Recursive Utility with Narrow Framing: Properties and Applications

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Contents

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my mother, Qiu Xiuhong,

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Introduction

When making multiple decision choices or evaluating multiple risks, individuals tend to consider one of them at a time, isolating it from other choices or risks, a phenomenon referred to as *narrow framing*; see for instance Tversky and Kahneman (1981) and Kahneman and Lovallo (1993). Since then, narrow framing has been found extensively both in experimental settings and in real lives. However, there was little work on building a broadly applicable model of narrow framing until Barberis and Huang (2009); see also Barberis et al. (2006), Barberis and Huang (2008a).[1]

The model proposed by Barberis and Huang (2009), which we refer to as the *BH model* in the following, is formulated by generalizing the classical recursive utility model (Epstein and Zin, 1989, Kreps and Porteus, 1978) in that (i) the risks that are evaluated in isolation by the individuals at the end of each period are assessed according to prospect theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992) so that the utility of gains and losses experienced by the individuals in these risks is calculated, (ii) the utility of gains and losses and the certainty

[1]In an earlier work by Barberis and Huang (2001), the authors propose an asset pricing model with narrow framing, in which the representative agent evaluates the investment gain and loss of each individual stock in isolation with consumption risk. This model, however, is specific for asset pricing, intractable in partial equilibrium settings, and inapplicable in the study of individuals' attitudes toward timeless gambles; see the discussion in Barberis and Huang (2008a, p. 210).

equivalent of the individuals' total utility from next period are added linearly with the weight for former to be a positive constant in the linear addition, and (iii) the sum of the utility of gains and losses and the certainty equivalent of the total utility from next period is aggregated with the individuals' consumption in the current period via an aggregation function, resulting in the individuals' total utility at the beginning of the current period. The certainty equivalent and aggregation function are chosen so that the so-called relative risk aversion degree (RRAD) and elasticity of intertemporal substitution (EIS) are constant, and the time horizon is usually set to be infinite. In particular, when the utility of gains and losses is set to be zero, the model of narrow framing degenerates into the classical recursive utility model.

The BH model provides an analytical framework to study the impact of narrow framing on decision making. In particular, this model has been successfully applied to explain individuals' attitude towards some monetary gambles that cannot be easily explained by many models of preferences and attitude towards large gambles, such as no-participation in the stock market; see Barberis et al. (2006) and Barberis and Huang (2008a, 2009). When applied to investment decision making, the narrow framing component of the BH model allows for utility from gains and losses in financial wealth, which are the foremost sources of utility when people invest. Therefore, the BH model has strong implications for portfolio selection and asset pricing, such as explaning high equity premia in the market; see for instance Barberis and Huang (2009), De Giorgi and Legg (2012), He and Zhou (2014), and Easley and Yang (2015).

Even with many successful applications, however, the existence and uniqueness of the BH model have not been established. Indeed, in the infinite-horizon setting, the agent's total utility is defined recursively without an end date, so its existence and uniqueness cannot be taken for granted. Surprisingly, even for the classical recursive utility, its existence and uniqueness have not been completely established; see Section 1.2.1.

In this thesis, we study the existence and uniqueness of the BH model of narrow framing, propose a new model of narrow framing that is better behaved than the BH model, and apply the new model to study an asset pricing problem with heterogeneous agents with different degrees of narrow framing and different degrees of loss aversion.

In Chapter 1, assuming constant EIS and RRAD, we study the existence and uniqueness of the agent's total utility in the model of narrow framing in a Markovian setting in which the market state, $\{X_t\}$, follows a Markovian process with the state space to be finite; see Section 1.2.1. This setting is not restrictive in terms of applying the model of narrow framing because (i) finite-state Markovian processes can be sufficiently flexible to describe financial data and (ii) we also allow for a sequence of market shocks, $\{Y_t\}$, that are conditionally independent of the market state and we do not impose any assumption on the shocks.

We prove that the agent's total utility in the model of narrow framing uniquely exists when her investment utility in each period is nonnegative, regardless of the values of the EIS and RRAD. When the investment utility becomes negative in some states, however, the total utility in the BH model can be nonexistent or nonunique even in some simple settings, such as in the setting in which the EIS is less than or equal to one and the state space of $\{X_t\}$ is singleton. In this case, we propose a sufficient condition under which the total utility in the model of narrow framing uniquely exists, and this condition is nearly necessary.

We also prove that if the total utility in the BH model uniquely exists, it can be obtained by applying the recursive equation defining the utility repeatedly with any positive utility value as the starting point. This result is not only computationally useful but also economically important: the infinite-horizon setting can be regarded as a limit of the finite-horizon setting when the number of periods in the latter is sent to infinity, and the utility in the latter setting converges to that in the former setting regardless of the terminal utility in the latter setting.

We then consider a portfolio selection model with narrow framing. We prove that a consumption-investment plan is optimal if and only if it, together with the value function of the portfolio selection problem, satisfies the dynamic programming equation. Moreover, we prove that the solution to the dynamic programming equation uniquely exists and can be computed by solving the equation recursively with any starting point. As a result, the portfolio selection problem in a finite-horizon setting approaches that in the infinite horizon setting when the number of periods in the former goes to infinity.

The analysis in Chapter 1 implies that the total utility process in the BH model may not exist. In Chapter 2, we show that when (i) an agent consumes a constant fraction of her wealth and invests another constant fraction of her wealth into some assets whose returns are independent and identically distributed (i.i.d.) over time, (ii) the EIS is less than or equal to one, and (iii) the utility of gains and losses experienced by the agent is negative, it is either the case in which the total utility process of the agent in the BH model does not exist or the case in which there exists two solutions to the recursive equation that defines the total utility process. We further show that the BH model may fail to define the total utility process even in finite-horizon setting. These findings reveal that one needs some restrictions on the parameters in the BH model to have a uniquely defined total process, but these restrictions are too tight for many financial applications when the economic agents in those applications are assumed to face negative utilities of gains and losses and to have EIS less than one; see for instance see for instance Barberis and Huang (2009), De Giorgi and Legg (2012).

Therefore, in Chapter 2, we propose a refinement of the BH model, referred to as the *GH model*. In this new model, the utility of gains and losses and the certainty equivalent of the individuals' total utility from next period are also added linearly, but instead of a constant, the weight for the utility of gains and losses is scaled in a sense that it is proportional to the certainty equivalent of the total utility from next period per unit wealth. The GH model defines a unique total utility process when the time horizon is finite, and by starting from any positive value and applying the recursive equation in the GH model repeatedly, we can obtain, as a limit, the total utility process in the infinite-horizon setting. Moreover, we show that the GH model is more tractable than the BH model when applied to the study of asset pricing and individuals' attitudes towards timeless gambles.

In Chapter 3, we apply the GH model to a heterogeneous-agent asset pricing model with two types of agents. The first type is Epstein-Zin (EZ) agents whose preferences are represented by the classical recursive utility. The second type is loss-averse (LA) agents whose preferences are represented by the GH model, so they are averse to the gain and loss incurred by the investment in risky assets. The agents can trade a risk-free asset with zero net supply and a risky stock with positive net supply in discrete time. The stock pays out independent and identically distributed (i.i.d.) dividends over time.

First, we show that when the LA-agents' loss aversion degree (LAD) is equal to the so-called equilibrium gain-loss ratio of the stock, they behave the same as the EZ-agents and thus the presence of the LA-agents in the economy does not affect equilibrium asset prices. This ratio is defined to be the ratio of the gain and loss of the stock relative to a reference point, and this reference point is endogenously determined in equilibrium: it is the equilibrium risk-free return in an economy with EZ-agents only. When the LA-agents' LAD is higher (lower) than the equilibrium gain-loss ratio, the LA-agents appear to be more (less) risk averse than the EZ-agents, and we propose a measure to quantify the risk aversion of the LA-agents.

Second, when the RRAD and EIS are equal to one, we prove the existence and uniqueness of the equilibrium. Moreover, when the economy is populated with only one EZ- and one LA-agent, we find that the LA-agent invests less (more) in the stock than the EZ-agent if and only if the LA-agent's LAD is higher (lower) than the equilibrium gain-loss ratio. Consequently, the conditional equity premium is increasing (decreasing) with respect to the wealth share of the LA-agent in the market when her LAD is higher (lower) than the equilibrium gain-loss ratio. In addition, we prove that the EZ-agent dominates the market in the long run unless the LA-agent's LAD is exactly equal to the equilibrium gain-loss ratio.

The readers can also refer to Guo and He (2016) for Chapter 1 of the thesis, Guo and He (2017b) for Chapter 2, and Guo and He (2017a) for Chapter 3.

Chapter 1

Recursive Utility with Narrow Framing: Existence and Uniqueness

1.1 Introduction

Barberis and Huang (2008a, 2009) and Barberis et al. (2006) propose a model of narrow framing in a discrete-time multiple-period setting in which an agent derives utility not only from her consumption stream but also from the investment gain and loss incurred by holding certain risky assets; the former is referred to as consumption utility and the latter is referred to as investment utility. The total utility of the agent is computed based on the classical recursive utility model (Epstein and Zin, 1989, Kreps and Porteus, 1978): the total utility of the agent's consumption and investment after time t is the aggregation of 1) her consumption at time t, 2) her investment utility in period from t to $t+1$, and 3) the time-t certainty equivalent of the agent's total utility of consumption and investment after time $t+1$. In particular, when the investment utility is set to be zero, the model of narrow framing degenerates into the classical recursive utility model.

Just as in the classical recursive utility model, the aggregation of different components of utility in the model of narrow framing is achieved by a function named aggregator and the certainty equivalent is computed under the expected utility theory. The aggregator thus measures the elasticity of intertemporal substitution (EIS) and the certainty equivalent measures the relative risk aversion degree (RRAD) of the agent. As in many applications of the classical recursive utility model to portfolio selection and asset pricing, in their model of narrow framing, Barberis and Huang (2008a, 2009) and Barberis et al. (2006) select a specific aggregator in which the EIS is constant and a specific certainty equivalent in which the RRAD is constant; see the exact forms in (1.2.2) and (1.2.3). Furthermore, the authors adopt an infinite-horizon setting. Both the specific choice of the aggregator and certainty equivalent and the infinite-horizon setting are known to be simple and helpful in obtaining close-form solutions to a variety of problems.

The model of narrow framing is successful in explaining some empirical findings, such as why people are averse to a small, independent gamble, even when the gamble is actuarially favorable; see for instance Barberis et al. (2006). The model of narrow framing is further extended by De Giorgi and Legg (2012) and He and Zhou (2014) with various applications, and these authors also assume constant EIS and RRAD and adopt the infinite-horizon setting. Even with many successful applications, however, the existence and uniqueness of the agent's total utility in the model of narrow framing have not been established. Indeed, in the infinite-horizon setting, the agent's total utility is defined recursively without an end date, so its existence and uniqueness cannot be taken for granted. Surprisingly, even for the classical recursive utility, its existence and uniqueness have not been completely established; see Section 1.2.1.

In the present chapter, assuming constant EIS and RRAD, we study the existence and uniqueness of the agent's total utility in the model of narrow framing in a Markovian setting. More precisely, we assume a Markovian process $\{X_t\}$ and a process $\{Y_t\}$ that is i.i.d. conditioning on

$\{X_t\}$. Thus, $\{X_t\}$ can be regarded as market states and $\{Y_t\}$ can be interpreted as random noise. The asset returns in period from t to $t+1$ are assumed to be functions of X_t, X_{t+1}, and Y_{t+1}, so the agent's consumption propensity and investment in the assets in that period are functions of X_t. We further assume that $\{X_t\}$ is irreducible and its state space is finite.

The Markovain setting here is the same as in Hansen and Scheinkman (2012), who show the existence of the classical recursive utility with non-unitary EIS and RRAD and the uniqueness of the utility with further conditions on the EIS and RRAD, except that we assume the state space to be finite; see Section 1.2.1. This setting is not restrictive in terms of applying the model of narrow framing because (i) finite-state Markovian processes can be sufficiently flexible to describe financial data and (ii) we do not impose any assumption on $\{Y_t\}$.

We prove that with the same condition as used by Hansen and Scheinkman (2012) to obtain the existence of the classical recursive utility, the agent's total utility in the model of narrow framing uniquely exists when her investment utility in each period is nonnegative, regardless of the values of the EIS and RRAD. When the investment utility becomes negative in some states, however, the total utility in the model of narrow framing can be non-existent or non-unique even in some simple settings, such as in the setting in which the EIS is less than or equal to one and the state space of $\{X_t\}$ is singleton. In this case, we propose a sufficient condition under which the total utility in the model of narrow framing uniquely exists, and this condition is nearly necessary.

We also prove that if the total utility in the model of narrow framing uniquely exists, it can be obtained by applying the recursive equation defining the utility repeatedly with any positive utility value as the starting point. This result is not only computationally useful but also economically important: the infinite-horizon setting can be regarded as a limit of the finite-horizon setting when the number of periods in the latter is sent to infinity, and the utility in the latter setting converges to that in the former setting regardless of the terminal utility in the latter setting.

We then consider a portfolio selection model with narrow framing. We prove that a consumption-investment plan is optimal if and only if it, together with the value function of the portfolio selection problem, satisfies the dynamic programming equation. Moreover, we prove that the solution to the dynamic programming equation uniquely exists and can be computed by solving the equation recursively with any starting point. As a result, the portfolio selection problem in a finite-horizon setting approaches that in the infinite horizon setting when the number of periods in the former goes to infinity.

The remainder of the chapter is organized as follows: In Section 1.2 we review the literature and in Section 1.3 we introduce the model of narrow framing that is used in the literature. In Section 1.4 we prove the existence and uniqueness of the total utility in the model of narrow framing in a finite-state Markovian setting and apply the results to a portfolio selection problem. In Section 1.5, we study the case where the state space is not finite. Section 1.6 concludes. Proofs are placed in Appendix 3.6.

1.2 Literature

1.2.1 Existence and Uniqueness of Recursive Utility

Recursive utility is a classical model for individual's preferences for discrete-time consumption streams; see Kreps and Porteus (1978) and Epstein and Zin (1989). In an infinite-horizon setting, the recursive utility of consumption stream C_t, $t = 0, 1, \dots$ derived by an agent is represented by V_t, $t = 0, 1, \dots$, where V_t stands for the utility of the consumption stream starting from time t, i.e., C_s, $s \geq t$. The recursive utility process $\{V_t\}$ is defined recursively:

$$V_t = H(C_t, \mathcal{M}_t(V_{t+1})), \quad t = 0, 1, \dots, \tag{1.2.1}$$

where $\mathcal{M}_t(X)$ stands for the *certainty equivalent* of random quantity X conditioning on the information at time t and $H(c, z)$ is an *aggregator*. There are many choices for the certainty equivalent and aggregator, but the following one, which was first proposed by Kreps and Porteus (1978), is popular due to its tractability in deriving asset pricing results (see e.g., Epstein and Zin, 1990, 1991):

$$H(c, z) := \begin{cases} [(1-\beta)c^{1-\rho} + \beta z^{1-\rho}]^{\frac{1}{1-\rho}}, & 0 < \rho \neq 1, \\ e^{(1-\beta)\ln c + \beta \ln z}, & \rho = 1, \end{cases} \tag{1.2.2}$$

$$\mathcal{M}_t(X) := u^{-1}\big(\mathbb{E}_t[u(X)]\big), \quad u(x) := \begin{cases} x^{1-\gamma}, & 0 < \gamma \neq 1, \\ \ln(x), & \gamma = 1, \end{cases} \tag{1.2.3}$$

where $\mathbb{E}_t$ stands for the expectation operator conditioning on the information at time t. In addition, $\beta \in (0,1)$ is a discount rate, γ stands for the relative risk aversion degree (RRAD), and $1/\rho$ is the elasticity of intertemporal substitution (EIS); see for instance Kreps and Porteus (1978) and Epstein and Zin (1989).

In the following, when $\rho \geq 1$, we set $H(c,0) = \lim_{z\downarrow 0} H(c,z) = 0$ and $H(0,z) = \lim_{c\downarrow 0} H(c,z) = 0$. As a result, $H(c,z)$ is well defined, takes real values, and continuous in $(c,z) \in [0,\infty)^2$. Similarly, when $\gamma = 1$, we define $u(0) := -\infty$ and $u^{-1}(-\infty) := 0$; when $\gamma > 1$, we define $u(0) := +\infty$ and $u^{-1}(+\infty) := 0$. As a result, $\mathcal{M}_t(X)$ is well defined for any nonnegative random variable X. Moreover, when $\gamma \geq 1$ and $X = 0$ with positive probability, $\mathcal{M}_t(X) = 0$.

Note that in the infinite-horizon setting the recursive utility process is defined recursively *without* a terminal condition, so the existence and uniqueness of this process is not automatically guaranteed. Epstein and Zin (1989) prove the existence when the aggregator is given by (1.2.2). Ma (1993, 1996, 1998) prove the existence and uniqueness of the recursive utility process by assuming that $H_z(c,z)$, the derivative of the aggregator $H(c,z)$ with respect to z, is bounded uniformly in c and z by a number strictly less than one.[1] However, this assumption does not hold for H as defined in (1.2.2) for any $\rho > 0$. Ozaki and Streufert (1996) prove the existence and uniqueness of the recursive utility process by assuming $H_z(c,z)$ to be uniformly bounded in c and z and a set of conditions.[2] However, these conditions are difficult to verify; see conditions

[1]See Assumption W4 in Ma (1993, p. 246) and Ma (1996, p. 568). In Ma (1998), the author assumes the recursive utility for deterministic consumption flows is well defined, but this requires $H_z(c,z)$ to be bounded by a number strictly less than one as well; see Footnote 5 of Ma (1998) and Assumption W5 in Lucas and Stokey (1984).

[2]In Ozaki and Streufert (1996, Theorem D), the authors assume that $\bar{\delta}$ and δ therein are finite, which is equiv-

N1–N12 in Ozaki and Streufert (1996, pp. 404–405); in addition, for H as defined in (1.2.2), $H_z(c, z)$ is not bounded when $\rho \leq 1$.

Marinacci and Montrucchio (2010) consider Thompson and Blackwell aggregators and study the existence and uniqueness of the recursive utility with these two types of aggregators. One can check that H as defined in (1.2.2) satisfies properties (W-i), (W-ii), and (W-iii) in Marinacci and Montrucchio (2010, p. 1783), satisfies property (W-iv) therein if and only if $\rho < 1$, and does not satisfy property (W-v) therein for any $\rho > 0$. Thus, H as defined in (1.2.2) with $\rho < 1$ is a Thompson aggregator, but the case in which $\rho \geq 1$ is neither Thompson nor Blackwell. Moreover, u as defined in (1.2.3) is constant relative risk averse (CRRA), i.e., $-xu''(x)/u'(x)$ is constant in x, so Theorem 3-(ii) of Marinacci and Montrucchio (2010) applies, showing that the recursive utility process uniquely exists if $\rho < 1$ and the growth rate of the consumption process $\{C_t\}$ is properly constrained. The case in which $\rho \geq 1$, however, is not studied by Marinacci and Montrucchio (2010).[3]

Hansen and Scheinkman (2012) assume that the consumption growth rate $C_{t+1}/C_t = \exp[\kappa(X_t, X_{t+1}, Y_{t+1})]$ for some function κ, where $\{X_t\}$ is a Markov process and the joint distribution of (X_{t+1}, Y_{t+1}) conditioned on (X_t, Y_t) depends only on X_t. They show that for H and $\mathcal{M}_t$ as defined, respectively, in (1.2.2) and (1.2.3) with $\rho \neq 1$ and $\gamma \neq 1$, the recursive utility process exists. They also show the uniqueness when $(1-\gamma)/(1-\rho) \geq 1$.

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alent to assuming that $H_z(c, z)$ is bounded; see pages 403–406 therein.

[3]Alternatively, one can consider the following transformation: $\tilde{V}_t := f(V_t)$, where $f(x) := x^{1-\rho}$ when $\rho \neq 1$ and $f(x) := \ln(x)$ when $\rho = 1$. Then, we have $\tilde{V}_t = \tilde{H}(C_t, \tilde{\mathcal{M}}_t(\tilde{V}_{t+1}))$ for a new aggregator $\tilde{H}$. However, $\tilde{H}(0, z)$ is finite only if $\rho < 1$, and the aggregators considered in Marinacci and Montrucchio (2010) are assumed to take real values for any $c, z \geq 0$, so the results in Marinacci and Montrucchio (2010) do not apply to the case $\rho \geq 1$ either even if we do the transformation.

1.2.2 A Recursive Utility Model with Narrow Framing

Barberis and Huang (2008a, 2009) and Barberis et al. (2006) consider a model of narrow framing: at time t, an agent consumes C_t and invests dollar amount $\Theta_{i,t}$ in asset i, $i = 0, 1, \dots, n$, where asset 0 is a risk-free asset and the other assets are risky. For simplicity, assume that short-selling of the risky assets is not allowed, i.e., $\Theta_{i,t} \geq 0$, $i = 1, \dots, n$. The agent's utility process $\{U_t\}$ is defined recursively as follows:

$$U_t = H\left(C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} b_i G_{i,t}\right), \quad t = 0, 1, \dots, \tag{1.2.4}$$

where H is an aggregator, $\mathcal{M}_t(X)$ is the certainty equivalent of X given information at time t, $b_i \geq 0$ is a constant, and $G_{i,t}$ stands for the utility of the gain and loss experienced by the agent for her investment in asset i. More precisely, in Barberis and Huang (2008a, 2009) and Barberis et al. (2006),

$$G_{i,t} = \mathbb{E}_t\left[\Theta_{i,t}(R_{i,t+1} - R_{f,t+1})\mathbf{1}_{R_{i,t+1} > R_{f,t+1}} + k\Theta_{i,t}(R_{i,t+1} - R_{f,t+1})\mathbf{1}_{R_{i,t+1} < R_{f,t+1}}\right]$$

for some $k \geq 1$, where $R_{i,t+1}$ and $R_{f,t+1}$ are the total returns of asset i and the risk-free asset, respectively, in period t to $t+1$. Indeed, $G_{i,t}$ represents the preference value of the agent's position in asset i under prospect theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992) with the reference point to be the risk-free return, utility function to be a piece-wise linear function in which parameter k measures the loss aversion degree of the agent, and no probability weighting. Thus, $G_{i,t}$ captures the agent's utility of the gain and loss for her investment in each individual risky asset due to narrow framing.

There are several variants and extensions of the model of narrow framing in the literature. De Giorgi and Legg (2012) generalize it by considering a piece-wise power utility function and nonlinear probability weighting functions. He and Zhou (2014) consider the case in which there is only one risky asset, but the reference point therein can be different from the risk-free return. In Barberis and Huang (2001), Barberis et al. (2001), and Li and Yang (2013a), b_i in (1.2.4) is set to be a constant proportion of a power transformation of the aggregate consumption in the market.

Although the model of narrow framing and its variants have been proven useful to explain individuals' choice under risk and the equity premium puzzle (Barberis and Huang, 2008a, 2009, Barberis et al., 2006, De Giorgi and Legg, 2012, He and Zhou, 2014), the existence and uniqueness of the utility process in these models have not been studied. The main objective of the present chapter is to establish the existence and uniqueness.

1.3 Model and Motivation

1.3.1 Model

Consider the following equation

$$V_t = H\left(c_t, \mathcal{M}_t(A_{t+1}V_{t+1}) + B_t\right), \quad t = 0, 1, \dots, \tag{1.3.1}$$

where the aggregator H and certainty equivalent $\mathcal{M}_t$ are given by (1.2.2) and (1.2.3), respectively. Here, $\{c_t\}$ stands for a consumption process, $\{A_t\}$ is a process that is used to model portfolio returns in portfolio selection problems, and $\{B_t\}$ is used to model the utility of investment gains and losses in the model of narrow framing. Our goal is to establish the existence and uniqueness of the solution $\{V_t\}$ to this equation. Compared to (1.2.1), which is the equation considered in the literature to study the existence and uniqueness of recursive utility, our formulation (1.3.1) has two additional terms A_{t+1} and B_t. The presence of B_t is motivated by the model of narrow framing, and the motivation for considering A_{t+1} will be clear momentarily.

Following Hansen and Scheinkman (2012), we consider equation (1.3.1) in a Markov environment. More precisely, we consider a Markov process $\{(X_t, Y_t)\}$ and suppose the following:

Assumption 1 *(i) $\{(X_t, Y_t)\}$ is a Markov process and the joint distribution of (X_{t+1}, Y_{t+1}) conditioned on (X_t, Y_t) depends only on X_t.*

(ii) Consumption dynamics evolve as

$$\log(c_{t+1}) - \log(c_t) + \log A_{t+1} = \kappa(X_t, X_{t+1}, Y_{t+1}), \quad t = 0, 1, \ldots$$

for some real-valued measurable function κ.

(iii) $B_t/c_t = \varpi(X_t), t = 0, 1, \ldots$ for some real-valued measurable function ϖ.

(iv) For any state x, $\mathbb{E}_t\left[u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})}\right)|X_t = x\right]$ exists.

Assumption 1-(i) is the same as Assumption 1-a) in Hansen and Scheinkman (2012): it posits that given $\{X_t\}$, $\{Y_t\}$ is an independent sequence. This assumption also implies that $\{X_t\}$ is a Markov process, and we denote its state space as $\mathbb{X}$. Assumption 1-(ii) and -(iii) are parallel to Assumption 1-b) in Hansen and Scheinkman (2012), which ensure a Markovian structure in equation (1.3.1). We assume the state space $\mathbb{X}$ to be a metric space, so the measurability in Assumption 1 is with respect to Borel σ-algebra of $\mathbb{X}$.

Dividing (1.3.1) by c_t on both sides and using the homogeneity of H, we obtain

$$V_t/c_t = H\Big(1, \mathcal{M}_t\big(A_{t+1}(c_{t+1}/c_t)(V_{t+1}/c_{t+1})\big) + B_t/c_t\Big), \quad t = 0, 1, \ldots \tag{1.3.2}$$

Thus, to solve equation (1.3.1), we only need to solve $\{V_t/c_t\}$ from (1.3.2). Moreover, because of Assumption 1, we restrict ourselves to Markovian solutions to (1.3.2), i.e., $V_t/c_t = f(X_t), t = 0, 1, \ldots$ for some function f. Then, the solution to equation (1.3.2) becomes the fixed point of operator $\mathbb{T}$ defined as

$$\mathbb{T}f(x) := H\left(1, u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})}f(X_{t+1})\right)|X_t = x\right]\right) + \varpi(x)\right), \quad x \in \mathbb{X}. \tag{1.3.3}$$

Denote $\mathcal{X}$ as the space of measurable functions on $\mathbb{X}$, $\mathcal{X}_+$ as the space of nonnegative measurable functions on $\mathbb{X}$, i.e., $\mathcal{X}_+ := \{f \in \mathcal{X} | f(x) \geq 0, x \in \mathbb{X}\}$, $\mathcal{X}_+^o$ as the space of nonnegative functions on $\mathbb{X}$ that are not zero, i.e., $\mathcal{X}_+^o := \{f \in \mathcal{X}_+ | f \neq 0\}$, and $\mathcal{X}_{++}$ as the space of positive functions on $\mathbb{X}$, i.e., $\mathcal{X}_{++} := \{f \in \mathcal{X} | f(x) > 0, x \in \mathbb{X}\}$. Recalling the definitions of H, u, and $\mathbb{T}$, we can see that the domain of $\mathbb{T}$ is contained in $\mathcal{X}_+$.

In the following, for any $f \in \mathcal{X}$, we denote f^+ as its positive part, i.e., $f^+(x) := \max(f(x), 0)$. For any $f_1, f_2 \in \mathcal{X}$, $f_1 \geq f_2$ means $f_1(x) \geq f_2(x), x \in \mathbb{X}$ and $f_1 > f_2$ means

$f_1(x) > f_2(x), x \in \mathbb{X}$. Denote $\mathbb{R}$ as the set of real numbers. For any $a \in \mathbb{R}$, it also denotes the function on $\mathbb{X}$ that takes value a in all states.

