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WHAT IS THE COMPLEXITY OF RELATED ELLIPTIC, PARABOLIC AND HYPERBOLIC PROBLEMS?

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Abstract

Traub and Woźniakowski have dealt with the complexity of some simple partial differential equations. They chose three model problems, and showed that the parabolic problem considered had significantly lower complexity than the elliptic problem, which in turn had significantly lower complexity from the hyperbolic problem considered. They asked whether this is true in general. We show that this is not the case. In fact, if L is a reasonably well-behaved elliptic operator, then the steady state equation $Lu = f$, the heat equation $\partial_t u + Lu = f$, and the wave equation $\partial_{tt} u + Lu = f$ all have roughly the same worst-case complexity over f satisfying certain boundary conditions and having Sobolev r-norm bounded by unity.

1. Introduction.

This paper deals with the complexity of "related" elliptic, parabolic and hyperbolic partial differential equations. (In this Introduction, we have to use terms such as complexity, minimal error, etc. without definition; they are defined rigorously later.)

Traub and Woźniakowski [6] have dealt with the complexity of three partial differential equations. The first problem was the heat equation on a thin rod of length π, with zero boundary data; the initial data was odd of period 2π, having an r-th derivative whose L_2-norm was bounded by unity. They found if the solution was to be considered at a fixed time t_0, then the n^{th} minimal error in the L_2-norm was $e^{-(n+1)^2 t_0}/(n+1)^r$ and the complexity of finding an ε-approximation was $\Theta((\frac{1}{t_0} \ln \frac{1}{\varepsilon})^{1/2})$ as $\varepsilon \to 0$.

The second problem was Laplace's equation on a square of length π in the x,y-plane, with zero boundary data on the west, south, and east sides of the square; the boundary data on the north side satisfied the same conditions as the initial data in the heat equation above. If the solution was to be considered along a line $y = y_0$ $(0 < y_0 < \pi)$, then they found the nth minimal L_2-error to behave asymptotically as $e^{-(n+1)(\pi-y_0)}/(n+1)^r$, and the complexity of finding an

ϵ-approximation to be $\Theta(\frac{1}{\pi-\gamma_0} \ln \frac{1}{\epsilon})$ as $\epsilon \to 0$.

The third problem was $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$, a first-order hyperbolic problem, with initial data of period 2π and mean value zero, the L_2-norm of whose rth derivative was bounded by unity. They considered the solution to be $u(x,t_0)$ for a fixed $t_0 > 0$, for $x \in [0,\pi]$. Here, the nth optimal L_2-error was found to be $(\left\lfloor\frac{n}{2}\right\rfloor+1)^{-r}$, and the complexity of finding an ϵ-approximation was $\Theta(\epsilon^{-1/r})$ as $\epsilon \to 0$.

Hence, they found examples of parabolic, elliptic, and hyperbolic problems for which the parabolic problem had significant smaller complexity than the elliptic problem, which in turn had significantly smaller complexity than the hyperbolic problem. They asked [6, p. 149] whether this was true in general, or whether this depended on these specially chosen examples.

In this paper we show that this phenomenon is not true in general.

We first note that the phenomenon noted above is norm-dependent. In fact, Traub and Woźniakowski showed that the nth minimal error for all three problems became $\Theta(n^{-r})$ as $n \to \infty$, when the error was measured in the $L_\infty(L_2)$ sense, and the complexity of finding an ϵ-approximation became $\Theta(\epsilon^{-1/r})$. However, this is a somewhat unnatural way of measuring the error for the elliptic problem.

Another difficulty is that the class of problem elements either played a different role (initial data for the parabolic problem, boundary data for the elliptic problem) or changed (odd functions for the parabolic and elliptic problems, functions with zero mean in the hyperbolic problem) when going from one problem to another. Hence the notion of "a class of problem elements with smoothness r" changed from problem to problem.

In this paper, we are interested in the inherent complexity of <u>related</u> elliptic, parabolic, and hyperbolic problems. That is, we let L be a reasonably nice elliptic operator, and consider the elliptic problem $Lu = f$, the parabolic problem $\partial_t u + Lu = f$, and the hyperbolic problem $\partial_{tt} u + Lu = f$ for f in the unit ball of $H_0^r(\Omega)$. Hence the problems are all related, and the problem element f plays the same role in all three problems. Error is measured in the $L_2(\Omega)$ norm for the elliptic problem, and either the $L_2(\Omega)$ norm at a fixed time t_0 or the $L_\infty(L_2)$ norm for the time-dependent problems.

Our main result is that, if the order of L is $2m$ and $\Omega \subset \mathbb{R}^N$, then for the elliptic and parabolic problems and for the hyperbolic problem solved over an interval in time the nth minimal error is $\Theta(n^{-(r+2m)/N})$ as $n \to \infty$, and the complexity of finding an ε-approximation is $\Theta(\varepsilon^{-N/(r+2m)})$ as $\varepsilon \to 0$. For the hyperbolic problem solved at a particular time t_0, the "Θ" becomes a "O"; moreover, this is (roughly speaking) the strongest statement possible, due to the possibility of a "fortunate" value of t_0 making the nth minimal error very small (in fact, even zero) for a small value of n.

It is important to point out that these results are mainly of theoretical interest. There are two reasons for this. The first is that we assume that the problem element f belong to $H_0^r(\Omega)$, and hence satisfy some boundary conditions; most authors only assume that $f \in H^r(\Omega)$, i.e., f satisfies no boundary conditions. The second reason is that we are mainly interested in the inherent (or intrinsic) complexity of these problems, which allows us to consider algorithms which may not be implementable in practice. (For example, we assume a model having infinite-precision arithmetic and for which exact information is available; if either of these assumptions were weakened, it may be the case that the complexity of these problems might be different. In addition, we consider algorithms using information that might not be available in many situations, such as the eigenvalues and eigenfunctions of L.) However, knowing the inherent complexity does provide a benchmark: it tells what price the user is paying for using non-optimal information or non-optimal algorithms. In addition, we are able to show that there are finite-element methods which are nearly optimal, so that if the usual "finite element information" [7] is available, these results become practical, as well as theoretical.

