Sato-Tate Problem for GL(3)

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ABSTRACT

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Based upon the work of Goldfeld and Kontorovich on the Kuznetsov trace formula of Maass forms for $\mathrm{SL}(3,\mathbb{Z})$, we prove a weighted vertical equidistribution theorem (with respect to the generalized Sato-Tate measure) for the p^{th} Hecke eigenvalue of Maass forms, with the rate of convergence. With a conjectured orthogonality relation between the Fourier coefficients of Maass forms for $\mathrm{SL}(N,\mathbb{Z})$ for $N \geq 4$, we generalize the above equidistribution theorem to $N \geq 4$.

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To My Parents

Chapter 1

Introduction

Let $\mathcal{N}$ be a positive integer and k a non-negative integer. We consider $S(\mathcal{N}, k)$ the space of holomorphic modular forms of level $\mathcal{N}$ and weight k for a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. For each positive integer n with $(n, \mathcal{N}) = 1$, the Hecke operator T_n acts on $S(\mathcal{N}, k)$. Let $\varphi \in S(\mathcal{N}, k)$ be an eigenfunction of all the Hecke operators T_n ($(n, \mathcal{N}) = 1$) and $a_\varphi(n)$ the eigenvalue of φ under the action of T_n. The Ramanujan conjecture states that for a prime number $p \nmid \mathcal{N}$

$$\left| \frac{a_\varphi(p)}{p^{\frac{k-1}{2}}} \right| \leq 2.$$

We define a measure on $\mathbb{R}$

$$d\mu_\infty^{(2)}(x) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}\, dx, & \text{when } |x| \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

called the Sato-Tate measure for $\mathrm{GL}(2)$, or the semi-circle measure. The Sato-Tate conjecture is a more refined statement about the statistics of the Hecke eigenvalues of non-CM holomorphic modular form φ of weight $k \geq 2$. It states that $\frac{a_\varphi(p)}{p^{\frac{k-1}{2}}}$ is an equidistributed sequence as $p \to \infty$ with respect to $\mu_\infty^{(2)}$. More precisely the Sato-Tate conjecture predicts that

$$\lim_{T \to \infty} \frac{\sum_{p \leq T} f\left(\frac{a_\varphi(p)}{p^{\frac{k-1}{2}}}\right)}{\sum_{p \leq T} 1} = \int_{\mathbb{R}} f \, d\mu_\infty^{(2)}$$

for any continuous test function $f : \mathbb{R} \to \mathbb{R}$. In recent years, there has been enormous progress toward proving this conjecture and its generalizations, most notably in [Barnet-Lamb-Gee-Geraghty, 2011] and [Barnet-Lamb-Geraghty-Harris-Taylor, 2011].

Considering this problem from another perspective, we can fix the prime number p and investigate the distribution of $\frac{a_\varphi(p)}{p^{\frac{k-1}{2}}}$ as φ runs over different modular forms. Most notably in [Serre, 1997], it is proved that $\frac{a_\varphi(p)}{p^{\frac{k-1}{2}}}$ is equidistributed with respect to the Plancherel measure

$$d\mu_p^{(2)} = \frac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2}\, d\mu_\infty^{(2)}$$

as φ runs over all Hecke eigenforms in $S(\mathcal{N}, k)$ and $\mathcal{N} + k \to \infty$ ($p \nmid \mathcal{N}$). Almost at the same time [Conrey-Duke-Farmer, 1997] obtained the same result independently for $\mathcal{N} = 1$. Recently, [Murty-Sinha, 2009] investigated the effective version of [Serre, 1997], which gives an explicit estimate on the rate of convergence.

From the same perspective of fixing p and varying φ, two earlier papers [Sarnak, 1987] and [Bruggeman, 1978] proved that $a_\varphi(p)$ is an equidistributed sequence with respect to the Plancherel measure $\mu_p^{(2)}$ by varying φ over all Hecke-Maass forms for $\mathrm{SL}(2,\mathbb{Z})$. As in [Murty-Sinha, 2009] an effective version of [Sarnak, 1987] appeared in [Lau-Wang, 2011]. Very recently [Shin-Templier, 2012] gave a highbrow generalization of [Serre, 1997] et al.

It is understandable that by fixing a prime number p instead of a Hecke eigenform φ we get the Plancherel measure instead of the Sato-Tate measure. Strikingly, if we give each Hecke eigenvalue $a_\varphi(p)$ a weight

$$\frac{1}{\operatorname*{Res}_{s=1} L(s, \varphi \times \tilde{\varphi})} \quad \left(\text{or } \frac{1}{L(1, \varphi, \mathrm{Ad})}\right)$$

and do the same statistics of fixing p and varying φ, the same Sato-Tate measure appears once again, instead of the Plancherel measure. In [Bruggeman, 1978] it is essentially proved that

$$\lim_{T\to\infty} \frac{\sum_{\lambda_\varphi \le T} \frac{f(a_\varphi(p))}{\operatorname*{Res}_{s=1} L(s,\varphi\times\tilde{\varphi})}}{\sum_{\lambda_\varphi \le T} \frac{1}{\operatorname*{Res}_{s=1} L(s,\varphi\times\tilde{\varphi})}} = \int_{\mathbb{R}} f\, d\mu_\infty^{(2)}$$

for any continuous test function $f : \mathbb{R} \to \mathbb{R}$, where φ runs over all Hecke-Maass forms for $\mathrm{SL}(2,\mathbb{Z})$ and λ_φ is the eigenvalue of φ under the action of the Laplace operator on the upper half plane. Later [Li, 2004] and [Gun-Murty-Rath, 2008] proved similar theorems for modular forms. The weight $1/\operatorname*{Res}_{s=1} L(s, \varphi \times \tilde{\varphi})$ appears naturally in the Petersson and Kuznetsov trace formulae.

The theory of Maass forms for $\mathrm{SL}(3,\mathbb{Z})$ has been studied since the 1980s. The definitions and results are summarized in [Goldfeld, 2006]. Let ϕ be a Maass form for $\mathrm{SL}(3,\mathbb{Z})$ which is an eigenfunction of all the Hecke operators T_n, where T_n corresponds to the union of double cosets

$\bigcup_{m_0^3 m_1^2 m_2 = n} \mathrm{SL}(3,\mathbb{Z}) \begin{pmatrix} m_0m_1m_2 & & \\ & m_0m_1 & \\ & & m_0 \end{pmatrix} \mathrm{SL}(3,\mathbb{Z})$. Let $a_\phi(p)$ be the Hecke eigenvalue of ϕ under the action of the Hecke operator T_p. In this context the Ramanujan conjecture predicts

$$a_\phi(p) \in \left\{ \sum_{i=1}^{3} e^{i\theta_i} : \sum_{i=1}^{3} \theta_i = 0,\ \theta_i \in \mathbb{R},\ i = 1,2,3 \right\}.$$

Note that $a_\phi(p)$ is not necessarily a real number. We define $\mu_\infty^{(3)}$, the Sato-Tate measure for GL(3), as the pushforward measure of the Haar measure of SU(3) by the trace map $\mathrm{Tr} : \mathrm{SU}(3) \to \mathbb{C}$. It is unlikely that this measure can be expressed explicitly by elementary functions. The Sato-Tate conjecture should be modified correspondingly, predicting that $a_\phi(p)$ is an equidistributed sequence as $p \to \infty$ with respect to the measure $\mu_\infty^{(3)}$.

Every Hecke-Maass form ϕ has spectral parameter $\nu^\phi = (\nu_1^\phi, \nu_2^\phi) \in \mathbb{C}^2$, and $\lambda_{\nu^\phi}(-\Delta_2) = 1 - 3(\nu_1^{\phi 2} + \nu_1^\phi \nu_2^\phi + \nu_2^{\phi 2})$ is the Laplace eigenvalue of ϕ.

Theorem 1.1 (**Main theorem**). *For $T \gg 1$, let $\{h_T : \mathbb{C}^2 \to \mathbb{R}\}$ be a family of test functions where h_T is essentially supported on $\{\nu = (\nu_1, \nu_2) : \lambda_\nu(-\Delta_2) \le T^2\}$ as in Definition 5.1. Let $a_\phi(p)$ be the p^{th} Hecke eigenvalue of a Hecke-Maass form ϕ. For any continuous test function $f : \mathbb{C} \to \mathbb{C}$, we have*

$$\lim_{T\to\infty} \frac{\sum_\phi f(a_\phi(p)) \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}}{\sum_\phi \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}} = \int_{\mathbb{C}} f \, d\mu_\infty^{(3)},$$

where $\sum_\phi$ sums over all Hecke-Maass forms for SL$(3,\mathbb{Z})$.

Our proof of Theorem 1.1 is based upon the orthogonality relation in [Goldfeld-Kontorovich, 2013] for Fourier coefficients of Maass forms for SL$(3,\mathbb{Z})$. Let $A_\phi(m_1, m_2)$ be the $(m_1, m_2)^{\mathrm{th}}$ Fourier coefficient of a Maass form ϕ. If we assume that ϕ is normalized so that $A_\phi(1,1) = 1$, then we have $A_\phi(p,1) = a_\phi(p)$ for all prime numbers p. Goldfeld and Kontorovich's orthogonality relation states

$$\frac{\sum_\phi A_\phi(m_1,m_2)\overline{A_\phi(n_1,n_2)} \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}}{\sum_\phi \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}} = \delta_{m_1,n_1}\delta_{m_2,n_2} + \mathcal{O}_{\{h_T\},\epsilon}((m_1m_2n_1n_2)^2 T^{\epsilon-2}) \tag{1.1}$$

as $T \to \infty$. A similar result was also obtained by [Blomer] and is equally usable to us. Combining the orthogonality relation with the Hecke relations we can compute the average of $a_\phi(p)^{k_1}\overline{a_\phi(p)}^{k_2}$

for non-negative integers k_1 and k_2. More precisely, we find that

$$\lim_{T\to\infty} \frac{\sum_{\phi} a_\phi(p)^{k_1}\overline{a_\phi(p)}^{k_2} \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}}{\sum_{\phi} \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}}$$

equals the number of random walks of some special type in the integer points of the first quadrant of the coordinate plane, which will be defined in later chapters. This type of random walk comes from the Hecke relations for $\mathrm{SL}(3,\mathbb{Z})$.

In the proof of Theorem 1.1 in the case of $f(z) = z^{k_1}\overline{z}^{k_2}$, we show that

$$\int_{\mathbb{C}} z^{k_1}\overline{z}^{k_2}\ d\mu_\infty^{(3)}$$

equals the multiplicity of the trivial representation in the tensor product representation $H^{\otimes k_1}\otimes\overline{H}^{\otimes k_2}$ where H is the defining representation $\mathrm{SU}(3) \hookrightarrow \mathrm{GL}(3,\mathbb{C})$. Relying on the Weyl character formula and two papers [Gessel-Zeilberger, 1992] and [Grabiner-Magyar, 1993] on the reflection method, we show that the multiplicity equals the number of random walks of certain type in a Weyl chamber of the root system A_2. By matching the random walk in the first quadrant and the random walk in a Weyl chamber, we obtain Theorem 1.1 when $f(z) = z^{k_1}\overline{z}^{k_2}$. Since all continuous functions on a compact set can be approximated by polynomials this completes the proof.

As in [Murty-Sinha, 2009], [Lau-Wang, 2011] and [Shin-Templier, 2012], we obtain an effective version of Theorem 1.1, which gives the rate of convergence, but only for a large family of special test functions.

Theorem 1.2 (**Rate of Convergence**). *For $T \gg 1$, let $\{h_T : \mathbb{C}^2 \to \mathbb{R}\}$ be a family of test functions where h_T is essentially supported on $\{\nu = (\nu_1,\nu_2) : \lambda_\nu(-\Delta_2) \le T^2\}$ as in Definition 5.1. Fix $\epsilon > 0$. For a power series $f(z) = \sum_{k_1,k_2\ge 0} a_{k_1,k_2} z^{k_1}\overline{z}^{k_2}$ which is absolutely convergent on $\{z : |z| \le p^2 + p^{-2} + 1\}$, we have*

$$\left| \frac{\sum_{\phi} f(a_\phi(p)) \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}}{\sum_{\phi} \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}} - \int_{\mathbb{C}} f\ d\mu_\infty^{(3)} \right| \ll_{\epsilon,f} T^{\epsilon-2},$$

where $\sum_{\phi}$ sums over all Hecke-Maass forms for $\mathrm{SL}(3,\mathbb{Z})$.

Our proof of Theorem 1.2 is based on the error terms $\mathcal{O}_{h,\epsilon}((m_1m_2n_1n_2)^2T^{\epsilon-2})$ in the orthogonality relation (Equation 1.1). By exploiting more equivalences between the random walk in the first quadrant and the random walk in a Weyl chamber, we get the error term accumulated by each monomial in the power series $\sum_{k_1,k_2\geq 0} a_{k_1,k_2} z^{k_1}\bar{z}^{k_2}$, with the help of its large convergence domain.

The mechanism we use in the proof of Theorem 1.1 can be generalized to $\mathrm{GL}(N)$ $(N \geq 2)$. In general we expect that the Kuznetsov trace formula would give an orthogonality relation for Fourier coefficients with the weight $1/\operatorname{Res}_{s=1} L(s, \phi \times \tilde{\phi})$, as in [Bruggeman, 1978], [Goldfeld-Kontorovich, 2013], [Blomer] et al. For GL(2) automorphic forms, both the Hecke relations and the Sato-Tate measure are simple enough to allow elementary techniques to connect them, as in [Bruggeman, 1978], [Gun-Murty-Rath, 2008] and [Li, 2004].

For $\mathrm{GL}(N)$ $(N \geq 4)$ the orthogonality relation is conjectured but not proved yet. Hopefully, a version of the orthogonality relation will appear in the near future for $\mathrm{GL}(N)$ $(N \geq 4)$. Let $\mu_\infty^{(N)}$ be the Sato-Tate measure for $\mathrm{GL}(N)$ which is the pushforward measure of the Haar measure of $\mathrm{SU}(N)$ by the trace map $\mathrm{Tr} : \mathrm{SU}(N) \to \mathbb{C}$.

