Columbia University

Department of Economics
Discussion Paper Series

Extraction of the Surplus in Standard Auctions

Revised July 2002

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Discussion Paper #:0102-73

Department of Economics
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August 2003

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Abstract

Crémer and McLean [2] and McAfee and Reny [4] showed that, in "nearly all auctions", the seller can offer a mechanism that obtains full rent extraction. Later, Robert [8] showed that the result fails in the presence of either limited liability or risk aversion. This paper provides yet another reason. It shows that the full rent extraction result fails if the seller is restricted to using auctions where the bidders' payments to the seller depend on the bids alone. Our interest for this problem is motivated by the fact that both the "standard model of auction" ([3]) as well as the most popular auctions display this feature. As a general matter, the proof shows that full rent extraction results fail whenever the mechanism uses only part of the information embodied in a player's type.

JEL classification number: D44

Keywords: auctions, surplus

1 Introduction

Auctions occupy an important place within economic theory due not only to the unquestionable importance of this procedure in economic life ([9], [7]), but also because an auction mechanism can be considered an archetype of a general mechanism design problem. In fact, problems like optimal taxation, regulation, monopolistic price discrimination, trade under asymmetric information, public good provision display, essentially, the same structure as an auction problem.

In a typical auction an individual, the seller, auctions off a single object to N bidders. The seller maximizes his expected revenue from the auction by designing a set of rules according to which the object is awarded to one of the

*I am grateful to Jaques Crémer, Rich McLean and Paolo Siconolfi for insightful comments and explanations. I also wish to thank the seminar participants at Columbia University and at the SED 2002 conference held at NYU. Partial financial support through a Columbia University Council Grant for Research in Social Sciences (summer 2000) is gratefully aknowledged.

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bidders. Each bidder privately knows how much the good is worth to him. Given such private information and the rules set by the seller, the strategic problem bidders face is modeled as a game with incomplete information. Hence, the seller's problem can be viewed as that of choosing the game of incomplete information that the bidders will play. An extensive literature (see [3] for a survey) has studied how different mechanisms perform from the viewpoint of the seller (first-price, second-price, Dutch and English auction, etc.) as well as conditions under which all these mechanisms give the same expected revenue.

It is clear that the presence of bidders' private information (about their own valuations, their beliefs about the opponents' valuations, etc.) constitutes the main hurdle the sellers faces in pursuing his objective. In fact, it seems reasonable to conjecture that the bidders' private information would lead them to enjoy positive rents and, hence, limit the seller's ability to maximize his revenue. Yet, contrary to this intuition, work by Crémer and McLean [2] and McAfee and Reny [4] has shown that – in "nearly all auctions" – the seller can offer a mechanism that obtains full rent extraction.

This begs an obvious question. Why we do not observe, in real-life auctions, a widespread use of the Crémer and McLean mechanism? Or, as Jacques Crémer once put it, "If I had to sell a good, I would not use one of this auctions. The puzzle is to understand why". Moreover, he went on by adding that "The literature has often given as a reason the fact that if there is little correlation, they could give rise to very high payments. This is not good enough. There are a number of circumstances in which correlation is quite high, and we still see some type of first price auction being used".[1]

Of course, the full surplus extraction result obtains in a certain formal setting. Consequently, one could seek an answer by questioning the ability of such a setting to faithfully describe real-life auctions. This is McAfee and Reny's take on the problem. In [4] (p. 400), they conclude that "*the result* casts doubt on the value of the current mechanism design paradigm as a model of institutional design".

Later, Robert [8] provided two other reasons for not using Crémer and McLean auctions. This time remaining within the setting of the current mechanism design paradigm. They are the bidders' limited liability and their risk-aversion.

In this paper, we provide yet another reason. If the seller's mechanism is constrained to depend only on part of the information incorporated in a bidder's type, then generic full surplus extraction no longer holds.

There are at least two motivations for addressing this type of problem. The first is that common auction procedures (first-price, second-price, Dutch and English auction, etc.) are restricted mechanisms. That is, they use only part of the information incorporated in the bidders' types. This is so because the bidders' payments to the seller depend on the bids alone (they are "standard auctions" in the terminology of McAfee and McMillan [3]). From this viewpoint, this paper says not only that those procedures cannot obtain the full surplus

[1] Private communication.

extraction (this is, of course, well-known) but also that no relatively simple procedure, namely a standard auction, can obtain it. In fact, as we shall see, full surplus extraction results critically hinge upon the bidders taking some other action in addition to their bids, and on the payments to the seller to depend on those as well.

The second motivation has to do with the observation that there might be an inherent inability for the seller to fully describe a bidder's type. For instance, this might be due to the seller's limited understanding of the various dimensions of the bidders' information or to the cost of gathering and processing all the necessary information. In all such circumstances, the seller would offer a restricted mechanism in the sense explained above.

