$J(x) = \begin{cases} \hat{h}(x) & \text{if } x \in (-\infty, d^*], \\ \hat{f}(x) > \hat{h}(x) & \text{if } x \in (d^*, +\infty), \end{cases}$$

and $\hat{f}(x)$ smooth pastes $\hat{h}(x)$ at d^*. Since both $\hat{h}(x)$ and $\hat{f}(x)$ are positive decreasing convex functions, it follows that $\hat{h}''(d^*) \le \hat{f}''(d^*)$. Observing that $\hat{h}''(d^*) = V''(d^*)$ and $\hat{f}''(d^*) = \frac{V(d^*)-d^*-c_b}{\hat{G}(d^*)}\hat{G}''(d^*)$, we have $V''(d^*) - \frac{V(d^*)-d^*-c_b}{\hat{G}(d^*)}\hat{G}''(d^*) \le 0$. Applying this to (2.3.14), we conclude that $d^{*\prime}(c_b) \le 0$. $\square$

We end this section with a special example in the OU model with no mean reversion.

Remark 2.3.9. *If we set $\mu = 0$ in (2.1.1), with r and $\hat{r}$ fixed, it follows that X reduces to a Brownian motion: $X_t = \sigma B_t$, $t \geq 0$. In this case, the optimal liquidation level b^* for problem (2.1.2) is*

$$b^* = c_s + \frac{\sigma}{\sqrt{2r}},$$

and the optimal entry level d^* for problem (2.1.3) is the root to the equation

$$\left(1 + \sqrt{\frac{\hat{r}}{r}}\right) e^{\frac{\sqrt{2r}}{\sigma}(d - c_s - \frac{\sigma}{\sqrt{2r}})} = \frac{\sqrt{2\hat{r}}}{\sigma}(d + c_b) + 1, \quad d \in (-\infty, b^*).$$

2.4 Incorporating Stop-Loss Exit

Now we consider the optimal entry and exit problems with a stop-loss constraint. For convenience, we restate the value functions from (2.1.4) and (2.1.5):

$$J_L(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-\hat{r}\nu}(V_L(X_\nu) - X_\nu - c_b) \right\}, \tag{2.4.1}$$

$$V_L(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r(\tau \wedge \tau_L)}(X_{\tau \wedge \tau_L} - c_s) \right\}. \tag{2.4.2}$$

After solving for the optimal timing strategies, we will also examine the dependence of the optimal liquidation threshold on the stop-loss level L.

2.4.1 Optimal Exit Timing

We first give an analytic solution to the optimal exit timing problem.

Theorem 2.4.1. *The optimal liquidation problem (2.4.2) with stop-loss level L admits the solution*

$$V_L(x) = \begin{cases} CF(x) + DG(x) & \text{if } x \in (L, b_L^*), \\ x - c_s & \text{otherwise,} \end{cases} \tag{2.4.3}$$

where

$$C = \frac{(b_L^* - c_s)G(L) - (L - c_s)G(b_L^*)}{F(b_L^*)G(L) - F(L)G(b_L^*)}, \quad D = \frac{(L - c_s)F(b_L^*) - (b_L^* - c_s)F(L)}{F(b_L^*)G(L) - F(L)G(b_L^*)}.$$

The optimal liquidation level b_L^* is found from the equation

$$[(L - c_s)G(b) - (b - c_s)G(L)]F'(b) + [(b - c_s)F(L) - (L - c_s)F(b)]G'(b)$$
$$= G(b)F(L) - G(L)F(b). \tag{2.4.4}$$

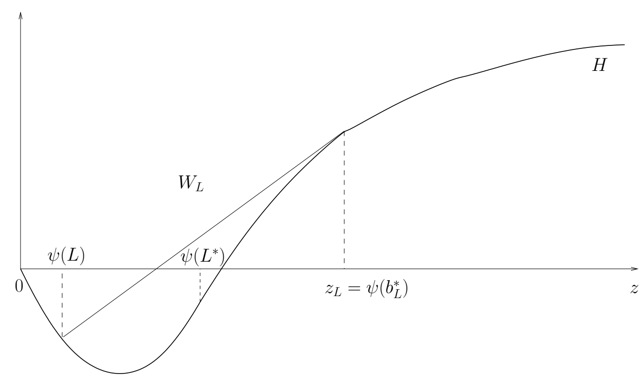


Figure 2.4: Sketch of W_L. On $[0, \psi(L)] \cup [z_L, +\infty)$, W_L coincides with H, and over $(\psi(L), z_L)$, W_L is a straight line tangent to H at z_L .

Proof. Due to the stop-loss level L, we consider the smallest concave majorant of $H(z)$, denoted by $W_L(z)$, over the restricted domain $[\psi(L), +\infty)$ and set $W_L(z) = H(z)$ for $z \in [0, \psi(L)]$.

From Lemma 2.3.2 and Figure 2.4, we see that $H(z)$ is convex over $(0, \psi(L^*)]$ and concave in $[\psi(L^*), +\infty)$. If $L \geq L^*$, then $H(z)$ is concave over $[\psi(L), +\infty)$, which implies that $W_L(z) = H(z)$ for $z \geq 0$, and thus $V_L(x) = x - c_s$ for $x \in \mathbb{R}$. On the other hand, if $L < L^*$, then $H(z)$ is convex on $[\psi(L), \psi(L^*)]$, and concave strictly increasing on $[\psi(L^*), +\infty)$. There exists a unique number $z_L > \psi(L^*)$ such that

$$\frac{H(z_L) - H(\psi(L))}{z_L - \psi(L)} = H'(z_L). \tag{2.4.5}$$

In turn, the smallest concave majorant admits the form:

$$W_L(z) = \begin{cases} H(\psi(L)) + (z - \psi(L))H'(z_L) & \text{if } \ z \in (\psi(L), z_L), \\ H(z) & \text{otherwise.} \end{cases} \tag{2.4.6}$$

Substituting $b_L^* = \psi^{-1}(z_L)$ into (2.4.5), we have from the LHS

$$\frac{H(z_L) - H(\psi(L))}{z_L - \psi(L)} = \frac{H(\psi(b_L^*)) - H(\psi(L))}{\psi(b_L^*) - \psi(L)} = \frac{\frac{b_L^* - c_s}{G(b_L^*)} - \frac{L - c_s}{G(L)}}{\frac{F(b_L^*)}{G(b_L^*)} - \frac{F(L)}{G(L)}} = C,$$

and the RHS

$$\begin{aligned} H'(z_L) &= \frac{G(\psi^{-1}(z_L)) - (\psi^{-1}(z_L) - c_s)G'(\psi^{-1}(z_L))}{F'(\psi^{-1}(z_L))G(\psi^{-1}(z_L)) - F(\psi^{-1}(z_L))G'(\psi^{-1}(z_L))} \\ &= \frac{G(b_L^*) - (b^* - c_s)G'(b_L^*)}{F'(b_L^*)G(b_L^*) - F(b_L^*)G'(b_L^*)}. \end{aligned}$$

Therefore, we can equivalently express (2.4.5) in terms of b_L^*:

$$\frac{(b_L^* - c_s)G(L) - (L - c_s)G(b_L^*)}{F(b_L^*)G(L) - F(L)G(b_L^*)} = \frac{G(b_L^*) - (b_L^* - c_s)G'(b_L^*)}{F'(b_L^*)G(b_L^*) - F(b_L^*)G'(b_L^*)},$$

which by rearrangement immediately simplifies to (2.4.4).

Furthermore, for $x \in (L, b_L^*)$, $H'(z_L) = C$ implies that

$$W_L(\psi(x)) = H(\psi(L)) + (\psi(x) - \psi(L))C.$$

Substituting this to $V_L(x) = G(x)W_L(\psi(x))$, the value function becomes

$$\begin{aligned} V_L(x) &= G(x)\big[H(\psi(L)) + (\psi(x) - \psi(L))C\big] \\ &= CF(x) + G(x)\big[H(\psi(L)) - \psi(L)C\big], \end{aligned}$$

which resembles (2.4.3) after the observation that

$$\begin{aligned} H(\psi(L)) - \psi(L)C &= \frac{L - c_s}{G(L)} - \frac{F(L)}{G(L)} \frac{(b_L^* - c_s)G(L) - (L - c_s)G(b_L^*)}{F(b_L^*)G(L) - F(L)G(b_L^*)} \\ &= \frac{(L - c_s)F(b_L^*) - (b_L^* - c_s)F(L)}{F(b_L^*)G(L) - F(L)G(b_L^*)} = D. \end{aligned}$$

□

We can interpret the investor's timing strategy in terms of three price intervals, namely, the liquidation region $[b_L^*, +\infty)$, the delay region (L, b_L^*), and the stop-loss region $(-\infty, L]$. In both liquidation and stop-loss regions, the value function $V_L(x) = x - c_s$, and therefore, the investor will immediately close out the position. From the proof of Theorem 2.4.1, if $L \geq L^* = \frac{\mu\theta + rc_s}{\mu + r}$ (see (2.3.1)), then $V_L(x) = x - c_s$, $\forall x \in \mathbb{R}$. In other words, if the stop-loss level is too high, then the delay region completely disappears, and the investor will liquidate immediately for every initial value $x \in \mathbb{R}$.

Corollary 2.4.2. *If $L < L^*$, then there exists a unique solution $b_L^* \in (L^*, +\infty)$ that solves* (2.4.4)*. If $L \geq L^*$, then $V_L(x) = x - c_s$, for $x \in \mathbb{R}$.*

The direct effect of a stop-loss exit constraint is forced liquidation whenever the price process reaches L before the upper liquidation level b_L^*. Interestingly, there is an additional indirect effect: a higher stop-loss level will induce the investor to *voluntarily* liquidate earlier at a lower take-profit level.

Proposition 2.4.3. *The optimal liquidation level b_L^* of* (2.4.2) *strictly decreases as the stop-loss level L increases.*

Proof. Recall that $z_L = \psi(b_L^*)$ and ψ is a strictly increasing function. Therefore, it is sufficient to show that z_L strictly decreases as $\tilde{L} := \psi(L)$ increases. As such, we denote $z_L(\tilde{L})$ to highlight its dependence on $\tilde{L}$. Differentiating (2.4.5) w.r.t. $\tilde{L}$ gives

$$z_L'(\tilde{L}) = \frac{H'(z_L) - H'(\tilde{L})}{H''(z_L)(z_L - \tilde{L})}. \tag{2.4.7}$$

It follows from the definitions of W_L and z_L that $H'(z_L) > H'(\tilde{L})$ and $z_L > \tilde{L}$. Also, we have $H''(z) < 0$ since H is concave at z_L. Applying these to (2.4.7), we conclude that $z_L'(\tilde{L}) < 0$. □

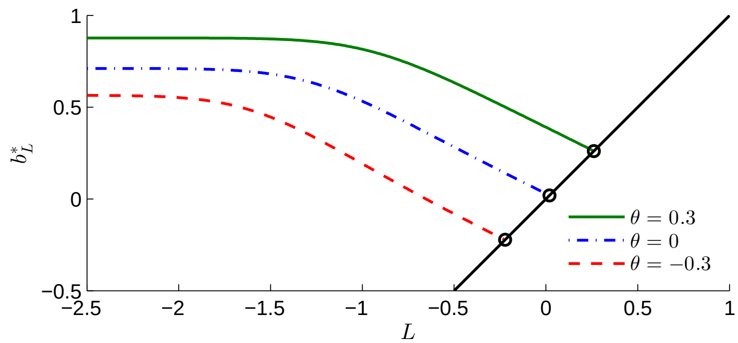


Figure 2.5: The optimal exit threshold b^*_L is strictly decreasing with respect to the stop-loss level L. The straight line is where $b^*_L = L$, and each of the three circles locates the critical stop-loss level L^*.

Figure 2.5 illustrates the optimal exit price level b^*_L as a function of the stop-loss levels L, for different long-run means θ. When b^*_L is strictly greater than L (on the left of the straight line), the delay region is non-empty. As L increases, b^*_L strictly decreases and the two meet at L^* (on the straight line), and the delay region vanishes.

Also, there is an interesting connection between cases with different long-run means and transaction costs. To this end, let us denote the value function by $V_L(x; \theta, c_s)$ to highlight the dependence on θ and c_s, and the corresponding optimal liquidation level by $b^*_L(\theta, c_s)$. We find that, for any $\theta_1, \theta_2 \in \mathbb{R}$, $c_1, c_2 > 0$, $L_1 \leq \frac{\mu\theta_1 + rc_1}{\mu + r}$, and $L_2 \leq \frac{\mu\theta_2 + rc_2}{\mu + r}$, the associated value functions and optimal liquidation levels satisfy the relationships:

$$V_{L_1}(x + \theta_1; \theta_1, c_1) = V_{L_2}(x + \theta_2; \theta_2, c_2), \tag{2.4.8}$$

$$b^*_{L_1}(\theta_1, c_1) - \theta_1 = b^*_{L_2}(\theta_2, c_2) - \theta_2, \tag{2.4.9}$$

as long as $\theta_1 - \theta_2 = c_1 - c_2 = L_1 - L_2$. These results (2.4.8) and (2.4.9) also hold in the case without stop-loss.

2.4.2 Optimal Entry Timing

We now discuss the optimal entry timing problem $J_L(x)$ defined in (2.4.1). Since $\sup_{x\in\mathbb{R}}(V_L(x) - x - c_b) \leq 0$ implies that $J_L(x) = 0$ for $x \in \mathbb{R}$, we can focus on the case with

$$\sup_{x\in\mathbb{R}}(V_L(x) - x - c_b) > 0, \tag{2.4.10}$$

and look for non-trivial optimal timing strategies.

Associated with reward function $\hat{h}_L(x) := V_L(x) - x - c_b$ from entering the market, we define the function $\hat{H}_L$ as in (2.2.9) whose properties are summarized in the following lemma.

Lemma 2.4.4. *The function $\hat{H}_L$ is continuous on $[0, +\infty)$, differentiable on $(0, \hat{\psi}(L)) \cup (\hat{\psi}(L), +\infty)$, twice differentiable on $(0, \hat{\psi}(L)) \cup (\hat{\psi}(L), \hat{\psi}(b_L^*)) \cup (\hat{\psi}(b_L^*), +\infty)$, and possesses the following properties:*

(i) $\hat{H}_L(0) = 0$. $\hat{H}_L(z) < 0$ *for* $z \in (0, \hat{\psi}(L)] \cup [\hat{\psi}(b_L^*), +\infty)$.

(ii) $\hat{H}_L(z)$ *is strictly decreasing for* $z \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(b_L^*), +\infty)$.

(iii) There exists some constant $\bar{d}_L \in (L, b_L^*)$ *such that* $(\mathcal{L} - \hat{r})\hat{h}_L(\bar{d}_L) = 0$, *and*

$$\hat{H}_L(z) \text{ is } \begin{cases} \text{convex} & \text{if } z \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(\bar{d}_L), +\infty), \\ \text{concave} & \text{if } z \in (\hat{\psi}(L), \hat{\psi}(\bar{d}_L)). \end{cases}$$

In addition, $\hat{z}_1 \in (\hat{\psi}(L), \hat{\psi}(\bar{d}_L))$, *where* $\hat{z}_1 := \arg\max_{z\in[0,+\infty)} \hat{H}_L(z)$.

Theorem 2.4.5. *The optimal entry timing problem* (2.4.1) *admits the solution*

$$J_L(x) = \begin{cases} P\hat{F}(x) & \text{if } x \in (-\infty, a_L^*), \\ V_L(x) - x - c_b & \text{if } x \in [a_L^*, d_L^*], \\ Q\hat{G}(x) & \text{if } x \in (d_L^*, +\infty), \end{cases} \tag{2.4.11}$$

where

$$P = \frac{V_L(a_L^*) - a_L^* - c_b}{\hat{F}(a_L^*)}, \quad Q = \frac{V_L(d_L^*) - d_L^* - c_b}{\hat{G}(d_L^*)}.$$

The optimal entry time is given by

$$\nu_{a_L^*, d_L^*} = \inf\{t \geq 0 \,:\, X_t \in [a_L^*, d_L^*]\}, \tag{2.4.12}$$

where the critical levels a_L^* and d_L^* satisfy, respectively,

$$\hat{F}(a)(V_L'(a) - 1) = \hat{F}'(a)(V_L(a) - a - c_b), \tag{2.4.13}$$

and

$$\hat{G}(d)(V_L'(d) - 1) = \hat{G}'(d)(V_L(d) - d - c_b). \tag{2.4.14}$$

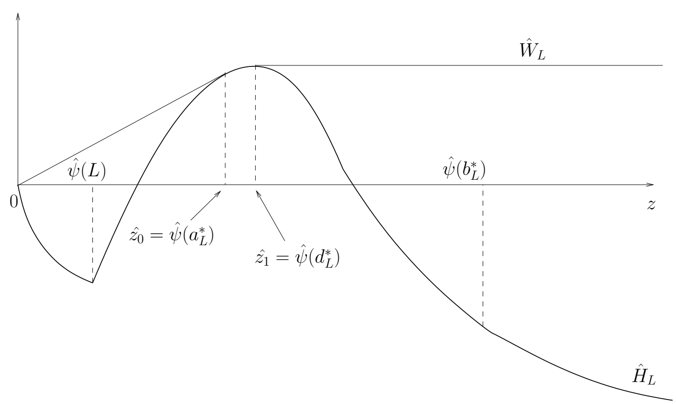


Figure 2.6: Sketches of $\hat{H}_L$ and $\hat{W}_L$. $\hat{W}_L$ is a straight line tangent to $\hat{H}_L$ at $\hat{z}_0$ on $[0, \hat{z}_0)$, coincides with $\hat{H}_L$ on $[\hat{z}_0, \hat{z}_1]$, and is equal to the constant $\hat{H}_L(\hat{z}_1)$ on $(\hat{z}_1, +\infty)$. Note that $\hat{H}_L$ is not differentiable at $\hat{\psi}(L)$.

Proof. We look for the value function of the form: $J_L(x) = \hat{G}(x)\hat{W}_L(\hat{\psi}(x))$, where $\hat{W}_L$ is the smallest non-negative concave majorant of $\hat{H}_L$. From Lemma 2.4.4 and the sketch of $\hat{H}_L$ in Figure 2.6, the maximizer of $\hat{H}_L$, $\hat{z}_1$, satisfies

$$\hat{H}_L'(\hat{z}_1) = 0. \tag{2.4.15}$$

Also there exists a unique number $\hat{z}_0 \in (\hat{\psi}(L), \hat{z}_1)$ such that

$$\frac{\hat{H}_L(\hat{z}_0)}{\hat{z}_0} = \hat{H}'_L(\hat{z}_0). \tag{2.4.16}$$

In turn, the smallest non-negative concave majorant admits the form:

$$\hat{W}_L(z) = \begin{cases} z\hat{H}'_L(\hat{z}_0) & \text{if } z \in [0, \hat{z}_0), \\ \hat{H}_L(z) & \text{if } z \in [\hat{z}_0, \hat{z}_1], \\ \hat{H}_L(\hat{z}_1) & \text{if } z \in (\hat{z}_1, +\infty). \end{cases}$$

Substituting $a_L^* = \hat{\psi}^{-1}(\hat{z}_0)$ into (2.4.16), we have

$$\frac{\hat{H}_L(\hat{z}_0)}{\hat{z}_0} = \frac{V_L(a_L^*) - a_L^* - c_b}{\hat{F}(a_L^*)},$$
$$\hat{H}'_L(\hat{z}_0) = \frac{\hat{G}(a_L^*)(V'_L(a_L^*) - 1) - \hat{G}'(a_L^*)(V_L(a_L^*) - a_L^* - c_b)}{\hat{F}'(a_L^*)\hat{G}(a_L^*) - \hat{F}(a_L^*)\hat{G}'(a_L^*)}.$$

Equivalently, we can express condition (2.4.16) in terms of a_L^*:

$$\frac{V_L(a_L^*) - a_L^* - c_b}{\hat{F}(a_L^*)} = \frac{\hat{G}(a_L^*)(V'_L(a_L^*) - 1) - \hat{G}'(a_L^*)(V_L(a_L^*) - a_L^* - c_b)}{\hat{F}'(a_L^*)\hat{G}(a_L^*) - \hat{F}(a_L^*)\hat{G}'(a_L^*)}.$$

Simplifying this shows that a_L^* solves (2.4.13). Also, we can express $\hat{H}'_L(\hat{z}_0)$ in terms of a_L^*:

$$\hat{H}'_L(\hat{z}_0) = \frac{\hat{H}_L(\hat{z}_0)}{\hat{z}_0} = \frac{V_L(a_L^*) - a_L^* - c_b}{\hat{F}(a_L^*)} = P.$$

In addition, substituting $d_L^* = \hat{\psi}^{-1}(\hat{z}_1)$ into (2.4.15), we have

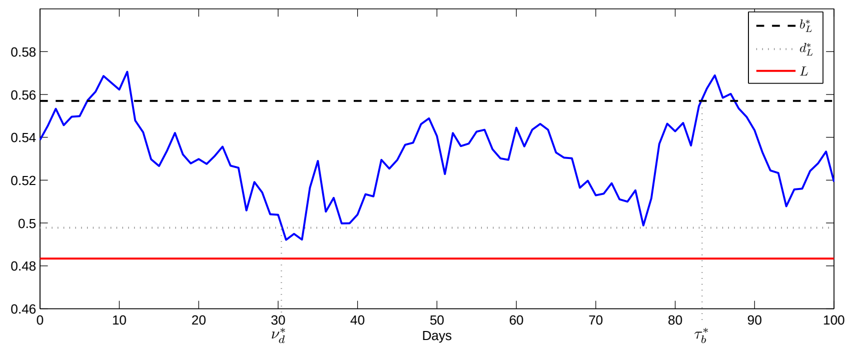
$$\hat{H}'_L(\hat{z}_1) = \frac{\hat{G}(d_L^*)(V'_L(d_L^*) - 1) - \hat{G}'(d_L^*)(V_L(d_L^*) - d_L^* - c_b)}{\hat{F}'(d_L^*)\hat{G}(d_L^*) - \hat{F}(d_L^*)\hat{G}'(d_L^*)} = 0,$$

which, after a straightforward simplification, is identical to (2.4.14). Also, $\hat{H}_L(\hat{z}_1)$ can be written as

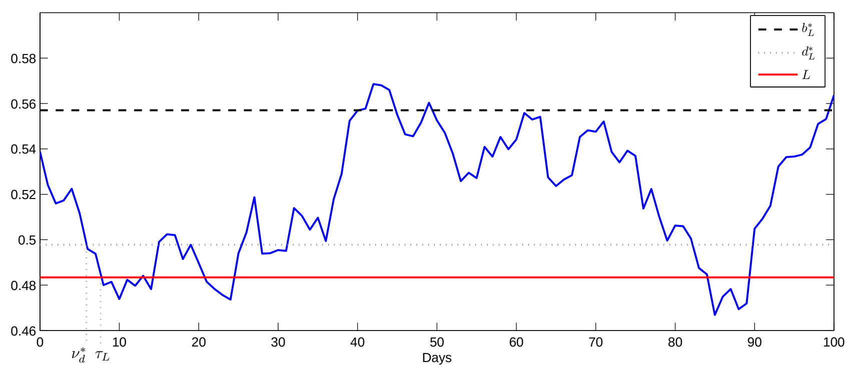
$$\hat{H}_L(\hat{z}_1) = \frac{V_L(d_L^*) - d_L^* - c_b}{\hat{G}(d_L^*)} = Q.$$

Substituting these to $J_L(x) = \hat{G}(x)\hat{W}_L(\hat{\psi}(x))$, we arrive at (2.4.11). □

Theorem 2.4.5 reveals that the optimal entry region is characterized by a price interval $[a_L^*, d_L^*]$ strictly above the stop-loss level L and strictly below the optimal exit level b_L^*. In particular, if the current asset price is between L and a_L^*, then it is optimal for the investor to wait even though the price is low. This is intuitive because if the entry price is too close to L, then the investor is very likely to be forced to exit at a loss afterwards. As a consequence, the investor's delay region, where she would wait to enter the market, is disconnected.



(a)



(b)

Figure 2.7: Simulated OU paths and exercise times. (a) The investor enters at $\nu_d^* = \inf\{t \geq 0 : X_t \leq d_L^*\}$ with $d_L^* = 0.4978$, and exit at $\tau_b^* = \inf\{t \geq \nu_d^* : X_t \geq b_L^*\}$ with $b_L^* = 0.5570$. (b) The investor enters at $\nu_d^* = \inf\{t \geq 0 : X_t \leq d_L^*\}$ but exits at stop-loss level $L = 0.4834$. Parameters: $\theta = 0.5388$, $\mu = 16.6677$, $\sigma = 0.1599$, $r = \hat{r} = 0.05$, and $c_s = c_b = 0.05$.

Figure 2.7 illustrates two simulated paths and the associated exercise times. We have chosen L to be 2 standard deviations below the long-run mean θ, with other parameters from our pairs trading example. By Theorem 2.4.5, the investor will enter the market at $\nu_{a_L^*,d_L^*}$ (see (2.4.12)). Since both paths start with $X_0 > d_L^*$, the investor waits to enter until the OU path reaches d_L^* from above, as indicated by ν_d^* in panels (a) and (b). After entry, Figure 2.7(a) describes the scenario where the investor exits voluntarily at the optimal level b_L^*, whereas in Figure 2.7(b) the investor is forced to exit at the stop-loss level L. These optimal levels are calculated from Theorem 2.4.5 and Theorem 2.4.1 based on the given estimated parameters.

Remark 2.4.6. *We remark that the optimal levels a_L^*, d_L^* and b_L^* are outputs of the models, depending on the parameters (μ, θ, σ) and the choice of stop-loss level L. Recall that our model parameters are estimated based on the likelihood maximizing portfolio discussed in Section 2.1.1. Other estimation methodologies and price data can be used, and may lead to different portfolio strategies (α, β) and estimated parameters values (μ, θ, σ). In turn, the resulting optimal entry and exit thresholds may also change accordingly.*

2.4.3 Relative Stop-Loss Exit

For some investors, it may be more desirable to set the stop-loss contingent on the entry level. In other words, if the value of X at the entry time is x, then the investor would assign a lower stop-loss level $x - \ell$, for some constant $\ell > 0$. Therefore, the investor faces the optimal entry timing problem

$$\mathcal{J}_\ell(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-\hat{r}\nu} (\mathcal{V}_\ell(X_\nu) - X_\nu - c_b) \right\},$$

where $\mathcal{V}_\ell(x) := V_{x-\ell}(x)$ (see (2.4.2)) is the optimal exit timing problem with stop-loss level $x - \ell$. The dependence of $V_{x-\ell}(x)$ on x is significantly more complicated than $V(x)$ or $V_L(x)$, making the problem much less tractable.

In Figure 2.8, we illustrate numerically the optimal timing strategies. The investor will still enter at a lower level d^*. After entry, the investor will wait to exit at either the stop-loss level $d^* - \ell$ or an upper level b^*.

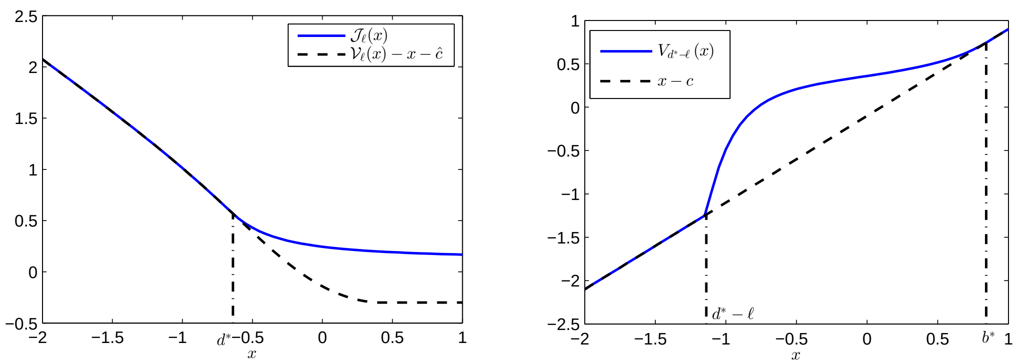


Figure 2.8: (Left) The optimal entry value function $\mathcal{J}_\ell(x)$ dominates the reward function $\mathcal{V}_\ell(x) - x - c_b$, and they coincide for $x \leq d^*$. (Right) For the exit problem, the stop-loss level is $d^* - \ell$ and the optimal liquidation level is b^*.