If we assume the orthogonality relation of type

$$\lim_{T\to\infty} \frac{\sum_{\phi} A_\phi(m_1, ..., m_{N-1})\overline{A_\phi(n_1, ..., n_{N-1})}\frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}}{\sum_{\phi} \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}} = \prod_{i=1}^{N-1} \delta_{m_i,n_i}$$

for some family of test functions h_T, we prove

$$\lim_{T\to\infty} \frac{\sum_{\phi} f(a_\phi(p))\frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}}{\sum_{\phi} \frac{h_T(\nu^\phi)}{\operatorname{Res}_{s=1} L(s,\phi\times\tilde{\phi})}} = \int_{\mathbb{C}} f \, d\mu_\infty^{(N)}$$

for any continuous test function $f : \mathbb{C} \to \mathbb{C}$. The idea of our proof is the same as the proof for GL(3). The set of integer points in the first quadrant is replaced with a lattice in higher dimensions. The root system A_2 is replaced by A_{N-1}. By matching the two types of random walk and approximation by polynomials we complete the proof.

Chapter 2

Historical Equidistribution Results on Hecke-Maass Forms for $\mathrm{SL}(2,\mathbb{Z})$

In this chapter we will recall the definitions of Maass forms for $\mathrm{SL}(2,\mathbb{Z})$ and review the historical results of Sarnak and of Bruggeman. This chapter is independent from the rest chapters. The main references are [Goldfeld, 2006], [Sarnak, 1987] and [Bruggeman, 1978].

Definition 2.1. Let $\mathfrak{H}^2$ be the upper half plane $\{z = x + iy : x, y \in \mathbb{R}, y > 0\}$. The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathfrak{H}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathfrak{H}^2$.

Definition 2.2. Let φ and φ' be two complex-valued functions on $\mathrm{SL}(2,\mathbb{Z}) \setminus \mathfrak{H}^2$. We define their inner product

$$< \varphi, \varphi' > = \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2} \varphi(z)\overline{\varphi'(z)}\, \frac{dxdy}{y^2}.$$

We define the L^2-norm

$$\| \varphi \|_2 = \sqrt{< \varphi, \varphi >}.$$

Definition 2.3. We define $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2)$ to be $\{\varphi : \mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2 \to \mathbb{C} : \| \varphi \|_2 < \infty\}$. We define

$\mathcal{L}^2_{\mathrm{cusp}}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2)$ to be the space of all $\varphi \in \mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2)$ such that φ is cuspidal, i.e.,

$$\int_0^1 \varphi(z)\, dx = 0.$$

Definition 2.4 (**Maass form for** $\mathrm{SL}(2,\mathbb{Z})$)**.** A Maass form of type $\nu \in \mathbb{C}$ for $\mathrm{SL}(2,\mathbb{Z})$ is a non-zero smooth function $\varphi \in \mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2)$ which satisfies

- $\varphi(\gamma z) = \varphi(z)$ for any $z \in \mathfrak{H}^2$ and any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$,
- φ is an eigenfunction of the Laplace operator, i.e.,
$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi(z) = (\frac{1}{4} - \nu^2)\varphi(z),$$
- φ is cuspidal or $\int_0^1 \varphi(z)\, dx = 0$.

Definition 2.5 (**Hecke operator**)**.** For $n > 0$, we define the Hecke operator T_n acting on $\mathcal{L}^2_{\mathrm{cusp}}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2)$ by

$$T_n\varphi(z) = \frac{1}{\sqrt{n}} \sum_{\substack{0\le b<d \\ ad=n}} \varphi\left(\frac{az+b}{d}\right).$$

Remark 2.6 (**Hecke-Maass form**)**.** The Hecke operators T_n $(n = 1, 2, 3, ...)$ commute with the Laplace operator $-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. All Hecke operators commute. We can simultaneously diagonalize the space $\mathcal{L}^2_{\mathrm{cusp}}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2)$ by these operators. The space $\mathcal{L}^2_{\mathrm{cusp}}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^2)$ has an orthogonal basis

$$\varphi_1, \varphi_2, \varphi_3, \dots.$$

Each φ_j is a Maass form of type ν_j and an eigenfunction of all Hecke operators T_n. We arrange the order of φ_j's by their Laplace eigenvalues so that

$$\frac{1}{4} - \nu_1^2 \le \frac{1}{4} - \nu_2^2 \le \frac{1}{4} - \nu_3^2 \le \dots.$$

These φ_j's are called Hecke-Maass forms.

Definition 2.7 (**Hecke eigenvalue**)**.** Let $a_j(n)$ be the eigenvalue of of φ_j under the Hecke operator T_n, i.e.,

$$T_n\varphi_j = a_j(n)\varphi_j.$$

Definition 2.8. We define the convolution L-function for a Hecke-Maass form φ_j to be

$$L(s,\varphi_j\times\tilde{\varphi}_j)=\zeta(2s)\sum_{n=1}^{\infty}\frac{|a_j(n)|^2}{n^s},$$

where $\zeta(s)=\sum\limits_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann-Zeta function. Let $\mathcal{L}_j$ be a number attached to each Hecke-Maass form φ_j and it is given by

$$\mathcal{L}_j=\operatorname*{Res}_{s=1}L(s,\varphi_j\times\tilde{\varphi}_j).$$

Definition 2.9 (the Sato-Tate measure and the Plancherel measure). We define the Sato-Tate measure for GL(2) as a measure on $\mathbb{R}$

$$d\mu_\infty^{(2)}(x)=\begin{cases}\frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,dx, & \text{if } |x|\le 2,\\ 0, & \text{otherwise.}\end{cases}$$

For a prime number p, we define the p-adic Plancherel measure for GL(2) as a measure on $\mathbb{R}$ given by

$$d\mu_p^{(2)}(x)=\begin{cases}\frac{(p+1)\sqrt{4-x^2}}{2\pi((p^{1/2}+p^{-1/2})^2-x^2)}\,dx, & \text{if } |x|\le 2,\\ 0, & \text{otherwise.}\end{cases}$$

Both measures are probability measure and supported on $[-2,2]$. Obviously we observe that

$$\lim_{p\to\infty}\mu_p^{(2)}=\mu_\infty^{(2)}.$$

Theorem 2.10 (Sarnak). *Let p be a prime number. For $j=1,2,3,...$, let $a_j(p)$ be the Hecke eigenvalue of the Hecke-Maass form φ_j as in Definition 2.7. The sequence*

$$a_1(p),a_2(p),a_3(p),...$$

is equidistributed with respect to the p-adic Plancherel measure $\mu_p^{(2)}$. In another word, for any continuous test function $f:\mathbb{R}\to\mathbb{R}$ we have

$$\lim_{T\to\infty}\frac{\sum\limits_{\frac{1}{4}-\nu_j^2\le T}f(a_j(p))}{\sum\limits_{\frac{1}{4}-\nu_j^2\le T}1}=\int_{\mathbb{R}}f\,d\mu_p^{(2)}.$$

Proof. See [Sarnak, 1987]. It is an application of the Selberg trace formula. □

Theorem 2.11 (Bruggeman). *Let p be a prime number. For $j = 1, 2, 3, ...$, let $a_j(p)$ be the Hecke eigenvalue of the Hecke-Maass form φ_j as in Definition 2.7. Define a family of test functions*

$$h_T(\nu) = e^{-\frac{\frac{1}{4}-\nu^2}{T}}$$

for $\nu \in \mathbb{C}$ and $T \gg 1$. For any continuous function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{T\to\infty} \frac{\sum_j f(a_j(p))h_T(\nu_j)}{\sum_j h_T(\nu_j)} = \int_{\mathbb{R}} f \, d\mu_p^{(2)}.$$

Proof. See [Bruggeman, 1978]. It is an application of the Selberg trace formula. □

Remark 2.12. The test function $h_T(\nu) = e^{-\frac{\frac{1}{4}-\nu^2}{T}}$ is essentially supported on $\{\nu : \frac{1}{4} - \nu^2 \leq T\}$. It acts as a characteristic function for the set $\{\nu : \frac{1}{4} - \nu^2 \leq T\}$ and it essentially counts the Hecke-Maass forms with eigenvalues no greater than T. Additionally we have the Weyl law

$$\sum_j h_T(\nu_j) \sim \frac{T}{12}.$$

Theorem 2.13 (Bruggeman). *Let p be a prime number. For $j = 1, 2, 3, ...$, let $a_j(p)$ be the Hecke eigenvalue of the Hecke-Maass form φ_j as in Definition 2.7. Let $\mathcal{L}_j$ be as in Definiton 2.8. For a continuous function $f : \mathbb{R} \to \mathbb{R}$, we have*

$$\lim_{T\to\infty} \frac{\sum_j f(a_j(p))\frac{h_T(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_T(\nu^{(j)})}{\mathcal{L}_j}} = \int_{\mathbb{R}} f \, d\mu_\infty^{(2)},$$

where h_T is the same as in Theorem 2.11. Additionally we have the "Weyl law"

$$\sum_j \frac{h_T(\nu_j)}{\mathcal{L}_j} \sim \frac{T}{4\pi}.$$

Proof. See [Bruggeman, 1978]. It is an application of the Kuznetsov trace formula. □

Remark 2.14. Theorem 2.10 and Theorem 2.11 are essentially the same. The later does not involve the notion of equidistributed sequences. We refer to Definition 4.5 for the definition of equidistributed sequences.

Remark 2.15. The contrasting difference between the Sato-Tate measure $\mu_\infty^{(2)}$ in Theorem 2.13 and the p-adic Plancherel measure $\mu_p^{(2)}$ in Theorem 2.10 and Theorem 2.11 indicates that the number

$$\mathcal{L}_j = \operatorname*{Res}_{s=1} L(s, \varphi_j \times \tilde{\varphi}_j)$$

which is attached to each Hecke-Maass form φ_j plays a key role. We can view Theorem 2.13 as a weighted version of the equidistribution problem. The sequence $\{a_j(p)\}$ is equidistributed with respect to the Sato-Tate measure $\mu_\infty^{(2)}$, if each $a_j(p)$ is given a weight $\frac{1}{\mathcal{L}_j}$. On the other hand, as in Theorem 2.10 and Theorem 2.11, the sequence $a_j(p)$ is equidistributed with respect to $\mu_p^{(2)}$, if each $a_j(p)$ given a uniform weight. The same thing happens in the papers [Serre, 1997], [Li, 2004], [Gun-Murty-Rath, 2008] and [Conrey-Duke-Farmer, 1997] in the case of holomorphic modular forms.

Chapter 3

Hecke-Maass Froms for $\mathrm{SL}(3,\mathbb{Z})$

In this chapter, we will summarize the definitions and some of the most important theorems of Maass forms for $\mathrm{SL}(3,\mathbb{Z})$. The main reference for this chapter is [Goldfeld, 2006].

Definition 3.1 (Generalized upper half plane). The generalized upper half plane $\mathfrak{H}^3$ is the set of all matrices $z = x \cdot y$ with

$$x = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, x_3 \in \mathbb{R}$ and $y_1, y_2 > 0$.

Remark 3.2. By the Iwasawa decomposition, we have

$$\mathrm{GL}(3,\mathbb{R}) = \mathfrak{H}^3 \cdot \mathrm{O}(3,\mathbb{R}) \cdot \mathbb{R}^\times \quad \text{and} \quad \mathfrak{H}^3 \cong \mathrm{GL}(3,\mathbb{R})/(\mathrm{O}(3,\mathbb{R}) \cdot \mathbb{R}^\times).$$

Definition 3.3. Let $\mathfrak{gl}(3,\mathbb{R})$ be the Lie algebra of $\mathrm{GL}(3,\mathbb{R})$. It consists of 3×3 real matrices with the Lie bracket give by

$$[\alpha, \beta] = \alpha \cdot \beta - \beta \cdot \alpha$$

for all $\alpha, \beta \in \mathfrak{gl}(3,\mathbb{R})$.

Definition 3.4. For $\alpha \in \mathfrak{gl}(3,\mathbb{R})$ and a smooth function $\phi : \mathrm{GL}(3,\mathbb{R}) \to \mathbb{C}$, we define

$$D_\alpha \phi(g) = \frac{\partial}{\partial t} \phi(g \exp(t\alpha))\Big|_{t=0}.$$

Definition 3.5 (Casimir operator). Let $E_{i,j} \in \mathfrak{gl}(3,\mathbb{C})$ be the matrix with 1 at the $(i,j)^{\text{th}}$ entry and 0 elsewhere, for $1 \le i,j \le 3$. We define three operators on smooth functions on $\mathrm{GL}(3,\mathbb{R})$

$$\Delta_1 = \sum_{i_1=1}^{3} D_{E_{i_1,i_1}}, \quad \Delta_2 = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3} D_{E_{i_1,i_2}} D_{E_{i_2,i_1}}, \quad \Delta_3 = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{i_2=1}^{3} D_{E_{i_1,i_2}} D_{E_{i_2,i_3}} D_{E_{i_3,i_1}}.$$

These three operators generate a commutative ring

$$\mathfrak{D}^3 = \mathbb{C}[\Delta_1, \Delta_2, \Delta_3],$$

and any element in $\mathfrak{D}^3$ is called a Casimir operator. Explicitly, we have

$$\Delta_2 = y_1^2 \frac{\partial}{\partial y_1^2} + y_2^2 \frac{\partial}{\partial y_2^2} - y_1 y_2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1^2(x_2^2 + y_2^2)\frac{\partial^2}{\partial x_3^2} + y_1^2 \frac{\partial}{\partial x_1^2} + y_2^2 \frac{\partial}{\partial x_2^2} + 2y_1^2 x_2 \frac{\partial^2}{\partial x_1 \partial x_3}$$

and we call $-\Delta_2$ the Laplace operator.

Lemma 3.6. *Any $\Delta \in \mathfrak{D}^3$ is well-defined for smooth functions on $\mathrm{SL}(3,\mathbb{Z}) \setminus \mathfrak{H}^3$, i.e., for*

$$\phi : \mathrm{SL}(3,\mathbb{Z}) \setminus \mathrm{GL}(3,\mathbb{R})/(\mathrm{O}(3,\mathbb{R}) \cdot \mathbb{R}^\times) \to \mathbb{C},$$

we have

$$(\Delta\phi)(\gamma \cdot g \cdot k \cdot \delta) = \Delta\phi(g)$$

for all $g \in \mathrm{GL}(3,\mathbb{R})$, $\gamma \in \mathrm{SL}(3,\mathbb{Z})$, $\delta \in \mathbb{R}^\times$ and $k \in \mathrm{O}(3,\mathbb{R})$.

