On a Spectral Bound for Congruence Subgroup Families in $SL_3(Z)$

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ABSTRACT

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Spectral bounds on Maass forms of congruence families in algebraic groups are important ingredients to proving almost prime results for these groups. Extending the work of Gamburd [Gamburd, 2002] and Magee [Magee, 2013], we produce a condition under which such a bound exists in congruence subgroup families of $SL_3(\mathbb{Z})$, uniformly and even when these groups are thin, i.e. of infinite index. The condition is analogous to the cusp and collar lemmas in Gamburd's work and is expected to hold for families whose Hausdorff dimension of the limit set is large enough.

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Chapter 1

Introduction

To state our result we introduce some basic notation. Let

$$\mathfrak{h}^3 = GL_3^+(\mathbb{R})/(\mathbb{R}_{>0}^\times SO_3(\mathbb{R}))$$

be the generalized upper half plane for GL_3. Via the Iwasawa decomposition, we express any z in $\mathfrak{h}^3$ as xy where

$$x = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for real numbers $y_1 > 0$, $y_2 > 0$, x_1, x_2, and x_3.

Let Λ be a finitely generated, Zariski dense subgroup of $SL_3(\mathbb{Z})$. Let Φ be a Λ-Maass form with complex spectral parameters $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ such that $\Re(\nu_1), \Re(\nu_2) \geq 1/3$, meaning that for any Δ in the center of the universal enveloping algebra for $\mathfrak{gl}_3(\mathbb{R})$, denoted $Z(\mathcal{U}(\mathfrak{gl}_3(\mathbb{R})))$, the function

$$I_\nu(z) = y_1^{\nu_1+2\nu_2} y_2^{2\nu_1+\nu_2}$$

has the same eigenvalue with respect to Δ, $\lambda(\Delta)$, as Φ. The algebra $Z(\mathcal{U}(\mathfrak{gl}_3(\mathbb{R}))))$ is generated by two Casimir elements, Δ_2 and Δ_3, of degree 2 and 3 respectively, whose action on $\mathcal{L}^2(\Lambda\backslash\mathfrak{h}^3)$ is defined explicitly in chapter 2. The spectral parameters ν_1 and ν_2 are related to the eigenvalues $\lambda(\Delta_2)$ and $\lambda(\Delta_3)$ by

$$\lambda(\Delta_2) = 3\nu_1^2 + 3\nu_1\nu_2 + 3\nu_2^2 - 3\nu_1 - 3\nu_2,$$

$$\lambda(\Delta_3) = 2\nu_1^3 + 3\nu_1^2\nu_2 - 3\nu_1\nu_2^2 - 2\nu_2^3 + 3\nu_1\nu_2 + 6\nu_2^2 - 2\nu_1 - 4\nu_2.$$

We would like to study Maass forms over congruence families of subgroups in $SL_3(\mathbb{Z})$. Let $\Gamma(p)$ be the pth principal congruence subgroup of $SL_3(\mathbb{Z})$, i.e. those matrices that are congruent to the identity modulo a prime p. We examine the family $\Lambda(p) = \Lambda \cap \Gamma(p)$ as p varies. We would like to bound the spectral parameters of Λ-newforms of level p for large primes p, where newforms of level p are $\Lambda(p)$-Maass forms that are not lifts of Λ-Maass forms.

The main result of this paper is a uniform bound on the real part of the sum of the spectral parameters for groups satisfying the following property.

Definition 1. *Let Λ be a Zariski dense subgroup of $SL_3(\mathbb{Z})$. The subgroup Λ is said to have the* $\mathcal{L}^2$-concentration property *when there is a compact subset $\mathcal{K}$ of $\Lambda\backslash\mathfrak{h}^3$ such that, if $\mathcal{K}_p$ is the preimage of $\mathcal{K}$ under the natural projection $\Lambda(p)\backslash\mathfrak{h}^n \to \Lambda\backslash\mathfrak{h}^n$; ν_0 is any real number greater than $1/3$; and Φ is any $\Lambda(p)$-Maass form with spectral parameters $\nu_1, \nu_2 > \nu_0$, then*

$$\int_{\mathcal{K}_p} |\Phi(z)|^2 d^*z >>_{\nu_0} \int_{\Lambda(p)\backslash\mathfrak{h}^n} |\Phi(z)|^2 d^*z$$

where the implied constant is independent of p, ν_1, ν_2, and Φ.

We anticipate that this property will hold for subgroups Λ that are *geometrically finite*, having a fundamental domain in $\mathfrak{h}^3$ with finitely many bounding geodesic sides, and that are "large enough". To determine if Λ is large enough, we examine the *limit set* of Λ, which is all limit points to the orbit of the identity under the action of Λ inside the compactification of $\mathfrak{h}^3$,

$$\overline{\mathfrak{h}^3} := M_n^{\geq 0}(\mathbb{R})/(\mathbb{R}_{>0}^{\times} SO_n(\mathbb{R})).$$

Here $M_n^{\geq 0}$ is the space of n by n real matrices with nonnegative determinant. We think of Λ as being large if the Hausdorff dimension of the limit set of Λ is close to the full dimension of the boundary of $\mathfrak{h}^3$,

$$\partial\mathfrak{h}^3 := \overline{\mathfrak{h}^3} \setminus \mathfrak{h}^3,$$

which has dimension 4. The Hausdorff dimension of the limit set is exactly 4 if and only if Λ is finite index.

We expect the $\mathcal{L}^2$-concentration property to hold when the Hausdorff dimension of the limit set is sufficiently large, as made explicit in the following conjecture.

Conjecture 1. *Let Λ be a Zariski dense, geometrically finite subgroup of $SL_3(\mathbb{Z})$. There exists some $\delta < 4$ such that if the Hausdorff dimension of the limit set of Λ in $\mathfrak{h}^3$ is at least δ then Λ has the $\mathcal{L}^2$-concentration property.*

The main theorem of this thesis bounds the spectral parameters for newforms as follows.

Theorem 1. *Let Λ be a Zariski dense, finitely-generated subgroup of $SL_3(\mathbb{Z})$ satisfying the $\mathcal{L}^2$-concentration property. For p sufficiently large, if Φ is a Λ-newform of level p having spectral parameters (ν_1, ν_2), then*

$$\Re(\nu_1) + \Re(\nu_2) \leq 7/6.$$

Ultimately, this should help establish a "spectral gap" between the largest possible value of $\Re(\nu_1) + \Re(\nu_2)$ among $\Lambda(p)$-Maass forms and the next largest. Note that spectral parameters $\nu_1 = \nu_2 = 2/3$ correspond to eigenvalue 0 with respect to Δ_2 and Δ_3, so to obtain a spectral gap for finite index Λ it is necessary to show that any Λ-newforms have $\Re(\nu_1 + \nu_2) \leq 4/3 - \epsilon$ for some $\epsilon > 0$. For infinite index Λ in $SL_3(\mathbb{Z})$, the constant function is not in $\mathcal{L}^2(\Lambda\backslash\mathfrak{h}^3)$ and thus will not be a Λ-Maass form (i.e. 0 will not be a part of the spectrum), so this will not be sufficient to show a spectral gap. However, if Λ is large enough we anticipate that there will be Λ-Maass forms with spectral parameters ν having $\Re(\nu_1) + \Re(\nu_2)$ close to 4/3.

Conjecture 2. *For any $\epsilon > 0$ there exists a $\delta < 4$ such that for any geometrically finite, Zariski dense subgroup of $SL_3(\mathbb{Z})$ with Hausdorff dimension of the limit set at least δ, there is a Λ-Maass form with spectral parameters (ν_1, ν_2) such that $\Re(\nu_1) + \Re(\nu_2) > 4/3 - \epsilon$.*

Thus a uniform bound on $\Re(\nu_1) + \Re(\nu_2)$ away from 4/3 should still yield a spectral gap for groups with Hausdorff dimension of the limit set close enough to 4.

Theorem 1 is a direct generalization Alexander Gamburd's analogous result for the upper half plane, or equivalently $GL_2^+(\mathbb{R})/(\mathbb{R}^\times_{>0} SO_2(\mathbb{R}))$. We can restate his theorem in the language of this paper as follows.

Theorem 2. (Gamburd, 2002) *Let Λ be a Zariski dense, geometrically finite subgroup of $SL_2(\mathbb{Z})$ such that the Hausdorff dimension of its limit set is greater than $5/6$. For p sufficiently large, if Φ is a Λ-newform of level p having spectral parameter ν (i.e. eigenvalue $\nu(1-\nu)$ with respect to the Laplacian and $\Re(\nu) \geq 1/2$) then*

$$\nu \leq 5/6.$$

In his paper, Gamburd shows that any Λ with the preceding properties must satisfy the analogue of the $\mathcal{L}^2$-concentration property for the upper-half plane, and so this property needs not be an explicit premise of the theorem. For subgroups of $SL_2(\mathbb{Z})$ with Hausdorff dimension of the limit set $\delta > 1/2$, it is also known that the greatest spectral parameter is δ (corresponding to the smallest eigenvalue $\delta(1-\delta)$) [Patterson, 1975]; and that the parameters of Maass forms for a fixed Λ are discrete, lying in the range (1/2,1] [Patterson, 1975 1976 1976]. Thus for $\delta > 5/6$, this result guarantees a "spectral gap", i.e. a positive infimum difference between the greatest spectral parameter of $\Lambda(p)$-Maass forms and all others over all primes p.

Gaps like this can be used to show "almost prime" results for subgroups of $SL_2(\mathbb{Z})$ as in [Bourgain *et al.*, 2010]. For example, they give the following theorem.

Theorem 3. (J. Bourgain, A. Gamburd, P. Sarnak, 2010) *Let Λ be a Zariski dense, geometrically finite subgroup of $SL_2(\mathbb{Z})$. Let v be a primitive vector in $\mathbb{Z}^2$. Let f be a polynomial in $\mathbb{Q}[x_1, x_2]$ such that f has integral valuation on all of Λv. For all positive integers r, let*

$$\mathcal{O}_r := \{w \in \Lambda v : f(w) \text{ has at most } r \text{ prime factors}\}.$$

Then for some r, $\mathcal{O}_r$ is Zariski dense in $\mathbb{Z}^2$.

This result does not have the stipulation that the Hausdorff dimension of the limit set is greater than 5/6, because the existence of a gap is proven more generally for subgroups of $SL_2(\mathbb{Z})$ within the paper. The spectral bound in [Bourgain *et al.*, 2010] is not uniform as Λ varies however. The uniformity in Theorem 2 has been exploited by Bourgain and Kontorovich to construct infinite index subgroups of $SL_2(\mathbb{Z})$ for which the r in Theorem 3 can be reduced to 1 in the situation that f is linear with integer coefficients [Bourgain and Kontorovich, 2010]. This shows, for example, that there are infinitely many primes appearing in a fixed entry across all matrices in such Λ. In [Bourgain *et al.*, 2010], Bourgain, Gamburd, and Sarnak show that the analogue of Theorem 3 holds for $SL_n(\mathbb{Z})$ with $n > 1$ contingent on the existence of a spectral gap. One could use Theorem 1 to create a condition for when this holds given $n = 3$. The uniformity of Theorem 1 may also be valuable in constructing subgroups of $SL_3(\mathbb{Z})$ and polynomials f for which r can be made small in this generalization.

Theorem 2 has already been generalized to subgroups of isometries acting on n-dimensional hyperbolic space by Michael Magee following a similar technique to Gamburd's [Magee, 2013]. But the analogous result for groups of higher rank, in particular for subgroups of $SL_n(\mathbb{Z})$ with $n > 2$, has not yet been found. We anticipate that similar results to Theorems 2 and 3 and the groups constructed in [Bourgain and Kontorovich, 2010] should exist for infinite index subgroups Λ of $SL_3(\mathbb{Z})$, and the main theorem of this paper is a step toward all of these results.

In chapter 2, we establish the terminology we will need for this paper and bound the dimension of newforms of level p for p sufficiently large. Then in chapter 3 we construct a family of automorphic kernels, $K_{p,T}$, whose traces we will take over a compact subset of $\Lambda(p)\backslash\mathfrak{h}^3$. In chapter 4 we show that the discrete part of the spectral trace of $K_{p,T}$ is bounded above by its geometric trace, establishing a pretrace inequality. In chapters 5 and 6, we estimate two important quantities associated with $\Lambda(p)$ that can be inserted into the pretrace inequality to help us bound the spectral parameters of newforms. These are the Selberg transform of a point-pair invariant used to construct $K_{p,T}$ and a lattice point count for the pth principal congruence subgroups of $SL_3(\mathbb{Z})$. In chapter 7 we bound the geometric side of the pretrace inequality. Then in chapter 8 we introduce the $\mathcal{L}^2$-concentration property, which in chapter 9 we use with the pretrace inequality to show that groups with this property satisfy the spectral bound in Theorem 1.

Part I

Spectral Bound

Chapter 2

Newforms for subgroups of $SL_n(\mathbb{Z})$

We are interested in Maass forms on the generalized upper half-space,

$$\mathfrak{h}^n = GL_n^+(\mathbb{R})/(\mathbb{R}_{>0}^\times SO_n(\mathbb{R})).$$

The Iwasawa decomposition for $GL_n^+(\mathbb{R})$ decomposes an arbitrary element g of $GL_n^+(\mathbb{R})$ uniquely as a product $z \cdot d \cdot \kappa$ for d in $\mathbb{R}_{>0}^\times$, κ in $SO_n(\mathbb{R})$, and $z = xy$, where x is unipotent and y is diagonal with positive entries and bottom rightmost entry 1. Following the notation of [Goldfeld, 2006], we will use the z from this decomposition as a system of representatives for $\mathfrak{h}^3$ and give standard coordinates to $\mathfrak{h}^n$ by labelling the entries of x and y as

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ 0 & 1 & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & x_{n-1,n} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \dots y_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & y_1 \dots y_{n-2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y_1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

for real numbers $y_i > 0$ and $x_{i,j}$. When working with an arbitrary z in $\mathfrak{h}^n$, we will assume it has the decomposition $z = xy$ with coordinates as above. For the special case of $n = 3$, which will be our primary focus, we take

$$x = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}$$

for convenience.