Recently, there has been much interest in "detail free" mechanisms (see Bergemann and Morris [1]), that is mechanisms that do not depend on the mechanism designer's knowledge of the probability distribution, P, over the bidders' types. This is a broader problem with, of course, interesting implications for the full surplus extraction question. Relatively to this latter, this paper should be viewed as complementing that line of research. The point, here, is that the problem of full surplus extraction goes beyond whether or not the mechanism designer knows P. If types are complex object – which they are if we think of a type as a sequence of beliefs of all orders – and if contracts are incomplete, in the sense that outcomes do not depend on all orders of beliefs, then full surplus extraction may not be possible even if the Crémer and McLean conditions are satisfied.

The paper is organized as follows. Section 2 describes the model used in this paper. Modulo variations in the terminology, this is the same as in Crémer and McLean. Section 3 reviews the mechanism of Crémer and McLean. There, we put special emphasis on the two critical features of the mechanism. That bidders are required to take some other actions in addition to making their bids and that the payments made to the seller be a function of these actions as well. In Section 4, we eliminate the latter possibility. We impose that, conforming to the "standard model of auctions" (McAfee and McMillan [3]), the payments to the seller depend on the bids alone. Our main result, established both in the case of a finite set of valuations and in the case of a compact set of those, is that there are open sets of auctions where the full extraction is not possible. Section 5 concludes. An Appendix contains all the proofs omitted in the main text.

2 The Model

In this section, we describe the model used in the paper. The description will be quick as the model is the same as Crémer and McLean's. The only difference is that we phrase it within the framework of Mertens and Zamir [6]. This is its proper *milieu*, and it will make the argument more transparent. Throughout the paper, the number of bidders is finite. The set of bidders is denoted by I, $I = \{1, 2, ..., N\}$. For simplicity, we deal with the independent private value case, only. All the reasonings can be effortlessly extended, but at the cost of tedious qualifications, to the common value case or to any common-private value mix. Since Crémer and McLean have remarked this in several occasions, there is no reason for doing so here.

Each bidders $i \in I$ has a valuation for the object $v \in V_i$, where V_i is a compact subset of the real line, which is denoted by R. The product $V^N = \overset{N}{\underset{1}{\times}} V_i$ is the basic domain of uncertainty for our problem. T_i denotes the set of types for bidder i, and $T^N = \underset{i}{\times} T_i$. Recall ([6]) that a type can be identified to a probability measure on $V^N \times T^{N-1}$, and observe that, in our special case, any of such measures must be the unit mass, $\delta(\cdot)$, on that type's valuation. Hence, by means of the identification $v \longmapsto \delta(v)$, the universal belief space generated by V^N is $\Omega = T^N = \times T_i$.

We restrict to problems that display a common prior P on T. Then, each type $t^i \in T_i$ can be derived from P as a conditional probability. We do so since the mechanism of Crémer and McLean is designed for these type of situations. Following Crémer and McLean, we have

Definition 1 *An information structure with basic domain of uncertainty V^N is a pair (T^N, P), where P is a probability measure on T^N.*[2] *The set of all information structures with basic domain of uncertainty V^N is the set of all pairs (T^N, P) as P varies among all possible probability measures on T^N. Such a set is denoted by $\mathcal{I}(V^N)$.*

For the sequel, it is useful to introduce an additional piece of notation. This is contained in the next definition.

Definition 2 *An information structure $(T^N, P) \in \mathcal{I}(V^N)$ is a finite information structure if P has finite support in T^N. The set of finite information structures in $\mathcal{I}(V^N)$ is denoted by $\mathcal{F}(V^N)$.*

Throughout the paper, the word generic refers to subsets of $\mathcal{I}(V^N)$, which is endowed with the weak-* topology.

The next definition is slightly different from Crémer and McLean's. It is so, because we need to distinguish between the case where payments depend on bids alone from the case dealt with by Crémer and McLean, where no such a restriction is imposed. For each $i \in I$, let A_i be a set, and let $a \in A = \times_i A_i$. One can interpret A_i as the set of possible "messages" that player i might send to the seller. Then,

Definition 3 *An auction with message space A is a collection of mappings $\{p_i, x_i\}_{i \in I}$, with $x_i : A \longrightarrow R$ and $p_i : A \longrightarrow R$ such that $p_i(a) \geq 0$ for all i and $a \in A$ and $\sum_{i \in I} p_i(a) \leq 1$ for all a.*

[2] Crémer and McLean call an information structure a triple (T^N, P, w_i), where w_i is a valuation function, i.e. a mapping $T_i \longrightarrow V_i$, which associate each type with his own valuation. I deleted the reference to w_i as a type's valuation is immediately derived from that type on the basis of the observation made in the previous paragraph.

The interpretation is that if players "announce" a, then bidder i pays an amount $x_i(a)$ to participate at the auction, and is awarded the object with probability $p_i(a)$.

The seller's problem consists of choosing $\left(A, \{p_i, x_i\}_{i \in I}\right)$ so to maximize his expected revenue. Bidder i's utility is given by $v_i(t) - x_i(a(t))$ if he is awarded the object, and by $-x_i(a(t))$ if he is not.

Crémer and McLean's result (see below) can be rephrased by saying that the seller can choose $A = T^N$ and mappings $\{p_i, x_i\}_{i \in I}$ so that full-surplus extraction obtains.