Proof. This is Proposition 2.3.1 of [Goldfeld, 2006]. □

Definition 3.7. For $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, we define a function on $\mathfrak{H}^3$,

$$I_\nu\left(\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = y_1^{1+\nu_1+2\nu_2} y_2^{1+\nu_2+2\nu_1}.$$

Lemma 3.8. *The function I_ν is an eigenfunction of Δ for any $\Delta \in \mathfrak{D}^3$.*

Proof. See Equation 6.1.1 of [Goldfeld, 2006]. □

Definition 3.9. For any $\Delta \in \mathfrak{D}^3$, we define $\lambda_\nu(\Delta)$ to be the eigenvalue in Lemma 3.8, i.e.,

$$\Delta I_\nu = \lambda_\nu(\Delta) I_\nu.$$

Definition 3.10 (**Upper triangular subgroups**). We define three upper triangular subgroups of $\mathrm{GL}(3,\mathbb{R})$:

$$U_{1,1,1} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad U_{1,2} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad U_{2,1} = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Definition 3.11 (**Inner product and L^2-norm**). Let ϕ and ϕ' be two functions on $\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3$. We define their inner product

$$< \phi, \phi' > = \int_{\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3} \phi(z)\overline{\phi'(z)}\ d^*z,$$

where $d^*z = dx_1dx_2dx_3\dfrac{dy_1dy_2}{(y_1y_2)^3}$. We define the L^2-norm

$$\| \phi \|_2 = \sqrt{\int_{\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3} |\phi(z)|^2\ d^*z}.$$

Definition 3.12. We define $\mathcal{L}^2(\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3)$ to be $\{\phi : \mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3 \to \mathbb{C}\ : \| \phi \|_2 < \infty\}$. We define $\mathcal{L}^2_{\mathrm{cusp}}(\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3)$ to be all $\phi \in \mathcal{L}^2(\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3)$ such that ϕ is cuspidal, i.e.,

$$\int_{\mathrm{SL}(3,\mathbb{Z})\cap U\backslash U} \phi(uz)\ du = 0, \text{ for } U = U_{1,1,1}, U_{1,2}, U_{2,1}.$$

Definition 3.13 (**Maass form**). Let $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. A Maass form of type ν for $\mathrm{SL}(3,\mathbb{Z})$ is a smooth function $\phi \in \mathcal{L}^2(\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3)$ which satisfies

- $\phi(\gamma z) = \phi(z)$ for any $z \in \mathfrak{H}^3$ and any $\gamma \in \mathrm{SL}(3,\mathbb{Z})$,
- for any Casimir operator $\Delta \in \mathfrak{D}^3$, we have $\Delta\phi(z) = \lambda_\nu(\Delta)\phi(z)$,
- ϕ is cuspidal or $\phi \in \mathcal{L}^2_{\mathrm{cusp}}(\mathrm{SL}(3,\mathbb{Z})\backslash\mathfrak{H}^3)$.

Definition 3.14 (**Jacquet's Whittaker function**). Let m_1 and m_2 be two integers. We define Jacquet's Whittaker function

$$W_{\mathrm{Jacquet}}(z; \nu, \psi_{m_1,m_2}) = \int_{U_3(\mathbb{R})} I_\nu(\omega_3 \cdot u \cdot z)\overline{\psi_{m_1,m_2}(u)}\ d^*u,$$

where $\omega_3 = \left(\begin{smallmatrix} & & 1 \\ & 1 & \\ 1 & & \end{smallmatrix}\right)$, $d^*u = du_1du_2du_3$ for $u = \left(\begin{smallmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $\psi_{m_1,m_2}\left(\left(\begin{smallmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = e^{2\pi i(m_1u_1+m_2u_2)}$.

Theorem 3.15 (**Fourier-Whittaker expansion**). *A Maass form ϕ of type ν for $\mathrm{SL}(3,\mathbb{Z})$ has Fourier-Whittaker expansion*

$$\phi(z) = \sum_{\gamma \in U_2(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1,m_2)}{|m_1 m_2|} W_{\text{Jacquet}}\left(\begin{pmatrix} |m_1 m_2| & & \\ & m_1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z; \nu, \psi_{1,\frac{m_2}{|m_2|}}\right),$$

where $A(m_1,m_2) \in \mathbb{C}$ is the $(m_1,m_2)^{th}$ Fourier coefficient and $U_2 = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$.

Proof. This is Theorem 5.3.2 and Equation 6.2.1 of [Goldfeld, 2006]. □

Definition 3.16 (**Hecke operators**). For $n = 1,2,3,...$, we define the Hecke operator T_n acting on $\mathcal{L}^2_{\text{cusp}}(\mathrm{SL}(3,\mathbb{Z})\backslash \mathfrak{H}^3)$ by

$$T_n\phi(z) = \frac{1}{n} \sum_{\substack{abc=n \\ 0 \le c_1, c_2 < c \\ 0 \le b_1 < b}} \phi\left(\begin{pmatrix} a & b_1 & c_1 \\ 0 & b & c_2 \\ 0 & 0 & c \end{pmatrix} z\right).$$

Theorem 3.17 (**Hecke relations**). *Let ϕ be a Maass form for $\mathrm{SL}(3,\mathbb{Z})$ as in Definition 3.13. Assume that ϕ is an eigenfunction of every Hecke operator T_n. If $A(1,1) = 0$, then ϕ vanishes identically. Assume $\phi \neq 0$ and it is normalized so that $A(1,1) = 1$. Then $T_n\phi = A(n,1)\phi$. Furthermore, we have the following multiplicativity relations*

$$A(n,1)A(m_1,m_2) = \sum_{\substack{b|m_1, a|m_2 \\ abc=n}} A(\frac{m_1 c}{b}, \frac{m_2 b}{a}),$$

$$A(1,n)A(m_1,m_2) = \sum_{\substack{a|m_1, b|m_2 \\ abc=n}} A(\frac{m_1 b}{a}, \frac{m_2 c}{b}),$$

and

$$A(m_1,m_2) = \overline{A(m_2,m_1)}.$$

Proof. This is Theorem 6.4.11 of [Goldfeld, 2006]. □

Remark 3.18 (**Hecke-Maass forms**). The Hecke operators commute with any Casimir operator $\Delta \in \mathfrak{D}^3$. All Hecke operators commute. We can simultaneously diagonalize the space $\mathcal{L}^2_{\text{cusp}}(\mathrm{SL}(3,\mathbb{Z})\backslash \mathfrak{H}^3)$ by these operators. The space $\mathcal{L}^2_{\text{cusp}}(\mathrm{SL}(3,\mathbb{Z})\backslash \mathfrak{H}^3)$ has an orthogonal basis

$$\phi_1, \phi_2, \phi_3, ...$$

Each ϕ_j is a Maass form of type $\nu^{(j)} = (\nu_1^{(j)}, \nu_2^{(j)}) \in \mathbb{C}^2$ and an eigenfunction of all Hecke operators T_n. They are normalized so that the $(1,1)^{\text{th}}$ Fourier coefficient is 1. We arrange the order of ϕ_j's by their eigenvalue under the Laplace operator $-\Delta_2$ (Definition 3.5) so that

$$0 < \lambda_1(-\Delta_2) \leq \lambda_2(-\Delta_2) \leq \lambda_3(-\Delta_2) \leq ...,$$

where $-\Delta_2\phi_j = \lambda_j(-\Delta_2)\phi_j$. These ϕ_j's are called Hecke-Maass forms.

Definition 3.19. We define $A_j(m_1, m_2)$ to be the $(m_1, m_2)^{\text{th}}$ Fourier coefficient of ϕ_j. Since ϕ_j is normalized we have $A_j(1, 1) = 1$ for all j.

Theorem 3.20 (Weyl law). *We have the following asymptotic formula*

$$\#\{j : \lambda_j(-\Delta_2) < T\} \sim cT^{\frac{5}{2}},$$

as $T \to \infty$ for some constant $c > 0$.

Proof. See [Miller, 2001]. □

Definition 3.21. We define the convolution L-function for a Hecke-Maass form ϕ_j to be

$$L(s, \phi_j \times \tilde{\phi}_j) = \zeta(3s) \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{|A_j(m_1, m_2)|^2}{m_1^{2s} m_2^{s}},$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann-Zeta function. Let $\mathcal{L}_j$ be a number attached to each Hecke-Maass form ϕ_j which is given by

$$\mathcal{L}_j = \operatorname*{Res}_{s=1} L(s, \phi_j \times \tilde{\phi}_j).$$

Chapter 4

Sato-Tate Problem and Equidistribution Theorems

4.1 Sato-Tate Measure for GL(N)

Definition 4.1 (Pushforward measure). Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces in which Σ_i is a σ-algebra for X_i. Let $F : X_1 \to X_2$ by a measurable map and a measure $\mu : \Sigma_1 \to [0, +\infty)$. The pushforward measure of μ by F is defined to be the measure $F_*(\mu) : \Sigma_2 \to [0, +\infty)$ given by

$$F_*(\mu)(B) = \mu(f^{-1}(B)) \text{ for any } B \in \Sigma_2.$$

The change of variables formula is given by

$$\int_{X_2} g \, dF_*(\mu) = \int_{X_1} (g \circ F) \, d\mu \tag{4.1}$$

for any integrable function g on (X_2, Σ_2) with respect to the measure $F_*(\mu)$.

Definition 4.2 (Sato-Tate measure for GL(N)). The Sato-Tate measure for GL(2) is a measure on $\mathbb{R}$ given by

$$d\mu_\infty^{(2)}(x) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}} \, dx, & \text{when } |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

When $N \geq 3$, the Sato-Tate measure for GL(N) is a measure on $\mathbb{C}$ given by

$$d\mu_\infty^{(N)}(z) = \begin{cases} d(\mathrm{Tr}_*(\omega))(z), & \text{when } |z| \le N, \\ 0, & \text{otherwise,} \end{cases}$$

where ω is the unique normalized Haar measure on $\mathrm{SU}(N)$ and the map

$$\mathrm{Tr} : \mathrm{SU}(N) \to \{z \in \mathbb{C} : |z| \le N\}$$

takes the trace of each element of $\mathrm{SU}(N)$ and $\mathrm{Tr}_*(\omega)$ is the pushforward measure of ω by Tr.

Remark 4.3. The Sato-Tate measure for $\mathrm{GL}(N)$ is supported inside $\{z \in \mathbb{C} : |z| \le N\}$, but we shall note that the support of this measure does not fill $\{z \in \mathbb{C} : |z| \le N\}$. Its support is the set

$$\left\{ \sum_{i=1}^{N} e^{i\theta_i} : \sum_{i=1}^{N} \theta_i = 0,\ \theta_i \in \mathbb{R},\ i = 1, 2, ..., N \right\}.$$

It is the area inside the curve

$$\left\{ (N-1)e^{-i\theta} - e^{iN\theta} : \theta \in \mathbb{R} \right\}.$$

In particular, for $N = 3$, the measure $\mu_\infty^{(3)}$ is supported on the area inside the delta-shaped curve

$$\{2e^{i\theta} + e^{-2i\theta} : \theta \in \mathbb{R}\}.$$

Remark 4.4. We shall note that the Sato-Tate measure for $\mathrm{GL}(2)$, or, the semi-circle measure

$$d\mu_\infty^{(2)}(x) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}\, dx, & \text{when } |x| \le 2, \\ 0, & \text{otherwise,} \end{cases}$$

is also a pushforward measure from a special unitary group. The unitary group

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \cos\theta e^{i\alpha} & -\sin\theta e^{-i\beta} \\ \sin\theta e^{i\beta} & \cos\theta e^{-i\alpha} \end{pmatrix} : \theta \in [0, \frac{\pi}{2}],\ \alpha \in [0, 2\pi),\ \beta \in [0, 2\pi) \right\}$$

has the unique normalized Haar measure

$$\frac{1}{2\pi^2} \cos\theta \sin\theta \, d\theta \, d\alpha \, d\beta.$$

The trace map $\mathrm{Tr} : \mathrm{SU}(2) \to [-2, 2]$ maps $\begin{pmatrix} \cos\theta e^{i\alpha} & -\sin\theta e^{-i\beta} \\ \sin\theta e^{i\beta} & \cos\theta e^{-i\alpha} \end{pmatrix}$ to $2\cos\theta\cos\alpha$. The limit

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \underset{2\cos\theta\cos\alpha \in [x, x+\Delta x]}{\int_0^{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}}} \frac{1}{2\pi^2} \cos\theta \sin\theta \, d\theta \, d\alpha \, d\beta \tag{4.2}$$

gives us the pushforward measure of the Haar measure by the trace map. Computing the limit in Equation 4.2, we get

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{1}{\Delta x} \underset{2\cos\theta\cos\alpha \in [x, x+\Delta x]}{\int_0^{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}}} \frac{1}{2\pi^2} \cos\theta \sin\theta \; d\theta \; d\alpha \; d\beta & \\
&= \frac{1}{2\pi^2} \int_0^{2\pi} d\beta \int_0^{\arccos\frac{|x|}{2}} 2\sin\theta\cos\theta \left| \frac{d}{dx} \arccos \frac{x}{2\cos\theta} \right| d\theta \\
&= \frac{1}{\pi} \int_0^{\arccos\frac{|x|}{2}} \frac{\sin\theta d\theta}{\sqrt{1 - (\frac{x}{2\cos\theta})^2}} \\
&= \frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}},
\end{aligned}$$

which is the same as in Definition 4.2.

4.2 Equidistribution

There are multiple definitions for equidistributed sequence in different settings. The following definition is taken from [Gun-Murty-Rath, 2008].

Definition 4.5 (**Equidistributed sequence**)**.** Let X be a Hausdorff space with a regular normalized Borel measure μ. We denote by $R(X)$ the space of continuous functions of compact support on X. A sequence $\{x_n\}$ of elements in X is called equidistributed with respect to the measure μ if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f \; d\mu$$

for all $f \in R(X)$.

Remark 4.6. In the cases we are interested in, the space X is $\mathbb{C}$, or $\mathbb{R}$. The Borel measure μ is the Sato-Tate measure given in Definition 4.2.

Chapter 5

Main Theorem

The recent paper [Goldfeld-Kontorovich, 2013] establishes an orthogonality relation for Fourier coefficients of Hecke-Maass forms for $\mathrm{SL}(3,\mathbb{Z})$, via the Kuznetsov trace formula.

Definition 5.1 (**Family of test functions**)**.** Let $\nu_3 = \nu_1 + \nu_2$. For $T \gg 1$ and fixed $R \geq 10$, we define a test function on $\mathbb{C}^2$

$$h_{T,R}(\nu) = e^{\frac{6(\nu_1^2+\nu_2^2+\nu_1\nu_2)}{T^2}} \frac{\left(\prod\limits_{1\leq i\leq 3} \Gamma(\frac{2+R+3\nu_i}{4})\Gamma(\frac{2+R-3\nu_i}{4})\right)^2}{\prod\limits_{1\leq i\leq 3} \Gamma(\frac{1+3\nu_i}{2})\Gamma(\frac{1-3\nu_i}{2})}.$$

Remark 5.2. The family of test functions $\{h_{T,R} : T \gg 1\}$ appears in the Kuznetsov trace formula and should not be confused with the test function f in the definition of equidistributed sequences.