We now outline the contents of this paper. In Section 2. we define the problems to be studied. In Section 3, we recall some terminology and results from [6] concerning optimal error algorithms. In Section 4. we use Hilbert scale techniques to replace the class of problem elements introduced in Section 2 by one that is equivalent, but easier to work with. In Section 5, we compute nth minimal errors for these related problems, and show that they are roughly the same. In Section 6, these results are used to show that the complexity of finding ε-approximations is roughly the same for all three problems. Finally, we summarize our results and pose some open questions in Section 7.

2. Related elliptic, parabolic, and hyperbolic problems.

In this Section, we will define the problems to be studied. We use the standard terminology and notations found in Agmon [1] for multi-indices, Sobolev spaces, etc.

Given a positive integer N, let $\Omega \subset \mathbb{R}^N$ be a bounded region with C^∞ boundary. Consider the formally self-adjoint, uniformly strongly 2m-th order elliptic operator L of the form

$$(Lv)(x) := \Sigma_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta v(x)),$$

with real valued functions $a_{\alpha\beta} \in C^\infty(\bar{\Omega})$ such that $a_{\alpha\beta} = a_{\beta\alpha}$. We additionally assume that

$$\text{(2.1)} \qquad Lv = 0 \text{ in } \Omega \text{ and } \partial_\nu^j v = 0 \ (0 \leq j \leq m-1) \text{ on } \partial\Omega \quad \text{implies} \quad v = 0 \text{ in } \Omega,$$

∂_ν denoting the outer normal derivative on $\partial\Omega$. For $r \geq 0$, recall that $H_0^r(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the $H^r(\Omega)$-norm.

We are interested in the complexity of the following problems:

(E) $\left\{ \begin{array}{l} \text{Given } f \in H_0^r(\Omega), \text{ find } u: \bar{\Omega} \to \mathbb{R} \text{ such that} \\ \quad Lu = f \quad \text{in} \quad \Omega \\ \quad \partial_\nu^j u = 0 \quad (0 \le j \le m-1) \quad \text{on} \quad \partial\Omega \\ \text{We define} \\ \quad S_E f := u. \end{array} \right.$

(Note that $S_E f$ is defined for all $f \in H_0^r(\Omega)$ because of (2.1).)

In the next two problems, we fix a value of $T > 0$.

(P) $\left\{ \begin{array}{l} \text{Given } f \in H_0^r(\Omega), \text{ find } u: \bar{\Omega} \times [0,T] \to \mathbb{R} \text{ such that} \\ \quad \partial_t u + Lu = f \quad \text{in} \quad \Omega \times (0,T) \\ \quad \partial_\nu^j u = 0 \quad (0 \le j \le m-1) \quad \text{on } \partial\Omega \times [0,T] \\ \quad u(\cdot,0) = 0 \quad \text{on} \quad \Omega. \\ \text{If we wish the solution for all } t \in [0,T], \text{ we} \\ \text{will consider} \\ \quad S_P f := u. \\ \text{If we are interested in the solution at a parti-} \\ \text{cular } t_0 \in [0,T] \text{ we will consider} \\ \quad S_{P,t_0} f := u(\cdot,t_0). \end{array} \right.$

$$
(H) \quad \left\{
\begin{array}{l}
\text{Given } f \in H_0^r(\Omega), \text{ find } u\colon \bar{\Omega} \times [0,T] \to \mathbb{R} \text{ such that} \\
\qquad \partial_{tt} u + Lu = f \quad \text{in} \quad \Omega \times (0,T) \\
\qquad \partial_\nu^j u = 0 \quad (0 \le j \le m-1) \quad \text{on} \quad \partial\Omega \times [0,T] \\
\qquad u(\cdot,0) = \partial_t u(\cdot,0) = 0 \quad \text{on} \quad \Omega. \\
\text{If we wish the solution for all } t \in [0,T], \text{ we} \\
\text{will consider} \\
\qquad S_H f := u \\
\text{while if we want the solution at a particular} \\
t_0 \in [0,T], \text{ we will be interested in} \\
\qquad S_{H,t_0} f := u(\cdot, t_0).
\end{array}
\right.
$$

The problems (E), (P), and (H) are said to be <u>related</u> since they all involve the same elliptic operator L. Note that for the choice $L = -\Delta$ (Δ is the N-dimensional Laplacian), (E) becomes Poisson's equation, while (P) and (H) become the heat and wave equations with a forcing term. respectively. Note also that $\lim_{t_0 \to \infty} S_{P,t_0} = S_E$, i.e., the solution to the elliptic problem is the steady-state solution of the parabolic problem.

3. Optimal-error algorithms and optimal information.

In this Section, we introduce some terminology and results from [6] concerning optimal-error algorithms using given information, as well as the selection of optimal information.

Let $S: \mathcal{F}_1 \to \mathcal{F}_2$ be a linear transformation of Banach spaces. We call S a solution operator. In this paper, we will let $\mathcal{F}_1 = H_0^r(\Omega)$; $\mathcal{F}_2$ will be $L_2(\Omega)$ for the choices $S = S_E$, $S = S_{P,t_0}$, and $S = S_{H,t_0}$, while $\mathcal{F}_2$ will be $L_\infty([0,T],L_2(\Omega))$ under the norm

$$\|u\|_{L_\infty(L_2)} := \operatorname*{ess\,sup}_{0\le t\le T} \|u(\cdot,t)\|_{L_2(\Omega)}$$

for the choices $S = S_P$ and $S = S_H$.

We let $\mathcal{F}_0 \subset \mathcal{F}_1$ be a set of problem elements. In this paper, we are mainly concerned with the choice $\mathcal{F}_0 = BH_0^r(\Omega)$, where $B\mathcal{X}$ denotes the unit ball of a Banach space $\mathcal{X}$. We will then refer to the problem $(S,\mathcal{F}_0)$.