As discussed in the Introduction, in Section 4 we study the problem of full surplus extraction for a special class of auctions. These are called standard auction ([3]), and are identified by the restriction that the bidders' payments to the seller depend on the bids alone. This is (up to inessential duplications of the seller's strategy space) a restriction on the nature of the message space that the seller allows. Formally, let B_i be the set of possible bids for player i, and let $B = \times_i B_i$. Then,

Definition 4 *We say that payments depend on bids alone if $A = B$. In such a case, we say that the auction is a "standard auction".*

3 The Mechanism of Crémer and McLean

In this section, we are going to quickly review the logic leading to the full surplus extraction result. There are two reasons for doing so. One is, of course, to make the paper as self-contained as possible. The other, more important, is that some of the consideration of the present section will come handy when, in Section 4, we study the problem for restricted mechanisms. Overall, this results in a more economic presentation.

Recall that according to Definition 3, each bidder i has to choose an action in a set A_i in order to participate at the auction. Then, the problem of full surplus extraction can be described as that of satisfying the following two requirements. For any $i \in I$: (1) Pick a set A_i, if any, so that the choice of bidder i reveal his type; (2) Associate to each i's bid and each chosen action a payment to the seller so that bidder i's expected payment equals his expected gain.

The idea of Crémer and McLean is to pick A_i equal to a subset of the set of lotteries over the other bidders' choices and bids. This is an especially virtuous choice. It produces the feature that bidder i's expected payment is linear in bidder i's probabilities over the other bidders' types. It is, then, an easy matter for Crémer and McLean to give conditions under which full surplus extraction obtains.

Formally, the game with incomplete information chosen by the seller is as follows. Each bidder i has to pick an element in a set $\mathcal{L}_i$, that is, one sets $A_i = \mathcal{L}_i$. A typical element $l_i \in \mathcal{L}_i$ is a mapping $l_i : \underset{j \neq i}{\times} (\mathcal{L}_j \times B_j) \longrightarrow R$, specifying bidder i's payment to the seller as a function of the other bidders' choices as well as their bids. Elements in $\mathcal{L}_i$ are called lotteries as they are such from the viewpoint of bidder i. The object is awarded to the highest bid (with the addition of a tie-breaking rule, which is irrelevant for our purposes). These rules define payoff functions $u_i : T_i \times \left(\underset{j}{\times} (\mathcal{L}_j \times B_j) \right) \to R$ for each bidder i. A strategy for bidder i is a mapping $\sigma_i : T_i \to \Delta(\mathcal{L}_i \times B_i)$.

Then, when T^N is a finite set, we have

Theorem 5 ***[Crémer and McLean [2]]*** *Generically (in the space of all the P's on T^N), the seller can choose the sets $\mathcal{L}_i$'s so that there exists an equilibrium of the corresponding game such that (I) Players play dominant strategies; (II) Full surplus extraction obtains.*

A similar result holds for the uncountable compact case (McAfee and Reny [4]).

To see how full surplus extraction obtains, let us begin by observing that since for each bidder the payment is determined by means of an $l_i \in \mathcal{L}_i$, bidder i's payment does not depend on his own bid. Hence, the rule that the object is awarded to the highest bid implies that it is a dominant strategy for bidder i to bid his own valuation.

Denote by t_k^i a type for bidder i who has valuation v_k^i for the object, and by g_k^i the expected gain gross of the payment that type t_k^i obtains *at this dominant-strategy profile.*

Under these circumstances, the seller's problem is solved if, for each bidder i, the seller can find a set of lotteries, $\mathcal{L}_i$, such that (at the above dominant strategies) the following is true for each i's type

$$\begin{array}{ll} E(l_k^i \;\mid\; t_k^i) = g_k^i & \\ E(l_z^i \;\mid\; t_k^i) > E(l_k^i \mid t_k^i), & z \neq k \end{array} \tag{1}$$

In words, when offered a choice among the lotteries in the set $\mathcal{L}_i$, type t_k^i picks lottery l_k^i and his (total) expected gain is zero.

For simplicity, we illustrate the argument in the two-bidder case. Let $T = \{t_1, t_2, ..., t_m\}$ be the set of bidder 1's types, and let $\Theta = \{\theta_1, \theta_2, ..., \theta_m\}$ be that of player 2.[3] In such a case, the common prior P is a matrix, $P = (p_{ij})$, on $T \times \Theta$. Denote by $p_k = P(\cdot \mid t_k)$ and $q_k = P(\cdot \mid \theta_k)$ the conditionals computed from P (the types).

