Dominating varieties by liftable ones

Remy van Dobben de Bruyn

Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

Columbia University

2018

Abstract

Dominating varieties by liftable ones

Remy van Dobben de Bruyn

Algebraic geometry in positive characteristic has a quite different flavour than in characteristic zero. Many of the pathologies disappear when a variety admits a lift to characteristic zero. It is known since the sixties [Ser61] that such a lift does not always exist. However, for applications it is sometimes enough to lift a variety dominating the given variety, and it is natural to ask when this is possible.

The main result of this dissertation is the construction of a smooth projective variety over any algebraically closed field of positive characteristic that cannot be dominated by another smooth projective variety admitting a lift to characteristic zero. We also discuss some cases in which a dominating liftable variety does exist.

This page intentionally left blank

Contents

Acknowledgements

First and foremost, I would like to heartily thank Johan de Jong for his endless generosity. Johan, although I didn't always listen to you, it is hard to imagine what this thesis would have looked like without your continuous encouragement and advice. You were always available when I was stuck, or when I wasn't but just wanted to rant (which you would then quickly shut down). I could not have wished for a better advisor.

I want to thank Robert Friedman, Michael Harris, Alena Pirutka, and Daniel Litt for agreeing to be on my defense committee, as well as for providing helpful suggestions.

Over the past few years, I have had the pleasure to learn mathematics from many professors, postdocs, graduate students, and undergraduate students, and I would like to thank everyone who helped me along the way. In particular, I thank Bhargav Bhatt, Raymond Cheng, Johan Commelin, Ashwin Deopurkar, Josse van Dobben de Bruyn, David Hansen, Kiran Kedlaya, Raju Krishnamoorthy, Shizhang Li, Daniel Litt, Milan Lopuhaä-Zwakenberg, Xuanyu Pan, Jason Starr, and Burt Totaro for helpful discussions related to this work and my other research projects, and Josse van Dobben de Bruyn for as many unhelpful ones.

An earlier version of this document only constructed a three-dimensional example. I want to thank Kiran Kedlaya, Frans Oort, Bhargav Bhatt, and Daniel Litt for encouraging me to think about the surface case. In particular, I would like to thank Bhargav Bhatt for the idea that lead to the improvement from Corollary 7.2.8 to Corollary 7.3.3; this is the improvement needed to get a surface.

Next, I would like to thank the large and very active group of graduate students and postdocs in algebraic geometry, number theory, and other areas at Columbia. It was a privilege to have so many fun and talented people around, and I hope to stay in touch with all of you, wherever your paths may take you.

During my time at Columbia, I have been very fortunate to have access to high quality music ensembles and undergraduate music courses. I would like to thank all students and faculty I interacted with for making me a more complete person. In particular, I would like to thank the Columbia Bach Society and its artistic director Kevin Lee for providing exactly the right mix of serious music making in an informal and sometimes somewhat silly setting.

I would also like to thank the Department of Mathematics for their generous allocation of resources, especially travel funding and delicious food.

Finally, I want to thank my brother Josse and my parents Else and Yvonne for their unconditional love and support, whatever part of the globe I may find myself on. Josse, I have said it before, but it is great to have you as a twin brother. I strongly recommend everyone to get a twin.

To Frans Oort,

whose mathematics has had a

great impact on my education

This page intentionally left blank

1. Introduction

1.1 Main problem

Let $k = \bar{\mathbb{F}}_p$, and let X be a smooth proper variety over k. A *lift*[1] of X to characteristic 0 is a smooth proper morphism $\mathcal{X} \to \operatorname{Spec} R$, where R is a DVR with residue field k such that the fraction field $K = \operatorname{Frac} R$ has characteristic 0, together with an isomorphism $\mathcal{X}_0 \cong X$. Serre showed [Ser61] that there exists a smooth projective threefold X that cannot be lifted to characteristic 0, and Mumford improved this to a surface [Mum61], [Ill05, §8.6].

The idea of Serre's construction is to start with a finite abelian group G that has an action $G \to \mathrm{PGL}_n(k)$ that cannot be lifted to a map $G \to \mathrm{PGL}_n(R)$. The action of G on $\mathbb{P}^{n-1}$ could have fixed points, but there exists a complete intersection $Y \subseteq \mathbb{P}^{n-1}$ on which G acts without fixed points. Then construct X as the quotient Y/G, and prove that if X lifts, then so does the action of G on $H^0(Y, \mathcal{O}_Y(1))$, contradicting the choice of the action $G \to \mathrm{PGL}_n(k)$.

In Serre's example, there is a finite étale map $Y \to X$ where Y is a complete intersection. In particular, Y can be lifted to characteristic 0, by just lifting the equations that define it. A natural question is the following.

Question 1. *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. Does there exist a smooth proper variety Z together with a dominant rational map $Z \dashrightarrow X$ such that Z admits a lift to characteristic 0?*

The main result, given in Theorem 2 below, is a negative answer to this question.

[1] We give a more general definition of a lift in Definition 6.1.1.

The variety X we construct is a (smooth projective) surface of general type.

In Theorem 6.3.1, we prove that if X is a surface of Kodaira dimension $\kappa \leq 1$, there does always exist a surface Z dominating X that can be lifted, at least if $\operatorname{char} k \geq 5$. Because every curve can be lifted as well, the example we construct is therefore in some sense the 'smallest' possible example.

1.2 Outline of the proof

Just like in Serre's example, the idea is to show that if a lift exists, there is some extra structure that can be lifted along. However, in Serre's example the extra structure is a finite étale morphism $Y \to X$ with a group action, and we are trying to prevent varieties above X from lifting. Therefore, we will instead lift along extra structure living *below* X. We prove the following result on lifting morphisms to genus ≥ 2 curves (see Theorem 5.4.4).

Theorem 1. *Let R be a DVR with residue field k and fraction field K. Assume that $\bar{K}$ is isomorphic to $\mathbb{C}$ and that k is algebraically closed. Let $\mathcal{X} \to \operatorname{Spec} R$ be a smooth proper morphism, and let C be a smooth projective curve of genus $g \geq 2$ over the residue field k of R. Let $\psi\colon \mathcal{X}_0 \to C$ be a morphism with $\psi_* \mathcal{O}_{\mathcal{X}_0} = \mathcal{O}_C$.*

Then there exists a generically finite extension $R \to R'$, a smooth projective curve $\mathcal{Y} \to \operatorname{Spec} R'$, a morphism $\phi\colon \mathcal{X} \times_R R' \to \mathcal{Y}$, and a commutative diagram

$$\begin{tikzcd} & \mathcal{X}_0 \arrow[dl, "\psi"'] \arrow[dr, "\phi_0"] & \\ C \arrow[rr, "\chi"'] & & \mathcal{Y}_0 , \end{tikzcd} \tag{1}$$

where χ is purely inseparable. In particular, χ is a power of the relative Frobenius if $\operatorname{char} k = p > 0$, and χ is an isomorphism if $\operatorname{char} k = 0$.

If $\operatorname{char} k = 0$, there is a deformation-theoretic proof using a vanishing theorem of Kollár [Kol86, Thm. 2.1(iii)]; see Theorem 5.5.1. However, if $\operatorname{char} k = p > 0$, then the obstruction space does not vanish in general, cf. Example 5.5.4.

Because there are no morphisms of degree > 1 between two curves of the same genus $g \geq 2$ in characteristic 0, one can show (see Remark 5.4.10) that if $\mathcal{X}_0 \to C$ lifts after composing with Frob^n, it cannot be lifted after composing with further powers of Frobenius. Thus, one cannot simply replace Kollár's vanishing theorems by vanishing theorems for Frobenius pullbacks [Ara04].

Instead, the proof of Theorem 1 uses a theorem of Simpson (Theorem 5.3.8) that says that non-rigid rank 2 local systems on $\mathcal{X}_{\bar{K}}$ are pulled back from a curve. We construct an auxiliary π_1-representation of the curve C, and pull it back to $\pi_1^{\text{ét},\ell}(\mathcal{X}_0) \cong \pi_1^{\text{ét},\ell}(\mathcal{X}_{\bar{K}})$ to obtain a local system on $\mathcal{X}_{\bar{K}}$. Then Simpson's theorem produces a morphism to a curve Y over $\bar{K}$. We prove that Y has good reduction $\mathcal{Y}$ over a generically finite extension R' of R, and that the special fibre $\mathcal{Y}_0$ is related to the original curve C by the commutative diagram (1).

To apply Theorem 1 to the liftability problem, we prove the following result in Corollary 7.2.8 (for $r \geq 4$) and its improvement Corollary 7.3.3 (for $r \geq 3$). The proof also relies on Lemma 7.1.6.

Lemma. *Let $r \geq 3$, and let $C_1 = \ldots = C_r$ be a supersingular curve of genus $g \geq 2$. Then there exists a very ample line bundle $\mathscr{L}$ on $\prod C_i$ with the following property: if $\mathcal{C}_i$ are curves lifting the C_i, then no multiple $\mathscr{L}^{\otimes m}$ for $m > 0$ can be lifted to $\prod \mathcal{C}_i$.*

The proof uses the decomposition (Theorem 4.4.10)

$$\operatorname{Pic}\left(\prod_i C_i\right) = \prod_i \operatorname{Pic}(C_i) \times \prod_{i<j} \operatorname{Hom}(\mathbf{Jac}_{C_i}, \mathbf{Jac}_{C_j}).$$

The coordinates corresponding to the homomorphisms $\phi_{ji}\colon \mathbf{Jac}_{C_i} \to \mathbf{Jac}_{C_j}$ can be arranged in a non-commutative diagram (for simplicity drawn if $r = 4$)

$$\begin{tikzcd} \mathbf{Jac}_{C_1} \arrow[r, "\phi_{21}"] \arrow[d, "\phi_{31}"'] \arrow[dr] & \mathbf{Jac}_{C_2} \arrow[d, "\phi_{42}"] \arrow[dl] \\ \mathbf{Jac}_{C_3} \arrow[r, "\phi_{43}"'] & \mathbf{Jac}_{C_4} \end{tikzcd} . \tag{2}$$

If we choose the ϕ_{ji} such that the compositions in diagram (2) corresponding to the loops

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] \\ \bullet & \bullet \arrow[ul] \end{tikzcd}, \qquad \begin{tikzcd} \bullet \arrow[d] & \bullet \\ \bullet \arrow[r] & \bullet \arrow[ul] \end{tikzcd}, \qquad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[dl] \\ \bullet \arrow[u] & \bullet \end{tikzcd}$$

generate $\mathrm{End}^\circ(\mathbf{Jac}_{C_1})$, then this prevents $\mathscr{L}$ from lifting. Indeed, if $\mathscr{L}$ lifts, then so does diagram (2), which gives simultaneous lifts of all endomorphisms of the supersingular abelian variety $\mathbf{Jac}_{C_1}$. This is impossible by a dimension count.

This argument is carried out in detail in Corollary 7.2.8, which relies on a theorem of Albert (Theorem 7.2.1) on the generation of absolutely semisimple algebras. This proof actually needs $r \geq 4$ curves, but we provide an improvement to $r \geq 3$ curves in Corollary 7.3.3 using a more refined argument.

The lemma together with Theorem 1 implies the main theorem (Theorem 7.4.3):

Theorem 2. *Let $\mathscr{L}$ be as in the lemma above, and let $X \in |\mathscr{L}^{\otimes n}|$ be a general smooth section for $n \gg 0$. If $k \subseteq k'$ is a field extension and Z is a smooth proper k'-variety with a dominant rational map $Z \dashrightarrow X \times_k k'$, then Z cannot be lifted to characteristic 0.*

Indeed, assume Z admits a lift $\mathcal{Z}$ to characteristic 0. Then Theorem 1 shows that the morphisms $Z \to C_i$ lift, up to taking its Stein factorisation and composing by a power of Frobenius. Assume for simplicity that we get actual morphisms $\mathcal{Z} \to \mathcal{C}_i$. Then the image of the morphism $\mathcal{Z} \to \prod \mathcal{C}_i$ is a lift of some multiple mX as a divisor, contradicting the choice of $\mathscr{L}$.

The actual proof is more difficult, because the morphisms $Z \to C_i$ only lift after taking Stein factorisation and up to a Frobenius twist. In the argument above, we realise $X \in |\mathscr{L}|$ on the one hand as a line bundle whose components (as in diagram (2)) generate $\operatorname{End}^\circ(\mathbf{Jac}_{C_1})$, but on the other hand as the image of the map $Z \to \prod C_i$. The intermediate notion that the ϕ_{ji} generate all endomorphisms coming from an isogeny factor of $\mathbf{Jac}_{C_i}$ (Definition 7.1.2) is stable under pullback (Lemma 7.1.7). However, the difficulty is that image is only well-behaved under pushforward.

To get around this problem, we want to choose the divisor $X \in |\mathscr{L}^{\otimes n}|$ in such a way that for any product covering $g\colon \prod C_i' \to \prod C_i$, the inverse image $g^{-1}(X) \subseteq \prod C_i'$ is still irreducible. Indeed, then Lemma 7.4.2 implies that any irreducible subvariety $V \subseteq \prod C_i'$ with image $X \subseteq \prod C_i$ is a member of $|g^*\mathscr{L}^{\otimes d}|$ for some $d \in \mathbb{Z}_{>0}$. Note that when we construct X, we do not yet know what the covers $C_i' \to C_i$ are, so we need to construct an X that $g^{-1}(X) \subseteq \prod C_i'$ is irreducible for *all* finite coverings $C_i' \to C_i$. We prove in Proposition 3.2.4 that this *product covering property* (Definition 3.2.2) holds for a general smooth section $X \in |\mathscr{L}^{\otimes n}|$ for $n \gg 0$.

1.3 Structure of the argument

Main construction

The construction outlined above is carried out in detail in the last chapter (Chapter 7). The main theorem (Theorem 2 above) is proved as Theorem 7.4.3. The construction of a line bundle on $\prod C_i$ that does not lift is carried out in Section 7.2 and Section 7.3, using Lemma 7.1.6. Because of the difficulties indicated at the end of the introduction, most results are stated in a more general setting than explained in the introduction; see Setup 7.1.1 and Definition 7.1.2. In particular, we need to control what happens when applying finite covers $C_i' \to C_i$; this happens in Lemma 7.1.7.

The penultimate chapter (Chapter 6) contains an introduction to lifting. We work with a more general definition of a lift than the one stated in the introduction; see Definition 6.1.1 and Lemma 6.1.3. We prove in Theorem 6.3.1 that surfaces of Kodaira dimension ≤ 1 can always be dominated by a liftable surface.

Main technical chapters

The methods that go into the proof are divided into abelian methods (Chapter 4) and anabelian methods (Chapter 5). The former is largely a survey of classical results about **Hom** schemes between abelian varieties, the Picard scheme of a product, and specialisation maps. The latter is built around Simpson's theorem (Theorem 5.3.8) and its application to lifting morphisms to a curve (Theorem 1 above, see Theorem 5.4.4).

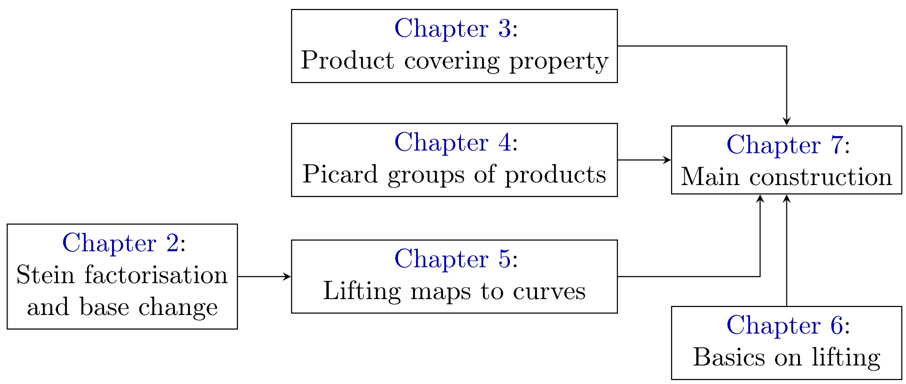
Auxiliary results

The proof of Theorem 1 relies on a study of Stein factorisation and base change (Chapter 2), which is of independent interest. We give conditions under which

Stein factorisation commutes with base change, as well as some counterexamples.

Finally, Chapter 3 contains some Bertini theorems that are needed to control what happens with $X \subseteq \prod C_i$ under pulling back along finite maps $g_i \colon C_i' \to C_i$, cf. the end of the introduction above. The main result is Proposition 3.2.4.

Leitfaden



1.4 Notation and terminology

In this document, a *variety* over a field k will always mean a geometrically integral, separated scheme of finite type over k. When we say *curve*, *surface*, *threefold*, etc., this is always understood to be a variety.

A scheme S has *equicharacteristic* 0 if for all $s \in S$, the residue field $\kappa(s)$ has characteristic 0. Equivalently, S is a $\mathbb{Q}$-scheme, i.e. all positive integers are invertible in $\mathcal{O}_S$.

A smooth proper variety X over $\bar{\mathbb{F}}_p$ is *supersingular* if there exists a model X_0 over a finite field $\mathbb{F}_q$ such that the eigenvalues of the q-power Frobenius on $H^*_{\text{ét}}(X, \mathbb{Q}_\ell)$ are all half integer powers of q. Note that if X is defined over a given finite field k, we might need to pass to a finite extension of k to get these Frobenius eigenvalues. By [Del74] and [Kro57, I], this is equivalent to the assertion that for any i, all slopes of the Frobenius action on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ are $i/2$.

In general, we will write $\operatorname{Mor}(X, Y)$ for the set of morphisms of schemes $X \to Y$. When working over a base S, everything is understood to be over that base. If A and B are group schemes over S, then $\operatorname{Hom}(A, B)$ will denote the set of homomorphisms of S-group schemes, to distinguish this from $\operatorname{Mor}(A, B)$. We use the same distinction for the scheme versions $\mathbf{Mor}(A, B)$ and $\mathbf{Hom}(A, B)$. The group $\operatorname{Hom}(A, B) \otimes \mathbb{Q}$ is denoted by $\operatorname{Hom}^\circ(A, B)$, and similarly for $\operatorname{End}^\circ(A)$ (this notation is recalled in Definition 4.3.5 for the reader's convenience).

A *pointed S-scheme* (X, σ) is an S-scheme X with a section $\sigma \colon S \to X$ of the structure morphism $X \to S$. A *morphisms of pointed S-schemes* $(X, \sigma) \to (Y, \tau)$ is a morphism $f \colon X \to Y$ such that $f \circ \sigma = \tau$. A *rational map* of integral schemes (S-schemes) $f \colon X \dashrightarrow Y$ is a morphism of schemes (S-schemes) $f \colon U \to Y$ defined on a dense open $U \subseteq X$.

A $\mathbb{Q}$-*rng* is a rng (i.e. a not necessarily commutative ring without unit) that is also a $\mathbb{Q}$-vector space.

Let $X \to S$ be a morphism between finite type k-schemes. Then a statement $\mathcal{P}$ *holds for a general member* X_s if there exists a dense open $U \subseteq S$ such that $\mathcal{P}$ holds for all X_s with $s \in U$. If this is the case, then a *general member* of the family is an X_s with $s \in U$, often assumed to be a *closed* point. If k is a finite field, there need not exist a general member that is defined over k.

If R is a DVR (discrete valuation ring) and $X \to \operatorname{Spec} R$ is a morphism of schemes, then we will denote by X_0 the special fibre and X_η the generic fibre. We sometimes write $\operatorname{Spec} R = \{0, \eta\}$ or $\operatorname{Spec} R = \{s, \eta\}$; in this case the special point s equals the point 0.

If X is a scheme and $\bar{x}$ a geometric point, then $\pi_1^{\text{ét}}(X, \bar{x})$ denotes the étale fundamental group, and $\pi_1^{\text{ét},\ell}(X, \bar{x})$ its maximal pro-ℓ quotient. We denote the topological fundamental group of a $\mathbb{C}$-variety X by $\pi_1^{\text{top}}(X, x)$. Unless otherwise specified, a representation $\rho\colon \pi_1^{\text{ét}}(X, \bar{x}) \to G$ (or its pro-ℓ-variant) to a profinite group G will always assumed to be continuous. We will omit mention of the base point $\bar{x}$ or x when it does not play an important role.

This page intentionally left blank

2. Stein factorisation and base change

Unfortunately, Stein factorisation does not commute with base change in general; see Lemma 2.1.4 and Remark 2.1.5. We collect some conditions under which it does (Corollary 2.2.3), as well as a counterexample (Example 2.3.3).

2.1 Stein factorisation after base change

Let $\phi\colon X \to Z$ be a proper morphism of schemes. The *Stein factorisation* of ϕ is

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

where $Y = \mathbf{Spec}\, \phi_* \mathcal{O}_X$. It has the property that $f_* \mathcal{O}_X = \mathcal{O}_Y$, and g is integral. If moreover Z is locally Noetherian, then g is finite. See [EGA3_{I}, Cor. 4.3.2] for the locally Noetherian case, and see [Stacks, Tag 03H2] for the general situation. The main lemma one proves is the following.

Lemma 2.1.1. *Let $f\colon X \to Y$ be a proper morphism such that $f_* \mathcal{O}_X = \mathcal{O}_Y$. Then f has geometrically connected fibres.*

Proof. See [*loc. cit.*]. □

The converse is not always true, as we will see below. However, we do get a converse under stronger hypotheses.

Lemma 2.1.2. *Let $f\colon X \to Y$ be a proper and flat morphism whose geometric fibres are reduced and connected. Then $f_* \mathcal{O}_X = \mathcal{O}_Y$.*

Proof. The $\kappa(y)$-varieties X_y satisfy $H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$. Cohomology and base change [Har77, Thm. III.12.11] now implies that $f_* \mathcal{O}_X$ is locally free of rank 1, and $f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \kappa(y) \cong \kappa(y)$ for all $y \in Y$. This implies that the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. $\square$

Setup 2.1.3. Let S be a scheme, let $\phi\colon X \to Z$ be a proper morphism of S-schemes, and let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be its Stein factorisation, i.e. f is proper with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and g is integral.

Lemma 2.1.4. *Let S, X, Y, and Z be as in Setup 2.1.3, and let $T \to S$ be arbitrary. If $Y' = \mathbf{Spec}\, f_{T,*}\mathcal{O}_{X_T}$, then we have a diagram*

$$X_T \to Y' \xrightarrow{h} Y_T \to Z_T.$$

The morphism h is integral with geometrically connected fibres, hence is radicial.

Proof. By definition, Y' is the Stein factorisation of f_T, which proves that h is integral. By Lemma 2.1.1, f has geometrically connected fibres, hence so does h. Hence, for each point $y \in Y_T$, the fibre $Y'_y \to y$ is geometrically connected and integral, hence consists of a single point y' whose residue extension $\kappa(y) \to \kappa(y')$ is purely inseparable. This forces h to be radicial [EGA1, Prop. 3.5.8]. $\square$

Remark 2.1.5. Alternatively, Y' could have been defined as $\mathbf{Spec}\, \phi_{T,*}\mathcal{O}_{X_T}$, since ϕ_T differs from f_T by the affine morphism g_T. Thus, Y' can also be viewed as the Stein factorisation of ϕ_T.

Comparing the Stein factorisation of ϕ with that of ϕ_T, we see that the only new part that can be introduced to the Stein factorisation of ϕ_T is the morphism h. Our main goal is to understand h, but unfortunately we can only prove some results when we put strong conditions on X and Y. The general question with geometric restrictions on X and Z is hard.

For the record, we state an easy case in which base change always holds.

Lemma 2.1.6. *Let S, X, Y, and Z be as in Setup 2.1.3. If $T \to S$ is flat, then the morphism h of Lemma 2.1.4 is an isomorphism.*

Proof. This follows since formation of $\phi_* \mathcal{O}_X$ commutes with flat base change. □

Corollary 2.1.7. *Let S, X, Y, and Z be as in Setup 2.1.3. Assume moreover that S is integral with generic point η, that Z is normal, and that each component of Z dominates S. If $\phi_{\eta,*}\mathcal{O}_{X_\eta} = \mathcal{O}_{Z_\eta}$, then $\phi_*\mathcal{O}_X = \mathcal{O}_Z$.*

Proof. The inclusion $\eta \to S$ is flat, hence Stein factorisation commutes with base change along $\eta \to S$. Then $g\colon Y \to Z$ is an integral morphism that is an isomorphism above $\eta \in S$. Thus, above each component Z_i of Z the map $g\colon g^{-1}(Z_i) \to Z_i$ is an integral morphism that is birational, hence it is an isomorphism since Z_i is normal and integral. □

Example 2.1.8. The assumptions of the corollary are satisfied for example if S is locally Noetherian, integral, and normal, and $Z \to S$ is a *normal morphism*, i.e. a flat morphism whose geometric fibres are normal and locally Noetherian. See [EGA4_{II}, Cor. 6.8.1] for the definition, and [EGA4_{II}, Cor. 6.5.4] for the proof that this implies that Z is normal.

2.2 Pushforward and base change

In this section, we will consider conditions on X and Y in Setup 2.1.3 that force the map h of Lemma 2.1.4 to be an isomorphism.

Setup 2.2.1. Let S and $f\colon X \to Y$ be as in Setup 2.1.3. We will moreover assume that X and Y are of finite type over S, with integral and normal geometric fibres $X_{\bar{s}}$ and $Y_{\bar{s}}$ for all $s \in S$.

Lemma 2.2.2. *Let S and $f\colon X \to Y$ be as in Setup 2.2.1. Then for every $s \in S$, the map $h\colon \mathbf{Spec}\, f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}} \to Y_{\bar{s}}$ of Lemma 2.1.4 sits in a commutative diagram*

$$\begin{tikzcd} Y_{\bar{s}}^{(p^{-n})} \arrow[rr, "\mathrm{Frob}^n_{Y_{\bar{s}}/\bar{s}}"] \arrow[dr] & & Y_{\bar{s}} \\ & \mathbf{Spec}\, f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}} \arrow[ur] & \end{tikzcd}$$

for some $n \in \mathbb{Z}_{\geq 0}$. Here, we understand that $n = 0$ if $\operatorname{char} \kappa(s) = 0$.

Proof. The morphism $\mathbf{Spec}\, f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}} \to Y_{\bar{s}}$ is finite and radicial. But both $X_{\bar{s}}$ and $Y_{\bar{s}}$ are normal integral varieties over the algebraically closed field $\kappa(\bar{s})$. Hence, $\mathbf{Spec}\, f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}}$ is the normalisation of $Y_{\bar{s}}$ in a purely inseparable field extension of $K(Y_{\bar{s}})$. Such an extension is dominated by $K(Y_{\bar{s}})^{1/p^n}$ for some n. The integral closure of $Y_{\bar{s}}$ in this further field extension is just $Y_{\bar{s}}^{(p^{-n})}$, since the latter is finite over $Y_{\bar{s}}$ and integrally closed since $Y_{\bar{s}}$ is. □

Corollary 2.2.3. *With the assumptions of Setup 2.2.1, if all points of S have characteristic 0, then $f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}} = \mathcal{O}_{Y_{\bar{s}}}$ for all $s \in S$.* □

In Example 2.3.3, we give examples in any positive characteristic p showing that n need not be 0 in Lemma 2.2.2, even if X and Y are *smooth* over S. There should exist similar counterexamples in mixed characteristic.