The group $GL_n^+(\mathbb{R})$ has a natural action on $\mathfrak{h}^n$ via the Iwasawa decomposition. For $\gamma \in GL_n^+(\mathbb{R})$ and z in $\mathfrak{h}^n$, $\gamma.z$ is defined to be the unique element of $\mathfrak{h}^n$ for which there is a (unique) $\kappa(\gamma, z)$ in $SO_n(\mathbb{R})$ and a d in $\mathbb{R}^\times$ such that

$$\gamma z = (\gamma.z) \cdot d \cdot \kappa(\gamma, z).$$

Usually we will be focused on the action of $SL_n(\mathbb{R})$, in which case d must be $\sqrt[n]{\det(z(\gamma.z)^{-1})}$ yielding a slightly more explicit formula.

$$\gamma z = (\gamma.z) \cdot \kappa(\gamma, z) \sqrt[n]{\det(z(\gamma.z)^{-1})} \tag{2.2}$$

The space $\mathfrak{h}^n$ is endowed with a measure μ, which is invariant under the action of $\Gamma = SL_n(\mathbb{R})$ and is given by

$$dz^* = \prod_{j=1}^{n-1} y_j^{\frac{-n^2+j^2+n-j-2}{2}} \prod_{j=1}^{n-1} dy_j \prod_{1 \le j < k \le n} dx_{j,k}.$$

The action of $SL_n(\mathbb{R})$ on $\mathfrak{h}^n$ serves as a generalization of the classical action of $SL_2(\mathbb{Z})$ on the upper half-plane, $\mathfrak{h}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z := \frac{az+b}{cz+d},$$

where $\mathfrak{h}$ is identified with $\mathfrak{h}^2$ via the map

$$x + iy \mapsto \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}.$$

Let Λ be a finitely generated, Zariski dense subgroup of Γ. Let f be a Λ-automorphic function, i.e. a smooth $f : \Lambda\backslash\mathfrak{h}^n \to \mathbb{C}$. A Λ-automorphic function can be lifted to a function $\widetilde{f} : GL_n^+(\mathbb{R}) \to \mathbb{C}^r$ by letting

$$\widetilde{f}(g) = f(g.I_n)$$

The universal enveloping algebra of $\mathfrak{gl}_n(\mathbb{R})$, denoted $\mathcal{U}(\mathfrak{gl}_n(\mathbb{R}))$, acts by differential operators on smooth functions $\widetilde{f} : GL_n^+(\mathbb{R}) \to \mathbb{C}^r$ as follows. For all $h \in \mathfrak{gl}_n(\mathbb{R})$ and $g \in GL_n^+(\mathbb{R})$,

$$(h.\widetilde{f})(g) = \lim_{t \to 0} \frac{\widetilde{f}(g \cdot e^{th}) - \widetilde{f}(g)}{t}.$$

The action of an arbitrary D in $\mathcal{U}(\mathfrak{gl}_n(\mathbb{R}))$ follows by extension. When D is in the center of the universal enveloping algebra, $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$, and is applied to the lift of a Λ-automorphic function, the resulting function is also a lift of a Λ-automorphic function. In this way, $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$ acts on Λ-automorphic functions. This action commutes with the left-regular action of $GL_n(\mathbb{R})$. Further, the elements of $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$ act self-adjointly on $\mathcal{L}^2(\Lambda\backslash\mathfrak{h}^n)$. Let $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))^*$ denote the dual space to $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$.

We are now in position to define a Λ-Maass form.

Definition 2. *A Λ-Maass form is a $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$*-eigenfunction, *$\Phi$, in $\mathcal{L}^2(\Lambda\backslash\mathfrak{h}^n)$, i.e. it has an* eigenvalue *λ in $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))^*$ such that for all Δ in $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$,*

$$\Delta.\Phi = \lambda(\Delta)\Phi.$$

Rather than parameterizing these forms by their $Z(\mathcal{U}(gl_n(\mathbb{R})))$-eigenvalue, it will be convenient to use a reparameterization by "spectral parameters".

Definition 3. *Let $\nu \in \mathbb{C}^{n-1}$. Consider the function $I_\nu : \mathfrak{h}^n \to \mathbb{C}$ given by*

$$I_\nu(z) := \prod_{j=1}^{n-1}\prod_{k=1}^{n-1} y_j^{b_{j,k}\nu_k}.$$

for

$$b_{j,k} := \begin{cases} jk & \text{if } j+k \leq n, \\ (n-j)(n-k) & \text{if } j+k \geq n. \end{cases}$$

As shown in [Goldfeld, 2006], I_ν is a $Z(\mathcal{U}(gl_n(\mathbb{R})))$-eigenfunction in the sense that there is a linear functional λ_ν in $Z(\mathcal{U}(gl_n(\mathbb{R})))^*$ such that for all differential operators Δ in $Z(\mathcal{U}(gl_n(\mathbb{R})))$,

$$\Delta.I_\nu = \lambda_\nu(\Delta)I_\nu.$$

If $\Phi : \Lambda\backslash\mathfrak{h}^n \to \mathbb{C}$ is a Λ-Maass form, then we say that it has spectral parameters *ν when for all differential operators Δ in $Z(\mathcal{U}(gl_n(\mathbb{R})))$*

$$\Delta.\Phi = \lambda_\nu(\Delta)\Phi$$

where λ_ν is still the eigenvalue for I_ν.

In the case of $n = 3$, we know that the $\mathbb{R}$-algebra $Z(\mathcal{U}(gl_3(\mathbb{R})))$ is generated by its two Casimir operators, which can be written explicitly as differential operators on $L^2(\mathfrak{h}^3)$ [Goldfeld, 2006].

$$\begin{aligned}
\Delta_2 := {} & y_1^2 \frac{\partial^2}{\partial y_1^2} + y_2^2 \frac{\partial^2}{\partial y_2} - y_1 y_2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1^2 \left(x_{1,2}^2 + y_{1,2}^2\right) \frac{\partial^2}{\partial x_{1,3}^2} \\
& + y_1^2 \frac{\partial^2}{\partial x_{2,3}^2} + y_2^2 \frac{\partial^2}{\partial x_1, 2^2} + 2 y_1^2 x_{1,2} \frac{\partial^2}{\partial x_{2,3} \partial x_{1,3}}, \\
\Delta_3 := {} & -y_1^2 y_2 \frac{\partial^3}{\partial y_1^2 \partial y_2} + y_1 y_2^2 \frac{\partial^3}{\partial y_1 \partial y_2^2} - y_1^3 y_2^2 \frac{\partial^3}{\partial x_{1,3}^2 \partial y_1} + y_1 y_2^2 \frac{\partial^3}{\partial x_{1,2}^2 \partial y_1} \\
& - 2 y_1^2 y_2 x_{1,2} \frac{\partial^3}{\partial x_{2,3} \partial x_{1,3} \partial y_2} + \left(-x_{1,2}^2 + y_2^2\right) y_1^2 y_2 \frac{\partial^3}{\partial x_{1,3}^2 y_2} \\
& - y_1^2 y_2 \frac{\partial^3}{\partial x_{2,3}^2 \partial y_2} + 2 y_1^2 y_2^2 \frac{\partial^3}{\partial x_{2,3} \partial x_{1,2} \partial x_{1,3}} + 2 y_1^2 y_2^2 x_{1,2} \frac{\partial^3}{\partial x_{1,2} \partial x_{1,3}^2} \\
& + y_1^2 \frac{\partial^2}{\partial y_1^2} - y_2^2 \frac{\partial^2}{\partial y_2^2} + 2 y_1^2 x_{1,2} \frac{\partial^2}{\partial x_{2,3} \partial x_{1,3}} \\
& + \left(x_{1,2}^2 + y_2^2\right) y_1^2 \frac{\partial^2}{\partial x_{1,3}^2} + y_1^2 \frac{\partial^2}{\partial x_{2,3}^2} - y_2^2 \frac{\partial^2}{\partial x_{1,2}^2}
\end{aligned}$$

Also for $n = 3$,

$$I_\nu = y_1^{\nu_1 + 2\nu_2} y_2^{2\nu_1 + \nu_2}. \tag{2.3}$$

We can explicitly compute its eigenvalues with Mathematica (short script in appendix).

$$\begin{aligned}
\lambda_\nu(\Delta_2) &= 3\nu_1^2 + 3\nu_1\nu_2 + 3\nu_2^2 - 3\nu_1 - 3\nu_2, \\
\lambda_\nu(\Delta_3) &= 2\nu_1^3 + 3\nu_1^2\nu_2 - 3\nu_1\nu_2^2 - 2\nu_2^3 + 3\nu_1\nu_2 + 6\nu_2^2 - 2\nu_1 - 4\nu_2,
\end{aligned}$$

which completely determine λ_ν. This allows us to explicitly move between eigenvalues and the corresponding spectral parameters. If we set

$$\begin{aligned}
\lambda_2 &:= \lambda_\nu(\Delta_2), \\
\lambda_3 &:= \lambda_\nu(\Delta_3),
\end{aligned}$$

then a convenient way of relating λ_2 and λ_3 to $\nu = (\nu_1, \nu_2)$ is by setting

$$\nu_3 := 1 - \nu_1 - \nu_2$$

and using the following polynomial.

$$\begin{aligned}
& \left(z - \nu_1\left(3\nu_1 - 2\right)\right)\left(z - \nu_2\left(3\nu_2 - 2\right)\right)\left(z - \nu_3\left(3\nu_3 - 2\right)\right) \\
&= z^3 - (1 + \lambda_2) z^2 + \left(\frac{\lambda_2^2}{4} + \frac{\lambda_2}{3}\right) z - \frac{1}{2}\lambda_2^3 - 3\lambda_3^2 + 9\lambda_2\lambda_3 - 6\lambda_2^2
\end{aligned}$$

Returning to arbitrary $n > 1$, we would like to study the Maass forms of not just one group, but of a congruence family of groups. For any prime p, let $\Gamma(p)$ be the pth principal congruence subgroup of $SL_n(\mathbb{Z})$, i.e. those matrices that are congruent to the identity matrix modulo p. We define the pth principal congruence subgroup of Λ to be

$$\Lambda(p) := \Lambda \cap \Gamma(p).$$

We want to study the spectral parameters of $\Lambda(p)$-Maass forms as p goes to infinity. Note that any Λ-Maass form lifts to a $\Lambda(p)$-Maass form. Any $\Lambda(p)$-Maass form that is not such a lift we will call a Λ-*newform of level* p. This varies slightly from standard notation where level usually refers to congruence families based on $\Gamma_0(p)$ rather than $\Gamma(p)$, but the idea is the same.

The vector space of all $\Lambda(p)$-Maass forms is a representation for $\Lambda/\Lambda(p)$, because $\Lambda(p)$ is a normal subgroup of Λ and because the action of $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$ commutes with the left-regular action of $GL_n(\mathbb{R})$. The resulting action of $\Lambda/\Lambda(p)$ preserves the eigenvalue of these forms, and so for each vector of spectral parameters ν of a $\Lambda(p)$-Maass form, there is a representation of $\Lambda/\Lambda(p)$ acting on all forms with these parameters. Let us call this space $\mathcal{M}(\Lambda, p, \nu)$. Now we have an equivalent way of interpreting a Λ-newform of level p and spectral parameters ν: as a $\Lambda(p)$-Maass form that generates a non-trivial subrepresentation of $\mathcal{M}(\Lambda, p, \nu)$.

The group $\Lambda/\Lambda(p)$ naturally embeds into $SL_n(\mathbb{F}_p)$, and by a result of Matthews, Vaserstein, and Weisfeiler, it is in fact an isomorphism for p sufficiently large [Matthews *et al.*, 1984]. Let Φ be a Λ-newform of level p and type ν for p sufficiently large, and let (π, V) be the representation of $SL_n(\mathbb{F}_p)$ that it generates. If we let $\hat{\pi}$ be the composition of π with the natural map $GL(V) \to \mathbb{P}GL(V)$, then the center of $SL_n(\mathbb{F}_p)$ is contained in the kernel of $\hat{\pi}$, so we can interpret $\hat{\pi}$ as a nontrivial representation of $\mathbb{P}SL_n(\mathbb{F}_p)$. The minimal dimension of such a representation (among much more related information) is computed by Tiep and Zalesskii and is summarized in the following theorem.

Theorem 4. (Tiep and Zalesskii, 1996) *Let $p > 3$ be a prime. If V is a non-trivial representation of $\mathcal{P}SL_n(\mathbb{F}_p)$ then*

$$\dim V \geq \begin{cases} \frac{p-1}{2}, & \text{if } n = 2 \\ \frac{p^n - p}{p-1}, & \text{if } n \geq 3 \end{cases}$$

As a result, we have the following lemma about newforms.

Lemma 1. *Let $n > 2$, p be a sufficient large prime, ν be the spectral parameters of a Λ-newform of level p, and $\mathcal{M}(\Lambda, p, \nu)$ be the space of $\Lambda(p)$-Maass forms of type ν and level p. Then*

$$\dim \mathcal{M}(\Lambda, p, \nu) \geq \frac{p^n - p}{p - 1}.$$

Our goal will be to show that for large primes p the type ν of any Λ-newform of level p has to be sufficiently small to compensate for the space $\mathcal{M}(\Lambda, p, \nu)$ being large as shown above. We do this by establishing a trace inequality that will let us compare the dimension of $\mathcal{M}(\Lambda, p, \lambda)$ to various other key quantities associated with Λ and its newforms. In the following chapter, we construct a useful family of automorphic kernels whose traces will be used for this inequality.