Now, suppose that bidder 2 is offered a set of lotteries $\Lambda = \{\lambda_1, ..., \lambda_m\}$, and that type θ_k picks lottery λ_k. Then, we must offer bidder 1 a set of lotteries $\left\{\tilde{l}_1, ..., \tilde{l}_m\right\}$ such that

$$\begin{array}{rclr} p_k \cdot \tilde{l}_k & = & g_k^1 & k = 1, ..., m \\ p_k \cdot \tilde{l}_j & > & g_k^1 & k \neq j \end{array}$$

[3] To shorten the exposition, we have assumed that T and Θ have the same cardinality. Obviously, the Crémer and McLean argument does not require such a condition.

or,

$$\begin{array}{rclr} p_k \cdot \tilde{l}_k & = & g_k^1 & k = 1, ..., m \\ p_j \cdot \tilde{l}_k & > & g_j^1 & k \neq j \end{array}$$

Equivalently, we must find m linear functionals, $\{l_1, ..., l_m\}$, such that

$$\begin{array}{rclr} p_k \cdot l_k & = & 0 & k = 1, ..., m \\ p_j \cdot l_k & > & 0 & k \neq j \end{array}$$

In other words, the linear functional l_k must separate p_k and $co\,\{p_1, ..., p_{k-1}, p_{k+1}, ...p_m\}$ [co denotes the convex hull], and, for each k, we must find a linear functional that does so.

It's clear that we can fulfill such a request as long as

$$p_k \cap co\,\{p_1, ..., p_{k-1}, p_{k+1}, ...p_m\} = \emptyset, \qquad \forall k \tag{2}$$

Let P_1 be the matrix of conditional probabilities of bidder 1. Since $(\det P_1 \neq 0) \Longrightarrow$ condition (2), generically we can do so. Hence, given bidder 2's choices, $\{\lambda_1, ..., \lambda_m\}$, and valuations, $\{v_1, ..., v_m\}$, we can use the components of the vector l_k to define a function

$$l_k(\lambda_j, v_j) \longmapsto R$$

which is our desired lottery.

Finally, we can proceed in a similar way for bidder 2.

A few comments are in order. First, it is worth remarking that Crémer and McLean constructively provide, in the way outlined above, a mechanism that leads to the full surplus extraction. One of the mechanism's virtues is that the actual state of the world the bidders' payment is contingent upon is verifiable not only by the bidders and the seller but also by an outside observer. This is so because the payment depends on the lotteries and the bids chosen by the other bidders, both of which are observable, and not on the other bidders' types (which are unobservable). Second, the wording "extraction of the surplus in dominant-strategy" used by Crémer and McLean is a bit misleading. In fact, while it is true that it is a dominant strategy for bidder i to bid his own valuation for the object, it also true that his strategy is more complex than just bidding. Bidder i is, in addition, required to pick a lottery in $\mathcal{L}_i$. When both components of bidder i's strategy are taken into account, it is no longer true that full surplus extraction obtains at a dominant-strategy profile. Finally, a consideration that will come useful in the next section. The full surplus extraction result depends in no way on the existence of a one-to-one relation between types and valuations. To see this, just picture two different types that have the same valuation for the object. Since, by assumption, the two types are different, so are the vectors of conditional probabilities that describe them. This is the only consideration that matters, as equation (2) makes it clear. As long as condition (2) is satisfied, full surplus extraction obtains. Put in a slightly different way, assume that condition (2) is satisfied. Even though the two types have the same valuation for the object, they will choose different lotteries because they have different beliefs about the other bidders' types. Hence, the seller will be able to distinguish between them.

4 The Case of Standard Auctions

Now, we are going to study what happens to the full surplus extraction result when the seller is restricted to using standard auctions. These are a subset of all possible auctions (Definitions 3 and 4), and are identified by the restriction that the bidders' payments to the seller depend on the bids alone. As the reader certainly recalls, this is equivalent to say that (up to inessential duplications of the seller's strategy space[4]) the bidders' message space coincides with space of the bids. In terms of the type of mechanisms devised by Crémer and McLean, this translates into a restriction on the domain of the lotteries. Elements in $\mathcal{L}_i$ are now mappings $l^i : \underset{j \neq i}{\times} B_j \longrightarrow R$. With this exception, all the other ingredient of the model are exactly as before. Our main result is the following.

Theorem 6 *When the seller is restricted to standard auctions, full surplus extraction is not a generic property in* $\mathcal{I}(V^N)$.

Let us begin with a simple, but useful, observation. The necessary and sufficient conditions to obtain full surplus extraction with a standard auction are of the same type as in Crémer and McLean. The only difference is that we have to consider the bidders' first-order beliefs in the place the bidders' types (the bidders' beliefs over the other bidders' types). To see this, begin by observing that, just like in the previous section, bidder i's payment does not depend on his own bid. Once again, the rule that the object is awarded to the highest bid implies that it is a dominant strategy for each bidder i to bid his own valuation. Since his payment to the seller is going to depend on the other players' bids only, it follows that (at such a dominant-strategy profile) the only beliefs that are relevant to determine player i's expected payment are his first-order beliefs, that is i's beliefs on the other players' valuations.

Now, we are going to split the argument into two parts. We will prove the theorem first for the case of a finite set of possible valuations, which is interesting in its own right. Then, we will extend the argument to the case of an uncountable compact set of possible valuations.

[4] The seller can, in principle, still allow for any message space. If A_i is one such (arbitrary) message space, the lotteries offered to bidder i are, just like before, mappings $l^i : \underset{j \neq i}{\times} (B_j \times A_j) \longrightarrow R$. However, the requirement that the bidders' payment depend on the bids alone implies that the values of such mappings are univoquely determined by the first argument, namely the bids. Hence, the use of the sets A_i's is inessential, and they can be discarded without loss.