2.5 Further Applications

The optimal trading problem studied herein is amenable for a number of extensions. We will conclude by briefly discussing the incorporation of a minimum holding period or a timing penalty.

2.5.1 Minimum Holding Period

Another timing constraint of practical interest is the minimum holding period. Recently, regulators and exchanges are contemplating to apply this rule to rein in high-frequency trading. This gives rise to the need to better understand the effect of this restriction on trading. Intuitively, a minimum holding period always delays the liquidation timing, but how does it influence the investor's timing to *enter* the market?

Suppose that once the investor enters the position, she is only allowed to liquidate after a pre-specified time period δ. The incorporation of a minimum holding period leads to the constrained optimal stopping problem

$$V^{\delta}(x) = \sup_{\tau \geq \delta} \mathbb{E}_x\{e^{-r\tau}(X_\tau - c_s)\} = \mathbb{E}_x\{e^{-r\delta}V(X_\delta)\},$$

where $V(x)$ the unconstrained problem in (2.1.2) with solution given in Theorem 2.3.3. The second equality follows from the strong Markov property of X and the optimality of $V(x)$. Compared to the unconstrained problem, the optimal liquidation timing for $V^{\delta}(x)$ is simply delayed by δ but otherwise identical to τ^* in (2.3.4). Also, by the supermartingale and non-negative property of $V(x)$, we see that $0 \leq V^{\delta}(x) \leq V(x)$ and $V^{\delta}(x)$ decreases with δ.

Turning to the optimal entry timing, the investor solves

$$J^{\delta}(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x\{e^{-\hat{r}\nu}(V^{\delta}(X_\nu) - X_\nu - c_b)\}. \tag{2.5.1}$$

The following result reflects the impact of the minimum holding period.

Proposition 2.5.1. *For every $x \in \mathbb{R}$, we have $J^{\delta}(x) \leq J(x)$ and $d^{\delta} \leq d^*$.*

Proof. As in Theorem 2.3.7, one can show that the optimal entry timing problem (2.5.1) admits the solution

$$J^{\delta}(x) = \begin{cases} V^{\delta}(x) - x - c_b & \text{if } x \in (-\infty, d^{\delta}], \\ \frac{V^{\delta}(d^{\delta}) - d^{\delta} - c_b}{\hat{G}(d^{\delta})}\hat{G}(x) & \text{if } x \in (d^{\delta}, +\infty), \end{cases}$$

where the optimal entry level d^{δ} is found from the equation

$$\hat{G}(d)(V^{\delta\prime}(d) - 1) = \hat{G}'(d)(V^{\delta}(d) - d - c_b).$$

To compare with the original case, we first define $h_2(x) = -x - c_b$,

$$\hat{H}^{\delta}(z) = \left(\frac{V^{\delta} + h_2}{\hat{G}}\right) \circ \hat{\psi}^{-1}(z),$$

and denote $\hat{W}^\delta(z)$ as the smallest concave majorant of $\hat{H}^\delta(z)$. Following the similar proof of Theorem 2.3.7, we can show that

$$\hat{W}^\delta(z) = \begin{cases} \hat{H}^\delta(z) & \text{if } z \in [0, \hat{z}^\delta], \\ \hat{H}^\delta(\hat{z}^\delta) & \text{if } z \in (\hat{z}^\delta, +\infty), \end{cases}$$

where $\hat{z}^\delta = \hat{\psi}(d^\delta)$ satisfies $\hat{H}^{\delta\prime}(\hat{z}^\delta) = 0$. Recall that $\hat{z} = \hat{\psi}(d^*)$ satisfies $\hat{H}'(\hat{z}) = 0$.

To show $d^\delta \leq d^*$, we examine the concavity of $\hat{H}^\delta$ and $\hat{H}$. Restating $\hat{H}$ in (2.2.16) in terms of h_2:

$$\hat{H}(z) = (\frac{V + h_2}{\hat{G}}) \circ \hat{\psi}^{-1}(z),$$

followed by differentiation, we have

$$\hat{H}''(z) = \frac{2}{\sigma^2 \hat{G}(x)(\hat{\psi}'(x))^2}(\mathcal{L} - \hat{r})(V + h_2)(x), \quad z = \hat{\psi}(x). \tag{2.5.2}$$

Similarly, (2.5.2) also holds for $\hat{H}^\delta$ with V replaced by V^δ. This leads us to analyze $(\mathcal{L} - \hat{r})(V + h_2)(x)$ and $(\mathcal{L} - \hat{r})(V^\delta + h_2)(x)$. As shown in Lemma 2.3.6 and Figure 2.3, $\hat{H}(z)$ is concave for $z \in (0, \hat{\psi}(\underline{b}))$, where $\underline{b} < L^*$ satisfies $(\mathcal{L} - \hat{r})(V + h_2)(x) = 0$, and $\hat{z} < \hat{\psi}(\underline{b})$.

Moreover, it follows from the supermartingale property of V that

$$\mathbb{E}_x\{e^{-rt}V^\delta(X_t)\} = \mathbb{E}_x\{e^{-r(t+\delta)}V(X_{t+\delta})\} \leq \mathbb{E}_x\{e^{-r\delta}V(X_\delta)\} = V^\delta(x).$$

From this and Proposition 5.9 in Dayanik and Karatzas [2003], we infer that $(\mathcal{L} - r)V^\delta(x) \leq 0$. In turn, for $x < \underline{b}$, we have

$$\begin{aligned}(\mathcal{L} - \hat{r})(V^\delta + h_2)(x) &= (\mathcal{L} - r)V^\delta(x) + (r - \hat{r})V^\delta(x) + (\mathcal{L} - \hat{r})h_2(x) \\ &\leq (r - \hat{r})V^\delta(x) + (\mathcal{L} - \hat{r})h_2(x) \\ &\leq (r - \hat{r})V(x) + (\mathcal{L} - \hat{r})h_2(x) = (\mathcal{L} - \hat{r})(V + h_2)(x),\end{aligned}$$

where the last equality follows from the fact that $(\mathcal{L}-r)V(x)=0$ for $x<b^*$, since W is a straight line for $z\leq\psi(b^*)$, and $\underline{b}<L^*<b^*$. Hence, for $z\in(0,\hat{\psi}(\underline{b}))$, $\hat{H}^{\delta''}(z)\leq\hat{H}''(z)\leq 0$ and $\hat{H}^{\delta}(z)$ is also concave.

Since $V(x)\geq V^{\delta}(x)\geq 0$, we have $\hat{H}(z)\geq\hat{H}^{\delta}(z)$ for $z\in(0,+\infty)$. Considering $\hat{H}(0)=\hat{H}^{\delta}(0)=0$ and $\hat{H}(0+),\hat{H}^{\delta}(0+)>0$, we have $\hat{H}'(0+)\geq\hat{H}^{\delta'}(0+)\geq 0$. This, along with $\hat{H}^{\delta''}(z)\leq\hat{H}''(z)\leq 0$ for $z\in(0,\hat{\psi}(\underline{b}))$, imply that $\hat{H}'(z)\geq\hat{H}^{\delta'}(z)$ for $z\in(0,\hat{\psi}(\underline{b}))$. So $\hat{H}^{\delta'}(\hat{z})\leq\hat{H}'(\hat{z})=0$. Considering $\hat{H}^{\delta'}(\hat{z}^{\delta})=0$ and the concavity of $\hat{H}^{\delta}$, we conclude that $\hat{z}^{\delta}\leq\hat{z}$, which by the monotonicity of $\hat{\psi}$ is equivalent to $d^{\delta}\leq d^*$.

To show $J^{\delta}(x)\leq J(x)$, it is equivalent to establish $\hat{W}^{\delta}(z)\leq\hat{W}(z)$ for all $z\in[0,\infty)$: (i) For $z\in[0,\hat{z}^{\delta}]$, this holds since $\hat{W}^{\delta}(z)=\hat{H}^{\delta}(z)$, $\hat{W}(z)=\hat{H}(z)$, and $\hat{H}^{\delta}(z)\leq\hat{H}(z)$. (ii) For $z\in(\hat{z}^{\delta},\hat{z}]$, $\hat{W}^{\delta}(z)=\hat{H}^{\delta}(\hat{z}^{\delta})\leq\hat{H}(\hat{z}^{\delta})\leq\hat{H}(z)=\hat{W}(z)$, where the last inequality follows from the fact that $\hat{H}'(z)\geq 0$ for $z\in(\hat{z}^{\delta},\hat{z}]$. (iii) For $z\in(\hat{z},+\infty)$, $\hat{W}^{\delta}(z)=\hat{H}^{\delta}(\hat{z}^{\delta})\leq\hat{H}(\hat{z})=\hat{W}(z)$. □

This means that the minimum holding period leads to a lower optimal entry level and lower value function as compared to the original value function J in (2.1.3).

2.5.2 Path-Dependent Risk Penalty

In addition to maximizing the expected liquidation value, a risk-sensitive investor may be concerned about the price fluctuation over time, and therefore, be willing to adjust her liquidation timing depending on the path behavior of prices. This motivates the incorporation of a *path-dependent* risk penalty up to the liquidation time τ. To illustrate this idea, we apply a penalty term of the form $\mathbb{E}_x\{\int_0^{\tau}e^{-ru}q(X_u)\,du\}$, where $q(x)$ could be any positive penalty function. This risk penalty only applies when the investor is holding the position, but not before entry. Hence, the investor solves the penalized optimal timing problems:

$$\mathcal{J}^q(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \Big\{ e^{-r\tau}(\mathcal{V}^q(X_\nu) - X_\nu - c_b) \Big\},$$
$$\mathcal{V}^q(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \Big\{ e^{-r\tau}(X_\tau - c_s) - \int_0^\tau e^{-ru} q(X_u)\, du \Big\}. \tag{2.5.3}$$

As a special case, let $q(x) \equiv q$, a strictly positive constant. Then, by computing the integral in (2.5.3),

$$\mathcal{V}^q(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E} \Big\{ e^{-r\tau}(X_\tau - (c_s - \frac{q}{r})) \Big\} - \frac{q}{r}. \tag{2.5.4}$$

This presents an interesting connection between the penalized problem $\mathcal{V}^q(x)$ in (2.5.4) and the unpenalized optimal stopping problem V in (2.1.2). Indeed, we observe that the penalty term amounts to reducing the transaction cost c_s by the positive constant $\frac{q}{r}$. In other words, the optimal stopping time τ_q^* for $\mathcal{V}^q(x)$ is identical to the optimal stopping time τ^* for $V(x)$ in (2.1.2) but with c_s replaced by $c_s - \frac{q}{r}$. Furthermore, since b^* is increasing in c_s, a higher penalty q lowers the optimal liquidation level. As for the entry problem $\mathcal{J}^q$, the solution is found from Theorem 2.3.7 by modifying the transaction cost to be $c_b + \frac{q}{r}$. More sophisticated path-dependent risk penalties can be considered under this formulation, including those based on the (integrated) shortfall with $q(x) = \rho((m - x)^+)$ where m is a constant benchmark and ρ is an increasing convex loss function (see e.g. [Föllmer and Schied, 2004, Chap 4.9]).

Chapter 3

Trading under Exponential OU Dynamics

In this chapter, we solve an optimal double stopping problem to determine the optimal times to enter and subsequently exit the market, when prices are driven by an exponential Ornstein-Uhlenbeck process. In addition, we analyze a related optimal switching problem that involves an infinite sequence of trades, and identify the conditions under which the double stopping and switching problems admit the same optimal entry and/or exit timing strategies. Among our results, we find that the investor generally enters when the price is low, but may find it optimal to wait if the current price is sufficiently close to zero. In other words, the continuation (waiting) region for entry is *disconnected*. Numerical results are provided to illustrate the dependence of timing strategies on model parameters and transaction costs.

In Section 3.1, we formulate both the optimal double stopping and optimal switching problems. Then, we present our analytical and numerical results in Section 3.2. The proofs of our main results are detailed in Section 3.3. Finally, Appendix C contains the proofs for a number of lemmas.

3.1 Problem Overview

Another widely used mean-reverting process is the exponential Ornstein-Uhlenbeck (XOU) process:

$$\xi_t = e^{X_t}, \qquad t \geq 0,$$

where X is the OU process defined in (2.1.1). In other words, X is the log-price of the positive XOU process ξ.

3.1.1 Optimal Double Stopping Problem

Given a risky asset with an XOU price process, we first consider the optimal timing to sell. If the share of the asset is sold at some time τ, then the investor will receive the value $\xi_\tau = e^{X_\tau}$ and pay a constant transaction cost $c_s > 0$. Denote by $\mathbb{F}$ the filtration generated by B, and $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times. To maximize the expected discounted value, the investor solves the optimal stopping problem

$$V^\xi(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau}(e^{X_\tau} - c_s) \right\}, \tag{3.1.1}$$

where $r > 0$ is the constant discount rate, and $\mathbb{E}_x\{\cdot\} \equiv \mathbb{E}\{\cdot|X_0 = x\}$.

The value function $V^\xi(x)$ represents the expected liquidation value associated with ξ. On the other hand, the current price plus the transaction cost constitute the total cost to enter the trade. Before even holding the risky asset, the investor can always choose the optimal timing to start the trade, or not to enter at all. This leads us to analyze the entry timing inherent in the trading problem. Precisely, we solve

$$J^\xi(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\nu}(V^\xi(X_\nu) - e^{X_\nu} - c_b) \right\}, \tag{3.1.2}$$

with the constant transaction cost $c_b > 0$ incurred at the time of purchase. In other words, the trader seeks to maximize the expected difference between the value function $V^\xi(X_\nu)$ and the current e^{X_ν}, minus transaction cost c_b. The value function $J^\xi(x)$ represents the maximum expected value of the investment opportunity in the price process ξ, with transaction costs c_b and c_s incurred, respectively, at entry and exit. For our analysis, the transaction costs c_b and c_s can be different. To facilitate presentation, we denote the functions

$$h_s^\xi(x) = e^x - c_s \quad \text{and} \quad h_b^\xi(x) = e^x + c_b. \tag{3.1.3}$$

If it turns out that $J^\xi(X_0) \leq 0$ for some initial value X_0, then the investor will not start to trade X (see Appendix A below). In view of the example in Appendix A, it is important to identify the trivial cases under any given dynamics. Under the XOU model, since $\sup_{x\in\mathbb{R}}(V^\xi(x) - h_b^\xi(x)) \leq 0$ implies that $J^\xi(x) \leq 0$ for $x \in \mathbb{R}$, we shall therefore focus on the case with

$$\sup_{x\in\mathbb{R}}(V^\xi(x) - h_b^\xi(x)) > 0, \tag{3.1.4}$$

and solve for the non-trivial optimal timing strategy.

3.1.2 Optimal Switching Problem

Under the optimal switching approach, the investor is assumed to commit to an infinite number of trades. The sequential trading times are modeled by the stopping times $\nu_1, \tau_1, \nu_2, \tau_2, \dots \in \mathcal{T}$ such that

$$0 \leq \nu_1 \leq \tau_1 \leq \nu_2 \leq \tau_2 \leq \dots.$$

A share of the risky asset is bought and sold, respectively, at times ν_i and τ_i, $i \in \mathbb{N}$. The investor's optimal timing to trade would depend on the initial position. Precisely, under the XOU model, if the investor starts with a zero position, then the first trading decision is when to *buy* and the corresponding optimal switching problem is

$$\tilde{J}^\xi(x) = \sup_{\Lambda_0} \mathbb{E}_x \left\{ \sum_{n=1}^{\infty} [e^{-r\tau_n} h_s^\xi(X_{\tau_n}) - e^{-r\nu_n} h_b^\xi(X_{\nu_n})] \right\}, \tag{3.1.5}$$

with the set of admissible stopping times $\Lambda_0 = (\nu_1, \tau_1, \nu_2, \tau_2, \dots)$, and the reward functions h_s^ξ and h_b^ξ defined in (3.1.3). On the other hand, if the investor is initially holding a share of the asset, then the investor first determines when to *sell* and solves

$$\tilde{V}^\xi(x) = \sup_{\Lambda_1} \mathbb{E}_x \left\{ e^{-r\tau_1} h_s^\xi(X_{\tau_1}) + \sum_{n=2}^{\infty} [e^{-r\tau_n} h_s^\xi(X_{\tau_n}) - e^{-r\nu_n} h_b^\xi(X_{\nu_n})] \right\}, \quad (3.1.6)$$

with $\Lambda_1 = (\tau_1, \nu_2, \tau_2, \nu_3, \dots)$.

In summary, the optimal double stopping and switching problems differ in the number of trades. Observe that any strategy for the double stopping problems (3.1.1) and (3.1.2) are also candidate strategies for the switching problems (3.1.6) and (3.1.5) respectively. Therefore, it follows that $V^\xi(x) \leq \tilde{V}^\xi(x)$ and $J^\xi(x) \leq \tilde{J}^\xi(x)$. Our objective is to derive and compare the corresponding optimal timing strategies under these two approaches.

3.2 Summary of Analytical Results

We first summarize our analytical results and illustrate the optimal trading strategies. The method of solutions and their proofs will be discussed in Section 3.3. We begin with the optimal stopping problems (3.1.1) and (3.1.2) under the XOU model.

3.2.1 Optimal Double Stopping Problem

We now present the result for the optimal exit timing problem under the XOU model. First, we obtain a bound for the value function V^ξ.

Lemma 3.2.1. *There exists a positive constant K^ξ such that, for all $x \in \mathbb{R}$, $0 \leq V^\xi(x) \leq e^x + K^\xi$.*

Theorem 3.2.2. *The optimal liquidation problem* (3.1.1) *admits the solution*

$$V^{\xi}(x) = \begin{cases} \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})} F(x) & \text{if } x < b^{\xi*}, \\ e^x - c_s & \text{if } x \geq b^{\xi*}, \end{cases} \tag{3.2.1}$$

where the optimal log-price level $b^{\xi*}$ for liquidation is uniquely found from the equation

$$e^b F(b) = (e^b - c_s) F'(b). \tag{3.2.2}$$

The optimal liquidation time is given by

$$\tau^{\xi*} = \inf\{\, t \geq 0 \,:\, X_t \geq b^{\xi*} \,\} = \inf\{\, t \geq 0 \,:\, \xi_t \geq e^{b^{\xi*}} \,\}.$$

We now turn to the optimal entry timing problem, and give a bound on the value function J^{ξ}.

Lemma 3.2.3. *There exists a positive constant $\hat{K}^{\xi}$ such that, for all $x \in \mathbb{R}$, $0 \leq J^{\xi}(x) \leq \hat{K}^{\xi}$.*

Theorem 3.2.4. *Under the XOU model, the optimal entry timing problem* (3.1.2) *admits the solution*

$$J^{\xi}(x) = \begin{cases} P^{\xi} F(x) & \text{if } x \in (-\infty, a^{\xi*}), \\ V^{\xi}(x) - (e^x + c_b) & \text{if } x \in [a^{\xi*}, d^{\xi*}], \\ Q^{\xi} G(x) & \text{if } x \in (d^{\xi*}, +\infty), \end{cases} \tag{3.2.3}$$

with the constants

$$P^{\xi} = \frac{V^{\xi}(a^{\xi*}) - (e^{a^{\xi*}} + c_b)}{F(a^{\xi*})}, \quad Q^{\xi} = \frac{V^{\xi}(d^{\xi*}) - (e^{d^{\xi*}} + c_b)}{G(d^{\xi*})},$$

and the critical levels $a^{\xi*}$ and $d^{\xi*}$ satisfying, respectively,

$$F(a)({V^{\xi}}'(a) - e^a) = F'(a)(V^{\xi}(a) - (e^a + c_b)), \tag{3.2.4}$$

$$G(d)({V^{\xi}}'(d) - e^d) = G'(d)(V^{\xi}(d) - (e^d + c_b)). \tag{3.2.5}$$

The optimal entry time is given by

$$\nu_{a^{\xi*},d^{\xi*}} := \inf\{t \geq 0 \,:\, X_t \in [a^{\xi*}, d^{\xi*}]\}.$$

In summary, the investor should exit the market as soon as the price reaches the upper level $e^{b^{\xi*}}$. In contrast, the optimal entry timing is the first time that the XOU price ξ enters the interval $[e^{a^{\xi*}}, e^{d^{\xi*}}]$. In other words, it is optimal to wait if the current price ξ_t is too close to zero, i.e. if $\xi_t < e^{a^{\xi*}}$. Moreover, the interval $[e^{a^{\xi*}}, e^{d^{\xi*}}]$ is contained in $(0, e^{b^{\xi*}})$, and thus, the continuation region for market entry is *disconnected*.

3.2.2 Optimal Switching Problem

We now turn to the optimal switching problems defined in (3.1.5) and (3.1.6) under the XOU model. To facilitate the presentation, we denote

$$f_s(x) := (\mu\theta + \frac{1}{2}\sigma^2 - r) - \mu x + rc_s e^{-x},$$
$$f_b(x) := (\mu\theta + \frac{1}{2}\sigma^2 - r) - \mu x - rc_b e^{-x}.$$

Applying the operator $\mathcal{L}$ (see (2.2.1)) to h_s^ξ and h_b^ξ (see (3.1.3)), it follows that $(\mathcal{L} - r)h_s^\xi(x) = e^x f_s(x)$ and $(\mathcal{L} - r)h_b^\xi(x) = e^x f_b(x)$. Therefore, f_s (resp. f_b) preserves the sign of $(\mathcal{L} - r)h_s^\xi$ (resp. $(\mathcal{L} - r)h_b^\xi$). It can be shown that $f_s(x) = 0$ has a unique root, denoted by x_s. However,

$$f_b(x) = 0 \tag{3.2.6}$$

may have no root, a single root, or two distinct roots, denoted by x_{b1} and x_{b2}, if they exist. The following observations will also be useful:

$$f_s(x) \begin{cases} > 0 & \text{if } x < x_s, \\ < 0 & \text{if } x > x_s, \end{cases} \quad \text{and} \quad f_b(x) \begin{cases} < 0 & \text{if } x \in (-\infty, x_{b1}) \cup (x_{b2}, +\infty), \\ > 0 & \text{if } x \in (x_{b1}, x_{b2}). \end{cases} \tag{3.2.7}$$

We first obtain bounds for the value functions $\tilde{J}^\xi$ and $\tilde{V}^\xi$.

Lemma 3.2.5. *There exists positive constants C_1 and C_2 such that*

$$0 \leq \tilde{J}^{\xi}(x) \leq C_1,$$
$$0 \leq \tilde{V}^{\xi}(x) \leq e^x + C_2.$$

The optimal switching problems have two different sets of solutions depending on the problem data.

Theorem 3.2.6. *The optimal switching problem* (3.1.5)-(3.1.6) *admits the solution*

$$\tilde{J}^{\xi}(x) = 0, \text{ for } x \in \mathbb{R}, \quad \text{and} \quad \tilde{V}^{\xi}(x) = \begin{cases} \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})} F(x) & \text{if } x < b^{\xi*}, \\ e^x - c_s & \text{if } x \geq b^{\xi*}, \end{cases} \tag{3.2.8}$$

where $b^{\xi*}$ satisfies (3.2.2)*, if any of the following mutually exclusive conditions holds:*

(i) There is no root or a single root to equation (3.2.6)*.*

(ii) There are two distinct roots to (3.2.6)*. Also*

$$\exists\, \tilde{a}^* \in (x_{b1}, x_{b2}) \quad \text{such that} \quad F(\tilde{a}^*)e^{\tilde{a}^*} = F'(\tilde{a}^*)(e^{\tilde{a}^*} + c_b), \tag{3.2.9}$$

and

$$\frac{e^{\tilde{a}^*} + c_b}{F(\tilde{a}^*)} \geq \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})}. \tag{3.2.10}$$

(iii) There are two distinct roots to (3.2.6) *but* (3.2.9) *does not hold.*

In Theorem 3.2.6, $\tilde{J}^{\xi} = 0$ means that it is optimal not to enter the market at all. On the other hand, if one starts with a unit of the underlying asset, the optimal switching problem reduces to a problem of optimal single stopping. Indeed, the investor will never re-enter the market after exit. This is identical to the optimal liquidation problem (3.1.1) where there is only a single (exit) trade. The optimal strategy in this case is the same as V^{ξ} in (3.2.1) – it is optimal to exit the market as soon as the log-price X reaches the threshold $b^{\xi*}$.

We also address the remaining case when none of the conditions in Theorem 3.2.6 hold. As we show next, the optimal strategy will involve both entry and exit thresholds.

Theorem 3.2.7. *If there are two distinct roots to* (3.2.6)*,* x_{b1} *and* x_{b2}*, and there exists a number* $\tilde{a}^* \in (x_{b1}, x_{b2})$ *satisfying* (3.2.9) *such that*

$$\frac{e^{\tilde{a}^*} + c_b}{F(\tilde{a}^*)} < \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})}, \tag{3.2.11}$$

then the optimal switching problem (3.1.5)-(3.1.6) *admits the solution*

$$\tilde{J}^{\xi}(x) = \begin{cases} \tilde{P}F(x) & \text{if } x \in (-\infty, \tilde{a}^*), \\ \tilde{K}F(x) - (e^x + c_b) & \text{if } x \in [\tilde{a}^*, \tilde{d}^*], \\ \tilde{Q}G(x) & \text{if } x \in (\tilde{d}^*, +\infty), \end{cases} \tag{3.2.12}$$

$$\tilde{V}^{\xi}(x) = \begin{cases} \tilde{K}F(x) & \text{if } x \in (-\infty, \tilde{b}^*), \\ \tilde{Q}G(x) + e^x - c_s & \text{if } x \in [\tilde{b}^*, +\infty), \end{cases} \tag{3.2.13}$$

where $\tilde{a}^*$ *satisfies* (3.2.9)*, and*

$$\tilde{P} = \tilde{K} - \frac{e^{\tilde{a}^*} + c_b}{F(\tilde{a}^*)},$$

$$\tilde{K} = \frac{e^{\tilde{d}^*} G(\tilde{d}^*) - (e^{\tilde{d}^*} + c_b)G'(\tilde{d}^*)}{F'(\tilde{d}^*)G(\tilde{d}^*) - F(\tilde{d}^*)G'(\tilde{d}^*)},$$

$$\tilde{Q} = \frac{e^{\tilde{d}^*} F(\tilde{d}^*) - (e^{\tilde{d}^*} + c_b)F'(\tilde{d}^*)}{F'(\tilde{d}^*)G(\tilde{d}^*) - F(\tilde{d}^*)G'(\tilde{d}^*)}$$

There exist unique critical levels $\tilde{d}^*$ *and* $\tilde{b}^*$ *which are found from the nonlinear system of equations:*

$$\frac{e^d G(d) - (e^d + c_b)G'(d)}{F'(d)G(d) - F(d)G'(d)} = \frac{e^b G(b) - (e^b - c_s)G'(b)}{F'(b)G(b) - F(b)G'(b)}, \tag{3.2.14}$$

$$\frac{e^d F(d) - (e^d + c_b)F'(d)}{F'(d)G(d) - F(d)G'(d)} = \frac{e^b F(b) - (e^b - c_s)F'(b)}{F'(b)G(b) - F(b)G'(b)}. \tag{3.2.15}$$

Moreover, the critical levels are such that $\tilde{d}^* \in (x_{b1}, x_{b2})$ *and* $\tilde{b}^* > x_s$.