Chapter 3

Constructing an Automorphic Kernel

There is a natural inner product on the space of n by n real matrices, $M_n(\mathbb{R})$, given by

$$\langle A, B\rangle := \mathrm{Tr}(AB^{\mathrm{tr}}) \text{ for all } A, B \in M_n(\mathbb{R}),$$

which induces a norm on these matrices,

$$||A|| := \sqrt{\langle A, A\rangle} = \sqrt{\sum_{i,j=1}^{n} a_{i,j}^2},$$

where $a_{i,j}$ is the entry in the ith row and jth column of A. This is simply the norm induced by embedding $M_n(\mathbb{R})$ in $\mathbb{R}^{n^2}$ by its entries. It is easy to verify via Cauchy Schwartz that

$$||AB|| \leq ||A||||B|| \text{ for all } A, B \in M_n(\mathbb{R}) \tag{3.1}$$

This norm has left and right translation invariance under the action of $O_n(\mathbb{R})$. This will make it ideal for constructing a *point-pair invariant* on $\mathfrak{h}^n$, i.e. a function $k : \mathfrak{h}^n \times \mathfrak{h}^n \to \mathbb{C}$ such that

$$k(\gamma.z, \gamma.w) = k(z, w) \text{ for all } z, w \in \mathfrak{h}^n, \gamma \in SL_n(\mathbb{R})$$

The significance of point-pair invariance for our purposes is tied to the following special case of a theorem due to Selberg.

Theorem 5. (Selberg, 1963) *Let* $k : \mathfrak{h}^n \times \mathfrak{h}^n \to \mathbb{C}$ *be a point-pair invariant. Then there exists a function* $h : Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R}))) \to \mathbb{C}$*, the* Selberg transform *of* k*, such that for all* $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$*-eigenfunctions* $\Phi : \mathfrak{h}^n \to \mathbb{C}$*,*

$$\int_{\mathfrak{h}_n} k(z,w)\Phi(w)d^*w := h(\nu)\Phi(z)$$

where ν *is the vector of spectral parameters for* Φ*.*

The Selberg transform $k \mapsto h$ is an algebra homomorphism taking convolution over $\mathfrak{h}^n$ to multiplication.

We begin with an auxiliary point-pair invariant given by the following formula.

$$u(z,w) := \det\left(\frac{w}{z}\right)^{2/n} ||w^{-1}z||^2 + \det\left(\frac{z}{w}\right)^{2/n} ||z^{-1}w||^2$$

The above function is well defined for w and z in $\mathfrak{h}^n$ because of the $O_n(\mathbb{R})$ biinvariance of the norm, and it gives us a notion of "multiplicative distance" on $\mathfrak{h}^n$. In fact, for $n = 2$, the usual hyperbolic distance on the upper-half plane is given by $\operatorname{arccosh}(u(z,w)/4)$. For general n, u is symmetric; its image is $[2n, \infty)$ attaining $2n$ exactly on the diagonal; and it satisfies a multiplicative triangle inequality (following from inequality 3.1),

$$u(z,w) \leq u(z,v)u(v,w) \text{ for all } z,w,v \in \mathfrak{h}^n. \tag{3.2}$$

We will construct a family of test functions from u, which we can vary to gain information from the pretrace inequality. Let $T > 2n$. We will define a cut-off function based on u called $\hat{k}_T : \mathfrak{h}^n \times \mathfrak{h}^n \to \mathbb{C}$.

$$\hat{k}_T(z,w) := \begin{cases} 1, & \text{if } u(z,w) \leq T \\ 0, & \text{if } u(z,w) > T \end{cases} \text{ for all } z,w \in \mathfrak{h}^n \tag{3.3}$$

The function $\hat{k}_T(z,w)$ inherits point-pair invariance from u and is compactly supported in w when z is fixed (and vice versa since it is symmetric). Let us call the support of $k_T(z,\cdot)$, $B(z,T)$, and its volume V_T, which is independent of z.

The point-pair invariant $\hat{k}_T$ will not be satisfactory for our purposes for two related reasons: its Selberg transform, $\hat{h}_T$, as described in theorem 5, will not be non-negative, and the integral operator we will construct from it later in this chapter, will not be non-negative either. To solve these problems, we use the convolution of $\hat{k}_T$ with its conjugate over $\mathfrak{h}^n$, i.e.

$$k_T(z,w) := \left(\hat{k}_T * \overline{\hat{k}_T}\right)(z,w) = \int_{\mathfrak{h}_n} \hat{k}_T(z,v)\hat{k}_T(v,w)d^*v \text{ for all } z,w \in \mathfrak{h}^n$$

From 3.2, we can see that the support of $k_T(z,\cdot)$ is contained in $B(z,T^2)$, and is thus compactly supported as well. Simply noting that $\hat{k}_T \leq 1$, we also see that k_T is bounded by V_T. If we think of u loosely as a distance, then $k_T(z,w)$ is the volume of the intersection of two balls of radius T, one centered at z and the other at w. We compute more precise bounds on k_T in chapter 7. If we let h_T be the Selberg transform of k_T, then because the Selberg transform is an algebra homomorphism,

$$h_T = \hat{h}_T\overline{\hat{h}_T} = |\hat{h}_T|^2,$$

and as a result, h_T will be non-negative, as desired.

From this test function, we can create a $\Lambda(p)$-autmorphic kernel, $K_{p,T} : \Lambda(p)\backslash\mathfrak{h}^n \times \Lambda(p)\backslash\mathfrak{h}^n \to \mathbb{C}$, given by

$$K_{p,T}(z,w) := \sum_{\gamma\in\Lambda(p)} k_T(\gamma.z,w) \text{ for all } z,w \in \Lambda(p)\backslash\mathfrak{h}^n.$$

The sum in the above construction is actually finite for a fixed z and w, since $SL_n(\mathbb{Z})$ acts discretely on $\mathfrak{h}^n$ with finite kernel, and k_T is compactly supported when one of its arguments is fixed. Likewise, $K_{p,T}$ is symmetric and inherits the property that it is compactly supported when one of its arguments is fixed.

We will also want to know that $K_{p,T}(I,\cdot)$ is bounded. Since k_T is bounded it is enough to show that there is an upper bound (independent of w) on the number of γ in $\Lambda(p)$ such that $k_T(\gamma.I,w)$ is non-zero. If γ and γ' are two such elements of $\Lambda(p)$ then we know from the support of k_T that

$$u(\gamma.I,w), u(\gamma'.I,w) \leq T^2.$$

Applying inequality 3.2,

$$T^4 \geq u(\gamma.I,\gamma'.I) = u(\gamma'^{-1}\gamma.I,I).$$

Since $\Lambda(p)$ acts discretely on $\mathfrak{h}^n$ with finite kernel, there will be only finitely many such γ' for a fixed γ.

We use $K_{p,T}$ as the kernel for an integral operator $I_{p,T} : \mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n) \to \mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n)$ given by

$$I_{p,T}(F)(z) := \int_{\Lambda(p)\backslash\mathfrak{h}^n} K_{p,T}(z,w)F(w)d^*w \text{ for all } F \in \mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n), z \in \Lambda(p)\backslash\mathfrak{h}^n.$$

The above integral converges for each z in $\Lambda(p)\backslash\mathfrak{h}^n$, because $K_{p,T}$ is compactly supported for fixed z. Further we can verify that $I_{p,T}(F)$ is in $\mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n)$ with the help of Minkowski's Inequality (in the fourth line below). Let $||\cdot||_p$ be the $\mathcal{L}^p$-norm on $\mathcal{C}(\Lambda(p)\backslash\mathfrak{h}^n)$. Then,

$$\begin{aligned}
||I_{p,T}(F)||_2 &= \sqrt{\int_{\Lambda(p)\backslash\mathfrak{h}^n} |I_{p,T}(F)(z)|^2 d^*z} \\
&= \sqrt{\int_{\Lambda(p)\backslash\mathfrak{h}^n} \left|\int_{\Lambda(p)\backslash\mathfrak{h}^n} K_{p,T}(z,w)F(w)d^*w\right|^2 d^*z} \\
&= \sqrt{\int_{\Lambda(p)\backslash\mathfrak{h}^n} \left|\int_{\Lambda(p)\backslash\mathfrak{h}^n} K_{p,T}(I,w)F(zw)d^*w\right|^2 d^*z} \\
&\leq \int_{\Lambda(p)\backslash\mathfrak{h}^n} \sqrt{\int_{\Lambda(p)\backslash\mathfrak{h}^n} |K_{p,T}(I,w)F(zw)|^2 \, d^*z} d^*w \\
&= \int_{\Lambda(p)\backslash\mathfrak{h}^n} |K_{p,T}(I,w)| \sqrt{\int_{\Lambda(p)\mathfrak{h}^n} |F(z)|^2 d^*z} d^*w \\
&= ||F||_2 ||K_{p,T}(I,\cdot)||_1,
\end{aligned}$$

which converges since F is in $\mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n)$, and $K_{p,T}(I,\cdot)$ is bounded with compact support.

The overarching tool of this paper will be to take the trace of this integral operator over a "large enough" compact subset, $\mathcal{K}_p$, of $\Lambda(p)\backslash\mathfrak{h}^n$,

$$\mathrm{Tr}_{\mathcal{K}_p} I_{p,T} := \int_{\mathcal{K}_p} K_{p,T}(z,z)d^*z,$$

and establish that this geometric trace is an upper bound for the discrete spectral trace (to be explained in the next chapter). In order for this inequality to hold, we need one more piece of important information about $I_{p,T}$, that it is a self-adjoint, non-negative operator, i.e. for all F and G in $\mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n)$,

$$\begin{aligned}
&\langle I_{p,T}F, G\rangle = \langle F, I_{p,T}G\rangle \text{ and} \\
&\langle I_{p,T}F, F\rangle \geq 0,
\end{aligned}$$

where the shown inner-product is the usual inner-product on $\mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n)$ given by

$$\langle F, G\rangle := \int_{\mathcal{K}_p} F(z)\overline{G(z)}d^*z \text{ for all } F, G \in \mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n).$$

It can be easily verified that $I_{p,T}$ is self-adjoint because its kernel, $K_{p,T}$, is real and symmetric, whereas its non-negativity comes from the following computation:

$$\begin{aligned}
\langle I_{p,T}F, F\rangle &= \int_{\mathcal{K}_p} I_{p,T}(F)(z)\overline{F(z)}d^*z \\
&= \int_{\Lambda(p)\backslash\mathfrak{h}^n}\int_{\Lambda(p)\backslash\mathfrak{h}^n} K_{p,T}(z,w)F(w)d^*w\overline{F(z)}d^*z \\
&= \int_{\Lambda(p)\backslash\mathfrak{h}^n}\int_{\Lambda(p)\backslash\mathfrak{h}^n}\sum_{\gamma\in\Lambda(p)} k_T(\gamma.z,w)F(w)\overline{F(z)}d^*wd^*z \\
&= \int_{\Lambda(p)\backslash\mathfrak{h}^n}\int_{\Lambda(p)\backslash\mathfrak{h}^n}\sum_{\gamma\in\Lambda(p)}\int_{\mathfrak{h}^n}\hat{k}(\gamma.z,v)\hat{k}(v,w)d^*vF(w)\overline{F(z)}d^*wd^*z \\
&= \sum_{\gamma\in\Lambda(p)}\int_{\mathfrak{h}^n}\overline{\int_{\Lambda(p)\backslash\mathfrak{h}^n}\hat{k}(\gamma.z,v)F(z)d^*z}\int_{\Lambda(p)\backslash\mathfrak{h}^n}\hat{k}(\gamma.w,v)F(\gamma.w)d^*wd^*v \\
&= \sum_{\gamma\in\Lambda(p)}\int_{\mathfrak{h}^n}\left|\int_{\Lambda(p)\backslash\mathfrak{h}^n}\hat{k}(\gamma.z,v)F(z)d^*z\right|^2 d^*v \geq 0
\end{aligned}$$

We are now ready to establish a pretrace inequality for $I_{p,T}$.

Chapter 4

Pretrace Inequality

We would like to compare the geometric and spectral trace of the integral operator, $I_{p,T}$, over some compact subset $\mathcal{K}_p$ of $\Lambda(p)\backslash\mathfrak{h}^n$. Unfortunately the continuous spectrum of $I_{p,T}$ contributes to the spectral trace in a way that is difficult to measure. We can avoid this problem by using the projection of $I_{p,T}$ onto the discrete portion of its spectral decomposition. This comes at a substantial cost. Rather than establishing an equality between the spectral trace and geometric trace, we will show that the discrete part of the spectral trace is bounded above by the geometric trace.

We begin by letting $\mathcal{L}^2_d(\Lambda(p)\backslash\mathfrak{h}^n)$ be the closure of the span of all $\Lambda(p)$-Maass forms. Note that because all Δ in $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$ are self-adjoint when acting on $\mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n)$, $\Lambda(p)$-Maass forms with different eigenvalues will be orthogonal. Let $\mathcal{L}^2_c(\Lambda(p)\backslash\mathfrak{h}^n)$ be the orthogonal complement to $\mathcal{L}^2_d(\Lambda(p)\backslash\mathfrak{h}^n)$. Let π_c be the orthogonal projection of $\mathcal{L}^2(\Lambda(p)\backslash\mathfrak{h}^n)$ onto $\mathcal{L}^2_c(\Lambda(p)\backslash\mathfrak{h}^n)$. The projection π_c is explicitly given by

$$\pi_c(\Psi) = \Psi - \sum_{(\Phi,\nu)} \langle \Psi, \Phi \rangle \Phi,$$

where the sum is over some orthonormal Hilbert basis of $\Lambda^2_d(\Lambda(p)\backslash\mathfrak{h}^n)$ consisting of $\Lambda(p)$-Maass forms Φ with corresponding types ν.