1.3.2 Motivation

We justify Assumption 1 by considering the model of narrow framing proposed by Barberis and Huang (2008a, 2009) and Barberis et al. (2006), i.e., model (1.2.4). Denote $\{W_t\}$ as the agent's wealth process corresponding to a consumption strategy, i.e., a consumption process $\{C_t\}$, and an investment strategy, i.e., the process of the dollar amount invested in asset i, $\{\Theta_{i,t}\}$, $i = 1, \ldots, n$. Then, the wealth dynamics evolve as

$$W_{t+1} = \left(W_t - C_t - \sum_{i=1}^{n} \Theta_{i,t}\right) R_{f,t+1} + \sum_{i=1}^{n} \Theta_{i,t} R_{i,t+1}, \quad t = 0, 1, \ldots$$

Because of the homogeneity of H, $\mathcal{M}_t$, and $G_{i,t}$, the agent's *utility per unit wealth*, U_t/W_t, satisfies

$$U_t/W_t = H\left(c_t, \mathcal{M}_t\Big((1-c_t)R_{p,t+1}(U_{t+1}/W_{t+1})\Big) + (1-c_t)\sum_{i=1}^{n} b_i \theta_{i,t} g_{i,t}\right) \tag{1.3.4}$$

where $c_t := C_t/W_t$ is the consumption propensity at time t, $\theta_{i,t} := \Theta_{i,t}/(W_t - C_t) \geq 0$ is the percentage allocation to risky asset i at time t, $i = 1, \ldots, n$,

$$R_{p,t+1} := R_{f,t+1} + \sum_{i=1}^{n} \theta_{i,t}(R_{i,t+1} - R_{f,t+1}) \tag{1.3.5}$$

is the portfolio return in period t to $t+1$, and

$$g_{i,t} = \mathbb{E}_t \left[(R_{i,t+1} - R_{f,t+1}) \mathbf{1}_{R_{i,t+1} > R_{f,t+1}} + k(R_{i,t+1} - R_{f,t+1}) \mathbf{1}_{R_{i,t+1} < R_{f,t+1}} \right]. \tag{1.3.6}$$

Suppose the total return rate of risky asset i in period t to $t+1$ is $R_{i,t+1} = r_i(X_t, X_{t+1}, Y_{t+1})$, $t = 0, 1, \ldots$, for some function r_i and the total return rate of the risk-free asset in period t to $t+1$ is $R_{f,t+1} = r_0(X_t)$, $t = 0, 1, \ldots$, for some function r_0. Because conditioning on (X_t, Y_t), the joint distribution of (X_{t+1}, Y_{t+1}) depends only on X_t, it is natural for the agent to consider Markovian strategies only, i.e., consider $c_t = c(X_t)$, $\theta_{i,t} = \theta_i(X_t)$, $t = 0, 1, \ldots, i = 1, \ldots, n$, for some functions c and θ_i's. Then, $g_{i,t}$ depends on X_t only. Moreover, $\{U_t\}$ is a solution to (1.2.4) if and only if $\{U_t/W_t\}$ is a solution to (1.3.1) with

$$A_{t+1} = (1 - c_t) R_{p,t+1}, \quad B_t = (1 - c_t) \sum_{i=1}^{n} b_i \theta_{i,t} g_{i,t}.$$

Furthermore, Assumption 1 holds in this example.

1.4 Existence and Uniqueness When the State Space is Finite

In this section, we study the existence and uniqueness of the solution to (1.3.1), i.e., of the fixed point of (0.16), when the state space of $\{X_t\}$ is finite. Thus, we impose

Assumption 2 *The state space for $\{X_t\}$ is finite and $\{X_t\}$ is irreducible.*

Hansen and Scheinkman (2012) consider a general Markov process when studying the solution to (1.2.1), but they assume the existence of Perron-Frobenius eigenvalue and eigenvector of a linear operator and the stochastic stability of $\{X_t\}$ after a change of measure. These assumptions are not easy to verify in general. Moreover, when applying model (1.3.1) to a decision problem, such as the portfolio selection example demonstrated in Section 1.3.2, it becomes even more difficult to find a feasible set of decisions so that these assumptions hold. Thus, we choose to focus mainly on the setting of finite-state Markov processes at the moment, and the setting of general Markov processes is discussed in Section 1.5.

Note that we assume $\{X_t\}$ to be irreducible. This assumption is necessary for the existence of the stationary distribution of $\{X_t\}$, which will be used in the following. Note also that we do not impose any assumptions on $\{Y_t\}$.

We also note that when $\mathbb{X}$ is finite, $\mathbb{T}$ is continuous. However, we cannot apply the classical Brouwer fixed point theorem to prove the existence and uniqueness of the fixed point of $\mathbb{T}$. First, the domain of $\mathbb{T}$ under consideration in the following, i.e., $\mathcal{X}_{++}$, is not compact. Second, the Brouwer theorem does not imply uniqueness of the fixed point. Third, the Brouwer theorem does not show how to compute the fixed point; we, however, will provide an easy algorithm to compute the fixed point.

1.4.1 Changing the Probability Measure

To prove the existence and uniqueness of the fixed point of $\mathbb{T}$, we follow Hansen and Scheinkman (2012) to do a change of probability measure based on the classical Perron-

Frobenius theory. To this end, consider the following operator

$$\mathbb{U}h(x) := \mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}\right)h(X_{t+1})|X_t = x\right], \quad x \in \mathbb{X}.$$

With Assumptions 1 and 2, this operator is well defined. Denote $\mathbf{P}$ as the transition matrix of $\{X_t\}$, i.e., $\mathbf{P}_{x,y} = \mathbb{P}(X_{t+1} = y|X_t = x)$, $x, y \in \mathbb{X}$. Define matrix $\tilde{\mathbf{P}}$ by $\tilde{\mathbf{P}}_{x,y} := \mathbf{P}_{x,y}\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}\right)|X_t = x, X_{t+1} = y\right]$, $x, y \in \mathbb{X}$.

Proposition 1 *Suppose Assumptions 1 and 2 hold.*

(i) Suppose $\gamma \neq 1$. Then, there exist $\eta > 0$ and $v \in \mathcal{X}_{++}$ such that

$$\mathbb{U}v(x) = \eta v(x), \quad x \in \mathbb{X}. \tag{1.4.1}$$

Moreover, η and v are the Perron-Frobenius eigenvalue and eigenvector of $\tilde{\mathbf{P}}$, respectively.

(ii) Suppose $\gamma = 1$. Then, there exist $\eta \in \mathbb{R}$ and $v \in \mathcal{X}$ such that

$$\mathbb{E}_t[\kappa(X_t, X_{t+1}, Y_{t+1})|X_t = x] = -\mathbb{E}_t[v(X_{t+1})|X_t = x] + v(x) + \eta, \quad x \in \mathbb{X}. \tag{1.4.2}$$

In addition,

$$\eta = \sum_{x\in\mathbb{X}} \pi_x \mathbb{E}_t[\kappa(X_t, X_{t+1}, Y_{t+1})|X_t = x],$$

where vector $(\pi_x)_{x\in\mathbb{X}}$ is the stationary distribution of $\{X_t\}$.

(iii) Define $\delta := u^{-1}(\eta)$*. Then,*

$$\delta = \max_{f \in \mathcal{X}_{++}} \min_{x \in \mathbb{X}} \frac{u^{-1}\Big(\mathbb{E}_t\left[u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} f(X_{t+1})\right) | X_t = x\right]\Big)}{f(x)}.$$

Proposition 1-(i) is the same as in Hansen and Scheinkman (2012, p. 11968), but Proposition 1-(ii) is new.[4] Proposition 1-(iii) transforms η obtained in Proposition 1-(i) and -(ii) into δ that is easy to use in the following. Moreover, it provides a representation of δ, which shows that δ is related to the certainty equivalent of $\exp[\kappa(X_t, X_{t+1}, Y_{t+1})]$. One can see that δ is decreasing with respect to the RRAD γ.

As we will see, δ is critical in proving the existence and uniqueness of the fixed point of $\mathbb{T}$. Thus, it is important to compute δ, i.e., to compute η. When $\gamma \neq 1$, η is the Perron-Frobenius eigenvalue of $\tilde{\mathbf{P}}$, so its computation has been studied extensively in the literature; see for instance Chanchana (2007). When $\gamma = 1$, η is actually the expectation of $\kappa(X_t, X_{t+1}, Y_{t+1})$ under the stationary distribution of $\{X_t\}$, which is also easy to compute.

1.4.2 The Case in Which ϖ is Nonnegative

Theorem 1 *Suppose Assumptions 1 and 2 hold. Assume* $\varpi(x) \geq 0$, $x \in \mathbb{X}$*. Recall* δ *as defined in Proposition 1 and assume* $\beta\delta^{1-\rho} < 1$*. Then, the fixed point of* $\mathbb{T}$ *in* $\mathcal{X}_{++}$ *uniquely exists. Moreover, for any* $f \in \mathcal{X}_{++}$, $\{\mathbb{T}^n f\}_{n \geq 0}$ *converges to the fixed point.*

Theorem 1 shows that when the state space of $\{X_t\}$ is finite and ϖ is nonnegative, the fixed

[4] Note that the notations in Hansen and Scheinkman (2012) are different from ours: η therein corresponds to $\ln \eta$ in the present chapter.

point of $\mathbb{T}$ in $\mathcal{X}_{++}$ and, thus, the recursive utility defined by (1.3.1) uniquely exist provided that $\beta\delta^{1-\rho} < 1$. Condition $\beta\delta^{1-\rho} < 1$ is the same as the one in Hansen and Scheinkman (2012, Proposition 6) where the authors study the existence and uniqueness of the classical recursive utility (without narrow framing).

Note that we restrict the domain of $\mathbb{T}$ to $\mathcal{X}_{++}$ although $\mathbb{T}$ is well defined on $\mathcal{X}_{+}$. This is because $\mathbb{T}$ can have nonpositive fixed points. For example, when $\rho \geq 1$ and $\varpi \equiv 0$, 0 is a fixed point of $\mathbb{T}$. When $\rho \geq 1$, $\gamma \geq 1$, the transition matrix of $\{X_t\}$ is positive, and $\varpi(x) = 0$ for some $x \in \mathbb{X}$, we can verify that $H(1, \varpi(x)), x \in \mathbb{X}$ is a fixed point of $\mathbb{T}$ but is not in $\mathcal{X}_{++}$. The fixed points in these two examples, however, are not economically meaningful to represent the agent's total utility of her consumption and investment: given a positive consumption stream and nonnegative investment utility, we expect the agent's total utility to be positive. Thus, we need to exclude such fixed points by restricting the domain of $\mathbb{T}$ to $\mathcal{X}_{++}$ and, by doing so, we obtain the uniqueness of the fixed point.

Theorem 1 also provides a simple algorithm to compute the fixed point: one can start from any positive function, e.g., a positive constant function, to do iteration, and one can obtain a sequence that eventually converges to the fixed point. This result provides another reason why nonpositive fixed points of $\mathbb{T}$, if exist, are not desirable: these fixed points cannot be obtained by a recursive algorithm with any positive starting point.

In the above algorithm, one can also choose a nonnegative function, i.e., $f \in \mathcal{X}_{+}$, as the starting point, provided that $\mathbb{T}^m f \in \mathcal{X}_{++}$ for some m. Such m exists (i) for any $f \in \mathcal{X}_{+}$ if $\rho < 1$ because $H(1,0) = (1-\beta)^{1/(1-\rho)} > 0$ and (ii) for any $f \in \mathcal{X}_{+}^{o}$ if $\gamma < 1$ because $\{X_t\}$ is irreducible and $u^{-1}(\mathbb{E}[u(Z)]) > 0$ for any nonnegative and nonzero random variable Z when $\gamma < 1$. If $\rho \geq 1$ and $\gamma \geq 1$, however, $\{\mathbb{T}^n f\}_{n\geq 0}$ may not converge to the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$. For instance, suppose $\mathbb{X}$ contains two elements, e.g., x_1 and x_2, the transition matrix of $\{X_t\}$ is positive, and $\varpi \equiv 0$. Consider $f \in \mathcal{X}_+^o$ such that $f(x_1) = 0$ and $f(x_2) > 0$. Note that $H(1,0) = 0$ because $\rho \geq 1$ and that $u^{-1}(\mathbb{E}[u(Z)]) = 0$ for any nonnegative random variable taking zero with positive probability because $\gamma \geq 1$. We then immediately obtain that $\mathbb{T}f = 0$ and thus the limit $\{\mathbb{T}^n f\}_{n\geq 0}$ is 0; i.e., this sequence does not converge to the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$.

The convergence of $\{\mathbb{T}^n f\}_{n\geq 0}$ to the fixed point of $\mathbb{T}$ for any positive f is economically important: it shows that a finite-horizon model of narrow framing, in which the utility at the terminal time is positive, converges to the infinite-horizon model when the number of periods in the former model goes to infinity. Moreover, the utility at the terminal time in the former model is irrelevant, provided that it is positive.

1.4.3 The Case in Which ϖ is Not Nonnegative

We first illustrate that when $\varpi(x) < 0$ for some $x \in \mathbb{X}$, $\mathbb{T}$ can have zero, one, or multiple fixed points, depending on the parameter values.

Example 1 *Suppose $\mathbb{X}$ is singleton. Then, operator $\mathbb{T}$ becomes a function on $[0, +\infty)$, and we denote this function as $T(f)$. In this case, δ defined in Proposition 1 becomes*

$$u^{-1}\left[\mathbb{E}\left(u(e^{\kappa(X_t, X_{t+1}, Y_{t+1})})\right)\right].$$

Then, function $T(f)$ can be written as

$$T(f) = H(1, \delta f + \varpi).$$

We assume $\beta\delta^{1-\rho} < 1$, and Theorem 1 shows that the fixed point of T in $(0, +\infty)$ uniquely exists when $\varpi \geq 0$. Next, we consider the case in which $\varpi < 0$.

It is obvious that the domain of T is $[-\varpi/\delta, +\infty)$. Straightforward computation yields

$$\lim_{f \downarrow -\varpi/\delta} T(f) = \begin{cases} (1-\beta)^{1/(1-\rho)}, & \rho < 1, \\ 0, & \rho \geq 1, \end{cases} \qquad \lim_{f \uparrow +\infty} T(f) = \begin{cases} +\infty, & \rho \leq 1, \\ (1-\beta)^{1/(1-\rho)}, & \rho > 1, \end{cases}$$

$$\lim_{f \downarrow -\varpi/\delta} T'(f) = \begin{cases} +\infty, & \rho \leq 1, \\ \left(\beta\delta^{1-\rho}\right)^{1/(1-\rho)}, & \rho > 1, \end{cases} \qquad \lim_{f \uparrow +\infty} T'(f) = \begin{cases} \left(\beta\delta^{1-\rho}\right)^{1/(1-\rho)}, & \rho < 1, \\ 0, & \rho \geq 1. \end{cases}$$

Moreover, T is strictly increasing and concave.

We first consider the case in which $\rho \geq 1$. Note that in this case $T(-\varpi/\delta) = 0 < -\varpi/\delta$. Because $T'(-\varpi/\delta) > 1$ and $T'(+\infty) < 1$, we conclude that except in a very special case in which the identity line is tangent to T, it is either the case in which T has no fixed point or the case in which T has two fixed points; see Figure 1.1.

Next, consider the case in which $\rho < 1$. If $T(-\varpi/\delta) = (1-\beta)^{1/(1-\rho)} \leq -\varpi/\delta$, we conclude, as in the case in which $\rho \geq 1$, that except in a very special case in which the identity line is tangent to T, it is either the case in which T has no fixed point or the case in which T has two fixed points. If $(1-\beta)^{1/(1-\rho)} > -\varpi/\delta$, then the fixed point exists and is unique; see Figure 1.2.

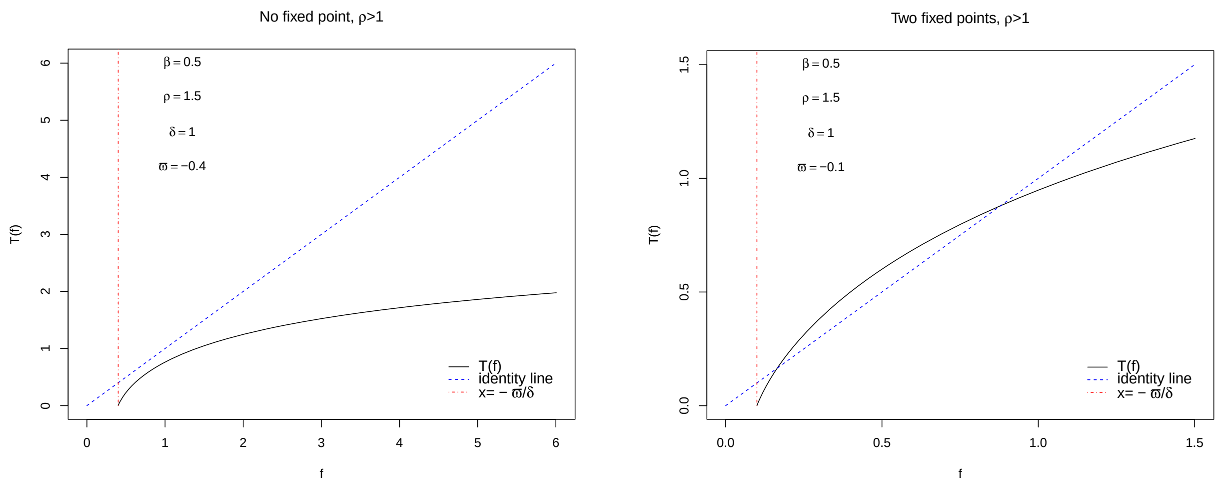

Figure 1.1: $T(f)$ in Example 1 without a fixed point (left panel) and with two fixed points (right panel) when $\rho > 1$. The solid lines in the two panels stand for $T(f)$ and the dashed lines stand for the identity function. Note that the domain of T is $[-\varpi/\delta, +\infty)$, as indicated by the dash-dotted lines.

Note that $\exp[\kappa(X_t, X_{t+1}, Y_{t+1})]$ *stands for the portfolio and consumption growth rate in the model of narrow framing in Section 1.3.2, so* δ *stands for the certainty equivalent of the portfolio and consumption growth rate and thus is decreasing with respect to the RRAD. On the other hand,* $-\varpi$ *stands for the disutility of loss. We can see that with* $\rho < 1$*, inequality* $(1-\beta)^{1/(1-\rho)} > -\varpi/\delta$ *holds if* β *is small,* δ *is large, and* $-\varpi$ *is small. Thus, we can conclude that the agent's total utility with narrow framing is well defined when her EIS is strictly larger than one, her time discounting is large, her portfolio and consumption growth rate is high, her RRAD is low, and her disutility of loss is small.*

Example 1 shows that we need some conditions on model parameters to establish the existence and uniqueness of the fixed point of $\mathbb{T}$ when ϖ is negative in some states.

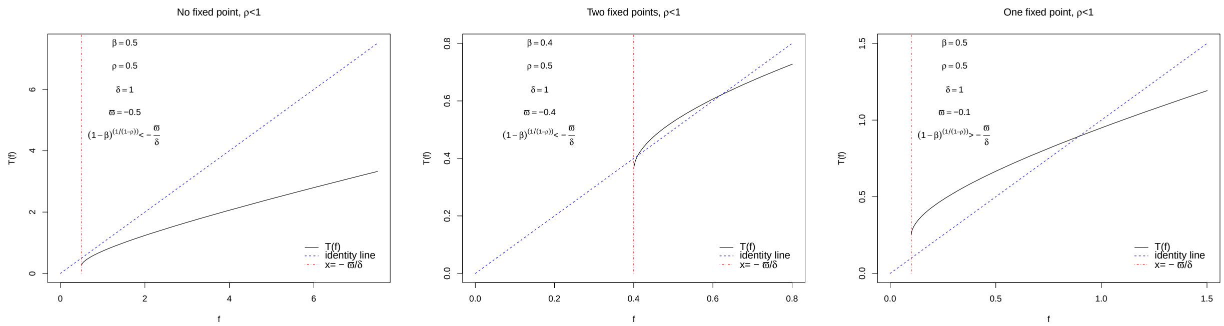

Figure 1.2: $T(f)$ in Example 1 without a fixed point (left panel), with two fixed points (middle panel), and with one fixed point (right panel) when $\rho < 1$. The solid lines in the two panels stand for $T(f)$ and the dashed lines stand for the identity function. Note that the domain of T is $[-\varpi/\delta, +\infty)$, as indicated by the dash-dotted lines. In the left and middle panels, $(1-\beta)^{1/(1-\rho)} < -\varpi/\delta$, and in the right panel, $(1-\beta)^{1/(1-\rho)} > -\varpi/\delta$.

Assumption 3 *Denote*

$$f_0(x) := H(1, \varpi^+(x)), \quad x \in \mathbb{X}. \tag{1.4.3}$$

Assume $\mathbb{T} f_0$ *is well defined, i.e.,*

$$u^{-1}\left[\mathbb{E}_t\left(u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} f_0(X_{t+1})\right)|X_t = x\right)\right] + \varpi(x) \geq 0, \quad x \in \mathbb{X},$$

and $\mathbb{T}^m f_0 > f_0$ *for some* $m \geq 1$.

Theorem 2 *Suppose Assumptions 1–3 hold. Assume* $\beta\delta^{1-\rho} < 1$*, where* δ *is defined as in Proposition 1. Then, the fixed point of* $\mathbb{T}$ *in its domain uniquely exists and is strictly larger than* f_0 *point-wisely. Moreover, for any* f *such that* $\mathbb{T} f$ *is well defined, sequence* $\{\mathbb{T}^n f\}_{n\geq 0}$ *converges to the fixed point of* $\mathbb{T}$.

Theorem 2 shows the existence and uniqueness of the fixed point of $\mathbb{T}$ when ϖ can go negative. Moreover, the calculation of the fixed point is easy: start from any f such that $\mathbb{T}f$ is well defined to do iteration. Then, the resulting sequence converges to the fixed point. As discussed in the case of nonnegative ϖ, this algorithm implies that a finite-horizon model of narrow framing converges to the infinite-horizon model when the number of periods in the former goes to infinity.

Assumption 3 is critical to obtain the existence and uniqueness of the fixed point of $\mathbb{T}$, so we discuss it in full details in the following:

(i) Note that if $\mathbb{T}f_0$ is well defined, we must have $\mathbb{T}f_0 \geq f_0$. However, this is insufficient to guarantee the uniqueness of the fixed point of $\mathbb{T}$. Indeed, in the setting of Example 1, when $\rho > 1$, if $(1-\beta)^{1/(1-\rho)} = -\bar{b}\zeta/\delta$, $\mathbb{T}f_0$ is well defined, and actually $\mathbb{T}f_0 = f_0$. We showed in the example that $\mathbb{T}$ has two fixed points and one of them is f_0, and both can represent the utility process. Thus, to guarantee the uniqueness, we need further conditions and Assumption 3 serves the purpose.

(ii) Assumption 3 implies that $\mathbb{T}f_0(x) > f_0(x)$ for some $x \in \mathbb{X}$. The reverse is also true when $\gamma < 1$ or $f_0 \in \mathcal{X}_{++}$. Indeed, for any $y \in \mathbb{X}$ such that the transition probability from y to x is positive, we have

$$u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}\mathbb{T}f_0(X_{t+1})\right)|X_t = y\right]\right) > u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}f_0(X_{t+1})\right)|X_t = y\right]\right).$$

As a result, $\mathbb{T}^2 f_0(y) = \mathbb{T}(\mathbb{T}f_0)(y) > \mathbb{T}f_0(y)$, and because of the irreducibility of $\{X_t\}$, we conclude that $\mathbb{T}^m f_0 > f_0$ for some $m \geq 1$.

(iii) When $\gamma \geq 1$ and $f_0(x) = 0$ for some $x \in \mathbb{X}$, which is the case if and only if $\varpi(x) \leq 0$ for the same x and $\rho \geq 1$, it is possible that $\mathbb{T}f_0$ is well defined, $\mathbb{T}f_0(x) > f_0(x)$ for *some* $x \in \mathbb{X}$, and the fixed point of $\mathbb{T}$ is *not* unique. For instance, consider $\{X_t\}$ with state space $\mathbb{X} = \{x_1, x_2, x_3\}$ such that $\mathbb{P}(X_{t+1} = x_3 | X_t = x_1) = 1$, $\mathbb{P}(X_{t+1} = x_3 | X_t = x_2) = 1$, $p_1 := \mathbb{P}(X_{t+1} = x_1 | X_t = x_3) > 0$, and $p_2 := \mathbb{P}(X_{t+1} = x_2 | X_t = x_3) > 0$. Then, $\{X_t\}$ is irreducible. Suppose $\kappa \equiv 0$. Suppose $\varpi(x_2) > 0$, $\varpi(x_3) > 0$, and $\varpi(x_1) := -H(1, \varpi(x_3)) < 0$. Then, one can verify that $\mathbb{T}f_0(x_1) = H(1, 0) = f_0(x_1) = 0$, $\mathbb{T}f_0(x_2) = H(1, f_0(x_3) + \varpi(x_2)) > f_0(x_2) > 0$, and $\mathbb{T}f_0(x_3) = H(1, \varpi(x_3)) = f_0(x_3) > 0$. Moreover, it is straightforward to see that $\mathbb{T}f_0$ is a fixed point of $\mathbb{T}$. On the other hand, suppose $\rho > 1$ and $\gamma > 1$. Then, straightforward calculation shows that

$$\left.\frac{d\mathbb{T}(f_0 + \epsilon \mathbf{1})(x_1)}{d\epsilon}\right|_{\epsilon=0} = H_z(1, 0) = \beta^{1/(1-\rho)},$$
$$\left.\frac{d\mathbb{T}(f_0 + \epsilon \mathbf{1})(x_2)}{d\epsilon}\right|_{\epsilon=0} = H_z(1, f_0(x_3) + \varpi(x_2)),$$
$$\left.\frac{d\mathbb{T}(f_0 + \epsilon \mathbf{1})(x_3)}{d\epsilon}\right|_{\epsilon=0} = H_z(1, \varpi(x_3)) p_1^{1/(1-\gamma)},$$

where $\mathbf{1}$ stands for the constant function taking value 1 and H_z is the partial derivative of $H(c, z)$ with respect to z. Because $\rho > 1$, $\gamma > 1$, and $\beta < 1$, with sufficiently small (but positive) p_1, $\varpi(x_2)$, and $\varpi(x_3)$, we have $\left.\frac{d\mathbb{T}(f_0+\epsilon\mathbf{1})(x_i)}{d\epsilon}\right|_{\epsilon=0} > 1$, $i = 1, 2, 3$. As a result, there exists $\epsilon > 0$ such that $\mathbb{T}(f_0 + \epsilon \mathbf{1}) \geq f_0 + \epsilon \mathbf{1}$. Consequently, $\{\mathbb{T}^n(f_0 + \epsilon \mathbf{1})\}_{n \geq 0}$ is increasing and converges because $\mathbb{T}f \leq (1 - \beta)^{1/(1-\rho)}$ for any f. It is obvious that the convergent point is a fixed point of $\mathbb{T}$ and is different from $\mathbb{T}f_0$ because $f_0(x_1) + \epsilon > 0 = \mathbb{T}f_0(x_1)$.

(iv) When $\rho \geq 1$, $\gamma \geq 1$, and the transition matrix of $\{X_t\}$ is positive, Assumption 3 does not hold and thus Theorem 2 cannot apply if $\varpi(x) < 0$ for some $x \in \mathbb{X}$. Indeed, in this case, we have

$$u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}f_0(X_{t+1})\right)|X_t = x\right]\right) = 0$$

because $f_0(x) = 0$ and $\mathbb{P}(X_{t+1} = x|X_t = x) > 0$, and $\gamma \geq 1$. Consequently, $\mathbb{T}f_0$ is not well defined because $\varpi(x) < 0$.

(v) Theorem 2 cannot cover Theorem 1. Indeed, suppose $\varpi \equiv 0$ and $\rho \geq 1$. Then, Assumption 3 does not hold, so Theorem 2 cannot apply. Theorem 1, however, can still apply. Therefore, Theorem 1 is more comprehensive than Theorem 2 when ϖ is nonnegative, and Theorem 2 is useful when ϖ goes negative.