The optimal strategy in Theorem 3.2.7 is described by the stopping times $\Lambda_0^* = (\nu_1^*, \tau_1^*, \nu_2^*, \tau_2^*, \dots)$, and $\Lambda_1^* = (\tau_1^*, \nu_2^*, \tau_2^*, \nu_3^*, \dots)$, with

$$\nu_1^* = \inf\{t \geq 0 : X_t \in [\tilde{a}^*, \tilde{d}^*]\},$$
$$\tau_i^* = \inf\{t \geq \nu_i^* : X_t \geq \tilde{b}^*\}, \quad \text{and} \quad \nu_{i+1}^* = \inf\{t \geq \tau_i^* : X_t \leq \tilde{d}^*\}, \quad \text{for } i \geq 1.$$

In other words, it is optimal to buy if the price is within $[e^{\tilde{a}^*}, e^{\tilde{d}^*}]$ and then sell when the price ξ reaches $e^{\tilde{b}^*}$. The structure of the buy/sell regions is similar to that in the double stopping case (see Theorems 3.2.2 and 3.2.4). In particular, $\tilde{a}^*$ is the same as $a^{\xi*}$ in Theorem 3.2.4 since the equations (3.2.4) and (3.2.9) are equivalent. The level $\tilde{a}^*$ is only relevant to the first purchase. Mathematically, $\tilde{a}^*$ is determined separately from $\tilde{d}^*$ and $\tilde{b}^*$. If we start with a zero position, then it is optimal to enter if the price ξ lies in the interval $[e^{\tilde{a}^*}, e^{\tilde{d}^*}]$. However, on all subsequent trades, we enter as soon as the price hits $e^{\tilde{d}^*}$ from above (after exiting at $e^{\tilde{b}^*}$ previously). Hence, the lower level $\tilde{a}^*$ becomes irrelevant after the first entry.

Note that the conditions that differentiate Theorems 3.2.6 and 3.2.7 are exhaustive and mutually exclusive. If the conditions in Theorem 3.2.6 are violated, then the conditions in Theorem 3.2.7 must hold. In particular, condition (3.2.9) in Theorem 3.2.6 holds if and only if

$$\left| \int_{-\infty}^{x_{b1}} \Psi(x) e^x f_b(x) dx \right| < \int_{x_{b1}}^{x_{b2}} \Psi(x) e^x f_b(x) dx, \tag{3.2.16}$$

where

$$\Psi(x) = \frac{2F(x)}{\sigma^2 \mathcal{W}(x)}, \quad \text{and} \quad \mathcal{W}(x) = F'(x)G(x) - F(x)G'(x) > 0. \tag{3.2.17}$$

Inequality (3.2.16) can be numerically verified given the model inputs.

3.2.3 Numerical Examples

We numerically implement Theorems 3.2.2, 3.2.4, and 3.2.7, and illustrate the associated entry/exit thresholds. In Figure 3.1 (left), the optimal entry levels $d^{\xi*}$ and $\tilde{d}^*$ rise, respectively, from 0.7425 to 0.7912 and from 0.8310 to 0.8850, as the speed of mean reversion μ increases from 0.5 to 1. On the other hand, the critical exit levels $b^{\xi*}$ and $\tilde{b}^*$ remain relatively flat over μ. As for the critical lower level $a^{\xi*}$ from the optimal double stopping problem, Figure 3.1 (right) shows that it is decreasing in μ. The same pattern holds for the optimal switching problem since the critical lower level $\tilde{a}^*$ is identical to $a^{\xi*}$, as noted above.

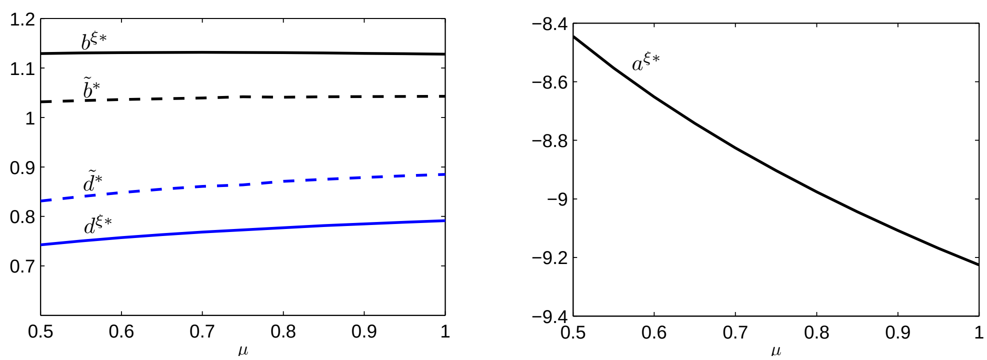


Figure 3.1: (Left) The optimal entry and exit levels vs speed of mean reversion μ. Parameters: $\sigma = 0.2$, $\theta = 1$, $r = 0.05$, $c_s = 0.02$, $c_b = 0.02$. (Right) The critical lower level of entry region $a^{\xi*}$ decreases monotonically from -8.4452 to -9.2258 as μ increases from 0.5 to 1. Parameters: $\sigma = 0.2$, $\theta = 1$, $r = 0.05$, $c_s = 0.02$, $c_b = 0.02$.

We now look at the impact of transaction cost in Figure 3.2. On the left panel, we observe that as the transaction cost c_b increases, the gap between the optimal switching entry and exit levels, $\tilde{d}^*$ and $\tilde{b}^*$, widens. This means that it is optimal to delay both entry and exit. Intuitively, to counter the fall in profit margin due to an increase in transaction cost, it is necessary to buy at a lower price and sell at a higher price to seek a wider spread. In comparison,

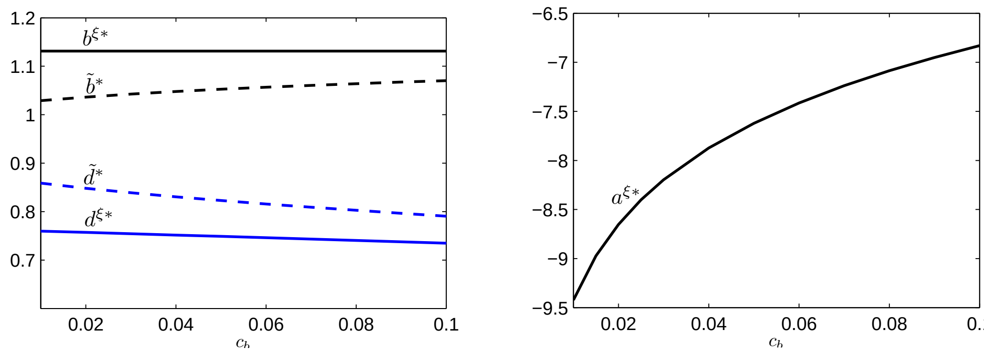


Figure 3.2: (Left) The optimal entry and exit levels vs transaction cost c_b. Parameters: $\mu = 0.6$, $\sigma = 0.2$, $\theta = 1$, $r = 0.05$, $c_s = 0.02$. (Right) The critical lower level of entry region $a^{\xi*}$ increases monotonically from -9.4228 to -6.8305 as c_b increases from 0.01 to 0.1. Parameters: $\mu = 0.6$, $\sigma = 0.2$, $\theta = 1$, $r = 0.05$, $c_s = 0.02$.

the exit level $b^{\xi*}$ from the double stopping problem is known analytically to be independent of the entry cost, so it stays constant as c_b increases in the figure. In contrast, the entry level $d^{\xi*}$, however, decreases as c_b increases but much less significantly than $\tilde{d}^*$. Figure 3.2 (right) shows that $a^{\xi*}$, which is the same for both the optimal double stopping and switching problems, increases monotonically with c_b.

In both Figures 3.1 and 3.2, we can see that the interval of the entry and exit levels, $(\tilde{d}^*, \tilde{b}^*)$, associated with the optimal switching problem lies within the corresponding interval $(d^{\xi*}, b^{\xi*})$ from the optimal double stopping problem. Intuitively, with the intention to enter the market again upon completing the current trade, the trader is more willing to enter/exit earlier, meaning a narrowed waiting region.

Figure 3.3 shows a simulated path and the associated entry/exit levels. As the path starts at $\xi_0 = 2.6011 > e^{\tilde{d}^*} > e^{d^{\xi*}}$, the investor waits to enter until the path reaches the lower level $e^{d^{\xi *}}$ (double stopping) or $e^{\tilde{d}^*}$ (switching) according to Theorems 3.2.4 and 3.2.7. After entry, the investor exits at the optimal level $e^{b^{\xi *}}$ (double stopping) or $e^{\tilde{b}^*}$ (switching). The optimal switching thresholds imply that the investor first enters the market on day 188 where the underlying asset price is 2.3847. In contrast, the optimal double stopping timing yields a later entry on day 845 when the price first reaches $e^{d^{\xi *}} = 2.1754$. As for the exit timing, under the optimal switching setting, the investor exits the market earlier on day 268 at the price $e^{\tilde{b}^*} = 2.8323$. The double stopping timing is much later on day 1160 when the price reaches $e^{b^{\xi *}} = 3.0988$. In addition, under the optimal switching problem, the investor executes more trades within the same time span. As seen in the figure, the investor would have completed two 'round-trip' (buy-and-sell) trades in the market before the double stopping investor liquidates for the first time.

3.3 Methods of Solution and Proofs

We now provide detailed proofs for our analytical results in Section 3.2 beginning with Theorems 3.2.2 and 3.2.4 for the optimal double stopping problems.

3.3.1 Optimal Double Stopping Problem

3.3.1.1 Optimal Exit Timing

To facilitate the presentation, we define the function H^ξ associated with the reward function h_s^ξ as in (2.2.9).

Lemma 3.3.1. *The function* H^ξ *is continuous on* $[0, +\infty)$, *twice differentiable on* $(0, +\infty)$ *and possesses the following properties:*

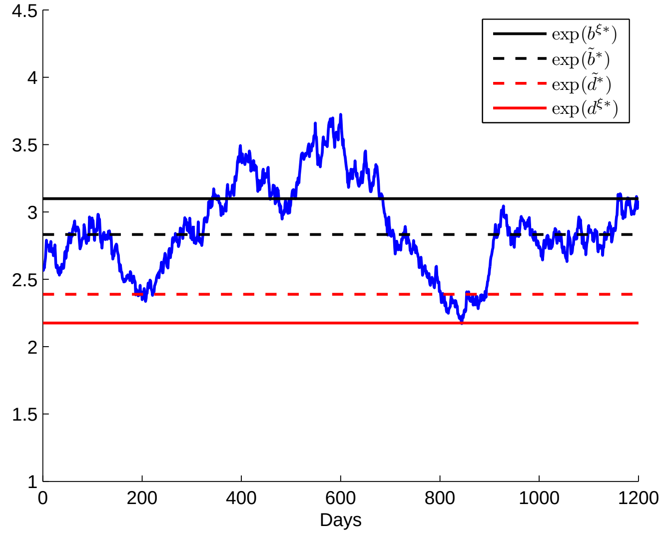


Figure 3.3: A sample exponential OU path, along with entry and exit levels. Under the double stopping setting, the investor enters at $\nu_{d^{\xi *}} = \inf\{t \geq 0 : \xi_t \leq e^{d^{\xi *}} = 2.1754\}$ with $d^{\xi *} = 0.7772$, and exit at $\tau_{b^{\xi *}} = \inf\{t \geq \nu_{d^{\xi *}} : \xi_t \geq e^{b^{\xi *}} = 3.0988\}$ with $b^{\xi *} = 1.1310$. The optimal switching investor enters at $\nu_{\tilde{d}^*} = \inf\{t \geq 0 : \xi_t \leq e^{\tilde{d}^*} = 2.3888\}$ with $\tilde{d}^* = 0.8708$, and exit at $\tau_{\tilde{b}^*} = \inf\{t \geq \nu_{\tilde{d}^*} : \xi_t \geq e^{\tilde{b}^*} = 2.8323\}$ with $\tilde{b}^* = 1.0411$. The critical lower threshold of entry region is $e^{a^{\xi *}} = 1.264 \cdot 10^{-4}$ with $a^{\xi *} = -8.9760$ (not shown in this figure). Parameters: $\mu = 0.8$, $\sigma = 0.2$, $\theta = 1$, $r = 0.05$, $c_s = 0.02$, $c_b = 0.02$.

(i) $H^\xi(0) = 0$, *and*

$$H^\xi(z) \begin{cases} < 0 & \text{if } z \in (0, \psi(\ln c_s)), \\ > 0 & \text{if } z \in (\psi(\ln c_s), +\infty). \end{cases}$$

(ii) $H^\xi(z)$ *is strictly increasing for* $z \in (\psi(\ln c_s), +\infty)$, *and* $H^{\xi\prime}(z) \to 0$ *as* $z \to +\infty$.

(iii)

$$H^{\xi}(z) \text{ is } \begin{cases} \text{convex} & \text{if } z \in (0, \psi(x_s)], \\ \text{concave} & \text{if } z \in [\psi(x_s), +\infty). \end{cases}$$

From Lemma 3.3.1, we see that H^{ξ} shares a very similar structure as H. Using the properties of H^{ξ}, we now solve for the optimal exit timing.

Proof of Theorem 3.2.2 We look for the value function of the form: $V^{\xi}(x) = G(x)W^{\xi}(\psi(x))$, where W^{ξ} is the smallest concave majorant of H^{ξ}. By Lemma 3.3.1, we deduce that H^{ξ} is concave over $[\psi(x_s), +\infty)$, strictly positive over $(\psi(\ln c_s), +\infty)$, and $H^{\xi\prime}(z) \to 0$ as $z \to +\infty$. Therefore, there exists a unique number $z^{\xi*} > \psi(x_s) \vee \psi(\ln c_s)$ such that

$$\frac{H^{\xi}(z^{\xi*})}{z^{\xi*}} = H^{\xi\prime}(z^{\xi*}). \tag{3.3.1}$$

In turn, the smallest concave majorant of H^{ξ} is given by

$$W^{\xi}(z) = \begin{cases} z\frac{H^{\xi}(z^{\xi*})}{z^{\xi*}} & \text{if } z \in [0, z^{\xi*}), \\ H^{\xi}(z) & \text{if } z \in [z^{\xi*}, +\infty). \end{cases}$$

Substituting $b^{\xi*} = \psi^{-1}(z^{\xi*})$ into (3.3.1), we have

$$\frac{H^{\xi}(z^{\xi*})}{z^{\xi*}} = \frac{H^{\xi}(\psi(b^{\xi*}))}{\psi(b^{\xi*})} = \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})},$$

and

$$\begin{aligned} H^{\xi\prime}(z^{\xi*}) &= \frac{e^{\psi^{-1}(z^{\xi*})}G(\psi^{-1}(z^{\xi*})) - (e^{\psi^{-1}(z^{\xi*})} - c_s)G'(\psi^{-1}(z^{\xi*}))}{F'(\psi^{-1}(z^{\xi*}))G(\psi^{-1}(z^{\xi*})) - F(\psi^{-1}(z^{\xi*}))G'(\psi^{-1}(z^{\xi*}))} \\ &= \frac{e^{b^{\xi*}}G(b^{\xi*}) - (e^{b^{\xi*}} - c_s)G'(b^{\xi*})}{F'(b^{\xi*})G(b^{\xi*}) - F(b^{\xi*})G'(b^{\xi*})}. \end{aligned}$$

Equivalently, we can express (3.3.1) in terms of $b^{\xi*}$:

$$\frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})} = \frac{e^{b^{\xi*}}G(b^{\xi*}) - (e^{b^{\xi*}} - c_s)G'(b^{\xi*})}{F'(b^{\xi*})G(b^{\xi*}) - F(b^{\xi*})G'(b^{\xi*})},$$

which is equivalent to (3.2.2) after simplification. As a result, we have

$$W^{\xi}(\psi(x)) = \begin{cases} \psi(x)\frac{H^{\xi}(z^{\xi*})}{z^{\xi*}} = \frac{F(x)}{G(x)}\frac{e^{b^{\xi*}}-c_s}{F(b^{\xi*})} & \text{if } x \in (-\infty, b^{\xi*}), \\ H^{\xi}(\psi(x)) = \frac{e^x - c_s}{G(x)} & \text{if } x \in [b^{\xi*}, +\infty). \end{cases}$$

In turn, the value function $V^{\xi}(x) = G(x)W^{\xi}(\psi(x))$ is given by (3.2.1).

3.3.1.2 Optimal Entry Timing

We can directly follow the arguments that yield Theorem 2.2.2, but with the reward as $\hat{h}^{\xi}(x) = V^{\xi}(x) - h^{\xi}_b(x) = V^{\xi}(x) - (e^x + c_b)$ and define $\hat{H}^{\xi}$ analogous to H:

$$\hat{H}^{\xi}(z) := \begin{cases} \frac{\hat{h}^{\xi}}{G} \circ \psi^{-1}(z) & \text{if } z > 0, \\ \lim\limits_{x \to -\infty} \frac{(\hat{h}^{\xi}(x))^+}{G(x)} & \text{if } z = 0. \end{cases}$$

We will look for the value function with the form: $J^{\xi}(x) = G(x)\hat{W}^{\xi}(\psi(x))$, where $\hat{W}^{\xi}$ is the smallest concave majorant of $\hat{H}^{\xi}$. The properties of $\hat{H}^{\xi}$ is given in the next lemma.

Lemma 3.3.2. *The function $\hat{H}^{\xi}$ is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, and twice differentiable on $(0, \psi(b^{\xi*})) \cup (\psi(b^{\xi*}), +\infty)$, and possesses the following properties:*

(i) $\hat{H}^{\xi}(0) = 0$, and there exists some $\underline{b}^{\xi} < b^{\xi*}$ such that $\hat{H}^{\xi}(z) < 0$ for $z \in (0, \psi(\underline{b}^{\xi})) \cup [\psi(b^{\xi*}), +\infty)$.

(ii) $\hat{H}^{\xi}(z)$ is strictly decreasing for $z \in [\psi(b^{\xi*}), +\infty)$.

(iii) Define the constant

$$x^{\xi*} = \theta + \frac{\sigma^2}{2\mu} - \frac{r}{\mu} - 1.$$

There exist some constants x_{b1} *and* x_{b2}*, with* $-\infty < x_{b1} < x^{\xi*} < x_{b2} < x_s$*, that solve* $f_b(x) = 0$*, such that*

$$\hat{H}^{\xi}(z) \text{ is } \begin{cases} \textit{convex} & \textit{if } y \in (0, \psi(x_{b1})) \cup (\psi(x_{b2}), +\infty) \\ \textit{concave} & \textit{if } z \in (\psi(x_{b1}), \psi(x_{b2})), \end{cases}$$

and $\hat{z}_1^{\xi} := \arg\max_{y \in [0,+\infty)} \hat{H}^{\xi}(y) \in (\psi(x_{b1}), \psi(x_{b2}))$.

Figure 3.4 gives a sketch of $\hat{H}^{\xi}$ according to Lemma 3.3.2, and illustrate the corresponding smallest concave majorant $\hat{W}^{\xi}$.

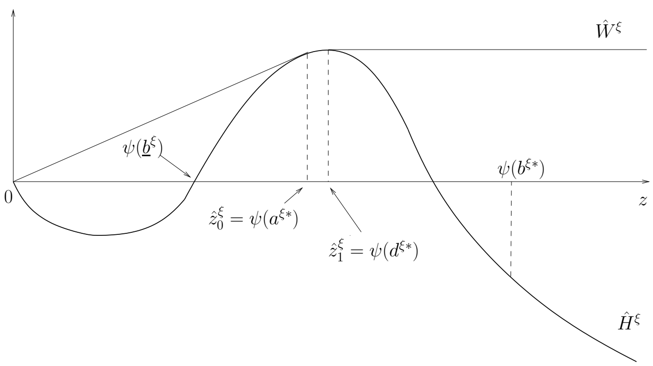


Figure 3.4: Sketches of $\hat{H}^{\xi}$ and $\hat{W}^{\xi}$. The smallest concave majorant $\hat{W}^{\xi}$ is a straight line tangent to $\hat{H}^{\xi}$ at $\hat{z}_0^{\xi}$ on $[0, \hat{z}_0^{\xi})$, coincides with $\hat{H}^{\xi}$ on $[\hat{z}_0^{\xi}, \hat{z}_1^{\xi}]$, and is equal to $\hat{H}^{\xi}(\hat{z}_1^{\xi})$ on $(\hat{z}_1^{\xi}, +\infty)$.

Proof of Theorem 3.2.4 As in Lemma 3.3.2 and Figure 3.4, by the definition of the maximizer of $\hat{H}^{\xi}$, $\hat{z}_1^{\xi}$ satisfies the equation

$$\hat{H}^{\xi\prime}(\hat{z}_1^{\xi}) = 0. \tag{3.3.2}$$

Also there exists a unique number $\hat{z}_0^{\xi} \in (x_{b1}, \hat{z}_1^{\xi})$ such that

$$\frac{\hat{H}^{\xi}(\hat{z}_0^{\xi})}{\hat{z}_0^{\xi}} = \hat{H}^{\xi\prime}(\hat{z}_0^{\xi}). \tag{3.3.3}$$

Using (3.3.2), (3.3.3) and Figure 3.4, $\hat{W}^{\xi}$ is a straight line tangent to $\hat{H}^{\xi}$ at $\hat{z}_0^{\xi}$ on $[0, \hat{z}_0^{\xi})$, coincides with $\hat{H}^{\xi}$ on $[\hat{z}_0^{\xi}, \hat{z}_1^{\xi}]$, and is equal to $\hat{H}^{\xi}(\hat{z}_1^{\xi})$ on $(\hat{z}_1^{\xi}, +\infty)$. As a result,

$$\hat{W}^{\xi}(z) = \begin{cases} z\hat{H}^{\xi\prime}(\hat{z}_0^{\xi}) & \text{if } z \in [0, \hat{z}_0^{\xi}), \\ \hat{H}^{\xi}(z) & \text{if } z \in [\hat{z}_0^{\xi}, \hat{z}_1^{\xi}], \\ \hat{H}^{\xi}(\hat{z}_1^{\xi}) & \text{if } z \in (\hat{z}_1^{\xi}, +\infty). \end{cases}$$

Substituting $a^{\xi*} = \psi^{-1}(\hat{z}_0^{\xi})$ into (3.3.3), we have

$$\frac{\hat{H}^{\xi}(\hat{z}_0^{\xi})}{\hat{z}_0^{\xi}} = \frac{V^{\xi}(a^{\xi*}) - (e^{a^{\xi*}} + c_b)}{F(a^{\xi*})},$$

and

$$\hat{H}^{\xi\prime}(\hat{z}_0^{\xi}) = \frac{G(a^{\xi*})(V^{\xi\prime}(a^{\xi*}) - e^{a^{\xi*}}) - G'(a^{\xi*})(V^{\xi}(a^{\xi*}) - (e^{a^{\xi*}} + c_b))}{F'(a^{\xi*})G(a^{\xi*}) - F(a^{\xi*})G'(a^{\xi*})}.$$

Equivalently, we can express condition (3.3.3) in terms of $a^{\xi*}$:

$$\frac{V^{\xi}(a^{\xi*}) - (e^{a^{\xi*}} + c_b)}{F(a^{\xi*})} = \frac{G(a^{\xi*})(V^{\xi\prime}(a^{\xi*}) - e^{a^{\xi*}}) - G'(a^{\xi*})(V^{\xi}(a^{\xi*}) - (e^{a^{\xi*}} + c_b))}{F'(a^{\xi*})G(a^{\xi*}) - F(a^{\xi*})G'(a^{\xi*})},$$

which is equivalent to (3.2.4) after simplification. Also, we can express $\hat{H}^{\xi\prime}(\hat{z}_0^{\xi})$ in terms of $a^{\xi*}$:

$$\hat{H}^{\xi\prime}(\hat{z}_0^{\xi}) = \frac{\hat{H}^{\xi}(\hat{z}_0^{\xi})}{\hat{z}_0^{\xi}} = \frac{V^{\xi}(a^{\xi*}) - (e^{a^{\xi*}} + c_b)}{F(a^{\xi*})} = P^{\xi}.$$

In addition, substituting $d^{\xi*} = \psi^{-1}(\hat{z}_1^{\xi})$ into (3.3.2), we have

$$\frac{G(d^{\xi*})(V^{\xi\prime}(d^{\xi*}) - e^{d^{\xi*}}) - G'(d^{\xi*})(V^{\xi}(d^{\xi*}) - (e^{d^{\xi*}} + c_b))}{F'(d^{\xi*})G(d^{\xi*}) - F(d^{\xi*})G'(d^{\xi*})} = 0,$$

which can be further simplified to (3.2.5). Furthermore, $\hat{H}^{\xi}(\hat{z}_1^{\xi})$ can be written in terms of $d^{\xi*}$:

$$\hat{H}^{\xi}(\hat{z}_1^{\xi}) = \frac{V^{\xi}(d^{\xi*}) - (e^{d^{\xi*}} + c_b)}{G(d^{\xi*})} = Q^{\xi}.$$

By direct substitution of the expressions for $\hat{W}^{\xi}$ and the associated functions, we obtain the value function in (3.2.3).

3.3.2 Optimal Switching Problem

Using the results derived in previous sections, we can infer the structure of the buy and sell regions of the switching problem and then proceed to verify its optimality. In this section, we provide detailed proofs for Theorems 3.2.6 and 3.2.7.

Proof of Theorem 3.2.6 (Part 1) First, with $h_s^\xi(x) = e^x - c_s$, we differentiate to get

$$\left(\frac{h_s^\xi}{F}\right)'(x) = \frac{(e^x - c_s)F'(x) - e^x F(x)}{F^2(x)}. \tag{3.3.4}$$

On the other hand, by Ito's lemma, we have

$$h_s^\xi(x) = \mathbb{E}_x\{e^{-rt}h_s^\xi(X_t)\} - \mathbb{E}_x\left\{\int_0^t e^{-ru}(\mathcal{L} - r)h_s^\xi(X_u)du\right\}.$$

Note that

$$\mathbb{E}_x\{e^{-rt}h_s^\xi(X_t)\} = e^{-rt}\left(e^{(x-\theta)e^{-\mu t}+\theta+\frac{\sigma^2}{4\mu}(1-e^{-2\mu t})} - c_s\right) \to 0 \quad \text{as} \quad t \to +\infty.$$

This implies that

$$\begin{aligned} h_s^\xi(x) &= -\mathbb{E}_x\left\{\int_0^{+\infty} e^{-ru}(\mathcal{L} - r)h_s^\xi(X_u)du\right\} \\ &= -G(x)\int_{-\infty}^x \Psi(s)(\mathcal{L} - r)h_s^\xi(s)ds \\ &\quad - F(x)\int_x^{+\infty} \Phi(s)(\mathcal{L} - r)h_s^\xi(s)ds, \end{aligned} \tag{3.3.5}$$

where Ψ is defined in (3.2.17) and

$$\Phi(x) := \frac{2G(x)}{\sigma^2\mathcal{W}(x)}.$$

The last line follows from Theorem 50.7 in Rogers and Williams [2000, p. 293]. Dividing both sides by $F(x)$ and differentiating the RHS of (3.3.5), we obtain

$$\begin{aligned}\left(\frac{h_s^\xi}{F}\right)'(x) &= -\left(\frac{G}{F}\right)'(x)\int_{-\infty}^{x}\Psi(s)(\mathcal{L}-r)h_s^\xi(s)ds\\ &\quad -\frac{G}{F}(x)\Psi(x)(\mathcal{L}-r)h_s^\xi(x) - \Phi(x)(\mathcal{L}-r)h_s^\xi(x)\\ &= \frac{\mathcal{W}(x)}{F^2(x)}\int_{-\infty}^{x}\Psi(s)(\mathcal{L}-r)h_s^\xi(s)ds = \frac{\mathcal{W}(x)}{F^2(x)}q(x),\end{aligned}$$

where

$$q(x) := \int_{-\infty}^{x}\Psi(s)(\mathcal{L}-r)h_s^\xi(s)ds.$$

Since $\mathcal{W}(x), F(x) > 0$, we deduce that $\left(\frac{h_s^\xi}{F}\right)'(x) = 0$ is equivalent to $q(x) = 0$. Using (3.3.4), we now see that (3.2.2) is equivalent to $q(b) = 0$.