4.1 Finite Set of Valuations

Here, our strategy for proving Theorem 6 is to show that a standard auction obtains full surplus extraction if and only if the information structure has a certain special property. Namely, the support of the common prior is such that there is a to a one-to-one relation between bidders' types and valuations. Then, we are going to show that the information structures with this property make up a negligible set.

Throughout this subsection, V is a finite set, $V = \{v_1, ..., v_k\}$. To avoid trivialities, we assume that $k > 1$. Also, recall that the space of possible types for a player is identified to the space of probability distributions on $V^N \times T^{N-1}$.

Because of the nature of the argument, which involves (essentially) only one bidder, two simplifying assumptions can be made for expositional purposes. First, one can present the proof for the two-bidder case only. The argument goes unchanged for the N-bidder case, but at the price of a more cumbersome notation. Therefore, we are going to focus on a finite 2-bidder information structure with common prior P, $(\Theta_1 \times \Theta_2, P)$. Here, $\Theta_1 \times \Theta_2$ denotes the support of P in $\Omega = T_1 \times T_2$, and Ω is the universal belief space. Second, one can treat players symmetrically by assuming that $\Theta_1 = \Theta_2 = \Theta$. For this is not the case, it suffices, in all of the reasonings below, to consider the player with the larger number of types.

Recall that since P is a common prior and $T_1 \times T_2$ is constructed as a projective limit ([6]), the beliefs of all orders of all players can be derived from P. In other words, for any j we have a mapping $P \longmapsto P_i^j$, which associates the common prior with the beliefs of order j of player i. In particular, i's first-order beliefs, P_i^1, are univoquely determined. In what follows, we write simply P^1 in the place of P_i^1 since we refer to one bidder only. In the 2-bidder case, both the common prior, P, and the first-order beliefs, P^1, are represented by means of a matrix (indexed by $\Theta \times \Theta$ and by $\Theta \times V$, respectively).

As observed above, the necessary an sufficient conditions for a standard auction to obtain full-surplus extraction are the same as in Crémer and McLean, just with first-order beliefs in the place the full specification of the bidder' beliefs. It is worth to record this formally.

Lemma 7 *Let $\varphi \in \mathcal{F}(V^N)$ be a finite information structure, and let $m = card(\Theta)$. Let P^1 be the matrix of first-order beliefs. Then, there exists a standard auction which obtains full surplus extraction iff no row in P^1 is a convex combination of the other rows.*

Obviously, this is the same argument that we saw for Crémer and McLean, but now restricted to first-order beliefs. The proof is in the Appendix, and is included only for completeness.

Given the content of the lemma, the next step is to inquire into the conditions under which no row in P^1 is a convex combination of the other rows. To this end, let us begin by observing that $card(\Theta) = m \geq k$. That is, the number of types is at least as big as the number of valuations. This is obvious as each valuation identifies at least one type. Hence, P^1 is an $m \times k$ matrix. From this, it is immediate to conclude that if $m \geq k+2$, then at least one row in P^1 is a convex combination of the others, and the conditions for full surplus extraction are not satisfied. In fact (see Appendix), such conditions are not satisfied even in the case $m = k+1$. Combining these observations we get to a useful intermediate result (again, see the Appendix for details).

Proposition 8 *Given a finite information structure $\varphi \in \mathcal{F}(V^N)$, there exists a standard auction which obtains full surplus extraction if and only if both the following conditions are satisfied:*
(a) P has full rank
(b) $v \neq v' \Longrightarrow \theta \neq \theta'$

Once we have the proposition, it is an easy matter to complete the proof of the theorem for the finite case.

Proof of Theorem 6 (finite case). The set $\mathcal{F}(V^N)$ of finite information structures is the set of all pairs (Ω, P), where $\Omega = T^N$ is the universal space of beliefs and P has finite support in Ω. Proposition 8 implies that the set of all auctions for which there exists a standard auction which obtains full surplus extraction can be identified to a subset of the probability distribution whose support is a set of $k \times k$ points. Hence, it is a subset of a linear space of dimension $(k \times k - 1)$, and is not dense in $\mathcal{F}(V^N)$, which has infinite dimension. From this, by observing that the set of probability distribution with finite support in Ω is dense in the set of all probability distributions on Ω ([5]), we have that, when we restrict to standard auctions, the full surplus extraction property in not generic in $\mathcal{I}(V^N)$. ■

In fact, the argument just given proves a stronger result. This is contained in the following corollary.

Corollary 9 *Generically in $\mathcal{I}(V^N)$, standard auctions do not obtain full surplus extraction.*

Proof. It suffices to observe that a subspace of finite dimension h is closed with empty interior in a space of dimension $z > h$. ■

The proof for the compact case is in the Appendix. It is worth pointing out, however that the extension of our reasoning to the case of a compact set of valuations does not respond to a mere technical need. In fact, the explanatory power of the model with a finite set of valuations is severely limited by the presumption that the seller as well as all the bidders have an exact knowledge of set of possible valuations. As soon as one admits that such a knowledge might not be exact, as it seems to be the case in most circumstances, one is naturally led to considering models with an uncountable set of possible valuations. This circumstance is clearly reflected by the widespread use of such a model in the economic literature. The reader should consult [4] for a thorough discussion.