Our main goal is to find, for $\varepsilon > 0$, an ε-approximation x to the problem $(S,\mathcal{F}_0)$. That is, given $f \in \mathcal{F}_0$, we wish to compute $x(f) \in \mathcal{F}_2$ such that

$$(3.1) \qquad \|Sf - x(f)\|_{\mathcal{F}_2} < \varepsilon \quad \forall f \in \mathcal{F}_0.$$

In order to do this, we must know some "information"

$\mathcal{N}f$ about the problem elements $f \in \mathcal{F}_0$. Hence, an <u>information operator</u> will be any linear operator

$$(3.2) \qquad \mathcal{N}: \mathcal{F}_1 \to \mathbb{R}^n.$$

(For example, if $\lambda_1, \ldots, \lambda_n$ are linear functionals on $\mathcal{F}_1$,

$$(3.3) \qquad \mathcal{N}f := [\lambda_1(f) \ \ldots \ \lambda_n(f)]^T \quad \forall f \in \mathcal{F}_1$$

is an information operator.) The number of "essential" pieces of information is referred to as the <u>cardinality</u>, i.e.,

$$\text{card } \mathcal{N} := \dim \mathcal{N}(\mathcal{F}_1) \equiv \text{codim ker } \mathcal{N}.$$

(Hence, if $\mathcal{N}$ is given by (3.3), card $\mathcal{N}$ is the number of linearly independent elements of $\{\lambda_1, \ldots, \lambda_n\}$.) Thus, if $\mathcal{N}$ is of the form (3.2), card $\mathcal{N} \leq n$.

<u>Remark 3.1</u>: Note that we only consider linear information which is <u>nonadaptive</u>, i.e., λ_i depends only on f and not upon the previous information. For the problems considered in this paper, adaptive and nonadaptive linear information are equally powerful, see [6, Chapter 2]. In addition, one might consider nonlinear information. It is known that <u>arbitrary</u> nonlinear information is <u>too</u> powerful, see [6, Chapter 7]. However, <u>continuous</u> nonlinear information is no more powerful than linear information (Kacewicz and Wasilkowski, private communication). $\square$

We now must use the information in an <u>algorithm</u>, i.e., a mapping $\varphi\colon \mathcal{N}(\mathcal{F}_1) \to \mathcal{F}_2$. The class of all such algorithms using $\mathcal{N}$ is denoted $\Phi(\mathcal{N})$. Given $\varphi \in \Phi(\mathcal{N})$, define its <u>error</u> to be

$$e(\varphi) := \sup_{f \in \mathcal{F}_0} \|Sf - \varphi(\mathcal{N}f)\|_{\mathcal{F}_2}$$

Then (3.1) yields that $e(\varphi) < \varepsilon$ iff φ gives an ε-approximation to $(S, \mathcal{F}_0)$.

Of course, we wish to use the information $\mathcal{N}$ as well as possible. That is, we wish to find an algorithm $\varphi^* \in \Phi(\mathcal{N})$ which is an <u>optimal-error</u> <u>algorithm</u>, i.e., such that

$$(3.4) \qquad e(\varphi^*) = \inf_{\varphi \in \Phi(\mathcal{N})} e(\varphi) =: e(\mathcal{N}, S, \mathcal{F}_0)$$

where $e(\mathcal{N}, S, \mathcal{F}_0)$ is the <u>optimal</u> <u>error</u> using $\mathcal{N}$ for the problem $(S, \mathcal{F}_0)$. It turns out that in our Hilbert space setting,

$$(3.5) \qquad e(\mathcal{N}, S, \mathcal{F}_0) = \sup_{z \in \mathcal{F}_0 \cap \ker \mathcal{N}} \|Sz\|_{\mathcal{F}_2}$$

and that there exists a <u>linear</u> optimal-error algorithm, i.e., there is a vector $\underset{\sim}{a} \in \mathbb{R}^n$ such that

$$\varphi^*(\mathcal{N}f) := \underset{\sim}{a} \cdot \mathcal{N}f \qquad \forall f \in \mathcal{F}_1$$

is an optimal error algorithm. (See [6, Chapter 3.)

Finally, we are interested in selecting optimal information of given cardinality. Let

$$(3.6)\qquad e(n,S,\mathfrak{F}_0) := \inf_{\#\mathcal{N}\leq n} e(\mathcal{N},S,\mathfrak{F}_0)$$

denote the nth <u>minimal error</u> for the problem $(S,\mathfrak{F}_0)$. Then an information operator $\mathcal{N}_n$ is said to be an nth <u>optimal information</u> if

$$(3.7)\qquad \text{card } \mathcal{N}_n \leq n \quad \text{and} \quad e(\mathcal{N}_n,S,\mathfrak{F}_0) = e(n,S,\mathfrak{F}_0).$$

Let Φ_n denote the class of all algorithms using information of cardinality at most n. If $\mathcal{N}_n$ is an nth optimal information for the problem $(S,\mathfrak{F}_0)$, and φ_n is an optimal-error algorithm using $\mathcal{N}_n$, then (3.4)-(3.8) yield

$$e(\varphi_n) = \inf_{\varphi\in\Phi_n} e(\varphi),$$

and we may refer to φ_n as an nth <u>minimal algorithm</u> for the problem $(S,\mathfrak{F}_0)$.