Theorem 5.3 (**Goldfeld-Kontorovich orthogonality relation**)**.** *Assume the Ramanujan conjecture at the infinite place, i.e., $\nu^{(j)} \in (i\mathbb{R})^2$ for all j. For four positive integers m_1, m_2, n_1, n_2, and $T \gg 1$, we have*

$$\sum_j A_j(m_1,m_2)\overline{A_j(n_1,n_2)}\frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} = \begin{cases} \sum\limits_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} + \mathcal{O}_{R,\epsilon}(T^{3+3R+\epsilon}(m_1m_2n_1n_2)^2), & \text{if } {m_1=n_1 \atop m_2=n_2}, \\ \mathcal{O}_{R,\epsilon}(T^{3+3R+\epsilon}(m_1m_2n_1n_2)^2), & \text{otherwise.} \end{cases}$$

For this family of test functions $\{h_{T,R} : T \gg 1\}$ we have the "Weyl law"

$$\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} \sim cT^{5+3R}$$

for some constant $c > 0$.

Proof. See [Goldfeld-Kontorovich, 2013]. □

Remark 5.4. A recent paper [Blomer] also establishes a similar orthogonality relation between Fourier coefficients of Hecke-Maass forms for $\mathrm{SL}(3,\mathbb{Z})$. It may also be used to prove a similar version of our main theorem.

Theorem 5.5 (**Main theorem**)**.** *For any continuous test function $f:\mathbb{C}\to\mathbb{C}$, we have*

$$\lim_{T\to\infty}\frac{\sum\limits_{j=1}^{\infty} f(A_j(p,1))\frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum\limits_{j=1}^{\infty}\frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}=\int_{\mathbb{C}} f\ d\mu_\infty^{(3)}.$$

Remark 5.6. We can interpret our main theorem in the context of the Sato-Tate conjecture and equidistribution. The Sato-Tate conjecture for GL(3) states that if ϕ_j is not a symmetric square lift from GL(2), the sequence

$$A_j(2,1),\ A_j(3,1),\ A_j(5,1),\quad ...,\quad A_j(p,1),\ ...,$$

is equidistributed with respect to the Sato-Tate measure $\mu_\infty^{(3)}$ for a fixed Hecke-Maass form ϕ_j. Considering the same problem in vertical perspective, we can fix the prime number p and vary ϕ_j. One can investigate the distribution of the sequence

$$A_1(p,1),\ A_2(p,1),\ A_3(p,1),\ ... \tag{5.1}$$

as in Theorem 2.10 and Theorem 2.11. But no such theorem has been proved yet. We conjecture that this sequence should be equidistributed with respect to a properly defined measure on $\mathbb{C}$ which depends on p. Our main theorem is a weighted version of this conjecture and essentially claims that Sequence 5.1 is equidistributed with respect to the Sato-Tate measure $\mu_\infty^{(3)}$, if each $A_j(p,1)$ is given a weight

$$\frac{1}{\mathcal{L}_j}=\frac{1}{\operatorname*{Res}\limits_{s=1} L(s,\phi_j\times\tilde{\phi}_j)}$$

Remark 5.7. We shall note that the Sato-Tate measure $\mu_\infty^{(3)}$ is supported inside the delta-shaped curve

$$\{2e^{i\theta}+e^{-2i\theta}:\theta\in\mathbb{R}\}$$

and this is equivalent to the Ramanujan conjecture at a finite prime p since $A_j(p,1)=\overline{A_j(1,p)}$.

Chapter 6

Hecke Relations and Random Walks in the First Quadrant

Let p be a fixed prime number. Let $A(m_1, m_2)$ be the $(m_1, m_2)^{\text{th}}$ Fourier coefficient of a Maass form ϕ as in Theorem 3.15. This chapter will heavily rely on the following three Hecke relations from Theorem 3.17:

$$A(n,1)A(m_1,m_2) = \sum_{\substack{b|m_1, a|m_2 \\ abc=n}} A\left(\frac{m_1 c}{b}, \frac{m_2 b}{a}\right), \tag{6.1}$$

$$A(1,n)A(m_1,m_2) = \sum_{\substack{a|m_1, b|m_2 \\ abc=n}} A\left(\frac{m_1 b}{a}, \frac{m_2 c}{b}\right) \tag{6.2}$$

and

$$A(m_1,m_2) = \overline{A(m_2,m_1)}. \tag{6.3}$$

We intend to use the above three relations to express $A(p,1)^{k_1}\overline{A(p,1)}^{k_2}$ as a linear combination of the $A(p^{i_1}, p^{i_2})$'s.

For example, by an easy application of the above relations, we obtain the following identities:

- $A(p,1)^2 = A(p^2,1) + A(1,p)$,
- $A(p,1)\overline{A(p,1)} = A(p,1)A(1,p) = A(1,1) + A(p,p)$,
- $\overline{A(p,1)}^2 = A(1,p)^2 = A(1,p^2) + A(p,1)$,
- $A(p,1)^3 = A(p,1)(A(p^2,1) + A(1,p)) = A(1,1) + 2A(p,p) + A(p^3,1)$,

- $A(p,1)^2\overline{A(p,1)} = A(p,1)(A(1,1)+A(p,p)) = 2A(p,1)+A(1,p^2)+A(p^2,p)$,
- $A(p,1)\overline{A(p,1)}^2 = 2A(1,p)+A(p^2,1)+A(p,p^2)$,
- $\overline{A(p,1)}^3 = A(1,1)+2A(p,p)+A(1,p^3)$,
- $A(p,1)^4 = A(p,1)(A(1,1)+2A(p,p)+A(p^3,1)) = 3A(p,1)+2A(1,p^2)+3A(p^2,p)+A(p^4,1)$.

The algorithm to express $A(p,1)^{k_1}\overline{A(p,1)}^{k_2}$ as a linear combination of $A(p^{i_1},p^{i_2})$'s is inductive. Assume that we have expressed $A(p,1)^{k_1-1}\overline{A(p,1)}^{k_2}$ as a linear combination of $A(p^{i_1},p^{i_2})$'s, i.e.,

$$A(p,1)^{k_1-1}\overline{A(p,1)}^{k_2} = \sum_{j_1,j_2} b_{j_1,j_2}A(p^{j_1},p^{j_2})$$

for some $b_{j_1,j_2}\in\mathbb{C}$. Multiply $A(p,1)$ on both sides and we get

$$A(p,1)^{k_1}\overline{A(p,1)}^{k_2} = \sum_{j_1,j_2} b_{j_1,j_2}A(p,1)A(p^{j_1},p^{j_2}).$$

For each $A(p,1)A(p^{j_1},p^{j_2})$ on the right hand side, we apply Equation 6.1 and it can be expressed as a sum of $A(p^{i_1},p^{i_2})$'s.

Similarly assume that we have expressed $A(p,1)^{k_1}\overline{A(p,1)}^{k_2-1}$ as a linear combination of $A(p^{i_1},p^{i_2})$'s, i.e.,

$$A(p,1)^{k_1}\overline{A(p,1)}^{k_2-1} = \sum_{j_1,j_2} b'_{j_1,j_2}A(p^{j_1},p^{j_2})$$

for some $b'_{j_1,j_2}\in\mathbb{C}$. Multiply $A(1,p)=\overline{A(p,1)}$ (Equation 6.3) on both sides and we get

$$A(p,1)^{k_1}\overline{A(p,1)}^{k_2} = \sum_{j_1,j_2} b'_{j_1,j_2}A(1,p)A(p^{j_1},p^{j_2}).$$

For each $A(1,p)A(p^{j_1},p^{j_2})$ on the right hand side, we apply Equation 6.2 and it can be expressed as a sum of $A(p^{i_1},p^{i_2})$'s.

This inductive algorithm allows us to express the coefficient before $A(p^{i_1},p^{i_2})$ as the number of random walks of certain type in the integer points of the first quadrant.

Definition 6.1. We define a random walk in the set

$$\mathcal{C}^{(3)} = \{(i_1,i_2)\in\mathbb{Z}^2 : i_1\geq 0, i_2\geq 0\},$$

which is the first quadrant of $\mathbb{Z}^2$ and the allowable steps are from

$$\mathcal{S}_1^{(3)} = \{e_1,\ e_2 - e_1,\ -e_2\},$$

where $e_1 =< 1, 0 >$ and $e_2 =< 0, 1 >$.

Remark 6.2. We consider random walks in $\mathcal{C}^{(3)}$ with allowable steps of $\mathcal{S}_1^{(3)}$. Because of the restriction of $\mathcal{C}^{(3)}$, any walk at the the boundary of $\mathcal{C}^{(3)}$ can take only one or two allowable steps from $\mathcal{S}_1^{(3)}$ as its next step. More specifically, a walk at the origin $(0, 0)$ can only take e_1 as its next step; a walk at a point of $\{(i_1, i_2) \in \mathbb{Z}^2 : i_1 > 0, i_2 = 0\}$ can only take e_1 or $e_2 - e_1$ as its next step; a walk at a point of $\{(i_1, i_2) \in \mathbb{Z}^2 : i_1 = 0, i_2 > 0\}$ can only take $-e_2$ or $e_2 - e_1$ as its next step.

Definition 6.3. Let $\lambda \in \mathcal{C}^{(3)}$. Let $q_\lambda^{(k)}$ be the number of walks from the origin $(0, 0)$ to λ of k steps which stay in $\mathcal{C}^{(3)}$.

Lemma 6.4. *We have*

$$q_\lambda^{(k+1)} = \begin{cases} q_{\lambda+e_2}^{(k)} + q_{\lambda-e_1}^{(k)} + q_{\lambda+e_1-e_2}^{(k)}, & \text{if } \lambda \in \{(i_1, i_2) \in \mathcal{C}^{(3)} : i_1 > 0, i_2 > 0\}, \\ q_{\lambda+e_2}^{(k)} + q_{\lambda-e_1}^{(k)}, & \text{if } \lambda \in \{(i_1, i_2) \in \mathcal{C}^{(3)} : i_1 > 0, i_2 = 0\}, \\ q_{\lambda+e_2}^{(k)} + q_{\lambda+e_1-e_2}^{(k)}, & \text{if } \lambda \in \{(i_1, i_2) \in \mathcal{C}^{(3)} : i_1 = 0, i_2 > 0\}, \\ q_{\lambda+e_2}^{(k)}, & \text{if } \lambda = (0, 0). \end{cases}$$

Proof. Any walk of $(k+1)$ steps from $(0, 0)$ to λ must be at either $\lambda + e_2$, $\lambda - e_1$ or $\lambda + e_1 - e_2$ after completing the k^{th} steps, if they are in $\mathcal{C}^{(3)}$. Thus, the number of walks of $(k + 1)$ steps from $(0, 0)$ to λ is the sum of the numbers of walks of k steps from $(0, 0)$ to $\lambda + e_2$, $\lambda - e_1$ and $\lambda + e_1 - e_2$, if any of them is in $\mathcal{C}^{(3)}$. □

Theorem 6.5. *We can express $A(p, 1)^k$ as a sum of $A(p^{i_1}, p^{i_2})$'s and the coefficient before $A(p^{i_1}, p^{i_2})$ is $q_{(i_1,i_2)}^{(k)}$, i.e.,*

$$A(p, 1)^k = \sum_{i_1, i_2 \geq 0} q_{(i_1,i_2)}^{(k)} A(p^{i_1}, p^{i_2}).$$

Proof. We shall note that the right hand side is a finite sum since for all except a finite number of (i_1, i_2)'s such that $q_{(i_1,i_2)}^{(k)} = 0$. For $k = 0$ and 1, this theorem is obvious. Assume this theorem is true for some $k > 0$, i.e.,

$$A(p, 1)^k = \sum_{i_1, i_2 \geq 0} q_{(i_1,i_2)}^{(k)} A(p^{i_1}, p^{i_2}).$$

Multiply $A(p,1)$ on both sides and we have

$$\begin{aligned} A(p,1)^{k+1} &= \sum_{i_1,i_2\geq 0} q^{(k)}_{(i_1,i_2)} A(p,1)A(p^{i_1},p^{i_2}) \\ &= \sum_{i_1,i_2>0} q^{(k)}_{(i_1,i_2)} (A(p^{i_1+1},p^{i_2}) + A(p^{i_1-1},p^{i_2+1}) + A(p^{i_1},p^{i_2-1})) \\ &\quad + \sum_{i_1=0,i_2>0} q^{(k)}_{(i_1,i_2)} (A(p^{i_1+1},p^{i_2}) + A(p^{i_1},p^{i_2-1})) \\ &\quad + \sum_{i_1>0,i_2=0} q^{(k)}_{(i_1,i_2)} (A(p^{i_1+1},p^{i_2}) + A(p^{i_1-1},p^{i_2+1})) + q^{(k)}_{(0,0)} A(p,1) \\ &= \sum_{i_1,i_2\geq 0} q^{(k+1)}_{(i_1,i_2)} A(p^{i_1},p^{i_2}). \end{aligned}$$

The last equality follows from Lemma 6.4. By induction, we complete the proof. □

Definition 6.6. We define another set of allowable steps

$$\mathcal{S}_2^{(3)} = \{e_2,\ e_1 - e_2,\ -e_1\}.$$

Remark 6.7. We shall now consider a random walk of $(k_1 + k_2)$ steps, where the first k_1 steps are from $\mathcal{S}_1^{(3)}$ and the remaining k_2 steps are from $\mathcal{S}_2^{(3)}$.

Definition 6.8. Let $\lambda \in \mathcal{C}^{(3)}$ and let $q_\lambda^{(k_1,k_2)}$ be the number of walks from the origin $(0,0)$ to λ confined in $\mathcal{C}^{(3)}$ of $(k_1 + k_2)$ steps in which the first k_1 steps are from $\mathcal{S}_1^{(3)}$ and the remaining k_2 steps are from $\mathcal{S}_2^{(3)}$.