5 Concluding remarks

In this paper, we have shown that the full surplus extraction result of Crémer and McLean fails when the seller is restricted to using standard auctions. Namely, auctions where the bidders' payments to the seller depend on the bids alone. Our interest for this case was motivated by the observations that the "standard model of auction" (McAfee and McMillan [3]) displays such a restriction and that so do the most popular auctions.

The failure of the full surplus extraction result had already been established in the case of limited liability and/or risk aversion [8]. Our result brings about another reason for such a failure. Because of the nature of its proof, it can be rephrased by saying that any mechanism that uses only part of the information embodied in a player's type does not obtain (generically) the full extraction of the surplus. The point can be easily appreciated if one focuses on the fundamental nature of the problem studied. Namely, the asymmetry in the information of the parties involved in an auction. By definition, a type of a player is everything that is known to him which is not common knowledge among the participants at the game. There are several reasons to think of this as a rather complicated object, which other parties, like a seller in an auction, have just a limited understanding of. Formally, a type is an infinite hierarchy of beliefs, and a limited description would correspond to truncated hierarchy. Then, our result says that in all such circumstances, the full surplus extraction result fails.

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APPENDIX

Proof of Lemma 7. Denote by p_j the j-th row of P^1, $j = 1, 2, ..., m$. The k-component vector p_j is a probability distribution on $V = \{v_1, ..., v_k\}$.

As explained in the above, in order to show that full surplus extraction obtains we need to show that there exist m linear functionals – $\tilde{l}_1, \tilde{l}_2, ..., \tilde{l}_m$ – such that

$$\begin{aligned} p_j\tilde{l}_j &= E(u \mid \theta_j) \\ p_j\tilde{l}_k &> E(u \mid \theta_j) \quad , \quad k \neq j \end{aligned} \tag{3}$$

or equivalently that there exist m linear functionals – $l_1, l_2, ..., l_m$ – such that

$$\begin{aligned} p_j l_j &= 0 \\ p_k l_j &> 0 \quad , \quad k \neq j \end{aligned} \tag{4}$$

Just like before, m linear functional, $l_1, l_2, ..., l_m$, with the desired properties exist if and only if

$$p_j \cap co\{p_1, ..., p_{j-1}, p_{j+1}, ..., p_m\} = \varnothing \quad , \quad j = 1, ..., m \tag{5}$$

In fact, if condition (5) is satisfied for p_j, then the desired linear functional l_j exists by any elementary separation theorem. The converse (necessity) is obvious. ■

Proof of Proposition 8

Recall that the common prior $P = (p_{ij})$ is a matrix on $\Theta \times \Theta$. The element p_{ij} gives the probability that bidder 1 is of type θ_i and bidder 2 is of type θ_j. Recall also that a type is a conditional probability on $V^2 \times \Theta$, and that his first order belief is a probability on V, computed as a marginal from his type. Therefore, bidder 1's first-order beliefs give the probability that bidder 2 has valuation v_j given that bidder 1 is of type θ_i. Denote by $\mathbf{1}_{v_l}$ the map on Θ that takes value 1 if type θ has valuation v_l, and 0 otherwise. Clearly, player 1's first-order beliefs (the elements of the matrix P^1) are computed from P according to the formula

$$\text{Prob}(v_l \mid \theta_i) = \frac{\sum\limits_{j=1}^{k} \mathbf{1}_{v_l} p_{ij}}{\sum\limits_{j=1}^{k} p_{ij}} \tag{6}$$

Let $\{J_l\}_1^k$ be a partition of the set of types, Θ, so that two types, θ and θ', are in the same class if they have the same valuation ($v = v' = v_l$, for some l). Then,

Lemma 10 *A row in P^1 is a convex combination of the other rows if $\exists \lambda \in R^m$, $\lambda \neq 0$, such that*

$$\lambda \left(\sum_{j \in J_l} c_j \right) = 0 \quad , \quad l = 1, ..., k$$

where c_j is the j-th column in P (the common prior).

If the n-th row is one of such rows, then λ is given by

$$\lambda = \left(-\beta_1 \frac{p_{\cdot n}}{p_{\cdot 1}}, -\beta_2 \frac{p_{\cdot n}}{p_{\cdot 2}}, ..., \underset{n\text{-}th\ pos}{1}, ..., -\beta_m \frac{p_{\cdot n}}{p_{\cdot m}} \right) \tag{7}$$

where $p_{\cdot j}$ is the sum of the elements of the j-th row in P, and $\sum\limits_{j \neq n} \beta_j = 1$.

Conversely, if $\exists \lambda$ satisfying (7), then the n-th row in P^1 is a convex combination of the other rows.