Next, it follows from (3.2.7) that

$$q'(x) = \Psi(x)(\mathcal{L}-r)h_s^\xi(x)\begin{cases} > 0 & \text{if } x < x_s, \\ < 0 & \text{if } x > x_s. \end{cases} \tag{3.3.6}$$

This, together with the fact that $\lim_{x\to-\infty} q(x) = 0$, implies that there exists a unique $b^{\xi*}$ such that $q(b^{\xi*}) = 0$ if and only if $\lim_{x\to+\infty} q(x) < 0$. Next, we show that this inequality holds. By the definition of h_s^ξ and F, we have

$$\begin{aligned}&\frac{h_s^\xi(x)}{F(x)} = \frac{e^x - c_s}{F(x)} > 0 \quad \text{for } x > \ln c_s, \qquad \lim_{x\to+\infty}\frac{h_s^\xi(x)}{F(x)} = 0,\\ &\left(\frac{h_s^\xi}{F}\right)'(x) = \frac{\mathcal{W}(x)}{F^2(x)}\int_{-\infty}^{x}\Psi(s)(\mathcal{L}-r)h_s^\xi(s)ds = \frac{\mathcal{W}(x)}{F^2(x)}q(x).\end{aligned} \tag{3.3.7}$$

Since q is strictly decreasing in $(x_s, +\infty)$, the above hold true if and only if $\lim_{x\to+\infty} q(x) < 0$. Therefore, we conclude that there exits a unique $b^{\xi*}$ such that $e^bF(b) = (e^b - c_s)F'(b)$. Using (3.3.6), we see that

$$b^{\xi*} > x_s \quad \text{and} \quad q(x) > 0 \quad \text{for all} \quad x < b^{\xi*}. \tag{3.3.8}$$

Observing that $e^{b^{\xi*}}, F(b^{\xi*}), F'(b^{\xi*}) > 0$, we can conclude that $h_s^{\xi}(b^{\xi*}) = e^{b^{\xi*}} - c_s > 0$, or equivalently $b^{\xi*} > \ln c_s$.

We now verify by direct substitution that $\tilde{V}^{\xi}(x)$ and $\tilde{J}^{\xi}(x)$ in (3.2.8) satisfy the pair of variational inequalities:

$$\min\{r\tilde{J}^{\xi}(x) - \mathcal{L}\tilde{J}^{\xi}(x), \tilde{J}^{\xi}(x) - (\tilde{V}^{\xi}(x) - h_b^{\xi}(x))\} = 0, \tag{3.3.9}$$

$$\min\{r\tilde{V}^{\xi}(x) - \mathcal{L}\tilde{V}^{\xi}(x), \tilde{V}^{\xi}(x) - (\tilde{J}^{\xi}(x) + h_s^{\xi}(x))\} = 0. \tag{3.3.10}$$

First, note that $\tilde{J}^{\xi}(x)$ is identically 0 and thus satisfies the equality

$$(r - \mathcal{L})\tilde{J}^{\xi}(x) = 0. \tag{3.3.11}$$

To show that $\tilde{J}^{\xi}(x) - (\tilde{V}^{\xi}(x) - h_b^{\xi}(x)) \geq 0$, we look at the disjoint intervals $(-\infty, b^{\xi*})$ and $[b^{\xi*}, \infty)$ separately. For $x \geq b^{\xi*}$, we have

$$\tilde{V}^{\xi}(x) - h_b^{\xi}(x) = -(c_b + c_s),$$

which implies $\tilde{J}^{\xi}(x) - (\tilde{V}^{\xi}(x) - h_b^{\xi}(x)) = c_b + c_s \geq 0$. When $x < b^{\xi*}$, the inequality

$$\tilde{J}^{\xi}(x) - (\tilde{V}^{\xi}(x) - h_b^{\xi}(x)) \geq 0$$

can be rewritten as

$$\frac{h_b^{\xi}(x)}{F(x)} = \frac{e^x + c_b}{F(x)} \geq \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})} = \frac{h_s^{\xi}(b^{\xi*})}{F(b^{\xi*})}. \tag{3.3.12}$$

To determine the necessary conditions for this to hold, we consider the derivative of the LHS of (3.3.12):

$$\begin{aligned}\left(\frac{h_b^{\xi}}{F}\right)'(x) &= \frac{\mathcal{W}(x)}{F^2(x)} \int_{-\infty}^{x} \Psi(s)(\mathcal{L} - r)h_b^{\xi}(s)ds \\ &= \frac{\mathcal{W}(x)}{F^2(x)} \int_{-\infty}^{x} \Psi(s)e^s f_b(s)ds.\end{aligned} \tag{3.3.13}$$

If $f_b(x) = 0$ has no roots, then $(\mathcal{L} - r)h_b^{\xi}(x)$ is negative for all $x \in \mathbb{R}$. On the other hand, if there is only one root $\tilde{x}$, then $(\mathcal{L}-r)h_b^{\xi}(\tilde{x}) = 0$ and $(\mathcal{L}-r)h_b^{\xi}(x) <$

0 for all other x. In either case, $h_b^\xi(x)/F(x)$ is a strictly decreasing function and (3.3.12) is true.

Otherwise if $f_b(x) = 0$ has two distinct roots x_{b1} and x_{b2} with $x_{b1} < x_{b2}$, then

$$(\mathcal{L} - r)h_b^\xi(x) \begin{cases} < 0 & \text{if } x \in (-\infty, x_{b1}) \cup (x_{b2}, +\infty), \\ > 0 & \text{if } x \in (x_{b1}, x_{b2}). \end{cases} \tag{3.3.14}$$

Applying (3.3.14) to (3.3.13), the derivative $(h_b^\xi/F)'(x)$ is negative on $(-\infty, x_{b1})$ since the integrand in (3.3.13) is negative. Hence, $h_b^\xi(x)/F(x)$ is strictly decreasing on $(-\infty, x_{b1})$. We further note that $b^{\xi*} > x_s > x_{b2}$. Observe that on the interval (x_{b1}, x_{b2}), the intergrand is positive. It is therefore possible for $(h_b^\xi/F)'$ to change sign at some $x \in (x_{b1}, x_{b2})$. For this to happen, the positive part of the integral must be larger than the absolute value of the negative part. In other words, (3.2.16) must hold. If (3.2.16) holds, then there must exist some $\tilde{a}^* \in (x_{b1}, x_{b2})$ such that $(h_b^\xi/F)'(\tilde{a}^*) = 0$, or equivalently (3.2.9) holds:

$$\left(\frac{h_b^\xi}{F}\right)'(\tilde{a}^*) = \frac{h_b^{\xi'}(\tilde{a}^*)}{F(\tilde{a}^*)} - \frac{h_b^\xi(\tilde{a}^*)F'(\tilde{a}^*)}{F^2(\tilde{a}^*)} = \frac{e^{\tilde{a}^*}}{F(\tilde{a}^*)} - \frac{(e^{\tilde{a}^*} + c_b)F(\tilde{a}^*)'}{F^2(\tilde{a}^*)}.$$

If (3.2.9) holds, then we have

$$\left|\int_{-\infty}^{x_{b1}} \Psi(x)e^x f_b(x)dx\right| = \int_{x_{b1}}^{\tilde{a}^*} \Psi(x)e^x f_b(x)dx.$$

In addition, since

$$\int_{\tilde{a}^*}^{x_{b2}} \Psi(x)e^x f_b(x)dx > 0,$$

it follows that

$$\left|\int_{-\infty}^{x_{b1}} \Psi(x)e^x f_b(x)dx\right| < \int_{x_{b1}}^{x_{b2}} \Psi(x)e^x f_b(x)dx.$$

This establishes the equivalence between (3.2.9) and (3.2.16). Under this condition, h_b^ξ/F is strictly decreasing on $(x_{b1}, \tilde{a}^*)$. Then, either it is strictly increasing on $(\tilde{a}^*, b^{\xi*})$, or there exists some $\bar{x} \in (x_{b2}, b^{\xi*})$ such that $h_b^\xi(x)/F(x)$ is strictly increasing on $(\tilde{a}^*, \bar{x})$ and strictly decreasing on $(\bar{x}, b^{\xi*})$. In both cases, (3.3.12) is true if and only if (3.2.10) holds.

Alternatively, if (3.2.16) doesn't hold, then by in (3.3.13), the integral $(h_b^\xi/F)'$ will always be negative. This means that the function $h_b^\xi(x)/F(x)$ is strictly decreasing for all $x \in (-\infty, b^{\xi*})$, in which case (3.3.12) holds.

We are thus able to show that (3.3.9) holds, in particular the minimum of 0 is achieved as a result of (3.3.11). To prove (3.3.10), we go through a similar procedure. To check that

$$(r - \mathcal{L})\tilde{V}^\xi(x) \geq 0$$

holds, we consider two cases. First when $x < b^{\xi*}$, we get

$$(r - \mathcal{L})\tilde{V}^\xi(x) = \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})}(r - \mathcal{L})F(x) = 0.$$

On the other hand, when $x \geq b^{\xi*}$, the inequality holds

$$(r - \mathcal{L})\tilde{V}^\xi(x) = (r - \mathcal{L})h_s^\xi(x) > 0,$$

since $b^{\xi*} > x_s$ (the first inequality of (3.3.8)) and (3.2.7).

Similarly, when $x \geq b^{\xi*}$, we have

$$\tilde{V}^\xi(x) - (\tilde{J}^\xi(x) + h_s^\xi(x)) = h_s^\xi(x) - h_s^\xi(x) = 0.$$

When $x < b^{\xi*}$, the inequality holds:

$$\tilde{V}^\xi(x) - (\tilde{J}^\xi(x) + h_s^\xi(x)) = \frac{h_s^\xi(b^{\xi*})}{F(b^{\xi*})}F(x) - h_s^\xi(x) \geq 0,$$

which is equivalent to $\frac{h_s^\xi(x)}{F(x)} \leq \frac{h_s^\xi(b^{\xi*})}{F(b^{\xi*})}$, due to (3.3.7) and (3.3.8).

Proof of Theorem 3.2.7 (Part 1) Define the functions

$$q_G(x, z) = \int_x^{+\infty} \Phi(s)(\mathcal{L} - r)h_b^\xi(s)ds - \int_z^{+\infty} \Phi(s)(\mathcal{L} - r)h_s^\xi(s)ds,$$
$$q_F(x, z) = \int_{-\infty}^{x} \Psi(s)(\mathcal{L} - r)h_b^\xi(s)ds - \int_{-\infty}^{z} \Psi(s)(\mathcal{L} - r)h_s^\xi(s)ds.$$

where We look for the points $\tilde{d}^* < \tilde{b}^*$ such that

$$q_G(\tilde{d}^*, \tilde{b}^*) = 0, \quad \text{and} \quad q_F(\tilde{d}^*, \tilde{b}^*) = 0.$$

This is because these two equations are equivalent to (3.2.14) and (3.2.15), respectively.

Now we start to solve the equations by first narrowing down the range for $\tilde{d}^*$ and $\tilde{b}^*$. Observe that

$$\begin{aligned} q_G(x,z) &= \int_x^z \Phi(s)(\mathcal{L}-r)h_b^\xi(s)ds + \int_z^\infty \Phi(s)[(\mathcal{L}-r)(h_b^\xi(s) - h_s^\xi(s)]ds \\ &= \int_x^z \Phi(s)(\mathcal{L}-r)h_b^\xi(s)ds - r(c_b+c_s)\int_z^\infty \Phi(s)ds \\ &< 0, \end{aligned} \tag{3.3.15}$$

for all x and z such that $x_{b2} \le x < z$. Therefore, $\tilde{d}^* \in (-\infty, x_{b2})$.

Since $b^{\xi*} > x_s$ satisfies $q(b^{\xi*}) = 0$ and $\tilde{a}^* < x_{b2}$ satisfies (3.2.9), we have

$$\begin{aligned} \lim_{z\to+\infty} q_F(x,z) &= \int_{-\infty}^x \Psi(s)(\mathcal{L}-r)h_b^\xi(s)ds - q(b^{\xi*}) - \int_{b^{\xi*}}^{+\infty} \Psi(s)(\mathcal{L}-r)h_s^\xi(s)ds \\ &> 0, \end{aligned}$$

for all $x \in (\tilde{a}^*, x_{b2})$. Also, we note that

$$\frac{\partial q_F}{\partial z}(x,z) = -\Psi(z)(\mathcal{L}-r)h_s^\xi(z) \begin{cases} < 0 & \text{if } z < x_s, \\ > 0 & \text{if } z > x_s, \end{cases} \tag{3.3.16}$$

and

$$\begin{aligned} q_F(x,x) &= \int_{-\infty}^x \Psi(s)(\mathcal{L}-r)\left[h_b^\xi(s) - h_s^\xi(s)\right] ds \\ &= -r(c_b+c_s)\int_{-\infty}^x \Psi(s)ds < 0. \end{aligned} \tag{3.3.17}$$

Then, (3.3.16) and (3.3.17) imply that there exists a unique function $\beta : [\tilde{a}^*, x_{b2}) \mapsto \mathbb{R}$ s.t. $\beta(x) > x_s$ and

$$q_F(x, \beta(x)) = 0. \tag{3.3.18}$$

Differentiating (3.3.18) with respect to x, we see that

$$\beta'(x) = \frac{\Psi(x)(\mathcal{L} - r)h_b^\xi(x)}{\Psi(\beta(x))(\mathcal{L} - r)h_s^\xi(\beta(x))} < 0,$$

for all $x \in (x_{b1}, x_{b2})$. In addition, by the facts that $b^{\xi*} > x_s$ satisfies $q(b^{\xi*}) = 0$, $\tilde{a}^*$ satisfies (3.2.9), and the definition of q_F, we have

$$\beta(\tilde{a}^*) = b^{\xi*}.$$

By (3.3.15), we have $\lim_{x\uparrow x_{b2}} q_G(x, \beta(x)) < 0$. By computation, we get that

$$\begin{aligned}
\frac{d}{dx} q_G(x, \beta(x)) &= -\frac{\Phi(x)\Psi(\beta(x)) - \Phi(\beta(x))\Psi(x)}{\Psi(\beta(x))}(\mathcal{L} - r)h_b^\xi(x) \\
&= -\Psi(x)\left[\frac{G(x)}{F(x)} - \frac{G(\beta(x))}{F(\beta(x))}\right](\mathcal{L} - r)h_b^\xi(x) < 0,
\end{aligned}$$

for all $x \in (x_{b1}, x_{b2})$. Therefore, there exists a unique $\tilde{d}^*$ such that $q_G(\tilde{d}^*, \beta(\tilde{d}^*)) = 0$ if and only if

$$q_G(\tilde{a}^*, \beta(\tilde{a}^*)) > 0.$$

The above inequality holds if (3.2.11) holds. Indeed, direct computation yields the equivalence:

$$\begin{aligned}
& q_G(\tilde{a}^*, \beta(\tilde{a}^*)) \\
&= \int_{\tilde{a}^*}^{+\infty} \Phi(s)(\mathcal{L} - r)h_b^\xi(s)ds - \int_{b^{\xi*}}^{+\infty} \Phi(s)(\mathcal{L} - r)h_s^\xi(s)ds \\
&= -\frac{h_b^\xi(\tilde{a}^*)}{F(\tilde{a}^*)} - \frac{G(b^{\xi*})}{F(b^{\xi*})}\int_{-\infty}^{b^{\xi*}} \Psi(s)(\mathcal{L} - r)h_s^\xi(s)ds - \int_{b^{\xi*}}^{+\infty} \Phi(s)(\mathcal{L} - r)h_s^\xi(s)ds \\
&= -\frac{e^{\tilde{a}^*} + c_b}{F(\tilde{a}^*)} + \frac{e^{b^{\xi*}} - c_s}{F(b^{\xi*})}.
\end{aligned}$$

When this solution exists, we have

$$\tilde{d}^* \in (x_{b1}, x_{b2}) \text{ and } \tilde{b}^* := \beta(\tilde{d}^*) > x_s.$$

Next, we show that the functions $\tilde{J}^\xi$ and $\tilde{V}^\xi$ given in (3.2.12) and (3.2.13) satisfy the pair of VIs in (3.3.9) and (3.3.10). In the same vein as the proof for the Theorem 3.2.6, we show

$$(r-\mathcal{L})\tilde{J}^\xi(x) \geq 0$$

by examining the 3 disjoint regions on which $\tilde{J}^\xi(x)$ assume different forms. When $x < \tilde{a}^*$,

$$(r-\mathcal{L})\tilde{J}^\xi(x) = \tilde{P}(r-\mathcal{L})F(x) = 0.$$

Next, when $x > \tilde{d}^*$,

$$(r-\mathcal{L})\tilde{J}^\xi(x) = \tilde{Q}(r-\mathcal{L})G(x) = 0.$$

Finally for $x \in [\tilde{a}^*, \tilde{d}^*]$,

$$(r-\mathcal{L})\tilde{J}^\xi(x) = (r-\mathcal{L})(\tilde{K}F(x) - h_b^\xi(x)) = -(r-\mathcal{L})h_b^\xi(x) > 0,$$

as a result of (3.3.14) since $\tilde{a}^*, \tilde{d}^* \in (x_{b1}, x_{b2})$.

Next, we verify that

$$(r-\mathcal{L})\tilde{V}^\xi(x) \geq 0.$$

Indeed, we have $(r-\mathcal{L})\tilde{V}^\xi(x) = \tilde{K}(r-\mathcal{L})F(x) = 0$ for $x < \tilde{b}^*$. When $x \geq \tilde{b}^*$, we get the inequality $(r-\mathcal{L})\tilde{V}^\xi(x) = (r-\mathcal{L})(\tilde{Q}G(x)+h_s^\xi(x)) = (r-\mathcal{L})h_s^\xi(x) > 0$ since $\tilde{b}^* > x_s$ and due to (3.2.7).

It remains to show that $\tilde{J}^\xi(x) - (\tilde{V}^\xi(x) - h_b^\xi(x)) \geq 0$ and $\tilde{V}^\xi(x) - (\tilde{J}^\xi(x) + h_s^\xi(x)) \geq 0$. When $x < \tilde{a}^*$, we have

$$\begin{aligned}\tilde{J}^\xi(x) - (\tilde{V}^\xi(x) - h_b^\xi(x)) &= (\tilde{P} - \tilde{K})F(x) + (e^x + c_b) \\ &= -F(x)\frac{e^{\tilde{a}^*} + c_b}{F(\tilde{a}^*)} + (e^x + c_b) \geq 0.\end{aligned}$$

This inequality holds since we have shown in the proof of Theorem 3.2.6 that $\frac{h_b^\xi(x)}{F(x)}$ is strictly decreasing for $x < \tilde{a}^*$. In addition,

$$\tilde{V}^\xi(x) - (\tilde{J}^\xi(x) + h_s^\xi(x)) = F(x)\frac{e^{\tilde{a}^*} + c_b}{F(\tilde{a}^*)} - (e^x - c_s) \geq 0,$$

since (3.3.6) (along with the ensuing explanation) implies that $\frac{h_s^\xi(x)}{F(x)}$ is increasing for all $x \leq \tilde{a}^*$.

In the other region where $x \in [\tilde{a}^*, \tilde{d}^*]$, we have

$$\begin{aligned}
&\tilde{J}^\xi(x) - (\tilde{V}^\xi(x) - h_b^\xi(x)) = 0,\\
&\tilde{V}^\xi(x) - (\tilde{J}^\xi(x) + h_s^\xi(x)) = h_b^\xi(x) - h_s^\xi(x) = c_b + c_s \geq 0.
\end{aligned}$$

When $x > \tilde{b}^*$, it is clear that

$$\begin{aligned}
&\tilde{J}^\xi(x) - (\tilde{V}^\xi(x) - h_b^\xi(x)) = h_b^\xi(x) - h_s^\xi(x) = c_b + c_s \geq 0,\\
&\tilde{V}^\xi(x) - (\tilde{J}^\xi(x) + h_s^\xi(x)) = 0.
\end{aligned}$$

To establish the inequalities for $x \in (\tilde{d}^*, \tilde{b}^*)$, we first denote

$$\begin{aligned}
g_{\tilde{J}^\xi}(x) &:= \tilde{J}^\xi(x) - (\tilde{V}^\xi(x) - h_b^\xi(x)) = \tilde{Q}G(x) - \tilde{K}F(x) + h_b^\xi(x)\\
&= F(x)\int_{\tilde{d}^*}^{x} \Phi(s)(\mathcal{L} - r)h_b^\xi(s)ds - G(x)\int_{\tilde{d}^*}^{x} \Psi(s)(\mathcal{L} - r)h_b^\xi(s)ds,\\
g_{\tilde{V}^\xi}(x) &:= \tilde{V}^\xi(x) - (\tilde{J}^\xi(x) + h_s^\xi(x)) = \tilde{K}F(x) - \tilde{Q}G(x) - h_s^\xi(x)\\
&= F(x)\int_{x}^{\tilde{b}^*} \Phi(s)(\mathcal{L} - r)h_s^\xi(s)ds - G(x)\int_{x}^{\tilde{b}^*} \Psi(s)(\mathcal{L} - r)h_s^\xi(s)ds.
\end{aligned}$$

In turn, we compute to get

$$\begin{aligned}
g'_{\tilde{J}^\xi}(x) &= F'(x)\int_{\tilde{d}^*}^{x} \Phi(s)(\mathcal{L} - r)h_b^\xi(s)ds - G'(x)\int_{\tilde{d}^*}^{x} \Psi(s)(\mathcal{L} - r)h_b^\xi(s)ds,\\
g'_{\tilde{V}^\xi}(x) &= F'(x)\int_{x}^{\tilde{b}^*} \Phi(s)(\mathcal{L} - r)h_s^\xi(s)ds - G'(x)\int_{x}^{\tilde{b}^*} \Psi(s)(\mathcal{L} - r)h_s^\xi(s)ds.
\end{aligned}$$

Recall the definition of x_{b2} and x_s, and the fact that $G' < 0 < F'$, we have $g'_{\tilde{J}^\xi}(x) > 0$ for $x \in (\tilde{d}^*, x_{b2})$ and $g'_{\tilde{V}^\xi}(x) < 0$ for $x \in (x_s, \tilde{b}^*)$. These, together with the fact that $g_{\tilde{J}^\xi}(\tilde{d}^*) = g_{\tilde{V}^\xi}(\tilde{b}^*) = 0$, imply that

$$g_{\tilde{J}^\xi}(x) > 0 \text{ for } x \in (\tilde{d}^*, x_{b2}), \text{ and } g_{\tilde{V}^\xi}(x) > 0 \text{ for } x \in (x_s, \tilde{b}^*).$$

Furthermore, since we have

$$g_{\tilde{J}^\xi}(\tilde{b}^*) = c_b + c_s \geq 0, \quad g_{\tilde{V}^\xi}(\tilde{d}^*) = c_b + c_s \geq 0, \tag{3.3.19}$$

and

$$(\mathcal{L} - r)g_{\tilde{J}^\xi}(x) = (\mathcal{L} - r)h_b^\xi(x) < 0 \text{ for all } x \in (x_{b2}, \tilde{b}^*),$$
$$(\mathcal{L} - r)g_{\tilde{V}^\xi}(x) = -(\mathcal{L} - r)h_s^\xi(x) < 0 \text{ for all } x \in (\tilde{d}^*, x_s). \tag{3.3.20}$$

In view of inequalities (3.3.19)–(3.3.20), the maximum principle implies that $g_{\tilde{J}^\xi}(x) \geq 0$ and $g_{\tilde{V}^\xi}(x) \geq 0$ for all $x \in (\tilde{d}^*, \tilde{b}^*)$. Hence, we conclude that $\tilde{J}(x) - (\tilde{V}(x) - h_b^\xi(x)) \geq 0$ and $\tilde{V}(x) - (\tilde{J}(x) + h_s^\xi(x)) \geq 0$ hold for $x \in (\tilde{d}^*, \tilde{b}^*)$.

Proof of Theorems 3.2.6 and 3.2.7 (Part 2) We now show that the candidate solutions in Theorems 3.2.6 and 3.2.7, denoted by $\tilde{j}^\xi$ and $\tilde{v}^\xi$, are equal to the optimal switching value functions $\tilde{J}^\xi$ and $\tilde{V}^\xi$ in (3.1.5) and (3.1.6), respectively. First, we note that $\tilde{j}^\xi \leq \tilde{J}^\xi$ and $\tilde{v}^\xi \leq \tilde{V}^\xi$, since $\tilde{J}^\xi$ and $\tilde{V}^\xi$ dominate the expected discounted cash low from any admissible strategy.

Next, we show the reverse inequaities. In Part 1, we have proved that $\tilde{j}^\xi$ and $\tilde{v}^\xi$ satisfy the VIs (3.3.9) and (3.3.10). In particular, we know that $(r - \mathcal{L})\tilde{j}^\xi \geq 0$, and $(r - \mathcal{L})\tilde{v}^\xi \geq 0$. Then by Dynkin's formula and Fatou's lemma, as in Øksendal [2003, p. 226], for any stopping times ζ_1 and ζ_2 such that $0 \leq \zeta_1 \leq \zeta_2$ almost surely, we have the inequalities

$$\mathbb{E}_x\{e^{-r\zeta_1}\tilde{j}^\xi(X_{\zeta_1})\} \geq \mathbb{E}_x\{e^{-r\zeta_2}\tilde{j}^\xi(X_{\zeta_2})\}, \tag{3.3.21}$$
$$\mathbb{E}_x\{e^{-r\zeta_1}\tilde{v}^\xi(X_{\zeta_1})\} \geq \mathbb{E}_x\{e^{-r\zeta_2}\tilde{v}^\xi(X_{\zeta_2})\}. \tag{3.3.22}$$

For $\Lambda_0 = (\nu_1, \tau_1, \nu_2, \tau_2, \dots)$, noting that $\nu_1 \leq \tau_1$ almost surely, we have

$$\tilde{j}^\xi(x) \geq \mathbb{E}_x\{e^{-r\nu_1}\tilde{j}^\xi(X_{\nu_1})\} \tag{3.3.23}$$

$$\geq \mathbb{E}_x\{e^{-r\nu_1}(\tilde{v}^\xi(X_{\nu_1}) - h_b^\xi(X_{\nu_1}))\} \tag{3.3.24}$$

$$\geq \mathbb{E}_x\{e^{-r\tau_1}\tilde{v}^\xi(X_{\tau_1})\} - \mathbb{E}_x\{e^{-r\nu_1}h_b^\xi(X_{\nu_1})\} \tag{3.3.25}$$

$$\geq \mathbb{E}_x\{e^{-r\tau_1}(\tilde{j}^\xi(X_{\tau_1}) + h_s^\xi(X_{\tau_1}))\} - \mathbb{E}_x\{e^{-r\nu_1}h_b^\xi(X_{\nu_1})\} \tag{3.3.26}$$

$$= \mathbb{E}_x\{e^{-r\tau_1}\tilde{j}^\xi(X_{\tau_1})\} + \mathbb{E}_x\{e^{-r\tau_1}h_s^\xi(X_{\tau_1}) - e^{-r\nu_1}h_b^\xi(X_{\nu_1})\}, \tag{3.3.27}$$

where (3.3.23) and (3.3.25) follow from (3.3.21) and (3.3.22) respectively. Also, (3.3.24) and (3.3.26) follow from (3.3.9) and (3.3.10) respectively. Observe that (3.3.27) is a recursion and $\tilde{j}^\xi(x) \geq 0$ in both Theorems 3.2.6 and 3.2.7, we obtain

$$\tilde{j}^\xi(x) \geq \mathbb{E}_x\left\{\sum_{n=1}^{\infty}[e^{-r\tau_n}h_s^\xi(X_{\tau_n}) - e^{-r\nu_n}h_b^\xi(X_{\nu_n})]\right\}.$$

Maximizing over all Λ_0 yields that $\tilde{j}^\xi(x) \geq \tilde{J}^\xi(x)$. A similar proof gives $\tilde{v}^\xi(x) \geq \tilde{V}^\xi(x)$.