Lemma 6.9. *We have*

$$q_\lambda^{(k_1,k_2+1)} = \begin{cases} q_{\lambda+e_1}^{(k_1,k_2)} + q_{\lambda-e_2}^{(k_1,k_2)} + q_{\lambda+e_2-e_1}^{(k_1,k_2)}, & \text{if } \lambda \in \{(i_1,i_2) \in \mathcal{C}^{(3)} : i_1 > 0, i_2 > 0\}, \\ q_{\lambda+e_1}^{(k_1,k_2)} + q_{\lambda+e_2-e_1}^{(k_1,k_2)}, & \text{if } \lambda \in \{(i_1,i_2) \in \mathcal{C}^{(3)} : i_1 > 0, i_2 = 0\}, \\ q_{\lambda+e_1}^{(k_1,k_2)} + q_{\lambda-e_2}^{(k_1,k_2)}, & \text{if } \lambda \in \{(i_1,i_2) \in \mathcal{C}^{(3)} : i_1 = 0, i_2 > 0\}, \\ q_{\lambda+e_1}^{(k_1,k_2)}, & \text{if } \lambda = (0,0). \end{cases}$$

Proof. This proof is essentially the same of that of Lemma 6.4. □

Theorem 6.10. *We can express $A(p,1)^{k_1}\overline{A(p,1)}^{k_2}$ as a sum of $A(p^{i_1},p^{i_2})$'s and the coefficient before $A(p^{i_1},p^{i_2})$ is $q_{(i_1,i_2)}^{(k_1,k_2)}$, i.e.,*

$$A(p,1)^{k_1}\overline{A(p,1)}^{k_2} = \sum_{i_1,i_2\geq 0} q_{(i_1,i_2)}^{(k_1,k_2)} A(p^{i_1},p^{i_2}).$$

Proof. In a manner similar to Theorem 6.5, this proof is based on induction. If $k_2 = 0$, this theorem is reduced to Theorem 6.5. Assume this theorem is valid for $k_2 \geq 0$, i.e.,

$$A(p,1)^{k_1}\overline{A(p,1)}^{k_2} = \sum_{i_1,i_2\geq 0} q^{(k_1,k_2)}_{(i_1,i_2)} A(p^{i_1},p^{i_2}).$$

Multiply $A(1,p) = \overline{A(p,1)}$ (see Equation 6.3) on both sides of the above identity to obtain

$$\begin{aligned}
A(p,1)^{k_1}\overline{A(p,1)}^{k_2+1} &= \sum_{i_1,i_2\geq 0} q^{(k_1,k_2)}_{(i_1,i_2)} A(1,p)A(p^{i_1},p^{i_2}) \\
&= \sum_{i_1,i_2>0} q^{(k_1,k_2)}_{(i_1,i_2)} (A(p^{i_1},p^{i_2+1}) + A(p^{i_1+1},p^{i_2-1}) + A(p^{i_1-1},p^{i_2})) \\
&\quad + \sum_{i=0,i_2>0} q^{(k_1,k_2)}_{(i_1,i_2)} (A(p^{i_1},p^{i_2+1}) + A(p^{i_1+1},p^{i_2-1})) \\
&\quad + \sum_{i>0,i_2=0} q^{(k_1,k_2)}_{(i_1,i_2)} (A(p^{i_1},p^{i_2+1}) + A(p^{i_1-1},p^{i_2})) + q^{(k_1,k_2)}_{(0,0)} A(1,p) \\
&= \sum_{i_1,i_2\geq 0} q^{(k_1,k_2+1)}_{(i_1,i_2)} A(p^{i_1},p^{i_2}).
\end{aligned}$$

The last equality follows from Lemma 6.9. By induction, we complete the proof. ☐

Chapter 7

Lie Theory and Random Walks in a Weyl Chamber

In this chapter we will summarize some basic concepts and results of Lie theory. Our main goal is to relate the following integral

$$\int_{\mathbb{C}} z^{k_1}\overline{z}^{k_2}\ d\mu_{\infty}^{(N)}$$

to some random walks in a Weyl chamber, via the decomposition of tensor power representations. A number of text books have treated the theory of finite-dimensional complex linear representations of Lie groups and Lie algebras, such as [Bump, 2004] and [Hall, 2003]. For random walks in a Weyl chamber, our main references are [Gessel-Zeilberger, 1992] and [Grabiner-Magyar, 1993].

Definition 7.1. Let (π, V) be a representation of a Lie group G, where V is finite-dimensional complex vector space and $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism. By abuse of language, we will also refer to (π, V) as V, on many occasions. In our context a representation is always finite-dimensional and linear over $\mathbb{C}$.

Definition 7.2. We define the character $\chi : G \to \mathbb{C}$ of a representation (π, V) by taking $\chi(g)$ to be the trace of the linear map $\pi(g) : V \to V$.

Proposition 7.3. *Let G be a compact Lie group and dg its unique normalized Haar measure such that $\int_G\ dg = 1$. If (π, V) is a representation of G and χ its character, then we have*

$$\int_G \chi(g)\ dg = dim_{\mathbb{C}}\left(V^G\right)$$

where $V^G = \{v \in V : g \cdot v = v \text{ for any } g \in G\}$.

Proof. This is Proposition 2.8 of [Bump, 2004]. □

Remark 7.4. We have

$$\int_G \chi(g)\, dg$$

is the multiplicity of the trivial representation in the decomposition of π into a direct sum of irreducible representations.

Definition 7.5. Let $\iota : \mathrm{SU}(N) \hookrightarrow \mathrm{GL}(N, \mathbb{C})$ be the standard inclusion map. This inclusion map defines a representation (ι, H), where $H = \mathbb{C}^N$ is an N-dimensional complex vector space. The Lie group $\mathrm{SU}(N)$ acts on H by matrix multiplication on the left if we assume $H = \mathbb{C}^N$ as column vectors. Let $(\bar{\iota}, \overline{H})$ be the contragredient representation of (ι, H).

Theorem 7.6. *We have*

$$\int_{\mathbb{C}} z^{k_1} \overline{z}^{k_2}\, d\mu_\infty^{(N)} = dim_{\mathbb{C}} \left((H^{\otimes k_1} \otimes \overline{H}^{\otimes k_2})^G \right),$$

which is the multiplicity of the trivial representation in $H^{\otimes k_1} \otimes \overline{H}^{\otimes k_2}$.

Proof. Obviously the trace of the representation (ι, H) is Tr in Definition 4.2. For two representations V_1 and V_2 with characters χ_1 and χ_2, the character of the tensor product representation $V_1 \otimes V_2$ is $\chi_1 \chi_2$. For a representation V with character χ, the character of its contragredient representation is $\overline{\chi}$. Inductively, we prove that the character of the representation $H^{\otimes k_1} \otimes \overline{H}^{\otimes k_2}$ is $\mathrm{Tr}^{k_1} \overline{\mathrm{Tr}^{k_2}}$. Let ω be the unique normalized Haar measure on $\mathrm{SU}(N)$. Recalling Definition 4.2, we have

$$\int_{\mathbb{C}} z^{k_1} \overline{z}^{k_2}\, d\mu_\infty^{(N)} = \int_{\mathrm{SU}(N)} \mathrm{Tr}^{k_1} \overline{\mathrm{Tr}^{k_2}}\, d\omega,$$

which is the change of variable formula in Definition 4.1. By Proposition 7.3, we complete the proof. □

Theorem 7.7. *There is a one-to-one correspondence between the finite-dimensional complex representations of* $\mathrm{SU}(N)$ *and the finite-dimensional complex representations of the Lie algebra* $\mathfrak{sl}(N, \mathbb{C})$. *This correspondence keeps irreducibility.*

Proof. Since SU(N) is simply connected, the finite-dimensional representations of SU(N) are in one-to-one correspondence with the finite-dimensional representations of the Lie algebra $\mathfrak{su}(N)$. The complex representations of $\mathfrak{su}(N)$ are in one-to-one correspondence with the complex representations of the complexified Lie algebra $\mathfrak{su}(N)_{\mathbb{C}} = \mathfrak{sl}(N, \mathbb{C})$. (This proof is modified from p.127 of [Hall, 2003].) □

Definition 7.8. Let (π, V) be a representation of SU(N). We use the same symbol V or (π, V) for the corresponding $\mathfrak{sl}(N, \mathbb{C})$ representation in Theorem 7.7. Therefore we will study representations of the Lie algebra $\mathfrak{sl}(N, \mathbb{C})$ instead of representations of the Lie group SU(N).

7.1 Basic Lie Algebra

Definition 7.9. Let $\mathfrak{g}$ be a semi-simple complex Lie algebra. Let $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be its Lie bracket. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A weight of $\mathfrak{g}$ is a complex linear map from $\mathfrak{h}$ to $\mathbb{C}$. Let $\mathfrak{h}^*$ be the set of all weights of $\mathfrak{g}$. Let $\mathbf{0}$ be the zero weight of $\mathfrak{h}^*$.

Definition 7.10. For $\lambda \in \mathfrak{h}^*$, we define

$$\mathfrak{g}_\lambda = \{a \in \mathfrak{g} : [h, a] = \lambda(h)a \text{ for all } h \in \mathfrak{h}\}.$$

Definition 7.11. We call a non-zero $\lambda \in \mathfrak{h}^*$ a root if $\mathfrak{g}_\lambda$ is not empty. One can show that $\mathfrak{g}_\lambda$ is one-dimensional for λ is a root. Let Φ be the set of all roots.

Definition 7.12. Let $(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be the inner product on $\mathfrak{h}^*$ which is inherited from the Killing form on $\mathfrak{g}$.

Definition 7.13. Let $\mathfrak{h}_0^*$ be the **real** subspace of $\mathfrak{h}^*$ spanned by all the roots in Φ. The Euclidean space $\mathfrak{h}_0^*$, along with its inner product $(\cdot, \cdot)$ and the roots Φ, forms the root system of the Lie algebra $\mathfrak{g}$.

Definition 7.14. Let Λ be the set of integral weights

$$\Lambda = \left\{ \mu \in \mathfrak{h}_0^* : \left(\mu, \frac{2\lambda}{(\lambda, \lambda)} \right) \in \mathbb{Z} \text{ for all } \lambda \in \Phi \right\}.$$

Definition 7.15. Let σ_λ be the reflection through the hyperplane orthogonal to a root λ, i.e.,

$$\sigma_\lambda(\mu) = \mu - 2\frac{(\mu, \lambda)}{(\lambda, \lambda)}\lambda$$

for $\mu \in \mathfrak{h}_0^*$.

Definition 7.16. Let W be the Weyl group of this root system and W is generated by σ_λ $(\lambda \in \Phi)$.

Definition 7.17. Assume we have picked up a set of positive roots $\Phi^+ \subset \Phi$. Let

$$\rho = \frac{1}{2}\sum_{\lambda \in \Phi^+} \lambda$$

be the half sum of all positive roots. Assume we have picked up a set of simple positive roots with respect to the chosen set of positive roots Φ^+. Every positive root can be written as a sum of simple roots.

Definition 7.18. Let $\mathbf{E}$ be the free $\mathbb{Z}$-module on the set of symbols $\{e^\lambda : \lambda \in \Lambda\}$. It consists of all the formal sums

$$\sum_{\lambda \in \Lambda} n_\lambda e^\lambda$$

with $n_\lambda \in \mathbb{Z}$ such that $n_\lambda = 0$ for all but finitely many λ. It is a ring with the multiplication

$$\left(\sum_{\lambda \in \Lambda} n_\lambda e^\lambda\right)\left(\sum_{\mu \in \Lambda} m_\mu e^\mu\right) = \sum_{\nu \in \Lambda}\left(\sum_{\lambda + \mu = \nu} n_\lambda m_\mu\right) e^\nu.$$

Definition 7.19. The Weyl group W acts on Λ and $\mathbf{E}$. Let $\mathbf{E}^W$ be the invariant elements of $\mathbf{E}$ under the action of W.

Definition 7.20. The length of a Weyl group element is the length of the shortest word representing that element in terms of the standard generators σ_λ for $\lambda \in \Phi$. Let $\operatorname{sgn}(w) = (-1)^{\text{length of } w}$ be the sign of $w \in W$.

Definition 7.21. For each finite-dimensional representation V of the Lie algebra $\mathfrak{g}$, we define

$$\operatorname{ch}(V) = \sum_{\lambda \in \Lambda} \dim(V_\lambda) e^\lambda \in \mathbf{E},$$

in which $V_\lambda = \{v \in V : h \cdot v = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}$.

Lemma 7.22. *Let V_1 and V_2 be two representations of $\mathfrak{g}$. We have*

$$\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1)\mathrm{ch}(V_2)$$

and

$$\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2).$$

Proof. See [Bump, 2004] or [Hall, 2003]. □

Definition 7.23. If $x \in \mathbf{E}$ and $\lambda \in \Lambda$, we define

$$x\Big|_{e^\lambda}$$

to be the coefficient of e^λ in x. Namely, if $x = \sum_\nu n_\nu e^\nu$, we have $x\Big|_{e^\lambda} = n_\lambda$.

Definition 7.24. Let $C = \{\mu \in \mathfrak{h}_0^* : (\mu, \lambda) \geq 0 \text{ for all } \lambda \in \Phi^+\}$ be the Weyl chamber defined by the positive roots of Φ^+. Let $C^\circ = \{\mu \in \mathfrak{h}_0^* : (\mu, \lambda) > 0 \text{ for all } \lambda \in \Phi^+\}$ be the interior of that Weyl chamber.

Definition 7.25. The hyperplane $\{\lambda \in \mathfrak{h}_0^* : (\lambda, \alpha) = 0\}$ orthogonal to a root $\alpha \in \Phi$ is called a Weyl chamber wall if $\{\lambda : (\lambda, \alpha) = 0\} \cap C^\circ = \emptyset$.

Theorem 7.26 (Weyl character formula). *Let λ be a weight in $\Lambda \cap C$. Let V_λ be the (irreducible) highest weight representation of λ. We have the following formula*

$$\mathrm{ch}(V_\lambda) = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho+\lambda)}}{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho)}}.$$

Proof. This is Chapter 25 of [Bump, 2004]. □

Theorem 7.27. *Every finite-dimensional representation V is a direct sum of finitely many V_μ's, where V_μ is the highest weight representation of the weight $\mu \in \Lambda \cap C$. Namely, for a finite-dimensional representation V, we have*

$$V = \bigoplus_{\mu_i \in \Lambda \cap C} V_{\mu_i}^{\oplus m_i}$$

where m_i is a non-negative integer and $m_i = 0$ for all but finitely i. The number m_i is the multiplicity of V_{μ_i} in V.

Proof. This follows from the assumption that $\mathfrak{g}$ is semi-simple. □

The following lemma is taken from [Grabiner-Magyar, 1993].