1.4.4 Portfolio Selection with Narrow Framing

1.4.4.1 Model

Consider the problem of portfolio selection with narrow framing discussed in Sections 1.2.2 and 1.3.2. The agent's total utility U_t is given by (1.2.4) and thus her total utility per unit wealth U_t/W_t satisfies (1.3.4), where $R_{p,t+1}$ and $g_{i,t}$ are given as in (1.3.5) and (1.3.6), respectively. Suppose the total return rate of risky asset i in period t to $t+1$ is $R_{i,t+1} = r_i(X_t, X_{t+1}, Y_{t+1})$ for some function r_i and the total return rate of the risk-free asset in period t to $t+1$ is $R_{f,t+1} = r_0(X_t)$ for some function r_0. Suppose the agent chooses consumption propensity $c_t = c(X_t)$ and portfolio $\theta_t = (\theta_1(X_t), \ldots, \theta_n(X_t))'$ at time t for some functions c and θ_i's. Then, the agent's total utility per unit wealth $U_t/W_t = F_{c,\theta}(X_t)$, where $F_{c,\theta}$ is a fixed point of

$$\mathbb{V}_{c,\theta}F(x) := H\Big(c(x), u^{-1}\left(\mathbb{E}_t\left[u\big((1-c(x))R_p(x, X_{t+1}, Y_{t+1})F(X_{t+1})\big)|X_t = x\right]\right) \\ +(1-c(x))\sum_{i=1}^{n} b_i\theta_i(x)g_i(x)\Big), \quad x \in \mathbb{X}$$

with $R_p(X_t, X_{t+1}, Y_{t+1}) := r_0(X_t) + \sum_{i=1}^{n}\theta_i(X_t)(r_i(X_t, X_{t+1}, Y_{t+1}) - r_0(X_t))$ and

$$g_i(x) := \mathbb{E}_t\left[(R_{i,t+1} - R_{f,t+1})\mathbf{1}_{R_{i,t+1}>R_{f,t+1}} + k(R_{i,t+1} - R_{f,t+1})\mathbf{1}_{R_{i,t+1}<R_{f,t+1}}|X_t = x\right], x \in \mathbb{X}.$$

For any c and θ such that $0 < c(X_t) < 1$ and $R_p(X_t, X_{t+1}, Y_{t+1}) > 0$, F is a fixed point of $\mathbb{V}_{c,\theta}$ if and only if $f(x) := F(x)/c(x)$ is a fixed point of $\mathbb{T}_{c,\theta}$, where

$$\mathbb{T}_{c,\theta}f(x) := H\left(1, u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa_{c,\theta}(X_t, X_{t+1}, Y_{t+1})}f(X_{t+1})\right)|X_t = x\right]\right) + \varpi_{c,\theta}(x)\right), \quad x \in \mathbb{X} \tag{1.4.4}$$

and

$$\kappa_{c,\theta}(X_t, X_{t+1}, Y_{t+1}) := \ln c(X_{t+1}) - \ln c(X_t) + \ln(1 - c(X_t)) + \ln R_p(X_t, X_{t+1}, Y_{t+1}), \tag{1.4.5}$$

$$\varpi_{c,\theta}(X_t) := \frac{1 - c(X_t)}{c(X_t)}\sum_{i=1}^{n} b_i\theta_i(X_t)g_i(X_t). \tag{1.4.6}$$

Denote δ in Proposition 1 as $\delta_{c,\theta}$ when κ and ϖ therein are set to be $\kappa_{c,\theta}$ and $\varpi_{c,\theta}$, respectively.

For each $x \in \mathbb{X}$, consider a set $I_x \subset (0,1)$ and a set $J_x \subset \mathbb{R}^n$. Define

$$\mathcal{A} := \{(c,\theta) | c(x) \in I_x, \theta(x) \in J_x, x \in \mathbb{X}\}.$$

Assumption 4 *For each $(c,\theta) \in \mathcal{A}$, $R_p(X_t, X_{t+1}, Y_{t+1}) > 0$ and $\beta\delta_{c,\theta}^{1-\rho} < 1$. Moreover, for each $(c,\theta) \in \mathcal{A}$, it is either the case in which $\varpi_{c,\theta}(x) \geq 0, x \in \mathbb{X}$ or the case in which $\mathbb{T}_{c,\theta} f_{0,c,\theta}$ with $f_{0,c,\theta}(x) := H(1, \varpi^{+}_{c,\theta}(x)), x \in \mathbb{X}$ is well defined and $\mathbb{T}^m_{c,\theta} f_{0,c,\theta} > f_{0,c,\theta}$ for some $m \geq 1$.*

With Assumptions 2 and 4 in place, Theorems 1 and 2 show that the fixed point of $\mathbb{T}_{c,\theta}$ in $\mathcal{X}_{++}$ uniquely exists for any $(c,\theta) \in \mathcal{A}$. Thus, if the agent consumes $C_s = c(X_s)W_s$ and invests $\Theta_{i,s} = \theta_i(X_s)(W_s - C_s)$ dollars in risky asset i, $i = 1, \dots, n$ at time $s \geq t$, her utility U_t is well defined. As a result, the following portfolio selection problem

$$\max_{(\{C_s\}_{s \geq t}, \{\Theta_s\}_{s \geq t}) \in \mathcal{B}_t} U_t \tag{1.4.7}$$

is well defined, where

$$\mathcal{B}_t := \{(\{C_s\}_{s \geq t}, \{\Theta_s\}_{s \geq t}) | C_s = c(X_s)W_s, \Theta_s = \theta(X_s)(W_s - C_s) \text{ for some } (c,\theta) \in \mathcal{A}\}.$$

It is obvious that problem (1.4.7) is equivalent to

$$\max_{(c,\theta) \in \mathcal{A}} F_{c,\theta}(x), \quad \forall x \in \mathbb{X}. \tag{1.4.8}$$

1.4.4.2 Dynamic Programming

The dynamic programming equation associated with the portfolio selection problem (1.4.8) can be heuristically derived as

$$\Phi(x) = \mathbb{W}\Phi(x), \quad x \in \mathbb{X}, \tag{1.4.9}$$

where

$$\mathbb{W}\Phi(x) := \max_{\bar{c} \in I_x} H\left(\bar{c}, (1-\bar{c}) \max_{\bar{\theta} \in J_x} D_\Phi(x, \bar{\theta})\right), \quad x \in \mathbb{X}, \tag{1.4.10}$$

$$D_\Phi(x, \bar{\theta}) := \sum_{i=1}^n \bar{\theta}_i b_i g_i(x)$$
$$+ u^{-1}\left(\mathbb{E}_t\Big[u\Big(\big(r_0(x) + \sum_{i=1}^n \bar{\theta}_i (r_i(x, X_{t+1}, Y_{t+1}) - r_0(x))\big)\Phi(X_{t+1})\Big)|X_t = x\Big]\right). \tag{1.4.11}$$

Proposition 2 *Suppose Assumptions 2 and 4 hold. Suppose $\Phi \in \mathcal{X}_{++}$ is a solution to (1.4.9). Then, $\Phi(x) \geq F_{c,\theta}(x), x \in \mathbb{X}$ for any $(c, \theta) \in \mathcal{A}$. Moreover, if there exists $(c^*, \theta^*) \in \mathcal{A}$ such that $(c^*(x), \theta^*(x))$ is a maximizer of (1.4.10) for each $x \in \mathbb{X}$, then (c^*, θ^*) and Φ are a maximizer and the optimal value, respectively, of (1.4.8).*

Proposition 2 shows that the solution to the dynamic programming equation, if exists, is the solution to (1.4.8).

Proposition 3 *Suppose Assumptions 2 and 4 hold, and for each $x \in \mathbb{X}$, I_x and J_x are compact. Then, the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ uniquely exists. Moreover, the following are true:*

(i) If $\varpi_{c,\theta} \geq 0$ for any $(c, \theta) \in \mathcal{A}$, then $\{\mathbb{W}^n\Phi\}_{n\geq 0}$ converges to the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ for any $\Phi \in \mathcal{X}_{++}$.

(ii) If $\varpi_{c,\theta}(x) < 0$ for some $x \in \mathbb{X}$ and some $(c, \theta) \in \mathcal{A}$, then $\{\mathbb{W}^n\Phi\}_{n\geq 0}$ converges to the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ for any Φ such that $\mathbb{W}\Phi$ is well defined and in particular for

$$\Phi(x) := \max_{\bar{c}\in I_x} H\left(\bar{c}, (1-\bar{c}) \max_{\bar{\theta}\in J_x} \big(\sum_{i=1}^{n} \bar{\theta}_i b_i g_i(x)\big)^+\right). \tag{1.4.12}$$

Proposition 3 shows the existence and uniqueness of the solution to the dynamic programming equation when the control sets I_x and J_x are compact. Moreover, it shows that starting from any Φ that is positive when $\varpi_{c,\theta} \geq 0$ for any $(c, \theta) \in \mathcal{A}$ or that makes $\mathbb{W}\Phi$ well defined in other cases, by applying the dynamic programming equation repeatedly, one eventually obtains the solution to the equation. This result shows that the optimal consumption and portfolio in a finite-horizon model converges to those in the infinite-horizon model when the number of periods in the former goes to infinity.

Note that $D_\Phi(x, \bar{\theta})$ is strictly concave in $\bar{\theta}$ for each x and Φ and $H(\bar{c}, (1-\bar{c})z)$ is strictly concave in $\bar{c}$ for any given $z \geq 0$. Thus, for each x and Φ, the maximization problem in the right-hand side of the dynamic programming equation (1.4.9), i.e., in (1.4.10), can be easily solved. As a result, $\mathbb{W}\Phi$ can be easily computed, and once we find the fixed point of $\mathbb{W}$, the optimal control (c^*, θ^*) can also be solved easily.

Finally, when $\varpi_{c,\theta}(x) < 0$ for some $x \in \mathbb{X}$ and some $(c, \theta) \in \mathcal{A}$, equation (1.4.12) provides a simple choice of Φ such that $\mathbb{W}\Phi$ is well defined and thus $\{\mathbb{W}^n\Phi\}_{n\geq 0}$ converges to the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$. This Φ is easy to compute: note that $\left(\sum_{i=1}^{n} \bar{\theta}_i b_i g_i(x)\right)^+$ is convex in θ_i

and $H(\bar{c}, (1-\bar{c})z)$ is concave in $\bar{c}$ for any $z > 0$, so the maximization in $\bar{\theta}_i$ and $\bar{c}$ can be easily computed.

1.4.4.3 Example

We consider a market with a risky stock and a risk-free asset. We can regard the stock as the market portfolio and we set the length of each period to be one year. To construct the return of the stock, we assume that the stock pays a dividend every year and the dividend growth rates are i.i.d. following the distribution given as in Table 1.1.

Table 1.1: Distribution of the dividend growth rate. The distribution is assumed to be the same as in Table I of Chapman and Polkovnichenko (2009), which is obtained using the historical gross consumption growth from 1949 to 2006.

State	1	2	3	4	5	6	7	8	9
Outcome	0.976	0.993	1.002	1.011	1.019	1.028	1.037	1.045	1.054
Probability	0.03	0.03	0.10	0.16	0.24	0.19	0.13	0.09	0.03

We assume the market is governed by a two-state Markovian process $\{X_t\}$ that takes values in $\mathbb{X} = \{0, 1\}$. We assume the price-dividend ratio at time t to be $\varphi(X_t)$ and the risk-free total return rate in period t to $t+1$ to be $r_0(X_t)$; i.e., both are functions of X_t. As a result, the total return rate of the stock in period t to $t+1$ is

$$r(X_t, X_{t+1}, Y_{t+1}) = Y_{t+1}(\varphi(X_{t+1}) + 1)/\varphi(X_t),$$

where Y_{t+1} refers to the dividend growth rate in period t to $t+1$.

We assume the transition matrix of $\{X_t\}$ to be

$$\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}.$$

We also set the risk-free total return rate and the price-dividend ratio to be

$$r_0(X_t) = \begin{pmatrix} 1.03 \\ 1.03 \end{pmatrix}, \quad \varphi(X_t) = \begin{pmatrix} 30.25 \\ 39.75 \end{pmatrix},$$

respectively, so that the mean and variance of the stock return under the stationary distribution of $\{X_t\}$ are 6% and 15%, respectively, and consequently the equity premium is 3%.

We set the loss aversion degree $k = 1.5$, so $g(X_t)$, which measures the investment utility, is

$$g(X_t) = \mathbb{E}_t\big[\nu\left(r(X_t, X_{t+1}, Y_{t+1}) - r_0(X_t)\right)|X_t\big] = \begin{pmatrix} 0.1532 \\ -0.0551 \end{pmatrix},$$

where $\nu(x) := x\mathbf{1}_{x>0} + kx\mathbf{1}_{x\leq 0}$. Finally, we set $\beta = 0.95$, $b = 0.001$, $\rho = 0.5$, and $\gamma = 8$.

Consider the feasible set $I_x = [0.1\%, 10\%]$ and $J_x = [0, 100\%]$, $x = 0, 1$. We can show that Assumption 4 holds for any $(c, \theta) \in \mathcal{A}$.[5] Then, we apply Propositions 2 and 3 to calculate the

[5]In general, given $\mathcal{A}$, Assumption 4 is not straightforward to verify, but in the two-state case, the verification can be done. The verification result is available from the authors upon request.

optimal consumption and portfolio and the value function, and the results are shown as follows:

$$c^*(X_t) = \begin{pmatrix} 3.65\% \\ \\ 4.62\% \end{pmatrix}, \quad \theta^*(X_t) = \begin{pmatrix} 100\% \\ \\ 19.0\% \end{pmatrix}, \quad \Phi(X_t) = \begin{pmatrix} 0.0686 \\ \\ 0.0541 \end{pmatrix}.$$

1.5 When the State Space is Not Finite

In this section, we study the existence and uniqueness of the solution to (1.3.1), i.e., the fixed point of $\mathbb{T}$, when the state space of $\{X_t\}$ is not finite. We consider only the case in which ϖ is nonnegative for two reasons. First, by imposing a similar condition to Assumption 3, we can prove that the fixed point of $\mathbb{T}$ exists, but we do not have uniqueness, so we chose not to present the results here.

Proposition 4 *Suppose Assumption 1 holds, $\rho \neq 1$, and $\varpi(x) \geq 0, x \in \mathbb{X}$. Suppose the results in Proposition 1-(i) and -(ii) hold.*

(i) When $\gamma \neq 1$, f is a fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$ if and only if $g := f^{1-\rho} v^{-1/\alpha}$ is a fixed point of

$$\begin{aligned} \mathbb{S}g(x) &:= (1-\beta)v(x)^{-\frac{1}{\alpha}} \\ &+ \beta\delta^{1-\rho}\left\{\left[\tilde{\mathbb{E}}_t\left(g(X_{t+1})^\alpha | X_t = x\right)\right]^{\frac{1}{\alpha(1-\rho)}} + \delta^{-1}\varpi(x)v(x)^{-1/(1-\gamma)}\right\}^{1-\rho}, \quad x \in \mathbb{X} \end{aligned}$$

in the same space, where $\alpha := (1-\gamma)/(1-\rho)$ and $\tilde{\mathbb{E}}$ is the expectation operator corresponding to probability measure $\tilde{\mathbb{P}}$ that is obtained by a change of measure using the Radon-Nikodym density $M_{t+1} := \eta^{-1} e^{(1-\gamma)\kappa(X_t, X_{t+1}, Y_{t+1})} v(X_{t+1})/v(X_t)$. Assume $\beta\delta^{1-\rho} < 1$ and the sta-

tionary distribution of $\{X_t\}$ *under* $\tilde{\mathbb{P}}$ *exists. For* $\alpha \geq 1$*, denote space* $\mathcal{X}_+$ *when equipped with the* L^α *norm under the stationary distribution of* $\{X_t\}$ *under* $\tilde{\mathbb{P}}$ *as* $\tilde{L}^\alpha_+(\mathbb{X})$*. Then, if* $\alpha \geq 1$ *and* $\mathbb{S}0 \in \tilde{L}^\alpha_+(\mathbb{X})$*,* $\mathbb{S}$ *is a contraction mapping in* $\tilde{L}^\alpha_+(\mathbb{X})$ *and its unique fixed point is positive. If* $\alpha < 1$ *and* $\mathbb{S}0 \in \tilde{L}^{\alpha'}_+(\mathbb{X})$ *for some* $\alpha' \geq 1$*, then the limit of* $\{\mathbb{S}^n 0\}$ *exists, belongs to* $\tilde{L}^{\alpha'}_+(\mathbb{X})$*, and is the minimum fixed point of* $\mathbb{S}$ *in* $\mathcal{X}_{++}$*.*

(ii) When $\gamma = 1$*,* f *is a fixed point of* $\mathbb{T}$ *in* $\mathcal{X}_{++}$ *if and only if* $g := f^{1-\rho}e^{-(1-\rho)v}$ *is a fixed point of*

$$\mathbb{S}g(x) := (1-\beta)e^{-(1-\rho)v(x)} + \beta\delta^{1-\rho}\left\{\left[e^{\mathbb{E}_t(\ln g(X_{t+1})|X_t=x)}\right]^{\frac{1}{1-\rho}} + \delta^{-1}\varpi(x)e^{-v(x)}\right\}^{1-\rho}, \quad x \in \mathbb{X}$$

in the same space. Assume $\beta\delta^{1-\rho} < 1$ *and the stationary distribution of* $\{X_t\}$ *exists. Suppose for some* $\alpha' \geq 1$*,* $\mathbb{S}0 \in L^{\alpha'}_+(\mathbb{X})$*, the space* $\mathcal{X}$ *equipped with the* $L^{\alpha'}$ *norm under the stationary distribution of* $\{X_t\}$*. Then, the limit of* $\{\mathbb{S}^n 0\}$ *exists, belongs to* $L^{\alpha'}_+(\mathbb{X})$*, and is the minimum fixed point of* $\mathbb{S}$ *in* $\mathcal{X}_{++}$*.*

Proposition 4 is completely parallel to Hansen and Scheinkman (2012, Proposition 6): when $\rho \neq 1$ and ϖ is nonnegative, if (i) η, v, and δ in Proposition 1 are well defined and $\beta\delta^{1-\rho} < 1$ and (ii) the stationary distribution of $\{X_t\}$ exists after a specific change of measure, then the fixed point of $\mathbb{T}$ exists. Moreover, if $(1-\gamma)/(1-\rho) \geq 1$, then the fixed point is unique and $\{\mathbb{T}^n f\}_{n\geq 0}$ converges to the fixed point for any positive f. Note that just as in Hansen and Scheinkman (2012), we do not have uniqueness when $(1-\gamma)/(1-\rho) < 1$.

Proposition 5 *Suppose Assumption 1 holds and the stationary distribution of* $\{X_t\}$ *exists. Suppose* $\rho = \gamma = 1$ *and* $\varpi(x) \geq 0, x \in \mathbb{X}$*. Then,* f *is a fixed point of* $\mathbb{T}$ *in* $\mathcal{X}_{++}$ *if and only if* $g := \ln f$ *is a fixed point of* $\mathbb{S}$ *in* $\mathcal{X}$*, where*

$$\mathbb{S}g(x) := \beta \ln \left[e^{\mathbb{E}_t(\kappa(X_t, X_{t+1}, Y_{t+1})|X_t = x)} e^{\mathbb{E}_t(g(X_{t+1})|X_t = x)} + \varpi(x) \right], \quad x \in \mathbb{X}. \tag{1.5.1}$$

Denote $L^1(\mathbb{X})$ *as* $\mathcal{X}$ *equipped with the* L^1 *norm under the stationary distribution of* $\{X_t\}$ *and assume there exists* $g \in L^1(\mathbb{X})$ *such that* $\mathbb{S}g \in L^1(\mathbb{X})$*. Then,* $\mathbb{S}$ *is a contraction mapping on* $L^1(\mathbb{X})$ *and thus the fixed point of* $\mathbb{S}$ *uniquely exists in* $L^1(\mathbb{X})$*.*

Proposition 5 shows the existence and uniqueness of the fixed point of $\mathbb{T}$ when $\rho = \gamma = 1$ provided that $\{X_t\}$ has a stationary distribution and ϖ is nonnegative. Moreover, because $\mathbb{S}$ as defined in (1.5.1) is a contraction mapping, $\{\mathbb{S}^n g\}$ converges to the fixed point of $\mathbb{S}$ for any g and as a result $\{\mathbb{T}^n f\}$ converges to the fixed point of $\mathbb{T}$ for any positive f.

Finally, we show that the solution to (1.3.1) uniquely exists when $\gamma = \rho = 1$ even in a non-Markovian setting.

Proposition 6 *Suppose* $\rho = \gamma = 1$, $A_t > 0$*, and* $B_t \geq 0$*. Then,* $\{V_t\}$ *is a positive solution to* (1.3.1) *if and only if* $\{\ln(V_t/c_t)\}$ *is a fixed point of*

$$(\mathcal{S}Z)_t := \beta \ln \left[e^{\mathbb{E}_t(\ln c_{t+1} - \ln c_t + \ln A_{t+1})} e^{\mathbb{E}_t(Z_{t+1})} + (B_t/c_t) \right], \quad t = 0, 1, \dots \tag{1.5.2}$$

Moreover, if there exist $\alpha \in (\beta, 1)$ *and* $\{Z_t\} \in \mathcal{L}^{1,\alpha}$*, the space of* $\{\mathcal{F}_t\}$*-adapted processes with* *norm* $||Z|| := \sum_{t=0}^{\infty} \alpha^t \mathbb{E}(|Z_t|)$*, such that* $\mathcal{S}Z \in \mathcal{L}^{1,\alpha}$*, then* $\mathcal{S}$ *is a contraction mapping on* $\mathcal{L}^{1,\alpha}$ *and thus the fixed point of* $\mathcal{S}$ *on this space uniquely exists.*

1.6 Conclusion

We examined the model of narrow framing in the literature in which an agent derives not only consumption utility but also investment utility of gains and losses; in particular, this model degenerates into the classical recursive utility model when the investment utility is zero. Assuming constant EIS and RRAD, we studied the existence and uniqueness of the agent's total utility in the model of narrow framing.

We assumed a Markovian setting: The asset returns in period from t to $t+1$ are assumed to be functions of X_t, X_{t+1}, and Y_{t+1}, so the agent's consumption propensity and investment in the assets in that period are functions of X_t, where $\{X_t\}$ is a Markov process that represents market states and $\{Y_t\}$ is i.i.d. conditioning on $\{X_t\}$ and thus represents random noise. We further assumed that $\{X_t\}$ is irreducible and its state space is finite.

We proved that the agent's total utility in the model of narrow framing uniquely exists when her investment utility is nonnegative, regardless of the values of the EIS and RRAD. We then illustrated by an example that when the state space of $\{X_t\}$ is singleton and the EIS is less than or equal to one, the agent's total utility is either non-existent or non-unique if her investment utility is negative.

We then proposed a sufficient condition under which the total utility in the model of narrow framing with negative investment utility uniquely exists, and this condition is nearly necessary.

We also proved that if the total utility in the model of narrow framing uniquely exists, it can be obtained by applying the recursive equation defining the utility repeatedly with any positive utility value as the starting point.

Finally, we considered portfolio selection with narrow framing and proved that a consumption and portfolio plan is optimal if and only if it, together with the value function of the portfolio selection problem, satisfies the dynamic programming equation. Moreover, we proved that the solution to the dynamic programming equation uniquely exists and can be computed by solving equation recursively with any starting point.

Chapter 2

A New Preference Model That Allows for Narrow Framing

2.1 Introduction

When making multiple decision choices or evaluating multiple risks, individuals tend to consider one of them at a time, isolating it from other choices or risks, a phenomenon referred to as *narrow framing*; see for instance Tversky and Kahneman (1981) and Kahneman and Lovallo (1993). Since then, narrow framing has been found extensively both in experimental settings and in real lives. However, there was little work on building a broadly applicable model of narrow framing until Barberis and Huang (2009); see also Barberis et al. (2006), Barberis and Huang (2008a).[1]

The model proposed by Barberis and Huang (2009), which we refer to as the *BH model* in the following, is formulated by generalizing the classical recursive utility model (Epstein and Zin, 1989, Kreps and Porteus, 1978) in that (i) the risks that are evaluated in isolation by the individuals at the end of each period are assessed according to prospect theory (Kahneman and

[1]In an earlier work by Barberis and Huang (2001), the authors propose an asset pricing model with narrow framing, in which the representative agent evaluates the investment gain and loss of each individual stock in isolation with consumption risk. This model, however, is specific for asset pricing, intractable in partial equilibrium settings, and inapplicable in the study of individuals' attitudes toward timeless gambles; see the discussion in Barberis and Huang (2008a, p. 210).

Tversky, 1979, Tversky and Kahneman, 1992) so that the utility of gains and losses experienced by the individuals in these risks is calculated, (ii) the utility of gains and losses and the certainty equivalent of the individuals' total utility from next period are added linearly with the weight for former to be a positive constant in the linear addition, and (iii) the sum of the utility of gains and losses and the certainty equivalent of the total utility from next period is aggregated with the individuals' consumption in the current period via an aggregation function, resulting in the individuals' total utility at the beginning of the current period. The certainty equivalent and aggregation function are chosen so that the so-called relative risk aversion degree (RRAD) and elasticity of intertemporal substitution (EIS) are constant, and the time horizon is usually set to be infinite.

The BH model provides an analytical framework to study the impact of narrow framing on decision making. In particular, this model has been successfully applied to explain individuals' attitude towards some monetary gambles that cannot be easily explained by many models of preferences and attitude towards large gambles, such as no-participation in the stock market; see Barberis et al. (2006) and Barberis and Huang (2008a, 2009). When applied to investment decision making, the narrow framing component of the BH model allows for utility from gains and losses in financial wealth, which are the foremost sources of utility when people invest. Therefore, the BH model has strong implications for portfolio selection and asset pricing, such as explaning high equity premia in the market; see for instance Barberis and Huang (2009), De Giorgi and Legg (2012), He and Zhou (2014), and Easley and Yang (2015).

The analysis in Chapter 1, however, implies that the BH model is not robust in that the total utility process may not be uniquely defined in this model. Indeed, in the present chapter, we show that when (i) an agent consumes a constant fraction of her wealth and invests another constant fraction of her wealth into some assets whose returns are independent and identically distributed (i.i.d.) over time, (ii) the EIS is less than or equal to one, and (iii) the utility of gains and losses experienced by the agent is negative, it is either the case in which the total utility process of the agent in the BH model does not exist or the case in which there exists two solutions to the recursive equation that defines the total utility process.

On the one hand, the non-robustness of the BH model does not diminish its value in the literature. Indeed, as mentioned above, Barberis and Huang (2009) is the first serious attempt to build up a broadly applicable model of narrow framing and the BH model has been successfully applied in decision making, portfolio selection, and asset pricing. Moreover, in some settings, e.g., when the utility of gains and losses experienced by the agent is nonnegative, the total utility process in the BH model uniquely exists; see Chapter 1. On the other hand, the nonrobustness of the BH model needs to be addressed because the cases in which the BH model does not admit a unique total utility process are economically important. Indeed, in many applications of the BH model, the EIS takes a value that is less than one and the utility of gains and losses is negative; see for instance Barberis and Huang (2009) and De Giorgi and Legg (2012). Moreover, in those applications, the agents also consume constant fractions of their wealth and the asset returns are i.i.d.

In the present chapter, we propose a refinement of the BH model, referred to as the *GH model*, so as to obtain a robust model of narrow framing. In this new model, the utility of gains and losses and the certainty equivalent of the individuals' total utility from next period are also added linearly, but instead of a constant, the weight for the utility of gains and losses is scaled in a sense that it is proportional to the certainty equivalent of the total utility from next period per unit wealth.

We first show that even in the finite-horizon setting, the BH model may fail to define the total utility process. Intuitively, the failure of the BH model arises from the restriction that the aggregation of the utility of gains and losses and the certainty equivalent of the total utility from next period must be nonnegative. Imagine that in the BH model, because the weight for the utility of gains and losses is a constant, the magnitude of a negative utility of gains and losses can dominate the certainty equivalent of the total utility from next period, and thus the sum of these two can be negative. In the GH model, the weight for the utility of gains and losses is proportional to the the certainty equivalent of the total utility from next period per unit wealth, so in the aggregation of the utility of gains and losses and the certainty equivalent of the total utility from next period, the latter always dominates the former, ensuring that the aggregation is nonnegative. Thus, the GH model defines a unique total utility process whether the time horizon is finite or infinite. Moreover, the GH model in the finite-horizon setting with any given utility at the terminal time converges to that in the infinite-horizon setting as the number of periods in the former goes to infinity. This results also implies that by starting from any positive value and applying the recursive equation in the GH model repeatedly, we can obtain, as a limit, the total utility process in the infinite-horizon setting.