Remark 3.3.3. *If there is no transaction cost for entry, i.e. $c_b = 0$, then f_b, which is now a linear function with a non-zero slope, has one root x_0. Moreover, we have $f_b(x) > 0$ for $x \in (-\infty, x_0)$ and $f_b(x) < 0$ for $x \in (x_0, +\infty)$. This implies that the entry region must be of the form $(-\infty, d_0)$, for some number d_0. Hence, the continuation region for entry is the* connected *interval (d_0, ∞).*

Remark 3.3.4. *Let $\mathcal{L}^\xi$ be the infinitesimal generator of the XOU process $\xi = e^X$, and define the function $H_b(\varsigma) := \varsigma + c_b \equiv h_b^\xi(\ln \varsigma)$. In other words, we have the equivalence:*

$$(\mathcal{L}^\xi - r)H_b(\varsigma) \equiv (\mathcal{L} - r)h_b^\xi(\ln \varsigma).$$

Referring to (3.2.6) *and* (3.2.7)*, we have* either *that*

$$(\mathcal{L}^{\xi} - r)H_b(\varsigma) \begin{cases} > 0 & \textit{for } \varsigma \in (\varsigma_{b1}, \varsigma_{b2}), \\ < 0 & \textit{for } \varsigma \in (0, \varsigma_{b1}) \cup (\varsigma_{b2}, +\infty), \end{cases} \quad (3.3.28)$$

where $\varsigma_{b1} = e^{x_{b1}} > 0$ *and* $\varsigma_{b2} = e^{x_{b2}}$ *and* $x_{b1} < x_{b2}$ *are two distinct roots to* (3.2.6)*,* or

$$(\mathcal{L}^{\xi} - r)H_b(\varsigma) < 0, \quad \textit{for } \varsigma \in (0, \varsigma^*) \cup (\varsigma^*, +\infty), \quad (3.3.29)$$

where $\varsigma^* = e^{x_b}$ *and* x_b *is the single root to* (3.2.6)*. In both cases, Assumption 4 of Zervos et al. [2013] is violated, and their results cannot be applied. Indeed, they would require that* $(\mathcal{L}^{\xi} - r)H_b(\varsigma)$ *is strictly negative over a connected interval of the form* (ς_0, ∞)*, for some fixed* $\varsigma_0 \geq 0$*. However, it is clear from* (3.3.28) *and* (3.3.29) *that such a region is disconnected.*

In fact, the approach by Zervos et al. [2013] applies to the optimal switching problems where the optimal wait-for-entry region (in log-price) is of the form $(\tilde{d}^*, \infty)$*, rather than the* disconnected *region* $(-\infty, \tilde{a}^*) \cup (\tilde{d}^*, \infty)$*, as in our case with an XOU underlying. Using the new inferred structure of the wait-for-entry region, we have modified the arguments in Zervos et al. [2013] to solve our optimal switching problem for Theorems 3.2.6 and 3.2.7.*

Chapter 4

Trading under CIR Dynamics

In this chapter, we study the optimal double stopping and switching problems under a CIR model. We establish the conditions under which the double stopping and switching problems admit the same optimal starting and/or stopping strategies. We rigorously prove that the optimal starting and stopping strategies are of threshold type, and give the analytical expressions for the value functions in terms of confluent hypergeometric functions. In Section 4.1, we formulate both the optimal starting-stopping and optimal switching problems. Then, we present our analytical results and numerical examples in Section 4.2. The proofs of our main results are detailed in Section 4.3. Finally, Appendix D contains the proofs for a number of lemmas.

4.1 Problem Overview

We consider a CIR process $(Y_t)_{t\geq 0}$ that satisfies the SDE

$$dY_t = \mu(\theta - Y_t)\,dt + \sigma\sqrt{Y_t}\,dB_t, \tag{4.1.1}$$

with constants $\mu, \theta, \sigma > 0$. If $2\mu\theta \geq \sigma^2$ holds, which is often referred to as the Feller condition (see Feller [1951]), then the level 0 is inaccessible by Y.

If the initial value $Y_0 > 0$, then Y stays strictly positive at all times almost surely. Nevertheless, if $Y_0 = 0$, then Y will enter the interior of the state space immediately and stays positive thereafter almost surely. If $2\mu\theta < \sigma^2$, then the level 0 is a reflecting boundary. This means that once Y reaches 0, it immediately returns to the interior of the state space and continues to evolve. For a detailed categorization of boundaries for diffusion processes, we refer to Chapter 2 of Borodin and Salminen [2002] and Chapter 15 of Karlin and Taylor [1981].

4.1.1 Optimal Starting-Stopping Problem

Given a CIR process, we first consider the optimal timing to stop. If a decision to stop is made at some time τ, then the amount Y_τ is received and simultaneously the constant transaction cost $c_s > 0$ has to be paid. Denote by $\mathbb{F}$ the filtration generated by B, and $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times. The maximum expected discounted value is obtained by solving the optimal stopping problem

$$V^\chi(y) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_y\big\{e^{-r\tau}(Y_\tau - c_s)\big\}\,, \tag{4.1.2}$$

where $r > 0$ is the constant discount rate, and $\mathbb{E}_y\{\cdot\} \equiv \mathbb{E}\{\cdot|Y_0 = y\}$.

The value function V^χ represents the expected value from optimally stopping the process Y. On the other hand, the process value plus the transaction cost constitute the total cost to start. Before even starting, one needs to choose the optimal timing to start, or not to start at all. This leads us to analyze the starting timing inherent in the starting-stopping problem. Precisely, we solve

$$J^\chi(y) = \sup_{\nu\in\mathcal{T}} \mathbb{E}_y\big\{e^{-r\nu}(V^\chi(Y_\nu) - Y_\nu - c_b)\big\}\,, \tag{4.1.3}$$

with the constant transaction cost $c_b > 0$ incurred at the start. In other words, the objective is to maximize the expected difference between the value function $V^{\chi}(Y_\nu)$ and the current Y_ν, minus transaction cost c_b. The value function $J^{\chi}(y)$ represents the maximum expected value that can be gained by entering and subsequently exiting, with transaction costs c_b and c_s incurred, respectively, on entry and exit. For our analysis, the transaction costs c_b and c_s can be different. To facilitate presentation, we denote the functions

$$h_s(y) = y - c_s, \quad \text{and} \quad h_b(y) = y + c_b. \tag{4.1.4}$$

If it turns out that $J^{\chi}(Y_0) \leq 0$ for some initial value Y_0, then it is optimal not to start at all. Therefore, it is important to identify the trivial cases. Under the CIR model, since $\sup_{y\in\mathbb{R}_+}(V^{\chi}(y) - h_b(y)) \leq 0$ implies that $J^{\chi}(y) = 0$ for $y \in \mathbb{R}_+$, we shall therefore focus on the case with

$$\sup_{y\in\mathbb{R}_+} (V^{\chi}(y) - h_b(y)) > 0, \tag{4.1.5}$$

and solve for the non-trivial optimal timing strategy.

4.1.2 Optimal Switching Problem

Under the optimal switching approach, it is assumed that an infinite number of entry and exit actions take place. The sequential entry and exit times are modeled by the stopping times $\nu_1, \tau_1, \nu_2, \tau_2, \cdots \in \mathcal{T}$ such that

$$0 \leq \nu_1 \leq \tau_1 \leq \nu_2 \leq \tau_2 \leq \ldots.$$

Entry and exit decisions are made, respectively, at times ν_i and τ_i, $i \in \mathbb{N}$. The optimal timing to enter or exit would depend on the initial position. Precisely, under the CIR model, if the initial position is zero, then the first task is to determine when to *start* and the corresponding optimal switching problem is

$$\tilde{J}^{\chi}(y) = \sup_{\Lambda_0} \mathbb{E}_y \left\{ \sum_{n=1}^{\infty} [e^{-r\tau_n} h_s(Y_{\tau_n}) - e^{-r\nu_n} h_b(Y_{\nu_n})] \right\}, \tag{4.1.6}$$

with the set of admissible stopping times $\Lambda_0 = (\nu_1, \tau_1, \nu_2, \tau_2, \dots)$, and the reward functions h_b and h_s defined in (4.1.4). On the other hand, if we start with a long position, then it is necessary to solve

$$\tilde{V}^{\chi}(y) = \sup_{\Lambda_1} \mathbb{E}_y \left\{ e^{-r\tau_1} h_s(Y_{\tau_1}) + \sum_{n=2}^{\infty} [e^{-r\tau_n} h_s(Y_{\tau_n}) - e^{-r\nu_n} h_b(Y_{\nu_n})] \right\}, \quad (4.1.7)$$

with $\Lambda_1 = (\tau_1, \nu_2, \tau_2, \nu_3, \dots)$ to determine when to *stop*.

In summary, the optimal starting-stopping and switching problems differ in the number of entry and exit decisions. Observe that any strategy for the starting-stopping problem (4.1.2)-(4.1.3) is also a candidate strategy for the switching problem (4.1.6)-(4.1.7). Therefore, it follows that $V^{\chi}(y) \leq \tilde{V}^{\chi}(y)$ and $J^{\chi}(y) \leq \tilde{J}^{\chi}(y)$. Our objective is to derive and compare the corresponding optimal timing strategies under these two approaches.

4.2 Summary of Analytical Results

We first summarize our analytical results and illustrate the optimal starting and stopping strategies. The method of solutions and their proofs will be discussed in Section 4.3.

We consider the optimal starting-stopping problem followed by the optimal switching problem. First, we denote the infinitesimal generator of Y as

$$\mathcal{L}^{\chi} = \frac{\sigma^2 y}{2} \frac{d^2}{dy^2} + \mu(\theta - y)\frac{d}{dy},$$

and consider the ordinary differential equation (ODE)

$$\mathcal{L}^{\chi} u(y) = r u(y), \quad \text{for } y \in \mathbb{R}_+. \quad (4.2.1)$$

To present the solutions of this ODE, we define the functions

$$F^{\chi}(y) := M(\frac{r}{\mu}, \frac{2\mu\theta}{\sigma^2}; \frac{2\mu y}{\sigma^2}), \quad \text{and} \quad G^{\chi}(y) := U(\frac{r}{\mu}, \frac{2\mu\theta}{\sigma^2}; \frac{2\mu y}{\sigma^2}), \quad (4.2.2)$$

where

$$M(a,b;z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{b_n n!}, \qquad a_0 = 1,\ a_n = a(a+1)(a+2)\cdots(a+n-1),$$
$$U(a,b;z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b; z)$$

are the confluent hypergeometric functions of first and second kind, also called the Kummer's function and Tricomi's function, respectively (see Chapter 13 of Abramowitz and Stegun [1965] and Chapter 9 of Lebedev [1972]). As is well known (see Göing-Jaeschke and Yor [2003]), F^{χ} and G^{χ} are strictly positive and, respectively, the strictly increasing and decreasing continuously differentiable solutions of the ODE (4.2.1). Also, we remark that the discounted processes $(e^{-rt}F^{\chi}(Y_t))_{t\geq 0}$ and $(e^{-rt}G^{\chi}(Y_t))_{t\geq 0}$ are martingales.

In addition, recall the reward functions defined in (4.1.4) and note that

$$(\mathcal{L}^{\chi} - r)h_b(y) \begin{cases} > 0 & \text{if } y < y_b, \\ < 0 & \text{if } y > y_b, \end{cases} \tag{4.2.3}$$

and

$$(\mathcal{L}^{\chi} - r)h_s(y) \begin{cases} > 0 & \text{if } y < y_s, \\ < 0 & \text{if } y > y_s, \end{cases} \tag{4.2.4}$$

where the critical constants y_b and y_s are defined by

$$y_b := \frac{\mu\theta - rc_b}{\mu + r} \quad \text{and} \quad y_s := \frac{\mu\theta + rc_s}{\mu + r}. \tag{4.2.5}$$

Note that y_b and y_s depend on the parameters μ, θ and r, as well as c_b and c_s respectively, but not σ.

4.2.1 Optimal Starting-Stopping Problem

We now present the results for the optimal starting-stopping problem (4.1.2)-(4.1.3). As it turns out, the value function V^{χ} is expressed in terms of F^{χ}, and J^χ in terms of V^χ and G^χ. The functions F^χ and G^χ also play a role in determining the optimal starting and stopping thresholds.

First, we give a bound on the value function V^χ in terms of $F^\chi(y)$.

Lemma 4.2.1. *There exists a positive constant K^χ such that, for all $y \geq 0$, $0 \leq V^\chi(y) \leq K^\chi F^\chi(y)$.*

Theorem 4.2.2. *The value function for the optimal stopping problem* (4.1.2) *is given by*

$$V^\chi(y) = \begin{cases} \frac{b^{\chi*}-c_s}{F^\chi(b^{\chi*})} F^\chi(y) & \text{if } y \in [0, b^{\chi*}), \\ y - c_s & \text{if } y \in [b^{\chi*}, +\infty). \end{cases}$$

Here, the optimal stopping level $b^{\chi*} \in (c_s \vee y_s, \infty)$ is found from the equation

$$F^\chi(b) = (b - c_s) F^{\chi\prime}(b). \tag{4.2.6}$$

Therefore, it is optimal to stop as soon as the process Y reaches $b^{\chi*}$ from below. The stopping level $b^{\chi*}$ must also be higher than the fixed cost c_s as well as the critical level y_s defined in (4.2.5).

Now we turn to the optimal starting problem. Define the reward function

$$\hat{h}^\chi(y) := V^\chi(y) - (y + c_b). \tag{4.2.7}$$

Since F^χ, and thus V^χ, are convex, so is $\hat{h}^\chi$, we also observe that the reward function $\hat{h}^\chi(y)$ is decreasing in y. To exclude the scenario where it is optimal never to start, the condition stated in (4.1.5), namely, $\sup_{y \in \mathbb{R}+} \hat{h}^\chi(y) > 0$, is now equivalent to

$$V^\chi(0) = \frac{b^{\chi*} - c_s}{F^\chi(b^{\chi*})} > c_b, \tag{4.2.8}$$

since $F^\chi(0) = 1$.

Lemma 4.2.3. *For all $y \geq 0$, the value function satisfies the inequality $0 \leq J^\chi(y) \leq (\frac{b^{\chi*}-c}{F^\chi(b^{\chi*})} - c_b)^+$.*

Theorem 4.2.4. *The optimal starting problem* (4.1.3) *admits the solution*

$$J^{\chi}(y) = \begin{cases} V^{\chi}(y) - (y + c_b) & \text{if } y \in [0, d^{\chi *}], \\ \frac{V^{\chi}(d^{\chi *}) - (d^{\chi *} + c_b)}{G^{\chi}(d^{\chi *})} G^{\chi}(y) & \text{if } y \in (d^{\chi *}, +\infty). \end{cases}$$

The optimal starting level $d^{\chi *} > 0$ is uniquely determined from

$$G^{\chi}(d)(V^{\chi\prime}(d) - 1) = G^{\chi\prime}(d)(V^{\chi}(d) - (d + c_b)). \tag{4.2.9}$$

As a result, it is optimal to start as soon as the CIR process Y falls below the strictly positive level $d^{\chi *}$.

4.2.2 Optimal Switching Problem

Now we study the optimal switching problem under the CIR model in (4.1.1).

Lemma 4.2.5. *For all $y \geq 0$, the value functions $\tilde{J}^{\chi}$ and $\tilde{V}^{\chi}$ satisfy the inequalities*

$$0 \leq \tilde{J}^{\chi}(y) \leq \frac{\mu\theta}{r},$$
$$0 \leq \tilde{V}^{\chi}(y) \leq y + \frac{2\mu\theta}{r}.$$

We start by giving conditions under which it is optimal not to start ever.

Theorem 4.2.6. *Under the CIR model, if it holds that*

(i) $y_b \leq 0$, *or*

(ii) $y_b > 0$ *and* $c_b \geq \frac{b^{\chi *} - c_s}{F^{\chi}(b^{\chi *})}$,

with $b^{\chi *}$ given in (4.2.6), *then the optimal switching problem* (4.1.6)-(4.1.7) *admits the solution*

$$\tilde{J}^{\chi}(y) = 0 \qquad \text{for} \quad y \geq 0, \tag{4.2.10}$$

and

$$\tilde{V}^{\chi}(y) = \begin{cases} \frac{b^{\chi *}-c_s}{F^{\chi}(b^{\chi *})} F^{\chi}(y) & \textit{if } y \in [0, b^{\chi *}), \\ y - c_s & \textit{if } y \in [b^{\chi *}, +\infty). \end{cases} \tag{4.2.11}$$

Conditions (i) and (ii) depend on problem data and can be easily verified. In particular, recall that y_b is defined in (4.2.5) and is easy to compute, furthermore it is independent of σ and c_s. Since it is optimal to never enter, the switching problem is equivalent to a stopping problem and the solution in Theorem 4.2.6 agrees with that in Theorem 4.2.2. Next, we provide conditions under which it is optimal to enter as soon as the CIR process reaches some lower level.

Theorem 4.2.7. *Under the CIR model, if*

$$y_b > 0 \quad \textit{and} \quad c_b < \frac{b^{\chi *} - c_s}{F^{\chi}(b^{\chi *})}, \tag{4.2.12}$$

with $b^{\chi *}$ given in (4.2.6)*, then the optimal switching problem* (4.1.6)-(4.1.7) *admits the solution*

$$\tilde{J}^{\chi}(y) = \begin{cases} P^{\chi} F^{\chi}(y) - (y + c_b) & \textit{if } y \in [0, \tilde{d}^{\chi *}], \\ Q^{\chi} G^{\chi}(y) & \textit{if } y \in (\tilde{d}^{\chi *}, +\infty), \end{cases} \tag{4.2.13}$$

and

$$\tilde{V}^{\chi}(y) = \begin{cases} P^{\chi} F^{\chi}(y) & \textit{if } y \in [0, \tilde{b}^{\chi *}), \\ Q^{\chi} G^{\chi}(y) + (y - c_s) & \textit{if } y \in [\tilde{b}^{\chi *}, +\infty), \end{cases} \tag{4.2.14}$$

where

$$P^{\chi} = \frac{G^{\chi}(\tilde{d}^{\chi *}) - (\tilde{d}^{\chi *} + c_b) G^{\chi\prime}(\tilde{d}^{\chi *})}{F^{\chi\prime}(\tilde{d}^{\chi *}) G^{\chi}(\tilde{d}^{\chi *}) - F^{\chi}(\tilde{d}^{\chi *}) G^{\chi\prime}(\tilde{d}^{\chi *})},$$
$$Q^{\chi} = \frac{F^{\chi}(\tilde{d}^{\chi *}) - (\tilde{d}^{\chi *} + c_b) F^{\chi\prime}(\tilde{d}^{\chi *})}{F^{\chi\prime}(\tilde{d}^{\chi *}) G^{\chi}(\tilde{d}^{\chi *}) - F^{\chi}(\tilde{d}^{\chi *}) G^{\chi\prime}(\tilde{d}^{\chi *})}.$$

There exist unique optimal starting and stopping levels $\tilde{d}^{\chi*}$ *and* $\tilde{b}^{\chi*}$*, which are found from the nonlinear system of equations:*

$$\frac{G^\chi(d) - (d + c_b)G^{\chi\prime}(d)}{F^{\chi\prime}(d)G^\chi(d) - F^\chi(d)G^{\chi\prime}(d)} = \frac{G^\chi(b) - (b - c_s)G^{\chi\prime}(b)}{F^{\chi\prime}(b)G^\chi(b) - F^\chi(b)G^{\chi\prime}(b)},$$
$$\frac{F^\chi(d) - (d + c_b)F^{\chi\prime}(d)}{F^{\chi\prime}(d)G^\chi(d) - F^\chi(d)G^{\chi\prime}(d)} = \frac{F^\chi(b) - (b - c_s)F^{\chi\prime}(b)}{F^{\chi\prime}(b)G^\chi(b) - F^\chi(b)G^{\chi\prime}(b)}.$$

Moreover, we have that $\tilde{d}^{\chi*} < y_b$ *and* $\tilde{b}^{\chi*} > y_s$.

In this case, it is optimal to start and stop an infinite number of times where we start as soon as the CIR process drops to $\tilde{d}^{\chi*}$ and stop when the process reaches $\tilde{b}^{\chi*}$. Note that in the case of Theorem 4.2.6 where it is never optimal to start, the optimal stopping level $b^{\chi*}$ is the same as that of the optimal stopping problem in Theorem 4.2.2. The optimal starting level $\tilde{d}^{\chi*}$, which only arises when it is optimal to start and stop sequentially, is in general not the same as $d^{\chi*}$ in Theorem 4.2.4.

We conclude the section with two remarks.

Remark 4.2.8. *Given the model parameters, in order to identify which of Theorem 4.2.6 or Theorem 4.2.7 applies, we begin by checking whether* $y_b \leq 0$*. If so, it is optimal not to enter. Otherwise, Theorem 4.2.6 still applies if* $c_b \geq \frac{b^{\chi*} - c_s}{F^\chi(b^{\chi*})}$ *holds. In the other remaining case, the problem is solved as in Theorem 4.2.7. In fact, the condition* $c_b < \frac{b^{\chi*} - c_s}{F^\chi(b^{\chi*})}$ *implies* $y_b > 0$ *(see the proof of Lemma 4.3.3 in the Appendix). Therefore, condition* (4.2.12) *in Theorem 4.2.7 is in fact identical to* (4.2.8) *in Theorem 4.2.4.*

Remark 4.2.9. *To verify the optimality of the results in Theorems 4.2.6 and 4.2.7, one can show by direct substitution that the solutions* $(\tilde{J}^\chi, \tilde{V}^\chi)$ *in* (4.2.10)-(4.2.11) *and* (4.2.13)-(4.2.14) *satisfy the variational inequalities:*

$$\min\{r\tilde{J}^\chi(y) - \mathcal{L}^\chi \tilde{J}^\chi(y), \tilde{J}^\chi(y) - (\tilde{V}^\chi(y) - (y + c_b))\} = 0,$$
$$\min\{r\tilde{V}^\chi(y) - \mathcal{L}^\chi \tilde{V}^\chi(y), \tilde{V}^\chi(y) - (\tilde{J}^\chi(y) + (y - c_s))\} = 0.$$

Indeed, this is the approach used by Zervos et al. [2013] for checking the solutions of their optimal switching problems.

4.2.3 Numerical Examples

We numerically implement Theorems 4.2.2, 4.2.4, and 4.2.7, and illustrate the associated starting and stopping thresholds. In Figure 4.1 (left), we observe the changes in optimal starting and stopping levels as speed of mean reversion increases. Both starting levels $d^{\chi *}$ and $\tilde{d}^{\chi *}$ rise with μ, from 0.0964 to 0.1219 and from 0.1460 to 0.1696, respectively, as μ increases from 0.3 to 0.85. The optimal switching stopping level $\tilde{b}^{\chi *}$ also increases. On the other hand, stopping level $b^{\chi *}$ for the starting-stopping problem stays relatively constant as μ changes.

In Figure 4.1 (right), we see that as the stopping cost c_s increases, the increase in the optimal stopping levels is accompanied by a fall in optimal starting levels. In particular, the stopping levels, $b^{\chi *}$ and $\tilde{b}^{\chi *}$ increase. In comparison, both starting levels $d^{\chi *}$ and $\tilde{d}^{\chi *}$ fall. The lower starting level and higher stopping level mean that the entry and exit times are both delayed as a result of a higher transaction cost. Interestingly, although the cost c_s applies only when the process is stopped, it also has an impact on the timing to *start*, as seen in the changes in $d^{\chi *}$ and $\tilde{d}^{\chi *}$ in the figure.

In Figure 4.1, we can see that the continuation (waiting) region of the switching problem $(\tilde{d}^{\chi *}, \tilde{b}^{\chi *})$ lies within that of the starting-stopping problem $(d^{\chi *}, b^{\chi *})$. The ability to enter and exit multiple times means it is possible to earn a smaller reward on each individual start-stop sequence while maximizing aggregate return. Moreover, we observe that optimal entry and exit levels of the starting-stopping problem is less sensitive to changes in model parameters than the entry and exit thresholds of the switching problem.

Figure 4.2 shows a simulated CIR path along with optimal entry and exit levels for both starting-stopping and switching problems. Under the starting-stopping problem, it is optimal to start once the process reaches $d^{\chi*} = 0.0373$ and to stop when the process hits $b^{\chi*} = 0.4316$. For the switching problem, it is optimal to start once the process values hits $\tilde{d}^{\chi*} = 0.1189$ and to stop when the value of the CIR process rises to $\tilde{b}^{\chi*} = 0.2078$. We note that both stopping levels $b^{\chi*}$ and $\tilde{b}^{\chi*}$ are higher than the long-run mean $\theta = 0.2$, and the starting levels $d^{\chi*}$ and $\tilde{d}^{\chi*}$ are lower than θ. The process starts at $Y_0 = 0.15 > \tilde{d}^{\chi*}$, under the optimal switching setting, the first time to enter occurs on day 8 when the process falls to 0.1172 and subsequently exits on day 935 at a level of 0.2105. For the starting-stopping problem, entry takes place much later on day 200 when the process hits 0.0306 and exits on day 2671 at 0.4369. Under the optimal switching problem, two entries and two exits will be completed by the time a single entry-exit sequence is realized for the starting-stopping problem.

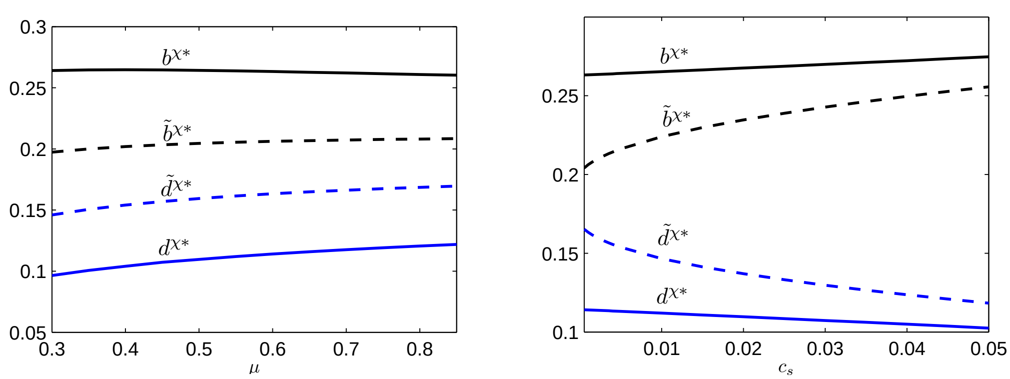


Figure 4.1: (Left) The optimal starting and stopping levels vs speed of mean reversion μ. Parameters: $\sigma = 0.15$, $\theta = 0.2$, $r = 0.05$, $c_s = 0.001$, $c_b = 0.001$. (Right) The optimal starting and stopping levels vs transaction cost c_s. Parameters: $\mu = 0.6$, $\sigma = 0.15$, $\theta = 0.2$, $r = 0.05$, $c_b = 0.001$.

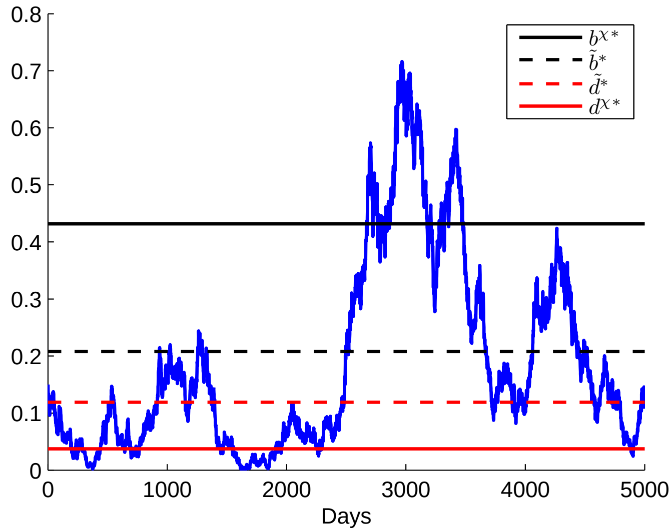


Figure 4.2: A sample CIR path, along with starting and stopping levels. Under the starting-stopping setting, a starting decision is made at $\nu_{d^{\chi*}} = \inf\{t \geq 0 : Y_t \leq d^{\chi*} = 0.0373\}$, and a stopping decision is made at $\tau_{b^{\chi*}} = \inf\{t \geq \nu_{d^{\chi*}} : Y_t \geq b^{\chi*} = 0.4316\}$. Under the optimal switching problem, entry and exit take place at $\nu_{\tilde{d}^{\chi*}} = \inf\{t \geq 0 : Y_t \leq \tilde{d}^{\chi*} = 0.1189\}$, and $\tau_{\tilde{b}^{\chi*}} = \inf\{t \geq \nu_{\tilde{d}^{\chi*}} : Y_t \geq \tilde{b}^{\chi*} = 0.2078\}$ respectively. Parameters: $\mu = 0.2$, $\sigma = 0.3$, $\theta = 0.2$, $r = 0.05$, $c_s = 0.001$, $c_b = 0.001$.

4.3 Methods of Solution and Proofs

We now provide detailed proofs for our analytical results in Section 4.2 beginning with the optimal starting-stopping problem. Our main result here is Theorem 4.3.1 which provides a mathematical characterization of the value function, and establishes the optimality of our method of constructing the solution.

4.3.1 Optimal Starting-Stopping Problem

We first describe the general solution procedure for the stopping problem V^{χ}, followed by the starting problem J^{χ}.