Lemma 7.28. *The multiplicity m_i of V_{μ_i} in Theorem 7.27 can be obtained by*

$$m_i = \sum_{w \in W} \operatorname{sgn}(w) \operatorname{ch}(V) \bigg|_{e^{\mu_i + \rho - w(\rho)}} .$$

Proof. By Lemma 7.22, we have

$$\operatorname{ch}(V) = \sum_j m_j \operatorname{ch}(V_{\mu_j}).$$

Multiply $\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)}$ on both sides, we get

$$\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)} \operatorname{ch}(V) = \sum_j m_j \sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)} \operatorname{ch}(V_{\mu_j}). \tag{7.1}$$

By Theorem 7.26, we have

$$\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)} \operatorname{ch}(V_{\mu_j}) = \sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho + \mu_j)}.$$

And we know that for $\mu_i \in \Lambda \cap C$

$$\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho + \mu_j)} \bigg|_{e^{\rho + \mu_i}} = \begin{cases} 1, & \text{if } \mu_i = \mu_j, \\ 0, & \text{if } \mu_i \neq \mu_j. \end{cases}$$

Thus, taking $\Big|_{e^{\rho+\mu_i}}$ on both sides of Equation 7.1, we get $m_i = \sum_{w \in W} \operatorname{sgn}(w) \left(e^{w(\rho)} \operatorname{ch}(V) \right) \Big|_{e^{\rho + \mu_i}}$ $= \sum_{w \in W} \operatorname{sgn}(w) \operatorname{ch}(V) \Big|_{e^{\mu_i + \rho - w(\rho)}}$. □

7.2 Reflection Method

D. André developed the reflection method (a.k.a. the reflection principle) to solve Bertrand's ballot problem. The same method is applied to random walk (of discrete time) and Brownian motion (of continuous time).

The reflection method is generalized to random walks in a Weyl chamber in [Gessel-Zeilberger, 1992]. André's reflection on the straight line is replaced with the Weyl group action. This is an elegant method, which covers both classical ballot problem and the hook-length formula for Young tableau. We are going to present the theorem of Gessel and Zeilberger in detail in our context. We are not going to apply their theorem directly but we will slightly modify their theorem and its proof for our purpose.

Definition 7.29. For $i = 1$ and 2, let Σ_i be a finite subset of Λ which is invariant under the action of the Weyl group W.

We will consider random walks in the lattices Λ and $\Lambda \cap C^\circ$. We shall note that the latter is the set of integral weights that lie strictly inside the positive Weyl chamber and it excludes all weights on the Weyl chamber walls. Each allowable step is taken either in Σ_1 or in Σ_2.

Definition 7.30. If $\lambda, \nu \in \Lambda$, let

$$\mathrm{WALK}_{(k_1,k_2)}(\lambda \to \nu)$$

be the number of walks of (k_1+k_2) steps from λ to ν with the first k_1 steps in Σ_1 and the remaining k_2 steps in Σ_2.

Definition 7.31. If $\lambda, \nu \in \Lambda \cap C^\circ$, let

$$\mathrm{WALK}^W_{(k_1,k_2)}(\lambda \to \nu)$$

be the number of walks of $(k_1 + k_2)$ steps from λ to ν that always stay strictly in $\Lambda \cap C^\circ$, with the first k_1 steps in Σ_1 and the remaining k_2 steps in Σ_2.

Lemma 7.32. *We have*

$$\mathrm{WALK}_{(k_1,k_2)}(\lambda \to \nu) = \left(\sum_{\sigma\in\Sigma_1} e^\sigma\right)^{k_1} \left(\sum_{\sigma\in\Sigma_1} e^\sigma\right)^{k_2}\Bigg|_{e^{-\lambda+\nu}}.$$

Proof. Obviously we have $\mathrm{WALK}_{(k_1,k_2)}(\lambda \to \nu) = \mathrm{WALK}_{(k_1,k_2)}(\mathbf{0} \to \nu - \lambda)$. The polynomials $\left(\sum_{\sigma\in\Sigma_1} e^\sigma\right)$ and $\left(\sum_{\sigma\in\Sigma_1} e^\sigma\right)$ are the generating functions for the random walks in $\mathrm{WALK}_{(k_1,k_2)}(\mathbf{0} \to \nu - \lambda)$. Each term in

$$\underbrace{\left(\sum_{\sigma\in\Sigma_1} e^\sigma\right) \cdots \left(\sum_{\sigma\in\Sigma_1} e^\sigma\right)}_{k_1} \underbrace{\left(\sum_{\sigma\in\Sigma_2} e^\sigma\right) \cdots \left(\sum_{\sigma\in\Sigma_2} e^\sigma\right)}_{k_2}$$

corresponds to a walk with $(k_1 + k_2)$ steps with the first k_1 steps in Σ_1 and the remaining k_2 steps in Σ_2. □

The following definition of reflectable random walks is key to the success of the reflection method. It is due to [Gessel-Zeilberger, 1992].

Definition 7.33. We say a random walk starting from λ is **reflectable** if for each simple root α_i, there is a real number k_i, such that $(\alpha_i, \sigma) = \pm k_i$ or 0 for all allowable steps σ and (α_i, λ) is an integer multiple of k_i.

Remark 7.34. The previous definition guarantees that a reflectable random walk cannot leave the Weyl chamber C from its interior C° before landing on a Weyl chamber wall at some step. We will only consider reflectable random walks, i.e., the set of allowable steps and the start point are properly chosen to satisfy the conditions in the previous definition. Because we have two sets of allowable steps, we need to specify the meaning of being reflectable. We require that for each simple root α_i, there is a real number k_i, such that $(\alpha_i, \sigma) = \pm k_i$ or 0 for all steps $\sigma \in \Sigma_1 \cup \Sigma_2$ and (α_i, λ) is an integer multiple of k_i, where the random walk starts from λ.

Theorem 7.35 (Gessel-Zeilberger). *Let $\lambda, \nu \in \Lambda \cap C^\circ$ and λ, Σ_1 and Σ_2 are properly chosen so that Remark 7.34 is satisfied. We have the equality*

$$\mathrm{WALK}^W_{(k_1,k_2)}(\lambda \to \nu) = \sum_{w \in W} \mathrm{sgn}(w) \mathrm{WALK}_{(k_1,k_2)}(w(\lambda) \to \nu).$$

Proof. The following proof is due to [Gessel-Zeilberger, 1992] but modified to our case. Let $\mathrm{WALK}^\bullet_{(k_1,k_2)}(\cdot \to \cdot)$ be the number of walks which appear in $\mathrm{WALK}_{(k_1,k_2)}(\cdot \to \cdot)$ and touch at least one wall of the Weyl chamber C. Obviously we have

$$\mathrm{WALK}^\bullet_{(k_1,k_2)}(\lambda \to \nu) + \mathrm{WALK}^W_{(k_1,k_2)}(\lambda \to \nu) = \mathrm{WALK}_{(k_1,k_2)}(\lambda \to \nu),$$

and

$$\mathrm{WALK}^\bullet_{(k_1,k_2)}(w(\lambda) \to \nu) = \mathrm{WALK}_{(k_1,k_2)}(w(\lambda) \to \nu)$$

for w not the identity in W. We claim that

$$\sum_{w \in W} \mathrm{sgn}(w) \mathrm{WALK}^\bullet_{(k_1,k_2)}(w(\lambda) \to \nu) = 0. \tag{7.2}$$

We will prove that all these walks which appears on the left side of Equation 7.2 can be divided into cancelling pairs. We put the positive roots in Φ^+ into a total order. This total order can be chosen arbitrarily but it must be fixed. Let a walk from $w(\lambda)$ to ν on the left side of Equation 7.2 be

$$\delta_1, \delta_2, \delta_3, \quad ..., \quad \delta_{k_1+k_2},$$

where $\delta_1, \delta_2, ..., \delta_{k_1} \in \Sigma_1$ and $\delta_{k_1+1}, \delta_{k_1+2}, ..., \delta_{k_1+k_2} \in \Sigma_2$. After the j^{th} step δ_j, the walk is at $w(\lambda) + \sum_{i=1}^{j} \delta_j$. Because ν is in $\Lambda \cap C^\circ$, we can find r the largest number such that after the r^{th} step δ_r, the walk is on a Weyl chamber wall. After the $(r+1)^{\text{th}}$ step, it stays strictly in $\Lambda \cap C^\circ$ and will never leave $\Lambda \cap C^\circ$ again. After the r^{th} step, the walk is on a Weyl chamber wall but it may be at the intersection of several Weyl chamber walls. Among the positive roots corresponding to these walls, we pick up the largest positive root α_i according to the total order we have assigned for Φ^+. We pair the above walk with the random walk from $\sigma_{\alpha_i} w(\lambda)$ to ν with steps

$$\sigma_{\alpha_i}(\delta_1), \sigma_{\alpha_i}(\delta_2), \sigma_{\alpha_i}(\delta_3), \quad ..., \quad \sigma_{\alpha_i}(\delta_r), \delta_{r+1}, \delta_{r+2}, \quad ..., \quad \delta_{k_1+k_2},$$

where σ_{α_i} is the reflection through the wall orthogonal to α_i. The signs of these two walks are $\text{sgn}(w)$ and $\text{sgn}(\sigma_{\alpha_i} w)$ respectively and they cancel each other. □

7.3 Special Linear Lie Algebra $\mathfrak{sl}(N, \mathbb{C})$

We need to apply the results of the previous sections to the special case of $\mathfrak{sl}(N, \mathbb{C})$. Let N be an integer greater than 1. The special linear Lie algebra of order N, or $\mathfrak{sl}(N, \mathbb{C})$, is the Lie algebra of $N \times N$ matrices of zero trace, with the Lie bracket $[X, Y] := XY - YX$. It is a semi-simple complex Lie algebra. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(N, \mathbb{C})$ is that of diagonal matrices of zero trace. It is spanned by $E_{i,i} - E_{i+1,i+1}$ for $i = 1, 2, ..., N-1$, where $E_{i,j}$ is the matrix with 1 at the $(i, j)^{\text{th}}$ entry and 0 elsewhere.

The Lie algebra $\mathfrak{sl}(N, \mathbb{C})$ has the root system of type A_{N-1}. We can identify $\mathfrak{h}_0^*$ with

$$\left\{ (x_1, x_2, ..., x_N) \in \mathbb{R}^N : \sum_{i=1}^{N} x_i = 0 \right\}.$$

Let ϵ_i be the vector in $\mathbb{R}^N$ with 1 at the i^{th} entry and 0 elsewhere. We have the set of roots $\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$ with the root $\epsilon_i - \epsilon_j$ corresponding to $E_{i,j} \in \mathfrak{sl}(N, \mathbb{C})$. Let the set of positive roots be

$\Phi^+ = \{\epsilon_i - \epsilon_j : i < j\}$. We pick up $\epsilon_i - \epsilon_{i+1}$ for $i = 1, 2, ..., N-1$ as the simple roots for Φ^+. The half sum of positive roots is $\rho = \frac{\sum_{i=1}^{N}(N-2i+1)\epsilon_i}{2}$. The inner product for this root system is the restriction of the standard scalar product of $\mathbb{R}^N$ to the subspace $\left\{(x_1, x_2, ..., x_N) \in \mathbb{R}^N : \sum_{i=1}^{N} x_i = 0\right\}$. Again we let $\mathbf{0}$ be the zero weight of $\mathfrak{h}_0^* = \left\{(x_1, x_2, ..., x_N) \in \mathbb{R}^N : \sum_{i=1}^{N} x_i = 0\right\}$.

The representation (ι, H) of $\mathrm{SU}(N)$ in Definition 7.5 corresponds to the highest weight representation of the weight $\epsilon_1 - \frac{1}{N}\sum_{j=1}^{N}\epsilon_j$ and hence we have

$$\mathrm{ch}(H) = \sum_{i=1}^{N} e^{\epsilon_i - \frac{1}{N}\sum_{j=1}^{N}\epsilon_j}.$$

Since $\overline{H}$ is the contragredient of H, we have

$$\mathrm{ch}(\overline{H}) = \sum_{i=1}^{N} e^{\frac{1}{N}\sum_{j=1}^{N}\epsilon_j - \epsilon_i}.$$

Let $\Sigma_1 = \left\{\epsilon_i - \frac{1}{N}\sum_{j=1}^{N}\epsilon_j : i = 1, 2, ..., N\right\}$ and $\Sigma_2 = \left\{\frac{1}{N}\sum_{j=1}^{N}\epsilon_j - \epsilon_i : i = 1, 2, ..., N\right\}$. With this pair of allowable steps Σ_1 and Σ_2, we apply Theorem 7.35 to the case $\lambda = \nu = \rho$. Hence we have

$$\begin{aligned}
\mathrm{WALK}^{W}_{(k_1,k_2)}(\rho \to \rho) &= \sum_{w\in W} \mathrm{sgn}(w)\mathrm{WALK}_{(k_1,k_2)}(w(\rho) \to \rho) \\
&= \sum_{w\in W} \mathrm{sgn}(w) \left(\mathrm{ch}(H)^{k_1}\mathrm{ch}(\overline{H})^{k_2}\right)\Big|_{e^{\rho - w(\rho)}} \\
&= \sum_{w\in W} \mathrm{sgn}(w)\mathrm{ch}(H^{\otimes k_1} \otimes \overline{H}^{\otimes k_2})\Big|_{e^{\mathbf{0}+\rho-w(\rho)}} \\
&= \text{the multiplicity of trivial representation in } H^{\otimes k_1} \otimes \overline{H}^{\otimes k_2} \\
&= \int_{\mathbb{C}} z^{k_1}\overline{z}^{k_2}\, d\mu_{\infty}^{(N)}.
\end{aligned}$$

The second to the last equality follows from Lemma 7.28 and we note that the highest weight representation of the zero weight $\mathbf{0}$ is the trivial representation. The main result that we will use from this chapter is the equality

$$\int_{\mathbb{C}} z^{k_1}\overline{z}^{k_2}\, d\mu_{\infty}^{(N)} = \mathrm{WALK}^{W}_{(k_1,k_2)}(\rho \to \rho). \tag{7.3}$$

7.4 More about $\mathfrak{sl}(3,\mathbb{C})$

For $\mathfrak{sl}(3,\mathbb{C})$, we can make a more detailed and pictorial explanation in this section. As in [Hall, 2003], we use the following basis for $\mathfrak{sl}(3,\mathbb{C})$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

A Cartan subalgebra $\mathfrak{h}$ is spanned by H_1 and H_2. The root system of $\mathfrak{sl}(3,\mathbb{C})$ can be identified with $\mathbb{R}^2$ with our two chosen simple roots

$$\alpha_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{2}\right) \text{ and } \alpha_2 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right),$$

with α_1 corresponding to X_1 and α_2 corresponding to X_2. The set of positive roots is $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$. The simple roots α_1 and α_2 define a Weyl chamber

$$C = \left\{(i_1, i_2) \in \mathbb{R}^2 : x \geq 0, -x \leq \sqrt{3}y \leq x\right\}.$$

The half sum of positive roots is $\rho = \left(\sqrt{2}, 0\right)$. The inner product of this root system is the standard scalar product of $\mathbb{R}^2$.