When asset returns are i.i.d. and portfolio and consumption strategies are constant, the total utility per unit wealth in the GH model becomes a constant and thus the weight for the utility of gains and losses in the GH model is also a constant. The total utility process in the GH model, however, cannot be computed from the BH model using a recursive algorithm no matter how one chooses the weight for the utility of gains and losses in the BH model.

When the GH model is applied to portfolio selection, the resulting dynamic programming equation admits a unique solution. Moreover, this solution can be computed by applying the equation repeatedly with any starting point. For the BH model, one can also derive a dynamic programming equation heuristically. We show that this equation can have multiple solutions. When one applies the dynamic programming equation repeatedly, different starting points can lead to different solutions. Even worse, different solutions to the equation lead to different portfolios solved from the dynamic programming equation.

Finally, we apply the GH model to study individuals' attitudes toward risk and portfolio selection and asset pricing implications. We first show that similar to the BH model, the GH model can explain an aversion to a small, independent, actuarially favorable gamble and acceptance of a large, independent gamble over a reasonable range of wealth levels. We then apply to the GH model to portfolio selection and find that it can explain why many households do not participate in the stock market. We also apply the GH model to asset pricing in a production-consumption economy that is studied in Barberis and Huang (2009) and find that this model can generate large equity premia. In all these applications, the GH model is more tractable than the BH model.

The remainder of the chapter is organized as follows: In Section 2.2 we review the BH model, propose the GH model, and show in various aspects why the GH model is more robust than the BH model. In Section 2.3, we apply the GH model to study individuals' attitudes toward timeless gambles. In Section 2.4, we show that the GH model can explain why many households do not participate in the stock market. In Section 2.5, we study the asset pricing implication of the GH model. Section 2.6 concludes. In Appendix 3.6, we show the existence and uniqueness of the total utility process in the GH model and of the solution to the dynamic programming equation when this model is applied to portfolio selection in a finite-state Markovian setting. All proofs are placed in Appendix 3.6.

2.2 A Refinement of Barberis and Huang's Model of Narrow Framing

2.2.1 Barberis and Huang's model of Narrow Framing

Consider an agent who consumes C_t at time t. Denote U_t as the total utility of the agent from time t. In the BH model, the agent's total utility is defined recursively by

$$U_t = H\left(C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} \bar{b}_i G_{i,t}\right), \tag{2.2.1}$$

where

$$H(c,z) := \begin{cases} [(1-\beta)c^{1-\rho} + \beta z^{1-\rho}]^{\frac{1}{1-\rho}}, & 0 < \rho \neq 1, \\ e^{(1-\beta)\ln c + \beta \ln z}, & \rho = 1 \end{cases} \tag{2.2.2}$$

is an aggregation function with $1/\rho$ and $\beta \in (0,1)$ standing for the EIS and discount rate of the agent, respectively,

$$\mathcal{M}_t(X) := u^{-1}\big(\mathbb{E}_t[u(X)]\big), \quad u(x) := \begin{cases} x^{1-\gamma}/(1-\gamma), & 0 < \gamma \neq 1, \\ \ln(x), & \gamma = 1 \end{cases} \tag{2.2.3}$$

stands for the certainty equivalent of X with γ representing the RRAD of the agent, $G_{i,t}$ stands for the agent's utility for risk i that is evaluated in isolation from other risks, such as consumption and investment risk, and $\bar{b}_i$ is a nonnegative constant.

Here and hereafter, $\mathbb{E}_t$ and $\mathcal{M}_t$ stand for the expectation and certainty equivalent, respectively, that are computed based on the information at time t. If X is independent of the information at time t, we simply drop the subscript t when calculating its expectation and certainty equivalent. In the following, when $\rho \geq 1$, we set $H(c,0) = \lim_{z\downarrow 0} H(c,z) = 0$ and $H(0,z) = \lim_{c\downarrow 0} H(c,z) = 0$. As a result, $H(c,z)$ is well defined, takes real values, and continuous in $(c,z) \in [0,\infty)^2$. Similarly, when $\gamma \geq 1$, we define $u(0) := -\infty$ and $u^{-1}(-\infty) := 0$. As a result, $\mathcal{M}_t(X)$ is well defined for any nonnegative random variable X. Moreover, when $\gamma \geq 1$ and $X = 0$ with a positive probability, $\mathcal{M}_t(X) = 0$.

Examples of risks that are evaluated in isolation include a monetary gamble that is offered to the agent and the gain and loss incurred by holding a stock. Barberis and Huang (2009) assume that the agent employs prospect theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992) to evaluate these risks. More precisely, suppose the gain and loss experienced in one of these risks is X, then the utility for taking this risk is $\mathbb{E}_t[\nu(X)]$, where

$$\nu(x) := x\mathbf{1}_{\{x\geq 0\}} + \lambda x\mathbf{1}_{\{x<0\}} \tag{2.2.4}$$

for some $\lambda \geq 1$ that represents the loss aversion degree (LAD) of the agent. Here and hereafter, for simplicity we follow Barberis and Huang (2009) not to consider probability and diminishing sensitivity in the agent's evaluation of gains and losses. These two features, however, can be easily incorporated; see Section 3.6.

We show in Chapter 1 that the BH model is not robust in that the total utility process in this model may not uniquely exist. Suppose the agent consumes a constant fraction c of her wealth, so the remaining $(1-c)$ fraction of her wealth is used for investment. Suppose θ fraction of her investment is in a stock and the remaining is in a risk-free asset. Suppose the agent frames the investment in the stock separately and uses the risk-free return as a reference point to calculate the gain and loss she experiences from holding the stock.[2] Denote W_t as the agent's wealth at time t. Denote the gross return rate of the stock in period t to $t+1$ as $R_{S,t+1}$ and assume $R_{S,t+1}$'s to be i.i.d. Assume the gross return rate of the risk-free asset is constant over time and

[2]Here and hereafter, we assume for simplicity that the agent uses the risk-free return as her reference point; general reference points are discussed in Section 3.6.

denote it as R_f. Then, the agent's gain and loss for holding the stock in period t to $t+1$ is $(1-c)W_t\theta(R_{t+1}-R_f)$ and the resulting utility is

$$G_t = \mathbb{E}_t\left[\nu\big((1-c)W_t\theta(R_{S,t+1}-R_f)\big)\right] = (1-c)W_t\mathbb{E}\left[\nu\big(\theta(R_{S,t+1}-R_f)\big)\right].$$

In consequence, the agent's total utility process $\{U_t\}$ in the BH model for this consumption and investment strategy is defined recursively by

$$U_t = H\left(cW_t, \mathcal{M}_t\left(U_{t+1}\right) + \bar{b}(1-c)W_t\mathbb{E}\left[\nu\big(\theta(R_{S,t+1}-R_f)\big)\right]\right) \tag{2.2.5}$$

for some constant $\bar{b} > 0$. Dividing both sides of the above equation by W_t and recalling that $W_{t+1} = (1-c)W_t\left(R_f + \theta(R_{S,t+1}-R_f)\right)$, we obtain

$$\frac{U_t}{W_t} = H\left(c, (1-c)\mathcal{M}_t\left(\frac{U_{t+1}}{W_{t+1}}R_{p,t+1}\right) + \bar{b}(1-c)\mathbb{E}\left[\nu\big(\theta(R_{S,t+1}-R_f)\big)\right]\right), \tag{2.2.6}$$

where $R_{p,t+1} := R_f + \theta(R_{S,t+1}-R_f)$ stands for the gross return of the agent's portfolio in period t to $t+1$.

Because the fraction of wealth for consumption and the fraction of wealth for investment in the stock are constant over time, the risk-free return rate is constant over time, and the stock return rates are i.i.d. over time, it is expected that U_t/W_t is a constant over time and we denote this constant as Ψ. Denote

$$\delta := (1-c)\mathcal{M}_t\left(R_{p,t+1}\right), \quad \zeta := (1-c)\mathbb{E}\left[\nu\big(\theta(R_{S,t+1}-R_f)\big)\right]/c. \tag{2.2.7}$$

Then, we conclude from (2.2.6) that

$$\Psi = H(c, \delta\Psi + c\bar{b}\zeta). \tag{2.2.8}$$

Theorem 1 shows that when $\beta\delta^{1-\rho} < 1$ and $\zeta \geq 0$, (2.2.8) has a unique solution in $(0, +\infty)$. Moreover, starting from any positive number for Ψ and applying the recursive equation (2.2.8) repeatedly, the resulting sequence converges to the solution to this equation. When $\zeta < 0$, however, Example 1 shows that the solution to (2.2.8) can be nonunique or nonexistent even when $\beta\delta^{1-\rho} < 1$. The following proposition expands the observation therein.

Proposition 7 *Suppose $\beta\delta^{1-\rho} < 1$ and $\zeta < 0$. Then, the right-hand side of (2.2.8), denoted as $\mathbb{V}(\Psi)$, is well defined only for $\Psi \geq -c\bar{b}\zeta/\delta$.*

(i) If $\rho < 1$ and $-\bar{b}\zeta/\delta < (1-\beta)^{1/(1-\rho)}$, then there exists a unique solution, denoted as Ψ^*, to (2.2.8) in $[-c\bar{b}\zeta/\delta, +\infty)$.

(ii) If $\rho < 1$ and $(1-\beta)^{1/(1-\rho)} < -\bar{b}\zeta/\delta < (1-\beta)^{1/(1-\rho)}\left[1-\left(\beta\delta^{1-\rho}\right)^{1/\rho}\right]^{-\rho/(1-\rho)}$, then there exists two solutions, denoted as $\Psi_1^* < \Psi_2^*$, to (2.2.8) in $(-c\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-c\bar{b}\zeta/\delta, \Psi_1^*)$, there exists positive integer n_0 such that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$; for any $\Psi > \Psi_1^*$, $\{\mathbb{V}^n(\Psi)\}_{n\geq 1}$ converges to Ψ_2^*.

(iii) If $\rho < 1$ and $-\bar{b}\zeta/\delta > (1-\beta)^{1/(1-\rho)}\left[1-\left(\beta\delta^{1-\rho}\right)^{1/\rho}\right]^{-\rho/(1-\rho)}$, then there does not exist any solution to (2.2.8) in $[-c\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-c\bar{b}\zeta/\delta, +\infty)$, there exists positive integer n_0 such that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$.

(iv) If $\rho \geq 1$ and $-\bar{b}\zeta/\delta < (1-\beta)^{1/(1-\rho)}\left[1-\left(\beta\delta^{1-\rho}\right)^{1/\rho}\right]^{-\rho/(1-\rho)}$, where the right-hand side is defined to be $(1-\beta)(\beta\delta)^{\beta/(1-\beta)}$ when $\rho = 1$, then there exists two solutions, denoted as $\Psi_1^* < \Psi_2^*$, to (2.2.8) in $(-c\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-c\bar{b}\zeta/\delta, \Psi_1^*)$, there exists positive integer n_0 such that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$; for any $\Psi > \Psi_1^*$, $\{\mathbb{V}^n(\Psi)\}_{n\geq 1}$ converges to Ψ_2^*.

(v) If $\rho \geq 1$ and $-\bar{b}\zeta/\delta > (1-\beta)^{1/(1-\rho)}\left[1-\left(\beta\delta^{1-\rho}\right)^{1/\rho}\right]^{-\rho/(1-\rho)}$, where the right-hand side is defined to be $(1-\beta)(\beta\delta)^{\beta/(1-\beta)}$ when $\rho = 1$, then there does not exist any solution to (2.2.8) in $[-c\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-c\bar{b}\zeta/\delta, +\infty)$, there exists positive integer n_0 such that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$.

Proposition 7 is illustrated by Figure 2.1, with the five panels from top to bottom and from left to right representing cases (i)–(v) in the proposition, respectively. Proposition 7 confirms the finding in Example 1 that the solution to (2.2.8) can be nonunique or nonexist when $\zeta < 0$. In particular, when $\rho \geq 1$, for *any* negative value of ζ and *any* positive value of $\bar{b}$, it is either the case in which the solution to (2.2.8) does not exist or the case in which the solution is not unique.[3] Thus, the BH model fails to define the total utility process of an agent who derives a

[3]Here, to simplify exposition, we exclude a marginal case in which $\bar{b}\zeta = -\delta(1-\beta)^{1/(1-\rho)}\left[1-\left(\beta\delta^{1-\rho}\right)^{1/\rho}\right]^{-\rho/(1-\rho)}$, where the right-hand side is defined to be $(1-\beta)(\beta\delta)^{\beta/(1-\beta)}$ when $\rho = 1$. In this case, (2.2.8) has a unique solution, Ψ^*, in $(-c\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-c\bar{b}\zeta/\delta, \Psi^*)$, there exists positive integer n_0 such that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$; for any $\Psi \geq \Psi^*$, $\{\mathbb{V}^n(\Psi)\}_{n\geq 1}$ converges to Ψ^*.

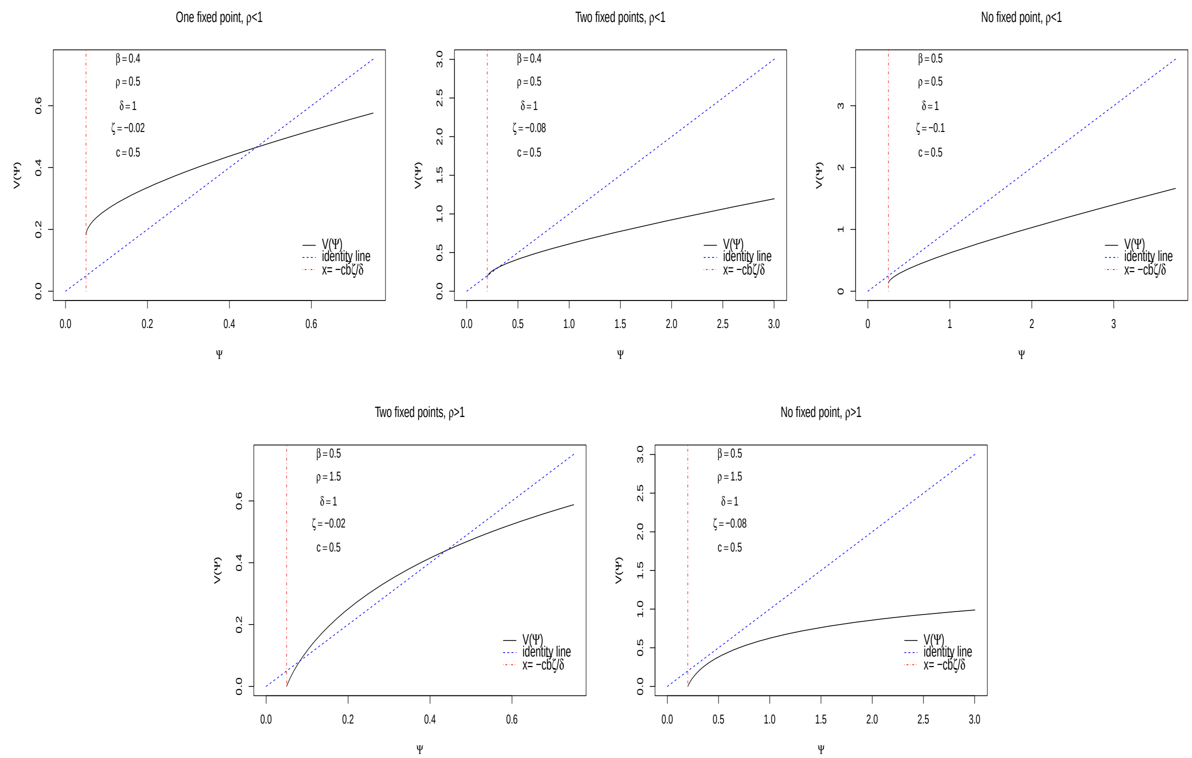

Figure 2.1: Plots of $\mathbb{V}(\Psi) = H(c, \delta\Psi + c\bar{b}\zeta)$ when $\beta\delta^{1-\rho} < 1$ and $\zeta < 0$. The top three panels correspond to cases (i)–(iii) in Proposition 7, respectively, and the bottom two panels correspond to cases (iv) and (v) in Proposition 7, respectively. In each panel, the solid line stands for $\mathbb{V}(\Psi)$, the dashed diagonal line represents the identity function, and the dash-dotted vertical line indicates that the domain of $\mathbb{V}$ is $[-c\bar{b}\zeta/\delta, +\infty)$.

negative utility, i.e., a disutility, of gains and losses and whose EIS is less than or equal to one, no matter how small the disutility is and no matter how small the weight of the utility of gains and losses is. This failure is essential for the BH model because many applications of the BH model in the literature, such as those in Barberis et al. (2006) and Barberis and Huang (2009), assume EIS to be less than one and a disutility of the gain and loss incurred by holding a stock.

One may argue that when an agent derives a sufficiently negative utility of gains and losses for a strategy, the nonexistence of her total utility process means that the strategy is not favorable. When the disutility of gains and losses is sufficiently small, Proposition 7-(iv) shows that with $\rho \geq 1$, the total utility process exists but is not unique. In this case, one possible remedy is to simply choose one of the utility processes, such as the maximal. Such choice, however, seems to be arbitrary. Furthermore, Proposition 7-(iv) shows that applying the recursive equation (2.2.8) repeatedly does not necessarily lead to its solution. Even worse, Proposition 7 shows that for any given negative utility of gains and losses, even in a finite-horizon setting, the agent's total utility may not exist. Indeed, consider the total utility process defined by (2.2.5) in $n+1$ periods. Recall ζ, δ, $\mathbb{V}$, and Ψ_i^*, $i = 1, 2$ in Proposition 7-(iv). Suppose that at the terminal time, $U_T = \Psi W_T$ for some $\Psi \in (-c\bar{b}\zeta/\delta, \Psi_1^*)$. Then, the agent's total utility at time 0 is $W_0 \mathbb{V}^{n+1}(\Psi)$. Proposition 7-(iv) shows that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$, so $\mathbb{V}^{n_0+1}(\Psi)$ does not exist. In consequence, when $n \geq n_0$, the agent's total utility at time 0 does not exist.

Note that the BH model has been used in the literature even though the total utility process in this model does not necessarily exist. For instance, in Barberis et al. (2006) and Barberis and Huang (2009), ρ is set to be larger than one and the utility of gains and losses is negative. Typically, the utility per unit capital is solved from (2.2.8) by starting from some $\Psi > 0$ and applying the right-hand side of the equation repeatedly. The successful application of the BH model in Barberis et al. (2006) and Barberis and Huang (2009) implies that the authors set $\bar{b}$ sufficiently small so that case-(iv) of Proposition 7 is in effect. Proposition 7-(iv) then implies that for the recursive algorithm to work, the authors must choose starting point $\Phi > \Phi_1^*$ and the resulting solution to (2.2.8) must be Φ_2^*.

2.2.2 A Refinement of Barberis and Huang's Model

Intuitively, the nonrobustness of the BH model arises from the restriction that the aggregation of the utility of gains and losses and the certainty equivalent of the total utility from next period must be nonnegative. In the BH model, the weight for the utility of gains and losses is constant over time. If the utility of gains and losses derived by the agent is constant over periods and its magnitude dominates the utility of immediate consumption in every period, then the agent's total utility is reduced by a constant amount in every period. In consequence, the sum of the utility of gains and losses and the certainty equivalent of the total utility from next period can eventually become negative, and thus the total utility fails to exist.

The GH model resolves the issue of nonexistence or nonuniqueness of the total utility process in the BH model by modeling the weight for the utility of gains and losses to be proportional to the certainty equivalent of the total utility per unit wealth from next period. More precisely, the agent's total utility process $\{U_t\}$ in the GH model is defined recursively by

$$U_t = H\left(C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} b_i \frac{\mathcal{M}_t(U_{t+1})}{\mathcal{M}_t(W_{t+1})} G_{i,t}\right), \tag{2.2.9}$$

where $b_i \geq 0$ is a constant, $i = 1, \ldots, n$. Alternatively, (2.2.9) can be written as

$$U_t = H\left(C_t, \left(1 + \sum_{i=1}^{n} \frac{b_i}{\mathcal{M}_t(W_{t+1})} G_{i,t}\right) \mathcal{M}_t(U_{t+1})\right). \tag{2.2.10}$$

This reformulation shows that in the BH model experiencing a loss (a gain) in risks that are evaluated separately reduces (increases) the certainty equivalent of the total utility *proportion-*

ally.

In Appendix 3.6, we show that in a finite-state Markovian setting in which randomness is driven by a finite-state Markov process $\{X_t\}$ representing the market state and by an independent time series $\{Y_t\}$ representing random shocks, the total utility process $\{U_t\}$ in the GH model uniquely exists provided that a nonnegativity condition and a growth condition hold; see Theorem 5 in Appendix 3.6. The nonnegativity condition, $\mathcal{M}_t(W_{t+1}) + \sum_{i=1}^{n} b_i G_{i,t} > 0$, stipulates that the agent's disutility of the losses experienced in risks in certain period that are evaluated separately cannot exceed the certainty equivalent of the agent's wealth at the end of the same period. In consequence, we can observe from (2.2.10) that if the agent's total utility at time $t+1$ is positive, her total utility at time t remains positive. The nonnegativity condition holds if $G_{i,t}$ is not largely negative or if b_i is sufficiently small. For gambles that yield gains and losses such that the nonnegativity condition does not hold, the agent should not take them because the disutility of the losses experienced in these gambles is overwhelming. On the other hand, the growth condition is standard; similar conditions also appear in the expected utility theory so as to make the sum of the expected discounted utility of consumption in infinite number of periods converge.

As we will see in Sections 2.3–2.5, these two conditions hold in various applications of the GH model with reasonable model parameters. In particular, these conditions can hold when the agent's EIS is larger than one, the agent's utility of gains and losses is negative, and asset returns are i.i.d. Thus, the GH model is more robust than the BH model in defining the total utility process uniquely.

Theorem 5 also shows that to compute the total utility per unit wealth U_t/W_t, which is a deterministic function of the market state $f(X_t)$, one can start from any positive function of the market state and apply the right-hand side of (2.2.9) repeatedly, and this recursive algorithm leads to the function f. This result not only provides an easy algorithm to compute the total utility process in the GH model but also implies that the total utility in the finite-horizon setting converges to that in the infinite-horizon setting when the number of periods in the former setting goes to infinity. In particular, in contrast to the BH model, the total utility process in the GH model is also well defined in the finite-horizon setting.

Barberis and Huang (2001), Barberis et al. (2001), and Li and Yang (2013a) consider the following preference model for the representative agent in their equilibrium asset pricing studies:

$$\mathbb{E}_t\left[\sum_{s=t}^{\infty}\left(\beta^s u(C_s)+\beta^{s+1}u'(\bar{C}_s)\sum_{i=1}^{n}\hat{b}_i G_{i,s}\right)\right], \tag{2.2.11}$$

where $\hat{b}_i$'s are constants and $\{\bar{C}_t\}$ is the aggregate consumption process in the whole economy. The idea of scaling the weight for the utility of gains and losses in the GH model is similar to using the ad-hoc factor $u'(\bar{C}_s)$ in (2.2.11). Indeed, in the asset pricing models studied by Barberis and Huang (2001), Barberis et al. (2001), and Li and Yang (2013a), the aggregate consumption is equal to the consumption of the representative agent. On the other hand, as argued by Barberis and Huang (2008a, p.210), the model (2.2.11) does not admit an explicit value function and thus is difficult to use to study individuals' attitudes towards timeless gambles and is intractable when applied to portfolio selection. The GH model, however, is tractable when applied to studying individuals' attitudes towards timeless gambles and to portfolio selection and asset pricing; see

Sections 2.3–2.5.

2.2.3 Connection between the GH Model and the BH Model

Consider an agent who consumes a constant fraction c of her wealth. For the remaining wealth, the agent invests θ fraction in a stock and the rest in a risk-free asset. Suppose that the agent frames the investment in the stock separately and uses the risk-free return as a reference point to calculate the gain and loss she experiences from holding the stock. Denote the gross return rate of the stock in period t to $t+1$ as $R_{S,t+1}$ and assume $R_{S,t+1}$'s to be i.i.d. Assume the gross return rate of the risk-free asset is constant over time and denote it as R_f. Recall δ and ζ in (2.2.7) and assume $\beta\delta^{1-\rho} < 1$. Suppose $\rho \geq 1$ and $\zeta < 0$. Then, Proposition 7 shows that the total utility process of the agent in the BH model is either nonexistent or nonunique if the agent frames the investment in the stock separately (i.e., if $\bar{b} > 0$).

Now, suppose the agent's preferences are represented by the GH model. Assume $\delta + cb\zeta > 0$ and $\beta(\delta + cb\zeta)^{1-\rho} < 1$, which actually imply $\beta\delta^{1-\rho} < 1$ because $\zeta < 0$ and $\rho \geq 1$. Then, according to Theorem 5, the total utility process in the GH model uniquely exists. Moreover, the total utility per unit wealth in the GH model, denoted as Ψ^*, is the unique solution to

$$\Psi = H\big(c, \delta\Psi + cb\zeta\Psi\big). \tag{2.2.12}$$

Now, define $\bar{b} := b\Psi^*$. By comparing (2.2.8) and (2.2.12), we conclude that Ψ^* is also a solution to (2.2.8). Then, according to Proposition 7-(iv) and -(v), one can expect that

$-\bar{b}\zeta/\delta < (1-\beta)^{1/(1-\rho)}\left[1-(\beta\delta^{1-\rho})^{1/\rho}\right]^{-\rho/(1-\rho)}$, where the right-hand side is defined to be $(1-\beta)(\beta\delta)^{\beta/(1-\beta)}$ when $\rho = 1$, in which case (2.2.8) has two solutions $\Psi_1^* < \Psi_2^*$. We want to see whether Ψ^* is equal to Ψ_1^* or Ψ_2^*.

Proposition 8 *Suppose $\rho \geq 1$, $\zeta < 0$, $\delta + cb\zeta > 0$, and $\beta(\delta + cb\zeta)^{1-\rho} < 1$. Let Ψ^* be the unique solution to* (2.2.12)*. Define $\bar{b} := b\Psi^*$ and recall* (2.2.8) *that defines the total utility process in the BH model.*

(i) If $cb\zeta < (\delta\beta)^{1/\rho} - \delta$, then $\Psi^* = \Psi_1^*$, where $\Psi_1^* < \Psi_2^*$ are the two solutions to (2.2.8) *in Proposition* 7*-(iv).*

(ii) If $cb\zeta > (\delta\beta)^{1/\rho} - \delta$, then $\Psi^* = \Psi_2^*$, where $\Psi_1^* < \Psi_2^*$ are the two solutions to (2.2.8) *in Proposition* 7*-(iv).*

Because $\zeta < 0$ and $\rho \geq 1$, $\beta(\delta + cb\zeta)^{1-\rho} < 1$ implies $\beta\delta^{1-\rho} < 1$, i.e., implies $\beta^{1/(\rho-1)} < (\delta\beta)^{1/\rho}$. Moreover, $\beta(\delta + cb\zeta)^{1-\rho} < 1$ if and only if $cb\zeta > \beta^{1/(\rho-1)} - \delta$. Thus, both cases (i) and (ii) of Proposition 8 are non-redundant. In other words, the solution to (2.2.12) in the GH model can be the smaller one or the larger one of solutions to (2.2.8), depending on model parameters. The left and right panels of Figure 2.2 plot cases (i) and (ii), respectively, of Proposition 8. Recall that the smaller solution to (2.2.8) is not computable in a sense that it cannot be obtained by applying the right-hand side of (2.2.8) repeatedly with any starting points (unless the starting point happens to be this fixed point). Thus, the total utility per unit wealth in the GH model may not correspond to any computable total utility per unit wealth in the BH model even if this quantity is constant.