Using Theorem 5, we can see that each $\Lambda(p)$-Maass form, Φ, is also an eigenfunction of $I_{p,T}$, because for all z in $\Lambda(p)\backslash\mathfrak{h}^n$,

$$\begin{aligned}(I_{p,T}\Phi)(z) &= \int_{\Lambda(p)\backslash\mathfrak{h}^n} K_{p,T}(z,w)\Phi(w)d^*w \\ &= \int_{\Lambda(p)\backslash\mathfrak{h}^n} \sum_{\gamma\in\Lambda(p)} k_T(\gamma.z,w)\Phi(w)d^*w \\ &= \sum_{\gamma\in\Lambda(p)} \int_{\Lambda(p)\backslash\mathfrak{h}^n} k_T(z,\gamma^{-1}.w)\Phi(\gamma^{-1}.w)d^*w \\ &= \int_{\mathfrak{h}^n} k_T(z,w)\Phi(w)d^*w \\ &= h_T(\nu)\Phi(z),\end{aligned}$$

where h_T is the Selberg transform of k_T. Let Ψ be in $\mathcal{L}_c^2(\Lambda(p)\backslash\mathfrak{h}^n)$. Because $I_{p,T}$ is self-adjoint we see that

$$\begin{aligned}\langle I_{p,T}\Psi,\Phi\rangle &= \langle \Psi, I_{p,T}\Phi\rangle \\ &= \langle \Psi, h_T(\nu)\Phi\rangle \\ &= \overline{h_T(\nu)}\langle\Psi,\Phi\rangle = 0,\end{aligned}$$

so we also know that $I_{p,T}$ preserves $\mathcal{L}_c^2(\Lambda(p)\backslash\mathfrak{h}^n)$. Consequently, π_c and $I_{p,T}$ commute. Since the composition of two commuting, non-negative operators is also non-negative [Lusternick and Sobolev, 1965], it follows that $\pi_c I_T$ is non-negative. This operator can be written explicitly as the integral operator over $\Lambda(p)\backslash\mathfrak{h}^n$ with kernel

$$L_{p,T}(z,w) := K_{p,T}(z,w) - \sum_{(\Phi,\nu)} h_T(\nu)\Phi(z)\overline{\Phi(w)} \text{ for all } z,w\in\Lambda(p)\backslash\mathfrak{h}^n. \tag{4.1}$$

We would like to show that because $\pi_c I_T$ is non-negative, its kernel, $L_{p,T}$, is non-negative on the diagonal of $\Lambda(p)\backslash\mathfrak{h}^n\times\Lambda(p)\backslash\mathfrak{h}^n$. To see this consider the "ball" of radius Y about some point z in $\Lambda(p)\backslash\mathfrak{h}^n$, by which we mean

$$B(z,Y) := \{w\in\Lambda(p)\backslash\mathfrak{h}^n : u(z,\gamma.w)\le Y \text{ for some } \gamma\in\Lambda(p)\}. \tag{4.2}$$

This set is compact and thus has some finite volume, V_Y, with respect to d^*w (independent of z). If we define a cut-off function,

$$\phi_{z,Y}(w) := \begin{cases} V_Y^{-1} & \text{if } w\in B(z,Y), \\ 0 & \text{if } w\notin B(z,Y),\end{cases}$$

then

$$L_{p,T}(z,z) = \lim_{Y \to 2n} \langle \pi_c I_{p,T}(\phi_{z,Y}), \phi_{z,Y} \rangle,$$

which is always non-negative, since $\pi_c I_{p,T}$ is. Returning to the definition of L, 4.1,

$$\sum_{(\phi,\nu)} h_T(\nu)|\Phi(z)|^2 \leq K_{p,T}(z,z).$$

By integrating this over an appropriate compact subset, $\mathcal{K}_p$, of $\Lambda(p)\backslash\mathfrak{h}^n$, the left hand side will be the discrete spectral trace of $I_{p,T}$ over $\mathcal{K}_p$, and the right hand side will be the geometric trace of $I_{p,T}$ over $\mathcal{K}_p$.

$$\sum_{\Phi,\nu} h_T(\nu) \int_{\mathcal{K}_p} |\Phi(z)|^2 d^*z \leq \int_{\mathcal{K}_p} K_{p,T}(z,z) d^*z \tag{4.3}$$

Note that we do not anticipate this inequality to be close to an equality because the entire contribution of the continuous spectrum of $I_{p,T}$ has been removed from the spectral side.

Chapter 5

Estimating a Selberg Transform

We will require an estimate for the Selberg transform $\hat{h}_T$ (from Theorem 5) of our test function $\hat{k}_T$ (3.3). Our goal in this is to prove the following bounds on $\hat{h}_T(\nu)$ serving as an asymptotic estimate of $\hat{h}_T(\nu)$ as $T \to \infty$.

Theorem 6. *Let $\nu_1, \nu_2 \geq 1/3$. Then for all $\epsilon > 0$,*

$$T^{3\nu_1/2+3\nu_2/2} <<_\nu \hat{h}_T(\nu) <<_{\nu,\epsilon} T^{3\nu_1/2+3\nu_2/2+\epsilon}$$

where the implied constants depend only on ν and ϵ.

A particularly useful case of the above theorem is when $\nu_1 = \nu_2 = 2/3$, which corresponds to the eigenvalue $\lambda = 0$. By setting $\Phi = 1$ in Theorem 5, we see that $\hat{h}_T(\nu)$ is the volume of $B(z,T)$ as defined in 4.2. Theorem 6 then tells us that

$$T^2 << \text{Vol } B(z,T) <<_\epsilon T^{2+\epsilon} \tag{5.1}$$

where the implied constants depend only on ϵ.

We begin to prove Theorem 6 with the help of the auxiliary function I_ν from definition 2.3. We create an integral formula for $\hat{h}_T$ by setting $\Phi = I_\nu$ in Theorem 5.

$$\hat{h}_T(\nu) = \hat{h}_T(\nu) I_\nu(I) = \int_{\mathfrak{h}^3} \hat{k}_T(I,z) I_\nu(z) d^*z$$

where I is the identity matrix in $GL_3(\mathbb{R})$. Let w be the long element of the Weyl group for $GL_3(\mathbb{R})$, i.e.

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and consider the transformation $\mu : GL_3^+(\mathbb{R}) \to GL_3^+(\mathbb{R})$ given by

$$\mu(z) := w^{\mathrm{tr}} z^{-1} w.$$

The transformation μ is an automorphism and an involution, which descends to an isometry of $\mathfrak{h}^3$. Explicitly, in our standard coordinates for $\mathfrak{h}^3$,

$$\mu(x_1, x_2, x_3, y_1, y_2) = (-x_1, x_1x_3 - x_2, -x_3, y_2, y_1).$$

It allows us to rewrite the condition $\hat{k}(z, I) = 1$ (equivalently $u(z, I) \leq T$) as

$$||z||^2 \det z^{-2/3} + ||\mu(z)||^2 \det z^{2/3} \leq T,$$

since $|| \cdot ||$ is invariant under transposition and left or right translation by $O_3(\mathbb{R})$. From this it is clear that $h_T(\nu_1, \nu_2) = h_T(\nu_2, \nu_1)$. Finally, we will make the convenient coordinate change

$$y_1' := y_1^{1/3} y_2^{2/3},$$
$$y_2' := y_1^{2/3} y_1^{1/3}.$$

All together we obtain the formula

$$\hat{h}_T(\nu) = 3 \int_{z \in \mathfrak{h}^3, ||z||^2 \det z^{-2/3} + ||\mu(z)||^2 \det z^{2/3} \leq T} y_1'^{3\nu_1 - 3} y_2'^{3\nu_2 - 3} dx dy' \tag{5.2}$$

where

$$dx = dx_1 dx_2 dx_3,$$
$$dy' = dy_1' dy_2'.$$

Let us compute $||z||^2 \det z^{-2/3}$ explicitly in the coordinates x_1, x_2, x_3, y_1', and y_2'.

$$\begin{aligned}
||z||^2 \det z^{-2/3} &= \mathrm{Tr}\left({}^{\mathrm{tr}}zz\right)(\det z)^{-2/3} \\
&= \mathrm{Tr}\left({}^{\mathrm{tr}}y^{\mathrm{tr}}xxy\right)(\det z)^{-2/3} \\
&= \mathrm{Tr}\left(y^{\mathrm{tr}}y^{\mathrm{tr}}xx\right)(\det z)^{-2/3} \\
&= \mathrm{Tr}\left(\begin{pmatrix} y_1^2y_2^2 & 0 & 0 \\ 0 & y_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * & * \\ * & 1+x_1^2 & * \\ * & * & 1+x_2^2+x_3^2 \end{pmatrix}\right) y_1^{-4/3}y_2^{-2/3} \\
&= y_1'^2 + \frac{y_2'^2}{y_1'^2}\left(1+x_1^2\right) + \frac{1}{y_2'^2}\left(1+x_2^2+x_3^2\right)
\end{aligned}$$

Thus we can write the expression $||z||^2 \det z^{-2/3} + ||\mu(z)||^2 \det z^{2/3}$ in these coordinates as well.

$$\begin{aligned}
u(z,I) =& y_1'^2 + \frac{1}{y_1'^2} + y_2'^2 + \frac{1}{y_2'^2} + \frac{y_1'^2}{y_2'^2} + \frac{y_2'^2}{y_1'^2} + \left(\frac{y_2'^2}{y_1'^2} + \frac{1}{y_1'^2}\right)x_1^2 \\
&+ \left(\frac{y_1'^2}{y_2'^2} + \frac{1}{y_2'^2}\right)x_3^2 + \frac{1}{y_2'^2}x_2^2 + \frac{1}{y_1'^2}\left(x_1x_3 - x_2\right)^2 \\
=& y_1'^2 + y_1'^{-2} + y_2'^2 + y_2'^{-2} + \frac{y_1'^2}{y_2'^2} + \frac{y_2'^2}{y_1'^2} + \left(\frac{y_2'^2}{y_1'^2} + \frac{1}{y_1'^2}\right)x_1^2 \\
&+ \left(\sqrt{\frac{1}{y_1'^2} + \frac{1}{y_2'^2}}x_2 - \frac{x_1x_3y_2'}{y_1'\sqrt{y_1'^2 + y_2'^2}}\right)^2 + \left(\frac{1}{y_2'^2} + \frac{y_1'^2}{y_2'^2} + \frac{x_1^2}{y_1'^2 + y_2'^2}\right)x_3^2
\end{aligned}$$

If we let

$$\begin{aligned}
y_3' &= \frac{y_1'}{y_2'}, \\
x_1' &= \sqrt{\frac{y_2'^2}{y_1'^2} + \frac{1}{y_1'^2}}x_1, \\
x_2' &= \sqrt{\frac{1}{y_1'^2} + \frac{1}{y_2'^2}}x_2 - \frac{x_1x_3y_2'}{y_1'\sqrt{y_1'^2 + y_2'^2}}, \\
x_3' &= \sqrt{\frac{1}{y_2'^2} + \frac{y_1'^2}{y_2'^2} + \frac{x_1^2}{y_1'^2 + y_2'^2}}x_3,
\end{aligned}$$

then we can succinctly write

$$u(z,I) = y_1'^2 + y_1'^{-2} + y_2'^2 + y_2'^{-2} + y_3'^2 + y_3'^{-2} + x_1'^2 + x_2'^2 + x_3'^2.$$

Finally, we solve for x_1, x_2, and x_3 to compute $\frac{\partial x}{\partial x'}$.

$$\begin{aligned}
x_1 &= \frac{x_1' y_1'}{\sqrt{1+y_2'^2}} \\
x_2 &= \frac{y_1' y_2' x_2'}{\sqrt{y_1'^2+y_2'^2}} + \frac{y_2'^2 x_1 x_3}{y_1'^2+y_2'^2} \\
x_3 &= \frac{y_2'\sqrt{y_1'^2+y_2'^2}\, x_3'}{\sqrt{\left(1+y_1'^2\right)\left(y_1'^2+y_2'^2\right)+\frac{x_1'^2 y_1'^2 y_2'^2}{1+y_2'^2}}} \\
\left|\frac{\partial x}{\partial x'}\right| &= \frac{y_1' y_2'}{\sqrt{y_1'^2+y_1'^{-2}+y_2'^2+y_2'^{-2}+y_3'^2+y_3'^{-2}+2+x_1'^2}}
\end{aligned}$$

Now we can give an integral expression for $\hat{h}_T(\nu)$ in terms of x' and y'.

$$\hat{h}_T(\nu) = 3\int_{\sum_{j=1}^3\left(y_j'^2+y_j'^{-2}+x_j'^2\right)\le T} \frac{y_1'^{3\nu_1-2} y_2'^{3\nu_2-2}}{\sqrt{\sum_{j=1}^3\left(y_j'^2+y_j'^{-2}\right)+2+x_1'^2}} dx\,dy_1'\,dy_2'$$

We will use a technique similar to [Duke *et al.*, 1993], Appendix A, to estimate this integral as $T\to\infty$. Let us use $\hat{h}_\nu(X)$ in place of $\hat{h}_X(\nu)$ for the moment and define

$$\begin{aligned}
H_\nu(t) &:= \int_0^\infty e^{-tX} d\hat{h}_\nu(X) \\
&= 3\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-t\sum_{j=1}^3\left(y_j'^2+y_j'^{-2}+x_j'^2\right)} \frac{y_1'^{3\nu_1-2} y_2'^{3\nu_2-2}}{\sqrt{\sum_{j=1}^3\left(y_j'^2+y_j'^{-2}\right)+x_1'^2+2}} dx'\,dy_1'\,dy_2' \\
&= \frac{3\pi}{t}\int_0^\infty\int_0^\infty e^{-t\sum_{j=1}^3\left(y_j'^2+y_j'^{-2}\right)} y_1'^{3\nu_1-2} y_2'^{3\nu_2-2}\int_{-\infty}^\infty \frac{e^{-tx^2}}{\sqrt{\sum_{j=1}^3\left(y_j'^2+y_j'^{-2}\right)+x^2+2}} dx\,dy_1'\,dy_2'.
\end{aligned} \tag{5.3}$$

The asymptotic estimate of $H_\nu(t)$ as $t\to 0$ will give an estimate for $\hat{h}_T(\nu)$ as $T\to\infty$ via the following Tauberian theorem, which can be found in [Widder, 1946] on page 192.