Corollary 2.2.4. *With the assumptions of Setup 2.2.1, if the fibres of $Y \to S$ are 1-dimensional, then for every $s \in S$ the map $h\colon \mathbf{Spec}\, f_{\bar{s},*}\mathcal{O}_{X_{\bar{s}}} \to Y_{\bar{s}}$ is a power of Frobenius.*

Proof. It is the normalisation morphism in a finite purely inseparable extension of the function field $K(Y_{\bar{s}})$ (compare the proof of Lemma 2.2.2). But the only purely inseparable extensions of a transcendence degree 1 field extension $K(Y_{\bar{s}})$ over an algebraically closed field $\kappa(\bar{s})$ are of the form $K(Y_{\bar{s}})^{1/p^n}$ for some n. □

2.3 Counterexample in positive characteristic

One can ask whether Corollary 2.2.3 is also true in characteristic $p > 0$ or in mixed characteristic. In Example 2.3.3, we construct a counterexample in arbitrary positive characteristic p, where in fact $X \to S$ and $Y \to S$ are *smooth*.

Lemma 2.3.1. *Let C be an elliptic curve over an algebraically closed field k, and let $f\colon C' \to C$ be a nontrivial α_p-torsor. Then C' is an elliptic curve.*

Proof. A nontrivial α_p-torsor over a normal integral scheme is integral, because it is Zariski-locally given by adjoining a p-th root of a non-p-power. Moreover, $f_*\mathcal{O}_{C'}$ has a filtration by coherent subsheaves whose subquotients are $\mathcal{O}_C$, hence

$$\chi(C', \mathcal{O}_{C'}) = \chi(C, f_*\mathcal{O}_{C'}) = p \cdot \chi(C, \mathcal{O}_C) = 0. \tag{2.3.1}$$

If C' is not smooth, then its normalisation is $\mathbb{P}^1$. By Riemann–Hurwitz, there are no dominant maps from $\mathbb{P}^1$ to an elliptic curve (even in characteristic p). Thus, C' is smooth, hence an elliptic curve by (2.3.1). $\square$

Remark 2.3.2. The result is not true over non-algebraically closed fields k unless one assumes that the torsor is *geometrically* nontrivial. This is the basis for our example: we will construct a family of geometrically nontrivial α_p-torsors degenerating to an α_p-torsor that is nontrivial but comes from the ring extension $\bar{\mathbb{F}}_p[x] \to \bar{\mathbb{F}}_p[\sqrt[p]{x}]$, hence picks up a finite part in its Stein factorisation.

Example 2.3.3. Let $R = \bar{\mathbb{F}}_p[t]$, and let $S = \operatorname{Spec} R$. Set $Y = \operatorname{Spec} \bar{\mathbb{F}}_p[t, x] \cong \mathbb{A}^1_R$. Let E be a supersingular elliptic curve over $\bar{\mathbb{F}}_p$, and consider the constant elliptic curve $\mathcal{E} = E \times Y \to Y$.

The short exact sequence $0 \to \alpha_p \to \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}} \to 0$ on the flat site of $\mathcal{E}$ gives a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(E,\mathcal{O}_E)[t,x] \arrow[r, "\mathrm{Frob}^*"] & H^0(E,\mathcal{O}_E)[t,x] \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (Z) [near end]\tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}, "\delta"] \\ H^1(\mathcal{E},\alpha_p) \arrow[r] & H^1(E,\mathcal{O}_E)[t,x] \arrow[r, "\mathrm{Frob}^*"] & H^1(E,\mathcal{O}_E)[t,x]\,, \end{tikzcd}$$

where $H^0(E,\mathcal{O}_E)$ and $H^1(E,\mathcal{O}_E)$ are 1-dimensional vector spaces over $\bar{\mathbb{F}}_p$. Since E is supersingular, the Frobenius action on H^1 is zero. Let $\eta \in H^1(E,\mathcal{O}_E)$ be a nonzero element, and choose a lift $\beta \in H^1(\mathcal{E},\alpha_p)$ of $\eta t \in H^1(E,\mathcal{O}_E)[t,x]$. Then the fibre β_0 of β over $t = 0 \in S$ maps to 0 in $H^1(E,\mathcal{O}_E)[x]$, so $\beta_0 = \delta(f)$ for some $f \in H^0(E,\mathcal{O}_E)[x]$. Replacing β by $\beta + \delta(x - f)$, we may assume that $\beta_0 = \delta(x)$. Let $X \to \mathcal{E}$ be the α_p-torsor corresponding to β, and let $f \colon X \to Y$ be the map $X \to \mathcal{E} \to Y$.

We will prove that $X \to S$ is smooth (Proposition 2.3.8), that $f_*\mathcal{O}_X = \mathcal{O}_Y$ (Corollary 2.3.6), and that this is not true in the fibre above 0 (Lemma 2.3.9).

Lemma 2.3.4. *For any $y \in Y$ not lying above $0 \in S$, the $\kappa(y)$-variety X_y is an elliptic curve (in particular, it is smooth and geometrically connected).*

Proof. Let $s \neq 0$ be the image of y in S, and consider $\beta|_{\mathcal{E}_{\bar{y}}} \in H^1(\mathcal{E}_{\bar{y}},\alpha_p)$. It is nonzero, since its image in $H^1(\mathcal{E}_{\bar{y}},\mathcal{O}_{\mathcal{E}_{\bar{y}}}) = H^1(E,\mathcal{O}_E) \otimes_{\bar{\mathbb{F}}_p} \kappa(\bar{y})$ is ηs. Hence, the α_p-torsor $X_{\bar{y}} \to \mathcal{E}_{\bar{y}}$ is nontrivial. The result now follows from Lemma 2.3.1. □

Corollary 2.3.5. *The morphism $f \colon X \to Y$ is smooth above $S \setminus \{0\}$.*

Proof. It is flat with smooth fibres, hence smooth [$\mathrm{EGA4}_{\mathrm{IV}}$, Thm. 17.5.1]. □

Corollary 2.3.6. *The morphism $f \colon X \to Y$ satisfies $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Proof. Lemma 2.1.2 and Lemma 2.3.4 imply that $f_{S\setminus\{0\},*}\mathcal{O}_{X_{S\setminus\{0\}}} = \mathcal{O}_{Y_{S\setminus\{0\}}}$. Then Corollary 2.1.7 implies that $f_*\mathcal{O}_X = \mathcal{O}_Y$, since Y is normal. □

Lemma 2.3.7. *The fibre X_0 is isomorphic to $E \times \operatorname{Spec} \bar{\mathbb{F}}_p[\sqrt[p]{x}]$.*

Proof. By construction, we have $\beta_0 = \delta(x)$. Hence the α_p-torsor $X_0 \to \mathcal{E}_0$ is given by adjoining a p-th root of $x \in H^0(\mathcal{E}_0, \mathcal{O}_{\mathcal{E}_0}) = H^0(E, \mathcal{O}_E)[x]$. $\square$

Proposition 2.3.8. *The morphism $X \to S$ is smooth.*

Proof. Clearly $X \to S$ is flat, since $X \to \mathcal{E}$ and $\mathcal{E} \to S$ are. Thus, again by [EGA4$_{\mathrm{IV}}$, Thm. 17.5.1], it suffices to prove that all fibres are smooth. The morphism $X_{S \setminus \{0\}} \to S \setminus \{0\}$ is smooth, by Corollary 2.3.5 and since $Y \to S$ is smooth. Finally, the fibre X_0 is smooth since it equals $E \times \operatorname{Spec} \bar{\mathbb{F}}_p[\sqrt[p]{x}]$ by Lemma 2.3.7. $\square$

Thus, $f\colon X \to Y$ is a proper morphism of smooth S-schemes with $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Lemma 2.3.9. *The morphism $f_0\colon X_0 \to Y_0$ does not satisfy $f_{0,*}\mathcal{O}_{X_0} = \mathcal{O}_{Y_0}$.*

Proof. This is the morphism $E \times \operatorname{Spec} \bar{\mathbb{F}}_p[\sqrt[p]{x}] \to \operatorname{Spec} \bar{\mathbb{F}}_p[x]$. The pushforward of the structure sheaf corresponds to the finite extension $\bar{\mathbb{F}}_p[x] \to \bar{\mathbb{F}}_p[\sqrt[p]{x}]$. $\square$

This page intentionally left blank

3. Bertini theorems for products

3.1 Two Bertini theorems

The main results of this section are Lemma 3.1.2 and Lemma 3.1.3. We start with a well-known lemma; the proof is included for the reader's convenience.

Lemma 3.1.1. *Let $f\colon X \to Y$ be a projective morphism of Noetherian schemes, and let $\mathcal{O}(1)$ be a relatively ample sheaf. Let $\mathscr{F}$ be a coherent sheaf on X. Then there exists $n_0 \in \mathbb{Z}_{>0}$ such that*

$$H^i(X_y, \mathscr{F}_y(n)) = 0$$

for all $y \in Y$, all $n \geq n_0$, and all $i > 0$.

Proof. By generic flatness [EGA4$_{\mathrm{II}}$, Thm. 6.9.1], there is a nonempty open $U \subseteq Y$ such that $\mathscr{F}|_{f^{-1}(U)}$ is flat over U. By a Noetherian induction, we obtain a stratification of Y by locally closed subsets V_i over which $\mathscr{F}$ is flat. The question only concerns the fibres of f, and the result for each member of a finite covering of Y proves the result for Y. Thus, replacing Y by $\coprod V_i$, we may assume $\mathscr{F}$ is flat over Y. Similarly, we may assume that Y is integral and affine.

Now Serre vanishing [EGA3$_{\mathrm{I}}$, Thm. 2.2.1] shows that there exists n_0 such that $R^i f_* \mathscr{F}(n) = 0$ for all $n \geq n_0$ and all $i > 0$. Then cohomology and base change [Har77, Thm. III.12.11] and a descending induction on i show that the natural map $R^i f_* \mathscr{F}(n) \otimes \kappa(y) \to H^i(X_y, \mathscr{F}_y(n))$ is an isomorphism, which proves the lemma. $\square$

Lemma 3.1.2. *Let $f\colon X \to Y$ be a smooth projective morphism of relative dimension d of k-schemes, let $e = \dim Y$, and let H be an ample divisor on X.*

Then there exists $n_0 \in \mathbb{Z}_{>0}$ such that for all $n \geq n_0$ and for a general divisor $D \in |nH|$, each fibre $D \cap X_y$ has at most e singular points.

In particular, each fibre has isolated singularities.

Proof. Since $\mathcal{O}_X(H)$ is ample on X, it is also f-ample [EGA2, Prop. 4.6.13(v)]. Write X_Y^{e+1} for the $(e+1)$-fold fibre product $X \times_Y \ldots \times_Y X$, and let $T \subseteq X_Y^{e+1}$ be the open set where all coordinates are pairwise distinct. The map $\pi \colon T \to Y$ is smooth of relative dimension $d(e+1)$, so $\dim T = de+d+e$. For $i \in \{1, \ldots, e+1\}$, write $\Delta_{0,i} = \{(x_0, \ldots, x_{e+1}) \in X \times_Y T \mid x_0 = x_i\}$. Let $Z \subseteq X \times_Y T$ be the subvariety given by the ideal sheaf

$$\mathcal{I}_Z := \mathcal{I}^2_{\Delta_{0,1}} \cdots \mathcal{I}^2_{\Delta_{0,e+1}},$$

so that the fibre over $t = (x_1, \ldots, x_{e+1}) \in T$ is the finite subscheme $Z_t \subseteq X_{\pi(t)}$ given by $\mathfrak{m}^2_{x_1} \cdots \mathfrak{m}^2_{x_{e+1}}$. Applying Lemma 3.1.1 to the morphism $X \times_Y T \to T$ and the sheaf $\mathcal{I}_Z$, we conclude that there exists $n_0 \in \mathbb{Z}_{>0}$ such that the map

$$H^0\left(X_y, \mathcal{O}_{X_y}(nH|_{X_y})\right) \to H^0\left(Z_t, \mathcal{O}_{Z_t}(nH|_{X_y})\right) \tag{3.1.1}$$

is surjective for all $y \in Y$, all $t \in T_y$, and all $n \geq n_0$.

Consider the incidence scheme $W_n \subseteq |nH| \times T$ given by

$$W_n = \left\{ (D,t) \in |nH| \times T \;\middle|\; Z_t \subseteq D\big|_{X_{\pi(t)}} \right\}.$$

Write π_1 and π_2 for the projections to $|nH| \cong \mathbb{P}^M$ and T respectively. The fibre of π_2 above any $t \in T$ has codimension $\ell(Z_t) = (e+1)(d+1)$ in $|nH|$ by surjectivity of (3.1.1). Hence, W_n has codimension $(e+1)(d+1)$ in each component of $|nH| \times T$. Since $|nH| \times T$ has dimension $M+de+d+e$, we conclude that W_n has dimension $M-1$, hence π_1 cannot be dominant. The complement of the image is an open $U \subseteq |nH|$ above which the required property holds. $\square$

The case $Y = \operatorname{Spec} k$ (so $e = 0$) recovers an asymptotic version of the classical Bertini theorem on smooth hyperplane sections. The proof is essentially the same. In the classical version, surjectivity holds already for $n = 1$ when H is very ample, because H separates tangent vectors.

Lemma 3.1.3. *Let X be a projective k-variety with an ample line bundle H, let T be a k-scheme of finite type, and let $Z \subseteq T \times X$ be a subscheme such that* $\dim Z_t \geq 1$ *for all $t \in T$. Then there exists $n_0 \in \mathbb{Z}_{>0}$ such that for every $n \geq n_0$, a general member of $|nH|$ does not contain any member Z_t.*

Proof. We reduce to the case that Z is flat over T and T is integral by the same argument as in the proof of Lemma 3.1.1. Then the Hilbert polynomial $\chi(Z_t, \mathcal{O}(nH))$ is independent of t [EGA3$_{\mathrm{II}}$, Thm. 7.9.4]. Since $\dim Z_t \geq 1$, the Hilbert polynomial has degree ≥ 1. Hence, there exists $n_0 \in \mathbb{Z}_{>0}$ such that $\chi(Z_t, \mathcal{O}(nH)) > \dim T + 1$ for all $n \geq n_0$. By Lemma 3.1.1, after enlarging n_0 if necessary, we also have

$$H^1(X, \mathcal{I}_{Z_t}(nH)) = 0, \qquad\qquad H^i(Z_t, \mathcal{O}_{Z_t}(nH)) = 0$$

for all $t \in T$, all $n \geq n_0$, and all $i > 0$. Now consider the incidence scheme

$$Y_n = \left\{ (D,t) \in |nH| \times T \;\middle|\; Z_t \subseteq D \right\}.$$

Write π_1 and π_2 for its projections to $|nH| \cong \mathbb{P}^M$ and T respectively. If $n \geq n_0$, then the fibre of π_2 above any $t \in T$ has dimension

$$
\begin{aligned}
h^0(X, \mathcal{O}_X(nH)) - h^0(Z_t, \mathcal{O}_{Z_t}(nH)) &= h^0(X, \mathcal{O}_X(nH)) - \chi(Z_t, \mathcal{O}_{Z_t}(nH)) \\
&< h^0(X, \mathcal{O}_X(nH)) - \dim T - 1.
\end{aligned}
$$

Hence, Y_n has dimension strictly smaller than $h^0(X, \mathcal{O}_X(nH)) - 1 = M$, so π_1 cannot be dominant. □

Remark 3.1.4. The family Z of all hypersurfaces in $\mathbb{P}^m$ of degree $\leq d$ shows that even if H is very ample, the multiple needed to avoid a family of subvarieties can be arbitrarily large, and depends on Z as well as on X and H.

Remark 3.1.5. The assumption on the dimension of the fibres cannot be dropped: no very ample linear system avoids all points in X.

Remark 3.1.6. If k is finite, there need not exist a general member of $|nH|$ that is defined over k. It would be interesting to see if there are analogues of Lemma 3.1.2 and Lemma 3.1.3 for a set of positive density (cf. [Poo04]).

3.2 Products of finite coverings

We will work in the following setup.

Setup 3.2.1. Let k be a field, and let $C_1, \ldots, C_r$ be smooth projective curves over k. We will write $X = C_1 \times \ldots \times C_r$, with projection maps $\pi_i \colon X \to C_i$. We will consider finite coverings $f_i \colon C_i' \to C_i$ of the C_i by smooth projective curves C_i', and in this case we will write $X' = C_1' \times \ldots \times C_r'$, with the map $f \colon X' \to X$. The projections $X' \to C_i'$ are also denoted by π_i.

We are interested in the following property for divisors $D \subseteq X$.

Definition 3.2.2. Let $C_1, \ldots, C_r$ and X be as in Setup 3.2.1. Then we say that an effective divisor $D \subseteq X$ *satisfies the product covering property* if for all finite coverings $C_i' \to C_i$ of the C_i by smooth projective curves, the inverse image $D' = f^{-1}(D) \subseteq X'$ is geometrically irreducible. In particular, D itself is geometrically irreducible.

Using the Bertini theorems from the previous section, we will show that a sufficiently general divisor D satisfies this property; see Proposition 3.2.4.

Lemma 3.2.3. *Let $r \geq 2$, and let $D \subseteq C_1 \times \ldots \times C_r$ be an effective ample divisor. Assume that D is geometrically normal and does not contain $\pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j)$ for any $i \neq j$ and any $x_i \in C_i$, $x_j \in C_j$, and that $D \cap \pi_i^{-1}(x_i)$ is generically smooth for all i and all $x_i \in C_i$. Then D satisfies the product covering property.*

Proof. Since all statements are geometric, we may assume k is algebraically closed. Let $f_i \colon C_i' \to C_i$ be finite coverings by smooth projective curves. If f_i is purely inseparable, then it is radicial, hence a homeomorphism. This does not affect irreducibility, so we only have to treat the case that the f_i are separable, i.e. generically étale.

The inverse image $D' = f^{-1}(D)$ is ample since D is [EGA2, Cor. 6.6.3], hence D' is connected since $r \geq 2$ [Har77, Cor. III.7.9]. Since D' is a divisor in a regular scheme, it is Cohen–Macaulay [Stacks, Tag 02JN]. We will show that the assumptions on D imply that D' is regular in codimension 1. Then Serre's criterion implies that D' is normal [EGA4_{II}, Thm. 5.8.6]. Then D' is integral, since it is normal and connected [EGA4_{II}, 5.13.5].

Now let $x' \in D'$ be a point of codimension 1, and consider the image x_i' of x' in C_i'. Let η_i' be the generic point of C_i'. Let x, x_i, and η_i be the images of x' in

X, of x_i' in C_i, and of η_i' in C_i respectively. Consider the set

$$I = \left\{ i \in \{1, \ldots, r\} \;\middle|\; x_i' = \eta_i' \right\} = \left\{ i \;\middle|\; x_i = \eta_i \right\}.$$

If $|I| > 2$, then $\overline{\{x'\}} \subseteq \pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j) \cap \pi_k^{-1}(x_k)$, contradicting the fact that x' has codimension 1 in D'. If $|I| = 2$, then x' is the generic point of $\pi_i^{-1}(x_i') \cap \pi_j^{-1}(x_j')$, hence D contains $\pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j)$, contradicting the assumptions on D. Hence, $|I| \leq 1$. If $|I| = 0$, then x maps to η_i for each i, hence $\mathcal{O}_{D,x}$ contains the fields $\kappa(\eta_i)$ for all i. Since f_i is separable, the field extension $\mathcal{O}_{C_i,\eta_i} \to \mathcal{O}_{C_i',\eta_i'}$ is étale. Hence, x is in the étale locus of $D' \to D$, so $\mathcal{O}_{D',x'}$ is a regular ring since $\mathcal{O}_{D,x}$ is [EGA4$_{\mathrm{IV}}$, Prop. 17.5.8].

Finally, if $|I| = 1$, then x is the generic point of a component of $D \cap \pi_i^{-1}(x_i)$, and similarly for x'. As in the case $|I| = 0$, the extensions $C_j' \to C_j$ for $j \neq i$ do not affect normality at x, so we may assume that $C_j' = C_j$ for $j \neq i$. Then the natural map $D' \cap \pi_i^{-1}(x_i') \to D \cap \pi_i^{-1}(x_i)$ is an isomorphism (since $\kappa(x_i) \to \kappa(x_i')$ is an isomorphism, because k is algebraically closed).

Consider the local homomorphism $\mathcal{O}_{C_i',x_i'} \to \mathcal{O}_{D',x'}$. It is flat because every irreducible component of D' dominates C_i'. Moreover, the fibre $\mathcal{O}_{D',x'}/\mathfrak{m}_{x_i'}\mathcal{O}_{D',x'}$ is a field, as $D' \cap \pi_i^{-1}(x_i') = D \cap \pi_i^{-1}(x_i)$ is generically smooth by the assumptions on D. Since $\mathcal{O}_{C_i',x_i'}$ is regular and $\mathcal{O}_{C_i',x_i'} \to \mathcal{O}_{D',x'}$ is flat and local, we conclude that $\mathcal{O}_{D',x'}$ is a regular ring [EGA4$_{\mathrm{II}}$, Prop. 6.5.1]. $\square$

Proposition 3.2.4. *Let $C_1, \ldots, C_r$ be smooth projective curves over k, and assume $r \geq 3$. Let H be an ample divisor on $X = C_1 \times \ldots \times C_r$. Then there exists $n_0 \in \mathbb{Z}_{>0}$ such that for all $n \geq n_0$, a general divisor $D \in |nH|$ satisfies the product covering property of Definition 3.2.2.*

Proof. There exists n_0 such that for all $n \geq n_0$, the divisor nH is very ample.

Then a general $D \in |nH|$ is smooth, so in particular geometrically normal. By Lemma 3.1.2 (increasing n_0 if necessary), for a general D all fibres $D \cap \pi_i^{-1}(x_i)$ for $x_i \in C_i$ have isolated singularities; in particular they are generically smooth. By Lemma 3.1.3 (again increasing n_0 if needed), a general D does not contain the families $\pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j)$, which have dimension ≥ 1 since $r \geq 3$. Then Lemma 3.2.3 shows that these D satisfy the product covering property. $\square$

Remark 3.2.5. On the other hand, for $r \leq 2$ no divisor has the product covering property. For $r \leq 1$ this is obvious. For $r = 2$, we claim that for any effective $D \subseteq C_1 \times C_2$, there exist coverings $f_i \colon C_i' \to C_i$ such that $f^{-1}(D)$ is reducible.

Indeed, we may assume D is irreducible. Then it dominates either C_1 or C_2; say that it dominates C_2. The generic points of $f^{-1}(D)$ are then the points of $D \cap (C_1' \times \operatorname{Spec} K(C_2'))$, so it suffices to consider the divisor $D \cap (C_1 \times \operatorname{Spec} K(C_2))$. Without loss of generality we may assume that it has degree $d > 1$; if the degree is 1 we first replace C_1 by a cover. If it has degree $d > 1$, then after a degree d extension of C_2 it splits off a rational point, hence becomes reducible.

This argument also shows that there is no hope for more general covers of $C_1 \times \ldots \times C_r$. Indeed, a suitable covering of $K(C_2 \times \cdots \times C_r)$ will split off a rational point, as in the argument for $r = 2$.

Example 3.2.6. The conclusion of Proposition 3.2.4 is not true for *all* smooth divisors $D \in |nH|$. For example, let $r = 3$, $C_i = \mathbb{P}^1$ with coordinates $[x_i : y_i]$, and let D be given by $x_1x_2x_3 - y_1y_2y_3 \in H^0((\mathbb{P}^1)^3, \mathcal{O}(1)^{\boxtimes 3})$. Consider the affine charts associated with inverting one of $\{x_i, y_i\}$ for each i. Then the local equations are $xyz - 1$ and $xy - z$, both of which define a smooth surface.

However, if we take the covers given by $C_i' = \mathbb{P}^1$ with map $C_i' \to C_i$ given by $[x_i : y_i] \mapsto [x_i^2 : y_i^2]$, then D' splits as $V(x_1x_2x_3 - y_1y_2y_3) \cup V(x_1x_2x_3 + y_1y_2y_3)$.

So even when D is smooth (in *arbitrary* characteristic), it does not automatically satisfy the product covering property. This D violates the assumptions of Lemma 3.2.3 because it contains $\pi_1^{-1}([0:1]) \cap \pi_2^{-1}([1:0])$.

4. Abelian methods

4.1 Extension of rational morphisms

We recall a few classical results that we will use repeatedly.

Definition 4.1.1. An *abelian scheme* $A \to S$ is a smooth proper group scheme over S with geometrically connected fibres.

Theorem 4.1.2 (Weil)**.** *Let S be a normal Noetherian scheme, let $X \to S$ be smooth, and let $A \to S$ be an abelian scheme. If $f \colon X \dashrightarrow A$ is a rational map of S-schemes, then f extends uniquely to a morphism $f \colon X \to A$ of S-schemes.*

Proof. This follows from [BLR, Thm. 4.4.1]. □

Corollary 4.1.3 (Néron extension property of abelian schemes)**.** *Let S be the spectrum of a DVR R with fraction field K, and let $A \to S$ be an abelian scheme. If X is a smooth S-scheme, then*

$$\operatorname{Hom}_S(X, A) \xrightarrow{\sim} \operatorname{Hom}_K(X_K, A_K).$$

Proof. A K-morphism $X_K \to A_K$ is a rational morphism $X \dashrightarrow A$ of S-schemes, hence it extends (uniquely) by Theorem 4.1.2. □

Corollary 4.1.4. *Let S be a normal Noetherian scheme, let $X \to S$ be smooth, and let $C \to S$ be a smooth proper curve over S with a section σ such that the fibres C_s have genus ≥ 1. If $f \colon X \dashrightarrow C$ is a rational map of S -schemes, then f extends (uniquely) to a morphism $f \colon X \to C$ of S-schemes.*

Proof. The section σ gives an Abel–Jacobi map $C \to \mathbf{Pic}^0_{C/S}$ of S-schemes, which is a closed immersion since the C_s have genus ≥ 1. Composing the rational map $X \dashrightarrow C$ with the Abel–Jacobi map $C \to \mathbf{Pic}^0_{C/S}$ gives a rational map $X \dashrightarrow \mathbf{Pic}^0_{C/S}$, which extends to a morphism by Theorem 4.1.2. $\square$

See Section 4.4 for a rephrasing of the Abel–Jacobi map in terms of the relative Albanese $\mathbf{Alb}^1_{C/S}$.

We give an alternative argument when $S = \operatorname{Spec} k$ is the spectrum of a field k, because we need a variant of this argument at some point as well. This proof does not use that C has a section.