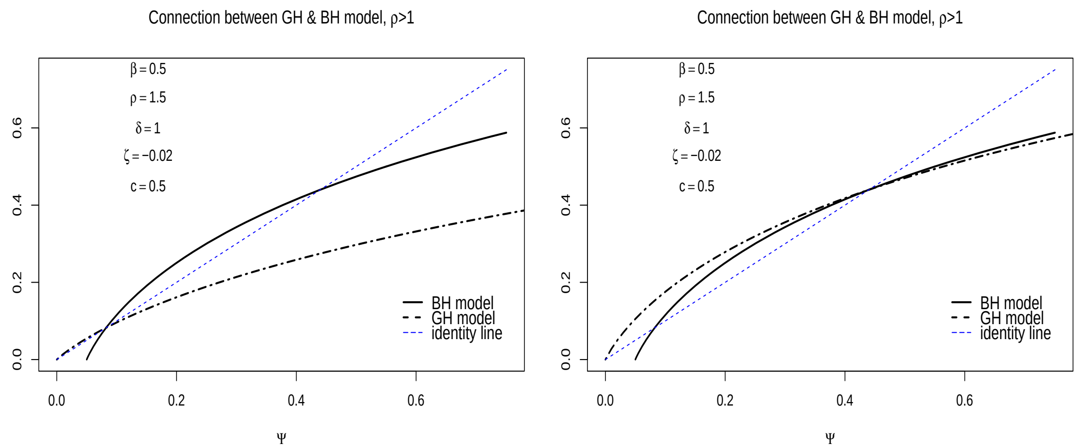

Figure 2.2: Plots of $H(c, \delta\Psi + cb\zeta\Psi)$ in the GH model and $H(c, \delta\Psi + c\bar{b}\zeta)$ in the BH model with $\bar{b} := b\Psi^*$, where Ψ^* is the unique fixed point of the former. In each of the two panels, the solid line stands for $H(c, \delta\Psi + c\bar{b}\zeta)$, the dash-dotted line stands for $H(c, \delta\Psi + cb\zeta\Psi)$, the dashed diagonal line stands for the identical function. The left panel plots the case in which the fixed point of $H(c, \delta\Psi + cb\zeta\Psi)$ is the same as the smaller fixed point of $H(c, \delta\Psi + c\bar{b}\zeta)$ and the right panel plots the case in which the fixed point of $H(c, \delta\Psi + cb\zeta\Psi)$ is the same as the larger fixed point of $H(c, \delta\Psi + c\bar{b}\zeta)$.

2.2.4 Dynamic Programming

2.2.4.1 Dynamic Programming in the GH Model

An important application of a model of preferences is portfolio selection. Assuming a finite-state Markovian setting in which randomness is driven by a finite-state Markov process $\{X_t\}$ representing the market state and by an independent time series $\{Y_t\}$ representing random shocks, we prove that portfolio selection problems in the GH model can be solved by dynamic programming; see Theorem 6 in Appendix 3.6. More precisely, the optimal utility per unit wealth is the unique solution to the dynamic programming equation (0.24), and the optimal portfolio and consumption can be obtained by solving a maximization problem; see equations (0.25)–(0.26). Moreover, the solution to the dynamic programming equation can be obtained by a recursive algorithm that starts from any positive value and then applies the equation repeatedly. This algorithm not only provides us with a method to compute the solution but also shows that the optimal portfolio and consumption in a finite-horizon portfolio selection model converge to those in an infinite-horizon model when the number of periods in the former goes to infinity.

The maximization problem we need to solve to obtain the optimal portfolio is easy to deal with when there is only one market state or when the agent's RRAD is one; see the discussion following Theorem 6 and in particular equation (0.27). The first special case is commonly assumed in the literature. Indeed, in all their applications of the BH model in Barberis et al. (2006) and Barberis and Huang (2009), the authors assume asset returns are i.i.d. over time, which simply means that there is only one market state. In the second special case, we need to assume that the agent's RRAD to be one. In Chapter 3, we propose a measure, referred to as implied RRAD, to compute the overall risk aversion degree of an agent whose preferences are represented by the GH model. It turns out that the implied RRAD is insensitive to the RRAD γ for reasonable values of b, i.e., for $b \geq 1$; see Table **??** therein. Thus, in the GH model, the RRAD γ is not very critical in determining the agent's overall risk aversion degree, so assuming it to be one should have little impact on portfolio selection and asset pricing results in this model.

2.2.4.2 Dynamic Programming in the BH Model

We have already showed the total utility process in the BH model can be nonexistent or nonunique. This model, however, has been applied to portfolio selection in the literature without being aware of the issue of nonexistence or nonuniqueness of the total utility process. The approach taken in the literature to portfolio selection with the BH model is to derive the dynamic programming equation heuristically and solve it numerically. In the following, we study whether this approach works.

We consider a simple economy in which an agent can invest in a risk-free asset with constant risk-free gross return R_f and in a stock with i.i.d. gross returns $R_{S,t+1}$'s. Thus, it is reasonable to assume that the agent's strategy is to consume a constant fraction c of her wealth W_t at time t, invest a constant fraction θ of the remaining wealth $(1-c)W_t$ in the stock and the remaining in the risk-free asset. Suppose the agent can choose $c \in I$ for some compact subset I of $(0,1)$ and $\theta \in J$ for some compact subset J of $[0,+\infty)$ such that the portfolio return $R_f + \theta(R_{S,t+1} - R_f) > 0$ for any $\theta \in J$, so short selling is not allowed. Assume the agent derives a negative utility of the gain and loss experienced by each dollar invested in the stock, in which the reference point is set to be the risk-free return; i.e., $\mathbb{E}[\nu(R_{S,t+1} - R_f)] < 0$. Suppose the agent's preferences are represented by the BH model. If we take the issue of nonexistence and nonuniqueness of the total utility process in the BH model aside, we can derive the dynamic programming equation heuristically:

$$\Phi = \max_{c\in I, \theta\in J} H\left(c, (1-c)\left(\Phi\mathcal{M}\big(R_f + \theta(R_{S,t+1} - R_f)\big) + \bar{b}\theta\mathbb{E}[\nu(R_{S,t+1} - R_f)]\right)\right), \tag{2.2.13}$$

where Φ stands for the optimal total utility per unit capital of the agent. Note that the right-hand side of (2.2.13) is not well defined for any $\theta > 0$ and $\Phi \geq 0$ if $\mathbb{E}[\nu(R_{S,t+1} - R_f)] < 0$. Noting that the maximization in c and in θ can be separated, however, we can rewrite (2.2.13) as

$$\Phi = \max_{c \in I} H\left(c, (1-c)\max_{\theta \in J}\left(\Phi \mathcal{M}\big(R_f + \theta(R_{S,t+1} - R_f)\big) + \bar{b}\theta \mathbb{E}[\nu(R_{S,t+1} - R_f)]\right)\right). \tag{2.2.14}$$

Now, the right-hand side of (2.2.14) is well defined for any $\Phi \geq 0$ if $0 \in J$, i.e., if investing only in the risk-free asset is feasible. Thus, (2.2.14) is a better formulation than (2.2.13) as the dynamic programming equation in the BH model. Moreover, the following proposition shows that (2.2.14) admits a solution provided that a growth condition holds.

Proposition 9 *Suppose* $0 \in J$ *and*

$$\beta \max_{c \in I, \theta \in J}\left[(1-c)\mathcal{M}\big(R_f + \theta(R_{S,t+1} - R_f)\big)\right]^{1-\rho} < 1. \tag{2.2.15}$$

Then, the solution to (2.2.14) *exists.*

Although the dynamic programming equation (2.2.14) admits a solution as shown in Proposition 9, the solution is not unique in general. This is not surprising because the total utility process in the BH model for each consumption-investment strategy can be non-unique. The non-uniqueness of the solution to (2.2.14) can even result in non-uniqueness of the optimal portfolio solved from this equation because the maximization in θ in this equation depends on Φ.

We provide an example to illustrate the issue of non-uniqueness of the solution to (2.2.14).

Set $\beta = 0.5$, $\rho = 1.5$, $\gamma = 0.25$, $\lambda = 5$, $b = 10$, $R_f = 1$, and

$$R_{S,t+1} = \begin{cases} 10.3911, & \text{with probability } 0.1, \\ 0.78983, & \text{with probability } 0.9. \end{cases}$$

Set $I = \{0.1, 0.5\}$ and $J = \{0, 3\}$. One can see that for any $\theta \in J$, the portfolio return $R_f + \theta(R_{S,t+1} - R_f) > 0$. Moreover, $\mathcal{M}(R_f + \theta(R_{S,t+1} - R_f))$ is 1 when $\theta = 0$ and is 2 when $\theta = 3$. In consequence, straightforward calculation yields that condition (2.2.15) holds.

Figure 2.3 plots in the solid line the right-hand side of the dynamic programming equation (2.2.14) as a function of Φ. The intersections of this line with the dashed diagonal line, which represents the identity function, in the region $(0, +\infty)$ are the solutions to (2.2.14). We can see that there are three solutions: $\Phi_1^* = 0.17157$, $\Phi_2^* = 0.2447$, and $\Phi_3^* = 0.30236$. Moreover, we solved that the optimal consumption-investment strategy corresponding to Φ_1^* is $c^* = 0.5, \theta^* = 0$, but that corresponding to Φ_2^* and Φ_3^* is $c^* = 0.5, \theta^* = 3$. Furthermore, from Figure 2.3, we can observe that with any starting point in $(0, \Phi_2^*)$, e.g., 0.1, applying the dynamic programming equation (2.2.14) repeatedly leads to Φ_1^* and thus the corresponding optimal consumption-investment strategy $c^* = 0.5, \theta^* = 0$; with any starting point in $(\Phi_2^*, +\infty)$, e.g., 0.35, however, this algorithm leads to Φ_3^* and thus the corresponding optimal consumption-investment strategy $c^* = 0.5, \theta^* = 3$. Therefore, this example shows that the dynamic programming equation in the BH model can have multiple solutions, corresponding to different portfolios. Moreover, when solving the equation using a recursive algorithm, the resulting solution depends heavily on the choice of the starting point.

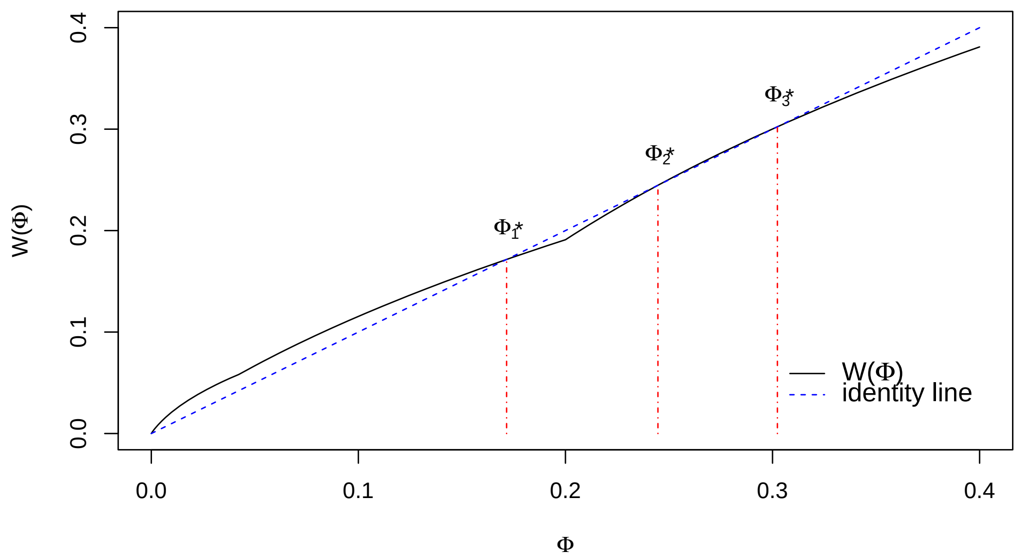

Figure 2.3: Plot of $\mathbb{W}(\Phi)$, the right-hand side of the dynamic programming equation (2.2.14) in the BH model as a function of Φ. The solid line represents $\mathbb{W}(\Phi)$ and the dashed diagonal line represents the identity function, so their intersections in $(0, +\infty)$ represent the solutions to (2.2.14). The parameter values are as follows: $\beta = 0.5$, $\rho = 1.5$, $\gamma = 0.25$, $\lambda = 5$, $b = 10$, $R_f = 1$, $R_{S,t+1} = 10.3911$ with probability 0.1 and $R_{t+1} = 0.78983$ with probability 0.9, $I = \{0.1, 0.5\}$, and $J = \{0, 3\}$. There are three intersection points: $\Phi_1^* = 0.17157$, $\Phi_2^* = 0.2447$, and $\Phi_3^* = 0.30236$, corresponding to optimal consumption-investment strategies $(c^* = 0.5, \theta^* = 0)$, $(c^* = 0.5, \theta^* = 3)$, and $(c^* = 0.5, \theta^* = 3)$, respectively.

2.2.5 Discussions on Alternative Modeling of Preferences with Narrow Framing

In this section, we discuss two alternatives to weight the utility of gains and losses, leading to two variants of the GH model. We show that each of the variants has its own issues.

2.2.5.1 Alternative I

The first variant of the GH model, referred to as GH-I model, is as follows

$$U_t = H\left(C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} b_i \mathcal{M}_t\left(\frac{U_{t+1}}{W_{t+1}}\right) G_{i,t}\right). \tag{2.2.16}$$

Note that when U_t/W_t is a constant in t or when $\gamma = 1$, we have

$$\mathcal{M}_t\left(\frac{U_{t+1}}{W_{t+1}}\right) = \frac{\mathcal{M}_t\left(U_{t+1}\right)}{\mathcal{M}_t\left(W_{t+1}\right)},$$

showing that the GH-I model is the same as the GH model in this case. In general, however, these two models can be different.

Appendix 3.6 shows that if we formulate the dynamic programming equation for the GH-I model heuristically, the resulting equation is the same as that in the GH model when there is only one market state or when the agent's RRAD is one. In general, the dynamic programming equations in the GH model and in the GH-I model can be different, and the latter appear to be simpler because the maximization problem therein for deriving the optimal portfolio is easier to solve. However, in contrast to the GH model, we do not know whether the solution to the dynamic programming equation in the GH-I model uniquely exists. Even worse, we do not know whether the total utility process in the GH-I model uniquely exists.

2.2.5.2 Alternative II

The second variant of the GH model, referred to as GH-II model, is as follows

$$U_t = H\left(C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} b_i \frac{\mathcal{M}_t\left(U_{t+1}\right)}{W_t - C_t} G_{i,t}\right). \quad (2.2.17)$$

Compared to the GH model, we replace $\mathcal{M}_t(U_{t+1})/\mathcal{M}_t(W_{t+1})$, the scaling factor in the weight for the utility of gains and losses, with $\mathcal{M}_t\left(U_{t+1}\right)/(W_t - C_t)$.

We can prove that the total utility process in the GH-II model uniquely exists and the solution to the dynamic programming equation in a portfolio selection problem with the GH-II model uniquely exists. Furthermore, a standard recursive algorithm can be used to find the total utility process and the solution to the dynamic programming equation with any positive starting point. However, when there is only one market state or when the agent's RRAD is one, the resulting dynamic programming equation in the GH-II model is more complicated than in the GH model.

2.3 Attitudes towards Timeless Gambles

One of the important implications for the BH model is to explain individuals' attitudes toward timeless gambles that are not easily explained by other models of preferences under risk. In this section, we show that the GH model is also able to explain these attitudes. Moreover, the GH model turns out to be more tractable than the BH model in explaining these attitudes.

2.3.1 Model

Following Barberis et al. (2006) and Barberis and Huang (2009), we suppose the current time is t and the agent faces a timeless gamble that is a 50:50 bet to gain $\$x$ and lose $\$y$. Denote the payoff of this gamble as ξ and assume it to be independent of other risks. Denote the agent's wealth at time t as W_t.

Following Barberis et al. (2006) and Barberis and Huang (2009), we assume that the agent is offered the gamble before she decides how much to consume and invest at time t and consider various approaches for the agent to evaluate the gamble.

In the first approach, the outcome of the timeless gamble is revealed in an infinitesimal time period Δt, and other risks have not been revealed in this period. The agent then applies a recursive equation over this time step to calculate her utility at t. In this case, the agent must prefer to decide the consumption and investment strategy after the gamble outcome is revealed, so the agent makes the first consumption immediately after $t + \Delta t$. Denote the agent's wealth at time $t+\Delta t$ as $W_{t+\Delta t}$. After deciding whether to accept the timeless gamble, the agent needs to decide her consumption propensity c_s (i.e., the fraction of her wealth that is used for consumption) and percentage allocation to n risky assets $\boldsymbol{\theta}_s = (\theta_{1,s}, \ldots, \theta_{n,s})'$, $s = t+\Delta t, t+1, t+2, \ldots$ Because Δt is infinitesimally small and the payoff of the timeless gamble is independent of other risks, the agent's optimal consumption propensity and percentage allocation to risky assets and optimal utility per unit wealth are independent of whether the agent accepts the gamble; indeed, these variables should be dependent only on market state variables. Denote the optimal utility per unit wealth at time $t + \Delta t$ as $\Phi_{t+\Delta t}$. Then, the agent's optimal utility at $t + \Delta t$ becomes

$W_{t+\Delta t}\Phi_{t+\Delta t}$. Applying the recursive equation over this time step, we obtain the agent's utility at t, if she accepts the gamble, as follows:

$$\tilde{H}\left(0, \mathcal{M}_t(W_{t+\Delta t}\Phi_{t+\Delta t}) + b\frac{\mathcal{M}_t(W_{t+\Delta t}\Phi_{t+\Delta t})}{\mathcal{M}_t(W_{t+\Delta t})}G(\xi)\right),$$

where $G(\xi) = \mathbb{E}_t[\nu(\xi)] = (x - \lambda y)/2$, $b \geq 0$ is a constant, and $\tilde{H}$ is an aggregator. Here, $\tilde{H}$ differs from H because the agent is discounting and aggregating utilities in a smaller time period Δt. We assume that $\tilde{H}$ is strictly increasing, homogeneous, and $\tilde{H}(0, z) > 0$ for any $z > 0$; the particular form of $\tilde{H}$ does not matter.

Because the only uncertainty that is resolved in the period t to $t + \Delta t$ is the outcome of the timeless gamble and the agent's optimal utility per unit wealth does not depend on this outcome, we conclude that $\Phi_{t+\Delta t}$ is known given information at t. Then, we conclude, together with the homogeneity of $\tilde{H}$, that the agent's total utility at time t is

$$\Phi_{t+\Delta t}\tilde{H}\left(0, \mathcal{M}_t(W_{t+\Delta t}) + bG(\xi)\right) = \Phi_{t+\Delta t}\tilde{H}\left(0, \mathcal{M}_t(W_t + \xi) + bG(\xi)\right).$$

Similarly, if the agent does not accept the gamble, her total utility at t is

$$\Phi_{t+\Delta t}\tilde{H}\left(0, \mathcal{M}_t(W_t)\right) = \Phi_{t+\Delta t}\tilde{H}\left(0, W_t\right).$$

In consequence, the agent accepts the gamble if and only if

$$\mathcal{M}_t \left(1 + \frac{\xi}{W_t}\right) + bG\left(\frac{\xi}{W_t}\right) > 1. \tag{2.3.1}$$

Note that condition (2.3.1) differs from eq. (28) in Barberis and Huang (2009) in that it is independent of the optimal utility per unit wealth $\Phi_{t+\Delta t}$ and thus is independent on the consumption and investment opportunities faced by the agent. With such independence, we do not need to assume any model for the agent's consumption and investment when studying her attitude towards timeless gambles that are evaluated immediately and separately from other risks. In this sense, the GH model is more tractable than the BH model.

In the second approach, the gamble outcome is revealed at time $t + 1$ and thus the agent evaluates the gamble over the same time interval she uses to evaluate her other risks, i.e., at time $t + 1$. In consequence, if the agent accepts the gamble, she needs to decide how much to consume and invest at time t before observing the gamble outcome at time $t + 1$. Denote W_{t+1} and Φ_{t+1} as the wealth and optimal utility per unit wealth of the agent at time $t + 1$, respectively. Again, Φ_{t+1} is independent of whether the agent accepts the gamble because the gamble outcome is independent of other risks. As in Barberis and Huang (2009), we assume in this case that the agent takes a fixed portfolio over time that generates return series $R_{p,s+1}$ in period s to $s+1$, $s \geq t$ and she does not derive any utility of gains and losses from the portfolio return. Then, if the agent accepts the gamble and consumes $c_t W_t$ at time t, her utility becomes

$$H\left(c_t W_t, \mathcal{M}_t\left(\left((1-c_t)W_t R_{p,t+1}+\xi\right)\Phi_{t+1}\right)+b\frac{\mathcal{M}_t(W_{t+1}\Phi_{t+1})}{\mathcal{M}_t(W_{t+1})}G(\xi)\right)$$
$$=W_t H\left(c_t, \mathcal{M}_t\left(\left((1-c_t)R_{p,t+1}+\frac{\xi}{W_t}\right)\Phi_{t+1}\right)+b\frac{\mathcal{M}_t(W_{t+1}\Phi_{t+1})}{\mathcal{M}_t(W_{t+1})}G(\frac{\xi}{W_t})\right).$$

The agent then maximizes the utility by choosing c_t optimally. Similarly, if the agent does not accept the gamble and consumes $c_t W_t$ at time t, her utility becomes

$$H\left(c_t W_t, \mathcal{M}_t\left(\left((1-c_t)W_t R_{p,t+1}\right)\Phi_{t+1}\right)\right)=W_t H\left(c_t, \mathcal{M}_t\left(\left((1-c_t)R_{p,t+1}\right)\Phi_{t+1}\right)\right).$$

Therefore, the agent accepts the gamble if and only if

$$\max_{c_t} H\left(c_t, \mathcal{M}_t\left(\left((1-c_t)R_{p,t+1}+\frac{\xi}{W_t}\right)\Phi_{t+1}\right)+b\frac{\mathcal{M}_t(W_{t+1}\Phi_{t+1})}{\mathcal{M}_t(W_{t+1})}G(\frac{\xi}{W_t})\right)$$
$$>\max_{c_t} H\left(c_t, \mathcal{M}_t\left(\left((1-c_t)R_{p,t+1}\right)\Phi_{t+1}\right)\right)$$

If we further assume that $R_{p,s+1}, s \geq t$ are i.i.d., Φ_{t+1} becomes a constant Φ, so the agent accepts the gamble if and only if

$$\max_{c_t} H\left(c_t, \Phi\left[\mathcal{M}_t\left(\left((1-c_t)R_{p,t+1}+\frac{\xi}{W_t}\right)\right)+bG(\frac{\xi}{W_t})\right]\right)$$
$$>\max_{c_t} H\left(c_t, \Phi\mathcal{M}_t\left(\left((1-c_t)R_{p,t+1}\right)\right)\right). \tag{2.3.2}$$

Moreover, because we assume that $R_{p,s+1}, s \geq t$ are i.i.d., the agent's optimal total utility per unit wealth should be a constant over time if she does not accept the gamble. In consequence, Φ can be solved in this case by

$$\Phi = \max_{c_t} H\left(c_t, \Phi \mathcal{M}_t\left(((1-c_t)R_{p,t+1})\right)\right).$$

Rigorous establishment of the above dynamic programming equation can be found in Corollary 8.

In the third approach, the gamble outcome is also revealed at time $t+1$ and thus the agent also evaluates the gamble at time $t+1$ as in the second case, but the agent makes a portfolio decision and, as a consequence of narrow framing, she derives a utility of gains and losses from some of her investment in risky assets. In consequence, if agent accepts the gamble, consumes c_tW_t at time t, and invests $\theta_{i,t}(1-c_t)W_t$ in risky asset i, $i=1,\ldots,n$, her total utility at time t is

$$\begin{aligned}
&H\Bigg(c_tW_t, \mathcal{M}_t\left(((1-c_t)W_tR_{p,t+1}+\xi)\,\Phi_{t+1}\right) \\
&\qquad + \frac{\mathcal{M}_t(W_{t+1}\Phi_{t+1})}{\mathcal{M}_t(W_{t+1})}\left[bG(\xi) + \sum_{i=1}^n b_i G(\theta_{i,t}(1-c_t)W_tR_{i,t+1})\right]\Bigg) \\
=&W_tH\Bigg(c_t, \mathcal{M}_t\left(\left((1-c_t)R_{p,t+1}+\frac{\xi}{W_t}\right)\Phi_{t+1}\right) \\
&\qquad + \frac{\mathcal{M}_t(W_{t+1}\Phi_{t+1})}{\mathcal{M}_t(W_{t+1})}\left[bG(\frac{\xi}{W_t}) + \sum_{i=1}^n b_i G(\theta_{i,t}(1-c_t)R_{i,t+1})\right]\Bigg),
\end{aligned}$$

where $R_{p,t+1} := \left(1-\sum_{i=1}^n \theta_{i,t}\right)R_{f,t+1} + \boldsymbol{\theta}_t'\mathbf{R}_{t+1}$ stands for the gross return of the agent's portfolio in period t to $t+1$. Similarly, if the agent does not accept the gamble, consumes c_tW_t at time t, and invests $\theta_{i,t}(1-c_t)W_t$ in risky asset i, $i=1,\dots,n$, her utility at time t is

$$W_t H\left(c_t, \mathcal{M}_t\left((1-c_t)R_{p,t+1}\Phi_{t+1}\right) + \frac{\mathcal{M}_t(W_{t+1}\Phi_{t+1})}{\mathcal{M}_t(W_{t+1})}\left[\sum_{i=1}^{n} b_i G(\theta_{i,t}(1-c_t)R_{i,t+1})\right]\right).$$

We further assume that $\mathbf{R}_{s+1} := (R_{1,s+1},\dots,R_{n,s+1}), s \geq t$ are i.i.d. and $R_{f,s+1}, s \geq t$ are constant. Then Φ_{s+1}, $s \geq t$ becomes a constant Φ that can be solved by

$$\Phi = \max_{c_t,\boldsymbol{\theta}_t} H\left(c_t, \Phi\left[\mathcal{M}_t\left((1-c_t)R_{p,s+1}\right) + \sum_{i=1}^{n} b_i G(\theta_{i,t}(1-c_t)R_{i,t+1})\right]\right);$$

see Corollary 8. In consequence, the agent accepts the gamble if and only if

$$\max_{c_t,\boldsymbol{\theta}_t} H\Bigg(c_t, \Phi\Bigg[\mathcal{M}_t\left(\left((1-c_t)R_{p,t+1} + \frac{\xi}{W_t}\right)\right) + bG(\frac{\xi}{W_t}) + \sum_{i=1}^{n} b_i G(\theta_{i,t}(1-c_t)R_{i,t+1})\Bigg]\Bigg) > \Phi. \tag{2.3.3}$$

The first two approaches are discussed in Barberis and Huang (2009) while the third approach in which the agent also makes a portfolio decision is new. We consider the third approach because some agents may actively manage their portfolios.

2.3.2 Examples

2.3.2.1 Example I

We first consider an example presented in Barberis and Huang (2009): an agent who, at time t, has wealth of \$500,000, is offered a timeless gamble, a 50:50 bet to gain \$200 or lose \$100 and the gamble outcome is independent of other risks. Suppose $\beta = 0.98$, $\gamma = \rho = 1.5$.

As discussed in Section 2.3.1, there are three possible approaches in which how the agent frames the gamble. In the first approach, the agent accepts the gamble if and only if (2.3.1) holds. Notes that this condition does not depend on the agent's consumption and investment strategy. In the second approach, the agent accepts the gamble if and only if (2.3.2) holds. Following Barberis and Huang (2009), we assume $R_{p,t+1}$ to be i.i.d. over time and $\ln R_{p,t+1}$ to follow normal distribution with mean 4% and standard deviation 3%. In the third approach, the agent accepts the gamble if and only if (2.3.3) holds. Note that this approach has not been discussed in the literature. We assume that there is only one risky asset to invest (i.e., $n = 1$) and set $b_1 = b$. Assume that $R_{1,t+1}$ is i.i.d. over time and $\ln R_{1,t+1}$ follows normal distribution with mean 4% and standard deviation 3%. Set $R_{f,t+1} = 1.027449$ so that with $b = 5$ and $\lambda = 3$, when the agent does not accept the gamble, her optimal percentage allocation to the stock is 100% and consequently the gross return of her optimal portfolio is the same as that in the second approach.