Theorem 7. *If h is a non-decreasing real function such that the integral,*

$$H(t) = \int_0^\infty e^{-tX} dh(X),$$

converges for all $t>0$, and there are some $A, B>0$ such that

$$H(t) \sim \frac{A}{t^B} \text{ as } t\to 0^+,$$

then,

$$h(X) \sim \frac{AX^B}{\Gamma(B+1)} \text{ as } t \to \infty.$$

It is thus sufficient to estimate $H_\nu(t)$ as $t \to 0^+$.

In order to estimate $H_\nu(t)$, we will use modified Bessel functions of the second kind with complex order ν, denoted by K_ν. The function K_ν is a solution to the differential equation,

$$z^2 \frac{\partial^2 K_\nu(z)}{\partial z^2} + z\frac{\partial K_\nu(z)}{\partial z} - \left(z^2 + \nu^2\right) K_\nu(z) = 0,$$

and has the explicit integral formula,

$$\begin{aligned} K_\nu(z) &= \int_0^\infty e^{-z\cosh(u)} \cosh(\nu u) du \\ &= \int_{-\infty}^\infty e^{-z\cosh(u)} e^{\nu u} du. \end{aligned}$$

Let us examine the innermost integral of equation 5.3, setting

$$\Sigma = \frac{1}{2}\sum_{j=1}^{3}\left(y_j'^2 + y_j'^{-2}\right) + 1,$$

and making the substitution

$$x = \sqrt{\Sigma(\cosh(u) - 1)}.$$

$$\begin{aligned} \int_{-\infty}^\infty \frac{e^{-tx^2}}{\sqrt{2\Sigma + x^2}} dx &= 2\int_0^\infty \frac{e^{-tx^2}}{\sqrt{2\Sigma + x^2}} dx \\ &= \int_0^\infty e^{-t\Sigma(\cosh(u)-1)} du \\ &= \frac{e^{t\Sigma}}{2}\int_{-\infty}^\infty e^{-t\Sigma\cosh(u)} du \\ &= \frac{e^{t\Sigma}}{2} K_0(t\Sigma) \end{aligned}$$

Inserting this into equation 5.3 yields the following expression for $H_\nu(t)$.

$$\begin{aligned} H_\nu(t) &= \frac{3\pi e^{2t}}{2t}\int_0^\infty\int_0^\infty e^{-t\Sigma} K_0(t\Sigma) y_1'^{3\nu_1-2} y_2'^{3\nu_2-2} dy_1' y_2' \\ &= \frac{3\pi e^{2t}}{2t}\int_0^\infty\int_0^\infty e^{-t\Sigma} K_0(t\Sigma) e^{\frac{3\nu_1-1}{2}u_1 + \frac{3\nu_2-1}{2}u_2} du_1 du_2 \end{aligned} \tag{5.4}$$

where in the last line we use the substitution

$$y_1 = e^{u_1/2},$$

$$y_2 = e^{u_2/2},$$

and as a result

$$\Sigma = \cosh(u_1) + \cosh(u_2) + \cosh(u_1 - u_2) + 1.$$

We will estimate $H_\nu(t)$ by bounding $K_0(t)$ and using the computation of the simpler integral in the following lemma.

Lemma 2. *For $\nu_1, \nu_2 > 0$, let*

$$G_\nu(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-t(\cosh(u_1)+\cosh(u_2)+\cosh(u_1-u_2))+u_1\nu_1+u_2\nu_2} du_1 du_2. \tag{5.5}$$

Then, there exists an $\epsilon > 0$, which depends on ν_1 and ν_2, such that as $t \to 0^+$,

$$G_\nu(t) = \frac{1}{8}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2} \Gamma(\nu_1)\Gamma(\nu_2) + \mathcal{O}_\nu\left(t^{-\nu_1-\nu_2+\epsilon}\right)$$

where the implied constant is independent of t.

Proof. By continuity, we may assume that $\mathbb{Z}$, $\nu_1 + \mathbb{Z}$, and $\nu_2 + \mathbb{Z}$ are pairwise disjoint. Choose a c such that $0 < c < \min\{\{\nu_1\}, 1 - \{\nu_2\}\}$, where $\{\nu\} = \nu - \lfloor \nu \rfloor$ for $\lfloor \cdot \rfloor$, the greatest integer function.

We start with an observation about $K_s(t)$. If for any real number u, we let

$$f_t(u) := e^{-t\cosh(u)},$$

then we find that $K_s(t)$ is the bilateral Laplace transform of f_t at s, i.e.

$$\int_{-\infty}^{\infty} f_t(u) e^{su} du = K_s(t)$$

Taking the inverse bilateral Laplace transform (using a transform of the Mellin Inversion Theorem), we obtain a contour integral for f_t

$$f_t(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_s(t) e^{-su} ds$$

where c is any real number, since K_s is entire in s. We will replace $e^{-t\cosh(u_1-u_2)}$ in the definition of $G_\nu(t)$ 5.5 with the corresponding contour integral.

$$\begin{aligned}
G_\nu(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-t(\cosh(u_1)+\cosh(u_2))+u_1\nu_1+u_2\nu_2} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_s(t) e^{s(u_2-u_1)} ds du_1 du_2 \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_s(t) \int_{-\infty}^{\infty} e^{-t\cosh(u_1)+(\nu_1-s)u_1} du_1 \int_{-\infty}^{\infty} e^{-t\cosh(u_2)+(\nu_2+s)u_2} du_2 ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_s(t) K_{\nu_1-s}(t) K_{\nu_2+s}(t) ds
\end{aligned}$$

Additionally, we use the following series expansion for K_s [Watson, 1962].

$$K_s(t) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-t^2/4)^k}{k!}\left(\left(\frac{t}{2}\right)^s\Gamma(-s-k) - \left(\frac{t}{2}\right)^{-s}\Gamma(s-k)\right)$$

Then

$$\begin{aligned}
G_\nu(t) &= \frac{1}{16\pi i}\int_{c-i\infty}^{c+i\infty}\begin{pmatrix} \sum_{k=0}^{\infty}\frac{(-t^2/4)^k}{k!}\left(\left(\frac{t}{2}\right)^s\Gamma(-s-k) - \left(\frac{t}{2}\right)^{-s}\Gamma(s-k)\right) \\ \cdot\sum_{l=0}^{\infty}\frac{(-t^2/4)^l}{l!}\left(\left(\frac{t}{2}\right)^{\nu_1-s}\Gamma(-\nu_1+s-l) - \left(\frac{t}{2}\right)^{-\nu_1+s}\Gamma(\nu_1-s-l)\right) \\ \cdot\sum_{m=0}^{\infty}\frac{(-t^2/4)^m}{m!}\left(\left(\frac{t}{2}\right)^{\nu_2+s}\Gamma(-\nu_2-s-k) - \left(\frac{t}{2}\right)^{-\nu_2-s}\Gamma(\nu_2+s-k)\right)\end{pmatrix} ds \\
&= \frac{1}{8}\sum_{k,l,m=0}^{\infty}\frac{(-t^2/4)^{k+m+l}}{k!l!m!} \\
&\quad\cdot\begin{pmatrix} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{\nu_1+\nu_2+s}\Gamma(-s-k)\Gamma(-\nu_1+s-l)\Gamma(-\nu_2-s-m)ds \\ -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{-\nu_1+\nu_2+3s}\Gamma(-s-k)\Gamma(\nu_1-s-l)\Gamma(-\nu_2-s-m)ds \\ +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2+s}\Gamma(-s-k)\Gamma(\nu_1-s-l)\Gamma(\nu_2+s-m)ds \\ +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{-\nu_1+\nu_2+s}\Gamma(s-k)\Gamma(\nu_1-s-l)\Gamma(-\nu_2-s-m)ds \\ +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{\nu_1-\nu_2-s}\Gamma(-s-k)\Gamma(-\nu_1+s-l)\Gamma(\nu_2+s-m)ds \\ +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{\nu_1+\nu_2-s}\Gamma(s-k)\Gamma(-\nu_1+s-l)\Gamma(-\nu_2-s-m)ds \\ -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{\nu_1-\nu_2-3s}\Gamma(s-k)\Gamma(-\nu_1+s-l)\Gamma(\nu_2+s-m)ds \\ +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2-s}\Gamma(s-k)\Gamma(\nu_1-s-l)\Gamma(\nu_2+s-m)ds \end{pmatrix}
\end{aligned}$$

By shifting the real part of the first four contour integrals toward infinity and shifting the final four toward negative infinity, we obtain an expansion for each integral from the residues of the integrand. In particular, we use that Γ is meromorphic having (simple) poles only at $-j$ with residue $(-1)^j/j!$ for all non-negative integers j.