Second proof (if $S = \operatorname{Spec} k$). Let $\bar{\Gamma}_f \subseteq X \times C$ be the closure of the graph, so that we get a commutative diagram

$$\begin{tikzcd} & \bar{\Gamma}_f \arrow[dl, "\pi"'] \arrow[dr, "\bar{f}"] & \\ X \arrow[rr, dashed, "f"'] & & C\,. \end{tikzcd}$$

Since π is birational and X is smooth, the fibres of π are rationally connected [Mur58]. Since $g(C) \geq 1$, there are no nonconstant maps $\mathbb{P}^1 \to C$, hence $\bar{f}$ contracts the fibres of π. Since π is proper, the universal property of contractions [EGA2, Lem. 8.11.1] implies that $\bar{f}$ factors through X. $\square$

4.2 Hom schemes of abelian schemes

We collect some proofs of well-known facts about homomorphisms between abelian schemes over an arbitrary base S. In particular, Theorem 4.2.8 shows that the **Mor** scheme between abelian schemes is always representable. Note that we do *not* assume $A \to S$ to be locally projective (so a priori the $\mathscr{M}or$ functor is only representable as an algebraic space; see Lemma 4.2.2).

Remark 4.2.1. Given two abelian schemes A, B over S, we write $\operatorname{Hom}_S(A, B)$

for the set of morphisms of S-group schemes $A \to B$, and $\mathrm{Mor}_S(A, B)$ for the set of all morphisms of S-schemes, and similarly for the corresponding sheaves $\mathscr{H}om$ and $\mathscr{M}or$ and their representing representing objects $\mathbf{Hom}$ and $\mathbf{Mor}$.

Lemma 4.2.2. *Let A and B be abelian schemes over S. Then the functor*

$$\begin{aligned} \mathscr{H}om_S(A, B) \colon (\underline{\mathrm{Sch}}\,/S)^{\mathrm{op}} &\longrightarrow \underline{\mathrm{Set}} \\ T &\longmapsto \mathrm{Hom}_T(A_T, B_T) \end{aligned}$$

is representable by an algebraic space $\mathbf{Hom}_S(A, B)$ that is separated and locally of finite presentation over S.

Proof. The functor $\mathscr{M}or_S(A, B)$ parametrising all S-morphisms $A \to B$ is representable by an open subspace $\mathbf{Mor}_S(A, B)$ of the algebraic space $\mathbf{Hilb}_{A \times_S B/S}$ that is locally of finite presentation over S; see [Stacks, Tags 0D1B and 0D1C]. Moreover, $\mathbf{Hilb}_{A \times_S B/S}$ is separated over S by [Stacks, Tag 0DM7]. Then it follows that $\mathbf{Mor}_S(A, B)$ is separated, since open immersions are separated and the composition of separated morphisms is separated.

A morphism of S-schemes $f \colon A \to B$ is a morphism of group schemes if and only if the diagram

$$\begin{array}{ccc} A \times_S A & \xrightarrow{f \times f} & B \times_S B \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes. Thus, in the $\mathbf{Mor}$ space, the morphisms of abelian schemes are cut out by the inverse image of $\Delta_{\mathbf{Mor}_S(A \times_S A, B)}$ under the morphism

$$\begin{aligned} \mathbf{Mor}_S(A, B) &\longrightarrow \mathbf{Mor}_S(A \times_S A, B) \times_S \mathbf{Mor}_S(A \times_S A, B) \\ f &\longmapsto (f \circ \mu, \mu \circ (f \times f)). \end{aligned}$$

Since $\mathbf{Mor}_S(A \times_S A, B)$ is separated by the same argument as above, we conclude that $\mathbf{Hom}_S(A, B)$ is a closed subspace of $\mathbf{Mor}_S(A, B)$. $\square$

In Corollary 4.2.4, we will show that $\mathbf{Hom}_S(A, B)$ is in fact representable by a scheme. We will use this in Theorem 4.2.8 to prove that $\mathbf{Mor}_S(A, B)$ is also representable by a scheme.

Lemma 4.2.3. *Let A and B be abelian schemes over S. Then $\mathbf{Hom}_S(A, B)$ is locally quasi-finite over S.*

Proof. By [Stacks, Tag 06RW], we have to show that for each $\operatorname{Spec} k \to S$ where k is a field, the space $|\mathbf{Hom}_S(A, B) \times_S \operatorname{Spec} k|$ is discrete. Since $\mathbf{Hom}$ commutes with base change, we have to show that $|\mathbf{Hom}_k(A_k, B_k)|$ is discrete. We may assume k is algebraically closed. Since abelian varieties over a field are projective, this $\mathbf{Hom}$ space is representable by a scheme [FGA, TDTE IV, 4.c].

Let T be any connected finite type closed subscheme of $\mathbf{Hom}_k(A_k, B_k)$, and consider the map $f \colon A_T \to B_T$ over T corresponding to $T \to \mathbf{Hom}_k(A_k, B_k)$. Let $t_0 \in T$ be a closed point, and consider $g = f - f_{t_0}$. Since g_{t_0} is constant 0, the rigidity lemma [GIT, Prop. 6.1(3)] implies that g factors through $A_T \to T$. Hence, g is constant 0, and thus $f = f_{t_0}$. Therefore, the closed immersion $T \to \mathbf{Hom}_k(A_k, B_k)$ factors through the structure morphism $T \to \operatorname{Spec} k$, which forces T to be a point. $\square$

Corollary 4.2.4. *Let A and B be abelian schemes over S. Then $\mathbf{Hom}_S(A, B)$ is representable by a scheme.*

Proof. An algebraic space that is separated and locally quasi-finite over a scheme is a scheme [Stacks, Tag 03XX]. $\square$

Lemma 4.2.5. *Let A and B be abelian schemes over S. Then $\mathbf{Hom}_S(A, B)$ is* *unramified*[1] *over* S.

Proof. A locally finitely presented morphism is unramified if and only if each fibre is [EGA4$_{\text{IV}}$, Cor. 17.4.2]. Since **Hom** commutes with base change, we may assume that $S = \operatorname{Spec} k$ where k is a field. We may assume that k is algebraically closed, since the property of a morphism being unramified is fpqc local on the target [Stacks, Tag 02VM]. Now the argument of the proof of Lemma 4.2.3 shows that any connected finite type subscheme of $\mathbf{Hom}_k(A_k, B_k)$ is isomorphic to $\operatorname{Spec} k$. □

Remark 4.2.6. If we already knew that $\mathbf{Hom}_S(A, B)$ were representable by a scheme, we could have skipped Lemma 4.2.3 and proved Lemma 4.2.5 directly. However, we used the quasi-finiteness to argue the representability. Note that directly verifying whether an algebraic space is unramified over S is a little bit more cumbersome [Stacks, Tag 03ZH].

Lemma 4.2.7. *Let A and B be abelian schemes over S. Then every finitely presented closed subscheme $Z \subseteq \mathbf{Hom}_S(A, B)$ is finite over S.*

Proof. The question is local on S, so we may assume S is affine. Because the **Hom** functor commutes with arbitrary base change and everything is finitely presented, we may assume S is Noetherian by a standard limit argument. If $T = \operatorname{Spec} R$ is an S-scheme, where R is a DVR with fraction field K, then the map

$$\operatorname{Hom}_R(A_R, B_R) \to \operatorname{Hom}_K(A_K, B_K)$$

is an isomorphism by the Néron extension property (Corollary 4.1.3).

Hence, the same is true when we restrict to the closed subspace Z. This proves

[1] In the sense of EGA [EGA4$_{\text{IV}}$, Def. 17.3.1]. The Stacks project calls this *G-unramified* [Stacks, Tag 02G3]; the only difference is the finite presentation vs. finite type assumption.

the valuative criterion of properness for Z, where we may restrict to the DVR case since S is Noetherian and Z of finite type [EGA2, Thm. 7.3.8]. Since $\mathbf{Hom}_S(A,B)$ is locally quasi-finite over S by Lemma 4.2.3, this forces Z to be finite over S [EGA3_{I}, Prop. 4.4.2]. □

Theorem 4.2.8. *Let A and B be abelian schemes over S. Then the natural transformation*

$$\begin{aligned} \psi\colon\ \mathbf{Mor}_S(A,B) &\to \mathbf{Hom}_S(A,B) \times B \\ \{f\colon A_T \to B_T\} &\mapsto (f - f(0), f(0)) \end{aligned}$$

is an isomorphism. In particular, $\mathbf{Mor}_S(A,B)$ is representable by a scheme that is separated and locally of finite presentation over S. Every finitely presented closed subscheme of $\mathbf{Mor}_S(A,B)$ is proper over S.

Proof. Any T-morphism $g\colon A_T \to B_T$ mapping 0 to 0 is a morphism of group schemes [GIT, Prop. 6.4]. In particular $f - f(0)$ is indeed in $\mathbf{Hom}_S(A,B)(T)$ as claimed. One easily checks that an inverse to ψ is given by $(g,h) \mapsto g+h$, where h is viewed as the constant map $A_T \to T \to B_T$ given by $h\colon T \to B_T$. This proves the first statement, and the others follow from Corollary 4.2.4, Lemma 4.2.2, and Lemma 4.2.7. □

Remark 4.2.9. The proof above also works when A and B are abelian algebraic spaces, with the conclusion that $\mathbf{Mor}_S(A,B)$ is representable by an algebraic space. However, Raynaud proved [FC90, Thm. I.1.9] that every abelian algebraic space is actually a scheme, so this does not give a more general statement.

4.3 Specialisation of endomorphisms

Definition 4.3.1. Let $S = \operatorname{Spec} R$ be the spectrum of a DVR R, with fraction field K and residue field k. Let T be an S-scheme satisfying the valuative criterion of properness. Then we define a specialisation map

$$\mathrm{sp}\colon T(K) \to T(k)$$

as the composition of the isomorphism $T(K) \xleftarrow{\sim} T(R)$ coming from the valuative criterion of properness with the natural map $T(R) \to T(k)$.

Lemma 4.3.2. *This map is functorial in R and T.* □

The main examples for us are $T = \mathbf{Mor}_S(A, B)$ and $T = \mathbf{Hom}_S(A, B)$, where A and B are abelian schemes over S. These both satisfy the valuative criterion of properness; see Lemma 4.2.7 and Theorem 4.2.8.

Remark 4.3.3. Because $T = \mathbf{Hom}_S(A, B)$ is actually unramified over S, one can also define a specialisation map $\mathrm{sp}\colon T(\kappa(\bar{s})) \to T(\kappa(\bar{s}'))$ for any specialisation $s \rightsquigarrow s'$ in S by going through the strict Henselisation at s'. This gives a more general notion of specialisation, at the expense of passing to the algebraic closure of the residue fields.

Corollary 4.3.4. *For abelian schemes A and B over S, the specialisation map*

$$\mathrm{sp}\colon \operatorname{Hom}(A_K, B_K) \to \operatorname{Hom}(A_k, B_k)$$

is an injective group homomorphism. Given a third abelian scheme C over S, *the diagram*

$$\begin{array}{ccc} \mathrm{Hom}(B_K, C_K) \times \mathrm{Hom}(A_K, B_K) & \xrightarrow{\circ} & \mathrm{Hom}(A_K, C_K) \\ {\scriptstyle \mathrm{sp}\times\mathrm{sp}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{sp}} \\ \mathrm{Hom}(B_k, C_k) \times \mathrm{Hom}(A_k, B_k) & \xrightarrow{\circ} & \mathrm{Hom}(A_k, C_k) \end{array} \tag{4.3.1}$$

commutes.

Proof. The map is the one from Definition 4.3.1. It is a group homomorphism by Lemma 4.3.2, because addition is given by a morphism of schemes

$$\mathbf{Hom}_S(A, B) \times_S \mathbf{Hom}_S(A, B) \to \mathbf{Hom}_S(A, B).$$

Similarly, diagram (4.3.1) commutes since composition is given by a morphism of schemes. Since $\mathbf{Hom}_S(A, B)$ is unramified (Lemma 4.2.5) and separated (Lemma 4.2.2), the specialisation map is injective [EGA4$_{\mathrm{IV}}$, Prop. 17.4.9] □

Definition 4.3.5. Given abelian varieties A and B over a field k, we write $\mathrm{Hom}^\circ(A, B)$ for $\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$. We note that there is an obvious analogue of Corollary 4.3.4 for Hom° instead of Hom.

Lemma 4.3.6. *Let k be an algebraic extension of a finite field, and let A be an abelian variety of dimension g over k. Then*

$$2g \le \dim \mathrm{End}^\circ(A) \le 4g^2.$$

Moreover, the dimension equals $4g^2$ if and only if A is isogenous to a power of a supersingular elliptic curve all of whose endomorphisms are defined over k.

Proof. For k finite, this is [Tat66, Thm. 3.2]. Since any finite-dimensional subspace of $\mathrm{End}^\circ(A)$ is defined over a finite field, the general case follows. □

Remark 4.3.7. If k is any field in characteristic $p > 0$, and A is an abelian variety isogenous to a power of a supersingular elliptic curve E all of whose endomorphisms are defined over k, then $\dim \mathrm{End}^\circ(A) = 4g^2$.

Indeed, since E is supersingular, we have $\dim \mathrm{End}^\circ(E_{\bar{k}}) = 4$, so the same holds over k since all endomorphisms are defined over k. In fact, such E have to be defined over a finite field, see e.g. [Sil09, Thm. V.3.1(a)]. In fact, there exists a finite field $\mathbb{F}$ such that E and all endomorphisms of E are defined over $\mathbb{F}$, since $\mathrm{End}(E)$ is a finitely generated group.

Lemma 4.3.8. *Let k be a field of characteristic 0, and let A be an abelian variety of dimension g over k. Then*

$$1 \leq \dim \mathrm{End}^\circ(A) \leq 2g^2.$$

Moreover, the dimension equals $2g^2$ if and only if A is isogenous to a power of a CM elliptic curve all of whose endomorphisms are defined over k.

Proof. First assume $k = \mathbb{C}$, and let $A \simeq A_1^{n_1} \times \ldots \times A_r^{n_r}$ be a decomposition up to isogeny in pairwise non-isogenous simple factors A_i of dimension g_i. Then

$$\mathrm{End}^\circ(A) \cong \prod_{i=1}^{r} M_{n_i}(\mathrm{End}^\circ(A_i)).$$

Since A_i is simple, the ring $\mathrm{End}^\circ(A_i)$ is a division algebra ("Schur's lemma"). On the other hand, it acts on $H_1(A_i, \mathbb{Q}) \cong \mathbb{Q}^{2g_i}$; say that it has dimension r_i as $\mathrm{End}^\circ(A_i)$-vector space. Then $2g_i = r_i \cdot \dim \mathrm{End}^\circ(A_i)$, so $\dim \mathrm{End}^\circ(A_i) \leq 2g_i$.

Hence, we get

$$\begin{aligned}\dim \operatorname{End}^\circ(A) &\le \sum_{i=1}^r 2g_i n_i^2 \le 2\sum_{i=1}^r g_i^2 n_i^2 \\ &\le 2\biggl(\sum_{i=1}^r g_i n_i\biggr)^2 = 2g^2\end{aligned}$$

The second inequality is an equality if and only if $g_i = 1$ for all i. The first inequality is an equality if and only if $\dim \operatorname{End}^\circ(A_i) = 2g_i = 2$ for all i, i.e. A_i is a CM elliptic curve. Finally, the last inequality is an equality if and only if $r = 1$. That is, $A = A_1^g$, where A_1 is a CM elliptic curve. This proves the result for $k = \mathbb{C}$, and the general result follows from a Lefschetz principle argument. □

Corollary 4.3.9. *Let A be an abelian scheme over a mixed characteristic DVR R such that A_k is isogenous to a power of a supersingular elliptic curve all of whose endomorphisms are defined over k. Then the specialisation map*

$$\mathrm{sp}\colon \operatorname{End}^\circ(A_K) \to \operatorname{End}^\circ(A_k)$$

is not surjective.

Proof. This follows from Remark 4.3.7 and Lemma 4.3.8, and the fact that the specialisation map is $\mathbb{Q}$-linear by Corollary 4.3.4. □

4.4 Picard scheme of a product

In this section, we want to compute the Picard scheme of a product in terms of the Picard schemes of each of the factors. We will work with the *relative Albanese* $\mathbf{Alb}^1_{X/S}$. The map $X \to \mathbf{Alb}^1_{X/S}$ is initial among maps from X to torsors under abelian schemes over S [FGA, TDTE VI, §3].

We will first treat the case of binary products in Lemma 4.4.5 and its corollaries. For the general case, we will assume for simplicity that each of the factors has a section. The main result is Theorem 4.4.10.

Setup 4.4.1. Let S be a scheme, and let $f_i \colon X_i \to S$ $(1 \leq i \leq r)$ be proper flat morphisms of finite presentation, such that $f_{i,*}\mathcal{O}_{X_i} = \mathcal{O}_S$ holds universally.

We let $f \colon X \to S$ be the fibre product $X_1 \times_S \ldots \times_S X_r$. Then $f_*\mathcal{O}_X = \mathcal{O}_S$ holds universally. We will also assume that $\mathbf{Pic}_{X_i/S}$ is representable and contains an abelian scheme over S whose support is $\mathbf{Pic}^0_{X_i/S}$.

Remark 4.4.2. Under these assumptions, $\mathbf{Alb}^0_{X_i/S}$ exists as a scheme, and $\mathbf{Alb}^1_{X_i/S}$ as an algebraic space. If f_i has a section, then $\mathbf{Alb}^1_{X_i/S} = \mathbf{Alb}^0_{X_i/S}$, so in particular it is a scheme.

Still under the same assumptions, existence of $\mathbf{Alb}^1_{X_i/S}$ as a scheme is stated without proof in [FGA, TDTE VI, Thm. 3.3(iii)]. However, not every torsor under an abelian scheme is representable [Ray70, Ex. XIII.3.2]. It is not clear if $\mathbf{Alb}^1$ is always representable by a scheme, but Raynaud's example suggests that this need not be true in general.

Example 4.4.3. If all f_i are flat and locally projective with integral geometric fibres, then $\mathbf{Pic}_{X_i/S}$ is representable by a scheme [FGA, TDTE V, Thm. 3.1]. The assumption on $\mathbf{Pic}^0$ of Setup 4.4.1 is for example satisfied if $\mathbf{Pic}^0_{X_i/S}$ is smooth over S (e.g. if S has equicharacteristic 0), or when S is a field.

Remark 4.4.4. Under the assumptions of Setup 4.4.1, formation of $\mathbf{Pic}_{X_i/S}$, $\mathbf{Pic}^0_{X_i/S}$, $\mathbf{Alb}^0_{X_i/S}$, and $\mathbf{Alb}^1_{X_i/S}$ commutes with arbitrary base change of S. For $\mathbf{Pic}$, this follows since sheafification on the big fppf site commutes with base change. For $\mathbf{Pic}^0$, it follows from the definition [SGA6, Exp. XIII, 4.1]. Finally, for $\mathbf{Alb}^0$ and $\mathbf{Alb}^1$ this is [FGA, TDTE VI, Thm. 3.3(iii)].

Lemma 4.4.5. *Let $f_i \colon X_i \to S$ and $f \colon X \to S$ be as in Setup 4.4.1, and assume $r = 2$. Then there exists a canonical short exact sequence*

$$0 \to \mathbf{Pic}_{X_1/S} \to \mathbf{Pic}_{X/S} \xrightarrow{\psi} \mathbf{Mor}_S(X_1, \mathbf{Pic}_{X_2/S}) \to 0$$

of sheaves on $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$.

Proof. The Grothendieck spectral sequence for the composition of the pushforwards along $X \to X_1 \to S$ gives the exact sequence of low degree terms

$$\begin{tikzcd} 0 \arrow[r] & R^1 f_{1,*}(\pi_{1,*}\mathcal{O}_X^\times) \arrow[r] & R^1 f_* \mathcal{O}_X^\times \arrow[r, "\psi"] & f_{1,*}(R^1 \pi_{1,*}\mathcal{O}_X^\times) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & R^2 f_{1,*}(\pi_{1,*}\mathcal{O}_X^\times) \arrow[r] & R^2 f_* \mathcal{O}_X^\times\,. & \end{tikzcd} \tag{4.4.1}$$

Since $f_{2,*}\mathcal{O}_{X_2} = \mathcal{O}_S$ holds universally, we have $\pi_{1,*}\mathcal{O}_X^\times = \mathcal{O}_{X_1}^\times$. Hence, the first two terms are $\mathbf{Pic}_{X_1/S}$ and $\mathbf{Pic}_{X/S}$ respectively. The third term has T-points given by

$$\begin{aligned} f_{1,*}(R^1\pi_{1,*}\mathcal{O}_X^\times)(T) &= f_{1,*}(\mathbf{Pic}_{X/X_1})(T) = \mathbf{Pic}_{X/X_1}(X_1 \times T) \\ &= \mathbf{Pic}_{X_2/S}(X_1 \times T) = \mathbf{Mor}_S(X_1, \mathbf{Pic}_{X_2/S})(T). \end{aligned}$$

Surjectivity of ψ may be checked fppf-locally, and formation of the **Pic** and **Mor** schemes commutes with base change. Hence, we may assume f_2 has a section, so $\pi_1 \colon X \to X_1$ has a section as well. Then the map $R^2 f_{1,*}\mathcal{O}_{X_1}^\times \to R^2 f_* \mathcal{O}_X^\times$ has a section, so it is injective. Surjectivity of ψ then follows from (4.4.1). □

Lemma 4.4.6. *Let S, X_i, f_i, X, and f be as in Setup 4.4.1, and assume $r = 2$. Then any section σ_2 of f_2 induces a splitting of the short exact sequence of Lemma 4.4.5.*

Proof. The section induces a retraction $(\sigma_2)^*_{X_1}\colon \mathbf{Pic}_{X/S} \to \mathbf{Pic}_{X_1/S}$. $\square$

Lemma 4.4.7. *Let S, X_i, f_i, X, and f be as in Setup 4.4.1, and assume $r = 2$. Then any section σ_1 of f_1 induces an isomorphism*

$$\mathbf{Mor}_S(X_1, \mathbf{Pic}_{X_2/S}) \xrightarrow{\sim} \mathbf{Pic}_{X_2/S} \times \mathbf{Mor}_S\big((X_1, \sigma_1), (\mathbf{Pic}^0_{X_2/S}, 0)\big)$$
$$\alpha \mapsto (\alpha \circ \sigma_1, \alpha - \alpha \circ \sigma_1).$$

Proof. Note that $\alpha - \alpha \circ \sigma_1$ maps the point $\sigma_1(s)$ of the fibre $(X_1)_s$ to $\mathbf{Pic}^0_{X_2/S}$, hence the entire fibre lands in $\mathbf{Pic}^0_{X_2/S}$ since X_1 the fibre $(X_1)_s$ is geometrically connected. The inverse is given by $(\alpha, \beta) \mapsto \alpha + \beta$ (compare Theorem 4.2.8). $\square$

Corollary 4.4.8. *Let S, X_i, f_i, X, and f be as in Setup 4.4.1, and assume $r = 2$. If both f_i have sections σ_i, then there is a noncanonical isomorphism*

$$\mathbf{Pic}_{X/S} \xrightarrow{\sim} \mathbf{Pic}_{X_1/S} \times \mathbf{Pic}_{X_2/S} \times \mathbf{Hom}_S(\mathbf{Alb}^1_{X_1/S}, \mathbf{Pic}^0_{X_2/S}).$$

Moreover, $\mathbf{Alb}^1_{X_1/S}$ is noncanonically isomorphic to the trivial torsor $\mathbf{Alb}^0_{X_1/S}$.

Here, $\mathbf{Hom}$ means homomorphisms of torsors under abelian schemes or, in case of abelian schemes, homomorphisms of abelian schemes.

Proof. By Lemma 4.4.6 and Lemma 4.4.7, the sections σ_1 and σ_2 induce an isomorphism

$$\mathbf{Pic}_{X/S} \xrightarrow{\sim} \mathbf{Pic}_{X_1/S} \times \mathbf{Pic}_{X_2/S} \times \mathbf{Mor}_S\big((X_1, \sigma_1), (\mathbf{Pic}^0_{X_2/S}, 0)\big).$$

By the universal property of the Albanese, the last term is isomorphic to $\mathbf{Hom}_S(\mathbf{Alb}^1_{X_1/S}, \mathbf{Pic}^0_{X_2/S})$. This proves the required isomorphism, and the final statement follows since X_1 has a section, hence so does $\mathbf{Alb}^1_{X_1/S}$ since X_1 maps to it. $\square$

Corollary 4.4.9. *Let S, X_i, f_i, X, and f be as in Setup 4.4.1, and assume $r = 2$. Then $\mathbf{Pic}^0_{X/S} = \mathbf{Pic}^0_{X_1/S} \times \mathbf{Pic}^0_{X_2/S}$, and the same holds for $\mathbf{Alb}^0$ and $\mathbf{Alb}^1$.*

Proof. Consider the pullback map $\psi\colon \mathbf{Pic}^0_{X_1/S} \times \mathbf{Pic}^0_{X_2/S} \to \mathbf{Pic}^0_{X/S}$. Since both X_i are flat of finite presentation over S, they have a section fppf locally. To show ψ is an isomorphism, it suffices to do so fppf-locally, so we may assume X_i have a section. Then Corollary 4.4.8 gives an isomorphism

$$\mathbf{Pic}_{X/S} \xrightarrow{\sim} \mathbf{Pic}_{X_1/S} \times \mathbf{Pic}_{X_2/S} \times \mathbf{Hom}_S(\mathbf{Alb}^0_{X_1/S}, \mathbf{Pic}^0_{X_2/S}).$$

But the **Hom** term is unramified over S by Lemma 4.2.5. Hence, the identity component is just S, so ψ is an isomorphism. This proves the statement about $\mathbf{Pic}^0$.

The statement about $\mathbf{Alb}^0$ follows since it is the dual of the (unique) abelian subscheme of $\mathbf{Pic}$ whose support is $\mathbf{Pic}^0$. The statement about $\mathbf{Alb}^1$ follows by checking that the map $X_1 \times X_2 \to \mathbf{Alb}^1_{X_1/S} \times \mathbf{Alb}^1_{X_2/S}$ satisfies the universal property of $\mathbf{Alb}^1_{X/S}$. □

Theorem 4.4.10. *Let $f_i\colon X_i \to S$ and $f\colon X \to S$ be as in Setup 4.4.1. Assume that each f_i has a section σ_i. Then there is a noncanonical isomorphism*

$$\mathbf{Pic}_{X/S} \cong \prod_{i=1}^{r} \mathbf{Pic}_{X_i/S} \times \prod_{i<j} \mathbf{Hom}_S(\mathbf{Alb}^0_{X_i/S}, \mathbf{Pic}^0_{X_j/S}).$$

Proof. We proceed by induction. The case $r \leq 1$ is trivial, and the case $r = 2$ follows from Corollary 4.4.8. For the general case, note that Corollary 4.4.8 gives an isomorphism

$$\mathbf{Pic}_{X/S} \cong \mathbf{Pic}_{X_1/S} \times \mathbf{Pic}_{X'/S} \times \mathbf{Hom}_S(\mathbf{Alb}^0_{X_1/S}, \mathbf{Pic}^0_{X'/S}),$$

where $X' = X_2 \times \ldots \times X_r$. Then the result for X follows inductively from that for X', along with Corollary 4.4.9. $\square$

Remark 4.4.11. Note that the argument above does not use the theorem of the cube to carry out the induction, but instead relies on the computation of $\mathbf{Pic}^0$ of a product. One can use this to give an alternative (albeit somewhat cumbersome) proof of the theorem of the cube.