In order to see an example of nth optimal information and n nth minimal algorithm, we consider a problem studied in Chapter 6 of [6]. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{z_n\}_{n=1}^{\infty}$ and let $\{\gamma_n\}_{n=1}^{\infty}$ be a set of real numbers with $0 < \gamma_1 \leq \gamma_2 \leq \cdots$. Let

$$\mathfrak{F}_1 = \{f \in \mathcal{H}: \|Tf\| < \infty\},$$

where

(3.8) $\quad Tf := \Sigma_{i=1}^{\infty} \gamma_i (f, z_i)_{\mathcal{H}} z_i,$

and let $\mathcal{F}_2 = \mathcal{H}$, and define the solution operator $S: \mathcal{F}_1 \to \mathcal{F}_2$ by

(3.9) $\quad Sf := \Sigma_{i=1}^{\infty} \sigma_i (f, z_i)_{\mathcal{H}} z_i.$

Let $\lambda_i = \sigma_i/\gamma_i$ and suppose that λ_{i_j} is the jth largest value of $\{\lambda_i\}_{i=1}^{\infty}$, i.e.

$$\lambda_{i_1} \geq \lambda_{i_2} \geq \cdots \geq 0.$$

Let

(3.10) $\quad \mathcal{F}_0 = B\mathcal{F}_1 = \{f \in \mathcal{H}: \|Tf\| \leq 1\}.$

Then Theorem 6.6.1 of [6] yields

<u>Lemma 3.1:</u>

(i) $e(n, S, \mathcal{F}_0) = \lambda_{i_{n+1}}$,

(ii) $\mathcal{N}_n f := [(f, z_{i_1}) \ \ldots \ (f, z_{i_n})]^T$ is an nth optimal information for the problem $(S, \mathcal{F}_0)$,

and

(iii) $\phi_n(\mathcal{N}_n f) := \Sigma_{j=1}^{n} \sigma_{i_j} (f, z_{i_j})_{\mathcal{H}} z_{i_j}$ is an nth minimal algorithm for the problem $(S, \mathcal{F}_0)$. $\square$

4. A Hilbert scale.

As indicated above, we are interested in nth optimal (or nearly optimal) algorithms for the problem $(S, BH_0^r(\Omega))$, S being one of the solution operators $S_E, S_{P,t_0}, \bar{S}_P, S_{H,t_0}$, or S_H. Our task would be simplified if the problem $(S_E, BH_0^r(\Omega))$ could be easily written in the form (3.9), (3.10). In this Section, we will use Hilbert scale techniques to reduce the problem $(S_E, BH_0^r(\Omega))$ to such a form.

Let $\{z_n\}_{n=1}^{\infty}$ be $L_2(\Omega)$-orthonormal eigenfunctions for L, i.e.,

$$Lz_n = \beta_n z_n \quad \text{in } \Omega$$

$$\partial_\nu^j z_n = 0 \quad (0 \leq j \leq m-1) \quad \text{on } \partial\Omega,$$

where the $\{\beta_n\}_{n=1}^{\infty}$ are ordered so that $0 < \beta_1 \leq \beta_2 \leq \cdots$. Then [1, p. 289] $\{z_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L_2(\Omega)$. Moreover, there is a constant $c_N > 0$ such that

(4.1) $$\beta_n \sim c_N n^{2m/N} \text{ as } n \to \infty.$$

(Agmon [1, Theorem 4.6]), and so $\lim_{n\to\infty} \beta_n = +\infty$.

Let

$$M := \left(\frac{1}{\beta_1} L\right)^{1/2m}.$$

<u>Lemma 4.1</u>: M is a self-adjoint operator in $L_2(\Omega)$, with

$$(Mu,u)_0 \geq \|u\|_0^2$$

and so

$$\|u\|_0 \leq \|Mu\|_0.$$

<u>Proof</u>: We need only establish the first inequality, since it implies the second. Let $u = \Sigma_{n=1}^{\infty} \alpha_n z_n$. Then

$$Mu = \Sigma_{n=1}^{\infty} \left(\frac{\beta_n}{\beta_1}\right)^{1/2m} \alpha_n z_n$$

and so $0 < \beta_1 \leq \beta_2 \leq \ldots$ yields

$$(Mu,u)_0 = \Sigma_{n=1}^{\infty} \left(\frac{\beta_n}{\beta_1}\right)^{1/2m} \alpha_n^2 \geq \Sigma_{n=1}^{\infty} \alpha_n^2 = \|u\|_0^2. \quad \square$$

For $r \geq 0$, $u \in C_0^{\infty}(\Omega)$, define

$$|||u|||_r := \|M^r u\|_0 = \frac{1}{\beta_1^{r/2m}} \|L^{r/2m} u\|_0.$$

We let $H_{r,0}(\Omega)$ denote the closure of $C_0^{\infty}(\Omega)$ under the norm $|||\cdot|||_r$. Then Lemma 4.1 and the results in Section 9 of Krein and Petunin [4] imply that $\{H_{r,0}(\Omega): r \in \mathbb{R}\}$ is a Hilbert scale.

We wish to show that for $r \geq 0$, $H_0^r(\Omega) = H_{r,0}(\Omega)$, and that $\|\cdot\|_r$ and $|||\cdot|||_r$ are equivalent norms on $H_0^r(\Omega)$. We first consider the case where $r = 2k_0 m$, k_0 being a nonnegative integer. To do this, we first must establish

<u>Lemma 4.2</u>: For $0 \leq j \leq k_0$, there exist $c_1(j)$, $c_2(j) > 0$ such that for any $u \in C_0^{\infty}(\Omega)$,

$$c_1(j)\|L^{k_0}u\|_0 \le \|L^{(k_0-j)}u\|_{2mj} \le c_2(j)\|L^{k_0}u\|_0.$$

__Proof__ (by induction): The case $j = 0$ is trivial, with $c_1(0) = c_2(0) = 1$. Suppose the Lemma holds for $j = j_0-1 \ge 0$. Let $u \in C_0^\infty(\Omega)$, and set $v := L^{k_0-(j_0-1)}u$. Since $u \in C_0^\infty(\Omega)$, $v \in C_0^\infty(\Omega)$ satisfies the Dirichlet boundary conditions for L. From elliptic regularity theory [4, Theorem 5.6], there exist positive constants $\alpha_i = \alpha_i(2m(j_0-1))$ $(i = 1,2)$, independent of v, such that

$$\alpha_1(2m(j_0-1))\|v\|_{2m(j_0-1)} \le \|Lv\|_{2mj_0}$$
$$\le \alpha_2(2m(j_0-1))\|v\|_{2m(j_0-1)}.$$