The left, middle, and right panels of Figure 2.1 show the ranges of the values of b and λ for which the agent rejects the gamble using the first, second, and third approaches, respectively. The ranges are highlighted by $+$ signs. Figure 2.1 shows that the specific approach the agent uses to frame the gamble has little effect on whether she rejects the gamble. Moreover, with reasonable parameter values, i.e., with $\lambda \geq 2.5$ and $b \geq 2$, the agent rejects the gamble. Our results are consistent with those in Barberis and Huang (2009) using the BH model. We also compute the ranges of the values of b and λ for different values of the standard deviation of ln $R_{p,t+1}$, such as 17.32%, and the results are almost the same.

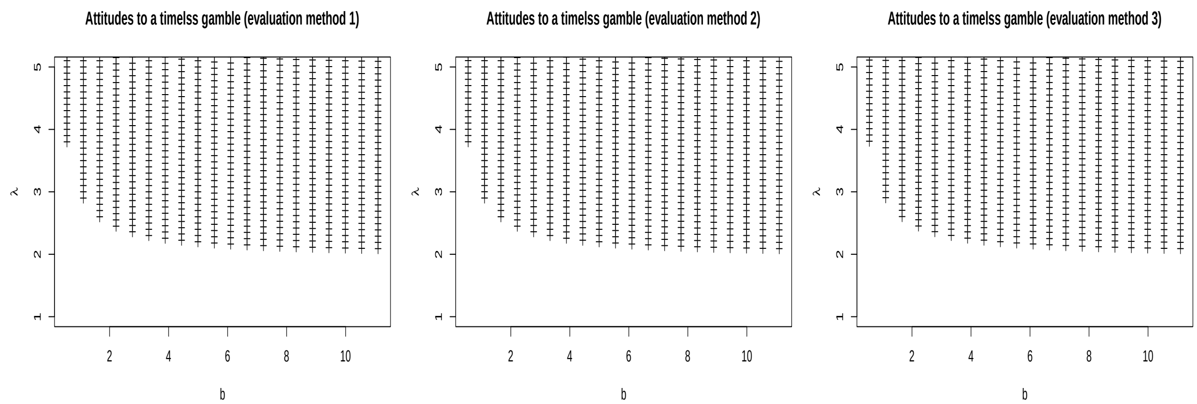

Figure 2.1: Ranges of the values of b and λ for which the agent rejects a timeless gamble that is a 50:50 bet to gain \$200 or lose \$100 and whose outcome is independent of other risks that are faced by the agent. The left, middle, and right panels correspond to three different approaches used by the agent to frame the timeless gamble, in which the agent rejects the gamble if and only if (2.3.1), (2.3.2), and (2.3.3) hold, respectively. We set the agent's current wealth to be \$500,000 and $\beta = 0.98, \gamma = \rho = 1.5$. In the second approach, we assume $R_{p,t+1}$ to be i.i.d. over time and ln $R_{p,t+1}$ to follow normal distribution with mean 4% and variance 3%. In the third approach, we assume that there is only one risky asset to invest (i.e., $n = 1$), set $b_1 = b$, assume that $R_{1,t+1}$ is i.i.d. over time and ln $R_{1,t+1}$ follows normal distribution with mean 4% and standard deviation 3%, and set $R_{f,t+1} = 1.027449$. The ranges are highlighted by $+$ signs.

2.3.2.2 Example II

We then consider the example presented in Barberis et al. (2006, p. 1071). In this example, we consider two gambles. The first gamble, denoted as G_S, is a 50:50 bet to gain \$550 and lose $\$500$. The second gamble, denoted as G_L, is a 50:50 bet to gain \$20,000,000 and lose \$10,000. As argued by Barberis et al. (2006), it is reasonable to posit that individuals tend to reject G_S for wealth levels $W_t \leq \$1,000,000$ and accept G_L for wealth levels $W_t \geq \$100,000$.

As in Section 2.3.1, a typical individual may use three approaches to evaluate G_S and G_L. We first show theoretically that when using the first approach to evaluate timeless gambles, the

GH model can explain the rejection of G_S and the acceptance of G_L.

Proposition 10 *Consider an agent who faces a timeless gamble with payoff ξ that is independent of other risks and evaluate this gamble using the GH model. Suppose $\mathbb{E}[\xi] > 0$ and the agent uses the first approach to evaluate the gamble. Denote $\xi_+ := \max(\xi, 0)$ and $\xi_- = \max(-\xi, 0)$.*

(i) If $\lambda \geq 1 + \big(1 + 1/b\big)\big(\mathbb{E}[\xi]/\mathbb{E}[\xi_-]\big)$, then the agent rejects ξ at any wealth level W_t.

(ii) If $\lambda < 1 + \big(1 + 1/b\big)\big(\mathbb{E}[\xi]/\mathbb{E}[\xi_-]\big)$, then the agent accepts ξ if and only if $W_t \geq W^*$, where $W^* \in (0, +\infty)$ is uniquely determined by $\mathcal{M}(1 + \xi/W^*) + b\mathbb{E}[\nu(\xi/W^*)] = 0$.

Proposition 10 shows that for any two gambles ξ_i, $i = 1, 2$ such that $\mathbb{E}[\xi_i] > 0, i = 1, 2$ and $\mathbb{E}[\xi_{1,+}]/\mathbb{E}[\xi_{1,-}] < \mathbb{E}[\xi_{2,+}]/\mathbb{E}[\xi_{2,-}]$, where $\xi_{i,+}$ and $\xi_{i,-}$ denote the positive and negative parts of ξ_i, respectively, there exists a range of the values of λ for which the agent rejects ξ_1 at any wealth level and accepts ξ_2 at sufficiently high wealth levels. This observation indicates that the GH model can possibly explain the rejection of G_S for wealth levels $W_t \leq \$1,000,000$ and the acceptance of G_L at for wealth levels $W_t \geq \$100,000$.

Figure 2.2 plots by $+$ signs the ranges of the values of b and λ for which the agent rejects G_S for wealth levels $W_t \leq \$1,000,000$ and accepts G_L at for wealth levels $W_t \geq \$100,000$. The left, middle, and right panels correspond to the first, second, and third approaches the agent uses to evaluate the gambles, respectively. The values of other parameters are the same as those in Figure 2.1. We can observe that the specific approach the agent uses has little impact on whether the agent rejects or accepts G_S and G_L, and with reasonable parameter values, the BH model predicts the rejection of G_S for wealth levels $W_t \leq \$1,000,000$ and the acceptance of

G_L at for wealth levels $W_t \geq \$100,000$. We also compute the ranges of the values of b and λ for different values of the standard deviation of ln $R_{p,t+1}$, such as 17.32%, and the results are almost the same.

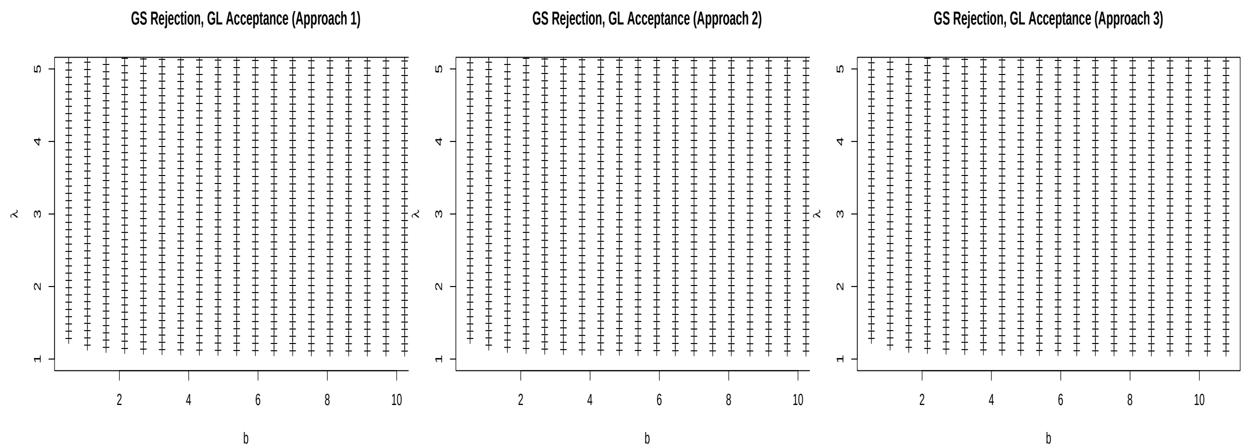

Figure 2.2: Ranges of the values of b and λ for which the agent rejects G_S for wealth levels $W_t \leq \$1,000,000$ and accepts G_L at for wealth levels $W_t \geq \$100,000$. The left, middle, and right panels correspond to three different approaches used by the agent to frame a timeless gamble, in which the agent rejects the gamble if and only if (2.3.1), (2.3.2), and (2.3.3) hold, respectively. We set $\beta = 0.98$ and $\gamma = \rho = 1.5$. In the second approach, we assume $R_{p,t+1}$ to be i.i.d. over time and ln $R_{p,t+1}$ to follow normal distribution with mean 4% and variance 3%. In the third approach, we assume that there is only one risky asset to invest (i.e., $n = 1$), set $b_1 = b$, assume that $R_{1,t+1}$ is i.i.d. over time and ln $R_{1,t+1}$ follows normal distribution with mean 4% and standard deviation 3%, and set $R_{f,t+1} = 1.027449$. The ranges are highlighted by $+$ signs.

2.4 Non-participation in the Stock Market

Barberis et al. (2006) consider an agent who, at the start of each period, has a fixed fraction $\bar{\theta}_N$ of her wealth invested in a nonfinancial asset with gross return $R_{N,s+1}$, $s \geq t$. The agent decides the fraction $\theta_{F,s+1}$ of her wealth invested in a stock with gross return $R_{F,s+1}$, $s \geq t$ and thus the remaining fraction $1 - \bar{\theta}_{N,s+1} - \theta_{F,s+1}$ invested in a risk-free asset with gross return $R_{f,s+1}$,

$s \geq t$. Assume $(\log R_{N,s+1}, \log R_{F,s+1})$, $s \geq t$ are i.i.d. and follow joint normal distribution with mean vector and covariance matrix to be

$$\begin{pmatrix} \mu_N \\ \mu_F \end{pmatrix}, \quad \begin{pmatrix} \sigma_N^2 & \omega\sigma_N\sigma_F \\ \omega\sigma_N\sigma_F & \sigma_F^2 \end{pmatrix}$$

respectively. Assume $R_{f,t+1}$ to be a constant. Suppose the agent derives a utility of gains and losses that are experienced by the investment in the stock. The agent needs to decide the consumption propensity and fraction of her wealth invested in the stock at each time. Because the gross returns of the nonfinancial asset and the stock are i.i.d. over time and the risk-free gross return is constant, the agent's optimal consumption propensity and percentage allocation to the stock must be constant as well. Moreover, the optimal utility per unit capital of the agent is also constant. Thus, the agent faces the following decision problem

$$\Phi = \max_{c,\theta_F} H\Bigg(c, (1-c)\Phi\Big[\mathcal{M}_t\left(\left(R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1}) + \theta_F(R_{F,t+1} - R_{f,t+1})\right)\right) \\ + bG(\theta_F(R_{F,t+1} - R_{f,t+1})\Big]\Bigg),$$

where Φ stands for the agent's optimal utility per unit capital; see Corollary 8.

Following Barberis et al. (2006), we want to know when the agent's optimal allocation to the stock is non-positive. Because

$$f(\theta_F) := \mathcal{M}_t\left(\left(R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1}) + \theta_F(R_{F,t+1} - R_{f,t+1})\right)\right) + bG(\theta_F(R_{F,t+1} - R_{f,t+1}))$$

is strictly concave in θ_F, this is the case if and only if $f'(0+) \leq 0$, i.e.,

$$\frac{\mathbb{E}[u'((R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1}))(R_{F,t+1} - R_{f,t+1})]}{u'(\mathcal{M}_t(R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1})))} + bG(R_{F,t+1} - R_{f,t+1}) \leq 0. \quad (2.4.1)$$

Note that this inequality does not depend on the discount rate, EIS, or the optimal utility per unit capital Φ of the agent. Thus, whether the agent's optimal allocation to the stock is non-positive is easier to check in the GH model than in the BH model.

Note that because f is concave in θ_F, the non-positivity of the optimal θ_F implies that the agent does not participate in the stock market when short selling is not allowed. When short-selling is allowed, however, condition (2.4.1) is insufficient to imply the non-participation in the stock market. Indeed, we need one more condition, $f'(0-) \geq 0$, which is equivalent to

$$\frac{\mathbb{E}[u'((R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1}))(R_{F,t+1} - R_{f,t+1})]}{u'(\mathcal{M}_t(R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1})))} - bG(R_{f,t+1} - R_{F,t+1}) \geq 0, \quad (2.4.2)$$

to conclude the nonparticipation in the stock market.

Note that with a higher λ, both (2.4.1) and (2.4.2) are more likely to hold, so it is more likely for the agent to have a non-positive allocation to the stock or not to participate in the stock market. The left panel of Figure 2.1 plots the threshold of λ, as a function of γ and b, for which the agent has a non-positive allocation to the stock (i.e., condition (2.4.1) holds). The right panel of Figure 3 plots a similar threshold for the agent not to participate in the stock market (i.e., conditions (2.4.1) and (2.4.2) hold). In both plots, we follow Barberis et al. (2006) to set $\mu_F = 6\%$, $\sigma_F = 20\%$, $\mu_N = 4\%$, $\sigma_N = 3\%$, $\bar{\theta}_N = 0.75$, $\omega = 0.1$, and $R_{f,t+1} = 1.02$. We can observe that for reasonable parameter values, the agent in the GH model does not participate in the stock market.

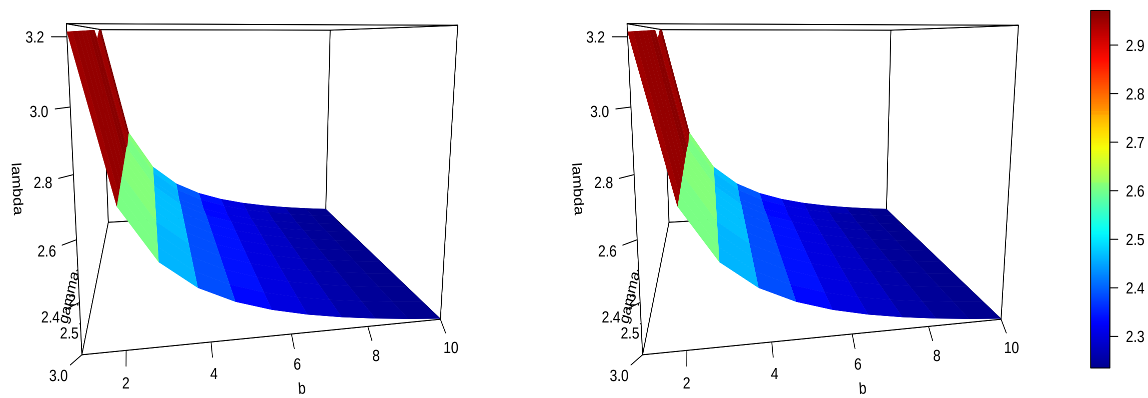

Figure 2.1: Thresholds of λ as functions of γ and b for which the agent has a non-positive allocation to the stock (left panel) and for which the agent does not participate in the stock market (right panel). In the plot, we set $\mu_F = 6\%$, $\sigma_F = 20\%$, $\mu_N = 4\%$, $\sigma_N = 3\%$, $\bar{\theta}_N = 0.75$, $\omega = 0.1$, and $R_{f,t+1} = 1.02$.

2.5 Asset Pricing

In the following, we consider a consumption-production equilibrium setting in Barberis and Huang (2009) and use the GH model to study the impact of narrow framing on asset prices.

Consider an economy with three assets. The first asset is a risk-free asset with zero net supply. The second asset is a non-financial asset, such as housing wealth or human capital, that has positive net supply. The third asset is a risky stock that has positive net supply. At each time t, the agent chooses consumption amount C_t and the remaining is used for investment, in which $\theta_{N,t}$ fraction is invested in the non-financial asset, $\theta_{S,t}$ fraction is invested in the stock, and the remaining is invested in the risk-free asset.

Following Barberis and Huang (2009), we consider an equilibrium in which (i) the risk-free gross return is a constant R_f; (ii) consumption growth C_{t+1}/C_t and stock gross return $R_{S,t+1}$ in period t to $t+1$ are distributed as

$$\log\left(C_{t+1}/C_t\right) = g_C + \sigma_C \varepsilon_{C,t+1}, \quad \log(R_{S,t+1}) = g_S + \sigma_S \varepsilon_{S,t+1},$$

where $(\varepsilon_{C,t+1}, \varepsilon_{S,t+1})$'s are i.i.d. over time and follow normal distribution with mean vector $(0,0)$ and covariance matrix $(1, \rho_{CS}; \rho_{CS}, 1)$; (iii) the consumption-wealth ratio is a constant over time; and (iv) the fraction of total wealth made up by the stock market is a constant ι, i.e.,

$$\frac{S_t}{S_t + N_t} = \iota, \quad \forall t,$$

where S_t and N_t are the value of the stock market and of the non-financial asset, respectively.

Barberis and Huang (2009, Appendix A.3) show that the above equilibrium can be embedded in a general consumption-production economy, so in the following we focus on solving asset prices in this equilibrium.

Proposition 11 *The equilibrium risk-free gross return R_f and the stock gross return $R_{S,t+1}$ in the GH model are solved together with the consumption-wealth ratio c^* from the following three* *equations:*

$$\frac{\mathbb{E}\left[u'\big(C_{t+1}/C_t\big)R_{S,t+1}\right]}{u'\left(\mathcal{M}\big(C_{t+1}/C_t\big)\right)} - R_f\frac{\mathbb{E}\left[u'\big(C_{t+1}/C_t\big)\right]}{u'\left(\mathcal{M}\big(C_{t+1}/C_t\big)\right)} + bG\big(R_{S,t+1}-R_f\big) = 0, \tag{2.5.1}$$

$$\frac{1}{1-c^*}\mathbb{E}\left[u'\left(\frac{C_{t+1}}{C_t}\right)\frac{C_{t+1}}{C_t}\right] = (1-\iota)R_f\mathbb{E}\left[u'\left(\frac{C_{t+1}}{C_t}\right)\right] + \iota\mathbb{E}\left[u'\left(\frac{C_{t+1}}{C_t}\right)R_{S,t+1}\right], \tag{2.5.2}$$

$$1-c^* = \left\{\beta\left[\frac{1}{1-c^*}\mathcal{M}\left(\frac{C_{t+1}}{C_t}\right) + \frac{\iota}{1-\iota}\frac{1}{1-c^*}\frac{\mathbb{E}\left[u'\big(C_{t+1}/C_t\big)\big(C_{t+1}/C_t\big)\right]}{u'\left(\mathcal{M}\big(C_{t+1}/C_t\big)\right)} - \frac{\iota}{1-\iota}\frac{\mathbb{E}\left[u'\big(C_{t+1}/C_t\big)R_{S,t+1}\right]}{u'\left(\mathcal{M}\big(C_{t+1}/C_t\big)\right)}\right]^{1-\rho}\right\}^{1/\rho}, \tag{2.5.3}$$

provided that $-(1-c^*)b\iota G(R_{S,t+1}-R_f) < \mathcal{M}(C_{t+1}/C_t)$ *and* $\beta\big[\mathcal{M}(C_{t+1}/C_t) + (1-c^*)b\iota G(R_{S,t+1}-R_f)\big]^{1-\rho} < 1$.

The equilibrium equations (2.5.1)–(2.5.3) are easy to solve: we can first solve $1-c^*$ in terms of g_S from (2.5.3), then solve R_f in terms of g_S from (2.5.2), and finally solve g_S from (2.5.1). Following Barberis and Huang (2009), we set $\rho = 1.5$, $\beta = 0.98$, $\iota = 0.3$, $g_c = 0.0184$, $\sigma_c = 0.0379$, $\sigma_s = .020$, $\rho_{cs} = .10$, and λ to be 2 or 3. We set γ to be 1.5 or 5, the former being used in Barberis and Huang (2009). Finally, we vary b from 0 to 10. Table 2.1 presents the net return rate of the risk-free asset and the equity premium (i.e., the expected excess return of the stock). We can see that our results are similar to those in Barberis and Huang (2009, Table 4), showing that the GH model can be used to explain a high equity premium due to narrow framing. We also note that for $b \geq 2$, the equity premium is insensitive to the value of γ.

Finally, when there is only one risky asset in the market and the aggregate consumption is equal to the aggregate dividend paid out by this risky asset, the equilibrium asset pricing

Table 2.1: Risk-free net return and equity premium in a consumption-production economy. Model parameters are set to be $\rho = 1.5$, $\beta = 0.98$, $\iota = 0.3$, $g_C = 0.0184$, $\sigma_C = 0.0379$, $\sigma_S = .020$, and $\rho_{CS} = .10$.

γ	λ	b	$R_f - 1$ (%)	$\mathbb{E}_t[R_{S,t+1}] - R_f$ (%)
1.5	2	0	4.65	0.19
1.5	2	2	2.74	4.42
1.5	2	4	2.45	5.06
1.5	2	6	2.34	5.31
1.5	2	8	2.27	5.45
1.5	3	0	4.65	0.19
1.5	3	2	1.45	7.26
1.5	3	4	1.04	8.16
1.5	3	6	0.88	8.51
1.5	3	8	0.80	8.70
5	2	0	3.94	0.65
5	2	2	2.20	4.52
5	2	4	1.94	5.10
5	2	6	1.84	5.34
5	2	8	1.78	5.46
5	3	0	3.94	0.65
5	3	2	0.93	7.33
5	3	4	0.55	8.18
5	3	6	0.40	8.52
5	3	8	0.31	8.70

equations (2.5.1)–(2.5.3) can be simplified. This special case has been studied in Proposition 2.

2.6 Conclusions

Barberis and Huang (2009) propose the BH model to allow for narrow framing in decision making. This model opens the door of studying the impact of narrow framing on decision making both analytically and systematically, and it has been successfully applied to explain individual's attitudes toward timeless gambles, nonparticipation of households in the stock market, and high equity premia in the market. We found, however, that the BH model is not perfect because the agent's total utility process in this model may not be well defined when the utility of the gain and loss experienced by the agent is negative. Moreover, we showed that even though one can derive the dynamic programming equation in the BH model heuristically, this equation may have multiple solutions, leading to different optimal consumption-investment strategies.

The issue of non-existence or non-uniqueness of the total utility process and the solution to the dynamic programming equation in the BH model arises because the weight for the utility of gains and losses used by the agent is a constant over time. In consequence, a negative utility of gains and losses in each period can drive the agent's total utility to be negative, resulting in the aforementioned ill-behaviors of the BH model.

We proposed the GH model, a refinement of the BH model, in which the weight for the utility of gains and losses in each period is scaled by the agent's total utility per unit wealth from next period. It turns out that in the GH model, the agent's total utility process uniquely exists and the solution to the dynamic programming equation in this model uniquely exists as well. Moreover, both the total utility process and the solution to the dynamic programming equation can be computed by a recursive algorithm with any starting point. We also showed that even when the agent's total utility per unit wealth is a constant so that the weight for the utility of gains and losses in the GH model is also a constant, the GH model differs from the BH model in that the total utility process in the former cannot be obtained from the latter.

We applied the GH model to explain individuals' attitudes toward timeless gambles, non-participation of households in the stock market, and high equity premia in the market. We found that the GH model is as powerful as the BH model in these explanations. More strikingly, the GH model is more tractable than the BH model in these applications.

Chapter 3

Equilibrium Asset Pricing with Epstein-Zin and Loss-Averse Investors

3.1 Introduction

In classical consumption-based asset pricing models, investors are assumed to maximize their preferences for consumption and the preferences are represented by expected utility theory (EUT) (von Neumann and Morgenstern, 1947) and, more generally, by recursive utility theory (Epstein and Zin, 1989). In the past decades, researchers have suggested that in addition to consumption, investors derive utility also from investment gains and losses as a consequence of narrow framing; moreover, investors are loss averse as assumed in cumulative prospect theory (CPT) (Kahneman and Tversky 1979 and Tversky and Kahneman 1992). Several preference models that feature narrow framing have then been proposed to combine the utilities of consumption and of investment gains and losses and been applied to asset pricing with homogeneous loss-averse investors, and these models have been proven effective to explain time-series and cross-sectional asset returns that are empirically observed in the market; see for instance Barberis and Huang (2008a, 2009), Barberis et al. (2006), De Giorgi and Legg (2012), and He and

Zhou (2014).

In a notable work by Easley and Yang (2015), the authors consider equilibrium asset pricing with two types of agents: EZ-agents whose preferences for consumption are represented by the recursive utility theory with constant relative risk aversion degree (RRAD) and elasticity of intertemporal substitution (EIS) and LA-agents who are concerned with the utility of investment gains and losses in addition to the consumption utility, and this utility is measured by CPT. The agents can trade a risk-free asset with zero net supply and a risky stock with positive net supply in discrete time. The stock pays out independent and identically distributed (i.i.d.) dividends over time and the dividend in each period follows a binomial distribution. After solving individual portfolio selection problems and characterizing equilibrium conditions, the authors show by simulation that the LA-agents will be driven out of the market in the long run if their portfolios are further away from those of log investors than the EZ-agents' portfolios. Moreover, the authors show that the market selection process can be slow in terms of wealth shares but can be fast in terms of price impact.

The objective of the present chapter is to obtain the main results in Easley and Yang (2015) analytically, rather than numerically, in certain settings. In consequence, we are able to answer several important questions theoretically, such as whether the equilibrium asset prices exist, whether the LA-agents hold less risky assets than the EZ-agents, and whether the EZ-agents dominate the market in the long run. Note that Easley and Yang (2015) uses the BH model to represent the LA-agents' preferences. Because we already showed in Chapter 2 that the GH model is better behaved than the BH model, we use the GH model to represent the LA-agents' preferences. Moreover, we assume the same RRAD and EIS for the EZ- and LA-agents so as to focus on the impact of the heterogeneity in the degrees of narrow framing and loss aversion on asset prices. On the other hand, the stock dividend in our model can follow any distribution.

First, we show that when the LA-agents' loss aversion degree (LAD) is equal to the so-called equilibrium gain-loss ratio of the stock, they behave the same as the EZ-agents and thus the presence of the LA-agents in the economy does not affect equilibrium asset prices. This ratio is defined to be the ratio of the gain and loss of the stock relative to a reference point, and this reference point is endogenously determined in equilibrium: it is the equilibrium risk-free return in an economy with EZ-agents only. When the LA-agents' LAD is higher (lower) than the equilibrium gain-loss ratio, the LA-agents appear to be more (less) risk averse than the EZ-agents, and we propose a measure to quantify the risk aversion of the LA-agents.

second, when the RRAD and EIS are equal to one, we prove the existence and uniqueness of the equilibrium. Moreover, when the economy is populated with only one EZ- and one LA-agent, we find that the LA-agent invests less (more) in the stock than the EZ-agent if and only if the LA-agent's LAD is higher (lower) than the equilibrium gain-loss ratio. Consequently, the conditional equity premium is increasing (decreasing) with respect to the wealth share of the LA-agent in the market when her LAD is higher (lower) than the equilibrium gain-loss ratio. In addition, we prove that the EZ-agent dominates the market in the long run unless the LA-agent's LAD is exactly equal to the equilibrium gain-loss ratio.

Our study is also related to the following literature: The preference representation of the LA-agents in our model is similar to the BH model as used in Barberis and Huang (2001, 2008a, 2009), Barberis et al. (2001, 2006), De Giorgi and Legg (2012), and He and Zhou (2014). These studies either consider asset pricing with a representative agent or consider the portfolio selection of a single agent, whereas the present chapter considers asset pricing with heterogeneous agents. Multi-period portfolio selection of a single agent concerning her preferences for wealth represented by CPT has also been widely studied in the literature; see for instance Barberis and Xiong (2009), Gomes (2005), Shi, Cui and Li (2015), Shi, Cui, Yao and Li (2015), and many others. In these works, however, the authors assume the agent not to derive consumption utility, and they do not study asset pricing.