$$
\begin{aligned}
G_\nu(t) &= \frac{1}{8}\sum_{k,l,m=0}^{\infty}\frac{(-t^2/4)^{k+m+l}}{k!l!m!} \\
&\cdot\left(\begin{array}{l}
\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{\nu_1+\nu_2+j}\frac{(-1)^{j+k}}{(j+k)!}\Gamma(-\nu_1+j-l)\Gamma(-\nu_2-j-m) \\
+\sum_{j=0}^{\lfloor\nu_1\rfloor+l}\left(\frac{t}{2}\right)^{\nu_1+\{\nu_1\}+\nu_2+j}\frac{(-1)^{\lfloor\nu_1\rfloor+l-j}}{(\lfloor\nu_1\rfloor+l-j)!}\Gamma(-\{\nu_1\}-j-k)\Gamma(-\{\nu_1\}-\nu_2-j-m) \\
+\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{\nu_1+\lfloor\nu_2\rfloor+j}\frac{(-1)^{\lfloor\nu_2\rfloor+j+m}}{(\lfloor\nu_2\rfloor+j+m)!}\Gamma(\{\nu_2\}-j-k)\Gamma(-\nu_1-\{\nu_2\}+j-l) \\
-\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{-\nu_1+\nu_2+3j}\frac{(-1)^{j+k}}{(j+k)!}\Gamma(\nu_1-j-l)\Gamma(-\nu_2-j-m) \\
\sum_{j=\max\{0,\lfloor\nu_1\rfloor-l\}}^{\infty}\left(\frac{t}{2}\right)^{-\lfloor\nu_1\rfloor+2\{\nu_1\}+\nu_2+3j}\frac{(-1)^{-\lfloor\nu_1\rfloor+l+j}}{(-\lfloor\nu_1\rfloor+l+j)!} \\
\qquad\qquad\qquad\cdot\Gamma(-\{\nu_1\}-j-k)\Gamma(-\{\nu_1\}-\nu_2-j-m) \\
-\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{\nu_1+\lfloor\nu_2\rfloor-2\{\nu_2\}+3j}\frac{(-1)^{\lfloor\nu_2\rfloor+j+m}}{(\lfloor\nu_2\rfloor+j+m)!}\Gamma(\{\nu_2\}-j-k)\Gamma(\nu_1+\{\nu_2\}-j-l) \\
+\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2+j}\frac{(-1)^{j+k}}{(j+k)!}\Gamma(\nu_1-j-l)\Gamma(-\nu_2+j-m) \\
+\sum_{j=\max\{0,\lfloor\nu_1\rfloor-l\}}^{\infty}\left(\frac{t}{2}\right)^{-\lfloor\nu_1\rfloor-\nu_2+j}\frac{(-1)^{-\lfloor\nu_1\rfloor+l+j}}{(-\lfloor\nu_1\rfloor+l+j)!}\Gamma(-\{\nu_1\}-j-k)\Gamma(\{\nu_1\}-\nu_2+j-m) \\
+\sum_{j=1}^{m-\lfloor\nu_2\rfloor}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2-\{\nu_2\}+j}\frac{(-1)^{m-\lfloor\nu_2\rfloor-j}}{(m-\lfloor\nu_2\rfloor-j)!}\Gamma(\{\nu_2\}-j-k)\Gamma(\nu_1+\{\nu_2\}-j-l) \\
+\sum_{j=1}^{k}\left(\frac{t}{2}\right)^{-\nu_1+\nu_2+j}\frac{(-1)^{k-j}}{(k-j)!}\Gamma(\nu_1-j-l)\Gamma(-\nu_2-j-m) \\
+\sum_{j=\max\{0,\lfloor\nu_1\rfloor-l\}}^{\infty}\left(\frac{t}{2}\right)^{-\lfloor\nu_1\rfloor+\nu_2+j}\frac{(-1)^{-\lfloor\nu_1\rfloor+l+j}}{(-\lfloor\nu_1\rfloor+l+j)!}\Gamma(\{\nu_1\}+j-k)\Gamma(-\{\nu_1\}-\nu_2-j-m) \\
+\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{-\nu_1+\lfloor\nu_2\rfloor+j}\frac{(-1)^{\lfloor\nu_2\rfloor+j+m}}{(\lfloor\nu_2\rfloor+j+m)!}\Gamma(-\{\nu_2\}+j-k)\Gamma(\nu_1+\{\nu_2\}-j-l) \\
+\sum_{j=0}^{k}\left(\frac{t}{2}\right)^{\nu_1-\nu_2+j}\frac{(-1)^{k-j}}{(k-j)!}\Gamma(-\nu_1-j-l)\Gamma(\nu_2-j-m) \\
+\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{-\lfloor\nu_1\rfloor-\nu_2+j}\frac{(-1)^{\lfloor\nu_1\rfloor+j+l}}{(\lfloor\nu_1\rfloor+j+l)!}\Gamma(-\{\nu_1\}+j-k)\Gamma(\{\nu_1\}+\nu_2-j-m) \\
+\sum_{j=\max\{0,\lfloor\nu_2\rfloor-m\}}^{\infty}\left(\frac{t}{2}\right)^{\nu_1-\lfloor\nu_2\rfloor-j}\frac{(-1)^{-\lfloor\nu_2\rfloor+j+m}}{(-\lfloor\nu_2\rfloor+j+m)!}\Gamma(\{\nu_2\}+j-k)\Gamma(-\nu_1-\{\nu_2\}-j-l) \\
+\sum_{j=0}^{\infty}\left(\frac{t}{2}\right)^{\nu_1+\nu_2+j}\frac{(-1)^{k+j}}{(k+j)!}\Gamma(-\nu_1-j-l)\Gamma(-\nu_2+j-m) \\
+\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{\lfloor\nu_1\rfloor+\nu_2+j}\frac{(-1)^{\lfloor\nu_1\rfloor+j+l}}{(\lfloor\nu_1\rfloor+j+l)!}\Gamma(\{\nu_1\}-j-k)\Gamma(\{\nu_1\}+\nu_2-j-m) \\
+\sum_{j=0}^{\lfloor\nu_2\rfloor+m}\left(\frac{t}{2}\right)^{\nu_1+\nu_2+\{\nu_2\}+j}\frac{(-1)^{\lfloor\nu_2\rfloor-j+m}}{(\lfloor\nu_2\rfloor-j+m)!}\Gamma(-\{\nu_2\}-j-k)\Gamma(-\nu_1-\{\nu_2\}-j-l) \\
-\sum_{j=0}^{\infty}\left(\frac{t}{2}\right)^{\nu_1-\nu_2+3j}\frac{(-1)^{j+k}}{(j+k)!}\Gamma(-\nu_1-j-l)\Gamma(-\nu_2+j-m) \\
-\sum_{j=1}^{\infty}\left(\frac{t}{2}\right)^{\lfloor\nu_1\rfloor-2\{\nu_1\}-\nu_2+3j}\frac{(-1)^{\lfloor\nu_1\rfloor+j+l}}{(\lfloor\nu_1\rfloor+j+l)!}\Gamma(\{\nu_1\}-j-k)\Gamma(\{\nu_1\}+\nu_2-j-m) \\
-\sum_{j=0}^{\lfloor\nu_2\rfloor-m}\left(\frac{t}{2}\right)^{\nu_1-\lfloor\nu_2\rfloor+2\{\nu_2\}+3j}\frac{(-1)^{-\lfloor\nu_2\rfloor+j+m}}{(-\lfloor\nu_2\rfloor+j+m)!}\Gamma(-\{\nu_2\}-j-k)\Gamma(-\nu_1-\{\nu_2\}-j-l) \\
+\sum_{j=0}^{\infty}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2+j}\frac{(-1)^{j+k}}{(j+k)!}\Gamma(\nu_1+j-l)\Gamma(\nu_2-j-m) \\
+\sum_{j=\max\{1,-\lfloor\nu_1\rfloor+l\}}^{\infty}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2-\{\nu_1\}+j}\frac{(-1)^{-\lfloor\nu_1\rfloor-j+l}}{(-\lfloor\nu_1\rfloor-j+l)!}\Gamma(\{\nu_1\}-j-k)\Gamma(\nu_2+\{\nu_1\}-j-m) \\
+\sum_{j=\max\{0,\lfloor\nu_2\rfloor-m\}}^{\infty}\left(\frac{t}{2}\right)^{-\nu_1-\lfloor\nu_2\rfloor+j}\frac{(-1)^{-\lfloor\nu_2\rfloor+j+m}}{(-\lfloor\nu_2\rfloor+j+m)!}\Gamma(-\{\nu_2\}-j-k)\Gamma(\nu_1+\{\nu_2\}+j-l)
\end{array}\right) \\
&= \frac{1}{8}\left(\frac{t}{2}\right)^{-\nu_1-\nu_2}\Gamma(\nu_1)\Gamma(\nu_2)+\mathcal{O}_\nu\left(t^{-\nu_1-\nu_2+\epsilon}\right)
\end{aligned}
$$

The main term of the final estimate comes from the third to last sum in the $j = k = l = m = 0$ term. All other terms have a greater power of t^{-1} and are relegated to the error. The constant ϵ can be made explicit in terms of ν_1 and ν_2. □

In order to estimate the integral from equation 5.4, we will also need bounds on K_0.

Lemma 3. *Let* $0 < \epsilon < 1/2$. *Then for all* $z > 0$,

$$e^{-(1+\epsilon)z} <<_\epsilon K_0(z) <<_\epsilon z^{-\epsilon} e^{-z}$$

where the implied constants depend only on ϵ.

Proof. We use some well known estimates for K_0, which can be found in [Watson, 1962]. As $z \to \infty$,

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}.$$

Thus,

$$e^{-(1+\epsilon)z} <<_\epsilon K_0(z) << z^{-\epsilon} e^{-z} \text{ as } z \to \infty.$$

For $z \to 0^+$,

$$K_0(z) \sim -\log(z/2) - \gamma$$

where γ is the Euler-Mascheroni constant. Thus,

$$1 << K_0(z) <<_\epsilon z^{-\epsilon} \text{ as } z \to 0^+.$$

For all $z > 0$, $K_0(z)$ is positive; therefore, we can combine these inequalities to conclude the lemma. □

We can now use the preceding two lemmas to estimate the integral in equation 5.4 as $t \to 0^+$. We begin by giving a lower bound for $H_\nu(t)$. For any real ϵ such that $0 < \epsilon < 1/2$,

$$\begin{aligned}
H_\nu(t) &= \frac{3\pi e^{2t}}{2t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-t\Sigma} K_0(t\Sigma) e^{\frac{3\nu_1 - 1}{2} u_1 + \frac{3\nu_2 - 1}{2} u_2} du_1 du_2 \\
&>>_\epsilon \frac{e^{2t}}{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(2+\epsilon)t\Sigma} e^{\frac{3\nu_1 - 1}{2} u_1 + \frac{3\nu_2 - 1}{2} u_2} du_1 du_2 \\
&= \frac{e^{-\epsilon t}}{t} G_{\frac{3\nu_1 - 1}{2}, \frac{3\nu_2 - 1}{2}}((2+\epsilon)t) \\
&>>_{\epsilon,\nu} e^{-\epsilon t} t^{-3\nu_1/2 - 3\nu_2/2}.
\end{aligned}$$

For the upper bound, we obtain

$$\begin{aligned}
H_\nu(t) &= \frac{3\pi e^{2t}}{2t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-t\Sigma}K_0(t\Sigma)e^{\frac{3\nu_1-1}{2}u_1+\frac{3\nu_2-1}{2}u_2}du_1du_2\\
&<<_\epsilon \frac{e^{2t}}{t^{1+\epsilon}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2t\Sigma}e^{\frac{3\nu_1-1}{2}u_1+\frac{3\nu_2-1}{2}u_2}du_1du_2\\
&= t^{-1-\epsilon}G_{\frac{3\nu_1-1}{2},\frac{3\nu_2-1}{2}}(2t)\\
&<<_{\epsilon,\nu} t^{-3\nu_1/2-3\nu_2/2-\epsilon}.
\end{aligned}$$

Theorem 6 follows from Theorem 7 with the preceding bounds on $H_\nu(t)$. Since the Selberg transform of k is $h = \hat{h}^2$, we obtain a similar estimate for h:

$$T^{3\nu_1+3\nu_2} <<_\nu h_T(\nu) <<_{\nu,\epsilon} T^{3\nu_1+3\nu_2+\epsilon}. \tag{5.6}$$

Chapter 6

Lattice Point Count for $\Gamma(p)$

We will need an elementary, crude lattice point count for $\Gamma(p)$, the pth principal congruence subgroup of $SL_3(\mathbb{Z})$. In particular we will be counting the number of matrices A in $\Gamma(p)$ such that

$$||A||, ||A^{-1}|| \leq T \tag{6.1}$$

where

$$||A|| := \sqrt{\mathrm{Tr}(AA^{\mathrm{tr}})}.$$

Let us call this number $\widetilde{N}(p, T)$.

Lemma 4. *Let $T \geq p$. Then for all $\epsilon > 0$,*

$$\widetilde{N}(p, T) <<_\epsilon \frac{T^{6+\epsilon}}{p^9} + \frac{T^{5+\epsilon}}{p^6} + \frac{T^{4+\epsilon}}{p^4}$$

where the implied constant is independent of T and P (depends only on ϵ).

Proof. Let A be in $\Gamma(p)$ and consider its characteristic polynomial $q(z) = z^3 + q_2 z^2 + q_1 z - 1$. If I is the identity in $\Gamma(p)$ then

$$A - I - pzI \equiv 0 \pmod{p}$$

for any integer z, and so

$$q(1 + pz) = \det(A - I - pzI) \equiv 0 \pmod{p^3}.$$

Examining the coefficients of

$$q(1 + pz) = p^3 z^3 + (q_2 + 3)p^2 z^2 + (q_1 + 2q_2 + 3)pz + (q_1 + q_2),$$

we obtain two useful congruence relations:

$$q_1 \equiv -q_2 \pmod{p^3}$$
$$q_2 \equiv -3 \pmod{p^2}$$

Naming the entries of A and A^{-1} as

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} a'_{1,1} & a'_{1,2} & a'_{1,3} \\ a'_{2,1} & a'_{2,2} & a'_{2,3} \\ a'_{3,1} & a'_{3,2} & a'_{3,3} \end{pmatrix},$$

we explicitly obtain

$$a'_{i,j} = (-1)^{i+j} A_{j,i}$$

where $A_{j,i}$ is the determinant of the minor of A formed by removing the jth row and ith column. Additionally, we can express the coefficients of the characteristic polynomial of A as

$$q_2 = -a_{1,1} - a_{2,2} - a_{3,3}$$
$$q_1 = a'_{1,1} + a'_{2,2} + a'_{3,3}.$$

As a result we can write our congruence relations in terms of the entries of A and A^{-1}:

$$a_{1,1} + a_{2,2} + a_{3,3} \equiv a'_{1,1} + a'_{2,2} + a'_{3,3} \pmod{p^3} \tag{6.2}$$
$$a_{1,1} + a_{2,2} + a_{3,3} \equiv 3 \pmod{p^2} \tag{6.3}$$

The bound given in this lemma utilizes only two other constraints. The first is that 6.1 necessitates

$$|a_{i,j}|, |a'_{i,j}| \leq T \tag{6.4}$$

for all i and j. The second is simply the congruence relations on the entries of A characterizing $\Gamma(p)$, namely,

$$a_{i,j} \equiv a'_{i,j} \equiv \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \pmod{p} \tag{6.5}$$

The key to this count is that for $\{i, j, k\} = \{1, 2, 3\}$, the entries $a_{i,i}$, $a_{j,j}$, and $a'_{k,k}$ almost determine $a_{i,j}$ and $a_{j,i}$, in the sense that for any $\epsilon > 0$ there are $<<_\epsilon T^\epsilon$ possible values for $a_{i,j}$ and $a_{j,i}$ given $a_{i,i}$, $a_{j,j}$, and $a'_{k,k}$ *unless* $a'_{k,k} = a_{i,i}a_{j,j}$. The reason for this is that

$$a_{i,j}a_{j,i} = a_{i,i}a_{j,j} - a'_{k,k}.$$

This quantity is at most T^2 in absolute value, so when $a'_{k,k} \neq a_{i,i}a_{j,j}$, the righthand side is nonzero, and consequently there are $<<_\epsilon T^\epsilon$ ways to express it as a product of two integers ($a_{i,j}$ and $a_{j,i}$).