That is,

$$\alpha_1(2m(j_0-1))\|L^{k_0-(j_0-1)}u\|_{2m(j_0-1)} \le \|L^{k_0-j_0}u\|_{2mj_0} \tag{4.2}$$
$$\le \alpha_2(2m(j_0-1))\|L^{k_0-(j_0-1)}u\|_{2m(j_0-1)}.$$

But the induction hypotheses yields

$$c_1(j_0-1)\|L^{k_0}u\|_0 \le \|L^{k_0-(j_0-1)}u\|_{2m(j_0-1)} \tag{4.3}$$
$$\le c_2(j_0-1)\|L^{k_0}u\|_0.$$

Setting

$$c_i(j_0) := c_i(j_0-1)\alpha_i(2m(j_0-1)) \quad (i = 1,2),$$

we find that (4.2) and (4.3) yield

$$c_1(j_0)\|L^{k_0}u\|_0 \le \|L^{k_0-j_0}u\|_{2mj_0}$$
$$\le c_2(j_0)\|L^{k_0}u\|_0,$$

as needed to complete the induction. □

Lemma 4.2: $H_0^{2k_0m}(\Omega) = H_{2k_0m,0}(\Omega)$, and the norms $\|\cdot\|_{2k_0m}$ and $|||\cdot|||_{2k_0m}$ are equivalent.

Proof: Let $u \in C_0^\infty(\Omega)$. Then

(4.4) $$|||u|||_{2k_0m} = \frac{1}{\beta_1^{k_0}}\|L^{k_0}u\|_0.$$

Letting $j = k_0$ in Lemma 4.1, we have

(4.5) $$c_1(k_0)\|L^{k_0}u\|_0 \le \|u\|_{2k_0m} \le c_2(k_0)\|L^{k_0}u\|_0.$$

Hence (4.4) and (4.5) yield

$$\beta_1^{k_0}c_1(k_0)|||u|||_{2k_0m} \le \|u\|_{2k_0m} \le \beta_1^{k_0}c_2(k_0)|||u|||_{2k_0m}.$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^{2k_0m}(\Omega)$ and $H_{2k_0m,0}(\Omega)$, the result follows. □

We now prove equivalence for the general case.

Theorem 4.1: For any $r \ge 0$, $H_0^r(\Omega) = H_{r,0}(\Omega)$, and the norms $\|\cdot\|_r$ and $|||\cdot|||_r$ are equivalent.

Proof: Given $r \ge 0$, choose a nonnegative integer k_0 such

$$\varphi_n(\eta_n f) := \Sigma_{j=1}^{n} \sigma_{i_j}(f, z_{i_j})_0 z_{i_j}.$$

Then

$$\alpha_1 \lambda_{i_{n+1}} \leq e(\varphi_n) \leq \alpha_2 \lambda_{i_{n+1}}$$

so that

$$e(\varphi_n) = \Theta(\lambda_{i_{n+1}}) = \Theta(\rho_{i_{n+1}} i_{n+1}^{-(r+2m)/N}) \quad \text{as } n \to \infty,$$

and φ_n is (to within a constant factor, independent of n) an nth optimal algorithm for the problem $(S, BH_0^n(\Omega))$.

<u>Proof:</u> To see (i), let $z \in \ker \eta \cap \mathfrak{F}_0$. Then (4.6) yields that $\alpha_1 z \in \ker \eta \cap BH_0^r(\Omega)$. Hence (3.5) yields

$$\alpha_1 \|Sz\| = \|S(\alpha_1 z)\| \leq e(\eta, S, BH_0^r(\Omega)).$$

Since $z \in \ker \eta \cap \mathfrak{F}_0$ is arbitrary, this inequality, along with (3.5), yields $\alpha_1 e(\eta, S, \mathfrak{F}_0) \leq e(\eta, S, BH_0^r(\Omega))$. The other inequality is proved analogously.

To establish the rest of the Theorem, note that $\mathfrak{F}_0$, as given in (4.7), is of the form (3.10), where $\gamma_i = \beta_i^{r/(2m)}$ in (3.8). Hence, the rest of the Theorem follows from (3.6), Lemma 3.1, (4.1), and (i) of this Theorem. $\square$

5. Minimal errors and algorithms for related problems.

In this Section, we use Theorem 4.2 to show that the nth minimal errors for the related problems are all $\Theta(n^{-(r+2m)/N})$, i.e., all roughly the same. Moreover, the same information is nearly nth optimal information for all these problems. We also give algorithms which are nearly nth optimal algorithms for these problems.

5.1 Elliptic problem $(S_E, BH_0^r(\Omega))$. Since $\{z_j\}_{j=1}^{\infty}$ are orthonormal eigenfunctions for L corresponding to the eigenvalues $\{\beta_j\}_{j=1}^{\infty}$ with $0 < \beta_1 \le \beta_2 \le \ldots$, we find

$$S_E z_j = \frac{1}{\beta_j} z_j,$$

so that

$$S_E f = \Sigma_{j=1}^{\infty} \frac{1}{\beta_j}(f, z_j)_0 z_j.$$

Hence Theorem 4.2 and (4.1) yield

Theorem 3.1:

(i) $\alpha_1 \beta_{n+1}^{-(r+2m)/(2m)} \le e(n, S_E, BH_0^r(\Omega)) \le \alpha_2 \beta_{n+1}^{-(r+2m)/(2m)}$.

so that

$$e(n, S, BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \quad \text{as} \quad n \to \infty.$$

(ii) Let

$$(5.1) \qquad \mathcal{N}_n f = [(f, z_1)_0 \ \ldots \ (f, z_n)_0]^T.$$

η_n is (to within a constant factor, independent of n) nth optimal information with

$$\alpha_1\beta_{n+1}^{-(r+2m)/(2m)} \leq e(\eta_n, S_E, BH_0^r(\Omega))$$

$$\leq \alpha_2\beta_{n+1}^{-(r+2m)/(2m)},$$

and so

$$e(\eta_n, S_E, BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as } n \to \infty.$$