4.3.1.1 Optimal Stopping Timing

A key step of our solution method involves the transformation

$$\phi(y) := -\frac{G^\chi(y)}{F^\chi(y)}, \quad y \geq 0. \tag{4.3.1}$$

With this, we also define the function

$$H^\chi(z) := \begin{cases} \frac{h_s}{F^\chi} \circ \phi^{-1}(z) & \text{if } z < 0, \\ \lim\limits_{y \to +\infty} \frac{(h_s(y))^+}{F^\chi(y)} & \text{if } z = 0, \end{cases} \tag{4.3.2}$$

where h_s is given in (4.1.4). We now prove the analytical form for the value function.

Theorem 4.3.1. *Under the CIR model, the value function V^χ of* (4.1.2) *is given by*

$$V^\chi(y) = F^\chi(y) W^\chi(\phi(y)), \tag{4.3.3}$$

with F^χ and ϕ given in (4.2.2) *and* (4.3.1) *respectively, and W^χ is the decreasing smallest concave majorant of H^χ in* (4.3.2).

Proof. We first show that $V^\chi(y) \geq F^\chi(y)W^\chi(\phi(y))$. Start at any $y \in [0, +\infty)$, we consider the first stopping time of Y from an interval $[a, b]$ with $0 \leq a \leq y \leq b \leq +\infty$. We compute the corresponding expected discounted reward

$$\begin{aligned}
&\mathbb{E}_y\{e^{-r(\tau_a \wedge \tau_b)} h_s(Y_{\tau_a \wedge \tau_b})\} \\
&= h_s(a)\mathbb{E}_y\{e^{-r\tau_a} 1_{\{\tau_a < \tau_b\}}\} + h_s(b)\mathbb{E}_y\{e^{-r\tau_b} 1_{\{\tau_a > \tau_b\}}\} \\
&= h_s(a)\frac{F^\chi(y)G^\chi(b) - F^\chi(b)G^\chi(y)}{F^\chi(a)G^\chi(b) - F^\chi(b)G^\chi(a)} + h_s(b)\frac{F^\chi(a)G^\chi(y) - F^\chi(y)G^\chi(a)}{F^\chi(a)G^\chi(b) - F^\chi(b)G^\chi(a)} \\
&= F^\chi(y)\left[\frac{h_s(a)}{F^\chi(a)}\frac{\phi(b) - \phi(y)}{\phi(b) - \phi(a)} + \frac{h_s(b)}{F^\chi(b)}\frac{\phi(y) - \phi(a)}{\phi(b) - \phi(a)}\right] \\
&= F^\chi(\phi^{-1}(z))\left[H^\chi(z_a)\frac{z_b - z}{z_b - z_a} + H^\chi(z_b)\frac{z - z_a}{z_b - z_a}\right],
\end{aligned}$$

where $z_a = \phi(a)$, $z_b = \phi(b)$. Since $V^\chi(y) \geq \sup_{\{a,b:a\leq y\leq b\}} \mathbb{E}_y\{e^{-r(\tau_a\wedge\tau_b)}h_s(Y_{\tau_a\wedge\tau_b})\}$, we have

$$\frac{V^\chi(\phi^{-1}(z))}{F^\chi(\phi^{-1}(z))} \geq \sup_{\{z_a,z_b:z_a\leq z\leq z_b\}} \left[H^\chi(z_a)\frac{z_b - z}{z_b - z_a} + H^\chi(z_b)\frac{z - z_a}{z_b - z_a}\right], \quad (4.3.4)$$

which implies that $V^\chi(\phi^{-1}(z))/F^\chi(\phi^{-1}(z))$ dominates the concave majorant of H^χ.

Under the CIR model, the class of interval-type strategies does not include all single threshold-type strategies. In particular, the minimum value that a can take is 0. If $2\mu\theta < \sigma^2$, then Y can reach level 0 and reflects. The interval-type strategy with $a = 0$ implies stopping the process Y at level 0, even though it could be optimal to wait and let Y evolve.

Hence, we must also consider separately the candidate strategy of waiting for Y to reach an upper level $b \geq y$ without a lower stopping level. The well-known supermartingale property of $(e^{-rt}V^\chi(Y_t))_{t\geq 0}$ (see Appendix D of Karatzas and Shreve [1998]) implies that $V^\chi(y) \geq \mathbb{E}_y\{e^{-r\tau}V^\chi(Y_\tau)\}$ for $\tau \in \mathcal{T}$. Then, taking $\tau = \tau_b$, we have

$$V^\chi(y) \geq \mathbb{E}_y\{e^{-r\tau_b}V^\chi(Y_{\tau_b})\} = V^\chi(b)\frac{F^\chi(y)}{F^\chi(b)},$$

or equivalently,

$$\frac{V^\chi(\phi^{-1}(z))}{F^\chi(\phi^{-1}(z))} = \frac{V^\chi(y)}{F^\chi(y)} \geq \frac{V^\chi(b)}{F^\chi(b)} = \frac{V^\chi(\phi^{-1}(z_b))}{F^\chi(\phi^{-1}(z_b))}, \quad (4.3.5)$$

which indicates that $V^\chi(\phi^{-1}(z))/F^\chi(\phi^{-1}(z))$ is *decreasing*. By (4.3.4) and (4.3.5), we now see that $V^\chi(y) \geq F^\chi(y)W^\chi(\phi(y))$, where W^χ is the *decreasing* smallest concave majorant of H^χ.

For the reverse inequality, we first show that

$$F^\chi(y)W^\chi(\phi(y)) \geq \mathbb{E}_y\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})W^\chi(\phi(Y_{t\wedge\tau}))\}, \quad (4.3.6)$$

for $y \in [0, +\infty)$, $\tau \in \mathcal{T}$ and $t \geq 0$. If the initial value $y = 0$, then the decreasing property of W^χ implies the inequality

$$\begin{aligned}\mathbb{E}_0\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})W^\chi(\phi(Y_{t\wedge\tau}))\} &\leq \mathbb{E}_0\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})\}W^\chi(\phi(0))\\ &= F^\chi(0)W^\chi(\phi(0)),\end{aligned}$$

where the equality follows from the martingale property of $(e^{-rt}F^\chi(Y_t))_{t\geq 0}$.

When $y > 0$, the concavity of W^χ implies that, for any fixed z, there exists an affine function $L^\chi_z(\alpha) := m^\chi_z\alpha + c^\chi_z$ such that $L^\chi_z(\alpha) \geq W^\chi(\alpha)$ for $\alpha \geq \phi(0)$ and $L^\chi_z(z) = W^\chi(z)$ at $\alpha = z$, with constants m^χ_z and c^χ_z. In turn, this yields the inequality

$$\begin{aligned}&\mathbb{E}_y\{e^{-r(\tau_0\wedge t\wedge\tau)}F^\chi(Y_{\tau_0\wedge t\wedge\tau})W^\chi(\phi(Y_{\tau_0\wedge t\wedge\tau}))\} &(4.3.7)\\ &\quad\leq \mathbb{E}_y\{e^{-r(\tau_0\wedge t\wedge\tau)}F^\chi(Y_{\tau_0\wedge t\wedge\tau})L^\chi_{\phi(y)}(\phi(Y_{\tau_0\wedge t\wedge\tau}))\}\\ &\quad= m^\chi_{\phi(y)}\mathbb{E}_y\{e^{-r(\tau_0\wedge t\wedge\tau)}F^\chi(Y_{\tau_0\wedge t\wedge\tau})\phi(Y_{\tau_0\wedge t\wedge\tau})\} + c^\chi_{\phi(y)}\mathbb{E}_y\{e^{-r(\tau_0\wedge t\wedge\tau)}F^\chi(Y_{\tau_0\wedge t\wedge\tau})\}\\ &\quad= -m^\chi_{\phi(y)}\mathbb{E}_y\{e^{-r(\tau_0\wedge t\wedge\tau)}G^\chi(Y_{\tau_0\wedge t\wedge\tau})\} + c^\chi_{\phi(y)}\mathbb{E}_y\{e^{-r(\tau_0\wedge t\wedge\tau)}F^\chi(Y_{\tau_0\wedge t\wedge\tau})\}\\ &\quad= -m^\chi_{\phi(y)}G^\chi(y) + c^\chi_{\phi(y)}F^\chi(y) &(4.3.8)\\ &\quad= F^\chi(y)L^\chi_{\phi(y)}(\phi(y))\\ &\quad= F^\chi(y)W^\chi(\phi(y)), &(4.3.9)\end{aligned}$$

where (4.3.8) follows from the martingale property of $(e^{-rt}F^\chi(Y_t))_{t\geq 0}$ and $(e^{-rt}G^\chi(Y_t))_{t\geq 0}$. If $2\mu\theta \geq \sigma^2$, then $\tau_0 = +\infty$ for $y > 0$. This immediately reduces (4.3.7)-(4.3.9) to the desired inequality (4.3.6).

On the other hand, if $2\mu\theta < \sigma^2$, then we decompose (4.3.7) into two terms:

$$\begin{aligned}&\mathbb{E}_y\{e^{-r(\tau_0\wedge t\wedge\tau)}F^\chi(Y_{\tau_0\wedge t\wedge\tau})W^\chi(\phi(Y_{\tau_0\wedge t\wedge\tau}))\}\\ &= \underbrace{\mathbb{E}_y\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})W^\chi(\phi(Y_{t\wedge\tau}))1_{\{t\wedge\tau\leq\tau_0\}}\}}_{\text{(I)}}\\ &+ \underbrace{\mathbb{E}_y\{e^{-r\tau_0}F^\chi(Y_{\tau_0})W^\chi(\phi(Y_{\tau_0}))1_{\{t\wedge\tau>\tau_0\}}\}}_{\text{(II)}}.\end{aligned}$$

By the optional sampling theorem and decreasing property of W^χ, the second term satisfies

$$\begin{aligned}(\text{II}) &= W^\chi(\phi(0))\mathbb{E}_y\{e^{-r\tau_0}F^\chi(Y_{\tau_0})1_{\{t\wedge\tau>\tau_0\}}\}\\ &\geq W^\chi(\phi(0))\mathbb{E}_y\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})1_{\{t\wedge\tau>\tau_0\}}\}\\ &\geq \mathbb{E}_y\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})W^\chi(\phi(Y_{t\wedge\tau}))1_{\{t\wedge\tau>\tau_0\}}\} =: (\text{II'}). \quad (4.3.10)\end{aligned}$$

Combining (4.3.10) with (4.3.9), we arrive at

$$F^\chi(y)W^\chi(\phi(y)) \geq (\text{I}) + (\text{II'}) = \mathbb{E}_y\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})W^\chi(\phi(Y_{t\wedge\tau}))\},$$

for all $y > 0$. In all, inequality (4.3.6) holds for all $y \in [0, +\infty)$, $\tau \in \mathcal{T}$ and $t \geq 0$. From (4.3.6) and the fact that W^χ majorizes H^χ, it follows that

$$\begin{aligned}F^\chi(y)W^\chi(\phi(y)) &\geq \mathbb{E}_y\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})W^\chi(\phi(Y_{t\wedge\tau}))\}\\ &\geq \mathbb{E}_y\{e^{-r(t\wedge\tau)}F^\chi(Y_{t\wedge\tau})H^\chi(\phi(Y_{t\wedge\tau}))\}\\ &\geq \mathbb{E}_y\{e^{-r(t\wedge\tau)}h_s(Y_{t\wedge\tau})\}. \quad (4.3.11)\end{aligned}$$

Maximizing (4.3.11) over all $\tau \in \mathcal{T}$ and $t \geq 0$ yields the reverse inequality $F^\chi(y)W^\chi(\phi(y)) \geq V^\chi(y)$. □

In summary, we have found an expression for the value function $V^\chi(y)$ in (4.3.3), and proved that it is sufficient to consider only candidate stopping times described by the first time Y reaches a single upper threshold or exits an interval. To determine the optimal timing strategy, we need to understand the properties of H^χ and its smallest concave majorant W^χ. To this end, we have the following lemma.

Lemma 4.3.2. *The function H^χ is continuous on $[\phi(0), 0]$, twice differentiable on $(\phi(0), 0)$ and possesses the following properties:*

(i) $H^\chi(0) = 0$, *and*

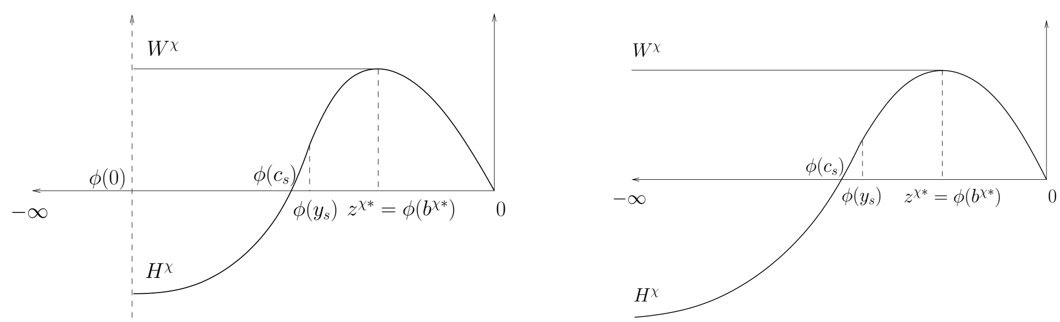
$$H^\chi(z)\begin{cases} < 0 & \text{if } z \in [\phi(0), \phi(c_s)),\\ > 0 & \text{if } z \in (\phi(c_s), 0).\end{cases} \quad (4.3.12)$$

(ii) $H^{\chi}(z)$ *is strictly increasing for* $z \in (\phi(0), \phi(c_s) \vee \phi(y_s))$.

(iii)

$$H^{\chi}(z) \text{ is } \begin{cases} \textit{convex} & \textit{if } z \in (\phi(0), \phi(y_s)], \\ \textit{concave} & \textit{if } z \in [\phi(y_s), 0). \end{cases}$$

In Figure 4.3, we see that H^{χ} is first increasing then decreasing, and first convex then concave. Using these properties, we now derive the optimal stopping timing.



(a) $2\mu\theta < \sigma^2$ (b) $2\mu\theta \geq \sigma^2$

Figure 4.3: Sketches of H^{χ} and W^{χ}. The function W^{χ} is equal to the constant $H^{\chi}(z^{\chi*})$ on $(\phi(0), z^{\chi*})$, and coincides with H^{χ} on $[z^{\chi*}, 0]$. Note that $-\infty < \phi(0) < 0$ if $2\mu\theta < \sigma^2$, and $\phi(0) = -\infty$ if $2\mu\theta \geq \sigma^2$.

Proof of Theorem 4.2.2 We determine the value function in the form: $V^{\chi}(y) = F^{\chi}(y)W^{\chi}(\phi(y))$, where W^{χ} is the decreasing smallest concave majorant of H^{χ}. By Lemma 4.3.2 and Figure 4.3, H^{χ} peaks at $z^{\chi*} > \phi(c_s) \vee \phi(y_s)$ so that

$$H^{\chi\prime}(z^{\chi*}) = 0. \tag{4.3.13}$$

In turn, the decreasing smallest concave majorant admits the form:

$$W^{\chi}(z) = \begin{cases} H^{\chi}(z^{\chi*}) & \text{if } z < z^{\chi*}, \\ H^{\chi}(z) & \text{if } z \geq z^{\chi*}. \end{cases} \tag{4.3.14}$$

Substituting $b^{\chi*} = \phi^{-1}(z^{\chi*})$ into (4.3.13), we have

$$\begin{aligned} H^{\chi\prime}(z^{\chi*}) &= \frac{F^{\chi}(\phi^{-1}(z^{\chi*})) - (\phi^{-1}(z^{\chi*}) - c_s)F^{\chi\prime}(\phi^{-1}(z^{\chi*}))}{F^{\chi\prime}(\phi^{-1}(z^{\chi*}))G^{\chi}(\phi^{-1}(z^{\chi*})) - F^{\chi}(\phi^{-1}(z^{\chi*}))G^{\chi\prime}(\phi^{-1}(z^{\chi*}))} \\ &= \frac{F^{\chi}(b^{\chi*}) - (b^{\chi*} - c_s)F^{\chi\prime}(b^{\chi*})}{F^{\chi\prime}(b^{\chi*})G^{\chi}(b^{\chi*}) - F^{\chi}(b^{\chi*})G^{\chi\prime}(b^{\chi*})}, \end{aligned}$$

which can be further simplified to (4.2.6). We can express $H^{\chi}(z^{\chi*})$ in terms of $b^{\chi*}$:

$$H^{\chi}(z^{\chi*}) = \frac{b^{\chi*} - c_s}{F^{\chi}(b^{\chi*})}. \tag{4.3.15}$$

Applying (4.3.15) to (4.3.14), we get

$$W^{\chi}(\phi(y)) = \begin{cases} H^{\chi}(z^{\chi*}) = \frac{b^{\chi*}-c_s}{F^{\chi}(b^{\chi*})} & \text{if } y < b^{\chi*}, \\ H^{\chi}(\phi(y)) = \frac{y-c_s}{F^{\chi}(y)} & \text{if } y \geq b^{\chi*}. \end{cases}$$

Finally, substituting this into the value function $V^{\chi}(y) = F^{\chi}(y)W^{\chi}(\phi(y))$, we conclude.

4.3.1.2 Optimal Starting Timing

We now turn to the optimal starting problem. Our methodology in Section 4.3.1.1 applies to general payoff functions, and thus can be applied to the optimal starting problem (4.1.3) as well. To this end, we apply the same transformation (4.3.1) and define the function

$$\hat{H}^{\chi}(z) := \begin{cases} \frac{\hat{h}^{\chi}}{F^{\chi}} \circ \phi^{-1}(z) & \text{if } z < 0, \\ \lim\limits_{y\to+\infty} \frac{(\hat{h}^{\chi}(y))^+}{F^{\chi}(y)} & \text{if } z = 0, \end{cases}$$

where $\hat{h}^{\chi}$ is given in (4.2.7). We then follow Theorem 4.2.2 to determine the value function J^{χ}. This amounts to finding the *decreasing* smallest concave majorant $\hat{W}^{\chi}$ of $\hat{H}^{\chi}$. Indeed, we can replace H^{χ} and W^{χ} with $\hat{H}^{\chi}$ and $\hat{W}^{\chi}$ in Theorem 4.2.2 and its proof. As a result, the value function of the optimal starting timing problem must take the form

$$J^{\chi}(y) = F^{\chi}(y)\hat{W}^{\chi}(\phi(y)).$$

To solve the optimal starting timing problem, we need to understand the properties of $\hat{H}^{\chi}$.

Lemma 4.3.3. *The function $\hat{H}^{\chi}$ is continuous on $[\phi(0), 0]$, differentiable on $(\phi(0), 0)$, and twice differentiable on $(\phi(0), \phi(b^{\chi*})) \cup (\phi(b^{\chi*}), 0)$, and possesses the following properties:*

(i) $\hat{H}^{\chi}(0) = 0$. *Let $\bar{d}^{\chi}$ denote the unique solution to $\hat{h}^{\chi}(y) = 0$, then $\bar{d}^{\chi} < b^{\chi*}$ and*

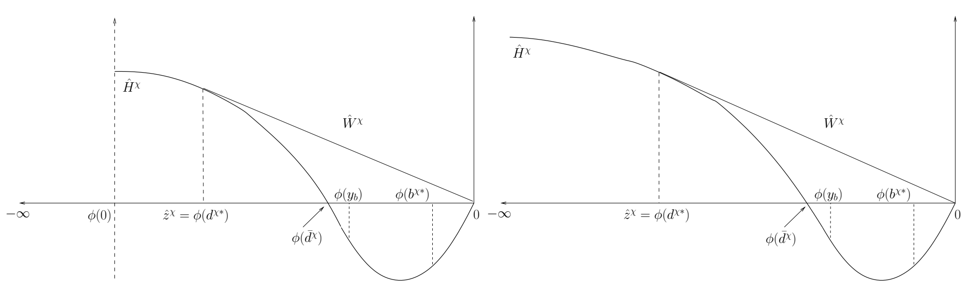
$$\hat{H}^{\chi}(z) \begin{cases} > 0 & \text{if } z \in [\phi(0), \phi(\bar{d}^{\chi})), \\ < 0 & \text{if } z \in (\phi(\bar{d}^{\chi}), 0). \end{cases}$$

(ii) $\hat{H}^{\chi}(z)$ *is strictly increasing for $z > \phi(b^{\chi*})$ and* $\lim_{z \to \phi(0)} \hat{H}^{\chi\prime}(z) = 0$.

(iii)

$$\hat{H}^{\chi}(z) \text{ is } \begin{cases} \text{concave} & \text{if } z \in (\phi(0), \phi(y_b)), \\ \text{convex} & \text{if } z \in (\phi(y_b), 0). \end{cases}$$

By Lemma 4.3.3, we sketch $\hat{H}^{\chi}$ in Figure 4.4.



(a) $2\mu\theta < \sigma^2$ (b) $2\mu\theta \geq \sigma^2$

Figure 4.4: Sketches of $\hat{H}^{\chi}$ and $\hat{W}^{\chi}$. The function $\hat{W}^{\chi}$ coincides with $\hat{H}^{\chi}$ on $[\phi(0), \hat{z}^{\chi}]$ and is a straight line tangent to $\hat{H}^{\chi}$ at $\hat{z}^{\chi}$ on $(\hat{z}^{\chi}, 0]$. Note that $-\infty < \phi(0) < 0$ if $2\mu\theta < \sigma^2$, and $\phi(0) = -\infty$ if $2\mu\theta \geq \sigma^2$.

Proof of Theorem 4.2.4 To determine the value function in the form: $J^\chi(y) = F^\chi(y)\hat{W}^\chi(\phi(y))$, we analyze the decreasing smallest concave majorant, $\hat{W}^\chi$, of $\hat{H}^\chi$. By Lemma 4.3.3 and Figure 4.3, we have $\hat{H}^{\chi\prime}(z) \to 0$ as $z \to \phi(0)$. Therefore, there exists a unique number $\hat{z}^\chi \in (\phi(0), \phi(b^{\chi*}))$ such that

$$\frac{\hat{H}^\chi(\hat{z}^\chi)}{\hat{z}^\chi} = \hat{H}^{\chi\prime}(\hat{z}^\chi). \tag{4.3.16}$$

In turn, the decreasing smallest concave majorant admits the form:

$$\hat{W}^\chi(z) = \begin{cases} \hat{H}^\chi(z) & \text{if } z \le \hat{z}^\chi, \\ z\frac{\hat{H}^\chi(\hat{z}^\chi)}{\hat{z}^\chi} & \text{if } z > \hat{z}^\chi. \end{cases} \tag{4.3.17}$$

Substituting $d^{\chi*} = \phi^{-1}(\hat{z}^\chi)$ into (4.3.16), we have

$$\frac{\hat{H}^\chi(\hat{z}^\chi)}{\hat{z}^\chi} = \frac{\hat{H}^\chi(\phi(d^{\chi*})}{\phi(d^{\chi*})} = -\frac{V^\chi(d^{\chi*}) - d^{\chi*} - c_b}{G^\chi(d^{\chi*})}, \tag{4.3.18}$$

and

$$\hat{H}^{\chi\prime}(\hat{z}^\chi) = \frac{F^\chi(d^{\chi*})(V^{\chi\prime}(d^{\chi*}) - 1) - F^{\chi\prime}(d^{\chi*})(V^\chi(d^{\chi*}) - (d^{\chi*} + c_b))}{F^{\chi\prime}(d^{\chi*})G^\chi(d^{\chi*}) - F^\chi(d^{\chi*})G^{\chi\prime}(d^{\chi*})}.$$

Equivalently, we can express condition (4.3.16) in terms of $d^{\chi*}$:

$$\begin{aligned} &-\frac{V^\chi(d^{\chi*}) - (d^{\chi*} + c_b)}{G^\chi(d^{\chi*})} \\ &= \frac{F^\chi(d^{\chi*})(V^{\chi\prime}(d^{\chi*}) - 1) - F^{\chi\prime}(d^{\chi*})(V^\chi(d^{\chi*}) - (d^{\chi*} + c_b))}{F^{\chi\prime}(d^{\chi*})G^\chi(d^{\chi*}) - F^\chi(d^{\chi*})G^{\chi\prime}(d^{\chi*})}, \end{aligned}$$

which shows $d^{\chi*}$ satisfies (4.2.9) after simplification.

Applying (4.3.18) to (4.3.17), we get

$$W^\chi(\phi(y)) = \begin{cases} \hat{H}^\chi(\phi(y)) = \frac{V^\chi(y) - (y + c_b)}{F^\chi(y)} & \text{if } y \in [0, d^{\chi*}], \\ \phi(y)\frac{\hat{H}^\chi(\hat{z}^\chi)}{\hat{z}^\chi} = \frac{V^\chi(d^{\chi*}) - (d^{\chi*} + c_b)}{G^\chi(d^{\chi*})}\frac{G^\chi(y)}{F^\chi(y)} & \text{if } y \in (d^{\chi*}, +\infty). \end{cases}$$

From this, we obtain the value function.

4.3.2 Optimal Switching Problem

Proofs of Theorems 4.2.6 and 4.2.7 Zervos et al. [2013] have studied a similar problem of trading a mean-reverting asset with fixed transaction costs, and provided detailed proofs using a variational inequalities approach. In particular, we observe that y_b and y_s in (4.2.3) and (4.2.4) play the same roles as x_b and x_s in Assumption 4 in Zervos et al. [2013], respectively. However, Assumption 4 in Zervos et al. [2013] requires that $0 \leq x_b$, this is not necessarily true for y_b in our problem. We have checked and realized that this assumption is not necessary for Theorem 4.2.6, and that $y_b < 0$ simply implies that there is no optimal starting level, i.e. it is never optimal to start.

In addition, Zervos et al. [2013] assume (in their Assumption 1) that the hitting time of level 0 is infinite with probability 1. In comparison, we consider not only the CIR case where 0 is inaccessible, but also when the CIR process has a reflecting boundary at 0. In fact, we find that the proofs in Zervos et al. [2013] apply to both cases under the CIR model. Therefore, apart from relaxation of the aforementioned assumptions, the proofs of our Theorems 4.2.6 and 4.2.7 are the same as that of Lemmas 1 and 2 in Zervos et al. [2013] respectively.

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Appendix A

Appendix for GBM Example

Let S be a geometric Brownian motion with drift and volatility parameters (μ, σ). In this case, the optimal exit problem is trivial. Indeed, if $\mu > r$, then

$$\begin{aligned}\mathcal{V}(s) &:= \sup_{\tau \in \mathcal{T}} \mathbb{E}_s\{e^{-r\tau}(S_\tau - c_s)\} \\ &\geq \sup_{t \geq 0} \left(\mathbb{E}_s\{e^{-rt}S_t\} - e^{-rt}c_s\right) \geq \sup_{t\geq 0} se^{(\mu-r)t} - c_s = +\infty.\end{aligned}$$

Therefore, it is optimal to take $\tau = +\infty$ and the value function is infinite.

If $\mu = r$, then the value function is given by

$$\begin{aligned}\mathcal{V}(s) &= \sup_{t\geq 0}\sup_{\tau\in\mathcal{T}} \mathbb{E}_s\{e^{-r(\tau\wedge t)}(S_{\tau\wedge t} - c_s)\} \\ &= s - c_s \inf_{t\geq 0}\inf_{\tau\in\mathcal{T}} \mathbb{E}_s\{e^{-r(\tau\wedge t)}\} = s, \qquad \text{(A.0.1)}\end{aligned}$$

where the second equality follows from the optional sampling theorem and that $(e^{-rt}S_t)_{t\geq 0}$ is a martingale. Again, the optimal value is achieved by choosing $\tau = +\infty$, but $\mathcal{V}(s)$ is finite in (A.0.1).

On the other hand, if $\mu < r$, then we have a non-trivial solution and exit timing:

$$\mathcal{V}(s) = \begin{cases} \left(\frac{c_s}{\eta-1}\right)^{1-\eta}\left(\frac{s}{\eta}\right)^{\eta} & \text{if } x < s^*, \\ s - c_s & \text{if } x \geq s^*, \end{cases}$$

where

$$\eta = \frac{\sqrt{2r\sigma^2 + (\mu - \frac{1}{2}\sigma^2)^2} - (\mu - \frac{1}{2}\sigma^2)}{\sigma^2} \quad \text{and} \quad s^* = \frac{c_s \eta}{\eta - 1} > c_s.$$

Therefore, it is optimal to liquidate as soon as S reaches level s^*. However, it is optimal *not* to enter because $\sup_{s \in \mathbb{R}_+}(\mathcal{V}(s) - s - c_b) \leq 0$, giving a zero value for the entry timing problem. Guo and Zervos [2010] provide a detailed study on this problem and its variation in the context of π options.