Proof. We saw that a generic element, p^1_{il}, in P^1 is of the form

$$p^1_{il} = \frac{\sum\limits_{j=1}^{k} \mathbf{1}_{v_l} p_{ij}}{\sum\limits_{j=1}^{k} p_{ij}} = \frac{\sum\limits_{j \in J_l} p_{ij}}{\sum\limits_{j=1}^{k} p_{ij}} \tag{8}$$

Suppose that row n in P^1 is a convex combination of the other rows. This means that there are $m-1$ numbers, $\{\beta_1, ..., \beta_{n-11}, \beta_{n+1}, ..., \beta_m\}$, such that $\beta_j \geq 0$ and $\sum\limits_{j \neq n} \beta_j = 1$ such that

$$p^1_{nz} = \sum_{j \neq n} \beta_j p^1_{jz} \qquad , \qquad z = 1, ..., k$$

or, using (8)

$$\frac{\sum\limits_{j \in J_z} p_{nj}}{\sum\limits_{j=1}^{k} p_{nj}} = \sum_{j \neq n} \beta_j \frac{\sum\limits_{l \in J_z} p_{jl}}{\sum\limits_{j=1}^{k} p_{jl}} \qquad , \qquad z = 1, ..., k$$

To ease the notation, denote by $p_{\cdot j}$ the sum of the elements of the j-th row in P, and set $p_{n,J_z} = \sum\limits_{j \in J_z} p_{nj}$. With this notation the above expression becomes

$$\frac{p_{n,J_z}}{p_{\cdot n}} = \sum_{j \neq n} \beta_j \frac{p_{j,J_z}}{p_{\cdot j}} \qquad , \qquad z = 1, ..., k$$

Rearranging, the preceding becomes

$$\lambda \left(\sum_{j \in J_z} c_j \right) = 0 \qquad , \qquad z = 1, ..., k$$

where

$$\lambda = \left(-\beta_1 \frac{p_{\cdot n}}{p_{\cdot 1}}, -\beta_2 \frac{p_{\cdot n}}{p_{\cdot 2}}, ..., \underset{n\text{-th pos}}{1}, ..., -\beta_m \frac{p_{\cdot n}}{p_{\cdot m}} \right)$$

The converse is obvious. ■

The proof of Proposition 8 rests on the following simple

Lemma 11 *Let $d_1, ..., d_k \in R^m$, $k < m$. Then, $\exists \lambda \in R^m$, $\lambda \neq 0$, such that*

$$\lambda d_l = 0 \qquad , \qquad l = 1, ..., k$$

Moreover, either λ is proportional to an element of the canonical basis of R^m or λ can be taken of the form

$$\lambda = \left(\frac{\alpha_j}{\sum_l d_{jl}} \right)$$

with $\sum_{j=1}^{m} \alpha_j = 0$.

Proof. Let $d_1, ..., d_k \in R^m$. By the dimension theorem and the orthogonal decomposition of Euclidean spaces, $k < m$ implies $\exists \lambda \in R^m$, $\lambda \neq 0$, such that $\lambda d_l = 0$, $l = 1, ..., k$.

First, suppose that $d_{jl} \neq 0$, $j = 1, ..., m$ and $l = 1, ..., k$. Let $\{\alpha_j\}_{j=1}^m$ be a set of numbers such that $\sum_{j=1}^{m} \alpha_j = 0$. Then, clearly the vector

$$\beta^l = (\beta_j^l) = \left(\frac{\alpha_j}{d_{jl}} \right)$$

solves the l-th equation, $l = 1, ..., k$.

Now, define a vector $\lambda \in R^m$ by

$$\lambda = (\lambda_j) = \left(\frac{1}{\sum_l d_{jl}} \sum_l d_{jl} \beta_j^l \right)$$

Then λ is of the form $\lambda = \left(\frac{\alpha_j}{\sum_l d_{jl}} \right)$, and for any l, $l = 1, ..., k$

$$\begin{aligned}
\sum_j \lambda_j d_{jl} &= \sum_j \left[\frac{1}{\sum_l d_{jl}} \sum_l d_{jl} \beta_j^l \right] d_{jl} \\
&= \sum_j \left[\frac{1}{\sum_l d_{jl}} \sum_l d_{jl} \frac{\alpha_j}{d_{jl}} \right] d_{jl} \\
&= \sum_j \left[\frac{1}{\sum_l d_{jl}} \alpha_j \sum_l d_{jl} \right] \\
&= \sum_j \alpha_j = 0
\end{aligned}$$

If $d_{jl} = 0$ for some j and l, then it is clear – by repeating the same reasoning or by simple geometric inspection – that a λ of the same form exists as long as $k < m - 1$ or at least one of the $d_1, ..., d_k$ has at least two components different from zero (this is obtained by simply setting in the above reasoning $\alpha_j = 0$ when $d_{jl} = 0$).

The only case where the above reasoning does not apply is when $k = m - 1$ and each d_j is proportional to a vector of the canonical basis of R^m. In such a case, our λ must be itself proportional to a vector of the canonical basis of R^m. ■

Proof of Proposition 8. Let $card(\Theta) = m$. The matrix P^1 of a bidder's first-order beliefs is an $m \times k$ matrix. Hence, its k columns are vectors in R^m. Let $m > k$. Then, it cannot be that all the vectors are vectors of the canonical basis in R^m (this would imply that at least one row in the $m \times m$ matrix P is 0, contradicting $\text{supp}(P) = \Theta \times \Theta$). Hence, by Lemmata 11 and 10, $\exists \lambda \in R^m$, $\lambda \neq 0$, $\lambda = \left(\frac{\alpha_j}{\sum_l d_{jl}} \right)$, such that $\lambda \left(\sum_{j \in J_l} c_j \right) = 0$, $l = 1, ..., k$. Hence, at least one row in P^1 is a convex combination of the others, and by Lemma 7 full surplus extraction does not obtain.