(iii) Let

$$\varphi_n(\eta_n f) := \Sigma_{j=1}^n \frac{1}{\beta_j}(f, z_j)_0 z_j.$$

Then φ_n is (to within a constant, independent of n) an nth minimal algorithm for the problem $(S_E, BH_0^r(\Omega))$ with

$$\alpha_1\beta_{n+1}^{-(r+2m)/(2m)} \leq e(\varphi_n \leq \alpha_2\beta_{n+1}^{-(r+2m)/(2m)},$$

and so

$$e(\varphi_n) = \Theta(n^{-(r+2m)/N}) \text{ as } \qquad n \to \infty. \qquad \Box$$

<u>Remark 5.1</u>: Of course, it will usually be difficult to determine the eigenfunctions and eigenvalues of operators L arising in practice. However, the main goal of this paper will be to show that related problems have the same inherent

complexity, so that this problem does not interfere with our main goal. On the other hand, it is worth pointing out that the finite-element method (FEM) using piecewise polynomial n-dimensional subspaces of $H_0^m(\Omega)$ (the degree of the polynomials being $\lceil r \rceil + 2m - 1$) will have error $\Theta(n^{-(r+2m)/N})$ provided a quasi-uniform family of triangulations is used; the proof follows from results of [3,5,7]. Hence, the FEM is (to within a constant) a minimal error algorithm. $\square$

<u>5.2 Parabolic problems</u> $(S_{P,t_0}, BH_0^r(\Omega))$ <u>and</u> $(S_P, BH_0^r(\Omega))$.

Using separation of variables, we see that

$$S_{P,t_0} f = \Sigma_{j=1}^{\infty} \frac{1}{\beta_j}(1 - e^{-\beta_j t_0})(f, z_j)_0 z_j.$$

Using Theorem 4.2 and (4.1), it is easy to establish

<u>Theorem 5.2:</u>

(i) $$\alpha_1(1-e^{-\beta_{n+1} t_0})\beta_{n+1}^{-(r+2m)/(2m)} \le e(n, S_{P,t_0}, BH_0^r(\Omega))$$
$$\le \alpha_2(1-e^{-\beta_{n+1} t_0})\beta_{n+1}^{-(r+2m)/(2m)}$$

and

$$\alpha_1 \beta_{n+1}^{-(r+2m)/(2m)} \le e(n, S_P, BH_0^r(\Omega))$$
$$\le \alpha_2 \beta_{n+1}^{-(r+2m)/(2m)},$$

so that

$$e(n, S_{P,t_0}, BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty$$

and

$$e(n, S_P, BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty.$$

(ii) Let η_n be given by (5.1). Then η_n is (to within a constant, independent of n) an nth optimal information for the problems $(S_{P,t_0}, BH_0^r(\Omega))$ and $(S_P, BH_0^r(\Omega))$, with

$$\alpha_1(1-e^{-\beta_{n+1}t_0})\beta_{n+1}^{-(r+2m)/(2m)} \le e(\eta_n, S_{P,t_0}, BH_0^r(\Omega))$$
$$\le \alpha_2(1-e^{-\beta_{n+1}t_0})\beta_{n+1}^{-(r+2m)/(2m)}$$

and

$$\alpha_1\beta_{n+1}^{-(r+2m)/(2m)} \le e(\eta_n, S_P, BH_0^r(\Omega)) \le \alpha_2\beta_{n+1}^{-(r+2m)/(2m)},$$

so that

$$e(\eta_n, S_{P,t_0}, BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty$$

and

$$e(\eta_n, S_P, BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty.$$

(iii) Let

$$\varphi_n(\eta_n f)(t) := \Sigma_{j=1}^n \frac{1}{\beta_j}\,(1-e^{-\beta_j t})(f, z_j)_0 z_j.$$

Then $\varphi_n(\cdot)(t_0)$ is (to within a constant, independent of n) an nth minimal algorithm for $(S_{P,t_0}, BH_0^r(\Omega))$, with

$$\alpha_1(1-e^{-\beta_{n+1}t_0})\beta_{n+1}^{-(r+2m)/(2m)} \leq e(\varphi_n(\cdot)(t_0))$$
$$\leq \alpha_2(1-e^{-\beta_{n+1}t_0})\beta_{n+1}^{-(r+2m)/(2m)},$$

so that

$$e(\varphi_n(\cdot)(t_0)) = \Theta(n^{-(r+2m)/N}) \quad \text{as} \quad n \to \infty.$$

Moreover, φ_n is (to within a constant, independent of n), an nth minimal algorithm for $(S_P, BH_0^r(\Omega))$, with

$$\alpha_1\beta_{n+1}^{-(r+2m)/(2m)} \leq e(\varphi_n) \leq \alpha_2\beta_{n+1}^{-(r+2m)/(2m)},$$

so that

$$e(\varphi_n) = \Theta(n^{-(r+2m)/N}) \quad \text{as} \quad n \to \infty. \quad \square$$

<u>Remark 5.2</u>: Once again, the algorithm presented in Theorem 5.2 will usually be difficult to realize in practice. However, the formula for S_{P,t_0} may be used to derive shift theorems for u and $\frac{\partial u}{\partial t}$. Following the results in [3, pp. 139 ff] this means that the FEM is optimal (to within a constant) for the parabolic problem. (Note that the techniques in [3] are only applied to the second-order problem. However, these

techniques are related to those of Wheeler [8], who maintains that they extend to the general 2n-th order problem.) $\square$

5.3 Hyperbolic problems $(S_{H,t_0}, BH_0^r(\Omega))$ and $(S_H, BH_0^r(\Omega))$.

Once again, using separation of variables, we find

$$S_{H,t_0}f = \Sigma_{j=1}^{\infty} \frac{1}{\beta_j}(1-\cos\sqrt{\beta_j}t_0)(f,z_j)_0 z_j.$$