Our count separates into two cases due to the preceding observation. In the first case, we assume that one of $a_{1,2}a_{2,1}$, $a_{1,3}a_{3,1}$ and $a_{2,3}a_{3,2}$ is nonzero. Since we are interested in counting only up to a constant independent of T and p, we may assume without loss of generality that $a_{1,2}a_{2,1} \neq 0$. In all other cases, we may conjugate to a matrix with $a_{1,2}a_{2,1} \neq 0$ via Weyl group sitting in $\Gamma(p)$ without changing the norm of the matrix or its inverse. This would only change the count by a constant multiple. By 6.4 and 6.5 alone there are at most $2T/p+1$ possible values for each of $a_{1,1}$, $a_{2,2}$. For a given $a_{1,1}$ and $a_{2,2}$ there are at most $2T/p^2+1$ possible values for $a_{3,3}$ based on 6.4 and 6.3. If $a_{2,3}a_{3,2} \neq 0$, then there are at most $2T/p+1$ possible choices for $a'_{1,1}$ given $a_{2,2}$ and $a_{3,3}$ and thus at most $<<_\epsilon T^{1+\epsilon}/p$ possible choices for $a_{2,3}$ and $a_{3,2}$ for any $\epsilon > 0$. If $a_{2,3}a_{3,2} = 0$ there are at most $4T/p+1$ possible choices for $a_{2,3}$ and $a_{3,2}$ by 6.4 and 6.5. Together there are $<<_\epsilon T^{1+\epsilon}/p$ possible choices for $a_{2,3}$ and $a_{3,2}$ given $a_{2,2}$ and $a_{3,3}$. Similarly, there are $<<_\epsilon T^{1+\epsilon}/p$ possible choices for $a_{1,3}$ and $a_{3,1}$ given $a_{1,1}$ and $a_{3,3}$. What is left is to determine the possible values of $a_{1,2}$ and $a_{2,1}$ given that neither is 0 and given all other entries of A. The other entries of A determine $a_{1,1}$, $a_{2,2}$, $a_{3,3}$, $a'_{1,1}$, and $a'_{2,2}$. Thus by 6.4 and 6.2, there are at most $2T/p^3+1$ possible values for $a'_{3,3}$. Consequently, there are

$$<<_\epsilon T^\epsilon \left(\frac{T}{p^3}+1\right)$$

possible values for $a_{1,2}$ and $a_{2,1}$. All together this tells us that for any $\epsilon > 0$ there are

$$<<_\epsilon \frac{T^{6+\epsilon}}{p^9} + \frac{T^{5+\epsilon}}{p^6} + \frac{T^{4+\epsilon}}{p^4}$$

matrices A in $\Gamma(p)$ such that $||A||, ||A^{-1}|| \leq T$ and $a_{1,2}a_{2,1} \neq 0$.

The preceding bound matches the bound given in the lemma, so our goal is to show that the matrices A with

$$a_{1,2}a_{2,1} = a_{1,3}a_{3,1} = a_{2,3}a_{3,2} = 0$$

provide an insignificant contribution to $N(p,T)$. Once again, conjugation by the Weyl group will significantly simplify the cases we need to consider. Without loss of generality, we can assume either

$$a_{2,1} = a_{3,1} = a_{3,2} = 0$$

(i.e. A is upper triangular) or

$$a_{1,3} = a_{2,1} = a_{3,2} = 0 \text{ while } a_{1,2}a_{2,3}a_{3,1} \neq 0. \tag{6.6}$$

In the former case, having determinant 1 guarantees all diagonal entries are ± 1. By 6.5 and 6.4 there are at most $2T/p+1$ possible choices for each of the above diagonal entries for a total of at most $32T^3/p^3$ upper triangular matrices A in $\Gamma(p)$ with $||A||, ||A^{-1}||$, which is insignificant compared to the bound given in the lemma. For the final case under constraint 6.6, we first observe that there are $2T/p+1$ possible values for each of $a_{1,1}$ and $a_{2,2}$. There are at most $2T/p^2+1$ possible values for $a_{3,3}$ given $a_{1,1}$ and $a_{2,2}$ using 6.3 and 6.4. Next, we compute the determinant of A as

$$\det(A) = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} = 1$$
$$a_{1,2}a_{2,3}a_{3,1} = 1 - a_{1,1}a_{2,2}a_{3,3}$$

Given $a_{1,1}$, $a_{2,2}$, and $a_{3,3}$, the right-hand side is determined. By 6.6, the left-hand side is nonzero and by 6.4 it has absolute value less than T^3. Thus, for any $\epsilon > 0$, there are $<<_\epsilon T^\epsilon$ ways to express the right-hand side as a product of three integers ($a_{1,2}$, $a_{2,3}$ and $a_{3,1}$). As a result there are

$$<<_\epsilon \frac{T^{3+\epsilon}}{p^4} + \frac{T^{2+\epsilon}}{p^2}$$

matrices A in $\Gamma(p)$ satisfying 6.6 and having $||A||, ||A^{-1}|| \leq T$. This is also insignificant compared to the bound on $N(p,T)$ given in the lemma, thus concluding the final case.

□

If we let $N(p,T)$ be the number of matrices γ in $\Gamma(p)$ such that $u(\gamma, I) \leq T$, then this constraint on γ requires that

$$||\gamma||, ||\gamma||^{-1} \leq \sqrt{T}.$$

Consequently, for any $\epsilon > 0$ and $T > 6$,

$$N(p,T) \leq \widetilde{N}(p, \sqrt{T}) <<_\epsilon \frac{T^{3+\epsilon}}{p^9} + \frac{T^{5/2+\epsilon}}{p^6} + \frac{T^{2+\epsilon}}{p^4} + 1. \tag{6.7}$$

This bound will be key to the following chapter.

Chapter 7

Geometric Trace Bound

Our goal in this chapter is to give an upper bound on the trace of the integral kernel K_T on a compact subset $\mathcal{K}_p$ of $\Lambda(p)\backslash\mathfrak{h}^3$, i.e.

$$\int_{\mathcal{K}_p} K_{T,p}(z,z)d^*z.$$

A lemma that we will need for this is a bound on $k_T(z,I)$ (for I the identity matrix) in terms of $u(z,I)$ and T.

Lemma 5. *For all z in $\mathfrak{h}^3$, $T > 6$, and $\epsilon > 0$,*

$$k_T(z,I) <<_\epsilon T^{2+\epsilon}/u(z,I)$$

Proof. The strategy of this proof is to notice that if

$$B(z,T) := \{w \in \mathfrak{h}^3 : u(z,w) \leq T\}$$

then

$$\begin{aligned} k_T(z,I) &= \int_{\mathfrak{h}^3} \hat{k}_T(z,w)\hat{k}_T(w,I)d^*w \\ &= \mathrm{Vol}(B(z,T) \cap B(I,T)). \end{aligned}$$

Our goal is to show that the region $B(z,T)\cap B(I,T)$ is covered $B(v, cT/\sqrt{u(z,I)})$ for finitely many points v in $\mathfrak{h}^3$ and some constant $c > 0$. The result then follows from inequality 5.1.

For any h and h' in $SO_3(\mathbb{R})$,

$$k_T(hzh',I) = k_T(z,h^{-1}.I) = k_T(z,I)$$

by point-pair invariance. Thus by the Cartan decomposition, we may assume without loss of generality that z is diagonal with diagonal entries z_1, z_2 and z_3 such that $|z_1| \geq |z_2| \geq |z_3|$ [Goldfeld and Hundley, 2011]. Let w be any element of $\mathfrak{h}^3$ such that $u(w,z) \leq T$ and $u(w,I) \leq T$ and let w_j be its jth row. Then

$$||zw||^2 = z_1^2|w_1|^2 + z_2^2|w_2|^2 + z_3^2|w_3|^2$$

where $|w_j|$ is the L^2-norm on the row vector w_j. Let τ be the permutation matrix,

$$\tau := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and let $\sqrt{z}$ be the diagonal matrix with diagonal entries $\sqrt{z_1}$, $\sqrt{z_2}$, and $\sqrt{z_3}$. Finally, let ρ be the permutation of 1, 2, and 3 such that $|w_{\rho(1)}| > |w_{\rho(2)}| > |w_{\rho(3)}|$ and associate it with the corresponding permutation matrix. Then by our choice of z,

$$\begin{aligned} &u(w,\rho^{-1}\sqrt{z})u(\sqrt{z},I) \\ &\quad= \left(z_1^{-1}|w_{\rho(1)}|^2 + z_2^{-1}|w_{\rho(2)}|^2 + z_3^{-1}|w_{\rho(3)}|^2 + z_1|w_{\rho(1)}|^{-2} + z_2|w_{\rho(2)}|^{-2} + z_3|w_{\rho(3)}|^{-2}\right) \\ &\qquad \cdot \left(z_1 + z_2 + z_3 + z_1^{-1} + z_2^{-1} + z_3^{-1}\right) \\ &\quad= \sum_{j=0}^{2} \left(u\left(\rho w, \sqrt{z}\tau^j\sqrt{z}\right) + u\left(\rho w, \sqrt{z}\tau^j\sqrt{z}^{-1}\right)\right) \\ &\quad\leq 3\left(u(w,z) + u(w,I)\right) \end{aligned}$$

Thus if w is some point inside $B(z,T)$ and $B(I,T)$, then

$$\begin{aligned} u(w,\rho^{-1}\sqrt{z}) &\leq \frac{6T}{u(\sqrt{z},I)} \\ &\leq \frac{6T}{\sqrt{u(z,I)}} \end{aligned}$$

where the final inequality comes from equation 3.2. This shows that w is contained in $B(v, cT/\sqrt{u(z,I)})$ for $v = \rho^{-1}\sqrt{z}$ with ρ as one of the six 3×3 permutation matrices and $c = 6$ as desired. □

Now we use this to bound the trace of $K_{T,p}$ on $\mathcal{K}_p$.

$$\begin{aligned}\int_{\mathcal{K}_p} K_{T,p}(z,z)d^*z &= \int_{\mathcal{K}_p} \sum_{\gamma\in\Lambda(p)} k_T(\gamma z,z)d^*z \\ &= \sum_{\delta\in\Lambda(p)\backslash\Lambda} \int_{\mathcal{K}} \sum_{\gamma\in\Lambda(p)} k_T(\gamma\delta z,\delta z)d^*z \\ &= \sum_{\delta\in\Lambda(p)\backslash\Lambda} \int_{\mathcal{K}} \sum_{\gamma\in\Lambda(p)} k_T(\delta^{-1}\gamma\delta z,z)d^*z \\ &= [\Lambda:\Lambda(p)] \sum_{\gamma\in\Lambda(p)} \int_{\mathcal{K}} k_T(\gamma z,z)d^*z \\ &\le [\Gamma:\Gamma(p)] \sum_{\gamma\in\Gamma(p)} \int_{\mathcal{K}} k_T(\gamma z,z)d^*z \qquad (7.1)\end{aligned}$$

Note that in the last step we have removed the dependence on Λ, which will allow our final bound to be uniform. We take advantage of the compactness of $\mathcal{K}$ to show that $k_T(\gamma z,z)$ is bounded by $k_{cT}(\gamma I,I)$ for some constant c, which depends only on $\mathcal{K}$ (not T, γ, or z). Let z be in $\mathcal{K}$ and v in $\mathfrak{h}^3$. Then $||v^{-1}\gamma z||^2$ is a homogeneous, positive definite quadratic form in the matrix entries of $v^{-1}\gamma$. Thus there exists a $c>0$ depending only on $\mathcal{K}$ (independent of v, z, and γ) such that

$$||v^{-1}\gamma z||^2 \ge ||v^{-1}\gamma I||^2/c$$

where I is the identity matrix in $\mathfrak{h}^3$. Similarly, one can choose c such that

$$||z^{-1}\gamma^{-1}v||^2 \le ||\gamma^{-1}vI||^2/c,$$

as well and consequently

$$u(\gamma z,v) \le u(\gamma I,v)/c.$$

It follows that

$$\hat{k}_T(\gamma z,v) \le \hat{k}_{cT}(\gamma I,v).$$

This leads to the desired bound on $k_T(\gamma z,z)$ via

$$\begin{aligned}k_T(\gamma z,z) &= \int_{\mathfrak{h}^3} \hat{k}_T(\gamma z,v)\hat{k}_T(v,z)d^*v \\ &\le \int_{\mathfrak{h}^3} \hat{k}_{cT}(\gamma I,v)\hat{k}_{cT}(v,I)d^*v \\ &= k_{cT}(\gamma I,I)\end{aligned}$$

Returning to inequality 7.1, we obtain

$$\int_{\mathcal{K}_p} K_{T,p}(z,z)d^*z << p^8 \sum_{\gamma\in\Gamma(p)} k_{cT}(\gamma I, I).$$

Using Lemma 5, we can bound the geometric trace in terms of T and p.

$$\begin{aligned}
&\int_{\mathcal{K}_p} K_{T,p}(z,z)d^*z \\
&\quad << p^8 \lim_{M\to\infty} \sum_{j=0}^{M-1} \left(N\left(p, 6+\frac{(2T-6)(j+1)}{M}\right) - N\left(p, 6+\frac{(2T-6)j}{M}\right)\right) \frac{T^{2+\epsilon}}{6+\frac{(2T-6)j}{M}} \\
&\quad = p^8 N(p,2T) + \lim_{M\to\infty} \sum_{j=0}^{M-1} \frac{(2T-6)}{M} N\left(p, 6+\frac{(2T-6)j}{M}\right) \frac{p^8 T^{2+\epsilon}}{\left(6+\frac{(2T-6)j}{M}\right)\left(6+\frac{(2T-6)(j+1)}{M}\right)} \\
&\quad = p^8 N(p,2T) + p^8 T^{2+\epsilon} \int_6^{2T} N(p,X) X^{-2} dX \\
&\quad <<_\epsilon p^8 \left(\frac{T^{3+\epsilon}}{p^9} + \frac{T^{5/2+\epsilon}}{p^6} + \frac{T^{2+\epsilon}}{p^4} + 1\right) + T^{2+\epsilon} \int_6^{2T} \left(\frac{X^{1+\epsilon}}{p} + X^{1/2+\epsilon}p^2 + X^\epsilon/p^4 + X^{-2}p^8\right) dX \\
&\quad << \frac{T^{4+2\epsilon}}{p} + T^{7/2+2\epsilon}p^2 + T^{3+2\epsilon}p^4 + T^{1+\epsilon}p^8 \qquad (7.2)
\end{aligned}$$

Chapter 8

The $\mathcal{L}^2$-concentration Property

We have mentioned that we would like to take the trace of the integral operator, $I_{p,T}$, over some compact subset $\mathcal{K}_p$ of $\Lambda(p)\backslash\mathfrak{h}^n$, but to this point we have not yet constructed an appropriate subset. There are two important properties that we desire from these compact subsets. We would like for them to be compatible with each other, essentially constructed from copies of a single compact subset of $\Lambda\backslash\mathfrak{h}^n$, and we would like them to be "large enough" to contain most of the L^2-mass of $\Lambda(p)$-Maass forms. These two properties are made explicit in the following definition.