It follows that a necessary condition for full surplus extraction is $m = k$, that is different valuations are associated to different types[5]. In such a case, P^1 is a $k \times k$ matrix, and by Lemma 7 a necessary and sufficient condition for full surplus extraction is that P^1 has full rank. In turn, this is the case if and only if the common prior P is itself a $k \times k$ matrix with full rank. ■

Proof of Theorem 6 (compact case). Suppose that generically in $\mathcal{I}(V^N)$ there exists a standard auction which obtains full surplus extraction (FSE). It follows at once from this assumption that there exists one information structure (Ω, P^*) such that P^* has full support and FSE obtains.

The set $\{(\Omega, P) \mid P \text{ has finite support}\}$ is dense in $\mathcal{I}(V^N)$. Hence, there is a sequence $\{(\Omega, P_j)\}_0^\infty$ such that

(i) $(\Omega, P_j) \longrightarrow (\Omega, P^*)$

(ii) for any integer j, P_j has finite support.

The assumption of genericity of FSE implies that we can construct the sequence $\{(\Omega, P_j)\}_0^\infty$ so that FSE obtains for (Ω, P_j), for any integer j. Moreover, without loss, we can assume that, along $\{(\Omega, P_j)\}_0^\infty$, $\text{Supp}(P_j) \subseteq \text{Supp}(P_{j+1})$ for each j [we can obtain this by simply constructing a new sequence with the desired property from $\{(\Omega, P_j)\}_0^\infty$].

With the sequence $\{(\Omega, P_j)\}_0^\infty$, there is associated the sequence $\left\{(P^1_{i,j})\right\}_0^\infty$ of first-order beliefs of player i, $i = 1, ..., N$.

By construction, FSE obtains on $\{(\Omega, P_j)\}_0^\infty$, $\forall j$. From the proof for the finite case, we know that for this to be true it must be true that for each j there

[5] Recall that one of the requirements in the construction of the type space is that a type knows his own valuation.

is a bijection

$$b_j : V_j \longrightarrow T_j$$

where V_j is a finite subset of V and T_j is a finite subset of T (the set of types for player i) [From the proof for the finite case, we know that for finite subsets of valuations, FSE obtains iff the mapping from valuations to first-order beliefs is a bijection. Then, with each first-order belief associate that type with that first order belief]. Recall that the consistency condition that a type knows his own valuation implies valuation v is associated to a type who has valuation v

$$b_j(v) = t_v \tag{9}$$

Now, each b_j satisfies (9), and since $T_j \subseteq T_{j+1}$ and b_j is a bijection

$$b_j \leq b_{j+1}$$

in the ordering of partial functions.

Let $\mathcal{B}$ be the set of partial functions defined on closed subsets of V which satisfy property (9). By Zorn's Lemma, $\mathcal{B}$ has a maximal element.

As $(\Omega, P_j) \longrightarrow (\Omega, P^*)$, $T_j \longrightarrow T$ and, along such a sequence, b_j can be extended to a maximal element in $\mathcal{B}$, which has necessarily V as domain.

Summarizing, along the sequence $(\Omega, P_j) \longrightarrow (\Omega, P^*)$ we can construct a bijection $b : V \longrightarrow T$ which satisfies property (9).

Because of property (9) and the construction of Ω, such a b is defined by

$$\begin{tikzcd} \Omega = V^N \times T^N \arrow[r, "t"] & T \\ V \arrow[u, "i"] \arrow[ur, "b"] & \end{tikzcd}$$

where i is the canonical injection. Hence, b can be taken to be continuous.

Now, let b be as defined above. Then, the following diagram commutes

$$\begin{tikzcd} & T \arrow[d] \\ & \Pi(V^N) \arrow[d, phantom, "\wr"] \\ & v \times \Pi(V^{N-1}) \arrow[d, "\pi_1"] \\ V \arrow[uuur, "b"] & V \arrow[l, "id"'] \end{tikzcd}$$

where $\sim$ denotes a homeomorphism. It follows that the mapping $T \longrightarrow V$ constructed on the right hand part of the diagram is the inverse of b, and it is continuous.

Therefore, along the sequence $(\Omega, P_j) \longrightarrow (\Omega, P^*)$, the mapping $b : V \longrightarrow T$ can be taken to be a homeomorphism.

But, Ω is V^N-based. Hence, T can be taken to be $\Pi(V^N \times T^{N-1})$, the set of probability measures on $V^N \times T^{N-1}$, and our construction implies that there is a homeomorphism $V \sim \Pi(V^N \times T^{N-1})$. Such a homeomorphism requires both V and T to be one-point spaces, contrary to our hypothesis. ■