Several studies on asset pricing with heterogeneous rational and irrational agents have been put forward in different settings. Del Vigna (2013) consider a single-period asset pricing model in which the agents have heterogeneous preferences represented by EUT and CPT. Assuming the asset returns to follow a multi-variate normal distribution, the authors prove the existence of the equilibrium. De Giorgi et al. (2011) and De Giorgi and Hens (2006) consider a similar model and study the equilibrium asset returns. De Giorgi et al. (2010) consider a single-period complete market with heterogeneous agents with CPT preferences and show that the equilibrium does not always exist. Xia and Zhou (2016) study the equilibrium in a single-period complete market in which the agents' preferences are represented by rank-dependent expected utility (Quiggin, 1982). The authors assume heterogeneous utility functions but homogeneous probability weighting functions for the agents and prove the existence of the equilibrium. All of the aforementioned papers assume a single-period setting and assume either normally distributed asset returns or complete markets. The present chapter uses a multi-period setting, and the dividend growth rate can follow any distribution.

Chapman and Polkovnichenko (2009) consider a single-period asset pricing model with two agents: one agent with EUT preferences and the other with non-EUT preferences, such as those with loss aversion. The authors show that the equilibrium equity premium with one EUT and one non-EUT agent is significantly lower than the equity premium in an economy with a single agent whose preferences are the average of those of the EUT and non-EUT agents. Li and Yang (2013b) consider an overlapping-generation model in which investors are loss averse and have heterogeneous beliefs, and use this model to examine the implications of prospect theory for the disposition effect, asset prices, and trading volume. Pasquariello (2014) consider a single-period equilibrium model in which loss-averse speculators, liquidity traders, and market makers trade a single risky asset. The present chapter differs from the aforementioned papers in that we use a standard multi-period consumption-based setting.

The remainder of the chapter is organized as follows: After introducing the asset pricing model in Section 3.2, we propose the equilibrium gain-loss ratio and quantify the risk aversion of the LA-agents in Section 3.3. In Section 3.4, we study portfolio selection, asset pricing, and market dominance when the EZ- and LA-agents have unitary RRAD and EIS. In Section 3.5, we extend all the results in the previous sections to the case in which the LA-agents' preferences for investment gains and losses involve probability weighting. Finally, Section 3.6 concludes and all proofs are placed in the Appendix 3.6.

3.2 Equilibrium Asset Pricing Model

3.2.1 The Market

We consider a discrete-time financial market with one stock and one risk-free asset. The net supplies of the stock and of the risk-free asset are one and zero, respectively. The stock distributes dividend D_t at each time t. The (gross) return rates of the stock and the risk-free asset in period t to $t+1$ are R_{t+1} and $R_{f,t+1}$, respectively. The following assumption on the dividend process is imposed throughout of the chapter:

Assumption 5 *The dividend growth rates* $Z_{t+1} := D_{t+1}/D_t$, $t = 0, 1, \ldots$ *are i.i.d.,* $\operatorname{essinf} Z_t > 0$, *and* $\mathbb{E}\left(Z_t\right) < \infty$.

Assuming an i.i.d. dividend process is common in many asset pricing models; see for instance Easley and Yang (2015). Assumption $\operatorname{essinf} Z_t > 0$ is necessary to induce heterogeneity in stock holding; otherwise, no agent in the market can invest more than her wealth in the stock.

3.2.2 Investors

Suppose there are m agents in the market. At each time t, with wealth $W_{i,t}$, agent i chooses consumption amount $c_{i,t}W_{i,t}$, dollar amount $\theta_{i,t}(W_{i,t} - c_{i,t}W_{i,t})$ to be invested in the stock, and the remaining dollar amount $(1-\theta_{i,t})(W_{i,t}-c_{i,t}W_{i,t})$ to be invested in the risk-free asset, where $c_{i,t}$ and $\theta_{i,t}$ stand for agent i's *consumption propensity* and *percentage allocation to the stock* at time t. In consequence, the wealth equation of the agent is $W_{i,t+1} = (1 - c_{i,t})W_{i,t}\big[R_{f,t+1} + \theta_{i,t}(R_{t+1} - R_{f,t+1})\big]$.

We assume that agent i evaluates the gain and loss of her investment in the stock in isolation with her consumption and her preferences are represented by the GH model. More precisely, agent i's total utility of her consumption $\{c_{i,t}W_{i,t}\}$ and investment gains and losses is represented by $\{U_{i,t}\}$, which is defined recursively by

$$U_{i,t} = H(c_{i,t}W_{i,t}, M(U_{i,t+1}|\mathscr{F}_t) + b_{i,t}G_{i,t}), \quad t \geq 0. \tag{3.2.1}$$

Here, $\mathscr{F}_t$ denotes the information available at time t, $M(U_{i,t+1}|\mathscr{F}_t)$ is the certainty equivalent of agent i's total utility at $t+1$, $G_{i,t}$ represents agent i's *investment utility*, i.e., her utility of the gain and loss incurred by her investment in the stock, $b_{i,t}$, defined as

$$b_{i,t} = b_i \frac{M\left(U_{i,t+1}|\mathscr{F}_t\right)}{M\left(W_{i,t+1}|\mathscr{F}_t\right)} \tag{3.2.2}$$

for some constant $b_i \geq 0$, is the weight of the investment utility in the total utility, and H is an aggregrator. We assume H and M to be defined in 2.2.2 and 2.2.3, respectively. In the following, if X is independent of $\mathscr{F}_t$, we simply write $M(X|\mathscr{F}_t)$ as $M(X)$.

We apply CPT (Kahneman and Tversky (1979) and Tversky and Kahneman 1992) to model the investment utility:

$$G_{i,t} := \mathbb{E}\left[\nu_i(W_{i,t+1} - (1 - c_{i,t})W_{i,t}R_{f,t+1})|\mathscr{F}_t\right], \tag{3.2.3}$$

where $\nu_i(x) := x\mathbf{1}_{x\geq 0} + K_i x \mathbf{1}_{x<0}$ for some $K_i \geq 1$. In other words, agent i sets the risk-free payoff $(1-c_{i,t})W_{i,t}R_{f,t+1}$ to be her *reference point* to distinguish gains and losses for her investment, and then the investment gain and loss are evaluated by CPT with piece-wise linear utility function ν_i. The parameter $K_i \geq 1$, referred to as *loss aversion degree* (LAD), models the empirical finding that individuals tend to be more sensitive to losses than to comparable gains. Note that following the settings in Barberis and Huang (2008a, 2009), Barberis et al. (2006) and Easley and Yang (2015), we do not consider diminishing sensitivity and probability weighting, another two important features of CPT, in our model of agents' investment utility at the moment. Following the ideas in De Giorgi and Legg (2012) and He and Zhou (2014), we can incorporate probability weighting in our model as well, and we find that the presence of probability weighting results in more tedious notations and calculations, but our main results remain unchange. Thus, we choose to focus our discussion on the case of no probability weighting and the asset pricing results with probability weighting will be presented in Section 3.5.

Because agents derive utility of investment gains and losses as a consequence of narrow framing, we refer to b_i as the *narrow framing degree* (NFD) of agent i. When $b_i = 0$, $\{U_{i,t}\}$ defined by (3.2.1) becomes recursive utility (Epstein and Zin, 1989, Kreps and Porteus, 1978). In this case, we call agent i an *EZ-agent*. When $b_i > 0$, agent i derives utility from two sources: consumption and investment gains and losses; moreover the agent is loss averse. In this case, we call agent i an *LA-agent*. The analysis in Section 2.3 shows that when $b_i \geq 1$ and the LAD is larger than 3, the LA-agent rejects the gamble.

In our model, we assume that the agents in the market are homogeneous in their RRAD and EIS, so these two quantities are not indexed by the agents' identities. However, the agents can be heterogeneous in their NFD and LAD. We use this setting because we want to focus ourselves on the impact of the heterogeneity in the degrees of narrow framing and loss aversion on asset prices.

In the following, whenever we consider homogeneous settings, such as the setting in which LA-agents have homogeneous LAD, we drop subscript i that stands for the agents' identities, e.g., we write K_i as K. We also denote $x_+ := \max(x, 0)$, $x_- := \max(-x, 0)$, and $\mathbf{1}_A$ as the indicator function of event A (which takes value 1 when A occurs and 0 otherwise).

3.2.3 Optimal Portfolio Selection

With preference representation (3.2.1), agent i's decision problem at t can be formulated as

$$\begin{aligned} \max_{\{c_{i,s}\}_{s\geq t}, \{\theta_{i,s}\}_{s\geq t}} \quad & U_{i,t} \\ \text{Subject to} \quad & W_{i,s+1} = (1 - c_{i,s}) W_{i,s} \left[R_{f,s+1} + \theta_{i,s} \left(R_{s+1} - R_{f,s+1} \right) \right], s \geq t. \end{aligned} \tag{3.2.4}$$

A strategy $\{(c_{i,t}, \theta_{i,t})\}$ is *feasible* if $0 < c_{i,t} < 1$, $M(W_{i,t+1}|\mathscr{F}_t) + b_i G_{i,t} > 0$, and some growth conditions hold so that $\{U_{i,t}\}$ uniquely exists; see Chapter 1. We do not formulate the feasible set of strategies explicitly for the sake of simplicity; rather, when presenting the solution to (3.2.4), we provide conditions for which the optimal strategy is feasible and thus a set of feasible strategies that contains the optimal one can be constructed so that the setup of the agent's decision problem is complete.

We apply dynamic programming to solve problem (3.2.4). According to Theorem 6, we obtain the following Bellman equation:

$$\Psi_{i,t} = \max_{c_{i,t},\theta_{i,t}} H\Bigg(c_{i,t}, (1-c_{i,t}) M(\Psi_{i,t+1} R_{i,t+1} | \mathscr{F}_t) \\ \times \Big(1 + b_i \left(M(R_{i,t+1}|\mathscr{F}_t) \right)^{-1} \mathbb{E}\left[\nu_i \left(\theta_{i,t}(R_{t+1} - R_{f,t+1}) \right) | \mathscr{F}_t \right] \Big) \Bigg), \quad (3.2.5)$$

where $R_{i,t+1} := R_{f,t+1} + \theta_{i,t}(R_{t+1} - R_{f,t+1})$ is the gross return of the agent's portfolio. We can observe that when $\Psi_{i,t+1}$ is a constant or when the agent's RRAD is one, we have

$$M(\Psi_{i,t+1} R_{i,t+1} | \mathscr{F}_t) = M(\Psi_{i,t+1} | \mathscr{F}_t) M(R_{i,t+1} | \mathscr{F}_t).$$

In consequence,

$$\begin{aligned} & M(\Psi_{i,t+1} R_{i,t+1} | \mathscr{F}_t) \Big(1 + b_i \left(M(R_{i,t+1}|\mathscr{F}_t) \right)^{-1} \mathbb{E}\left[\nu_i \left(\theta_{i,t}(R_{t+1} - R_{f,t+1}) \right) | \mathscr{F}_t \right] \Big) \\ = & M(\Psi_{i,t+1} | \mathscr{F}_t) \Big(M(R_{i,t+1}|\mathscr{F}_t) + b_i \mathbb{E}\left[\nu_i \left(\theta_{i,t}(R_{t+1} - R_{f,t+1}) \right) | \mathscr{F}_t \right] \Big), \end{aligned}$$

so the agent's optimal portfolio does not depend on $\Psi_{i,t+1}$, her optimal utility per unit wealth in the future. This property makes it possible to obtain equilibrium asset returns analytically when $\Psi_{i,t+1}$ is a constant or when the agent's RRAD is one.

3.2.4 Equilibrium

Denote P_t as the ex-dividend price of the stock at time t. Then, the gross return of the stock is $R_{t+1} = (P_{t+1} + D_{t+1})/P_t, t \geq 0$. We follow the standard definition of competitive equilibria in the literature; see for instance Yan (2008).

Definition 1 *A competitive equilibrium is a price system* $\{R_{f,t+1}, P_t\}$ *and a consumption-investment plan* $\{c_{i,t}, \theta_{i,t}\}$ *with the corresponding wealth processes* $\{W_{i,t}\}$, $i = 1, 2, \ldots, m$ *that satisfy*

(i) individual optimality: for each $i = 1, 2, \ldots, m$, $\{c_{i,t}, \theta_{i,t}\}$ *is the optimal consumption-investment plan of agent* i*;*

(ii) clearing of consumption: $\sum_{i=1}^{m} c_{i,t} W_{i,t} = D_t$*;*

(iii) clearing of the stock: $\sum_{i=1}^{m} \theta_{i,t}(W_{i,t} - c_{i,t}W_{i,t}) = P_t$*; and*

(iv) clearing of the risk-free asset: $\sum_{i=1}^{m}(1 - \theta_{i,t})\,(W_{i,t} - c_{i,t}W_{i,t}) = 0$.

Lemma 1 *A price system* $\{R_{f,t+1}, P_t\}$ *and a consumption-investment plan* $\{c_{i,t}, \theta_{i,t}\}$ *with corresponding wealth process* $\{W_{i,t}\}$, $i = 1, 2, \ldots, m$ *constitute a competitive equilibrium if and only if* $\{c_{i,t}, \theta_{i,t}\}$ *are the optimal choice of agent* i, $i = 1, 2, \ldots, m$, *and satisfy*

$$\sum_{i=1}^{m} \frac{c_{i,t}}{1 - c_{i,t}} Y_{i,t} = D_t/P_t, \quad \sum_{i=1}^{m} \theta_{i,t} Y_{i,t} = 1, \tag{3.2.6}$$

where $Y_{i,t} := (1 - c_{i,t})W_{i,t}/\sum_{j=1}^{m}\big((1 - c_{j,t})W_{j,t}\big)$ *stands for agent* i*'s post-consumption wealth share in the market at time* t *and* $(c_{i,t}/(1 - c_{i,t}))Y_{i,t}$ *is defined to be* $W_{i,t}/\sum_{j=1}^{m}\big((1 - c_{j,t})W_{j,t}\big)$ *when* $c_{i,t} = 1$.

3.3 Equilibrium Gain-Loss Ratio

Theorem 3 *Assume* $\alpha_{\mathrm{EZ}} := \beta(M(Z_{t+1}))^{1-\rho} < 1$. *Then,* $R_{t+1} = Z_{t+1}/\alpha_{\mathrm{EZ}}$ *and* $R_{f,t+1} = \mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)$ *are the equilibrium return rates of the stock and the risk-free asset, respectively, when the market is populated with EZ-agents only. Moreover, the equilibrium remains the same if the market is populated with not only EZ- but also LA-agents with LAD equal to*

$$K^* := \frac{\mathbb{E}\left[\left(R_{t+1} - \left(\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\right)\right)_+\right]}{\mathbb{E}\left[\left(R_{t+1} - \left(\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\right)\right)_-\right]} = \frac{\mathbb{E}\left[\left(Z_{t+1} - \left(\mathbb{E}\left(Z_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma}\right)\right)\right)_+\right]}{\mathbb{E}\left[\left(Z_{t+1} - \left(\mathbb{E}\left(Z_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma}\right)\right)\right)_-\right]}, \tag{3.3.1}$$

and in this case the EZ- and LA-agents have the same consumption-investment plan.

Theorem 3 provides the equilibrium asset returns when the market is populated with EZ-agents only.[1] More importantly, Theorem 3 shows that the presence of LA-agents in the market does not change the equilibrium if their LAD is equal to K^*. In addition, the EZ- and LA-agents have the same consumption-investment strategy in this case. The intuition behind this result is as follows: when the LAD is equal to K^*, the investment utility derived by the LA-agents is zero, so these agents behave the same as the EZ-agents. Moreover, the particular form of $b_{i,t}$ is actually irrelevant in obtaining Theorem 3 because if the LA-agent's investment utility is zero, it does not add to the agent's total utility however it is weighted.

We can see that K^* is the ratio of the expected gain and loss of the stock with the reference point $\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)$. Therefore, K^* is similar to the gain-loss ratios proposed

[1] Condition $\beta(M(Z_{t+1}))^{1-\rho} < 1$ in Theorem 3 is to ensure that the recursive utility of consuming the stock dividend is well-defined; see for instance Epstein and Zin (1989) and Hansen and Scheinkman (2012) for conditions in general settings.

by Bernardo and Ledoit (2000) and Cherny and Madan (2009). Note that the reference point $\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)$ is not arbitrarily chosen; it is the risk-free rate in an economy with EZ-agents only and thus is endogenously determined in equilibrium. This marks the difference of K^* from the gain-loss ratios in Bernardo and Ledoit (2000) and Cherny and Madan (2009) where the reference points are exogenously given. Therefore, we call K^* the *equilibrium gain-loss ratio* of the stock.

Note that K^* depends on γ, the RRAD of the agents in the market, but does not depend on $1/\rho$, the EIS of the agents. Moreover, the following proposition shows that K^* is always larger than 1 and is strictly increasing in γ. Therefore, the more risk averse the agents are, the larger the equilibrium gain-loss ratio is.

Proposition 12 $\mathbb{E}\left(Z_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma}\right)$ *is strictly decreasing in* $\gamma \geq 0$. *In consequence, the equilibrium gain-loss ratio* K^* *is strictly increasing in* $\gamma \geq 0$ *and is equal to 1 when* $\gamma = 0$.

Next, we numerically compute K^*. Note that K^* depends on the length of each period, consistent with the myopic loss aversion theory proposed by Benartzi and Thaler (1995). Following Benartzi and Thaler (1995), we set the length of each period to be one year. We use the same *annual* dividend growth data as in Chapman and Polkovnichenko (2009), which is reproduced in Table 3.1.

Table 3.2 shows the values of K^* with respect to different values of γ, assuming that the dividend growth-rate follows the distribution as specified in Table 3.1. Most empirical estimates of LAD are larger than 1.5 and thus larger than K^*.[2]

[2]See, for instance, Footnote 11 and Figure 5.1 of He and Kou (2016) for a summary of the empirical estimates

Table 3.1: Distribution of the annual dividend growth rate. The distribution is assumed to be the same as in Table I of Chapman and Polkovnichenko (2009), which is obtained using the historical gross consumption growth from 1949 to 2006.

State	1	2	3	4	5	6	7	8	9
Outcome	0.976	0.993	1.002	1.011	1.019	1.028	1.037	1.045	1.054
Probability	0.03	0.03	0.10	0.16	0.24	0.19	0.13	0.09	0.03

Table 3.2: Equilibrium gain-loss ratio K^* with respect to RRAD γ. The distribution of the dividend growth rate Z_{t+1} is assumed as in Table 3.1.

γ	0.5	0.9	.95	1.05	1.1	1.5	2	3	4	5
K^*	1.021	1.038	1.040	1.044	1.046	1.063	1.086	1.132	1.182	1.234

Theorem 3 suggests that the LA-agents are more (less) risk averse than the EZ-agents in equilibrium when the LA-agents' LAD is larger (less) than K^*. To further compare the risk attitudes of the EZ- and LA-agents, we formalize the idea in Easley and Yang (2015, Table 3) to provide a measure of risk aversion for the LA-agents through an equilibrium analysis.

Proposition 13 *Suppose the market is populated with homogeneous LA-agents. Assume that* $\alpha_{\text{LA}} := \beta \left(M(Z_{t+1})^\gamma \mathbb{E}[Z_{t+1}^{-\gamma}] r_{f,\text{LA}} \right)^{1-\rho} < 1$, *where* $r_{f,\text{LA}}$ *is the unique solution to*

$$M(Z_{t+1}) - r_{f,\text{LA}} M(Z_{t+1})^\gamma \mathbb{E}[Z_{t+1}^{-\gamma}] + b\mathbb{E}[\nu(Z_{t+1} - r_{f,\text{LA}})] = 0. \tag{3.3.2}$$

Then, the equilibrium exists with $R_{f,t+1} = r_{f,\text{LA}}/\alpha_{\text{LA}}$ *and* $R_{t+1} = Z_{t+1}/\alpha_{\text{LA}}$. *Moreover,* $r_{f,\text{LA}} = \mathbb{E}[Z_{t+1}^{1-\gamma}]/\mathbb{E}[Z_{t+1}^{-\gamma}]$ *when* $K = K^*$, $r_{f,\text{LA}}$ *is decreasing in* K, *and the monotonicity becomes strict when* $b > 0$. *Finally,* $r_{f,\text{LA}}$ *is strictly decreasing (increasing) in* b *when* $K > K^*$ $(K < K^*)$.

Proposition 13 provides the equilibrium asset returns in an economy populated with homo-

of LAD.

geneous LA-agents. We can then compute the *continuously compounded equity premium*

$$\log \mathbb{E}[R_{t+1}] - \log R_{f,t+1} = \log \mathbb{E}[Z_{t+1}] - \log r_{f,\text{LA}} \tag{3.3.3}$$

and find $\gamma' > 0$ such that this equity premium is equal to $\log \mathbb{E}[Z_{t+1}] - \log \left(\mathbb{E}[Z_{t+1}^{1-\gamma'}]/\mathbb{E}[Z_{t+1}^{-\gamma'}]\right)$, the continuously compounded equity premium in an economy populated with EZ-agents with RRAD γ' (see Theorem 3). The resulting γ' is the *implied RRAD* of the LA-agents in equilibrium. Note that the implied RRAD does not depend on the discount rate β or EIS $1/\rho$.

The notion of implied RRAD formalizes the analysis in Table 3 of Easley and Yang (2015) that is conducted to compare the risk attitudes of the LA- and EZ-agents. Because we obtain the equilibrium asset returns in closed form, we are able to compute the implied RRAD and analyze its properties easily. Indeed, Proposition 13 shows that $r_{f,\text{LA}} = \mathbb{E}[Z_{t+1}^{1-\gamma}]/\mathbb{E}[Z_{t+1}^{-\gamma}]$ when $K = K^*$, $r_{f,\text{LA}}$ is decreasing in K, and the monotonicity becomes strict when $b > 0$. Moreover, Proposition 12 shows that $\mathbb{E}[Z_{t+1}^{1-\gamma'}]/\mathbb{E}[Z_{t+1}^{-\gamma'}]$ is strictly decreasing in γ'. Thus, the implied RRAD is equal to γ when $K = K^*$, is increasing in K, and the monotonicity becomes strict when $b > 0$. Similarly, we can see that the implied RRAD is strictly increasing (decreasing) in b when $K > K^*$ ($K < K^*$).

We can also compute the *implied EIS* $1/\rho'$ that is determined by matching the price-dividend ratios. In other words, ρ' is determined by $\left(M(Z_{t+1})^{\gamma}\mathbb{E}[Z_{t+1}^{-\gamma}]r_{f,\text{LA}}\right)^{1-\rho} = \left(M'(Z_{t+1})\right)^{1-\rho'}$, where M' is the certainty equivalent with RRAD equal to γ'. Note that ρ' does not depend on the discount rate β.

We set $K = 2.25$, the estimate of LAD that was obtained by Tversky and Kahneman (1992), set the length of each period to be one year, use the dividend growth rate in Table 3.1, and then compute the implied RRAD and EIS of the LA-agents for different values of b, γ, and ρ. Table 3.3 shows the results. Note that for all the parameter values we use in Table 3.3, $K > K^*$. We can observe that because of narrow framing, the LA-agents derive negative utility for their investment in the stock and thus can become much more risk averse. The implied EIS of the LA-agents, however, is not very sensitive to the agents' NFD.

Table 3.3: Implied RRAD γ' and implied EIS $1/\rho'$ of the LA-agents. The length of each period is one year, the distribution of the dividend growth rate Z_{t+1} is given in Table 3.1, and $K = 2.25$.

b	0	0.10	0.33	1.00	3.00	5.00	10.00
				γ'			
$\gamma = 1$	1	3.45	7.00	11.61	15.57	16.63	17.61
$\gamma = 3$	3	5.13	8.22	12.51	15.87	16.92	17.82
$\gamma = 5$	5	6.82	9.49	13.25	16.20	17.10	17.89
				$1/\rho'$			
$1/\rho = 1$, any γ	1	1	1	1	1	1	1
$1/\rho = 0.33$, $\gamma = 1$	0.33	0.34	0.34	0.35	0.36	0.36	0.36
$1/\rho = 0.33$, $\gamma = 3$	0.33	0.34	0.34	0.35	0.36	0.36	0.36
$1/\rho = 0.33$, $\gamma = 5$	0.33	0.34	0.34	0.35	0.35	0.35	0.36
$1/\rho = 0.20$, $\gamma = 1$	0.20	0.20	0.21	0.21	0.22	0.22	0.22
$1/\rho = 0.20$, $\gamma = 3$	0.20	0.20	0.21	0.21	0.22	0.22	0.22
$1/\rho = 0.20$, $\gamma = 5$	0.20	0.20	0.21	0.21	0.21	0.22	0.22

3.4 Equilibrium Analysis with Unitary RRAD and EIS

In this section, we study the competitive equilibrium when $\rho = \gamma = 1$. In this case, in each period the agent's optimal consumption and investment depend only on the asset returns in that period. We first study the investment problem of a typical agent in the market. Then, we establish the existence and uniqueness of the equilibrium. Finally, we study the equilibrium with one EZ-agent and one LA-agent; in particular, we show that the EZ-agent dominates the market if the LA-agent's LAD is not equal to the equilibrium gain-loss ratio of the stock.

3.4.1 Optimal Portfolio

A typical agent in the market with unitary RRAD and EIS solves the following single-period portfolio choice problem:

$$\begin{aligned} \max_{\theta} \quad & \exp\left[\mathbb{E}(\ln(a+\theta(X-a)))\right] + b\mathbb{E}\left[\nu\left(\theta(X-a)\right)\right] \\ \text{subject to} \quad & a+\theta(X-a) \geq 0, \end{aligned} \tag{3.4.1}$$

where a and X stand for the returns of a risk-free asset and a stock, respectively. $\nu(x) := x\mathbf{1}_{x\geq 0} + Kx\mathbf{1}_{x<0}$ with $b \geq 0$ and $K \geq 1$ referring to the NFD and LAD of the agent, respectively.

Proposition 14 *Assume* $\underline{x} := \text{essinf}X > 0$ *and* $\mathbb{E}[X] < \infty$*. Denote* $\overline{x} := \text{esssup}X \in (\underline{x}, +\infty]$*. Suppose* $a \in (\underline{x}, \overline{x})$*. Then,* (3.4.1) *admits unique optimal solution* $\varphi(a; b, K)$*. Furthermore,*

(i) There exist $\overline{a}_l \in (\underline{x}, \mathbb{E}[X]]$ *and* $\underline{a}_s \in [\mathbb{E}[X], \overline{x}]$ *such that* $\varphi(a; b, K) > 0$ *if and only if* $a \in (\underline{x}, \overline{a}_l)$ *and* $\varphi(a; b, K) < 0$ *if and only if* $a \in (\underline{a}_s, \overline{x})$*. Moreover, there exists* $\underline{a}_l \in [\underline{x}, \overline{a}_l)$ *such that* $\varphi(a; b, K) = \overline{\theta}(a) := a/(a - \underline{x})$ *if and only if* $a \in (\underline{x}, \underline{a}_l]$*; when* $\overline{x} < +\infty$*, there exists* $\overline{a}_s \in (\underline{a}_s, \overline{x}]$ *such that* $\varphi(a; b, K) = \underline{\theta}(a) := a/(a - \overline{x})$ *if and only if* $a \in [\overline{a}_s, \overline{x})$*.*

(ii) For fixed $b \geq 0$ *and* $K \geq 1$*,* $\varphi(a; b, K)$ *is decreasing and continuous in* $a \in (\underline{x}, \overline{x})$ *and is strictly decreasing in* a *when* $\varphi(a; b, K) \neq 0$*. Moreover,* $\lim_{a\downarrow\underline{x}} \varphi(a; b, K) = +\infty$*,* $\lim_{a\uparrow\overline{x}} \varphi(a; b, K) = -\infty$ *when* $\overline{x} < +\infty$*, and* $\lim_{a\uparrow\overline{x}} \varphi(a; b, K) = 0$ *when* $\overline{x} = +\infty$*.*

(iii) For fixed $b > 0$ and $a \in (\underline{x}, \overline{x})$, $|\varphi(a; b, K)|$ is decreasing in K and the monotonicity becomes strict when $\varphi(a; b, K) \in (0, \overline{\theta}(a))$ or when $\varphi(a; b, K) \in (\underline{\theta}(a), 0)$.