Appendix B

Appendix for Chapter 2

B.1 Proof of Lemma 2.3.1 (Bounds of V)

First, observe that $F(-\infty) = G(+\infty) = 0$ and $F(+\infty) = G(-\infty) = +\infty$. The limit

$$\limsup_{x\to+\infty} \frac{(h(x))^+}{F(x)} = \limsup_{x\to+\infty} \frac{x - c_s}{F(x)} = \limsup_{x\to+\infty} \frac{1}{F'(x)} = 0.$$

Therefore, there exists some x_0 such that $(h(x))^+ < F(x)$ for $x \in (x_0, +\infty)$. As for $x \leq x_0$, $(h(x))^+$ is bounded above by the constant $(x_0 - c_s)^+$. As a result, we can always find a constant K such that $(h(x))^+ \leq KF(x)$ for all $x \in \mathbb{R}$.

By definition, the process $(e^{-rt}F(X_t))_{t\geq 0}$ is a martingale. This implies, for every $x \in \mathbb{R}$ and $\tau \in \mathcal{T}$,

$$KF(x) = \mathbb{E}_x\{e^{-r\tau}KF(X_\tau)\} \geq \mathbb{E}_x\{e^{-r\tau}(h(X_\tau))^+\} \geq \mathbb{E}_x\{e^{-r\tau}h(X_\tau)\}.$$

Therefore, $V(x) \leq KF(x)$. Lastly, the choice of $\tau = +\infty$ as a candidate stopping time implies that $V(x) \geq 0$.

B.2 Proof of Lemma 2.3.2 (Properties of H)

The continuity and twice differentiability of H on $(0,+\infty)$ follow directly from those of h, G and ψ. To show the continuity of H at 0, since $H(0) = \lim_{x\to-\infty}\frac{(x-c_s)^+}{G(x)} = 0$, we only need to show that $\lim_{y\to 0} H(z) = 0$. Note that $z = \psi(x) \to 0$, as $x \to -\infty$. Therefore,

$$\lim_{z\to 0} H(z) = \lim_{x\to-\infty}\frac{h(x)}{G(x)} = \lim_{x\to-\infty}\frac{x-c_s}{G(x)} = \lim_{x\to-\infty}\frac{1}{G'(x)} = 0.$$

We conclude that H is also continuous at 0.

(i) One can show that $\psi(x) \in (0,+\infty)$ for $x \in \mathbb{R}$ and is a strictly increasing function. Then property (i) follows directly from the fact that $G(x) > 0$.

(ii) By the definition of H,

$$H'(z) = \frac{1}{\psi'(x)}(\frac{h}{G})'(x) = \frac{h'(x)G(x) - h(x)G'(x)}{\psi'(x)G^2(x)}, \quad z = \psi(x).$$

Since both $\psi'(x)$ and $G^2(x)$ are positive, we only need to determine the sign of $h'(x)G(x) - h(x)G'(x) = G(x) - (x-c_s)G'(x)$.

Define $u(x) := (x-c_s) - \frac{G(x)}{G'(x)}$. Note that $u(x) + c_s$ is the intersecting point at x axis of the tangent line of $G(x)$, and $u'(x) = \frac{G(x)G''(x)}{(G'(x))^2}$. Since $G(\cdot)$ is a positive, strictly decreasing and convex function, $u(x)$ is strictly increasing and $u(x) < 0$ as $x \to -\infty$. Also, note that

$$u(c_s) = -\frac{G(c_s)}{G'(c_s)} > 0,$$
$$u(L^*) = (L^* - c_s) - \frac{G(x)}{G'(x)} = \frac{\mu}{r}(\theta - L^*) - \frac{G(L^*)}{G'(L^*)} = -\frac{\sigma^2}{2r}\frac{G''(L^*)}{G'(L^*)} > 0.$$

Therefore, there exists a unique root x^* that solves $u(x) = 0$, and $x^* < c_s \wedge L^*$, such that

$$G(x) - (x-c_s)G'(x) \begin{cases} < 0 & \text{if } x \in (-\infty, x^*), \\ > 0 & \text{if } x \in (x^*, +\infty). \end{cases}$$

Thus $H(z)$ is strictly decreasing if $z \in (0, \psi(x^*))$, and increasing otherwise.
(iii) By the definition of H,

$$H''(z) = \frac{2}{\sigma^2 G(x)(\psi'(x))^2}(\mathcal{L} - r)h(x), \quad z = \psi(x).$$

Since $\sigma^2, G(x)$ and $(\psi'(x))^2$ are all positive, we only need to determine the sign of $(\mathcal{L} - r)h(x)$:

$$\begin{aligned}(\mathcal{L} - r)h(x) &= \mu(\theta - x) - r(x - c_s) \\ &= (\mu\theta + rc_s) - (\mu + r)x \begin{cases} \geq 0 & \text{if } x \in (-\infty, L^*], \\ \leq 0 & \text{if } x \in [L^*, +\infty). \end{cases}\end{aligned}$$

Therefore, $H(z)$ is convex if $z \in (0, \psi(L^*)]$, and concave otherwise.

B.3 Proof of Lemma 2.3.5 (Bounds of J)

Since $\hat{F}(-\infty) = \hat{G}(+\infty) = 0$ and $\hat{F}(+\infty) = \hat{G}(-\infty) = +\infty$. Next, from the limit

$$\limsup_{x \to -\infty} \frac{\left(\hat{h}(x)\right)^+}{\hat{G}(x)} = \limsup_{x \to -\infty} \frac{\hat{h}(x)}{\hat{G}(x)} = 0,$$

we see that there exists some $\hat{x}_0$ such that $(\hat{h}(x))^+ < \hat{G}(x)$ for every $x \in (-\infty, \hat{x}_0)$. Since $(\hat{h}(x))^+$ is bounded between $[0, (V(\hat{x}_0) - \hat{x}_0 - c_b)^+]$ for $x \in [\hat{x}_0, +\infty)$, there exists some constant $\hat{K}$ such that $(\hat{h}(x))^+ \leq \hat{K}\hat{G}(x)$ for all $x \in \mathbb{R}$.

By the definition of $\hat{G}$, we can write $\hat{G}(x) = \mathbb{E}_x\{e^{-\hat{r}\tau}\hat{G}(X_\tau)\}$ for any $\tau \in \mathcal{T}$. This yields the inequality

$$\hat{K}\hat{G}(x) = \mathbb{E}_x\{e^{-\hat{r}\tau}\hat{K}\hat{G}(X_\tau)\} \geq \mathbb{E}_x\{e^{-\hat{r}\tau}(\hat{h}(X_\tau))^+\} \geq \mathbb{E}_x\{e^{-\hat{r}\tau}\hat{h}(X_\tau)\},$$

for every $x \in \mathbb{R}$ and every $\tau \in \mathcal{T}$. Hence, $J(x) \leq \hat{K}\hat{G}(x)$. Since $\tau = +\infty$ is a candidate stopping time, we have $J(x) \geq 0$.

B.4 Proof of Lemma 2.3.6 (Properties of $\hat{H}$)

We first show that $V(x)$ and $\hat{h}(x)$ are twice differentiable everywhere, except for $x = b^*$. Recall that

$$V(x) = \begin{cases} (b^* - c_s)\frac{F(x)}{F(b^*)} & \text{if } x \in (-\infty, b^*), \\ x - c_s & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{h}(x) = V(x) - x - c_b.$$

Therefore, it follows from (2.3.3) that

$$V'(x) = \begin{cases} (b^* - c_s)\frac{F'(x)}{F(b^*)} = \frac{F'(x)}{F'(b^*)} & \text{if } x \in (-\infty, b^*), \\ 1 & \text{if } x \in (b^*, +\infty), \end{cases}$$

which implies that $V'(b^*-) = 1 = V'(b^*+)$. Therefore, $V(x)$ is differentiable everywhere and so is $\hat{h}$. However, $V(x)$ is not twice differentiable since

$$V''(x) = \begin{cases} \frac{F''(x)}{F'(b^*)} & \text{if } x \in (-\infty, b^*), \\ 0 & \text{if } x \in (b^*, +\infty), \end{cases}$$

and $V''(b^*-) \neq V''(b^*+)$. Consequently, $\hat{h}(x) = V(x) - x - c_b$ is not twice differentiable at b^*.

The twice differentiability of $\hat{G}$ and $\hat{\psi}$ are straightforward. The continuity and differentiability of $\hat{H}$ on $(0, +\infty)$ and twice differentiability on $(0, \hat{\psi}(b^*)) \cup (\hat{\psi}(b^*), +\infty)$ follow directly. Observing that $\hat{h}(x) > 0$ as $x \to -\infty$, $\hat{H}$ is also continuous at 0 by definition. We now establish the properties of $\hat{H}$.

(i) First we prove the value of $\hat{H}$ at 0:

$$\begin{aligned} \hat{H}(0) &= \lim_{x\to-\infty} \frac{(\hat{h}(x))^+}{\hat{G}(x)} = \limsup_{x\to-\infty} \frac{\frac{(b^*-c_s)}{F(b^*)}F(x) - x - c_b}{\hat{G}(x)} \\ &= \limsup_{x\to-\infty} \frac{\frac{(b^*-c_s)}{F(b^*)}F'(x) - 1}{\hat{G}'(x)} = 0. \end{aligned}$$

Next, observe that $\lim_{x\to-\infty} \hat{h}(x) = +\infty$ and $\hat{h}(x) = -(c_s + c_b)$, for $x \in [b^*, +\infty)$. Since $F'(x)$ is strictly increasing and $F'(x) > 0$ for $x \in \mathbb{R}$, we have, for $x < b^*$,

$$\hat{h}'(x) = V'(x) - 1 = \frac{F'(x)}{F'(b^*)} - 1 < \frac{F'(b^*)}{F'(b^*)} - 1 = 0,$$

which implies that $\hat{h}(x)$ is strictly decreasing for $x \in (-\infty, b^*)$. Therefore, there exists a unique solution $\bar{d}$ to $\hat{h}(x) = 0$, and $\bar{d} < b^*$, such that $\hat{h}(x) > 0$ if $x \in (-\infty, \bar{d})$ and $\hat{h}(x) < 0$ if $x \in (\bar{d}, +\infty)$. It is trivial that $\hat{\psi}(x) \in (0, +\infty)$ for $x \in \mathbb{R}$ and is a strictly increasing function. Therefore, along with the fact that $\hat{G}(x) > 0$, property (i) follows directly.

(ii) With $z = \hat{\psi}(x)$, for $x > b^*$,

$$\hat{H}'(z) = \frac{1}{\hat{\psi}'(x)}(\frac{\hat{h}}{\hat{G}})'(x) = \frac{1}{\hat{\psi}'(x)}(\frac{-(c_s + c_b)}{\hat{G}(x)})' = \frac{1}{\hat{\psi}'(x)}\frac{(c_s + c_b)\hat{G}'(x)}{\hat{G}^2(x)} < 0,$$

since $\hat{\psi}'(x) > 0$, $\hat{G}'(x) < 0$, and $\hat{G}^2(x) > 0$. Therefore, $\hat{H}(z)$ is strictly decreasing for $z > \hat{\psi}(b^*)$.

(iii) By the definition of $\hat{H}$,

$$\hat{H}''(z) = \frac{2}{\sigma^2 \hat{G}(x)(\hat{\psi}'(x))^2}(\mathcal{L} - \hat{r})\hat{h}(x), \quad z = \hat{\psi}(x).$$

Since σ^2, $\hat{G}(x)$ and $(\hat{\psi}'(x))^2$ are all positive, we only need to determine the sign of $(\mathcal{L} - \hat{r})\hat{h}(x)$:

$$\begin{aligned}(\mathcal{L} - \hat{r})\hat{h}(x) &= \frac{1}{2}\sigma^2 V''(x) + \mu(\theta - x)V'(x) - \mu(\theta - x) - \hat{r}(V(x) - x - c_b) \\ &= \begin{cases} (r - \hat{r})V(x) + (\mu + \hat{r})x - \mu\theta + \hat{r}c_b & \text{if } x < b^*, \\ \hat{r}(c_s + c_b) > 0 & \text{if } x > b^*. \end{cases}\end{aligned}$$

To determine the sign of $(\mathcal{L} - \hat{r})\hat{h}(x)$ in $(-\infty, b^*)$, first note that $[(\mathcal{L} - \hat{r})\hat{h}](x)$ is a strictly increasing function in $(-\infty, b^*)$, since $V(x)$ is a strictly increasing function and $r \geq \hat{r}$ by assumption. Next note that for $x \in [L^*, b^*)$,

$$\begin{aligned}(\mathcal{L} - \hat{r})\hat{h}(x) &= (r - \hat{r})V(x) + (\mu + \hat{r})x - \mu\theta + \hat{r}c_b \\ &\geq (r - \hat{r})(x - c_s) + (\mu + \hat{r})x - \mu\theta + \hat{r}c_b \\ &= (r + \mu)x - (\mu\theta + rc_s) + \hat{r}(c_s + c_b) \\ &\geq (r + \mu)L^* - (\mu\theta + rc_s) + \hat{r}(c_s + c_b) = \hat{r}(c_s + c_b) > 0.\end{aligned}$$

Also, note that $(\mathcal{L} - \hat{r})\hat{h}(x) \to -\infty$ as $x \to -\infty$. Therefore, $(\mathcal{L} - \hat{r})\hat{h}(x) < 0$ if $x \in (-\infty, \underline{b})$ and $(\mathcal{L} - \hat{r})\hat{h}(x) > 0$ if $x \in (\underline{b}, +\infty)$ with $\underline{b} < L^*$ being the break-even point. From this, we conclude property (iii).

B.5 Proof of Lemma 2.4.4 (Properties of $\hat{H}_L$)

(i) The continuity of $\hat{H}_L(z)$ on $(0, +\infty)$ is implied by the continuities of $\hat{h}_L$, $\hat{G}$ and $\hat{\psi}$. The continuity of $\hat{H}_L(z)$ at 0 follows from

$$\begin{aligned}\hat{H}_L(0) &= \lim_{x \to -\infty} \frac{(\hat{h}_L(x))^+}{\hat{G}(x)} = \lim_{x \to -\infty} \frac{0}{\hat{G}(x)} = 0, \\ \lim_{z \to 0} \hat{H}_L(z) &= \lim_{x \to -\infty} \frac{\hat{h}_L}{\hat{G}}(x) = \lim_{x \to -\infty} \frac{-(c_s + c_b)}{\hat{G}(x)} = 0,\end{aligned}$$

where we have used that $z = \hat{\psi}(x) \to 0$ as $x \to -\infty$.

Furthermore, for $x \in (-\infty, L] \cup [b_L^*, +\infty)$, we have $V_L(x) = x - c_s$, and thus, $\hat{h}_L(x) = -(c_s + c_b)$. Also, with the facts that $\hat{\psi}(x)$ is a strictly increasing function and $\hat{G}(x) > 0$, property (i) follows.

(ii) By the definition of $\hat{H}_L$, since $\hat{G}$ and $\hat{\psi}$ are differentiable everywhere, we only need to show the differentiability of $V_L(x)$. To this end, $V_L(x)$ is differentiable at b_L^* by (2.4.3)-(2.4.4), but not at L. Therefore, $\hat{H}_L$ is differentiable for $z \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(L), +\infty)$.

In view of the facts that $\hat{G}'(x) < 0$, $\hat{\psi}'(x) > 0$, and $\hat{G}^2(x) > 0$, we have for

$x \in (-\infty, L) \cup [b_L^*, +\infty)$,

$$\hat{H}_L'(z) = \frac{1}{\hat{\psi}'(x)}(\frac{\hat{h}_L}{\hat{G}})'(x) = \frac{1}{\hat{\psi}'(x)}(\frac{-(c_s+c_b)}{\hat{G}(x)})' = \frac{(c_s+c_b)\hat{G}'(x)}{\hat{\psi}'(x)\hat{G}^2(x)} < 0.$$

Therefore, $\hat{H}_L(z)$ is strictly decreasing for $z \in (0, \hat{\psi}(L)) \cup [\hat{\psi}(b_L^*), +\infty)$.

(iii) Both $\hat{G}$ and $\hat{\psi}$ are twice differentiable everywhere, while $V_L(x)$ is twice differentiable everywhere except at $x = L$ and b_L^*, and so is $\hat{h}_L(x)$. Therefore, $\hat{H}_L(z)$ is twice differentiable on $(0, \hat{\psi}(L)) \cup (\hat{\psi}(L), \hat{\psi}(b_L^*)) \cup (\hat{\psi}(b_L^*), +\infty)$.

To determine the convexity/concavity of $\hat{H}_L$, we look at the second order derivative:

$$\hat{H}_L''(z) = \frac{2}{\sigma^2\hat{G}(x)(\hat{\psi}'(x))^2}(\mathcal{L} - \hat{r})\hat{h}_L(x),$$

whose sign is determined by

$$\begin{aligned}
&(\mathcal{L} - \hat{r})\hat{h}_L(x) \\
&= \frac{1}{2}\sigma^2 V_L''(x) + \mu(\theta - x)V_L'(x) - \mu(\theta - x) - \hat{r}(V_L(x) - x - c_b) \\
&= \begin{cases} (r - \hat{r})V_L(x) + (\mu + \hat{r})x - \mu\theta + \hat{r}c_b & \text{if } x \in (L, b_L^*), \\ \hat{r}(c_s + c_b) > 0 & \text{if } x \in (-\infty, L) \cup (b_L^*, +\infty). \end{cases}
\end{aligned}$$

This implies that $\hat{H}_L$ is convex for $z \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(b_L^*), +\infty)$.

On the other hand, the condition $\sup_{x\in\mathbb{R}} \hat{h}_L(x) > 0$ implies that

$$\sup_{z\in[0,+\infty)} \hat{H}_L(z) > 0.$$

By property (i) and twice differentiability of $\hat{H}_L(z)$ for $z \in (\hat{\psi}(L), \hat{\psi}(b_L^*))$, there must exist an interval $(\hat{\psi}(\underline{a}_L), \hat{\psi}(\bar{d}_L)) \subseteq (\hat{\psi}(L), \hat{\psi}(b_L^*))$ such that $\hat{H}_L(z)$ is concave, maximized at $\hat{z}_1 \in (\hat{\psi}(\underline{a}_L), \hat{\psi}(\bar{d}_L))$.

Furthermore, if $V_L(x)$ is strictly increasing on (L, b_L^*), then $(\mathcal{L} - \hat{r})\hat{h}_L(x)$ is also strictly increasing. To prove this, we first recall from Lemma 2.3.2 that $H(z)$ is strictly increasing and concave on $(\psi(L^*), +\infty)$. By Proposition 2.4.3, we have $b_L^* < b^*$, which implies $z_L < z^*$, and thus, $H'(z_L) > H'(z^*)$.

Then, it follows from (2.3.5), (2.3.6) and (2.4.6) that $W_L'(z) = H'(z_L) > H'(z^*) = W'(z)$ for $z \in (\psi(L), z_L)$. Next, since $W_L(z) = \frac{V_L}{G} \circ \psi^{-1}(z)$, we have

$$W_L'(z) = \frac{1}{\psi'(x)}(\frac{V_L}{G})'(x) = \frac{1}{\psi'(x)}(\frac{V_L'(x)G(x) - V_L(x)G'(x)}{G^2(x)}).$$

The same holds for $W'(z)$ with $V(x)$ replacing $V_L(x)$. As both $\psi'(x)$ and $G^2(x)$ are positive, $W_L'(z) > W'(z)$ is equivalent to $V_L'(x)G(x) - V_L(x)G'(x) > V'(x)G(x) - V(x)G'(x)$. This implies that

$$V_L'(x) - V'(x) = -\frac{G'(x)}{G(x)}(V(x) - V_L(x)) > 0,$$

since $G(x) > 0$, $G'(x) < 0$, and $V(x) > V_L(x)$. Recalling that $V'(x) > 0$, we have established that $V_L(x)$ is a strictly increasing function, and so is $(\mathcal{L} - \hat{r})\hat{h}_L(x)$. As we have shown the existence of an interval $(\hat{\psi}(\underline{a}_L), \hat{\psi}(\bar{d}_L)) \subseteq (\hat{\psi}(L), \hat{\psi}(b_L^*))$ over which $\hat{H}(z)$ is concave, or equivalently $(\mathcal{L} - \hat{r})\hat{h}_L(x) < 0$ with $x = \hat{\psi}^{-1}(z)$. Then by the strictly increasing property of $(\mathcal{L} - \hat{r})\hat{h}_L(x)$, we conclude $\underline{a}_L = L$ and $\bar{d}_L \in (L, b_L^*)$ is the unique solution to $(\mathcal{L} - \hat{r})\hat{h}_L(x) = 0$, and

$$(\mathcal{L} - \hat{r})\hat{h}_L(x) \begin{cases} < 0 & \text{if } x \in (L, \bar{d}_L), \\ > 0 & \text{if } x \in (-\infty, L) \cup (\bar{d}_L, b_L^*) \cup (b_L^*, +\infty). \end{cases}$$

Hence, we conclude the convexity and concavity of the function $\hat{H}_L$.

Appendix C

Appendix for Chapter 3

C.1 Proof of Lemma 3.2.1 (Bounds of V^ξ)

First, by Dynkin's formula, we have every $x \in \mathbb{R}$ and $\tau \in \mathcal{T}$,

$$\begin{aligned}\mathbb{E}_x\{e^{-r\tau}e^{X_\tau}\} - e^x &= \mathbb{E}_x\{\int_0^\tau e^{-rt}(\mathcal{L}-r)e^{X_t}dt\}\\ &= \mathbb{E}_x\{\int_0^\tau e^{-rt}e^{X_t}\left(\frac{\sigma^2}{2}+\mu\theta-r-\mu X_t\right)dt\}.\end{aligned}$$

The function $e^x\left(\frac{\sigma^2}{2}+\mu\theta-r-\mu x\right)$ is bounded above on $\mathbb{R}$. Let M be an upper bound, it follows that

$$\mathbb{E}_x\{e^{-r\tau}e^{X_\tau}\} - e^x \le M\mathbb{E}\{\int_0^\tau e^{-rt}dt\} \le M\mathbb{E}\{\int_0^{+\infty} e^{-rt}dt\} = \frac{M}{r} := K^\xi.$$

Since $h_s^\xi(x) = e^x - c_s \le e^x$, we have

$$\mathbb{E}_x\{e^{-r\tau}h_s^\xi(X_\tau)\} \le \mathbb{E}_x\{e^{-r\tau}e^{X_\tau}\} \le e^x + K^\xi.$$

Therefore, $V^\xi(x) \le e^x + K^\xi$. Lastly, the choice of $\tau = +\infty$ as a candidate stopping time implies that $V^\xi(x) \ge 0$.

C.2 Proof of Lemma 3.2.3 (Bounds of J^ξ)

From the limit

$$\limsup_{x\to-\infty}(\hat{h}^\xi(x))^+ = \limsup_{x\to-\infty}(V^\xi(x) - e^x - c_b)^+ = 0,$$

it follows that there exists some $\hat{x}_0^\xi$ such that $(\hat{h}^\xi(x))^+ \le \hat{K}_1$ for every $x \in (-\infty, \hat{x}_0^\xi)$ and some positive constant $\hat{K}_1$. Next, $(\hat{h}^\xi(x))^+$ is bounded by some positive constant $\hat{K}_2$ on the closed interval $[\hat{x}_0^\xi, b^{\xi*}]$. Also, $(\hat{h}^\xi(x))^+ = (V^\xi(x) - e^x - c_b)^+ = (-(c_s + c_b))^+ = 0$ for $x \ge b^{\xi*}$. Taking $\hat{K}^\xi = \hat{K}_1 \vee \hat{K}_2$, we have $(\hat{h}^\xi(x))^+ \le \hat{K}^\xi$ for all $x \in \mathbb{R}$. This yields the inequality

$$\mathbb{E}_x\{e^{-r\tau}\hat{h}^\xi(X_\tau)\} \le \mathbb{E}_x\{e^{-r\tau}(\hat{h}^\xi(X_\tau))^+\} \le \mathbb{E}_x\{e^{-r\tau}\hat{K}^\xi\} \le \hat{K}^\xi,$$

for every $x \in \mathbb{R}$ and every $\tau \in \mathcal{T}$. Hence, $J^\xi(x) \le \hat{K}^\xi$. The admissibility of $\tau = +\infty$ yields $J^\xi(x) \ge 0$.

C.3 Proof of Lemma 3.2.5 (Bounds of $\tilde{J}^\xi$ and $\tilde{V}^\xi$)

By definition, both $\tilde{J}^\xi(x)$ and $\tilde{V}^\xi(x)$ are nonnegative. Using Dynkin's formula,we have

$$\begin{aligned}\mathbb{E}_x\{e^{-r\tau_n}e^{X_{\tau_n}}\} - \mathbb{E}_x\{e^{-r\nu_n}e^{X_{\nu_n}}\} &= \mathbb{E}_x\{\int_{\nu_n}^{\tau_n} e^{-rt}(\mathcal{L}-r)e^{X_t}dt\} \\ &= \mathbb{E}_x\{\int_{\nu_n}^{\tau_n} e^{-rt}e^{X_t}\left(\frac{\sigma^2}{2} + \mu\theta - r - \mu X_t\right)dt\}.\end{aligned}$$

As we have pointed out in Section C.1, the function $e^x\left(\frac{\sigma^2}{2} + \mu\theta - r - \mu x\right)$ is bounded above on $\mathbb{R}$ and M is an upper bound. It follows that

$$\mathbb{E}_x\{e^{-r\tau_n}e^{X_{\tau_n}}\} - \mathbb{E}_x\{e^{-r\nu_n}e^{X_{\nu_n}}\} \le M\mathbb{E}_x\{\int_{\nu_n}^{\tau_n} e^{-rt}dt\}.$$

Since $e^x - c_s \leq e^x$ and $e^x + c_b \geq e^x$, we have

$$\begin{aligned}
&\mathbb{E}_x\{\sum_{n=1}^{\infty}[e^{-r\tau_n}h_s^{\xi}(X_{\tau_n}) - e^{-r\nu_n}h_b^{\xi}(X_{\nu_n})]\} \\
&\leq \sum_{n=1}^{\infty}\left(E\{e^{-r\tau_n}e^{X_{\tau_n}}\} - \mathbb{E}_x\{e^{-r\nu_n}e^{X_{\nu_n}}\}\right) \\
&\leq \sum_{n=1}^{\infty} M\mathbb{E}_x\{\int_{\nu_n}^{\tau_n} e^{-rt}dt\} \leq M\int_0^{\infty} e^{-rt}dt = \frac{M}{r} := C_1,
\end{aligned}$$

which implies that $0 \leq \tilde{J}^{\xi}(x) \leq C_1$. Similarly,

$$\begin{aligned}
&\mathbb{E}_x\{e^{-r\tau_1}h_s^{\xi}(X_{\tau_1}) + \sum_{n=2}^{\infty}[e^{-r\tau_n}h_s^{\xi}(X_{\tau_n}) - e^{-r\tau_n}h_b^{\xi}(X_{\tau_n})]\} \\
&\leq C_1 + \mathbb{E}_x\{e^{-r\tau_1}h_b^{\xi}(X_{\tau_1})\}.
\end{aligned}$$

Letting $\nu_1 = 0$ and using Dynkin's formula again, we have

$$\mathbb{E}_x\{e^{-r\tau_1}e^{X_{\tau_1}}\} - e^x \leq \frac{M}{r}.$$

This implies that

$$\tilde{V}^{\xi}(x) \leq C_1 + e^x + \frac{M}{r} := e^x + C_2.$$

C.4 Proof of Lemma 3.3.1 (Properties of H^{ξ})

The continuity and twice differentiability of H^{ξ} on $(0, +\infty)$ follow directly from those of h_s^{ξ}, G and ψ. On the other hand, we have $H^{\xi}(0) := \lim_{x\to-\infty} \frac{(h_s^{\xi}(x))^+}{G(x)} = \lim_{x\to-\infty} \frac{(e^x - c_s)^+}{G(x)} = \lim_{x\to-\infty} \frac{0}{G(x)} = 0$. Hence, the continuity of H^{ξ} at 0 follows from

$$\lim_{z\to 0} H^{\xi}(z) = \lim_{x\to-\infty} \frac{h_s^{\xi}(x)}{G(x)} = \lim_{x\to-\infty} \frac{e^x - c_s}{G(x)} = 0.$$

Next, we prove properties (i)-(iii) of H^{ξ}.