Corollary 4.4.12. *Let k be a field, and let $X_1, \ldots, X_r$ be projective k-varieties with $X_i(k) \neq \varnothing$, and let X be their product. Then there is a noncanonical isomorphism*

$$\operatorname{Pic}(X) \cong \prod_{i=1}^{r} \operatorname{Pic}(X_i) \times \prod_{i<j} \operatorname{Hom}_k(\mathbf{Alb}^0_{X_i/k}, \mathbf{Pic}^0_{X_j/k}).$$

Proof. The X_i satisfy the assumptions of Setup 4.4.1; see also Example 4.4.3. Moreover, $\operatorname{Pic}(X_i)$ equals $\mathbf{Pic}_{X_i/k}(k)$ when $X_i(k) \neq \varnothing$ [FGA, TDTE V, Cor. 2.3]. The result now follows from Theorem 4.4.10, again using that the X_i have a rational point. $\square$

Remark 4.4.13. It seems that in the absence of a section, the best one can hope for is a short exact sequence

$$0 \to \prod_{i=1}^{r} \mathbf{Pic}_{X_i/S} \to \mathbf{Pic}_{X/S} \to \prod_{i<j} \mathbf{Hom}_S(\mathbf{Alb}^1_{X_i/S}, \mathbf{Pic}^0_{X_j/S}) \to 0,$$

where the first map is given by multiplying the pullbacks. However, the author does not know how to construct such a sequence, even in the case $r = 2$.

We also address the functoriality of the isomorphism of Theorem 4.4.10.

Lemma 4.4.14. *Let $f_i \colon X_i \to S$ and $f \colon X \to S$ be as in Setup 4.4.1, and let $f'_i \colon X'_i \to S$ and $f' \colon X' \to S$ satisfy the same assumptions. Let $g_i \colon X'_i \to X_i$ be a* *morphism of S-schemes for each i, and let $g\colon X' \to X$ be the product morphism. Assume that each f_i (f'_i) has a section σ_i (σ'_i), and that $g_i \circ \sigma'_i = \sigma_i$. Under the isomorphism of Theorem 4.4.10, the pullback $\mathbf{Pic}_{X/S} \to \mathbf{Pic}_{X'/S}$ is given by*

$$\left(\left(\mathscr{L}_i\right)_i, \left(\phi_{ji}\right)_{i<j}\right) \mapsto \left(\left(g_i^*\mathscr{L}_i\right)_i, \left(g_j^*\phi_{ji}g_{i,*}\right)_{i<j}\right).$$

Proof. The coordinates corresponding to $\mathbf{Pic}_{X'_i/S}$ are given by pulling back along σ'_j for all $j \neq i$. Since $g_j\sigma'_j = \sigma_j$, these coordinates are given by $g_i^*\mathscr{L}_i$ where $\mathscr{L}_i$ is the pullback of $\mathscr{L}$ along σ_j for all $j \neq i$. The other coordinates are given by a line bundle on $X_i \times X_j$ that is trivial along $\sigma_i \times X_j$ and $X_i \times \sigma_j$. This corresponds to a map $X_i \to \mathbf{Pic}^0_{X_j/S}$, which factors through the Albanese. Pulling back to $X'_i \times X'_j$ then corresponds to the map

$$X'_i \xrightarrow{g_i} X_i \xrightarrow{\phi_{ji}} \mathbf{Pic}^0_{X_j/S} \xrightarrow{g_j^*} \mathbf{Pic}^0_{X'_j/S}.$$

The universal property of the Albanese gives a commutative diagram

$$\begin{tikzcd}
X'_i \arrow[r] \arrow[d, "g_i"'] & \mathbf{Alb}^0_{X'_i/S} \arrow[d, "g_{i,*}"] \\
X_i \arrow[r] & \mathbf{Alb}^0_{X_i/S}
\end{tikzcd},$$

where we write $\mathbf{Alb}^0$ instead of $\mathbf{Alb}^1$ because X_i and X'_i have a section. Moreover, the horizontal arrows are chosen to map the sections σ_i and σ'_i to 0. Thus, the pullback of ϕ_{ji} is $g_j^*\phi_{ij}g_{i,*}$. $\square$

Finally, we prove compatibility of the isomorphism of Theorem 4.4.10 with the specialisation maps of Definition 4.3.1.

Lemma 4.4.15. *Let $f_i\colon X_i \to S$ and $f\colon X \to S$ be as in Setup 4.4.1, where S is the spectrum of a DVR R. Assume that each f_i has a section σ_i. Then we* *get a commutative diagram*

$$\begin{tikzcd}
\operatorname{Pic}(X_K) \arrow[r, "\sim"] \arrow[d, "\mathrm{sp}"'] & \prod_{i=1}^{r} \operatorname{Pic}(X_{i,K}) \times \prod_{i<j} \operatorname{Hom}_K(\mathbf{Alb}^0_{X_{i,K}/K}, \mathbf{Pic}^0_{X_{i,K}/K}) \arrow[d, "\mathrm{sp}"] \\
\operatorname{Pic}(X_k) \arrow[r, "\sim"] & \prod_{i=1}^{r} \operatorname{Pic}(X_{i,k}) \times \prod_{i<j} \operatorname{Hom}_k(\mathbf{Alb}^0_{X_{i,k}/k}, \mathbf{Pic}^0_{X_{i,k}/k}) \, ,
\end{tikzcd}$$

where the specialisation maps are the ones from Definition 4.3.1 (applied componentwise on the right hand side).

Proof. The definition of the specialisation map uses the valuative criterion of properness, which acts componentwise on a product. Moreover, we are using the same section σ_i to define both horizontal isomorphisms of Theorem 4.4.10 and Corollary 4.4.12. $\square$

This page intentionally left blank

5. Anabelian methods

Our main input is a theorem of Simpson, see Theorem 5.3.8. We think of this as some sort of anabelian statement, and we use it to lift maps to curves. See Theorem 5.4.4 for the precise statement.

5.1 Representation and character schemes

Let G be an algebraic group (i.e. a finite type group scheme) over a field k, and let Γ be a (discrete) group. Under mild assumptions, there are k-schemes $R(\Gamma, G)$ and $M(\Gamma, G)$ parametrising homomorphisms $\Gamma \to G$ and conjugacy classes of homomorphisms $\Gamma \to G$ respectively.

The case $G = \mathrm{GL}_n$ and $\Gamma = \mathbb{Z}^{*n}$ corresponds to n-tuples of invertible matrices, which was studied in this language in [Pro76]. The case Γ finitely generated and $G = \mathrm{GL}_n$ is studied in [LM85]. The case SL_2 was studied in [BL83, §5] and [CS83, §1], both under the assumption that Γ is finitely generated. Finally, [Nak00] studied GL_n over $\operatorname{Spec} \mathbb{Z}$, without assumptions on Γ.

One can easily formulate much more general settings, like replacing k with any base S, and G with any group scheme over S. As far as the author is aware, a systematic study in the most general setting is still lacking from the literature. However, most applications are to representations (i.e. $G = \mathrm{GL}_n$) of fundamental groups of smooth proper k-varieties, so it seems that there is no need for a more general machinery (besides perhaps clarity and uniformity in exposition).

Remark 5.1.1. The terminology stems from [Sai96]. Since representation and character varieties are in general neither reduced nor irreducible, we will say representation/character *scheme* instead of *variety*.

Definition 5.1.2. Let G be an algebraic group over k, and let Γ be a group. Then the *representation functor* $R(\Gamma, G)$ is the functor

$$\begin{aligned}(\underline{\mathrm{Sch}}\,/k)^{\mathrm{op}} &\to \underline{\mathrm{Set}} \\ T &\mapsto \operatorname{Hom}(\Gamma, G(T)).\end{aligned}$$

In most reasonable cases, this is representable by a k-scheme $R(\Gamma, G)$, called the *representation scheme* of Γ with values in G.

Lemma 5.1.3. *Let G be an algebraic group over k, and let Γ be a group. If Γ is finitely generated or G is affine, then $R(\Gamma, G)$ is representable by a k-scheme.*

Proof. If Γ is presented as $\langle \{\gamma_i\}_{i \in I} \mid \{r_j\}_{j \in J} \rangle$, then $R(\Gamma, G)$ is given by

$$R(\Gamma, G)(T) = \left\{ (\gamma_i)_i \in G(T)^I \;\middle|\; r_j((\gamma_i)_i) = 1 \text{ for all } j \in J \right\}.$$

Since $G \to \operatorname{Spec} k$ is separated [$\mathrm{SGA3_I}$, Exp. $\mathrm{VI_A}$,0.3], we conclude that $R(\Gamma, G)$ is a closed subfunctor of G^I. Thus, the functor $R(\Gamma, G)$ is representable whenever the product G^I exists. This is true when G is affine [Stacks, Tag 0CNI][1], or when I is finite. □

Remark 5.1.4. On the other hand, if G is an elliptic curve over a field k, then the infinite product G^I does not exist, even as an algebraic space. Indeed, the argument of [Stacks, Tag 078E] for $\mathbb{P}^1$ can be generalised to any quasi-projective k-variety that is not affine, by replacing the covering $\mathrm{SL}_2 \to \mathbb{P}^1$ with Jouanolou's trick [Jou73, Lemme 1.5], and the sheaf $\mathcal{O}(-2, \ldots, -2)$ with some quasi-coherent sheaf with nonzero higher cohomology, which exists by [EGA2, Thm. 5.2.1]

Thus, if Γ is a free group on infinitely many generators and G is an elliptic curve

[1] This also follows from [$\mathrm{EGA4_{III}}$, §8.2], but it is never explicitly stated there.

over a field k, then $R(\Gamma, G)$ is not representable by an algebraic space.

Definition 5.1.5. Let Γ be a group, and let G be an algebraic group over k. Then define the conjugation action of G on the functor $R(\Gamma, G)$ by

$$\begin{aligned} G(T) \times R(\Gamma, G)(T) &\to R(\Gamma, G)(T) \\ (g, \rho) &\mapsto (\gamma \mapsto g\rho(\gamma)g^{-1}). \end{aligned}$$

If $R(\Gamma, G)$ is representable, this is an action of G on the scheme $R(\Gamma, G)$. The quotient stack $[R(\Gamma, G)/G]$ is denoted $\mathscr{M}(\Gamma, R)$, and is called the *character stack* of Γ with values in G.

Definition 5.1.6. Let Γ be a group, and let G be a reductive algebraic group over k. In particular G is affine, hence so is $R(\Gamma, G)$. Then the *character scheme* $M(\Gamma, G)$ is the GIT quotient $R(\Gamma, G)/\!\!/G$.

The map $R(\Gamma, G) \to M(\Gamma, G)$ is a uniform categorical quotient, and if $\operatorname{char} k = 0$ then it is in fact a universal categorical quotient [GIT, Thm. 1.1]. However, it is not typically a geometric quotient:

Example 5.1.7. Let $\Gamma = \mathbb{Z}$, and let $G = \mathrm{GL}_2$ over $\operatorname{Spec} \mathbb{C}$. Then $R(\Gamma, G)$ is just G, with the conjugation action. The matrices

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

are all conjugate for $t \neq 0$, and their orbit closure contains the identity matrix. Thus, we see that not all orbits are closed, so $R(\Gamma, G) \to M(\Gamma, G)$ is not a geometric quotient.

However, this problem disappears if we only consider absolutely irreducible representations.

Definition 5.1.8. Let $\rho\colon \Gamma \to GL_n(T)$ be a representation. Then we say that ρ is *absolutely irreducible* if $t^*\rho$ is irreducible for every map $t\colon \operatorname{Spec} k \to T$. Equivalently, for every point $t \in T$, denoting by $\bar{t}$ an algebraic closure, the representation $\bar{t}^*\rho$ is irreducible.

Lemma 5.1.9. *Let $\rho\colon \Gamma \to \mathrm{GL}_n(R)$ be a representation, where R is a ring. The following are equivalent:*

(1) *ρ is absolutely irreducible;*

(2) *$\rho(\Gamma)$ generates $M_n(R)$ as R-algebra;*

(3) *$\rho(\Gamma)$ spans $M_n(R)$ as R-module.*

Proof. Implications (2) $\Leftrightarrow$ (3) follow since $\rho(\Gamma)$ is closed under multiplication. Statements (1) and (3) can be checked after tensoring with $\kappa(\mathfrak{p})$ for all primes $\mathfrak{p}$ in R, and are insensitive to field extension, so we may assume $R = k$ is an algebraically closed field. Moreover, we clearly have (2) $\Rightarrow$ (1), and the converse follows from Burnside's theorem [Bou12, Cor. 2 of Prop. 5.3.4][1]. $\square$

Definition 5.1.10. We say that a representation $\rho\colon \Gamma \to \mathrm{SL}_n(T)$ is *absolutely irreducible* if the induced representation $\Gamma \to \mathrm{GL}_n(T)$ is absolutely irreducible. We can use the criteria from Lemma 5.1.9 to check this. There are similar notions for other standard groups, like Sp_{2n}, etc.

Lemma 5.1.11. *Let $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$ over some field k. Then the subfunctor $R(\Gamma, G)^{\mathrm{irr}}$ of $R(\Gamma, G)$ of absolutely irreducible representations is an open subfunctor.*

[1] The numbering in Bourbaki is not consistent with earlier editions.

Proof. (Following [Nak00, §3].) We use criterion (3) in Lemma 5.1.9: given an n^2-tuple $\gamma_1, \dots, \gamma_{n^2}$ of elements of Γ, the elements $\rho(\gamma_i)$ span $M_n(R)$ as R-module if and only if a certain $n^2 \times n^2$ determinant does not vanish. This is a standard open of $R(\Gamma, G)$, and $R(\Gamma, G)^{\mathrm{irr}}$ is the union of these standard opens running over all n^2-tuples in Γ. $\square$

Remark 5.1.12. Since the determinants used in the proof of Lemma 5.1.11 are equivariant under the conjugation action, they are G-equivariant elements of $\Gamma(R(\Gamma, G), \mathcal{O})$. Hence, in fact the open subfunctor $R(\Gamma, G)^{\mathrm{irr}}$ is pulled back from an open subscheme $M(\Gamma, G)^{\mathrm{irr}} \subseteq M(\Gamma, G)$ (remember that the latter is just the GIT quotient, cf. Definition 5.1.6). Moreover, $R(\Gamma, G)^{\mathrm{irr}} \to M(\Gamma, G)^{\mathrm{irr}}$ is a uniform categorical quotient, since the same holds for $R(\Gamma, G) \to M(\Gamma, G)$.

Theorem 5.1.13 (Nakamoto)**.** *Let $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$ over a field k. Then the map $R(\Gamma, G)^{\mathrm{irr}} \to M(\Gamma, G)^{\mathrm{irr}}$ is a PGL_n-torsor, hence a universal geometric quotient. Moreover, the functions on $M(\Gamma, G)$ are generated by the coefficients of the characteristic polynomials of $\rho(\gamma)$, where $\rho \colon \Gamma \to G(R(\Gamma, G))$ is the universal representation.*

Proof. For $G = \mathrm{GL}_n$, this is [Nak00, Cor. 6.8 and Rmk. 6.9]. For $G = \mathrm{SL}_n$, this is [Nak00, Thm. 6.18]. We note that $\mathrm{PSL}_n = \mathrm{PGL}_n$ on the level of schemes, i.e. the sheaf quotient of SL_n by μ_n is represented by PGL_n. $\square$

Remark 5.1.14. In fact, the results from [Nak00] hold over $\operatorname{Spec} \mathbb{Z}$. Instead of GIT, the quotient is constructed using characteristic polynomials, cf. the last statement of Theorem 5.1.13.

One could be tempted to replace GIT by the good moduli spaces of [Alp13], but we note that GL_n and SL_n are not linearly reductive over fields of positive characteristic. In that light, the results of [Nak00] are all the more remarkable.

Definition 5.1.15. Let Γ be a group, let G be an algebraic group over k, and let $\rho\colon \Gamma \to G(\ell)$ be a representation, where $k \subseteq \ell$ is a field extension. Then ρ is *rigid* if it corresponds to an isolated point in the character scheme $M(\Gamma, G)$.

This terminology is inconsistent with the standard usage of the word *rigid* in deformation theory, where it is usually an infinitesimal criterion. Rigid representations need not be infinitesimally rigid.

Lemma 5.1.16. *Let Γ be a finitely generated group, let $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$ over a field k, and let $\rho_0\colon \Gamma \to G(\ell)$ be a representation, where $k \subseteq \ell$ is a field extension. Then the following are equivalent:*

(1) *ρ_0 is non-rigid;*

(2) *there exists an integral k-scheme T with an ℓ-point t, a representation $\rho\colon \Gamma \to G(T)$ such that $t^*\rho = \rho_0$, and an element $\gamma \in \Gamma$ such that not all coefficients of the characteristic polynomial $P_{\rho(\gamma)} \in \Gamma(T, \mathcal{O}_T)[X]$ are integral over k.*

The scheme T in (2) *can be chosen to be of finite type over k.*

Proof. By Theorem 5.1.13, the coefficients of the characteristic polynomials of $\rho_{\mathrm{univ}}(\gamma)$ for $\gamma \in \Gamma$ give the coordinates of an affine embedding $M(\Gamma, G) \subseteq \mathbb{A}^S$ for some set S. Since Γ is finitely generated, $M(\Gamma, G)$ is of finite type over k [GIT, Thm. 1.1], so we may assume that S is finite. Note that ρ_0 is rigid if and only if its connected component in $M(\Gamma, G)$ is Artinian, local, and finite over k.

Hence, if ρ_0 is rigid and (T, t) is an integral k-scheme as in (2), then T maps to a point in $M(\Gamma, G)$. Hence, the scheme-theoretic image of T in $M(\Gamma, G)$ is finite over k, so the same goes for any coordinate projection $M(\Gamma, G) \to \mathbb{A}^1$ corresponding to a coefficient of the characteristic polynomial $P_{\rho_{\mathrm{univ}}(\gamma)}$ for $\gamma \in \Gamma$.

This means that the corresponding coefficient of $P_{\rho(\gamma)}$ is integral over k, which proves (2) $\Rightarrow$ (1).

Conversely, if ρ_0 is non-rigid, then let T be an irreducible component through ρ_0. Then T is positive dimensional, hence $\Gamma(T, \mathcal{O}_T)$ contains elements that are not integral over k. Since it is generated by coefficients of characteristic polynomials $P_{\rho(\gamma)}$ for $\gamma \in \Gamma$, we conclude that one such coefficient is not integral over k. This proves (1) $\Rightarrow$ (2). The final statement follows since $M(\Gamma, G)$ is of finite type. $\square$

Remark 5.1.17. The representation ρ of (2) has the property that the fibres near ρ_0 are not conjugate to ρ_0. The converse is true if ρ_0 is absolutely irreducible, but not in general. This has to do with the fact that $R(\Gamma, G) \to M(\Gamma, G)$ is not a geometric quotient, cf. Example 5.1.7. However, $R(\Gamma, G)^{\mathrm{irr}} \to M(\Gamma, G)^{\mathrm{irr}}$ is a geometric quotient, by Theorem 5.1.13.

Remark 5.1.18. It seems plausible that there could exist non-finitely generated groups Γ such that $M(\Gamma, G)$ contains an isolated point ρ_0 such that $\kappa(\rho_0)$ is not algebraic over k. Then ρ_0 is rigid when viewed in $M(\Gamma, G_k)$, but not rigid in $M(\Gamma, G_{\kappa(\rho_0)})$, since $(\rho_0, \rho_0) \in \operatorname{Spec} \kappa(\rho_0) \otimes_k \kappa(\rho_0)$ is not an isolated point. The author does not know such an example.

On the other hand, if Γ is finitely generated, then the notion of rigidity for an ℓ-point does not depend on the choice of the base field k. Indeed, an isolated point is then necessarily defined over a finite extension of k, and so remains isolated after base change.

5.2 Construction of local systems on curves

We construct auxiliary local systems that will be used in Theorem 5.4.4.

Remark 5.2.1. If C is a smooth proper curve of genus g over an algebraically closed field k and ℓ is a prime invertible in k, then [SGA1, Exp. XIII, Cor. 2.12] gives

$$\pi_1^{\text{ét},\ell}(C) \cong \left(\mathbb{Z}^{*2g}/\langle\prod_{i=1}^{g}[a_i,b_i]\rangle\right)^{\wedge_\ell}.$$

Hence, if $g \geq 2$, there exists a quotient $\pi_1^{\text{ét},\ell}(C) \twoheadrightarrow \mathbb{Z}_\ell^{*2}$, where $\mathbb{Z}_\ell^{*2}$ is the pro-ℓ completion of the free group $\mathbb{Z}^{*2}$. For example, consider the map $\mathbb{Z}^{*2g} \to \mathbb{Z}^{*2}$ mapping a_1 and a_2 to free generators, and all other a_i and b_i to 1. Note that this maps $\prod[a_i,b_i]$ to 1 since all b_i go to 1, so it induces a surjection

$$\mathbb{Z}^{*2g}/\langle\prod[a_i,b_i]\rangle \twoheadrightarrow \mathbb{Z}^{*2}.$$

Taking the ℓ-adic completion gives the required map.

Lemma 5.2.2. *Let ℓ be a prime, and $m,n \in \mathbb{Z}_{>0}$. Then the subgroup*

$$U_{m,n} := 1 + (\ell^m, t^n)M_2(\mathbb{Z}_\ell[[t]]) \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell[[t]])$$

is a pro-ℓ group, which is torsion-free if $\ell > 2$ or $m \geq 2$.

Proof. For the first statement, note that $U_{m,n}$ is the kernel of the reduction map

$$\mathrm{GL}_2(\mathbb{Z}_\ell[[t]]) \to \mathrm{GL}_2(\mathbb{Z}[t]/(\ell^m,t^n)).$$

Hence, $U_{m,n}$ is an open normal subgroup. The quotient $U_{1,1}/U_{m,n}$ is identified with the kernel of $\mathrm{GL}_2(\mathbb{Z}[t]/(\ell^m,t^n)) \to \mathrm{GL}_2(\mathbb{Z}/\ell)$, hence has $\ell^{4(mn-1)}$ elements. The intersection of the $U_{m,n}$ is 1, so $U_{1,1}$ is pro-ℓ, hence so is each $U_{m,n}$. For the second statement, note that the exponential map gives a homeomorphism

$$\exp\colon (\ell^2,t)M_2(\mathbb{Z}_\ell[[t]]) \xrightarrow{\sim} 1 + (\ell^2,t)M_2(\mathbb{Z}_\ell[[t]]),$$

with the property that $\exp(x+y) = \exp(x)\exp(y)$ whenever x and y commute. In particular, its restriction to a cyclic subgroup is a group homomorphism, so the right hand side is torsion-free since the left hand side is. If $\ell > 2$, then we may replace ℓ^2 by ℓ. $\square$

Notation 5.2.3. Given a representation $\rho\colon G \to \mathrm{GL}_n(R)$, we will write $\bar{\rho}$ for the induced map $G \to \mathrm{PGL}_n(R)$, and similarly for SL_n and PSL_n.

Proposition 5.2.4. *Let ℓ be a prime. Then there exists a representation $\rho\colon \mathbb{Z}_\ell^{*2} \to \mathrm{SL}_2(\mathbb{Q}_\ell)$ with image landing in $\mathrm{SL}_2(\mathbb{Z}_\ell)$ such that for every finitely generated subgroup $G \subseteq \mathbb{Z}_\ell^{*2}$ whose closure is open, the representation $\rho|_G$ has the following properties:*

(1) *$\rho|_G$ is Zariski dense;*

(2) *$\rho|_G$ is absolutely irreducible;*

(3) *the image of $\bar{\rho}|_G\colon G \to \mathrm{PSL}_2(\mathbb{Q}_\ell)$ is torsion-free;*

(4) *$\rho|_G$ is not rigid.*

Proof. We will construct ρ as the mod t reduction of a representation

$$\psi\colon \mathbb{Z}_\ell^{*2} \to \mathrm{SL}_2(\mathbb{Z}_\ell[[t]]).$$

Since the subgroup $1 + (\ell^2, t)M_2(\mathbb{Z}_\ell[[t]]) \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell[[t]])$ is pro-ℓ, defining a representation from $\mathbb{Z}_\ell^{*2}$ into it is equivalent to defining a representation of $\mathbb{Z}^{*2}$, i.e. giving two elements corresponding to the free generators a and b of $\mathbb{Z}^{*2}$. Then let $\psi\colon \mathbb{Z}_\ell^{*2} \to \mathrm{SL}_2(\mathbb{Z}_\ell[[t]])$ be given by

$$a \mapsto \begin{pmatrix} 1+t & \ell^2 \\ 0 & (1+t)^{-1} \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 1+\ell^2 & 0 \\ \ell^2 & (1+\ell^2)^{-1} \end{pmatrix},$$

and let ρ be the mod t reduction.

Now let $G \subseteq \mathbb{Z}_\ell^{*2}$ be a finitely generated subgroup whose closure is open. In particular, $\bar{G}$ has finite index, so there exist $m, n \in \mathbb{Z}_{>0}$ with $a^m, b^n \in \bar{G}$. The Zariski closure of $\rho(G)$ contains the profinite closure of $\rho(G)$, hence it contains

$$\rho(a)^m = \begin{pmatrix} 1 & \ell^2 \\ 0 & 1 \end{pmatrix}^m, \qquad \rho(b)^n = \begin{pmatrix} 1+\ell^2 & 0 \\ \ell^2 & (1+\ell^2)^{-1} \end{pmatrix}^n.$$

One easily sees that these generate a Zariski dense subgroup, which proves (1). Then (2) follows from Lemma 5.1.9. Statement (3) follows from Lemma 5.2.2 and the observation that the image of a torsion-free subgroup of $\mathrm{SL}_2(\mathbb{Z}_\ell)$ in $\mathrm{PSL}_2(\mathbb{Z}_\ell)$ is torsion-free, since the kernel of $\mathrm{SL}_2(\mathbb{Z}_\ell) \to \mathrm{PSL}_2(\mathbb{Z}_\ell)$ is torsion.