Definition 4. *Let Λ be a Zariski dense subgroup of $SL_n(\mathbb{Z})$. The subgroup Λ is said to have the* $\mathcal{L}^2$-concentration property *when there is a compact subset $\mathcal{K}$ of $\Lambda\backslash\mathfrak{h}^n$ such that, if $\mathcal{K}_p$ is the preimage of $\mathcal{K}$ under the natural projection $\Lambda(p)\backslash\mathfrak{h}^n \to \Lambda\backslash\mathfrak{h}^n$, $\nu_0 > 1/n$, and Φ is any $\Lambda(p)$-Maass form with spectral parameters (ν_1, ν_2) satisfying $\nu_1, \nu_2 > \nu_0$, then*

$$\int_{\mathcal{K}_p} |\Phi(z)|^2 d^*z >>_{\nu_0} \int_{\Lambda(p)\backslash\mathfrak{h}^n} |\Phi(z)|^2 d^*z$$

where the implied constant is independent of p, ν_1, ν_2, and Φ.

That is to say, that the $\mathcal{L}^2$ norms over $\mathcal{K}_p$ of all $\Lambda(p)$-Maass forms (with spectral parameters at least some fixed amount larger than $1/n$) are at least some positive portion of its $\mathcal{L}^2$ norm on all of $\Lambda(p)\backslash\mathfrak{h}^n$, independent of the form and p. When our $\Lambda(p)$-Maass forms are normalized to have norm 1 on $\Lambda(p)\backslash\mathfrak{h}^n$, it is more simply put that their $\mathcal{L}^2$-norm on $\mathcal{K}_p$ is greater than some fixed positive constant. Assuming that Λ has the $\mathcal{L}^2$-concentration property, we can update the trace inequality from 4.3.

$$\begin{aligned}\int_{\mathcal{K}_p} K_{p,T}(z,z)d^*z &\geq \sum_{\phi,\nu} h_T(\nu) \int_{\mathcal{K}_p} |\Phi(z)|^2 d^*z \\ &>> \sum_{(\phi,\nu):\nu_j>\nu_0} h_T(\nu) \\ &= \sum_{\nu:\nu_j>\nu_0} \dim \mathcal{M}(\Lambda,p,\nu) h_T(\nu)\end{aligned}$$

where once again the implied constant is independent of p, the sums are over all sets of spectral parameters ν of $\Lambda(p)$-Maass forms (and all $\Lambda(p)$-Maass forms Φ having spectral parameters ν). If we suppose that ν is the vector of parameters for some Λ-newform of level p such that $\nu_1, \nu_2 > \nu_0$, then we can apply our bound on the dimension of $\mathcal{M}(\Lambda,p,\nu)$ from Lemma 1 to see that

$$p^{n-1} h_T(\lambda) << \int_{\mathcal{K}_p} K_{p,T}(z,z) d^*z. \tag{8.1}$$

This gives us a bound on $h_T(\nu)$ based on the geometric trace of $I_{p,T}$ over $K_{p,T}$, which we have already estimated in the previous chapter.

We would like to know then when a group has the $\mathcal{L}^2$-concentration property. For $n = 2$, this is addressed in [Gamburd, 2002]. A similar result for n-dimensional hyperbolic space, (taking $\Gamma = SO(n,1)$), is discussed in [Magee, 2013]. The $n = 2$ result can be stated with the help of two other properties of subgroups of $SL_n(\mathbb{Z})$, geometrically finiteness and limit sets. A group Λ is said to be *geometrically finite* if it has a fundamental domain in $\mathfrak{h}^n$ with finitely many bounding, geodesic sides. The *limit set* for Λ is all limit points of ΛI inside $M_n^{\geq 0}(\mathbb{R})/(\mathbb{R}^\times_{>0} SO_n(\mathbb{R}))$ where $M_n^{\geq 0}(\mathbb{R})$ is the space of all n by n matrices with non-negative determinant. Since Λ acts discretely on $\mathfrak{h}^n$, these limit points all lie in the space

$$\partial\mathfrak{h}^n := M_n^0(\mathbb{R})/(\mathbb{R}^\times_{>0} SO_n(\mathbb{R}))$$

where $M_n^0(\mathbb{R})$ is all n by n matrices with determinant 0. The space $\partial\mathfrak{h}^n$ has dimension $n(n+1)/2-2$, which is an upper bound on the Hausdorff dimension of the limit set for Λ. For $n = 2$, the result is

Theorem 8. *[Gamburd, 2002] Let Λ be a Zariski dense, geometrically-finite subgroup of $SL_2(\mathbb{Z})$. If the Hausdorff dimension of the limit set of Λ in $\mathfrak{h}^2$ is greater than $1/2$, then Λ has the $\mathcal{L}^2$-concentration property.*

This follows from two key lemmas, the "cusp lemma" and the "collar lemma". In essence, these lemmas identify boundary features of $\Lambda(p)\backslash\mathfrak{h}$, unbounded pieces of some fundamental domain for $\Lambda(p)\backslash\mathfrak{h}$, and show that each of these features individually satisfies the $\mathcal{L}^2$-concentration property.

An analogous result for $n > 2$ would be the following conjecture

Conjecture 3. *Let Λ be a Zariski dense, geometrically finite subgroup of $SL_n(\mathbb{Z})$. There exists some $\delta_n < n(n+1)/2 - 2$ such that if the Hausdorff dimension of the limit set of Λ in $\mathfrak{h}^n$ is at least δ_n then Λ has the $\mathcal{L}^2$-concentration property.*

The limit set for any finite-index subgroup Λ has dimension $n(n+1)/2 - 2$, so they would all have this property. A potential approach to proving this conjecture would be to identify all possible boundary features of $\Lambda(p)\backslash\mathfrak{h}^n$ and show the analogous "cusp/collar lemma" for each such feature.

Let us take for example a finite-index subgroup Λ of $\Gamma = SL_3(\mathbb{Z})$. To understand the boundary features of $\Lambda(p)\backslash\mathfrak{h}^3$, let us first examine $\Gamma\backslash\mathfrak{h}^3$. We can take the following set as a fundamental domain for $\Gamma\backslash\mathfrak{h}^3$ [Greenier, 1988].

$$\mathcal{F} = \{z \in \mathfrak{h}^3 : 1 \leq y_1^2 + x_3^2 \leq y_1^2 y_2^2 + x_1^2 y_1^2 + x_2^2, |x_j| \leq 1/2 \text{ for } 1 \leq j \leq 3\}$$

The boundary feature for this domain is its one cusp, defined up to choice of $Y \geq \sqrt{3}/2$ as

$$\mathcal{C}_Y = \{z \in \mathcal{F} : y_j > Y \text{ for } j = 1 \text{ or } 2\}.$$

This is the only boundary feature, as $\mathcal{F} \setminus \mathcal{C}_Y$ is compact. The constraints for $\mathcal{F}$ can be quite cumbersome, so it is convenient to use a Siegel set as an approximation [Goldfeld, 2006].

$$\Sigma := \left\{ z \in \mathfrak{h}^3 : |x_j| \leq 1/2, y_k \geq \frac{\sqrt{3}}{2} \text{ for } 1 \leq j \leq 3, 1 \leq k \leq 2 \right\}$$

The Siegel set is a good approximation in the sense that

1. $\mathfrak{h}^3 = \bigcup_{\gamma \in \Gamma} \gamma.\Sigma$,

2. For all z in $\mathfrak{h}^3$ there exist only finitely many γ in Γ such that $\gamma.z$ is in Σ.

Another way of saying this is that Σ contains a fundamental domain for Γ, and intersects finitely many translates of this fundamental domain by elements of Γ. The significance for us is that if we have a function F in $\mathcal{L}^2(\Gamma\backslash\mathfrak{h}^3)$ and extend it naturally as a function on $\mathfrak{h}^3$, then

$$\int_{\Gamma\backslash\mathfrak{h}^3} |F(z)|^2 d^*z \leq \int_{\Sigma} |F(z)|^2 d^*z << \int_{\Gamma\backslash\mathfrak{h}^3} |F(z)|^2 d^*z,$$

where the implied constant is independent of F, i.e. the $\mathcal{L}^2$-norm of F on Σ differs from its $\mathcal{L}^2$ norm on $\Gamma\backslash\mathfrak{h}^3$ only by some bounded constant scaling.

If we examine the cusp of Σ,

$$\widetilde{\mathcal{C}}_Y := \{z \in \Sigma : y_j \geq Y \text{ for } j = 1 \text{ or } 2\}$$

for any $Y \geq \sqrt{3}/2$, it similarly approximates the cusp of $\Gamma\backslash\mathfrak{h}^3$,

$$\int_{\mathcal{C}_Y} |F(z)|^2 d^*z \leq \int_{\widetilde{\mathcal{C}}_Y} |F(z)|^2 d^*z << \int_{\mathcal{C}_Y} |F(z)|^2 d^*z$$

Thus far we have focused only on the boundary feature of $\Lambda = \Gamma$, but all boundary features for Λ of finite index in Γ are essentially the same. Taking such a Λ and a system of representatives S for Γ/Λ, $S.\mathcal{F}$ is a fundamental domain for Λ. As such, the cusps of $\Lambda\backslash\mathfrak{h}^3$ have fundamental domains,

$$\{\gamma \in S : \gamma.\mathcal{C}_Y\},$$

and the complement of their union in $\Lambda\backslash\mathfrak{h}^3$ is compact. If we have some Λ-Maass form Φ with spectral parameters ν, then the behavior of Φ at a cusp $\gamma.\mathcal{C}_Y$ is the same as the behavior of $\gamma^{-1}.\Phi$ at $\mathcal{C}_Y$, where γ^{-1} acts by the left regular action. Because the action of $Z(\mathcal{U}(\mathfrak{gl}_n(\mathbb{R})))$ commutes with the left regular action of $SL_n(\mathbb{R})$, $\gamma^{-1}.\Phi$ is a $\gamma\Lambda\gamma^{-1}$-Maass form with spectral parameters ν. Thus if we wish to show that all finite index subgroups of Γ have the $\mathcal{L}^2$-concentration property, it will be enough to analyse their Maass-forms at the usual cusp for Γ, $\mathcal{C}_Y$.

The following is an analogue to the cusp lemma in this context, which would enable us to show that finite-index subgroups of Γ have the $\mathcal{L}^2$-concentration property.

Conjecture 4. *Let Λ be a finite index subgroup of $SL_3(\mathbb{Z})$, and let $\nu_0 > 1/3$. Then for all $\Lambda(p)$-Maass forms with spectral parameters $\nu_1, \nu_2 > \nu_0$,*

$$\int_{C_Y\backslash C_{2Y}} |\Phi(z)|^2 d^*z >>_{\nu_0} \int_{C_{2Y}} |\Phi(z)|^2 d^*z,$$

where the implied constant is independent of Φ and p.

The usual approach here is to decompose Φ into a series of simpler functions, in this case Whittaker functions; show that the Whittaker functions satisfy the collar lemma above; and then show that the Φ satisfies the collar lemma as a consequence.

Chapter 9

Concluding Proof of Main Theorem

We now combine the results of chapters 7 and 8. Let Λ be a Zariski dense subgroup of $\Gamma = SL_3(\mathbb{Z})$ with the $\mathcal{L}^2$-concentration property defined in 4. Let Φ be a Λ-newform of level p and spectral parameters ν. Then if p is sufficiently large, combining the inequalities 7.2 and 8.1 with the estimate of h_T from 5.6, we see that for all $\epsilon > 0$ and all $\nu_1, \nu_2 > 1/3$ (equivalently $\Re\nu_1, \Re\nu_2 \neq 1/3$),

$$p^2 T^{3\nu_1+3\nu_2} <<_\epsilon \frac{T^{4+\epsilon}}{p} + T^{7/2+\epsilon}p^2 + T^{3+\epsilon}p^4 + T^{1+\epsilon}p^8$$

Choosing to take $T = p^6$ optimizes the bound on ν_1 and ν_2. Solving for $\nu_1 + \nu_2$ and taking the limit as $\epsilon \to 0$ we find

$$\nu_1 + \nu_2 \leq 7/6$$

This bound is the main theorem, summarized below.

Theorem 9. *Let Λ be a Zariski dense subgroup of $SL_3(\mathbb{Z})$ with the $\mathcal{L}^2$-concentration property. Let Φ be a Λ-newform of level p and spectral parameters ν. Then for p sufficiently large,*

$$\Re\nu_1 + \Re\nu_2 \leq 7/6.$$

Part II

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Part III

Appendix

Appendix A

Relating Eigenvalues and Spectral Parameters

The following short Mathematica code generates the polynomial relations between eigenvalues and spectral parameters as seen in section 2. It uses the GL(n)Pack by Kevin A. Broughan, which is available at `http://www.math.waikato.ac.nz/~kab` [Broughan, 2006]. We generate polynomials l2 and l3 in v1 and v2 (representing the $SL_3(\mathbb{Z})$ spectral parameters ν_1 and ν_2). l2 and l3 are the eigenvalues with respect to Δ_2 and Δ_3, which correspond to ν_1 and ν_2.

```
v := {v1, v2}
z := DiagonalMatrix[{y[1]*y[2], y[1], 1}]
Iv := IFun[v, z]
l2 := Simplify[ApplyCasimirOperator[2, Iv, z]/Iv]
l3 := Simplify[ApplyCasimirOperator[3, Iv, z]/Iv]
```