The Weyl group W acting on $\mathbb{R}^2$ is generated by reflection with respect to the following three lines (Weyl chamber walls):

$$y = \frac{\sqrt{3}}{3}x, \quad y = -\frac{\sqrt{3}}{3}x, \quad x = 0.$$

It is a group of order 6 and is isomorphic to the permutation of three elements.

Define ϖ_1 to be $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{6}\right)$ and ϖ_2 to be $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}\right)$. The defining representation (ι, H) is the highest weight representation of ϖ_1 and its contragredient is the highest weight representation of

ϖ_2. Easily we see that $\Sigma_1 = \{\varpi_1, -\varpi_2, \varpi_2 - \varpi_1\}$ and $\Sigma_2 = \{\varpi_2, -\varpi_1, \varpi_1 - \varpi_2\}$. Both Σ_1 and Σ_2 are invariant under the action of W.

Theorem 7.36. *Let us recall $\mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)}$ defined in Definition 6.8. We have the equality*

$$\mathfrak{q}_{(0,0)}^{(k_1,k_2)} = \mathrm{WALK}_{(k_1,k_2)}^{W}(\rho \to \rho),$$

and more generally

$$\mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} = \mathrm{WALK}_{(k_1,k_2)}^{W}(\rho \to \lambda),$$

if $\lambda = \rho + i_1\left(\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2\right) + i_2\left(\frac{2}{3}\alpha_2 + \frac{1}{3}\alpha_1\right) \in \Lambda \cap C^\circ$ for $i_1, i_2 \geq 0$.

Proof. We will map the random walks in $\mathcal{C}^{(3)}$ with steps in $\mathcal{S}_1^{(3)}$ and $\mathcal{S}_2^{(3)}$ to the random walks in $\Lambda \cap C^\circ$ with steps in Σ_1 and Σ_2. The points in $\mathcal{C}^{(3)}$ are mapped to $\Lambda \cap C^\circ$ by taking $(i_1, i_2) \in \mathcal{C}^{(3)}$ to $\rho + i_1\varpi_1 + i_2\varpi_2 \in \Lambda \cap C^\circ$. This map is bijective between $\mathcal{C}^{(3)}$ and $\Lambda \cap C^\circ$. Correspondingly we map $\mathcal{S}_1^{(3)}$ to Σ_1 by $e_1 \mapsto \varpi_1$, $-e_2 \mapsto -\varpi_2$ and $e_2 - e_1 \mapsto \varpi_2 - \varpi_1$. We map $\mathcal{S}_2^{(3)}$ to Σ_2 by $e_2 \mapsto \varpi_2$, $-e_1 \mapsto -\varpi_1$ and $e_1 - e_2 \mapsto \varpi_1 - \varpi_2$. This map establishes the equivalence of the two random walks from different sides of $\mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} = \mathrm{WALK}_{(k_1,k_2)}^{W}(\rho \to \lambda)$. □

Chapter 8

Proof of the Main Theorem

We will finish the proof of Theorem 1.1 in this chapter. Let us fix a prime number p and recall that $A_j(p,1)$ is the eigenvalue of the Hecke-Maass form ϕ_j under the action of the Hecke operator T_p, i.e.,

$$T_p\phi_j = A_j(p,1)\phi_j.$$

Theorem 8.1. *We have the equality*

$$\lim_{T\to\infty} \frac{\sum_j A_j(p,1)^{k_1}\overline{A_j(p,1)}^{k_2} \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}} = \int_{\mathbb{C}} z^{k_1}\overline{z}^{k_2}\, d\mu_\infty^{(3)}.$$

Proof. Since $A_j(1,1)=1$, we have

$$A_j(p,1)^{k_1}\overline{A_j(p,1)}^{k_2} = \sum_{i_1,i_2\geq 0} \mathfrak{q}^{(k_1,k_2)}_{(i_1,i_2)} A_j(p^{i_1},p^{i_2}) = \sum_{i_1,i_2\geq 0} \mathfrak{q}^{(k_1,k_2)}_{(i_1,i_2)} A_j(p^{i_1},p^{i_2})\overline{A_j(1,1)}.$$

Applying Theorem 5.3 to $A_j(p^{i_1},p^{i_2})\overline{A_j(1,1)}$, we obtain

$$\begin{aligned}
\lim_{T\to\infty} \frac{\sum_j A_j(p,1)^{k}\overline{A_j(p,1)}^{k_2} \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}} &= \lim_{T\to\infty} \frac{\sum_j \sum_{i_1,i_2\geq 0} \mathfrak{q}^{(k_1,k_2)}_{(i_1,i_2)} A_j(p^{i_1},p^{i_2})\overline{A_j(1,1)} \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}} \\
&= \sum_{i_1,i_2\geq 0} \mathfrak{q}^{(k_1,k_2)}_{(i_1,i_2)} \lim_{T\to\infty} \frac{\sum_j A_j(p^{i_1},p^{i_2})\overline{A_j(1,1)} \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}} \\
&= \mathfrak{q}^{(k_1,k_2)}_{(0,0)}.
\end{aligned}$$

By Theorem 7.36 and Equation 7.3, we have

$$\mathfrak{q}_{(0,0)}^{(k_1,k_2)} = \mathrm{WALK}_{(k_1,k_2)}^{W}(\rho \to \rho) = \int_{\mathbb{C}} z^{k_1} \overline{z}^{k_2} \, d\mu_{\infty}^{(3)}$$

and hence we complete the proof. □

Corollary 8.2. *For any polynomial function*

$$f(z) = \sum_{k_1,k_2 \le D} a_{k_1,k_2} z^{k_1} \overline{z}^{k_2}, \quad a_{k_1,k_2} \in \mathbb{C},$$

we have the equality

$$\lim_{T\to\infty} \frac{\sum_j f(A_j(p,1)) \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}} = \int_{\mathbb{C}} f \, d\mu_{\infty}^{(3)}.$$

Proof. This theorem is obvious after Theorem 8.1. □

Proposition 8.3. *The Hecke eigenvalue $A_j(p,1)$ is bounded, i.e.,*

$$|A_j(p,1)| \le M$$

for some $M \ge 3$ (M may depend on p) and M does not depend on j.

Proof. This is Proposition 12.1.6 of [Goldfeld, 2006]. □

Theorem 8.4 (**Main theorem**). *For any continuous test function $f : \mathbb{C} \to \mathbb{C}$, we have the equality*

$$\lim_{T\to\infty} \frac{\sum_{j=1}^{\infty} f(A_j(p,1)) \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_{j=1}^{\infty} \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}} = \int_{\mathbb{C}} f \, d\mu_{\infty}^{(3)}.$$

Proof. We can find a continuous function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ such that the support of $\tilde{f}$ is contained in $\{z : |z| \le M+1\}$ and the restrictions of f and $\tilde{f}$ on $\{z : |z| \le M\}$ are identical. By Proposition 8.3, together with the fact that the Sato-Tate measure $\mu_{\infty}^{(3)}$ is supported inside $\{z : |z| \le 3\}$, we only need to prove the main theorem for $\tilde{f}$. Let $\mathcal{C}(\{z : |z| \le M+1\})$ be the space of all continuous functions on $\{z : |z| \le M+1\}$. We define a norm $\| \cdot \|_{\infty}$ on $\mathcal{C}(\{z : |z| \le M+1\})$ by

$$\| g \|_{\infty} = \max_{|z| \le M+1} |g(z)|$$

for $g \in \mathcal{C}(\{z : |z| \leq M+1\})$. For $T \gg 1$ and fixed $R \geq 10$, we define a linear functional

$$\mathbb{L}_T(g) = \frac{\sum_j g(A_j(p,1)) \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}$$

for $g \in \mathcal{C}(\{z : |z| \leq M+1\})$. We define another linear functional by

$$\mathbb{L}_\infty(g) = \int_{\mathbb{C}} g \, d\mu_\infty^{(3)}$$

for $g \in \mathcal{C}(\{z : |z| \leq M+1\})$. Both $\mathbb{L}_T$ and $\mathbb{L}_\infty$ are continuous under the norm $\| \cdot \|_\infty$ and we have the inequalities

$$|\mathbb{L}_T(g)| \leq \| g \|_\infty \text{ and } \quad |\mathbb{L}_\infty(g)| \leq \| g \|_\infty .$$

By the Stone-Weierstrass theorem $\tilde{f}$ can be approximated uniformly by polynomial functions on the compact set $\{z : |z| \leq M+1\}$, i.e., we can find a sequence of polynomial functions $\{f_n\}$ such that

$$\lim_{n\to\infty} \| \tilde{f} - f_n \|_\infty = 0.$$

For any $\epsilon > 0$, we can find n' such that $\| \tilde{f} - f_n \|_\infty \leq \frac{\epsilon}{3}$ for any $n > n'$. Since we already have

$$\lim_{T\to\infty} \mathbb{L}_T(f_{n'+1}) = \mathbb{L}_\infty(f_{n'+1})$$

by Corollary 8.2, we can find T' such that

$$|\mathbb{L}_T(f_{n'+1}) - \mathbb{L}_\infty(f_{n'+1})| < \frac{\epsilon}{3},$$

for any $T > T'$. For any $T > T'$, we have

$$\begin{aligned}
\left|\mathbb{L}_T(\tilde{f}) - \mathbb{L}_\infty(\tilde{f})\right| &\leq \left|\mathbb{L}_T(\tilde{f}) - \mathbb{L}_T(f_{n'+1})\right| + |\mathbb{L}_T(f_{n'+1}) - \mathbb{L}_\infty(f_{n'+1})| + \left|\mathbb{L}_\infty(f_{n'+1}) - \mathbb{L}_\infty(\tilde{f})\right| \\
&\leq 2 \| f - f_{n'+1} \|_\infty + \frac{\epsilon}{3} \\
&\leq \epsilon.
\end{aligned}$$

It follows that the limit $\lim_{T\to\infty} \mathbb{L}_T(\tilde{f}) = \lim_{T\to\infty} \frac{\sum_j \tilde{f}(A_j(p,1)) \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}$ exists and it equals $\mathbb{L}_\infty(\tilde{f}) = \int_{\mathbb{C}} \tilde{f} \, d\mu_\infty^{(3)}$. □

Chapter 9

Rate of Convergence

In this chapter we will prove an effective version of Theorem 8.4 which gives the rate of convergence of the limit in Theorem 8.4. Unfortunately we cannot prove this for every continuous test function, but only for a large class of convergent power series. We should also be aware that there can be multiple versions of rate of convergence for any Sato-Tate problem, as summarized in [Mazur, 2008].

Lemma 9.1. *We have the inequality*

$$\sum_{i_1,i_2\geq 0} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} p^{2i_1} p^{2i_2} \leq (p^2+p^{-2}+1)^{k_1+k_2}.$$

Proof. By the equivalence of the two random walks of Theorem 7.36, we have

$$\begin{aligned}
\sum_{i_1,i_2\geq 0} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} p^{2i_1} p^{2i_2} &= \sum_{(i_1,i_2)\in\mathcal{C}^{(3)}} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} p^{2i_1} p^{2i_2} \\
&= \sum_{\lambda\in\Lambda\cap C^\circ} \mathrm{WALK}_{(k_1,k_2)}^{W}(\rho\to\lambda) e^{\lambda-\rho}\Big|_{e^{\alpha_1}=e^{\alpha_2}=p^2}
\end{aligned}$$

(we shall note that $\lambda-\rho$ can be uniquely written as a linear combination of α_1 and α_2)

$$\begin{aligned}
&\leq \sum_{\lambda\in\Lambda} \mathrm{WALK}_{(k_1,k_2)}(\rho\to\lambda) e^{\lambda-\rho}\Big|_{e^{\alpha_1}=e^{\alpha_2}=p^2} \\
&= \mathrm{ch}(H)^{k_1}\mathrm{ch}(\overline{H})^{k_2}\Big|_{e^{\alpha_1}=e^{\alpha_2}=p^2} \\
&= (p^2+p^{-2}+1)^{k_1+k_2}.
\end{aligned}$$

Hence we complete the proof. ∎

Theorem 9.2 (Rate of Convergence). *Let us fix a number $\epsilon > 0$. For a power series $f(z) = \sum_{k_1,k_2\geq 0} a_{k_1,k_2} z^{k_1}\overline{z}^{k_2}$ which is absolutely convergent on $\{z : |z| \leq p^2 + p^{-2} + 1\}$, we have*

$$\left| \frac{\sum_j f(A_j(p,1)) \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}}{\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j}} - \int_{\mathbb{C}} f \, d\mu_\infty^{(3)} \right| \ll_{R,\epsilon,f} T^{\epsilon-2}.$$

Proof. By Theorem 5.3, we have

$$\begin{aligned}
&\sum_j \quad A_j(p,1)^{k_1} \quad \overline{A_j(p,1)}^{k_2} \, \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} \\
&\quad = \mathfrak{q}_{(0,0)}^{(k_1,k_2)} \quad \sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} + \sum_{i_1,i_2\geq 0} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} \mathcal{O}_{R,\epsilon}(T^{3+3R+\epsilon}|p^{i_1}p^{i_2}|^2) \\
&\quad = \mathfrak{q}_{(0,0)}^{(k_1,k_2)} \quad \sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} + \left(\sum_{i_1,i_2\geq 0} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} p^{2i_1}p^{2i_2} \right) \mathcal{O}_{R,\epsilon}(T^{3+3R+\epsilon}).
\end{aligned}$$

Therefore, by Theorem 5.3, we have

$$\begin{aligned}
&\sum_j \quad f(A_j(p,1)) \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} \\
&= \sum_j \sum_{k_1,k_2\geq 0} a_{k_1,k_2} A_j(p,1)^{k_1} \overline{A_j(p,1)}^{k_2} \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} \\
&= \sum_{k_1,k_2\geq 0} a_{k_1,k_2} \mathfrak{q}_{(0,0)}^{(k_1,k_2)} \sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} + \sum_{k_1,k_2\geq 0} |a_{k_1,k_2}| \sum_{i_1,i_2\geq 0} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} \mathcal{O}_{R,\epsilon}(T^{3+3R+\epsilon}|p^{i_1}p^{i_2}|^2) \\
&= \left(\int_{\mathbb{C}} f \, d\mu_\infty^{(3)} \right) \sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} + \left(\sum_{k_1,k_2\geq 0} |a_{k_1,k_2}| \sum_{i_1,i_2\geq 0} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} p^{2i_1}p^{2i_2} \right) \mathcal{O}_{R,\epsilon}(T^{3+3R+\epsilon}).
\end{aligned}$$

By Lemma 9.1, we have

$$\sum_{k_1,k_2\geq 0} |a_{k_1,k_2}| \sum_{i_1,i_2\geq 0} \mathfrak{q}_{(i_1,i_2)}^{(k_1,k_2)} p^{2i_1}p^{2i_2} \leq \sum_{k_1,k_2\geq 0} |a_{k_1,k_2}|(p^2 + p^{-2} + 1)^{k_1+k_2}$$

and combining with the "Weyl law"

$$\sum_j \frac{h_{T,R}(\nu^{(j)})}{\mathcal{L}_j} \sim cT^{5+3R}$$

in Theorem 5.3 we complete the proof. □

Chapter 10

Case of GL(N)

The mechanism that we use to prove the main theorem via the orthogonality relation can be generalized from GL(3) to GL(N) ($N \geq 2$). If any version of the orthogonality relation between Fourier coefficients of Hecke-Maass forms for SL($N, \mathbb{Z}$) is established, our mechanism can prove the corresponding version of Theorem 5.5.