By construction, $\mathrm{tr}(\psi(a^m)) = (1+t)^m + (1+t)^{-m}$, which is a non-constant element of $\mathbb{Z}_\ell[[t]]$. Let $c, d \in \mathbb{Z}_{>0}$ be such that the image of $\mathrm{tr}(\psi(a^m))$ in $\mathbb{Z}_\ell[[t]]/(\ell^c, t^d)$ is non-constant. Since $\bar{G}$ contains a^m and ψ is continuous, there exists $g \in G$ such that

$$\mathrm{tr}(\psi(g)) \equiv \mathrm{tr}(\psi(a^m)) \mod (\ell^c, t^d).$$

Hence, $\mathrm{tr}(\psi(g))$ is non-constant, hence not algebraic over $\mathbb{Q}_\ell$, so Lemma 5.1.16 proves that $\rho|_G$ is not rigid. The final statement follows by construction. $\square$

Corollary 5.2.5. *Let C be a curve of genus $g \geq 2$ over an algebraically closed field k, and let ℓ be a prime invertible in k. Then there exists a representation $\rho\colon \pi_1^{\text{ét}}(C) \to \mathrm{SL}_2(\mathbb{Q}_\ell)$ with image landing in $\mathrm{SL}_2(\mathbb{Z}_\ell)$ such that for every finitely generated subgroup $G \subseteq \pi_1^{\text{ét}}(C)$ whose closure is open, the representation $\rho|_G$ has the following properties:*

(1) $\rho|_G$ *is Zariski dense;*

(2) $\rho|_G$ *is absolutely irreducible;*

(3) *the image of* $\bar{\rho}|_G \colon G \to \mathrm{PSL}_2(\mathbb{Q}_\ell)$ *is torsion-free;*

(4) $\rho|_G$ *is not rigid.*

Proof. By Remark 5.2.1, there exists a surjection $\pi_1^{\text{ét}}(C) \twoheadrightarrow \mathbb{Z}_\ell^{*2}$. Now let $\rho' \colon \mathbb{Z}_\ell^{*2} \to \mathrm{SL}_2(\mathbb{Q}_\ell)$ be the representation constructed in Proposition 5.2.4, and let ρ be its pullback to $\pi_1^{\text{ét}}(C)$. For any finitely generated subgroup $G \subseteq \pi_1^{\text{ét}}(C)$ whose closure is open, the same properties hold for its image in $\mathbb{Z}_\ell^{*2}$. Hence, the result follows from Proposition 5.2.4. $\square$

5.3 Local systems and maps to curves

The main input is Theorem 5.3.8, which is a mild generalisation of a theorem by Simpson. We will use it in Section 5.4 to deduce a liftability statement for maps to curves.

We start by recalling some results on pushforward maps of fundamental groups.

Lemma 5.3.1. *Let* $f \colon X \to Y$ *be a dominant morphism that is locally of finite type between connected schemes, and assume that* Y *is geometrically unibranch. Let* $\bar{x}$ *be a geometric point of* X*, and let* $\bar{y} = f(\bar{x})$*. Then the pushforward* $f_* \colon \pi_1^{\text{ét}}(X, \bar{x}) \to \pi_1^{\text{ét}}(Y, \bar{y})$ *is an open map.*

Proof. Since Y is geometrically unibranch and connected, its reduction Y^{red} is integral, and if $\tilde{Y}$ denotes the normalisation of Y^{red} then the map $\tilde{Y} \to Y$ is radicial. Hence it induces an isomorphism on fundamental groups, also after base change. Thus, we may assume Y is normal and integral. Since X and Y are connected, the fundamental groups do not depend on the basepoints $\bar{x}, \bar{y}$. Thus, we may assume that $\bar{x}$ is a geometric generic point of an irreducible component

$X_1 \subseteq X$ dominating Y. We get the commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(K(X_1)) & \longrightarrow & \mathrm{Gal}(K(Y)) \\ \downarrow & & \downarrow \\ \pi_1^{\text{ét}}(X, \bar{x}) & \longrightarrow & \pi_1^{\text{ét}}(Y, \bar{y})\,. \end{array} \tag{5.3.1}$$

Since Y is normal, the right vertical arrow is surjective [SGA1, Exp. V, Prop. 8.2]. Note that the map $\pi_1^{\text{ét}}(X, \bar{x}) \to \pi_1^{\text{ét}}(Y, \bar{y})$ is always closed since it is a continuous map of profinite groups. To show that such a map is open, it suffices to prove that the image has finite index. In diagram (5.3.1), this property for the bottom map is implied by the same property for the top map.

Now $K(Y) \subseteq K(X_1)$ is a finitely generated field extension. Treating the finite separable, finite purely inseparable, and finitely generated purely transcendental cases separately, we get an open embedding, an isomorphism, and a surjection on Galois groups respectively. □

See also [Kol03, Lemma 10] for the case of varieties over an algebraically closed field, but our proof is easier and more general.

Remark 5.3.2. The statement is not true even for varieties over an algebraically closed field if we do not make some sort assumption on Y. Indeed, consider a nodal cubic curve C with its normalisation $f\colon \tilde{C} \to C$. Then f is dominant and finite, but $\pi_1^{\text{ét}}(\tilde{C}) = 0$ and $\pi_1^{\text{ét}}(C) \cong \hat{\mathbb{Z}}$. Thus, f_* is not an open map.

Remark 5.3.3. If f is proper with geometrically connected fibres, then f_* is actually surjective [SGA1, Exp. IX, Cor. 5.6].

Definition 5.3.4. Let k be a field, and $\mathcal{X}$ a finite type Deligne–Mumford stack over k. Then $\mathcal{X}$ is an *orbifold stack* if $\mathcal{X}$ is smooth over k and the stabiliser at the generic point of every component of $\mathcal{X}$ is trivial. If moreover $\dim \mathcal{X} = 1$, then $\mathcal{X}$ is an *orbicurve.*

See [BN06] for a more detailed discussion of DM curves and orbicurves, including a classification of the latter.

Remark 5.3.5. If $\mathcal{Y}$ is an orbicurve over $\mathbb{C}$, then it only has finitely many points $y_1, \ldots, y_n$ with nontrivial stabilisers. Then $Y \setminus \{y_1, \ldots, y_n\}$ is a curve over $\mathbb{C}$, and $\pi^{\mathrm{top}}(\mathcal{Y})$ is the quotient of $\pi_1^{\mathrm{top}}(Y \setminus \{y_1, \ldots, y_n\})$ by $\gamma_i^{n_i}$, where γ_i is a loop around y_i, and n_i is the orbifold degree at y_i.

Remark 5.3.6. The statement of Lemma 5.3.1 is true for a dominant morphism $f \colon X \to \mathcal{Y}$, where $\mathcal{Y}$ is an orbifold stack. Indeed, in this case $\mathcal{Y}$ has an open substack that is representable by a scheme U. The map $\pi_1^{\text{ét}}(U) \to \pi_1^{\text{ét}}(\mathcal{Y})$ is surjective since $\mathcal{Y}$ is normal, so the result for $\mathcal{Y}$ easily follows from that for U, by the same argument as the proof of Lemma 5.3.1.

Corollary 5.3.7. *Let k be a field, let X be a smooth k-variety, and let $\mathcal{Y}$ be a smooth proper orbicurve with infinite étale fundamental group. Then every rational map $X \dashrightarrow \mathcal{Y}$ extends to a morphism $X \to \mathcal{Y}$.*

Proof. There does not exist a nonconstant map $\mathbb{P}^1 \to \mathcal{Y}$: by Remark 5.3.6 the map $\pi_1^{\text{ét}}(\mathbb{P}^1) \to \pi_1^{\text{ét}}(\mathcal{Y})$ is open, which is impossible if $\pi_1^{\text{ét}}(\mathcal{Y})$ is infinite. Thus, the second proof of Corollary 4.1.4 carries through in this setting as well. □

Theorem 5.3.8 (Simpson)**.** *Let X be a smooth proper variety over $\mathbb{C}$. Let $\rho \colon \pi_1^{\mathrm{top}}(X) \to \mathrm{SL}_2(\mathbb{C})$ be a representation with Zariski-dense image. Then at least one of the following holds:*

(1) *ρ is rigid and is a subquotient of a variation of Hodge structure;*

(2) *there exists a map $f \colon X \to \mathcal{Y}$ where $\mathcal{Y}$ is an orbicurve, and a representation $\tau \colon \pi_1^{\mathrm{top}}(\mathcal{Y}) \to \mathrm{PSL}_2(\mathbb{C})$ such that $\bar{\rho} = \tau \circ f_*$.*

In case (2)*, if the image of* $\bar{\rho}$ *in* $\mathrm{PSL}_2(\mathbb{C})$ *is torsion-free, then we may take* $\mathcal{Y}$ *to be an actual curve* Y*.*

Remark 5.3.9. Since there exist rigid rank 2 local systems on curves, the two statements of Theorem 5.3.8 are not mutually exclusive.

Proof of Theorem. The first statement is [Sim91, Thm. 10] if X is smooth and projective; see also [CS08] for an alternative proof, a quasi-projective version, and a much more detailed discussion.

Now let X be smooth and proper, and let $\rho\colon \pi_1^{\mathrm{top}}(X) \to \mathrm{SL}_2(\mathbb{C})$ be a non-rigid representation with Zariski dense image. By Chow's lemma [EGA2, Thm. 5.6.1] and resolution of singularities [Hir64], there exists a smooth projective variety $\tilde{X}$ with a birational morphism $\pi\colon \tilde{X} \to X$.

Then π induces an isomorphism on fundamental groups; for étale fundamental groups this is [SGA1, Exp. X, Cor. 3.4], and for topological fundamental groups this is classical. From the projective case treated above, we get a morphism $\tilde{f}\colon \tilde{X} \to \mathcal{Y}$ where $\mathcal{Y}$ is an orbicurve and a representation $\tau\colon \pi_1^{\mathrm{top}}(\mathcal{Y}) \to \mathrm{PSL}_2(\mathbb{C})$ such that

$$\bar{\rho} \circ \pi_* = \tau \circ \tilde{f}_*.$$

But since the image of ρ is Zariski dense, the fundamental group of $\mathcal{Y}$ is infinite. Thus, by Corollary 5.3.7, the map $\tilde{f}\colon \tilde{X} \to \mathcal{Y}$ factors through a map $f\colon X \to \mathcal{Y}$. This proves the first statement in the smooth proper case.

The last statement follows from the description of $\pi_1^{\mathrm{top}}(\mathcal{Y})$ of Remark 5.3.5: if $\mathcal{Y} \to Y$ is the coarse space, then the kernel of the natural map $\pi_1^{\mathrm{top}}(\mathcal{Y}) \to \pi_1^{\mathrm{top}}(Y)$ is generated by torsion elements γ_i. Thus, if the image of τ in $\mathrm{PSL}_2(\mathbb{C})$ is torsion-free, then τ factors through $\pi_1^{\mathrm{top}}(Y)$. $\square$

Remark 5.3.10. Let $\rho\colon \pi_1^{\mathrm{top}}(X) \to \mathrm{SL}_2(\mathbb{C})$ be a Zariski-dense representation that is not rigid, such that the image in $\mathrm{PSL}_2(\mathbb{C})$ is torsion-free. Then the theorem gives a map $X \to Y$ to a curve such that $\bar{\rho}$ factors through $\pi_1^{\mathrm{top}}(Y)$. However, this Y is not unique, and we will measure the failure of uniqueness.

Corollary 5.3.11. *Let X be a normal proper variety over an algebraically closed field k. Let $\rho\colon \pi_1^{\text{ét}}(X) \to G$ be a homomorphism with infinite image. If ρ factors through two maps $f\colon X \to C$, $f'\colon X \to C'$ where C and C' are smooth curves, then it factors through a smooth curve C'' dominating both C and C'.*

Proof. By assumption, there exist morphisms $f\colon X \to C$ and $f'\colon X \to C'$ and maps $\tau\colon \pi_1^{\text{ét}}(C) \to G$ and $\tau'\colon \pi_1^{\text{ét}}(C') \to G$ such that $\rho = \tau \circ f_* = \tau' \circ f'_*$. Note f and f' are nonconstant since ρ is nontrivial, hence C and C' are proper since X is.

Now suppose that the induced map $X \to C \times C'$ is dominant. Then by Lemma 5.3.1, the map $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(C \times C')$ has open image. We have an isomorphism $\pi_1^{\text{ét}}(C \times C') = \pi_1^{\text{ét}}(C) \times \pi_1^{\text{ét}}(C')$ since C and C' are smooth and proper [SGA1, Exp. X, Cor. 1.7]. But the image of $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(C) \times \pi_1^{\text{ét}}(C')$ is contained in the subgroup

$$H = \left\{ (\alpha, \beta) \in \pi_1^{\text{ét}}(C) \times \pi_1^{\text{ét}}(C') \;\middle|\; \tau(\alpha) = \tau'(\beta) \right\}.$$

Since the image of ρ is infinite, this subgroup H has infinite index, contradicting the fact that $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(C) \times \pi_1^{\text{ét}}(C')$ has open image. Thus, we conclude that $X \to C \times C'$ is not dominant, hence its image is contained in a (possibly singular) curve D. Take C'' to be the normalisation of D, and note that ρ factors through C'' since it factors through C and X is normal. $\square$

Corollary 5.3.12. *With the assumptions of Corollary 5.3.11, if $f'_*\mathcal{O}_X = \mathcal{O}_{C'}$,* *then f factors through f':*

$$X \xrightarrow{f'} C' \to C.$$

Proof. The assumption on f' forces the map $C'' \to C'$ of the conclusion of Corollary 5.3.11 to be an isomorphism. $\square$

5.4 Lifting maps to curves

We will use the results from the previous sections to prove a liftability statement about maps to curves.

Setup 5.4.1. Let R be a DVR whose residue field k is algebraically closed (of arbitrary characteristic), such that the algebraic closure of the fraction field $K = \operatorname{Frac} R$ is isomorphic to $\mathbb{C}$. Equivalently, K is a field of characteristic 0 of cardinality the continuum.

Remark 5.4.2. Alternatively, one can demand that R is a ring of characteristic 0 of cardinality the continuum, since the cardinality of a domain equals that of its fraction field (remarkably, this is true for both finite and infinite domains, for different reasons). The field k is necessarily of cardinality at most the continuum.

Example 5.4.3. Let R be a complete DVR of mixed characteristic whose residue field k is countable and algebraically closed. Then R is a finite extension of $W(k)$. The latter has cardinality the continuum since $W(k) \cong k^{\mathbb{N}}$ as sets. Hence R has the same cardinality since it is a finite extension.

From Simpson's theorem (Theorem 5.3.8), we deduce the following theorem.

Theorem 5.4.4. *Let R, k, and K be as in Setup 5.4.1, and let $\mathcal{X} \to \operatorname{Spec} R$ be a smooth proper morphism. Let C be a smooth proper curve over k of genus $g \geq 2$ and $\psi \colon \mathcal{X}_0 \to C$ a morphism of k-varieties such that $\psi_* \mathcal{O}_{\mathcal{X}_0} = \mathcal{O}_C$. Then* *there exists a generically finite extension $R \to R'$ of DVRs, a smooth proper curve $\mathcal{Y}$ over R', a morphism $\phi\colon \mathcal{X} \times_R R' \to \mathcal{Y}$, and a commutative diagram*

$$\begin{tikzcd} & \mathcal{X}_0 \arrow[dl, "\psi"'] \arrow[dr, "\phi_0"] & \\ C \arrow[rr, "\chi"'] & & \mathcal{Y}_0 , \end{tikzcd}$$

where χ is purely inseparable. In particular, χ is a power of the relative Frobenius if $\operatorname{char} k = p > 0$*, and χ is an isomorphism if* $\operatorname{char} k = 0$.

Remark 5.4.5. Thus, given a lift of $\mathcal{X}_0$ over R, any map $\psi\colon \mathcal{X}_0 \to C$ for $g(C) \geq 2$ with $\phi_* \mathcal{O}_{\mathcal{X}_0} = \mathcal{O}_C$ can be lifted along with $\mathcal{X}_0$, up to an extension of R and a purely inseparable morphism.

Remark 5.4.6. The theorem does not require k to have characteristic $p > 0$. In equicharactersitic 0, the theorem is a deformation result for maps to smooth proper hyperbolic curves. In Section 5.5, we give a deformation-theoretic proof in equicharacteristic 0.

Proof of theorem. Let ℓ be a prime invertible in k, and let $\rho\colon \pi_1^{\text{ét}}(C) \to \mathrm{SL}_2(\mathbb{Q}_\ell)$ be the representation constructed in Corollary 5.2.5. Note that its image lands in $\mathrm{SL}_2(\mathbb{Z}_\ell)$. The map $\psi_*\colon \pi_1^{\text{ét}}(\mathcal{X}_0) \to \pi_1^{\text{ét}}(C)$ is surjective by Remark 5.3.3, and the specialisation map $\pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}}) \to \pi_1^{\text{ét}}(\mathcal{X}_0)$ is surjective [SGA1, Exp. X, Cor. 2.4]. Thus, the composite map $\pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}}) \to \pi_1^{\text{ét}}(C)$ is surjective. By abuse of notation, we will denote the pullback of ρ to $\pi_1^{\text{ét}}(\mathcal{X}_0)$ or $\pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}})$ by ρ as well, and we will write $\bar{\rho}$ for the respective representations to PSL_2 instead of SL_2.

We have $\bar{K} \cong \mathbb{C}$, so we get a $\mathbb{C}$-variety $\mathcal{X}_{\bar{K}}$. Then $\pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}})$ is the profinite completion of $\pi_1^{\text{top}}(\mathcal{X}_{\bar{K}})$ [SGA1, Exp. XII, Cor. 5.2]. If G denotes the image of $\pi_1^{\text{top}}(\mathcal{X}_{\bar{K}})$ in $\pi_1^{\text{ét}}(C)$, then G is a finitely generated dense subgroup of $\pi_1^{\text{ét}}(C)$. By Corollary 5.2.5, the pullback of ρ to $\pi_1^{\text{top}}(\mathcal{X}_{\bar{K}})$ is a Zariski-dense, absolutely irreducible, non-rigid representation whose image in $\mathrm{PSL}_2(\mathbb{Q}_\ell)$ is torsion-free.

Choosing an injection $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we get a representation $\rho\colon \pi_1^{\mathrm{top}}(\mathcal{X}_{\bar{K}}) \to \mathrm{SL}_2(\mathbb{C})$. Then Simpson's Theorem 5.3.8 implies that there exists a smooth proper curve $\bar{Y}$ over $\mathbb{C}$, a map $\phi\colon \mathcal{X}_{\bar{K}} \to \bar{Y}$, and a representation $\tau\colon \pi_1^{\mathrm{top}}(\bar{Y}) \to \mathrm{PSL}_2(\mathbb{C})$ such that $\bar{\rho} = \tau \circ \phi_*$. Replacing $\bar{Y}$ by $\mathbf{Spec}\, \phi_* \mathcal{O}_{\mathcal{X}_{\bar{K}}}$ if necessary, we may assume that ϕ satisfies $\phi_* \mathcal{O}_{\mathcal{X}_{\bar{K}}} = \mathcal{O}_{\bar{Y}}$. Then ϕ has connected geometric fibres, so the pushforward $\phi_*\colon \pi_1^{\mathrm{top}}(\mathcal{X}_{\bar{K}}) \to \pi_1^{\mathrm{top}}(\bar{Y})$ is surjective. Hence, τ lands in the profinite group $\mathrm{PSL}_2(\mathbb{Z}_\ell) \subseteq \mathrm{PSL}_2(\mathbb{C})$, so τ extends uniquely to a representation of the étale fundamental group, which by abuse of notation we denote by

$$\tau\colon \pi_1^{\text{ét}}(\bar{Y}) \to \mathrm{PSL}_2(\mathbb{Z}_\ell).$$

Now $\bar{Y}$ is defined over some finite extension K' of K, i.e. there exists a smooth proper curve Y over K' such that $Y \times_{K'} \bar{K}' \cong \bar{Y}$. Extending K' further, we may also assume that Y has a rational point, and the map $\phi\colon \mathcal{X}_{\bar{K}} \to \bar{Y}$ is defined over K'. We then have a $\mathrm{Gal}(\bar{K}'/K')$-equivariant surjection

$$\phi_*\colon \pi_1^{\text{ét},\ell}(\mathcal{X}_{\bar{K}}) \twoheadrightarrow \pi_1^{\text{ét},\ell}(Y_{\bar{K}'}). \tag{5.4.1}$$

As $\mathcal{X}_K$ has good reduction $\mathcal{X}$, the $\mathrm{Gal}(\bar{K}/K)$-action on $\pi_1^{\text{ét},\ell}(\mathcal{X}_{\bar{K}})$ is unramified. Let R' be the localisation of the integral closure of R in K' at any prime above $\mathfrak{m}_R$. Then the $\mathrm{Gal}(\bar{K}'/K)$-action on $\pi_1^{\text{ét},\ell}(\mathcal{X}_{\bar{K}})$ is unramified, so the same holds for the $\mathrm{Gal}(\bar{K}'/K')$-action on $\pi_1^{\text{ét},\ell}(\mathcal{Y}_{\bar{K}'})$ by the surjection (5.4.1). By a theorem of Takayuki Oda [Oda95, Thm. 3.2][1] ("Néron–Ogg–Shafarevich for curves"), this implies that Y has good reduction. Thus, there exists a smooth proper curve $\mathcal{Y} \to \operatorname{Spec} R'$ with generic fibre Y.

[1]Oda's paper only states the result over a number field, but the methods work over any DVR. See e.g. [Tam97, Thm. 0.8] for a proof over an arbitrary DVR.

We note that the genus of Y is at least 1, for otherwise τ would be trivial, contradicting the irreducibility of ρ. By Corollary 4.1.4 (here we use that Y has a rational point), the morphism $\phi\colon \mathcal{X}_{K'} \to Y$ extends (uniquely) to a morphism

$$\phi\colon \mathcal{X}_{R'} \to \mathcal{Y}.$$

The specialisation map on étale fundamental groups induces an isomorphism [SGA1, Exp. X, Cor. 3.9]

$$\mathrm{sp}\colon \pi_1^{\text{ét},\ell}(\bar{Y}) \xrightarrow{\sim} \pi_1^{\text{ét},\ell}(\mathcal{Y}_0).$$

This gives a representation $\tau\colon \pi_1^{\text{ét}}(\mathcal{Y}_0) \to \mathrm{PSL}_2(\mathbb{Q}_\ell)$ such that $\tau \circ \phi_{0,*} = \rho$ on $\pi_1^{\text{ét}}(\mathcal{X}_0)$. Moreover, the image is Zariski dense, so in particular infinite. Since $\psi_*\mathcal{O}_{\mathcal{X}_0} = \mathcal{O}_C$, Corollary 5.3.12 implies that ϕ_0 factors through ψ:

$$\begin{tikzcd} & \mathcal{X}_0 \arrow[dl, "\psi"'] \arrow[dr, "\phi_0"] & \\ C \arrow[rr, "\chi"'] & & \mathcal{Y}_0 . \end{tikzcd}$$

Thus, $\mathcal{X}_0 \to C \to \mathcal{Y}_0$ is the Stein factorisation of ϕ_0. On the other hand, by Corollary 2.1.7 we have $\phi_*\mathcal{O}_{\mathcal{X}_{R'}} = \mathcal{O}_{\mathcal{Y}}$, since this holds in the generic fibre and $\mathcal{Y}$ is normal (in fact, regular). Thus, Corollary 2.2.4 implies that χ is purely inseparable, hence a power of Frobenius. $\square$

For our main application to liftability, we want to have a version that does not require $\psi_*\mathcal{O}_{\mathcal{X}_0} = \mathcal{O}_C$. It is easy to deduce this from the theorem above:

Corollary 5.4.7. *Let R, k, and K be as in Setup 5.4.1, and let $\mathcal{X} \to \operatorname{Spec} R$ be a smooth proper morphism. Let C be a smooth proper curve over k of genus $g \geq 2$, and let $\psi\colon \mathcal{X}_0 \to C$ be any morphism of k-varieties. Then there exists a* *generically finite extension $R \to R'$ of DVRs, a smooth proper curve $\mathcal{Y}$ over R', a morphism $\phi\colon \mathcal{X} \times_R R' \to \mathcal{Y}$, and a commutative diagram*

$$\begin{tikzcd} & \mathcal{X}_0 \arrow[dl, "\psi"'] \arrow[d] \arrow[dr, "\phi_0"] & \\ & C' \arrow[dl] \arrow[dr, "\chi"'] & \\ C & & \mathcal{Y}_0 , \end{tikzcd}$$

where χ is purely inseparable. In particular, χ is a power of the relative Frobenius if $\operatorname{char} k = p > 0$*, and χ is an isomorphism if* $\operatorname{char} k = 0$*.*

Proof. If ψ is constant, there is nothing to prove, so we may assume it is dominant. Let $\psi' \colon \mathcal{X}_0 \to C'$ be the Stein factorisation of ψ. Note that $g(C') \geq 2$, for example because the pushforward $\mathbf{Jac}_{C'} \to \mathbf{Jac}_C$ is surjective and $g(C) \geq 2$. The result now follows from Theorem 5.4.4, applied to $\psi' \colon \mathcal{X}_0 \to C'$. $\square$

Remark 5.4.8. Instead of deducing Corollary 5.4.7 from Theorem 5.4.4, we could also prove it directly using the same strategy. This only requires the following modifications:

In the first paragraph of the proof, the map $\psi_* \colon \pi_1^{\text{ét}}(\mathcal{X}_0) \to \pi_1^{\text{ét}}(C)$ is now only open by Lemma 5.3.1. Then Corollary 5.2.5 still shows that the local system on G we get is non-rigid. In the final part of the proof, use Corollary 5.3.11 instead of Corollary 5.3.12.

Remark 5.4.9. In Theorem 5.5.1 below, we give a deformation-theoretic proof of Theorem 5.4.4 in equicharacteristic 0, relying on a vanishing theorem of Kollár [Kol86, Thm. 2.1]. This shows that in equicharacteristic 0, we do not need the generically finite extension $R \to R'$.

Remark 5.4.10. In mixed characteristic, the situation is a bit more mysterious.

Indeed, Theorem 5.4.4 shows that ψ can be lifted after postcomposition with a power of Frobenius. However, this power of Frobenius is *unique*.