We first consider the problem $(S_{H,t_0}, BH_0^r(\Omega))$. Let $\rho_i = 1 - \cos\sqrt{\beta_i}t_0$ and $\lambda = \rho_i\beta_i^{-(r+2m)/(2m)}$. Let λ_{i_j} be the jth largest value in the set $\{\lambda_i\}_{i=1}^{\infty}$, i.e.,

$$\lambda_{i_1} \geq \lambda_{i_2} \geq \cdots .$$

Using Theorem 4.2 and (4.1), we find

Theorem 5.3:

(i) $\alpha_1\lambda_{i_{n+1}} \leq e(n, S_{H,t_0}, BH_0^r(\Omega)) \leq \alpha_2\lambda_{i_{n+1}}$,

so that

$$e(n, S_{H,t_0}, BH_0^r(\Omega)) = \Theta(\lambda_{i_{n+1}}) = \Theta(\rho_{i_{n+1}} i_{n+1}^{-(r+2m)/N})$$

as $n \to \infty$.

(ii) Let η_n be given by

$$\eta_n f = [(f,z_{i_1})_0 \ \dots \ (f,z_{i_n})_0]^T.$$

Then

$$\alpha_1 \lambda_{i_{n+1}} \le e(\eta_n, S_{H,t_0}, BH_0^n(\Omega)) \le \alpha_2 \lambda_{i_{n+1}},$$

so that η_n is (to within a constant, independent of n) nth optimal information for the problem $(S_{H,t_0}, BH_0^r(\Omega))$, with

$$e(\eta_n, S_{H,t_0}, BH_0^r(\Omega)) = \Theta(\lambda_{i_{n+1}}) = \Theta(\rho_{i_{n+1}} i_{n+1}^{-(r+2m)/N})$$

as $n \to \infty$.

(iii) Let

$$\varphi_n(\eta_n f) := \Sigma_{j=1}^n \frac{1}{\beta_{i_j}}(1 - \cos\sqrt{\beta_{i_j}} t_0)(f, z_{i_j})_0 z_{i_j}.$$

Then

$$\alpha_1 \lambda_{i_{n+1}} \le e(\varphi_n) \le \alpha_2 \lambda_{i_{n+1}},$$

so that φ_n is (to within a constant, independent of n) an nth minimal algorithm for the problem $(S_{H,t_0}, BH_0^r(\Omega))$, with

$$e(\varphi_n) = \Theta(\lambda_{i_{n+1}}) = \Theta(\rho_{i_{n+1}} i_{n+1}^{-(r+2m)/N}) \quad \text{as } n \to \infty. \quad \square$$

Hence, the result for this problem is not quite as nice as in the previous proboems. First, note that this result tells us that the nth minimal algorithm depends strongly on the ordering of the λ_j, which depends strongly on the value of t_0. Hence, a slight change in t_0 can make a big change in the definition of φ_n, i.e., we have a certain amount of instability.

Next, it is easy to see that

$$(5.2) \qquad e(n, S_{H,t_0}, BH_0^r(\Omega)) = O(n^{-(r+2m)/N}) \quad \text{as } n \to \infty.$$

One would like to change the "O" to a "Θ", but this is not possible. In fact, it is possible that a fortunate choice of t_0 can yield zero error for the zero algorithm $\varphi \equiv 0$, as in

Example 5.1: Let $N = 1$, $\Omega = (0,\pi)$, $m = 1$, and $Lv := -v''$. Then $z_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$ is the nth orthonormal eigenfunction of L corresponding to the eigenvalue $\beta_n = n^2$. Hence

$$S_{H,t_0}f = \Sigma_{j=1}^{\infty} \frac{1}{j}(1 - \cos jt_0)(f,z_j)_0 z_j.$$

Suppose that $T \geq 2\pi$. Then setting $t_0 = 2\pi$, we see that $\cos t_0 = 1$, and so

$$S_{H,2\pi}f = 0 \qquad \forall f \in BH_0^r(\Omega).$$

Hence, the algorithm $\varphi \equiv 0$ has zero error when $t_0 = 2\pi$. $\square$

In order to avoid these difficulties, it is more natural to consider the problem $(S_H, BH_0^r(\Omega))$. Using Theorem 4.2 and (4.1), it is easy to establish

<u>Theorem 5.4</u>:

(i) $\alpha_1\beta_{n+1}^{(r+2m)/(2m)} \le e(n,S_H,BH_0^r(\Omega)) \le \alpha_2\beta_{n+1}^{(r+2m)/(2m)}$,

so that

$$e(n,S_H,BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty.$$

(ii) Let $\mathcal{N}_n$ be given by (5.1). Then $\mathcal{N}_n$ is (to within a constant, independent of n) an nth optimal information for the problem $(S_H,BH_0^r(\Omega))$, with

$$\alpha_1\beta_{n+1}^{(r+2m)/(2m)} \le e(\mathcal{N}_n,S_H,BH_0^r(\Omega)) \le \alpha_2\beta_{n+1}^{(r+2m)/(2m)},$$

so that

$$e(\mathcal{N}_n,S_H,BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty.$$

(iii) Let

$$\varphi_n(\mathcal{N}_n f)(t) := \Sigma_{j=1}^n \frac{1}{\beta_j}(1 - \cos\sqrt{\beta_j}\,t)(f,z_j)_0 z_j.$$

Then φ_n is (to within a constant factor, independent of n), an nth minimal algorithm for the problem $(S_H,BH_0^n(\Omega))$, with

$$\alpha_1\beta_{n+1}^{(r+2m)/(2m)} \le e(\varphi_n) \le \alpha_2\beta_{n+1}^{(r+2m)/(2m)},$$

so that

$$e(\varphi_n) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty. \quad \Box$$

Remark 5.3: As before, the algorithm given by this Theorem may be difficult to implement in practice. Again, it turns out that there is an FEM which has optimal error to within a constant. (See [4, Section 5.6] for a discussion of the second-order case; the 2m-th order case is analogous.) $\square$

5.4 Summary.

We may sum up the results of this Section in

Theorem 5.5:

(i) If S is any of S_E, S_{P,t_0}, S_P, or S_{H_1}, then

$$e(n,S,BH_0^r(\Omega)) = \Theta(n^{-(r+2m)/N}) \qquad \text{as} \quad n \to \infty.$$

(ii) We have

$$e(n,S_{H,t_0},BH^r(\Omega)) = O(n^{-(r+2m)/N}) \quad \text{as} \quad n \to \infty. \quad \square$$

6. Complexity of related problems.

In this Section, we show that the ε-complexity of the problems $(S, BH_0^r(\Omega))$ with S any of S_E, S_{P,t_0}, S_P, or S_H is $\Theta(\varepsilon^{-N/(r+2m)})$, while the ε-complexity of the problem $(S_{H,t_0}, BH_0^r(\Omega))$ is $O(\varepsilon^{-N/(r+2m)})$.