If the orthogonality relation can be given in the stronger form

$$\lim_{T\to\infty} \frac{\sum_{j\leq T} \frac{A_j(m_1,m_2)\overline{A_j(n_1,n_2)}}{\operatorname{Res}_{s=1} L(s,\phi_j\times\tilde{\phi_j})}}{\sum_{j\leq T} \frac{1}{\operatorname{Res}_{s=1} L(s,\phi_j\times\tilde{\phi_j})}} = \delta_{m_1,n_1}\delta_{m_2,n_2},$$

then we can prove the corresponding version of weighted equidistribution

$$\lim_{T\to\infty} \frac{\sum_{j\leq T} \frac{f(A_j(p,1))}{\operatorname{Res}_{s=1} L(s,\phi_j\times\tilde{\phi_j})}}{\sum_{j\leq T} \frac{1}{\operatorname{Res}_{s=1} L(s,\phi_j\times\tilde{\phi_j})}} = \int_{\mathbb{C}} f \, d\mu_\infty^{(3)}$$

for any continuous test function $f : \mathbb{C} \to \mathbb{C}$.

If any version of Goldfeld and Kontorovich's orthogonality relation is generalized to Maass forms for SL($N, \mathbb{Z}$), we can prove the corresponding weighted equidistribution theorem. We conjecture that it should be possible to obtain the orthogonality relation for GL(N) as an application of the Kuznetsov trace formula for Maass forms for SL($N, \mathbb{Z}$). Our mechanism also covers the case of GL(2) but more elementary techniques easily give the desired results as in [Bruggeman, 1978] etc.

As in the case of GL(3), there is a correspondence between the Hecke relations for SL($N, \mathbb{Z}$) and random walks in a Weyl chamber of the root system A_{N-1}. For the Hecke relations, our main reference is Chapter 9 of [Goldfeld, 2006]. For random walks in a Weyl chamber of the root system A_{N-1}, we refer to our previous chapters about the Lie theory and the reflection method.

Let us fix a prime number p and recall that $A_j(p,1,...,1)$ is the eigenvalue of the Hecke-Maass form ϕ_j under the action of the Hecke operator T_p, i.e.,

$$T_p\phi_j = A_j(p,1,...,1)\phi_j.$$

Definition 10.1. Let $A_j(m_1,...,m_{N-1})$ be the $(m_1,...,m_{N-1})^{\text{th}}$ Fourier coefficient of the j^{th} Hecke-Maass form ϕ_j for $\mathrm{SL}(N,\mathbb{Z})$, normalized so that $A_j(1,...,1)=1$. We have three Hecke relations

$$\begin{aligned}
A_j(n,1,...,1)A_j(m_1,...,m_{N-1}) &= \sum_{\substack{c_1|m_1,...,c_{N-1}|m_{N-1} \\ \prod_{l=1}^{N} c_l = n}} A_j\left(\frac{m_1c_N}{c_1}, \frac{m_2c_1}{c_2}, ..., \frac{m_{N-1}c_{N-2}}{c_{N-1}}\right), \\
A_j(1,...,1,n)A_j(m_1,...,m_{N-1}) &= \sum_{\substack{c_1|m_1,...,c_{N-1}|m_{N-1} \\ \prod_{l=1}^{N} c_l = n}} A_j\left(\frac{m_1c_2}{c_1}, \frac{m_2c_3}{c_2}, ..., \frac{m_{N-1}c_N}{c_{N-1}}\right), \\
A_j(m_1,...,m_{N-1}) &= \overline{A_j(m_{N-1},...,m_1)},
\end{aligned}$$

for positive integers $m_i (i=1,2,...,N-1)$ and n.

Let L_j^T be a non-negative number associated with $j=1,2,...$ and $T \gg 1$. We assume that $A_j(m_1,...,m_{N-1})$ and L_j^T satisfy the following properties:

- **Property I**: for a prime number p we can find some $M \geq N$ (M may depend on p) such that for all j, we have $|A_j(p,1,...,1)| \leq M$,
- **Property II**: $0 < \sum_j L_j^T < \infty$,
- **Property III**: We have orthogonality relation

$$\lim_{T\to\infty} \frac{\sum_j A_j(m_1,m_2,...,m_{N-1})\overline{A_j(n_1,n_2,...,n_{N-1})}L_j^T}{\sum_j L_j^T} = \begin{cases} 1, & \text{if } m_i = n_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 10.2. Property I is given by Proposition 12.1.6 of [Goldfeld, 2006]. For $N=2$ Proposition 4.1 of [Bruggeman, 1978] gives a version of the Property III. For $N=3$ Theorem 5.3 gives a version of the Property III.

10.1 Two Types of Random Walks

As we did for GL(3), we will express $A_j(p,1,...,1)^{k_1}\overline{A_j(p,1,...,1)}^{k_2}$ as a linear combination of $A_j(p^{i_1},...,p^{i_{N-1}})$'s via the Hecke relations. The idea remains the same and the results are analogous.

Definition 10.3. We define random walks in

$$\mathcal{C}^{(N)} = \left\{(i_1,...,i_{N-1}) \in \mathbb{Z}^{N-1} : i_1,...,i_{N-1} \geq 0\right\}$$

with allowable steps either in

$$\mathcal{S}_1^{(N)} = \{e_1, -e_1+e_2, -e_2+e_3, ..., -e_{N-2}+e_{N-1}, -e_{N-1}\}$$

or in

$$\mathcal{S}_2^{(N)} = \{e_{N-1}, -e_{N-1}+e_{N-2}, -e_{N-2}+e_{N-3}, ..., -e_2+e_1, -e_1\},$$

where e_i is the vector with 1 at the i^{th} entry and 0 elsewhere.

Definition 10.4. Let $\lambda \in \mathcal{C}^{(N)}$. By abuse of terminology, we define $\mathfrak{q}_\lambda^{(k_1,k_2)}$ to be the number of walks from $(0,...,0)$ to λ in $\mathcal{C}^{(N)}$ of (k_1+k_2) steps, with the first k_1 steps in $\mathcal{S}_1^{(N)}$ and the remaining k_2 steps in $\mathcal{S}_2^{(N)}$.

Theorem 10.5. *We can express $A_j(p,1,...,1)^{k_1}\overline{A_j(p,1,...,1)}^{k_2}$ as a finite sum of $A_j(p^{i_1},...,p^{i_{N-1}})$'s, and the coefficient of $A_j(p^{i_1},...,p^{i_{N-1}})$ is $\mathfrak{q}_{(i_1,...,i_{N-1})}^{(k_1,k_2)}$, i.e.,*

$$A_j(p,1,...,1)^{k_1}\overline{A_j(p,1,...,1)}^{k_2} = \sum_{i_1,...,i_{N-1}\geq 0} \mathfrak{q}_{(i_1,...,i_{N-1})}^{(k_1,k_2)} A_j(p^{i_1},...,p^{i_{N-1}}).$$

Proof. As in GL(3), the proof is inductive. Firstly we apply induction on k_1 when $k_2 = 0$. Then we apply induction on k_2 for a fixed k_1. □

The other random walk is in a Weyl chamber of the root system A_{N-1}, which is associated with the Lie algebra $\mathfrak{sl}(N,\mathbb{C})$. We follow all the notations from Section 7.3. We consider random walks in $\Lambda \cap C^\circ$.

Theorem 10.6. *We have the following equality which matches the two types of random walks*

$$q^{(k_1,k_2)}_{(0,...,0)} = \mathrm{WALK}^W_{(k_1,k_2)}(\rho \to \rho).$$

Proof. We will map the random walks in $\mathrm{WALK}^W_{(k_1,k_2)}(\rho \to \rho)$ to the random walks in $q^{(k_1,k_2)}_{(0,...,0)}$. This map between Σ_i and $\mathcal{S}^{(N)}_i$ is given by

$$\epsilon_1 - \frac{1}{N}\sum_{j=1}^N \epsilon_j \mapsto e_1,\ \epsilon_i - \frac{1}{N}\sum_{j=1}^N \epsilon_j \mapsto -e_{i-1} + e_i \text{ for } i = 2, 3, ..., N-1,\ \epsilon_N - \frac{1}{N}\sum_{j=1}^N \epsilon_j \mapsto -e_{N-1},$$

and

$$-\epsilon_1 + \frac{1}{N}\sum_{j=1}^N \epsilon_j \mapsto -e_1,\ -\epsilon_i + \frac{1}{N}\sum_{j=1}^N \epsilon_j \mapsto e_{i-1} - e_i \text{ for } i = 2, 3, ..., N-1,\ -\epsilon_N + \frac{1}{N}\sum_{j=1}^N \epsilon_j \mapsto e_{N-1}.$$

The point ρ is mapped to $(0, ..., 0) \in \mathcal{C}^{(N)}$. All other points are matched correspondingly by

$$\rho + \sum_{i=1}^{N-1} l_i \sum_{i'=1}^{i} \epsilon_{i'} \mapsto (l_1, ..., l_{N-1}).$$

Therefore we establish the equivalence between the walks in $\mathrm{WALK}^W_{(k_1,k_2)}(\rho \to \rho)$ and those in $q^{(k_1,k_2)}_{(0,...,0)}$. □

10.2 Completion of the Proof

Theorem 10.7. *Assuming Property I, II and III, we have*

$$\lim_{T\to\infty} \frac{\sum_{j=1}^{\infty} A_j(p,1,...,1)^{k_1}\overline{A_j(p,1,...,1)}^{k_2} L_j^T}{\sum_{j=1}^{\infty} L_j^T} = \int_{\mathbb{C}} z^{k_1}\bar{z}^{k_2}\, d\mu^{(N)}_\infty.$$

Proof. Since $A_j(1, ..., 1) = 1$, we have

$$A_j(p,1,...,1)^{k_1}\overline{A_j(p,1,...,1)}^{k_2} = \sum_{i_1,i_2,...,i_{N-1}\geq 0} q^{(k_1,k_2)}_{(i_1,i_2,...,i_{N-1})} A_j(p^{i_1}, ..., p^{i_{N-1}})\overline{A_j(1,...,1)}.$$

Apply Theorem 10.5 and Property III to $A_j(p^{i_1}, ..., p^{i_{N-1}})\overline{A_j(1,...,1)}$, we obtain

$$\begin{aligned}
&\lim_{T\to\infty} \frac{\sum_j A_j(p,1,...,1)^{k_1}\overline{A_j(p,1,...,1)}^{k_2}L_j^T}{\sum_j L_j^T} \\
&= \lim_{T\to\infty} \frac{\sum_j \sum_{i_1,...,i_{N-1}\ge 0} \mathfrak{q}_{(i_1,...,i_{N-1})}^{(k_1,k_2)} A_j(p^{i_1},...,p^{i_{N-1}})\overline{A_j(1,...,1)}L_j^T}{\sum_j L_j^T} \\
&= \sum_{i_1,...,i_{N-1}\ge 0} \mathfrak{q}_{(i_1,...,i_{N-1})}^{(k_1,k_2)} \lim_{T\to\infty} \frac{\sum_j A_j(p^{i_1},...,p^{i_{N-1}})\overline{A_j(1,...,1)}L_j^T}{\sum_j L_j^T} \\
&= \mathfrak{q}_{(0,...,0)}^{(k_1,k_2)} \\
&= \mathrm{WALK}^W_{(k_1,k_2)}(\rho\to\rho) \\
&= \int_{\mathbb{C}} z^{k_1}\overline{z}^{k_2}\, d\mu_\infty^{(N)}.
\end{aligned}$$

The last equality is from Equation 7.3 and the second to the last equality follows from Theorem 10.6. □

Proposition 10.8. *Assume Property I, II and III. For any polynomial*

$$f(z) = \sum_{k_1,k_2\le D} a_{k_1,k_2} z^{k_1}\overline{z}^{k_2}, \qquad a_{k_1,k_2}\in\mathbb{C},$$

we have

$$\lim_{T\to\infty} \frac{\sum_{j=1}^{\infty} f(A_j(p,1,...,1))L_j^T}{\sum_{j=1}^{\infty} L_j^T} = \int_{\mathbb{C}} f\, d\mu_\infty^{(N)}.$$

Proof. This theorem is obvious after Theorem 10.7. □

Theorem 10.9 (Main theorem for GL(N)). *Assume Property I, II and III. For any continuous test function $f:\mathbb{C}\to\mathbb{C}$, we have*

$$\lim_{T\to\infty} \frac{\sum_{j=1}^{\infty} f\left(A_j(p,1,...,1)\right)L_j^T}{\sum_{j=1}^{\infty} L_j^T} = \int_{\mathbb{C}} f\, d\mu_\infty^{(N)}.$$

Proof. We can find a function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ such that the support of $\tilde{f}$ is contained in $\{z : |z| \leq M+1\}$ and the restrictions of f and $\tilde{f}$ on $\{z : |z| \leq M\}$ are identical. Because of Property I and the fact that the Sato-Tate measure $\mu_\infty^{(N)}$ is supported inside $\{z : |z| \leq N\}$, we only need to prove the main theorem for $\tilde{f}$. We complete the proof by the Stone-Weierstrass approximation on $\{z : |z| \leq M+1\}$. We omit the details of the proof because it is the same as in the proof of Theorem 8.4. $\square$

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