Indeed, assume $\phi\colon \mathcal{X} \to \mathcal{Y}$ is a lift of $\mathrm{Frob}^n \circ \psi$ and $\phi'\colon \mathcal{X} \to \mathcal{Y}'$ is a lift of $\mathrm{Frob}^{n+r} \circ \psi$ with $r > 0$ (without loss of generality, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Y}'$ are all defined over the same DVR R). We will also assume that $\phi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$, as was the case for the curve $\mathcal{Y}$ constructed in Theorem 5.4.4. We get a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\phi"'] \arrow[rr, bend left, "\phi'"] & \mathcal{Y} & \mathcal{Y}' \\
\mathcal{X}_0 \arrow[u, hook, squiggly] \arrow[r] & \mathcal{Y}_0 \arrow[u, hook, squiggly] \arrow[r, "\mathrm{Frob}^r"] & \mathcal{Y}'_0 . \arrow[u, hook, squiggly]
\end{tikzcd}$$

Let $\rho\colon \pi_1^{\text{ét}}(\mathcal{Y}'_0) \to \mathrm{SL}_2(\mathbb{Q}_\ell)$ be the representation constructed in Corollary 5.2.5. Pulling back to $\mathcal{Y}'_{\bar{K}}$ and $\mathcal{Y}_{\bar{K}}$ gives representations whose pullbacks to $\mathcal{X}_{\bar{K}}$ agree. Hence, Corollary 5.3.12 implies that $\phi'_{\bar{K}}$ factors through $\phi_{\bar{K}}$, i.e. we get a map $F\colon \mathcal{Y}_{\bar{K}} \to \mathcal{Y}'_{\bar{K}}$ such that $F \circ \phi_{\bar{K}} = \phi'_{\bar{K}}$. Extending R if necessary, this comes from a map $F\colon \mathcal{Y}_K \to \mathcal{Y}'_K$.

By Corollary 4.1.4, this extends to a map $F\colon \mathcal{Y} \to \mathcal{Y}'$ with $F \circ \phi = \phi'$. Hence $F_0 \circ \phi_0 = \mathrm{Frob}^r \circ \phi_0$, so $F_0 = \mathrm{Frob}^r$ because ϕ_0 is an epimorphism (this is easy to check by hand, but it also follows from [Stacks, Tag 023Q]). Thus, we have filled in the diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\phi"] & \mathcal{Y} \arrow[r, "F"] & \mathcal{Y}' \\
\mathcal{X}_0 \arrow[u, hook, squiggly] \arrow[r] & \mathcal{Y}_0 \arrow[u, hook, squiggly] \arrow[r, "\mathrm{Frob}^r"] & \mathcal{Y}'_0 . \arrow[u, hook, squiggly]
\end{tikzcd}$$

For all $s \in \operatorname{Spec} R$, the morphism $F_s\colon \mathcal{Y}_s \to \mathcal{Y}'_s$ is flat (since it is a dominant morphism of curves). Since $\mathcal{Y} \to \operatorname{Spec} R$ is flat, the fibrewise criterion of flatness [EGA4_{III}, Thm. 11.3.10] shows that F is flat; in particular $\deg F = \deg(\mathrm{Frob}^r)$. But by Riemann–Hurwitz there are no degree p^r maps (for $r > 0$) between two smooth proper hyperbolic curves of the same genus in characteristic 0.

Remark 5.4.11. Thus, we cannot simply replace Kollár's vanishing theorems by "Frobenius-vanishing theorems" (cf. [Ara04]). It would be interesting to see whether Theorem 5.4.4 is true without any Frobenii. This is related to the example constructed in Example 2.3.3, but there $\mathcal{Y} = \mathbb{A}^1_R$. Perhaps one can make a similar example where $\mathcal{Y}$ is a smooth proper hyperbolic curve.

5.5 Deformation theory of morphisms to curves

We will use a vanishing theorem of Kollár [Kol86, Thm. 2.1] to give a deformation-theoretic proof of Theorem 5.4.4 in characteristic 0.

Theorem 5.5.1. *Let k be a field of characteristic 0, let X and C be smooth proper varieties over k such that C is a curve of genus $g \geq 2$, and let $\psi\colon X \to C$ be a morphism such that $\psi_*\mathcal{O}_X = \mathcal{O}_C$. Then the natural map*

$$\mathrm{Def}_\psi \to \mathrm{Def}_C$$

is an isomorphism.

Remark 5.5.2. Observe that the statement is stronger than Theorem 5.4.4; for example, we no longer need a generically finite extension of DVRs $R \to R'$. We will see in Example 5.5.4 that our proof of Theorem 5.5.1 breaks down in positive characteristic. The author does not know if the stronger deformation-theoretic statement is still true in mixed characteristic.

Proposition 5.5.3. *Let X and Y be finite type k-schemes, and let $\psi\colon X \to Y$ be a morphism such that $\psi_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{Hom}_Y(\Omega_Y, R^1\psi_*\mathcal{O}_X) = 0$. Then the* *natural map*

$$\mathrm{Def}_\psi \to \mathrm{Def}_X$$

is an isomorphism.

Proof. See e.g. [Vak06, Thm. 5.1 (arXiv version)] or [BHPS13, Prop. 3.10]. □

Proof of Theorem 5.5.1. By Proposition 5.5.3, it suffices to show that

$$\mathrm{Hom}_C(\Omega_C, R^1\psi_*\mathcal{O}_X) = 0.$$

If $n = \dim X$, then [Kol86, Cor. 7.8] shows that $R^1\psi_*\mathcal{O}_X \cong \omega_C \otimes (R^{n-2}\psi_*\omega_X)^\vee$, hence

$$\begin{aligned}\mathrm{Hom}_C(\omega_C, R^1\psi_*\mathcal{O}_X) &= H^0(C, (R^{n-2}\psi_*\omega_X)^\vee) \\ &\cong H^1(C, \omega_C \otimes R^{n-2}\psi_*\omega_X)^\vee.\end{aligned}$$

If $\mathrm{char}\, k = 0$, this vanishes by [Kol86, Thm. 2.1(iii)], since ω_C is ample. □

We finish our discussion by exhibiting an example in positive characteristic where the obstruction space

$$\mathrm{Hom}_C(\Omega_C, R^1\psi_*\mathcal{O}_X)$$

of Proposition 5.5.3 does not vanish. This also gives a particular example where Kollár's vanishing result does not hold in positive characteristic.

Example 5.5.4. Let k be an algebraically closed field of characteristic $p > 2$. Let $f\colon D \to S = \mathbb{P}^1$ be the non-isotrivial family of supersingular compact type genus 2 curves as constructed in [Mor81], with Jacobian[1] $g\colon \mathcal{A} \to S$. Then

[1]Moret-Bailly constructs D as a family of divisors in $\mathcal{A}$, whence the notation.

[Mor81, II, Prop. 3.1] shows that

$$\mathrm{Lie}(\mathcal{A}/S) = \mathcal{O}_S(1) \oplus \mathcal{O}_S(-p).$$

Note that the geometric fibres of f are reduced and connected (because they are semistable curves), so $f_*\mathcal{O}_D = \mathcal{O}_S$ holds universally (Lemma 2.1.2). Let $h\colon C \to S = \mathbb{P}^1$ be a morphism from a smooth curve C of genus $g \geq 2$ that is ramified only in points $s \in S$ such that D_s is smooth, and assume that $d := \deg(h) > 4g - 4$. Consider the base change $X = D \times_S C$, with the map

$$\psi\colon X \to C.$$

Note that X is smooth. Indeed, D is smooth [Mor81, II, Prop. 2.5(iii)], and by construction every point of X is either étale over D or in the smooth locus of $\psi\colon X \to C$. The Jacobian of $X \to C$ is the base change $\mathcal{B} = \mathcal{A} \times_S C$, hence

$$\mathrm{Lie}(\mathcal{B}/C) = h^*\big(\mathcal{O}_S(1) \oplus \mathcal{O}_S(-p)\big).$$

Since $\mathbf{Pic}^0_{X/C} = \hat{\mathcal{B}}$, we conclude that

$$R^1\psi_*\mathcal{O}_X = \mathrm{Lie}(\hat{\mathcal{B}}/C) = \mathrm{Lie}(\mathcal{B}/C) = h^*\left(\mathcal{O}_S(1) \oplus \mathcal{O}_S(-p)\right).$$

But since $d = \deg(h) > 4g - 4$, we have $\deg h^*(\mathcal{O}_S(1)) = d > 4g - 4$, hence

$$\deg \mathscr{H}om_C(\omega_C, h^*(\mathcal{O}_S(1))) > 2g - 2.$$

We conclude that $\mathrm{Hom}_C(\omega_C, R^1\psi_*\mathcal{O}_X) \neq 0$, showing that the obstruction space of Proposition 5.5.3 does not always vanish in positive characteristic.

6. Lifting varieties

6.1 The general lifting problem

We use a very general definition of a lift of a proper scheme X over a field k of characteristic p. At the expense of enlarging the field k, it can be reduced to the case of a mixed characteristic DVR; see Lemma 6.1.3.

Definition 6.1.1. Let k be a field of characteristic $p > 0$, and let X be a proper k-scheme. Then a *lift* of X to characteristic 0 consists of the following data:

- An irreducible pointed scheme (S, η) such that $\kappa(\eta)$ has characteristic 0,
- A field Ω with an embedding $k \hookrightarrow \Omega$ and a morphism $\operatorname{Spec} \Omega \to S$,
- A flat proper morphism $\mathcal{X} \to S$,
- An isomorphism $\phi\colon \mathcal{X} \times_S \operatorname{Spec} \Omega \xrightarrow{\sim} X \times_{\operatorname{Spec} k} \operatorname{Spec} \Omega$.

By abuse of notation, we sometimes refer to $\mathcal{X} \to S$ or even $\mathcal{X}_\eta \to \operatorname{Spec} \kappa(\eta)$ as the lift of X.

The situation is summarised by the diagram

$$\begin{tikzcd}
X \times_k \Omega & \mathcal{X} \times_S \operatorname{Spec} \Omega \arrow[l, "\phi", "\sim"'] \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_\eta \arrow[l] \arrow[d] \\
& \operatorname{Spec} \Omega \arrow[r] & S & \operatorname{Spec} \kappa(\eta) \arrow[l]
\end{tikzcd},$$

where the two squares are pullbacks.

Remark 6.1.2. In Definition 6.1.1, if $\operatorname{char} k = \operatorname{char} \kappa(\eta)$ (either both 0 or both $p > 0$), then $\mathcal{X}$ is a *deformation* of X instead of a lift.

Lemma 6.1.3. *Let X be a proper k-scheme with $\operatorname{char} k = p > 0$. If there exists a lift of X, then there also exists a lift where $S = \operatorname{Spec} R$, with R a DVR that is* *essentially of finite type over* $\operatorname{Spec} \mathbb{Z}$*, and* η *is the generic point.*

Proof. If S has a point η of characteristic 0, then its generic point also has characteristic 0. Thus, we may assume η is the generic point. Let s be the image of $\operatorname{Spec} \Omega \to S$. Replacing S by the reduction of an open neighbourhood of s, we may assume that S is affine and integral. By a standard limit argument [EGA4$_{\text{III}}$, Prop. 8.9.1], we may assume that S is a domain of finite type over $\mathbb{Z}$.

Blowing up in the closure of $\{s\}$ gives a map $(S', s') \to (S, s)$ of pointed varieties, where S' is the blowup and s' the generic point of the exceptional divisor. Replacing Ω by a larger field if necessary, we can extend $\kappa(s) \to \Omega$ to $\kappa(s') \to \Omega$. Thus, we may replace (S, s) by (S', s'), so we may assume that s is a codimension 1 point (but S is no longer affine). Base change the family to the local ring $\mathcal{O}_{S,s}$, and normalise to get a DVR (again, this possibly requires enlarging Ω). $\square$

Remark 6.1.4. Thus, in Definition 6.1.1, we may assume $S = \operatorname{Spec} R$, where R is a DVR. The fraction field $K = \operatorname{Frac} R$ has characteristic 0, and both k and the residue field of R embed into Ω. We will write 0 (or s) and η for the special and generic point of $\operatorname{Spec} R$ respectively.

6.2 Lifting divisors and line bundles

We use a decidedly scheme-theoretic construction of various specialisation maps, using Definition 4.3.1 and Lemma 4.3.2. For a more intersection theoretic approach, see [SGA6, Exp. X, §7] or [Ful98, §20.3]. We work over a DVR R.

Remark 6.2.1. For a proper scheme $\mathcal{X}$ over R, the formalism of Definition 4.3.1 gives a specialisation map $\operatorname{sp}\colon \mathbf{Hilb}_{\mathcal{X}/R}(K) \to \mathbf{Hilb}_{\mathcal{X}/R}(k)$. Explicitly, this takes a subscheme $Z \subseteq \mathcal{X}_K$ and maps it to the special fibre of its (scheme-theoretic) closure $\bar{Z} \subseteq \mathcal{X}$.

Similarly, we obtain a specialisation map for $\mathbf{Div}_{\mathcal{X}/R}$, and one for $\mathbf{Pic}_{\mathcal{X}/R}$ if the latter is proper, e.g. when $\mathcal{X} \to \operatorname{Spec} R$ is smooth and proper with geometrically connected fibres [FGA, TDTE VI, Thm. 2.1].

These specialisation maps satisfy the expected compatibilities when going from divisors to line bundles. In Section 4.3 we studied specialisation of morphisms of abelian schemes, and by Lemma 4.4.15 the (noncanonical) isomorphism in Theorem 4.4.10 commutes with these specialisation maps.

Remark 6.2.2. However, we have $\mathbf{Pic}_{\mathcal{X}/R}(K) = \operatorname{Pic}(\mathcal{X}_{\bar{K}})^{\operatorname{Gal}(\bar{K}/K)}$, which need not equal $\operatorname{Pic}(\mathcal{X}_K)$ (and similarly for $\operatorname{Pic}(\mathcal{X}_0)$). A different method is needed to construct a specialisation map on Picard groups (as opposed to Picard schemes). We will give a definition under the additional hypothesis that $\mathcal{X}$ is smooth (and proper) over R; see [SGA6, Exp. X, §7] or for the general case.

Now all schemes in sight are regular, so we may replace the Picard group by the class group. Then [Har77, Prop. II.6.5] gives an exact sequence

$$\mathbb{Z}^{\pi_0(\mathcal{X}_0)} \to \operatorname{Cl}(\mathcal{X}) \to \operatorname{Cl}(\mathcal{X}_K) \to 0.$$

But in fact the first map vanishes, since all connected components of $\mathcal{X}_0$ are principal divisors in $\mathcal{X}$. Then define the specialisation map as the composition

$$\operatorname{Cl}(\mathcal{X}_K) \xleftarrow{\sim} \operatorname{Cl}(\mathcal{X}) \to \operatorname{Cl}(\mathcal{X}_0).$$

The careful reader should check that this definition satisfies the required compatibilities with the specialisation maps already defined.

Definition 6.2.3. Let $\mathcal{X}$ be a proper scheme over R. Then we say that a subvariety (resp. divisor, or line bundle if $\mathcal{X} \to \operatorname{Spec} R$ is also smooth) of $\mathcal{X}_0$

lifts to $\mathcal{X}$ if it is in the image of the specialisation map.

In particular, we deduce the following result.

Lemma 6.2.4. *Let $\mathcal{X}$ be a smooth proper scheme over a DVR R. Let $D \subseteq \mathcal{X}_0$ be an effective divisor. Consider the following statements.*

(1) *D lifts as a subvariety;*

(2) *D lifts as a divisor.*

(3) *$\mathcal{O}_{\mathcal{X}_0}(D)$ lifts as a line bundle.*

Then (1) and (2) are equivalent, and they imply (3).

Proof. For (1) $\Leftrightarrow$ (2), note that any lift of D as a subvariety has pure codimension 1, by the flatness assumption. Since everything is smooth, we do not have to worry about Weil divisors versus Cartier divisors. The commutative diagram

$$\begin{array}{ccc} \operatorname{Div}(\mathcal{X}_K) & \longrightarrow & \operatorname{Cl}(\mathcal{X}_K) \\ {\scriptstyle \mathrm{sp}}\downarrow & & \downarrow{\scriptstyle \mathrm{sp}} \\ \operatorname{Div}(\mathcal{X}_0) & \longrightarrow & \operatorname{Cl}(\mathcal{X}_0) \end{array}$$

immediately shows (2) $\Rightarrow$ (3). $\square$

Remark 6.2.5. The implication (3) $\Rightarrow$ (2) is false in general. We give two examples in equicharacteristic 0 or p (so *lift* should be replaced by *deformation*).

Example 6.2.6. Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be the n^{th} Hirzebruch surface. There exists a family $\mathcal{X} \to \operatorname{Spec} k[t]$ with $\mathcal{X}_0 \cong \mathbb{F}_2$ and $\mathcal{X}_t \cong \mathbb{F}_0$ for all $t \neq 0$. For example, we can use the nonzero element $\alpha \in \operatorname{Ext}^1_{\mathbb{P}^1_k}(\mathcal{O}(1), \mathcal{O}(-1))$ coming from the Koszul complex

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$$

of the generators $x, y \in \mathcal{O}(1)$ to obtain an element

$$t\alpha \in \operatorname{Ext}^1_{\mathbb{P}^1_k}(\mathcal{O}(1), \mathcal{O}(-1))[t] = \operatorname{Ext}^1_{\mathbb{P}^1_{k[t]}}(\mathcal{O}(1), \mathcal{O}(-1)).$$

This gives a rank 2 vector bundle $\mathscr{E}$ on $\mathbb{P}^1_{k[t]}$ whose fibre above 0 is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$, and all other fibres are $\mathcal{O} \oplus \mathcal{O}$. Taking $\mathcal{X} = \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1_{k[t]} \to \operatorname{Spec} k[t]$ gives a family whose special fibre is $\mathbb{F}_2$ and all other fibres are $\mathbb{F}_0$.

The specialisation map $\operatorname{sp}\colon \operatorname{Pic}(\mathcal{X}_\eta) \to \operatorname{Pic}(\mathcal{X}_0)$ is an isomorphism, for example because the generators of each fibre [Beau96, Prop. IV.1] can be defined in families. But $\mathbb{F}_2$ has a (-2)-curve [*loc. cit.*], whereas $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ does not. This curve can therefore not be deformed as a divisor, but its associated line bundle can be deformed.

Example 6.2.7. Similarly, there exists a family $\mathcal{X} \to \operatorname{Spec} k[t]_{(t)}$ whose generic fibre is the blowup of $\mathbb{P}^2$ at three points in general position, and the special fibre is the blowup of $\mathbb{P}^2$ at three points on a line.

For example, consider the open $U \subseteq (\mathbb{P}^1)^{[3]} \cong \mathbb{P}^3$ where the points are pairwise distinct. It has a strict closed subset Z where the three points are collinear, and we can choose a (germ of a) line hitting Z transversely. This gives a map $\operatorname{Spec} k[t]_{(t)} \to U$. Then blow up U at the universal points and pull back to $\operatorname{Spec} k[t]_{(t)}$.

The strict transform of the line through the three points is a (-2)-curve in the special fibre, and again the generic fibre has no (-2)-curves. But one can again see easily that the specialisation map $\operatorname{sp}\colon \operatorname{Pic}(\mathcal{X}_\eta) \to \operatorname{Pic}(\mathcal{X}_0)$ is an isomorphism. Thus we get a curve that cannot be deformed as a divisor, but its associated line bundle can be deformed.

6.3 Lifting surfaces of small Kodaira dimension

We use the (Enriques–)Bombieri–Mumford classification of surfaces in positive characteristic [Mum69b; BM77; BM76], as well as well-known liftability results by Deligne [Del81], Mumford [Mum69a], Norman and Oort [NO80], and Seiler [Sei88], to show the following result.

Theorem 6.3.1. *Let k be an algebraically closed field of characteristic $p \geq 5$, and let X be a smooth proper surface of Kodaira dimension $\kappa(X) \leq 1$. Then there exists a surjection $Z \to X$ from a smooth projective surface Z that can be lifted to a finite extension of $W(k)$.*

Before giving the proof, we recall the following corollary of the Bombieri–Mumford classification in positive characteristic:

Theorem 6.3.2 (Bombieri–Mumford)**.** *Let k be a field, and let X be a minimal surface over k. Then*

(1) $\kappa(X) = -\infty$ *if and only if X is ruled;*

(2) *if $\kappa(X) = 0$, then one of the following holds:*

 (i) *X is a K3 surface;*

 (ii) *X is an abelian surface;*

 (iii) *X is elliptic, or possibly quasi-elliptic if* $\operatorname{char} k \in \{2, 3\}$*;*

(3) *if $\kappa(X) = 1$, then X is elliptic, or possibly quasi-elliptic if* $\operatorname{char} k = 2$.

Proof. Parts (1) and (3) follow from the main result of [Mum69b], which also implies that $12K_X \equiv 0$ in case (2). Then the discussion before Thm. 5 in [BM77] gives a table of five possibilities for the numerical invariants. One of them cannot be realised [Thm. 5], and the the others correspond to K3 surfaces [Thm. 5], abelian surfaces [Thm. 6], bielliptic surfaces (or possibly quasi-bielliptic if $\operatorname{char} k \in \{2,3\}$) [Prop.], and Enriques surfaces [BM76]. Finally, it is proven in [BM76, Thm. 3] that every Enriques surface is elliptic, or possibly quasi-elliptic if $\operatorname{char} k = 2$. $\square$

Proof of Theorem 6.3.1. Let $X \to Y$ be a minimal model. By Theorem 6.3.2, since $\operatorname{char} k \geq 5$, we are in one of the following three cases:

(1) Y is ruled;

(2) Y is a K3 surface or abelian surface;

(3) Y is an elliptic surface.

In case (1), either $Y \cong \mathbb{P}^2$ (hence Y lifts to $W(k)$), or Y is given by a pair $(C, \mathscr{E})$ of a smooth projective curve C and a rank 2 vector bundle $\mathscr{E}$ on C. Then C lifts to a curve $\mathscr{C}$ over $W(k)$, because curves are unobstructed and algebraisable. Moreover, $\mathscr{E}$ is unobstructed as $\operatorname{Ext}^2_C(\mathscr{E}, \mathscr{E}) = 0$, and every formal lift is algebraisable by the Grothendieck existence theorem [EGA3_{I}, Cor. 5.1.6]. This gives a lift of Y over $W(k)$.

In case (2), Y lifts to a finite extension of $W(k)$ by [Del81, Cor. 1.8] if Y is a K3 surface, and by [Mum69a] and [NO80, Cor. 3.2] if Y is an abelian surface. Thus, in cases (1) and (2), we can lift Y over a finite extension R of $W(k)$.

Writing $X \to Y$ as a chain $X = X_0 \to X_1 \to \ldots \to X_r = Y$ of blowups in points $p_i \in X_i$, we will prove by descending induction that X_i can be lifted over R. Indeed, the result for $i = r$ follows from the above. If $\mathscr{X}_i$ is a lift of X_i over R, then by Hensel's lemma we can lift p_i to a point of $\mathscr{X}_i$, whose blowup gives a lift of X_{i-1}. This finishes the proof in cases (1) and (2).

In case (3), X is also an elliptic surface. Choosing a smooth curve $C' \subseteq X$ not contained in any fibre of π, we see that the elliptic surface $X' = X \times_C C' \to C'$

has a section. If $Z \to X'$ is a resolution of singularities, then $Z \to C'$ is a smooth elliptic surface with a section, hence Z lifts to $W(k)$ by [Sei88, Thm. 4.4]. $\square$

Remark 6.3.3. In the ruled and elliptic cases, we actually get a lift to $W(k)$, whereas in the K3 and abelian cases, we need a finite extension $W(k) \to R$. Only the elliptic case requires an actual cover $Z \to X$; the others can be lifted on the nose. We don't know if this stronger conclusion also holds in the elliptic case, but the answer to this question is probably known to experts.

Remark 6.3.4. The most essential use of the assumption $\operatorname{char} k \geq 5$ is that Seiler's result [Sei88] is only stated in this case. It seems plausible that for elliptic and quasi-elliptic fibrations in characteristic 2 and 3 a similar argument could work.

7. A variety that cannot be dominated by one that lifts

7.1 Lifting line bundles on products of curves

We will study line bundles on products of curves. We give a condition under which a line bundle $\mathscr{L}$ on $C_1 \times \ldots \times C_r$ has the property that for any lifts $\mathcal{C}_i$ of C_i, the line bundle $\mathscr{L}$ does not lift to $\prod \mathcal{C}_i$; see Lemma 7.1.6. This will be used in the counterexample to Question 1; see Construction 7.4.1.

Setup 7.1.1. Let k be a field, and let $C_1, \ldots, C_r$ be smooth projective curves over k with $C_i(k) \neq \varnothing$, and let X be their product. We will choose isomorphisms $\mathbf{Alb}^0_{C_i/k} \cong \mathbf{Pic}^0_{C_i/k}$, and denote both by $\mathbf{Jac}_{C_i}$.

The C_i are a particular instance of Setup 4.4.1, so Corollary 4.4.12 gives an isomorphism

$$\operatorname{Pic}(X) \cong \prod_{i=1}^{r} \operatorname{Pic}(C_i) \times \prod_{i<j} \operatorname{Hom}_k(\mathbf{Jac}_{C_i}, \mathbf{Jac}_{C_j}).$$

If $\mathscr{L} \in \operatorname{Pic}(X)$ then we write

$$\mathscr{L} = \Big(\big(\mathscr{L}_i\big)_i, \big(\phi_{ji}\big)_{i<j} \Big) \in \prod_{i=1}^{r} \operatorname{Pic}(C_i) \times \prod_{i<j} \operatorname{Hom}(\mathbf{Jac}_{C_i}, \mathbf{Jac}_{C_j}). \tag{7.1.1}$$

The ϕ_{ji} can be arranged into the following (non-commutative) diagram (for simplicity drawn when $r = 4$).

$$\begin{tikzcd}
\mathbf{Jac}_{C_1} \arrow[r, "\phi_{21}"] \arrow[d, "\phi_{31}"'] \arrow[dr] & \mathbf{Jac}_{C_2} \arrow[d, "\phi_{42}"] \arrow[dl] \\
\mathbf{Jac}_{C_3} \arrow[r, "\phi_{43}"'] & \mathbf{Jac}_{C_4} .
\end{tikzcd}$$

We set $\phi_{ij} = \phi_{ji}^{\top}$ (the Rosati transpose with respect to the principal polarisation coming from the theta-divisor), and we write $\phi_{i_1\dots i_m} = \phi_{i_1 i_2}\cdots\phi_{i_{r-1}i_m}$ for any $i_1,\dots,i_m$. For each i, write $E(\mathscr{L})_i$ for the $\mathbb{Q}$-subrng of $\mathrm{End}^\circ(\mathbf{Jac}_{C_i})$ generated by the endomorphisms $\phi_{i_1\dots i_m}$ of $\mathbf{Jac}_{C_{i_1}}$ with $i_1 = i_m = i$ and $m \geq 3$. It may or may not contain 1. Equivalently, it is the $\mathbb{Q}$-vector space generated by the $\phi_{i_1\dots i_m}$, since the set of such endomorphisms is closed under multiplication.

Definition 7.1.2. Let C_i and X be as in Setup 7.1.1, and let $\mathscr{L} \in \mathrm{Pic}(X)$. Then $\mathscr{L}$ *corresponds to an isogeny factor A of* $\mathbf{Jac}_{C_i}$ if there exists an isogeny factor

$$\mathbf{Jac}_{C_i} \underset{\iota}{\overset{\pi}{\rightleftarrows}} A \tag{7.1.2}$$

such that $E(\mathscr{L})_i = \iota\,\mathrm{End}^\circ(A)\pi$. Here, π is a surjective homomorphism and ι is an element of $\mathrm{Hom}^\circ(A, \mathbf{Jac}_{C_i})$ such that $\pi\iota = \mathrm{id}$.