(i) This follows trivially from the fact that $\psi(x)$ is a strictly increasing function and $G(x) > 0$.

(ii) By the definition of H^ξ,

$$H^{\xi\prime}(z) = \frac{1}{\psi'(x)}(\frac{h_s^\xi}{G})'(x) = \frac{[e^x G(x) - (e^x - c_s)G'(x)]}{\psi'(x)G^2(x)}, \quad z = \psi(x).$$

For $x \in (\ln c_s, +\infty)$, $e^x - c_s > 0$, $G'(x) < 0$, so $e^x G(x) - (e^x - c_s)G'(x) > 0$. Also, since both $\psi'(x)$ and $G^2(x)$ are positive, we conclude that $H^{\xi\prime}(z) > 0$ for $z \in (\psi(\ln c_s), +\infty)$.

The proof of the limit of $H^{\xi\prime}(z)$ will make use of property (iii), and is thus deferred until after the proof of property (iii).

(iii) By differentiation, we have

$$H^{\xi\prime\prime}(z) = \frac{2}{\sigma^2 G(x)(\psi'(x))^2}[(\mathcal{L} - r)h_s^\xi](x), \quad z = \psi(x).$$

Since $\sigma^2, G(x)$ and $(\psi'(x))^2$ are all positive, we only need to determine the sign of $(\mathcal{L} - r)h_s^\xi(x) = e^x f_s(x)$. Hence, property (iii) follows from (3.2.7).

To find the limit of $H^{\xi\prime}(z)$, we first observe that

$$\lim_{x\to+\infty} \frac{h_s^\xi(x)}{F(x)} = 0. \tag{C.4.1}$$

Indeed, we have

$$\begin{aligned}
\lim_{x\to+\infty} \frac{h_s^\xi(x)}{F(x)} &= \lim_{x\to+\infty} \frac{1}{e^{-x}F(x)} \\
&= \lim_{x\to+\infty} \left(\int_0^{+\infty} u^{\frac{r}{\mu}-1} e^{(\sqrt{\frac{2\mu}{\sigma^2}} - \frac{1}{u})xu - \sqrt{\frac{2\mu}{\sigma^2}}\theta u - \frac{u^2}{2}}\, du\right)^{-1} \\
&= \lim_{x\to+\infty} \left(\int_0^{\sqrt{\frac{\sigma^2}{2\mu}}} u^{\frac{r}{\mu}-1} e^{(\sqrt{\frac{2\mu}{\sigma^2}} - \frac{1}{u})xu - \sqrt{\frac{2\mu}{\sigma^2}}\theta u - \frac{u^2}{2}}\, du + \int_{\sqrt{\frac{\sigma^2}{2\mu}}}^{+\infty} u^{\frac{r}{\mu}-1} e^{(\sqrt{\frac{2\mu}{\sigma^2}} - \frac{1}{u})xu - \sqrt{\frac{2\mu}{\sigma^2}}\theta u - \frac{u^2}{2}}\, du\right)^{-1}.
\end{aligned}$$

Since the first term on the RHS is non-negative and the second term is strictly increasing and convex in x, the limit is zero.

Turning now to $H^{\xi\prime}(z)$, we note that

$$H^{\xi\prime}(z) = \frac{1}{\psi'(x)}(\frac{h_s^\xi}{G})'(x), \quad z = \psi(x).$$

As we have shown, for $z > \psi(\ln c_s) \wedge \psi(x_s)$, ${H^{\xi}}'(z)$ is a positive and decreasing function. Hence the limit exists and satisfies

$$\lim_{z\to+\infty} {H^{\xi}}'(z) = \lim_{x\to+\infty} \frac{1}{\psi'(x)} (\frac{h_s^{\xi}}{G})'(x) = c \geq 0. \tag{C.4.2}$$

Observe that $\lim_{x\to+\infty} \frac{h_s^{\xi}(x)}{G(x)} = +\infty$, $\lim_{x\to+\infty} \psi(x) = +\infty$, and $\lim_{x\to+\infty} \frac{(\frac{h_s^{\xi}(x)}{G(x)})'}{\psi'(x)}$ exists, and $\psi'(x) \neq 0$. We can apply L'Hopital's rule to get

$$\lim_{x\to+\infty} \frac{h_s^{\xi}(x)}{F(x)} = \lim_{x\to+\infty} \frac{\frac{h_s^{\xi}(x)}{G(x)}}{\frac{F(x)}{G(x)}} = \lim_{x\to+\infty} \frac{(\frac{h_s^{\xi}(x)}{G(x)})'}{\psi'(x)} = c. \tag{C.4.3}$$

Comparing (C.4.1) and (C.4.3) implies that $c = 0$. From (C.4.2), we conclude that $\lim_{z\to+\infty} {H^{\xi}}'(z) = 0$.

C.5 Proof of Lemma 3.3.2 (Properties of $\hat{H}^{\xi}$)

It is straightforward to check that $V^{\xi}(x)$ is continuous and differentiable everywhere, and twice differentiable everywhere except at $x = b^{\xi*}$. The same properties hold for $\hat{h}^{\xi}(x)$. Since both G and ψ are twice differentiable everywhere, the continuity and differentiability of $\hat{H}^{\xi}$ on $(0, +\infty)$ and twice differentiability on $(0, \psi(b^{\xi*})) \cup (\psi(b^{\xi*}), +\infty)$ follow directly.

To see the continuity of $\hat{H}^{\xi}(z)$ at 0, note that $V^{\xi}(x) \to 0$ and $e^x \to 0$ as $x \to -\infty$. Then we have

$$\hat{H}^{\xi}(0) := \lim_{x\to-\infty} \frac{(\hat{h}^{\xi}(x))^+}{G(x)} = \lim_{x\to-\infty} \frac{(V^{\xi}(x) - e^x - c_b)^+}{G(x)} = \lim_{x\to-\infty} \frac{0}{G(x)} = 0,$$

and $\lim_{z\to0} \hat{H}^{\xi}(z) = \lim_{x\to-\infty} \frac{\hat{h}^{\xi}(x)}{G(x)} = \lim_{x\to-\infty} \frac{-c_b}{G(x)} = 0$. There follows the continuity at 0.

(i) For $x \in [b^{\xi*}, +\infty)$, we have $\hat{h}^{\xi}(x) \equiv -(c_s + c_b) < 0$. Next, the limits $\lim\limits_{x\to-\infty} V^{\xi}(x) \to 0$ and $\lim\limits_{x\to-\infty} e^x \to 0$ imply that $\lim\limits_{x\to-\infty} \hat{h}^{\xi}(x) = V^{\xi}(x) - e^x - c_b \to -c_b < 0$. Therefore, there exists some $\underline{b}^{\xi}$ such that $\hat{h}^{\xi}(x) < 0$ for

$x \in (-\infty, \underline{b}^{\xi})$. For the non-trivial case in question, $\hat{h}^{\xi}(x)$ must be positive for some x, so we must have $\underline{b}^{\xi} < b^{\xi*}$. To conclude, we have $\hat{h}^{\xi}(x) < 0$ for $x \in (-\infty, \underline{b}^{\xi}) \cup [b^{\xi*}, +\infty)$. This, along with the facts that $\psi(x) \in (0, +\infty)$ is a strictly increasing function and $G(x) > 0$, implies property (i).

(ii) By differentiating $\hat{H}^{\xi}(z)$, we get

$$\hat{H}^{\xi\prime}(z) = \frac{1}{\psi'(x)}(\frac{\hat{h}^{\xi}}{G})'(x), \quad z = \psi(x).$$

To determine the sign of $\hat{H}^{\xi\prime}$, we observe that, for $x \geq b^{\xi*}$,

$$(\frac{\hat{h}^{\xi}(x)}{G(x)})' = (\frac{-(c_s + c_b)}{G(x)})' = \frac{(c_s + c_b)G'(x)}{G^2(x)} < 0.$$

Also, $\psi'(x) > 0$ for $x \in \mathbb{R}$. Therefore, $\hat{H}^{\xi}(z)$ is strictly decreasing for $z \geq \psi(b^{\xi*})$.

(iii) To study the convexity/concavity, we look at the second derivative

$$\hat{H}^{\xi\prime\prime}(z) = \frac{2}{\sigma^2 G(x)(\psi'(x))^2}(\mathcal{L} - r)\hat{h}^{\xi}(x), \quad z = \psi(x).$$

Since $\sigma^2, G(x)$ and $(\psi'(x))^2$ are all positive, we only need to determine the sign of $(\mathcal{L} - r)\hat{h}^{\xi}(x)$:

$$\begin{aligned}(\mathcal{L} - r)\hat{h}^{\xi}(x) &= \frac{\sigma^2}{2}(V^{\xi\prime\prime}(x) - e^x) + \mu(\theta - x)(V^{\xi\prime}(x) - e^x) - r(V^{\xi}(x) - e^x - c_b) \\ &= \begin{cases} [\mu x - (\mu\theta + \frac{\sigma^2}{2} - r)]e^x + rc_b & \text{if } x \in (-\infty, b^{\xi*}), \\ r(c_s + c_b) > 0 & \text{if } x \in (b^{\xi*}, +\infty). \end{cases}\end{aligned}$$

which suggests that $\hat{H}^{\xi}(z)$ is convex for $z \in (\psi(b^{\xi*}), +\infty)$.

Furthermore, for $x \in (x_s, b^{\xi*})$, we have

$$\begin{aligned}(\mathcal{L} - r)\hat{h}^{\xi}(x) &= [\mu x - (\mu\theta + \frac{\sigma^2}{2} - r)]e^x + rc_b \\ &= -e^x f_s(x) + r(c_s + c_b) > r(c_s + c_b) > 0,\end{aligned}$$

by the definition of x_s. Therefore, $\hat{H}^{\xi}(z)$ is also convex on $(\psi(x_s), \psi(b^{\xi*}))$. Thus far, we have established that $\hat{H}^{\xi}(z)$ is convex on $(\psi(x_s), +\infty)$.

Next, we determine the convexity of $\hat{H}^\xi(z)$ on $(0, \psi(x_s)]$. Denote $\hat{z}_1^\xi := \arg\max_{z\in[0,+\infty)} \hat{H}^\xi(z)$. Since $\sup_{x\in\mathbb{R}} \hat{h}^\xi(x) > 0$, we must have

$$\hat{H}^\xi(\hat{z}_1^\xi) = \sup_{z\in[0,+\infty)} \hat{H}^\xi(z) > 0.$$

By its continuity and differentiability, $\hat{H}^\xi$ must be concave at $\hat{z}_1^\xi$. Then, there must exist some interval $(\psi(a^{(0)}), \psi(d^{(0)}))$ over which $\hat{H}^\xi$ is concave and $\hat{z}_1^\xi \in (\psi(a^{(0)}), \psi(d^{(0)}))$.

On the other hand, for $x \in (-\infty, x_s]$,

$$((\mathcal{L}-r)\hat{h}^\xi)'(x) = [\mu x - (\mu\theta + \frac{\sigma^2}{2} - r - \mu)]e^x \begin{cases} < 0 & \text{if } x \in (-\infty, x^{\xi*}), \\ > 0 & \text{if } x \in (x^{\xi*}, x_s], \end{cases}$$

where $x^{\xi*} = \theta + \frac{\sigma^2}{2\mu} - \frac{r}{\mu} - 1$. Therefore, $(\mathcal{L}-r)\hat{h}^\xi(x)$ is strictly decreasing on $(-\infty, x^{\xi*})$, strictly increasing on $(x^{\xi*}, x_s]$, and is strictly positive at x_s and $-\infty$:

$$(\mathcal{L}-r)\hat{h}^\xi(x_s) = r(c_s + c_b) > 0 \quad \text{and} \quad \lim_{x\to-\infty}(\mathcal{L}-r)\hat{h}^\xi(x) = rc_b > 0.$$

If $(\mathcal{L}-r)\hat{h}^\xi(x^{\xi*}) = -\mu e^{x^{\xi*}} + rc_b < 0$, then there exist exactly two distinct roots to the equation $(\mathcal{L}-r)\hat{h}^\xi(x) = 0$, denoted as x_{b1} and x_{b2}, such that $-\infty < x_{b1} < x^{\xi*} < x_{b2} < x_s$ and

$$(\mathcal{L}-r)\hat{h}^\xi(x) \begin{cases} > 0 & \text{if } x \in (-\infty, x_{b1}) \cup (x_{b2}, x_s], \\ < 0 & \text{if } x \in (x_{b1}, x_{b2}). \end{cases}$$

On the other hand, if $(\mathcal{L}-r)\hat{h}^\xi(x^{\xi*}) = -\mu e^{x^{\xi*}} + rc_b \geq 0$, then $(\mathcal{L}-r)\hat{h}^\xi(x) \geq 0$ for all $x \in \mathbb{R}$, and $\hat{H}^\xi(z)$ is convex for all z, which contradicts with the existence of a concave interval. Hence, we conclude that $-\mu e^{x^{\xi*}} + rc_b < 0$, and (x_{b1}, x_{b2}) is the unique interval that $(\mathcal{L}-r)\hat{h}^\xi(x) < 0$. Consequently, $(a^{(0)}, d^{(0)})$ coincides with (x_{b1}, x_{b2}) and $\hat{z}_1^\xi \in (\psi(x_{b1}), \psi(x_{b2}))$. This completes the proof.

Appendix D

Appendix for Chapter 4

D.1 Proof of Lemma 4.2.1 (Bounds of V^χ)

First, the limit

$$\limsup_{y\to+\infty}\frac{(h_s^\chi(y))^+}{F^\chi(y)}=\limsup_{y\to+\infty}\frac{y-c_s}{F^\chi(y)}=\limsup_{y\to+\infty}\frac{1}{F^{\chi\prime}(y)}=0.$$

Therefore, there exists some y_0 such that $(h_s^\chi(y))^+ < F^\chi(y)$ for $y\in(y_0,+\infty)$. As for $y\le y_0$, $(h_s^\chi(y))^+$ is bounded above by the constant $(y_0-c_s)^+$. As a result, we can always find a constant K^χ such that $(h_s^\chi(y))^+\le K^\chi F^\chi(y)$ for all $y\in\mathbb{R}$.

By definition, the process $(e^{-rt}F^\chi(Y_t))_{t\ge0}$ is a martingale. This implies, for every $y\in\mathbb{R}_+$ and $\tau\in\mathcal{T}$,

$$K^\chi F^\chi(y)=\mathbb{E}_y\{e^{-r\tau}K^\chi F^\chi(Y_\tau)\}\ge\mathbb{E}_y\{e^{-r\tau}(h_s^\chi(Y_\tau))^+\}\ge\mathbb{E}_y\{e^{-r\tau}h_s^\chi(Y_\tau)\}.$$

Therefore, $V^\chi(y)\le K^\chi F^\chi(y)$. Lastly, the choice of $\tau=+\infty$ as a candidate stopping time implies that $V^\chi(y)\ge0$.

D.2 Proof of Lemma 4.2.3 (Bounds of J^χ)

As we pointed out in Section 4.3.1.2 that $\hat{h}^\chi(y)$ is decreasing in y, thus so is $(\hat{h}^\chi(y))^+$. We can conclude that $(\hat{h}^\chi(y))^+ \le (V^\chi(0)-c_b)^+ = (\frac{b^{\chi*}-c_s}{F^\chi(b^{\chi*})}-c_b)^+$. The rest of the proof is similar to that of Lemma 3.2.3, with $\hat{K}$ changed to $(\frac{b^{\chi*}-c_s}{F^\chi(b^{\chi*})}-c_b)^+$.

D.3 Proof of Lemma 4.2.5 (Bounds of $\tilde{J}^\chi$ and $\tilde{V}^\chi$)

By definition, both $\tilde{J}^\chi(y)$ and $\tilde{V}^\chi(y)$ are nonnegative. Using Dynkin's formula,we have

$$\begin{aligned}\mathbb{E}_y\{e^{-r\tau_n}Y_{\tau_n}\} - \mathbb{E}_y\{e^{-r\nu_n}Y_{\nu_n}\} &= \mathbb{E}_y\{\int_{\nu_n}^{\tau_n} e^{-rt}(\mathcal{L}^\chi - r)Y_t dt \\ &= \mathbb{E}_y\{\int_{\nu_n}^{\tau_n} e^{-rt}\left(\mu\theta - (r+\mu)Y_t\right)dt\}.\end{aligned}$$

For $y \ge 0$, the function $\mu\theta-(r+\mu)y$ is bounded by $\mu\theta$. It follows that

$$\mathbb{E}_y\{e^{-r\tau_n}Y_{\tau_n}\} - \mathbb{E}_y\{e^{-r\nu_n}Y_{\nu_n}\} \le \mu\theta\mathbb{E}_y\{\int_{\nu_n}^{\tau_n} e^{-rt}dt\}.$$

Since $y-c_s \le y$ and $y+c_b \ge y$, we have

$$\begin{aligned}&\mathbb{E}_y\{\sum_{n=1}^{\infty}[e^{-r\tau_n}h_s^\chi(Y_{\tau_n}) - e^{-r\nu_n}h_b^\chi(Y_{\nu_n})]\} \\ &\le \sum_{n=1}^{\infty}\left(E\{e^{-r\tau_n}Y_{\tau_n}\} - \mathbb{E}_y\{e^{-r\nu_n}Y_{\nu_n}\}\right) \\ &\le \sum_{n=1}^{\infty}\mu\theta\mathbb{E}_y\{\int_{\nu_n}^{\tau_n} e^{-rt}dt\} \le \mu\theta\int_0^\infty e^{-rt}dt = \frac{\mu\theta}{r}.\end{aligned}$$

This implies that $0 \le \tilde{J}^\chi(y) \le \frac{\mu\theta}{r}$. Similarly,

$$\mathbb{E}_y\{e^{-r\tau_1}h_s^\chi(Y_{\tau_1}) + \sum_{n=2}^{\infty}[e^{-r\tau_n}h_s^\chi(Y_{\tau_n}) - e^{-r\tau_n}h_b^\chi(Y_{\tau_n})]\} \le \frac{\mu\theta}{r} + \mathbb{E}_y\{e^{-r\tau_1}h_b^\chi(Y_{\tau_1})\}.$$

Letting $\nu_1 = 0$ and using Dynkin's formula again, we have

$$\mathbb{E}_y\{e^{-r\tau_1}Y_{\tau_1}\} - y \leq \frac{\mu\theta}{r}.$$

This implies that

$$\tilde{V}^\chi(y) \leq \frac{\mu\theta}{r} + y + \frac{\mu\theta}{r} := y + \frac{2\mu\theta}{r}.$$

D.4 Proof of Lemma 4.3.2 (Properties of H^χ)

(i) First, we compute

$$H^\chi(0) = \lim_{y\to+\infty} \frac{(h_s(y))^+}{F^\chi(y)} = \lim_{y\to+\infty} \frac{y - c_s}{F^\chi(x)} = \lim_{y\to+\infty} \frac{1}{F^{\chi\prime}(y)} = 0.$$

Using the facts that $F^\chi(y) > 0$ and $\phi(y)$ is a strictly increasing function, (4.3.12) follows.

(ii) We look at the first derivative of H^χ:

$$H^{\chi\prime}(z) = \frac{1}{\phi'(y)}(\frac{h_s}{F^\chi})'(y) = \frac{1}{\phi'(y)}\frac{F^\chi(y) - (y - c_s)F^{\chi\prime}(y)}{F^{\chi 2}(y)}, \quad z = \phi(y).$$

Since both $\phi'(y)$ and $F^{\chi 2}(y)$ are positive, it remains to determine the sign of $F^\chi(y) - (y - c_s)F^{\chi\prime}(y)$. Since $F^{\chi\prime}(y) > 0$, we can equivalently check the sign of $v(y) := \frac{F^\chi(y)}{F^{\chi\prime}(y)} - (y - c_s)$. Note that $v'(y) = -\frac{F^\chi(y)F^{\chi\prime\prime}(y)}{(F^{\chi\prime}(y))^2} < 0$. Therefore, $v(y)$ is a strictly decreasing function. Also, it is clear that $v(c_s) > 0$ and $v(y_s) > 0$. Consequently, $v(y) > 0$ if $y < (c_s \vee y_s)$ and hence, $H^\chi(z)$ is strictly increasing if $z \in (\phi(0), \phi(c_s) \vee \phi(y_s))$.

(iii) By differentiation, we have

$$H^{\chi\prime\prime}(z) = \frac{2}{\sigma^2 F^\chi(y)(\phi'(y))^2}(\mathcal{L}^\chi - r)h_s(y), \quad z = \phi(y).$$

Since $\sigma^2, F^\chi(y)$ and $(\phi'(y))^2$ are all positive, the convexity/concavity of H^χ

depends on the sign of

$$(\mathcal{L}^{\chi} - r)h_s(y) = \mu(\theta - y) - r(y - c_s)$$
$$= (\mu\theta + rc_s) - (\mu + r)y \qquad \begin{cases} \geq 0 & \text{if } y \in [0, y_s], \\ \leq 0 & \text{if } y \in [y_s, +\infty), \end{cases}$$

which implies property (iii).

D.5 Proof of Lemma 4.3.3 (Properties of $\hat{H}^{\chi}$)

It is straightforward to check that $V^{\chi}(y)$ is continuous and differentiable everywhere, and twice differentiable everywhere except at $y = b^{\chi *}$, and all these holds for $\hat{h}^{\chi}(y) = V^{\chi}(y) - (y + c_b)$. Both F^{χ} and ϕ are twice differentiable. In turn, the continuity and differentiability of $\hat{H}^{\chi}$ on $(\phi(0), 0)$ and twice differentiability of $\hat{H}^{\chi}$ on $(\phi(0), \phi(b^{\chi *})) \cup (\phi(b^{\chi *}), 0)$ follow.

To show the continuity of $\hat{H}^{\chi}$ at 0, we note that

$$\hat{H}^{\chi}(0) = \lim_{y \to +\infty} \frac{(\hat{h}^{\chi}(y))^+}{F^{\chi}(y)} = \lim_{y \to +\infty} \frac{0}{F^{\chi}(y)} = 0, \quad \text{and}$$
$$\lim_{z \to 0} \hat{H}^{\chi}(z) = \lim_{y \to +\infty} \frac{\hat{h}^{\chi}}{F^{\chi}}(y) = \lim_{y \to +\infty} \frac{-(c_s + c_b)}{F^{\chi}(y)} = 0.$$

From this, we conclude that $\hat{H}^{\chi}$ is also continuous at 0.

(i) First, for $y \in [b^{\chi *}, +\infty)$, $\hat{h}^{\chi}(y) \equiv -(c_s + c_b)$. For $y \in (0, b^{\chi *})$, we compute

$$V^{\chi\prime}(y) = \frac{b^{\chi *} - c_s}{F^{\chi}(b^{\chi *})} F^{\chi\prime}(y) = \frac{F^{\chi\prime}(y)}{F^{\chi\prime}(b^{\chi *})}, \quad \text{by (4.2.6).}$$

Recall that $F^{\chi\prime}(y)$ is a strictly increasing function and $\hat{h}^{\chi}(y) = V^{\chi}(y) - y - c_b$. Differentiation yields

$$\hat{h}^{\chi\prime}(y) = V^{\chi\prime}(y) - 1 = \frac{F^{\chi\prime}(y)}{F^{\chi\prime}(b^{\chi *})} - 1 < \frac{F^{\chi\prime}(b^{\chi *})}{F^{\chi\prime}(b^{\chi *})} - 1 = 0, \quad y \in (0, b^{\chi *}),$$

which implies that $\hat{h}^{\chi}(y)$ is strictly decreasing for $y \in (0, b^{\chi *})$. On the other hand, $\hat{h}^{\chi}(0) > 0$ as we are considering the non-trivial case. Therefore, there exists a unique solution $\bar{d}^\chi < b^{\chi*}$ to $\hat{h}^\chi(y) = 0$, such that $\hat{h}^\chi(y) > 0$ for $y \in [0, \bar{d}^\chi)$, and $\hat{h}^\chi(y) < 0$ for $y \in (\bar{d}^\chi, +\infty)$. With $\hat{H}^\chi(z) = (\hat{h}^\chi / F^\chi) \circ \phi^{-1}(z)$, the above properties of $\hat{h}^\chi$, along with the facts that $\phi(y)$ is strictly increasing and $F^\chi(y) > 0$, imply property (i).

(ii)With $z = \phi(y)$, for $y > b^{\chi*}$, $\hat{H}^\chi(z)$ is strictly increasing since

$$\hat{H}^{\chi\prime}(z) = \frac{1}{\phi'(y)}(\frac{\hat{h}^\chi}{F^\chi})'(y) = \frac{1}{\phi'(y)}(\frac{-(c_s + c_b)}{F^\chi(y)})' = \frac{1}{\phi'(y)}\frac{(c_s + c_b)F^{\chi\prime}(y)}{F^{\chi 2}(y)} > 0.$$

When $y \to 0$, because $(\frac{\hat{h}^\chi(y)}{F^\chi(y)})'$ is finite, but $\phi'(y) \to +\infty$, we have $\lim_{z \to \phi(0)} \hat{H}^{\chi\prime}(z) = 0$.

(iii) Consider the second derivative:

$$\hat{H}^{\chi\prime\prime}(z) = \frac{2}{\sigma^2 F(y)(\phi'(y))^2}(\mathcal{L}^\chi - r)\hat{h}^\chi(y).$$

The positivities of $\sigma^2, F^\chi(y)$ and $(\phi'(y))^2$ suggest that we inspect the sign of $(\mathcal{L}^\chi - r)\hat{h}^\chi(y)$:

$$(\mathcal{L}^\chi - r)\hat{h}^\chi(y) = \frac{1}{2}\sigma^2 y V^{\chi\prime\prime}(y) + \mu(\theta - y)V^{\chi\prime}(y) - \mu(\theta - y) - r(V^\chi(y) - (y + c_b))$$

$$= \begin{cases} (\mu + r)y - \mu\theta + rc_b & \text{if } y < b^{\chi*}, \\ r(c_s + c_b) > 0 & \text{if } y > b^{\chi*}. \end{cases}$$

Since $\mu, r > 0$ by assumption, $(\mathcal{L}^\chi - r)\hat{h}^\chi(y)$ is strictly increasing on $(0, b^{\chi*})$. Next, we show that $0 < y_b < y_s < b^{\chi*}$. By the fact that $F^{\chi\prime}(0) = \frac{r}{\mu\theta}$ and the assumption that $V^\chi(0) = \frac{b^{\chi*} - c_s}{F^\chi(b^{\chi*})} > c_b$, we have

$$V^{\chi\prime}(0) = \frac{b^{\chi*} - c_s}{F^\chi(b^{\chi*})}F^{\chi\prime}(0) = \frac{b^{\chi*} - c_s}{F^\chi(b^{\chi*})}\frac{r}{\mu\theta} > \frac{rc_b}{\mu\theta}.$$

In addition, by the convexity of V^χ and $V^{\chi\prime}(b^{\chi*}) = 1$, it follows that

$$\frac{rc_b}{\mu\theta} < V^{\chi\prime}(0) < V^{\chi\prime}(b^{\chi*}) = 1,$$

which implies $\mu\theta > rc_b$ and hence $y_b > 0$. By simply comparing the definitions of y_b and y_s, it is clear that $y_b < y_s$. Therefore, by observing that $(\mathcal{L}^\chi - r)\hat{h}^\chi(y_b) = 0$, we conclude $(\mathcal{L}^\chi - r)\hat{h}^\chi(y) < 0$ if $y \in [0, y_b)$, and $(\mathcal{L}^\chi - r)\hat{h}^\chi(y) > 0$ if $y \in (y_b, +\infty)$. This suggests the concavity and convexity of $\hat{H}^\chi$ as desired.