We use the model of computation discussed in Chapter 5 of [6]. (Informally, we assume that linear functionals can be computed in finite time and that the cost of an arithmetic operation is unity.) The complexity of the problem $(S, BH_0^r(\Omega))$ is then defined to be

$$\mathrm{comp}(\varepsilon, S, BH_0^r(\Omega)) := \inf\{\mathrm{comp}(\varphi) : e(\varphi) < \varepsilon\},$$

$\mathrm{comp}(\varphi)$ denoting the complexity of the algorithm φ. We are interested in optimal-complexity algorithms φ_ε such that

$$e(\varphi_\varepsilon) < \varepsilon \quad \text{and} \quad \mathrm{comp}(\varphi_\varepsilon) = \mathrm{comp}(\varepsilon, S, BH_E^r(\Omega)),$$

but we will settle for almost-optimal-complexity algorithms φ_ε such that

$$e(\varphi_\varepsilon) < \varepsilon \text{ and } \mathrm{comp}(\varphi_\varepsilon) = \Theta(\mathrm{comp}(\varepsilon, S, BH_0^r(\Omega))$$
$$\text{as } \varepsilon \to 0.$$

We need the following result from Chapter 5 of [5]:

<u>Lemma 6.1</u>: Define, for $\varepsilon > 0$, the ε-<u>cardinality</u> <u>number</u>

$$m(\varepsilon,S,BH_0^r(\Omega)) := \inf\{n \geq 0\colon e(n,S,BH_0^r(\Omega)) < \varepsilon\}.$$

Then the following hold:

(i) $\mathrm{comp}(\varepsilon,S,BH_0^r(\Omega)) = \Theta(m(\varepsilon,S,BH_0^r(\Omega)))$.

(ii) Let $n = \lceil m(\varepsilon,S,BH_0^r(\Omega))\rceil$ and let φ_ε be the nth minimal algorithm for the problem $(S,BH_0^r(\Omega))$. Then φ_ε is an almost-optimal complexity algorithm. $\square$

From Theorem 5.5, we have

$$m(\varepsilon,S,BH_0^r(\Omega)) = \Theta(\varepsilon^{-N/(r+2n)}) \text{ as } \varepsilon \to 0$$

for S any of S_E, S_{P,t_0}, S_P, or S_H, while

$$m(\varepsilon,S_{H,t_0},BH_0^r(\Omega)) = O(\varepsilon^{-N/(r+2m)}) \quad \text{as} \quad \varepsilon \to 0.$$

Hence, Lemma 6.1 yields

<u>Theorem 6.1:</u>

(i) Let S be one of S_E, S_{P,t_0}, S_P, or S_H. Then

$$\mathrm{comp}(\varepsilon,S,BH_0^r(\Omega)) = \Theta(\varepsilon^{-N/(r+2m)}) \quad \text{as} \quad \varepsilon \to 0.$$

(ii) $\mathrm{comp}(\varepsilon,S,BH_0^r(\Omega)) = O(\varepsilon^{-N/(r+2m)})$ as $\varepsilon \to 0$. $\square$

So the problems S_E, S_{P,t_0}, S_P, and S_H have the same complexity, which is no better than the complexity of the problem S_{H,t_0}.

7. Summary, extensions, and open questions.

We have shown that it is not generally true that parabolic problems are significantly easier than elliptic problems, and that elliptic problems are significantly easier than hyperbolic problems. In fact, we have shown that (under somewhat general circumstances) elliptic problems, parabolic problems, and hyperbolic problems solved over a time-interval all have the same complexity, while the complexity of hyperbolic problems solved to a particular time is not greater than that of the other problems.

This leads one to ask whether the result on the complexity of partial differential equations noted in [6] is an isolated result, or an example of a more general situation. We feel that the latter may be the case. Indeed, consider the parabolic problem

$$\partial_t u + Lu = 0 \quad \text{in} \quad \Omega \times (0,T)$$

$$\partial_\nu^j u = 0 \quad (0 \le j \le m-1) \quad \text{on} \quad \partial\Omega \times [0,T]$$

$$u(\cdot,0) = f \in H_0^r(\Omega) \quad \text{on} \quad \Omega$$

solved out to time $t = t_0$. Then the nth minimal error is $\Theta(e^{-n^{2m/N}t_0}n^{-r/N})$ as $n \to \infty$, and the ε-complexity is $\Theta((\frac{1}{t_0}\ln\frac{1}{\varepsilon})^{N/2m})$ as $\varepsilon \to 0$. (Note that changing the manner in which the data is used, i.e., initial data vs. a forcing term, drastically alters the complexity. We have also observed this for hyperbolic problems.) Unfortunately, we have not succeeded in generalizing the results in [6] for related elliptic and hyperbolic problems.

Finally, note that we required that the data lie in $H_0^r(\Omega)$ for various technical reasons. However, it is more usual to assume that the data is in $H^r(\Omega)$ (i.e., no boundary conditions are required for the data), especially for elliptic problems (see e.g. [5, Theorem 8.5] or the results of [7]). Do the results of this paper still hold when the data is in $H^r(\Omega)$, rather than $H_0^r(\Omega)$?

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