Equivalently, $E(\mathscr{L})_i = p\,\mathrm{End}^\circ(\mathbf{Jac}_{C_i})p$ for some idempotent $p \in \mathrm{End}^\circ(\mathbf{Jac}_{C_i})$. Indeed, isogeny factors as in (7.1.2) correspond to idempotents $p \in \mathrm{End}^\circ(\mathbf{Jac}_{C_i})$ by setting $p = \iota\pi$, and under this correspondence we have

$$\iota\,\mathrm{End}^\circ(A)\pi = p\,\mathrm{End}^\circ(\mathbf{Jac}_{C_i})p.$$

If $E(\mathscr{L})_i = \mathrm{End}^\circ(\mathbf{Jac}_{C_i})$, then we say that $\mathscr{L}$ *generates all endomorphisms of* $\mathbf{Jac}_{C_i}$. This is a special case of the above, where we take $A = \mathbf{Jac}_{C_i}$, or equivalently $p = \mathrm{id}$.

Remark 7.1.3. We will study how the $E(\mathscr{L})_i$ change under specialisation (Lemma 7.1.4) and under finite coverings $C_i' \to C_i$ of the curves (Lemma 7.1.7).

Note that $E(\mathscr{L})_i$ is only a $\mathbb{Q}$-rng, and not a ring in general; the reason we do not include the identity is because of the functoriality properties we will prove below. See in particular Lemma 7.1.4 (4), Lemma 7.1.7 (4), and Remark 7.1.8.

Lemma 7.1.4. *Let R be a DVR, and let $\mathcal{C}_i$ be smooth projective geometrically integral curves over $\operatorname{Spec} R$ that admit a section. Let $\mathcal{X}$ be their fibre product. Let $\mathscr{L}_K \in \operatorname{Pic}(\mathcal{C}_{i,K})$, and let $\mathscr{L}_0 \in \operatorname{Pic}(\mathcal{C}_{i,0})$ be its specialisation. If $\mathscr{L}_{i,K}$ ($\mathscr{L}_{i,0}$) and $\phi_{ji,K}$ ($\phi_{ji,0}$) denote the components of $\mathscr{L}_K$ ($\mathscr{L}_0$) as in (7.1.1), then*

(1) $\mathscr{L}_{i,0} = \operatorname{sp}(\mathscr{L}_{i,K})$ *for all i;*

(2) $\phi_{ji,0} = \operatorname{sp}(\phi_{ji,K})$ *for all i, j;*

(3) $\phi_{i_1\dots i_m,0} = \operatorname{sp}(\phi_{i_1\dots i_m,0})$ *for all $i_1, \dots, i_m$;*

(4) $E(\mathscr{L}_0)_i = \operatorname{sp}(E(\mathscr{L}_K)_i)$ *for all i. Hence, $E(\mathscr{L}_0)_i \cong E(\mathscr{L}_K)_i$ as $\mathbb{Q}$-rngs;*

(5) *If $\mathscr{L}_0$ corresponds to an isogeny factor $\mathcal{A}_0$ of $\mathbf{Jac}_{\mathcal{C}_{i,0}}$, then $\mathscr{L}_K$ corresponds to a unique isogeny factor $\mathcal{A}_K$ of $\mathbf{Jac}_{\mathcal{C}_{i,K}}$, and $\mathcal{A}$ is a lift of $\mathcal{A}_0$.*

Proof. The first two statements are Lemma 4.4.15. The third statement follows from the second since specialisation commutes with composition (Corollary 4.3.4). The first statement in (4) is immediate from (3), and the second follows since specialisation is injective, again by Corollary 4.3.4.

For the final statement, if $E(\mathscr{L}_0)_i = p\operatorname{End}^\circ(\mathbf{Jac}_{\mathcal{C}_{i,0}})p$ for some idempotent p, then (4) implies that $p = \operatorname{sp}(q)$ for some q. Since specialisation is injective, we conclude that such q is unique, and that q is an idempotent as well. Let $\mathcal{A}_K$ be the isogeny factor corresponding to q. Then $\mathcal{A}_K$ has good reduction by Néron–Ogg–Shafarevich [ST68, Thm. 1], since $\mathbf{Jac}_{\mathcal{C}_{i,K}}$ does. Let $\mathcal{A}$ be the Néron model over $\operatorname{Spec} R$.

Let (ι, π) correspond to the idempotent q as in (7.1.2). Then π extends uniquely to a morphism $\pi\colon \mathbf{Pic}_{\mathcal{C}_i/R} \to \mathcal{A}$ by Corollary 4.1.3. Similarly, if n is such that $n\iota \in \operatorname{Hom}(\mathcal{A}_K, \mathbf{Jac}_{\mathcal{C}_{i,K}})$, then $n\iota$ extends uniquely to a morphism $\mathcal{A} \to \mathbf{Pic}_{\mathcal{C}_i/R}$, which we also denote $n\iota$. The uniqueness property implies that $\pi_0\iota_0 = \mathrm{id}$ and $\iota_0\pi_0 = p$, which proves that p corresponds to the reduction $\mathcal{A}_0$.

Finally, if $\psi \in E(\mathscr{L}_K)_i$, then $q\psi = \psi = \psi q$, since this holds after applying sp by part (4), and since specialisation is injective by Corollary 4.3.4. Thus,

$$E(\mathscr{L}_K)_i \subseteq q \operatorname{End}^\circ(\mathbf{Jac}_{\mathcal{C}_{i,K}})q.$$

The reverse inclusion follows from the dimension count

$$\begin{aligned}\dim E(\mathscr{L}_K)_i = \dim E(\mathscr{L}_0)_i &= \dim \operatorname{End}^\circ(A_0) \\ &\geq \dim \operatorname{End}^\circ(A_K) = \dim(q \operatorname{End}^\circ(\mathbf{Jac}_{\mathcal{C}_{i,K}})q),\end{aligned}$$

again using injectivity from Corollary 4.3.4. This proves (5). ☐

Remark 7.1.5. We note that the converse of (5) is not true. For example, let $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{E}$ be an elliptic curve over R such that $\operatorname{End}^\circ(\mathcal{E}_K) = \mathbb{Q}$ but $\operatorname{End}^\circ(\mathcal{E}_0)$ is bigger (such examples exist in both pure characteristic 0 or p, or in mixed characteristic). Let ϕ_{21} be the identity, so that $E(\mathscr{L}_K)_1 = \mathbb{Q}$ and $E(\mathscr{L}_0)_1 = \mathbb{Q}$. Then $\mathscr{L}_K$ corresponds to the isogeny factor E_K, but $\mathscr{L}_0$ does not correspond to any isogeny factor, since $E(\mathscr{L}_0)_1$ is the subring $\mathbb{Q} \subsetneq \operatorname{End}^\circ(\mathcal{E}_0)$.

This gives a criterion for a line bundle on a product of curves that implies that it cannot be lifted along with the curves.

Lemma 7.1.6. *Let $C_1, \ldots, C_r$ be curves over k all of whose endomorphisms are defined over k. Let $\mathscr{L}$ be a line bundle on $\prod C_i$ that corresponds to a supersingular isogeny factor A of $\operatorname{End}^\circ(\mathbf{Jac}_{C_i})$ for some i (see Definition 7.1.2). If $\mathcal{C}_i$ are curves over a DVR R that lift the C_i, then no multiple $\mathscr{L}^{\otimes m}$ for $m > 0$ can be lifted to $\prod \mathcal{C}_i$.*

Proof. Note that $E(\mathscr{L}^{\otimes m})_i = E(\mathscr{L})_i$, so we may take $m = 1$. Suppose $\mathcal{C}_i$ are lifts of the C_i and $\widetilde{\mathscr{L}}$ is a lift of $\mathscr{L}$. By Lemma 7.1.4 (5), $\widetilde{\mathscr{L}}_K$ corresponds to a lift $\mathcal{A}_K$ of A, and Lemma 7.1.4 (4) shows that $\mathrm{End}^\circ(\mathcal{A}_K) \cong \mathrm{End}^\circ(A)$. But $A = \mathbf{Jac}_{C_i/k}$ is a supersingular abelian variety, so by Corollary 4.3.9 it is impossible to lift all its endomorphisms simultaneously. $\square$

In the next two sections, we will construct such line bundles, when $r \geq 4$ and $r \geq 3$ respectively; see Corollary 7.2.8 and Corollary 7.3.3.

Lemma 7.1.7. *Let C_i and X be as in Setup 7.1.1, and let C_i' and X' satisfy the same assumptions. Let $g_i \colon C_i' \to C_i$ be a finite morphism, and denote by $g \colon X' \to X$ the product. Let $\mathscr{L} \in \mathrm{Pic}(X)$, and let $\mathscr{L}' = g^*\mathscr{L}$. If $\mathscr{L}_i$ ($\mathscr{L}_i'$) and ϕ_{ji} (ϕ_{ji}') denote the components of $\mathscr{L}$ ($\mathscr{L}'$) as in (7.1.1), then*

(1) $\mathscr{L}_i' = g_i^*\mathscr{L}_i$ *for all* i;

(2) $\phi_{ji}' = g_j^*\phi_{ji}g_{i,*}$ *for all* i, j;

(3) $\phi_{i_1\ldots i_m}' = \deg(g_{i_2})\cdots\deg(g_{i_{m-1}}) \cdot g_{i_1}^*\phi_{i_1\ldots i_m}g_{i_m,*}$ *for all* $i_1, \ldots, i_m$;

(4) $E(\mathscr{L}')_i = g_i^*E(\mathscr{L})_i g_{i,*}$ *for all* i*. Hence,* $E(\mathscr{L}')_i \cong E(\mathscr{L})_i$ *as* $\mathbb{Q}$*-rngs;*

(5) *If $\mathscr{L}$ corresponds to an isogeny factor A of $\mathbf{Jac}_{C_i}$, then $\mathscr{L}'$ corresponds to the isogeny factor A of $\mathbf{Jac}_{C_i'}$;*

(6) *If $g(C_i') = g(C_i)$, then the converse of (5) holds as well.*

Proof. The first two statements are Lemma 4.4.14 if $i < j$; the case $i > j$ easily follows from that since $\phi_{ij} = \phi_{ji}^\top$ and $g_i^* = (g_{i,*})^\top$. The third statement follows from the second, because $g_{i,*}g_i^* \colon \mathbf{Jac}_{C_i} \to \mathbf{Jac}_{C_i}$ is multiplication by the degree of g_i. The fourth statement is immediate from the third. For (5), note that the pair $(\iota, \pi) = (\frac{1}{\deg(g_i)}g_i^*, g_{i,*})$ realises $\mathbf{Jac}_{C_i}$ as isogeny factor of $\mathbf{Jac}_{C_i'}$. The result then follows immediately from (4) and Definition 7.1.2. Finally, in (6) the isogeny pair $(\iota, \pi) = (\frac{1}{\deg(g_i)}g_i^*, g_{i,*})$ is an isomorphism. We may repeat the proof argument for (5) using the pair (ι^{-1}, π^{-1}). $\square$

Remark 7.1.8. We saw that $\mathbf{Jac}_{C_i}$ is an isogeny factor of $\mathbf{Jac}_{C_i'}$. If none of the other isogeny factors of $\mathbf{Jac}_{C_i'}$ is isomorphic to one of the factors of $\mathbf{Jac}_{C_i}$, then $\operatorname{End}^\circ(\mathbf{Jac}_{C_i})$ is a factor of the semisimple ring $\operatorname{End}^\circ(\mathbf{Jac}_{C_i'})$.

If this is not the case, then the inclusion $g_i^* \operatorname{End}^\circ(\mathbf{Jac}_{C_i}) g_{i,*} \subseteq \operatorname{End}^\circ(\mathbf{Jac}_{C_i'})$ can for example look like an inclusion of rngs (but not of rings)

$$M_n(D) \subseteq M_{n+1}(D)$$

for some division algebra D. Thus, $\operatorname{End}^\circ(\mathbf{Jac}_{C_i})$ is not a factor of $\operatorname{End}^\circ(\mathbf{Jac}_{C_i'})$.

7.2 Generation of separable algebras

We give an elementary geometric proof of the following well-known theorem.

Theorem 7.2.1 (Albert [Alb44])**.** *Let k be an infinite field, and let A be an absolutely semisimple k-algebra. Then A can be generated over k by two elements.*

Our main application is Corollary 7.2.8, which is an example of the situation of Lemma 7.1.6. In Section 7.3, we improve this result to Corollary 7.3.3; the only difference with Corollary 7.2.8 is the number of curves and the minimal genus needed. The current section is included because it is less technical, and suffices for the construction of a counterexample to Question 1.

Remark 7.2.2. Absolutely semisimple algebras are also known as separable algebras. They are finite products of matrix algebras over division rings whose centre is separable over k [Bou12, Thm. 13.3.1][1]. Hence, if k is perfect, then A is absolutely semisimple if and only if it is semisimple.

Definition 7.2.3. For a k-vector space V, write $\mathbb{A}(V)$ for the affine space of

[1]The numbering in Bourbaki is not consistent with earlier editions.

points of V. It's functor of points is $\mathbb{A}(V)(R) = V \otimes_k R$, and a choice of a basis gives an isomorphism to $\mathbb{A}^{\dim V}$.

Note that if A is a k-algebra, then the multiplication $\mathbb{A}(A) \times \mathbb{A}(A) \to \mathbb{A}(A)$ is a morphism of schemes. If $(A, (-)^\dagger)$ is a k-algebra with involution, then $(-)^\dagger \colon \mathbb{A}(A) \to \mathbb{A}(A)$ is a morphism of schemes (in fact, it is a linear map).

Lemma 7.2.4. *Let A be a finite-dimensional k-algebra, and let $r \in \mathbb{Z}_{\geq 0}$. Then the subfunctor $U_r \subseteq \mathbb{A}(A)^r$ given by $\{(x_i) \in A^r \otimes_k R \mid R\langle\{x_i\}\rangle = A \otimes_k R\}$ is a (possibly empty) open subfunctor.*

Proof. Let $\mathbb{N}^{*r}$ be the free monoid on r generators e_i, and ϕ the forgetful map

$$\phi \colon A^r \cong \mathrm{Hom}_{\underline{\mathrm{Mon}}}(\mathbb{N}^{*r}, A) \to \mathrm{Hom}_{\underline{\mathrm{Set}}}(\mathbb{N}^{*r}, A).$$

Since the multiplication $\mathbb{A}(A) \times \mathbb{A}(A) \to \mathbb{A}(A)$ is algebraic, this gives a morphism of schemes

$$\phi \colon \mathbb{A}(A)^r \to \mathbb{A}(A)^{\mathbb{N}^{*r}}.$$

Concretely, the coordinate of $\phi(x_1, \ldots, x_r)$ corresponding to $e_{i_1} \cdots e_{i_s} \in \mathbb{N}^{*r}$ is given by $x_{i_1} \cdots x_{i_s} \in A$.

For any set I, the locus $U_I \subseteq \mathbb{A}(A)^I$ of I-tuples that generate A as a k-vector space is Zariski open, because it is given by the nonvanishing of certain $n \times n$ minors, where $n = \dim A$. Thus, $U_r = \phi^{-1}(U_{\mathbb{N}^{*r}})$ is open as well. $\square$

Definition 7.2.5. In the matrix algebra $M_n(k)$, we will write ρ for the cyclic rotation matrix of order n given by $e_i \mapsto e_{i+1}$ for $i < n$ and $e_n \mapsto e_1$. We will write e_{ij} for the single-entry matrix that maps e_j to e_i and all other e_k to 0.

Lemma 7.2.6. *The matrix algebra $M_n(k)$ is generated over k by ρ and any* *single-entry matrix* e_{ij}.

Proof. Every $e_{i'j'}$ can be made as $\rho^a e_{ij} \rho^b$ for suitable a and b. $\square$

Proof of Theorem. Since k is infinite, the Zariski open $U_2 \subseteq \mathbb{A}(A)^2$ has a k-point if and only if it is (scheme-theoretically) nonempty. To show that U_2 is nonempty, we may replace k by its algebraic closure. Then A is a product of matrix algebras; say $A = \prod_{i=1}^m M_{n_i}(k)$. Let λ_i be pairwise distinct nonzero scalars for $i \in \{1, \ldots, m\}$ (we can do this because k is infinite). Let $x, y \in \prod_i M_{n_i}(k)$ be the elements $x = (x_i)_i$ and $y = (y_i)_i$ defined as follows. Set $x_i = \rho$ as in Definition 7.2.5 (understood to be the identity if $n_i = 1$). Set $y_i = \lambda_i$ if $n_i = 1$, and $y_i = \lambda_i + e_{12}$ if $n_i > 1$. We claim that x and y generate A, i.e. the k-subalgebra $B = k\langle x, y \rangle \subseteq A$ equals A.

Indeed, B surjects onto each factor $\pi_i \colon A \to M_{n_i}(k)$ by Lemma 7.2.6. The element $z_i = \prod_{j \neq i} (y - \lambda_j)^{n_j}$ maps to 0 under π_j for $j \neq i$. Under π_i, it maps to an upper triangular matrix that is invertible because $\lambda_j \neq \lambda_i$ for $j \neq i$. Choose a lift $w_i \in B$ of $\pi_i(z_i)^{-1}$ under the surjection $B \twoheadrightarrow M_{n_i}(k)$. Then the element $w_i z_i \in B$ is the i-th coordinate projector $(0, \ldots, 1, \ldots, 0)$. Since B surjects onto each factor and contains the coordinate projectors, it equals A. $\square$

Example 7.2.7. To see that the result is false for finite fields, consider the separable $\mathbb{F}_2$-algebra $A = \mathbb{F}_2^5$. Since any element $x \in A$ satisfies $x^2 = x$, the algebra it generates has dimension ≤ 2. Thus, any two elements of A can only generate a 4-dimensional subalgebra of A.

Corollary 7.2.8. *Let k be an extension of $\bar{\mathbb{F}}_p$, and let C be a supersingular curve over k of genus $g \geq 1$. Let $r \geq 4$, and set $C_i = C$ for all $i \in \{1, \ldots, r\}$. Then there exists a very ample line bundle $\mathscr{L}$ on $\prod C_i$ that generates all endomorphisms of $\mathbf{Jac}_{C_1}$ in the sense of Definition 7.1.2.*

Proof. As in Setup 7.1.1, we get an isomorphism

$$\operatorname{Pic}(X) \cong \prod_{i=1}^{r} \operatorname{Pic}(C_i) \times \prod_{i<j} \operatorname{Hom}_k(\mathbf{Jac}_{C_i}, \mathbf{Jac}_{C_j}),$$

and for any $\mathscr{L} \in \operatorname{Pic}(X)$ we write

$$\mathscr{L} = \Big(\big(\mathscr{L}_i\big)_i, \big(\phi_{ji}\big)_{i<j} \Big) \in \prod_{i=1}^{r} \operatorname{Pic}(C_i) \times \prod_{i<j} \operatorname{Hom}(\mathbf{Jac}_{C_i}, \mathbf{Jac}_{C_j}). \tag{7.2.1}$$

Since $r \geq 4$, part of this data sits in a (not necessarily commutative!) diagram

$$\begin{tikzcd}
\mathbf{Jac}_{C_1} \arrow[r, "\phi_{21}"] \arrow[d, "\phi_{31}"'] \arrow[dr] & \mathbf{Jac}_{C_2} \arrow[d, "\phi_{42}"] \arrow[dl] \\
\mathbf{Jac}_{C_3} \arrow[r, "\phi_{43}"'] & \mathbf{Jac}_{C_4}\ .
\end{tikzcd} \tag{7.2.2}$$

By Theorem 7.2.1, the separable algebra $\operatorname{End}^\circ(\mathbf{Jac}_{C_1})$ can be generated over $\mathbb{Q}$ by two elements x, y; without loss of generality $x, y \in \operatorname{End}(\mathbf{Jac}_{C_1})$ are integral. If we set $\phi_{42} = x$, $\phi_{43} = y$, and all other $\phi_{ji} = 1$, then the maps

$$\phi_{1421} = \phi_{41}^{\top}\phi_{42}\phi_{21} \colon \mathbf{Jac}_{C_1} \to \mathbf{Jac}_{C_1}$$
$$\phi_{1431} = \phi_{41}^{\top}\phi_{43}\phi_{31} \colon \mathbf{Jac}_{C_1} \to \mathbf{Jac}_{C_1}$$
$$\phi_{1321} = \phi_{31}^{\top}\phi_{32}\phi_{21} \colon \mathbf{Jac}_{C_1} \to \mathbf{Jac}_{C_1}$$

are given by x, y, and 1 respectively.

This corresponds to going around the following loops in diagram (7.2.2):

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[d] \\
\bullet & \bullet \arrow[ul]
\end{tikzcd}, \qquad
\begin{tikzcd}
\bullet \arrow[d] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[ul]
\end{tikzcd}, \qquad
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[dl] \\
\bullet \arrow[u] & \bullet
\end{tikzcd}.$$

If $p \in C(k)$ is a rational point, then the line bundle $\mathcal{O}(p)^{\boxtimes r}$ is very ample. Hence, for $d \gg 0$, the line bundle

$$\mathscr{L} = \Big(\big(\mathcal{O}_{C_i}(dP)\big)_i, \big(\phi_{ji}\big)_{i<j} \Big)$$

is very ample. Since x, y, and 1 generate $\mathrm{End}^\circ(\mathbf{Jac}_{C_1})$ as $\mathbb{Q}$-rng, the $\mathbb{Q}$-rng $E(\mathscr{L})_1$ of Definition 7.1.2 equals $\mathrm{End}^\circ(\mathbf{Jac}_{C_1})$. Thus, $\mathscr{L}$ generates all endomorphisms of $\mathbf{Jac}_{C_1}$. $\square$

Remark 7.2.9. The proof of Theorem 7.2.1 shows that A can actually be generated by two elements as k-rng, as opposed to as k-algebra. The only difference is whether one is allowed to use the identity element.

Indeed, the set where the words $x_{i_1} \cdots x_{i_s}$ for $s \geq 1$ (i.e. omitting the empty word 1) generate A is still Zariski open (the same proof as Lemma 7.2.4 applies). Moreover, in Lemma 7.2.6, we don't need the identity. Finally, in the proof of Theorem 7.2.1, the generators we chose automatically give the identity element: if n is a common multiple of the n_i, then $x^n = 1$.

Remark 7.2.10. Thus, in the proof of Corollary 7.2.8, we could have just used the two loops

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] \\ \bullet & \bullet \arrow[ul] \end{tikzcd}, \qquad \begin{tikzcd} \bullet \arrow[d] & \bullet \\ \bullet \arrow[r] & \bullet \arrow[ul] \end{tikzcd}.$$

However, using just one loop does not seem to be sufficient, because the k-subalgebra generated by one element is commutative and therefore does not equal A in general. This is why we needed $r \geq 4$ in this section. In the next section, we show that if $r \geq 3$, then using one loop and its inverse is still enough to generate all endomorphisms.

7.3 Generation by Rosati dual elements

The main result of this section is an improvement of Corollary 7.2.8, given below in Corollary 7.3.3.

The main difference is the number of curves required: it suffices to have $r \geq 3$ instead of $r \geq 4$ (see also Remark 7.2.10). The advantage of this is that we can make the example we produce in Theorem 7.4.3 a surface instead of a threefold.

This section is a bit more technical than the previous, and can be skipped by the reader for whom a threefold counterexample to Question 1 is sufficient.

Theorem 7.3.1. *Let (A, ϕ) be a polarised supersingular abelian variety of dimension $g \geq 2$ over a field k containing $\bar{\mathbb{F}}_p$. Then there exists an element $x \in \operatorname{End}^\circ(A)$ such that x and $x^\dagger = \phi^{-1}x^\top\phi$ generate $\operatorname{End}^\circ(A)$ as $\mathbb{Q}$-rng.*

Proof. Any supersingular abelian variety over a field containing $\bar{\mathbb{F}}_p$ is isogenous to E^g, where E is a supersingular elliptic curve. Then $D = \operatorname{End}^\circ(E)$ is the quaternion algebra over $\mathbb{Q}$ ramified only at p and ∞, and $\operatorname{End}^\circ(A) \cong M_g(D)$. Moreover, when A is supersingular, the Rosati involution on $\operatorname{End}^\circ(A)$ does not depend on the rational polarisation used [Eke87, Prop. 1.4.2], so we may assume that ϕ is the product polarisation. Then the Rosati involution on $M_g(D)$ is given by

$$\begin{aligned}(-)^\dagger \colon M_g(D) &\to M_g(D)\\ (a_{ij}) &\mapsto (a_{ji}^\dagger),\end{aligned}$$

where $a^\dagger = \operatorname{Trd}(a) - a$ is the Rosati involution on $D = \operatorname{End}^\circ(E)$.

By the same argument as Lemma 7.2.4, the set of elements $x \in \mathbb{A}(M_g(D))$ such that x and $x^\dagger$ generate $M_g(D)$ as $\mathbb{Q}$-rng is Zariski open. Thus, it has a $\mathbb{Q}$-point if and only if it is nonempty, i.e. if and only if it has a $\bar{\mathbb{Q}}$-point. Thus, it suffices to study $\mathrm{End}^\circ(A) \otimes_\mathbb{Q} \bar{\mathbb{Q}}$. The algebra $\mathrm{End}^\circ(A) \otimes_\mathbb{Q} \bar{\mathbb{Q}}$ is isomorphic to $M_{2g}(\bar{\mathbb{Q}})$, with involution $(-)^\dagger$ given by

$$\begin{pmatrix} \begin{pmatrix} a_{11} & b_{11} \\ c_{11} & d_{11} \end{pmatrix} & \cdots & \begin{pmatrix} a_{1g} & b_{1g} \\ c_{1g} & d_{1g} \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} a_{g1} & b_{g1} \\ c_{g1} & d_{g1} \end{pmatrix} & \cdots & \begin{pmatrix} a_{gg} & b_{gg} \\ c_{gg} & d_{gg} \end{pmatrix} \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} d_{11} & -b_{11} \\ -c_{11} & a_{11} \end{pmatrix} & \cdots & \begin{pmatrix} d_{g1} & -b_{g1} \\ -c_{g1} & a_{g1} \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} d_{1g} & -b_{1g} \\ -c_{1g} & a_{1g} \end{pmatrix} & \cdots & \begin{pmatrix} d_{gg} & -b_{gg} \\ -c_{gg} & a_{gg} \end{pmatrix} \end{pmatrix}.$$

Now consider the matrix

$$x = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix}.$$

We want to show that the $\bar{\mathbb{Q}}$-subrng B of $M_{2g}(\bar{\mathbb{Q}})$ generated by x and $x^\dagger$ is $M_{2g}(\bar{\mathbb{Q}})$. One easily computes

$$\begin{aligned} x^{2g-1} &= e_{1,2g}, \\ x^{2g-3} &= e_{1,2g-2} + e_{2,2g-1} + e_{3,2g}, \\ (x^{2g-3})^\dagger &= -\left(e_{2g-3,2} + e_{2g,1} + e_{2g-1,4}\right). \end{aligned}$$

Write $a = x^{2g-1}$ and $b = (x^{2g-3})^\dagger$, which makes sense because $g \geq 2$. Then $ab = -e_{11}$, hence $bab = -e_{2g,1}$. Hence, $x - bab$ is the rotation matrix ρ^{-1} from Definition 7.2.5. Thus, by Lemma 7.2.6, the matrix algebra $M_{2g}(\bar{\mathbb{Q}})$ is generated (as $\bar{\mathbb{Q}}$-rng) by $x - bab$ and a, for example. Thus, $B = M_{2g}(\bar{\mathbb{Q}})$. $\square$

Remark 7.3.2. The theorem is false for $g = 1$. Indeed, for any $x \in D$, we have $x^\dagger = \operatorname{Trd}(x) - x$, so in particular x and $x^\dagger$ commute. Therefore, the non-commutative algebra D can never be generated by an element and its Rosati transpose.

Corollary 7.3.3. *Let k be an extension of $\bar{\mathbb{F}}_p$, and let C be a supersingular curve over k of genus $g \geq 2$. Let $r \geq 3$, and set $C_i = C$ for all $i \in \{1, \ldots, r\}$. Then there exists a very ample line bundle $\mathscr{L}$ on $\prod C_i$ that generates all endomorphisms of $\mathbf{Jac}_{C_1}$ in the sense of Definition 7.1.2.*

Proof. The proof is the same as that of Corollary 7.2.8, with the following modifications:

Instead of diagram (7.2.2), we only have the diagram

$$\begin{tikzcd} & \mathbf{Jac}_{C_1} \arrow[dl, "\phi_{21}"'] \arrow[dr, "\phi_{31}"] & \\ \mathbf{Jac}_{C_2} \arrow[rr, "\phi_{32}"'] & & \mathbf{Jac}_{C_3} \end{tikzcd} \quad . \tag{7.3.1}$$

By Theorem 7.3.1, there exists $x \in \operatorname{End}^\circ(\mathbf{Jac}_{C_1})$ such that x and $x^\dagger$ generate $\operatorname{End}^\circ(\mathbf{Jac}_{C_1})$ as $\mathbb{Q}$-rng, where we take the Rosati involution with respect to the principal polarisation on $\mathbf{Jac}_{C_1}$ coming from the theta divisor. This means exactly that x and $x^\top$ generate $\operatorname{End}^\circ(\mathbf{Jac}_{C_1})$.

Now set $\phi_{21} = x$, and $\phi_{31} = \phi_{32} = 1$. Then the maps

$$\phi_{1321} = \phi_{31}^\top \phi_{32} \phi_{21} \colon\ \mathbf{Jac}_{C_1} \to \mathbf{Jac}_{C_1}$$
$$\phi_{1231} = \phi_{21}^\top \phi_{32}^\top \phi_{31} \colon\ \mathbf{Jac}_{C_1} \to \mathbf{Jac}_{C_1}$$

are given by x and $x^\top$ respectively. Pictorially, this corresponds to going around the following loops in diagram (7.3.1):

$$\begin{tikzcd} & \bullet \arrow[dl] & \\ \bullet \arrow[rr] & & \bullet \arrow[ul] \end{tikzcd}, \qquad \begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \arrow[ll] \end{tikzcd}.$$

The rest of the proof is identical to that of Corollary 7.2.8. $\square$

7.4 The construction

We give a negative answer to Question 1. To be precise, we construct smooth projective varieties X of every dimension ≥ 2 over $\bar{\mathbb{F}}_p$ for every prime p, such that no smooth projective variety Z dominating X can be lifted to characteristic 0. The construction is carried out in Construction 7.4.1, and we prove its properties in Theorem 7.4.3.

Construction 7.4.1. Let p be a prime, let $r \geq 3$ be an integer, and let $k = \bar{\mathbb{F}}_p$. Let $C_1 = \ldots = C_r = C$ be a supersingular curve over k of genus $g \geq 2$. For example, the Fermat curve $x^{q+1} + y^{q+1} + z^{q+1} = 0$ is supersingular whenever q is a power of p [SK79, Lem. 2.9]. Alternatively, a smooth member of Moret–Bailly's family [Mor81; Mor79] is a supersingular of curve genus 2.

Set $Y = C_1 \times \ldots \times C_r$. By Corollary 7.3.3 (or Corollary 7.2.8 if $r \geq 4$), there exists a very ample line bundle $\mathscr{L}$ on Y that generates all endomorphisms of $\mathbf{Jac}_{C_1}$ in the sense of Definition 7.1.2.

Finally, we define $X \subseteq Y$ as a smooth divisor in $|n\mathscr{L}|$ for $n \gg 0$ that satisfies the product covering property of Definition 3.2.2. Such a divisor exists by Proposition 3.2.4 and the usual Bertini smoothness theorem.

The following result will be useful in the proof.

Lemma 7.4.2. *Let $g\colon X \to Y$ be a finite flat morphism of finite type k-schemes. Let $V \subseteq X$ be an irreducible subscheme, and let $W = f(V)$. If $g^{-1}(W)$ is irreducible, then $g^*[W] = d \cdot [V]$ for some $d \in \mathbb{Z}_{>0}$.*

Proof. Note that W is irreducible since V is. Since specialisations lift along finite morphisms, we have $\dim(V) = \dim(W) = \dim(g^{-1}(W))$. Hence, V is a component of $g^{-1}(W)$. Since $g^{-1}(W)$ is irreducible, we conclude that $V = g^{-1}(W)$ holds set-theoretically. Therefore, $g^*[W]$ is a multiple of $[V]$. $\square$

Theorem 7.4.3. *Let X be as in Construction 7.4.1. If $k \subseteq k'$ is a field extension and Z is a smooth proper k'-variety with a dominant rational map $Z \dashrightarrow X \times_k k'$, then Z cannot be lifted to characteristic 0.*

Remark 7.4.4. Since X is a divisor in a product of $r \geq 3$ curves, we get examples in every dimension ≥ 2. Of course, if X is an example of dimension d and Y is any m-dimensional smooth projective variety, then $X \times Y$ is an example of dimension $d + m$.

Remark 7.4.5. Since curves are unobstructed, the result in dimension 2 is the best possible. Moreover, by Theorem 6.3.1, every smooth projective surface X of Kodaira dimension $\kappa(X) \leq 1$ can be dominated by a liftable surface, so our surface of general type is to some extent the 'easiest' example possible.

Proof of Theorem. By Corollary 4.1.4, the rational maps $Z \dashrightarrow X \to C_i$ can be extended to morphisms $\psi_i\colon Z \to C_i$. Hence, the rational map $Z \dashrightarrow X \subseteq \prod_i C_i$ extends to a morphism $f\colon Z \to X$.

Assume Z admits a lift to characteristic 0. By Lemma 6.1.3, there exists a lift $\mathcal{Z} \to S$ where $S = \operatorname{Spec} R$ with R a DVR that is essentially of finite type over $\operatorname{Spec} \mathbb{Z}$ (see Definition 6.1.1 for additional notation). Upon enlarging k'

and the residue field κ of R, we may assume that they both equal Ω. We may assume Ω is a countable algebraically closed field by a limit argument. Passing to the completion of R, we may assume that R is complete. Replacing (k, C_i) by $(k', C_i \times_k k')$, we may also assume that $k = k'$. Thus, the fields k, k', κ, and Ω are equal, and are a countable algebraically closed field containing $\bar{\mathbb{F}}_p$.

Now we are in Setup 5.4.1 (see Example 5.4.3). Applying Corollary 5.4.7 to the maps $\psi_i \colon Z \to C_i$ obtained by composing f with the projections $\pi_i \colon X \to C_i$, we conclude that there exist finite extensions of DVR's $R \to R_i'$, smooth proper curves $\mathcal{Y}_i \to \operatorname{Spec} R_i'$, morphisms $\phi_i \colon \mathcal{Z} \times_R R_i' \to \mathcal{Y}_i$ over R_i', and commutative diagrams

$$\begin{tikzcd}
 & Z \arrow[dl, "\psi_i"'] \arrow[d, "h_i"] \arrow[dr, "\phi_{i,0}"] & \\
C_i & \mathcal{Y}_{i,0}^{(p^{-n_i})} \arrow[l, "g_i"] \arrow[r, "\chi_i"'] & \mathcal{Y}_{i,0}\,,
\end{tikzcd}$$

where all χ_i are radicial. We may assume all R_i' are the same DVR R'. Replacing $(R, \mathcal{Z})$ by $(R', \mathcal{Z} \times_R R')$ we may also assume for simplicity that $R = R'$. We will write $C_i' = Y_{i,0}^{(p^{-n_i})}$, $Y' = \prod C_i'$, and $\mathcal{Y} = \prod \mathcal{Y}_i$ (recall that $Y = \prod C_i$; see Construction 7.4.1).

For each of the morphisms indexed by the subscript i, the same notation without the subscript is understood to be the product morphism. The product is taken over the target in all cases, and over the source when that also depends on the index i. For example, we get morphisms $g \colon Y' \to Y$, and $\phi \colon \mathcal{Z} \to \mathcal{Y}$, etcetera.

We then get the commutative diagram

$$\begin{tikzcd} & Z \arrow[ddl, "\psi"'] \arrow[d, "h"] \arrow[ddr, "\phi_0"] & \\ & Y' \arrow[dl, "g"] \arrow[dr, "\chi"'] & \\ Y & & \mathcal{Y}_0 . \end{tikzcd}$$

Consider the image $h(Z) \subseteq Y'$. It satisfies $g(h(Z)) = X$. Then $g^{-1}(X) \subseteq Y'$ is irreducible since X satisfies the product covering property of Definition 3.2.2 by Construction 7.4.1. Hence by Lemma 7.4.2, there exists some $d \in \mathbb{Z}_{>0}$ such that

$$g^*[X] = d \cdot [h(Z)].$$

Let $\mathcal{W}$ be the scheme theoretic image of $\phi\colon \mathcal{Z} \to \mathcal{Y}$, and let W' be the scheme-theoretic image of ϕ_0. Since $\mathcal{Z}$ is flat over R, the sheaf $\mathcal{O}_{\mathcal{Z}}$ is π-torsion free. Hence, the subsheaf $\mathcal{O}_{\mathcal{W}} \subseteq \phi_* \mathcal{O}_{\mathcal{Z}}$ is also π-torsion free, so $\mathcal{W}$ is flat over R. Therefore, $\mathcal{W}$ is a lift of $\mathcal{W}_0$ as a divisor. Moreover, since ϕ is proper, it is surjective onto its image, so $\mathcal{W}_0 = W'$ set-theoretically. But $\mathcal{Z}_0$ is integral, hence so is W', so $\mathcal{W}_0$ is a multiple of W'. Moreover, $\chi^{-1}(W')$ is irreducible since W' is and since χ is radicial. Thus, Lemma 7.4.2 gives $\chi^*[\mathcal{W}_0] = e \cdot [h(Z)]$ for some $e \in \mathbb{Z}_{>0}$. Hence

$$g^*[X] = \tfrac{d}{e} \cdot \chi^*[\mathcal{W}_0].$$

Note that $\mathscr{L}$ generates all endomorphisms of the supersingular abelian variety $\mathbf{Jac}_{C_1}$ by Construction 7.4.1. Applying Lemma 7.1.7 (5) to g and Lemma 7.1.7 (6) to χ, we conclude that $\mathcal{W}_0$ corresponds to the isogeny factor $A = \mathbf{Jac}_{C_1}$. Since $\mathcal{W}_0$ lifts as a divisor, by Lemma 6.2.4 the corresponding line bundle lifts as well. This contradicts Lemma 7.1.6, since $\mathcal{W}_0$ corresponds to a supersingular isogeny factor. $\square$

Remark 7.4.6. The proof of Corollary 7.3.3 shows that the set of $\mathscr{L} \in \mathrm{Pic}(Y)$ that generate all endomorphisms of $\mathbf{Jac}_{C_1}$ in the sense of Definition 7.1.2 form a Zariski open subset of $\mathrm{NS}(Y) \otimes \mathbb{Q}$. Similarly, the set of $X \in |n\mathscr{L}|$ that satisfy the product covering property of Definition 3.2.2 is Zariski open, and nonempty if $n \gg 0$. This shows that 'most' divisors in $Y = C_1 \times \ldots \times C_r$ give counterexamples to Question 1.

It is conceivable that there exists $\mathscr{L}$ generating all endomorphisms of $\mathbf{Jac}_{C_1}$ such that certain smooth members $D \in |n\mathscr{L}|$ can be dominated by a liftable variety Z. For such Z, the curves $C_i' \to C_i$ of the proof of Theorem 7.4.3 cannot all be equal to C_i, for otherwise the product covering property is not needed in the proof of Theorem 7.4.3 above. It would be interesting to study if there exist smooth members $D \in |n\mathscr{L}|$ such that $g^{-1}(D) \subseteq \prod C_i'$ for certain covers $g_i \colon C_i' \to C_i$ has a component that is liftable as divisor on $\prod C_i'$. This would show that the answer to Question 1 is not invariant under deformations of X.

Remark 7.4.7. Our methods do not address the weaker question of dominating X by a smooth proper variety Z that admits a *formal* lift to characteristic 0. Similarly, our methods do not answer Bhatt's question [Bha10, Rmk. 5.5.5] whether every smooth projective variety X can be dominated by a smooth proper variety Z that admits a lift to the length 2 Witt vectors $W_2(k)$.

Although it is possible that a closer analysis of the abelian methods of Chapter 4 give bounds on dimensions of endomorphism algebras over rings like $W_2(k)$, the main obstacle is that the methods of Chapter 5 rely on a deep theorem of Simpson in complex geometry (Theorem 5.3.8). Thus, the current method of proof is not suited for deformation-theoretic results. See also the additional remarks in Section 5.4 and Section 5.5; in particular Remark 5.4.9, Remark 5.4.10 and Remark 5.5.2.

References

[Alb44] A. A. Albert. *Two element generation of a separable algebra*. Bull. Amer. Math. Soc. 50 (1944), pp. 786–788. DOI: 10.1090/S0002-9904-1944-08238-4.

[Alp13] J. Alper. *Good moduli spaces for Artin stacks*. Ann. Inst. Fourier (Grenoble) 63.**6** (2013), pp. 2349–2402. DOI: 10.5802/aif.2833.

[Ara04] D. Arapura. *Frobenius amplitude and strong vanishing theorems for vector bundles*. Duke Math. J. 121.**2** (2004), pp. 231–267. DOI: 10.1215/S0012-7094-04-12122-0.

[BL83] H. Bass and A. Lubotzky. *Automorphisms of groups and of schemes of finite type*. Israel J. Math. 44.**1** (1983), pp. 1–22. DOI: 10.1007/BF02763168.

[Beau96] A. Beauville. *Complex algebraic surfaces*. Second edition. London Mathematical Society Student Texts **34**. Cambridge University Press, Cambridge, 1996. DOI: 10.1017/CBO9780511623936.

[BN06] K. Behrend and B. Noohi. *Uniformization of Deligne–Mumford curves*. J. Reine Angew. Math. 599 (2006), pp. 111–153. DOI: 10.1515/CRELLE.2006.080.

[Bha10] B. Bhatt. *Derived direct summands*. PhD thesis. 2010. URL: http://search.proquest.com/docview/527715525/.

[BHPS13] B. Bhatt, W. Ho, Z. Patakfalvi and C. Schnell. *Moduli of products of stable varieties*. Compos. Math. 149.**12** (2013), pp. 2036–2070. DOI: 10.1112/S0010437X13007288.

[BM76] E. Bombieri and D. Mumford. *Enriques' classification of surfaces in char. p. III*. Invent. Math. 35 (1976), pp. 197–232. DOI: 10.1007/BF01390138.

[BM77] E. Bombieri and D. Mumford. *Enriques' classification of surfaces in char. p. II*. In: *Complex analysis and algebraic geometry*. Iwanami Shoten, Tokyo, 1977, pp. 23–42.

[BLR] S. Bosch, W. Lütkebohmert and M. Raynaud. *Néron models*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **21**. Springer-Verlag, Berlin, 1990. DOI: 10.1007/978-3-642-51438-8.

[Bou12] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*. Second revised edition of the 1958 edition. Springer, Berlin, 2012. DOI: 10.1007/978-3-540-35316-4.

[CS08] K. Corlette and C. Simpson. *On the classification of rank-two representations of quasiprojective fundamental groups*. Compos. Math. 144.**5** (2008), pp. 1271–1331. DOI: 10.1112/S0010437X08003618.

[CS83] M. Culler and P. B. Shalen. *Varieties of group representations and splittings of* 3*-manifolds*. Ann. of Math. (2) 117.**1** (1983), pp. 109–146. DOI: 10.2307/2006973.

[Del81] P. Deligne. *Relèvement des surfaces* $K3$ *en caractéristique nulle*. In: *Surfaces algébriques (Orsay, 1976–78)*. Lecture Notes in Math. **868**. Springer, Berlin-New York, 1981, pp. 58–79. DOI: 10.1007/BFb0090646.

[Del74] P. Deligne. *La conjecture de Weil. I*. Inst. Hautes Études Sci. Publ. Math. **43** (1974), pp. 273–307. URL: http://www.numdam.org/item/PMIHES_1974__43__273_0.

[EGA1] A. Grothendieck. *Éléments de géométrie algébrique. I. Le langage des schémas*. Inst. Hautes Études Sci. Publ. Math. **4** (1960). URL: http://www.numdam.org/item/PMIHES_1960__4__5_0.

[EGA2] A. Grothendieck. *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*. Inst. Hautes Études Sci. Publ. Math. **8** (1961). URL: http://www.numdam.org/item/PMIHES_1961__8__5_0.

[$\mathrm{EGA3_I}$] A. Grothendieck. *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*. Inst. Hautes Études Sci. Publ. Math. **11** (1961). URL: http://www.numdam.org/item/PMIHES_1961__11__5_0.

[$\mathrm{EGA3_{II}}$] A. Grothendieck. *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II*. Inst. Hautes Études Sci. Publ. Math. **17** (1963). URL: http://www.numdam.org/item/PMIHES_1963__17__5_0.

[$\mathrm{EGA4_{II}}$] A. Grothendieck. *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*. Inst. Hautes Études Sci. Publ. Math. **24** (1965). URL: http://www.numdam.org/item/PMIHES_1965__24__5_0.

[$\mathrm{EGA4_{III}}$] A. Grothendieck. *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*. Inst. Hautes Études Sci. Publ. Math. **28** (1966). URL: http://www.numdam.org/item/PMIHES_1966__28__5_0.

[$EGA4_{IV}$] A. Grothendieck. *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. IV.* Inst. Hautes Études Sci. Publ. Math. **32** (1967). URL: http://www.numdam.org/item/PMIHES_1967__32__5_0.

[Eke87] T. Ekedahl. *On supersingular curves and abelian varieties.* Math. Scand. 60.**2** (1987), pp. 151–178. DOI: 10.7146/math.scand.a-12178.

[FC90] G. Faltings and C.-L. Chai. *Degeneration of abelian varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **22**. Springer-Verlag, Berlin, 1990. DOI: 10.1007/978-3-662-02632-8.

[FGA] A. Grothendieck. *Fondements de la géométrie algébrique.* Extraits du Séminaire Bourbaki, 1957–1962. Secrétariat mathématique, Paris, 1962.

[Ful98] W. Fulton. *Intersection theory.* 2nd edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **2**. Springer-Verlag, Berlin, 1998. DOI: 10.1007/978-1-4612-1700-8.

[GIT] D. Mumford, J. Fogarty and F. Kirwan. *Geometric invariant theory.* 3rd edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) **34**. Springer-Verlag, Berlin, 1994.

[Har77] R. Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics **52**. Springer-Verlag, New York-Heidelberg, 1977. DOI: 10.1007/978-1-4757-3849-0.

[Hir64] H. Hironaka. *Resolution of singularities of an algebraic variety over a field of characteristic zero. I.* Ann. of Math. (2) 79.**1** (1964), pp. 109–203. DOI: 10.2307/1970486.

[Ill05] L. Illusie. *Grothendieck's existence theorem in formal geometry. With a letter of Jean-Pierre Serre.* In: *Fundamental algebraic geometry.* Math. Surveys Monogr. **123**. Amer. Math. Soc., Providence, RI, 2005, pp. 179–233.

[Jou73] J. P. Jouanolou. *Une suite exacte de Mayer–Vietoris en K-théorie algébrique.* In: *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972).* Lecture Notes in Math. **341**. Springer, Berlin, 1973, pp. 293–316. DOI: 10.1007/BFb0067063.

[Kol86] J. Kollár. *Higher direct images of dualizing sheaves. I.* Ann. of Math. (2) 123.**1** (1986), pp. 11–42. DOI: 10.2307/1971351.

[Kol03] J. Kollár. *Rationally connected varieties and fundamental groups*. In: *Higher dimensional varieties and rational points (Budapest, 2001)*. Bolyai Soc. Math. Stud. **12**. Springer, Berlin, 2003, pp. 69–92. DOI: 10.1007/978-3-662-05123-8_4.

[Kro57] L. Kronecker. *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*. J. Reine Angew. Math. 53 (1857), pp. 173–175. DOI: 10.1515/crll.1857.53.173.

[LM85] A. Lubotzky and A. R. Magid. *Varieties of representations of finitely generated groups*. Mem. Amer. Math. Soc. 58.**336** (1985). DOI: 10.1090/memo/0336.

[Mor79] L. Moret-Bailly. *Polarisations de degré* 4 *sur les surfaces abéliennes*. C. R. Acad. Sci. Paris Sér. A-B 289.**16** (1979), A787–A790.

[Mor81] L. Moret-Bailly. *Familles de courbes et de variétés abéliennes sur* $\mathbb{P}^1$. Astérisque **86** (1981), pp. 109–140.

[Mum61] D. Mumford. *Letter to J.-P. Serre*. May 1961.

[Mum69a] D. Mumford. *Bi-extensions of formal groups*. In: *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*. Oxford Univ. Press, London, 1969, pp. 307–322.

[Mum69b] D. Mumford. *Enriques' classification of surfaces in char. p. I*. In: *Global Analysis (Papers in Honor of K. Kodaira)*. Univ. Tokyo Press, Tokyo, 1969, pp. 325–339.

[Mur58] J. P. Murre. *On a connectedness theorem for a birational transformation at a simple point*. Amer. J. Math. 80 (1958), pp. 3–15. DOI: 10.2307/2372818.

[Nak00] K. Nakamoto. *Representation varieties and character varieties*. Publ. Res. Inst. Math. Sci. 36.**2** (2000), pp. 159–189. DOI: 10.2977/prims/1195143100.

[NO80] P. Norman and F. Oort. *Moduli of abelian varieties*. Ann. of Math. (2) 112.**3** (1980), pp. 413–439. DOI: 10.2307/1971152.

[Oda95] T. Oda. *A note on ramification of the Galois representation on the fundamental group of an algebraic curve. II*. J. Number Theory 53.**2** (1995), pp. 342–355. DOI: 10.1006/jnth.1995.1095.

[Poo04] B. Poonen. *Bertini theorems over finite fields*. Ann. of Math. (2) 160.**3** (2004), pp. 1099–1127. DOI: 10.4007/annals.2004.160.1099.

[Pro76] C. Procesi. *The invariant theory of $n \times n$ matrices*. Advances in Math. 19.**3** (1976), pp. 306–381. DOI: 10.1016/0001-8708(76)90027-X.

[Ray70] M. Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics **119**. Springer-Verlag, Berlin-New York, 1970. DOI: 10.1007/BFb0059504.

[Sai96] K. Saito. *Character variety of representations of a finitely generated group in* SL_2. In: *Topology and Teichmüller spaces (Katinkulta, 1995)*. World Sci. Publ., River Edge, NJ, 1996, pp. 253–264.

[Sei88] W. K. Seiler. *Deformations of Weierstrass elliptic surfaces*. Math. Ann. 281.**2** (1988), pp. 263–278. DOI: 10.1007/BF01458433.

[Ser61] J.-P. Serre. *Exemples de variétés projectives en caractéristique p non relevables en caractéristique zéro*. Proc. Nat. Acad. Sci. U.S.A. 47 (1961), pp. 108–109.

[ST68] J.-P. Serre and J. Tate. *Good reduction of abelian varieties*. Ann. of Math. (2) 88 (1968), pp. 492–517. DOI: 10.2307/1970722.

[SGA1] A. Grothendieck. *Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 - Revêtements étales et groupe fondamental (SGA 1)*. Lecture Notes in Mathematics **224**. Springer-Verlag, Berlin-New York, 1971.

[$\mathrm{SGA3_I}$] M. Demazure and A. Grothendieck. *Séminaire de Géométrie Algébrique du Bois Marie 1962–64 - Schémas en groupes (SGA 3). I: Propriétés générales des schémas en groupes*. Lecture Notes in Mathematics **151**. Springer-Verlag, Berlin-New York, 1970.

[SGA6] A. Grothendieck et al. *Séminaire de Géométrie Algébrique du Bois Marie 1966–1967 - Théorie des intersections et théorème de Riemann–Roch (SGA 6)*. Lecture Notes in Mathematics **225**. Springer-Verlag, Berlin-New York, 1971.

[SK79] T. Shioda and T. Katsura. *On Fermat varieties*. Tōhoku Math. J. (2) 31.**1** (1979), pp. 97–115. DOI: 10.2748/tmj/1178229881.

[Sil09] J. H. Silverman. *The arithmetic of elliptic curves*. 2nd edition. Graduate Texts in Mathematics **106**. Springer, Dordrecht, 2009. DOI: 10.1007/978-0-387-09494-6.

[Sim91] C. T. Simpson. *The ubiquity of variations of Hodge structure*. In: *Complex geometry and Lie theory (Sundance, UT, 1989)*. Vol. 53. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1991, pp. 329–348. DOI: 10.1090/pspum/053/1141208.

[Stacks] The Stacks Project Authors. *Stacks project*. http://stacks.math.columbia.edu. 2005–2018.

[Tam97] A. Tamagawa. *The Grothendieck conjecture for affine curves*. Compos. Math. 109.**2** (1997), pp. 135–194. DOI: 10.1023/A:1000114400142.

[Tat66] J. Tate. *Endomorphisms of abelian varieties over finite fields*. Invent. Math. 2 (1966), pp. 134–144. DOI: 10.1007/BF01404549.

[Vak06] R. Vakil. *Murphy's law in algebraic geometry: badly-behaved deformation spaces*. Invent. Math. 164.**3** (2006), pp. 569–590. DOI: 10.1007/s00222-005-0481-9. arXiv: math/0411469.