(iv) For fixed $K \geq 1$ and $a \in (\underline{x}, \overline{x})$, $\max(\varphi(a; b, K), 0)$ is increasing in b if $\mathbb{E}[\nu(X - a)] \geq 0$ and decreasing in b if $\mathbb{E}[\nu(X - a)] \leq 0$; moreover, the monotonicity becomes strict when $\mathbb{E}[\nu(X - a)] \neq 0$ and $\varphi(a; b, K) \in (0, \overline{\theta}(a))$. Similarly, for fixed $K \geq 1$ and $a \in (\underline{x}, \overline{x})$, $\max(-\varphi(a; b, K), 0)$ is increasing in b if $\mathbb{E}[\nu(a - X)] \geq 0$ and decreasing in b if $\mathbb{E}[\nu(a - X)] \leq 0$; moreover, the monotonicity becomes strict when $\mathbb{E}[\nu(a-X)] \neq 0$ and $\varphi(a; b, K) \in (\underline{\theta}(a), 0)$.

Proposition 14-(i) shows that the agent takes a long position in the stock when the risk-free return is lower than a threshold $\overline{a}_l$ and even takes the maximum leverage $\overline{\theta}(a)$ when the risk-free return is sufficiently low, i.e., when $a \leq \underline{a}_l$. Similarly, the agent takes a short position when the risk-free return is higher than a threshold $\underline{a}_s$ and even takes the maximum leverage in short positions $\underline{\theta}(a)$ when the risk-free return is sufficiently high, i.e., when $a \geq \overline{a}_s$. Finally, when the risk-free return is in the range $[\overline{a}_l, \underline{a}_s]$, the agent does not invest in the stock. Moreover, this range contains the expected return of the stock $\mathbb{E}[X]$.

Proposition 14-(i) also reveals that in equilibrium, neither EZ- nor LA-agents short-sell the stock. Indeed, Proposition 14-(i) shows that regardless of the value of b, $\varphi(a; b, K) \geq 0$ when $a \leq \mathbb{E}[X]$ and $\varphi(a; b, K) \leq 0$ when $a \geq \mathbb{E}[X]$. In consequence, to clear the stock market in equilibrium, the equity premium, i.e., the stock's expected return less the risk-free rate, must be strictly positive, so short-selling cannot occur. A consequence of this observation is that the equilibrium asset prices remain the same even if short-selling is prohibited. The intuition behind this result is as follows: the LA-agents are risk averse because the LAD is larger than or equal to one, so they will hold the stock only when the equity premium is positive and short-sell the stock only when the risk premium is negative.

Proposition 14-(ii) shows that the optimal allocation to the stock, $\varphi(a; b, K)$, is continuous and decreasing in the risk-free rate a. Proposition 14-(iii) shows that the more loss averse the agent is (i.e., the larger K is), the smaller the long or short position she takes in the stock (i.e., the smaller $|\varphi(a; b, K)|$). Proposition 14-(iv) reveals that when the agent's NFD is higher, she takes a larger (smaller) position in the stock, whether the position is long or short, if she derives positive (negative) utility of investment gains and losses.

3.4.2 Existence and Uniqueness of Equilibrium

Theorem 4 *Suppose $\rho = \gamma = 1$. Then, the competitive equilibrium uniquely exists with stock return $R_{t+1} = Z_{t+1}/\beta$ and risk-free return $R_{f,t+1}$ uniquely determined by*

$$\sum_{i=1}^{m} \varphi(R_{f,t+1}; b_i, K_i) Y_{i,t} = 1, \tag{3.4.2}$$

where $Y_{i,t} \in [0, 1]$ is agent i's post-consumption wealth share at time t and φ is the optimal solution to (3.4.1) with $X = R_{t+1}$. Furthermore, essinf $R_{t+1} < R_{f,t+1} < \mathbb{E}[R_{t+1}]$.

Because $\rho = \gamma = 1$, every agent in the market consumes the same constant fraction of her wealth. As a result, the equilibrium price-dividend ratio must be a constant. In consequence, because Z_{t+1}'s are i.i.d., so are R_{t+1}'s. The risk-free rate is determined by (3.4.2), which is a clearing condition for the stock market.

Consider an economy with heterogeneous agents, referred to as the heterogeneous-agent economy, and another economy with a representative agent whose preferences are the average, weighted by wealth shares, of those of the agents in the heterogeneous-agent economy. Are asset prices in these two economies quantitatively similar? The equilibrium equation (3.4.2) sheds some light on the answer to this question. With heterogeneous EZ- and LA-agents, Theorem 4 shows that in equilibrium the stock return $R_{t+1} = Z_{t+1}/\beta$ and the risk-free return $R_{f,t+1}$ is determined by $\sum_{i=1}^{m} \varphi(R_{f,t+1}; b_i, K_i)Y_{i,t} = 1$. Now, suppose the current time is t and consider a representative agent whose preferences are represented by (3.2.1) with NFD $\sum_{i=1}^{m} Y_{i,t}b_i$ and LAD $\sum_{i=1}^{m} Y_{i,t}K_i$. Then, the risk-free return $\bar{R}_f$ in an economy with this representative agent only is determined by $\varphi(\bar{R}_f; \sum_{i=1}^{m} Y_{i,t}b_i, \sum_{i=1}^{m} Y_{i,t}K_i) = 1$. We can see that $R_{f,t+1}$ and $\bar{R}_f$ are the same if $\varphi(a; b, K)$ is linear in (b, K). Thus, whether the asset prices in these two economies are quantitatively similar depends on the degree of the linearity of $\varphi(a; b, K)$ in (b, K).

We plot $\varphi(a; b, K)$ with respect to (b, K) in Figure 3.1. The risk-free return a and stock return are set to be 1.0392 and Z_{t+1}/β, respectively, where $\beta = 0.98$ and the distribution of Z_{t+1} is given as in Table 3.1. We can observe that in the region of (b, K) such that $\varphi(a; b, K) \in (0, 10]$ (i.e., roughly in $b \in [0, 12]$ and $K \in [1.5, 2]$), $\varphi(a; b, K)$ is approximately linear in (b, K). Suppose that in the heterogeneous-agent economy, the wealth share of each agent is at least 10%. Then, the optimal percentage allocation of each agent to the stock is at most 10. Thus, if all agents in this economy invests some of their wealth in the stock, the approximate linearity of $\varphi(a; b, K)$ in (b, K) when $\varphi(a; b, K) \in (0, 10)$ (i.e., roughly in $b \in [0, 12]$ and $K \in [1.5, 2]$)

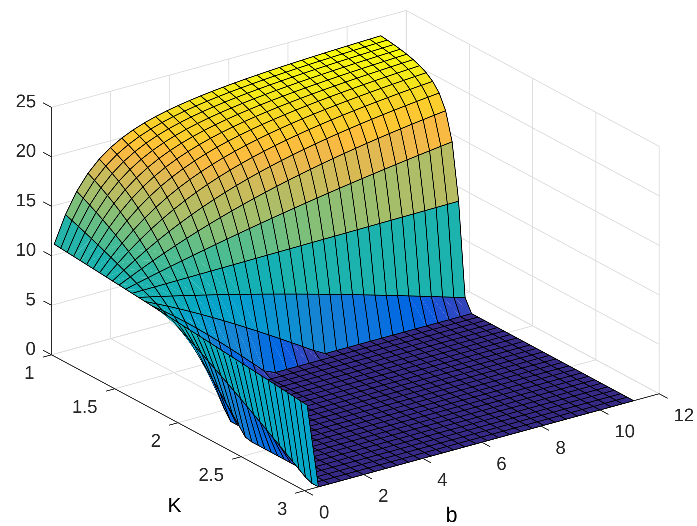

Figure 3.1: Optimal percentage allocation to the stock $\varphi(a; b, K)$ with respect to (b, K). The risk-free return a and stock return are set to be 1.0392 and Z_{t+1}/β, respectively where $\beta = 0.98$ and the distribution of Z_{t+1} is given as in Table 3.1.

implies that the $R_{f,t+1} \approx \bar{R}_f$. We also observe that $\varphi(a; b, K)$ is strongly nonlinear in $\varphi(a; b, K)$ in the region of (b, K) such that $\varphi(a; b, K) \in [0, 10]$ (i.e., roughly in $b \in [0, 12]$ and $K \in [1.5, 3]$, including the flat region). Thus, if a large fraction of the agents in the economy do not invest in the stock at all, $R_{f,t+1}$ and $\bar{R}_f$ can be significantly different. Thus, the asset returns in the heterogeneous-agent and representative-agent economies differ significantly if and only if a large fraction of agents in the first economy do not invest in the stock. This conclusion is consistent with the findings in Chapman and Polkovnichenko (2009).[3]

We can also compute agent i's total utility per unit wealth as a function of the wealth shares of all the agents in the market. In consequence, we can compute $b_{i,t}$ as defined by (3.2.2), which

[3]We also did numerical studies and reached similar conclusions as those in Chapman and Polkovnichenko (2009), so we chose not to report them.

is again a function of the wealth shares of the agents in the market.[4] For instance, suppose the economy consists of one EZ- and one LA-agent only. Assuming the distribution of Z_{t+1} as in Table 3.1, $\beta = 0.98$, $b = 5$, and $K = 2.25$, we plot $b_{i,t}$ as a function of the wealth share of the EZ-agent in Figure 3.2. We observe that $b_{i,t}$ is not very sensitive to Y_t.

Let us emphasize that even if $b_{i,t}$ in our model is nearly a constant, our model is better behaved than that proposed by Barberis and Huang (2009). First, as shown in Chapter 1, the total utility process in the former model always uniquely exists but is either nonunique or nonexistent in the latter when EIS is less than one. Second, as discussed in Section 3.2.3, the former is more tractable than the latter in terms of deriving portfolio selection and asset pricing results, even if we set aside the issue of nonexistence or nonuniqueness of the total utility process in the latter model.

3.4.3 Equilibrium Analysis with One EZ- and One LA-Agent

In this subsection, we consider the case in which there are only two agents in the market: one is an EZ-agent and the other is an LA-agent. We index the EZ-agent by $i = 0$ and set $b_0 = 0$, and we index the LA-agent by $i = 1$ and set $b_1 = b > 0$ and $K_1 = K \geq 1$. In addition, we denote $\nu(x) = x1_{x\geq 0} + Kx\mathbf{1}_{x<0}$.

Recall the equilibrium gain-loss ratio K^* as defined in (3.3.1). We denote $R_{f,\text{EZ}}$ and $R_{f,\text{LA}}$ as the risk-free returns when the market is populated with the EZ-agent only and with the

[4] Indeed, because $\gamma = 1$, we conclude from (3.2.2) that $b_{i,t} = b_i \exp\{\mathbb{E}[\ln(U_{i,t+1}/W_{i,t+1})|\mathscr{F}_t]\}$. The proof of Theorem 4 shows that $\ln(U_{i,t+1}/W_{i,t+1}) = \psi_i(\mathbf{Y}_{t+1})$, where $\mathbf{Y}_{t+1}$ refers to the vector of wealth shares of the agents in the market and ψ_i is solved from (0.65). In consequence, $b_{i,t} = b_i \exp\{\mathbb{E}[\psi_i(\mathbf{Y}_{t+1})|\mathscr{F}_t]\} = b_i \exp\{\mathbb{E}[\psi_i(h(\mathbf{Y}_t, Z_{t+1}))|\mathscr{F}_t]\}$, where h is defined by (0.64).

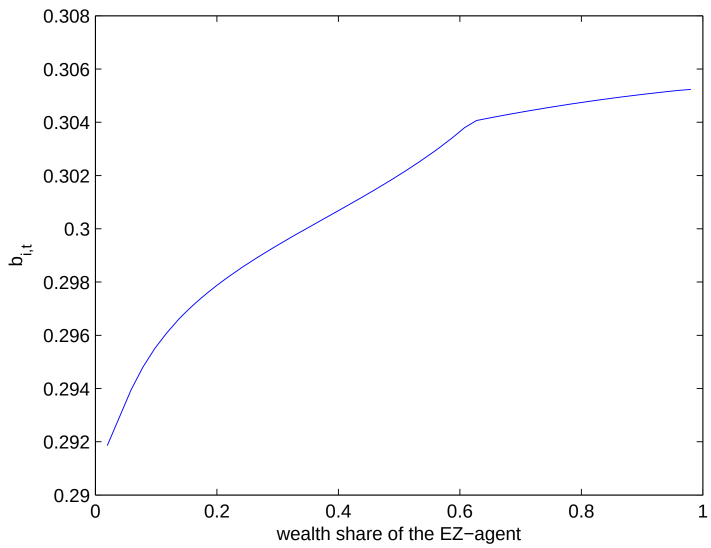

Figure 3.2: $b_{i,t}$ for an LA-agent with respect to the wealth share of an EZ-agent in an economy with these two agents. $\rho = \gamma = 1$, $\beta = 0.98$, the LA-agent's NFD $b = 5$ and LAD $K = 2.25$, and the distribution of the dividend growth rate Z_{t+1} is given as in Table 3.1.

LA-agent only, respectively. In view of Theorem 3 and Proposition 13 and noting that we are setting $\rho = \gamma = 1$, we conclude that $R_{f,\text{EZ}} = 1/\mathbb{E}[R_{t+1}^{-1}]$ and $R_{f,\text{LA}}$ uniquely solves $e^{\mathbb{E}(\ln R_{t+1})}\big(1 - R_{f,\text{LA}}\mathbb{E}(1/R_{t+1})\big) + b\mathbb{E}[\nu(R_{t+1} - R_{f,\text{LA}})] = 0$.

3.4.3.1 Optimal investment

Proposition 15 *Suppose $\rho = \gamma = 1$. Let $\theta_{0,t}^*$ and $\theta_{1,t}^*$ be the optimal percentage allocations to the stock of the EZ- and LA-agents, respectively.*

(i) If $K = K^*$, then $\mathbb{E}\left[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t\right] = 0$ and $\theta_{0,t}^* = \theta_{1,t}^* = 1$.

(ii) If $K < K^*$, then $\mathbb{E}\left[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t\right] > 0$ and $0 < \theta_{0,t}^* < 1 < \theta_{1,t}^*$.

(iii) If $K > K^*$, then $\mathbb{E}\left[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t\right] < 0$ and $\theta_{0,t}^* > 1 > \theta_{1,t}^* \geq 0$.

Proposition 15-(i) shows that the LA-agent invests the same amount in the stock as the EZ-agent when her LAD is K^*. This result is consistent with Theorem 3. Proposition 15-(ii) shows that the agent derives strictly positive investment utility if her LAD is strictly lower than K^*. In this case, the LA-agent is willing to hold more stocks than the EZ-agent. If her LAD is strictly higher than K^*, however, the agent derives strictly negative investment utility and thus holds less stocks than the EZ-agent. We also observe that that the EZ-agent always holds some stocks in equilibrium, but the LA-agent may choose not to hold any stocks.

3.4.3.2 Equity premium

The conditional *equity premium* is the expected return of the stock in excess of the risk-free return, i.e., $\mathrm{EP}_t := \mathbb{E}[R_{t+1}|\mathcal{F}_t] - R_{f,t+1}, t \geq 0$. Because both the EZ- and LA-agents consume the same constant fraction of their wealth, the dividend-price ratio of the stock in the equilibrium must be a constant as well. Thus, the stock returns are proportional to the dividends and are i.i.d. over time, independent of the market shares of the agents. In consequence, as we vary the wealth shares of the LA- and EZ-agents, the risk-free rate must vary accordingly to drive the equity premium so that the market remains in equilibrium. The following proposition reveals the dependence of the risk-free rate and thus the equity premium on the wealth share of the EZ-agent.

Proposition 16 *Suppose $\rho = \gamma = 1$. Denote Y_t as the wealth share of the LA-agent. Then, the following are true:*

(i) If $K = K^*$, then $R_{f,t+1} \equiv R_{f,\mathrm{EZ}}$ and $\mathrm{EP}_t = \mathbb{E}[R_{t+1}] - R_{f,\mathrm{EZ}}$.

(ii) If $K < K^*$, then $R_{f,t+1}$ is continuous and strictly decreasing with respect to $Y_t \in [0, 1]$. Consequently, EP_t *is continuous and strictly increasing with respect to $Y_t \in [0, 1]$.*

(iii) If $K > K^*$, then $R_{f,t+1}$ is continuous and strictly increasing with respect to $Y_t \in [0, 1]$. Consequently, EP_t *is continuous and strictly decreasing with respect to $Y_t \in [0, 1]$.*

Proposition 16 shows that when the LA-agent's LAD K happens to be K^*, she behaves the same as the EZ-agent, so the equity premium is the same as in an economy with EZ-agents only. When $K > K^*$, the LA-agent invests less in the stock than the EZ-agent, so the larger the wealth share the LA-agent has, the larger the equity premium is. When $K < K^*$, the LA-agent invests more in the stock than the EZ-agent, so the larger the LA-agent's wealth share is, the smaller the equity premium is.

3.4.3.3 Market dominance

We study whether the EZ-agent will dominate the market in the long run. Suppose that Y_t is the wealth share of the EZ-agent in the market at time t. The EZ-agent *becomes extinct* if $\lim_{t\to\infty} Y_t = 0$ almost surely, *survives* if extinction does not occur, and *dominates* the market if $\lim_{t\to\infty} Y_t = 1$ almost surely. The extinction, survival, and dominance of the LA-agent are defined similarly.

There is a vast literature on market dominance; see for instance the surveys by Blume and Easley (2009, 2010). In this literature, an alternative definition of market dominance is based on consumption shares rather than wealth shares. This alternative definition is the same as the one used here because the consumption rates of the EZ- and LA-agents are the same constant.

Proposition 17 *Suppose $\rho = \gamma = 1$. Suppose $K \neq K^*$. Then, the EZ-agent dominates the market.*

According to Theorem 3, when $K = K^*$, the LA-agent behaves the same as the EZ-agent, so neither of the two agents dominates the market. Otherwise, Proposition 17 shows that the EZ-agent drives the LA-agent out of the market.

The existing market dominance literature assumes that all market participants are expected utility maximizers or, more generally, EZ-agents with possibly heterogeneous RRAD, EIS, and beliefs; see for instance Blume and Easley (2009, 2010), Borovička (2013), and the references therein. The only work addressing both EZ- and LA-agents is Easley and Yang (2015), in which the authors illustrate numerically that the EZ-agents dominate the market when they have the same RRAD and EIS as the LA-agents. To our best knowledge, we are the first to prove the dominance when the RRAD and EIS of the EZ- and LA-agents are one. Note that in this case, the EZ-agent is an expected-utility maximizer with logarithmic utility who maximizes the growth rate of wealth. A well known insight into market dominance in the literature, such as in De Long et al. (1991) and Blume and Easley (1992), is that investors who maximize the growth rate of wealth dominate the market, but rigorous proofs need some conditions that do not hold in our setting. For instance, Theorem 5.2 in Blume and Easley (1992) assumes that the market is complete and the percentage allocation to each risky asset is bounded from below by a positive number. However, in our model, the LA-agent may not invest in the stock. Proposition 1 in Sandroni (2000) assumes some conditions that do not hold in our setting either. Thus, our result is new to the literature.

Because the EZ-agent dominates the market in the long run, the equilibrium asset prices in the long run are determined by the EZ-agent only, i.e., the LA-agent has no price impact in the long run. In particular, the long-run risk-free rate is $R_{f,\mathrm{EZ}}$. However, although the LA-agent theoretically becomes extinct in the market and has zero price impact after an infinite number of years, it is unclear whether this agent is negligible after a sufficiently long but finite time period, e.g., after 50 years. In the following, we assume the length of each period to be one year and the dividend growth rate distribution to be in Table 3.1, set $\rho = \gamma = 1$, $\beta = 0.98$, $b = 1$, and $K = 2.25 > K^*$, and use simulation to compute the wealth share $1 - Y_t$ and the price impact of the LA-agent for $t = 5$, 20, 50, and 200. Here, we define the price impact of the LA-agent to be $(\mathrm{EP}_t - \mathrm{EP}_{\mathrm{EZ}})/\mathrm{EP}_{\mathrm{EZ}}$, where $\mathrm{EP}_{\mathrm{EZ}}$ is the equity premium when the market is populated with the EZ-agent only. We set the wealth share of the LA-agent at the beginning to be 0.5 and 0.9, respectively, and simulate a thousand paths, along each of which the wealth share and price impact of the LA-agent are computed at each time (5, 20, 50, and 200). The mean and standard error (in brackets) of these two quantities are reported in Table 3.1. We observe that the wealth share of the LA-agent decreases slowly in time, showing that the market dominance force takes effect slowly. The price impact of the LA-agent, however, decreases quickly when the agent's initial wealth share is large. Our observation is consistent with those in Easley and Yang (2015).

3.5 Probability Weighting

To simplify the presentation of our model and to follow Barberis and Huang (2009) and Easley and Yang (2015), we did not consider probability weighting in the preferences of the agents in the market. Nonetheless, following De Giorgi and Legg (2012), we can incorporate probability

Table 3.1: Wealth share $1 - Y_t$ and price impact $(\mathrm{EP}_t - \mathrm{EP}_{\mathrm{EZ}})/\mathrm{EP}_{\mathrm{EZ}}$ of the LA-agent. Parameters are set to be $\beta = .98$, $b = 1$, $\rho = \gamma = 1$, and $K = 2.25 > K^*$. The dividend growth rate distribution is given as in Table 3.1. The initial wealth share of the LA-agent is 0.5 in the upper panel and 0.9 in the lower panel. A thousand scenarios of the stock return series are simulated. In each scenario, the EZ- and LA-agents follow their optimal strategies and their wealth shares and price impact in 5, 20, 50, and 200 years are computed. The mean and standard error (in brackets) of the wealth share and price impact of the LA-agent are reported.

		$1 - Y_0 = 0.5$		
Years	5	20	50	200
Wealth Share	0.499 [1.50E-03]	0.497 [1.80E-03]	0.492 [2.70E-03]	0.469 [6.00E-03]
Price Impact	1.141 [1.66E-04]	1.178 [3.70E-04]	1.172 [5.80E-04]	1.167 [1.10E-03]
		$1 - Y_0 = 0.9$		
Years	5	20	50	200
Wealth Share	0.893 [8.49E-04]	0.865 [1.80E-03]	0.830 [2.6E-03]	0.709 [4.00E-03]
Price Impact	9.258 [5.01E-02]	7.546 [8.35E-02]	6.274 [9.75E-02]	3.658 [8.95E-02]

weighting in our model.

To consider probability weighting, we replace $G_{i,t}$ in (3.2.1) with

$$\begin{aligned}\tilde{G}_{i,t} = &\int_0^{+\infty} \nu_i(x) d\big[-T_{i,+}\big(\mathbb{P}(W_{i,t+1} - (1-c_{i,t})W_{i,t}R_{f,t+1} > x|\mathscr{F}_t)\big)\big] \\ &+ \int_{-\infty}^{0} \nu_i(x) d\big[T_{i,-}\big(\mathbb{P}(W_{i,t+1} - (1-c_{i,t})W_{i,t}R_{f,t+1} \le x|\mathscr{F}_t)\big)\big], \end{aligned} \quad (3.5.1)$$

where $T_{i,\pm}$ stand for the probability weighting functions with respect to gains and losses, respectively, and satisfy

Assumption 6 *$T_{i,\pm}$ are continuous on $[0,1]$ and differentiable on $(0,1)$ with $T'_{i,\pm}(z) > 0, z \in (0,1)$, $T_{i,\pm}(0) = 0$, and $T_{i,\pm}(1) = 1$.*

The following further assumptions on the probability weighting functions will be needed in some of the following analysis:

Assumption 7 $K_i \geq \sup_{z\in(0,1)}[T'_{i,+}(z)/T'_{i,-}(z)]$.

Assumption 8 $K_i \geq \sup_{z\in(0,1)}[T'_{i,+}(1-z)/T'_{i,-}(z)]$.

Assumption 7 ensures that the optimization problem corresponding to agent i's portfolio choice is concave. This assumption is satisfied if we set the probability weighting functions with respect to gains and losses to be the same, i.e., $T_{i,+} = T_{i,-}$. In some estimates in the literature, such as those in Tversky and Kahneman (1992), show that $T_{i,+}$ is indeed approximately the same as $T_{i,-}$; moreover, in some applications of PT in finance, such as Barberis (2012), $T_{i,+}$ and $T_{i,-}$ are assumed to be the same.

Assumption 8 is needed to show that agent i's allocation to the stock is decreasing with respect to the risk-free return. Proposition 5 in He and Zhou (2011) shows that this assumption holds for $K = 2.25$ and $T_{i,\pm}(z) = z^{\delta}/\big(z^{\delta} + (1-z)^{\delta}\big)^{1/\delta}$ with $\delta = 0.61$ or 0.69, which are the parametric forms used by Tversky and Kahneman (1992) and the estimates obtained therein.

In the following, for any random variable X, denote

$$\mathcal{E}_{i,+}(X) := \int_0^{\infty} xd\left[-T_{i,+}\big(\mathbb{P}(X > x)\big)\right], \mathcal{E}_{i,-}(X) := -\int_{-\infty}^{0} xd\left[T_{i,-}\big(\mathbb{P}(X \leq x)\big)\right]. \quad (3.5.2)$$

Again, when the agents under consideration are homogeneous, we drop subscript i.

3.5.1 Equilibrium Gain-Loss Ratio in the Presence of Probability Weighting

The following two corollaries show that the results in Theorem 3 and Proposition 13 still hold in the presence of probability weighting.

Corollary 1 *Assume* $\alpha_{\rm EZ} := \beta(M(Z_{t+1}))^{1-\rho} < 1$*. Then,* $R_{t+1} = Z_{t+1}/\alpha_{\rm EZ}$ *and* $R_{f,t+1} = \mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)$ *are the equilibrium return rates of the stock and the risk-free asset, respectively, if the market is populated with EZ-agents only. Moreover, the equilibrium remains the same if the market is populated with not only EZ-agents but also LA-agents whose LAD and probability weighting functions satisfy (i) Assumptions 6-7, (ii)* $\mathcal{E}_{i,+}(Z_{t+1}) < +\infty$*, and (iii)*

$$K_i = \frac{\mathcal{E}_{i,+}\big(R_{t+1} - \mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\big)}{\mathcal{E}_{i,-}\big(R_{t+1} - \mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\big)}. \tag{3.5.3}$$

In this case, the EZ- and LA-agents take the same consumption-investment strategy. Moreover, when the LA-agents have homogeneous probability weighting functions, (iii) holds if and only if $K_i = \tilde{K}^*$*, where*

$$\tilde{K}^* := \frac{\mathcal{E}_{+}\big(R_{t+1} - \mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\big)}{\mathcal{E}_{-}\big(R_{t+1} - \mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\big)} = \frac{\mathcal{E}_{+}\big(Z_{t+1} - \mathbb{E}\left(Z_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma}\right)\big)}{\mathcal{E}_{-}\big(Z_{t+1} - \mathbb{E}\left(Z_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma}\right)\big)}. \tag{3.5.4}$$

Corollary 2 *Suppose the market is populated with homogeneous LA-agents such that Assumptions 6-7 hold and* $\mathcal{E}_+(Z_{t+1}) < +\infty$*. Assume that* $\tilde{\alpha}_{\rm LA} := \beta\big(M(Z_{t+1})^{\gamma}\mathbb{E}[Z_{t+1}^{-\gamma}]\tilde{r}_{f,\rm LA}\big)^{1-\rho} < 1$*, where*

$\tilde{r}_{f,\mathrm{LA}}$ is the unique solution to

$$M(Z_{t+1}) - \tilde{r}_{f,\mathrm{LA}} M(Z_{t+1})^{\gamma} \mathbb{E}[Z_{t+1}^{-\gamma}] + b\big[\mathcal{E}_+(Z_{t+1} - \tilde{r}_{f,\mathrm{LA}}) - K\mathcal{E}_-(Z_{t+1} - \tilde{r}_{f,\mathrm{LA}})\big] = 0. \quad (3.5.5)$$

Then, the market equilibrium exists with $R_{f,t+1} = \tilde{r}_{f,\mathrm{LA}}/\tilde{\alpha}_{\mathrm{LA}}$ and $R_{t+1} = Z_{t+1}/\tilde{\alpha}_{\mathrm{LA}}$. Moreover, $\tilde{r}_{f,\mathrm{LA}} = \mathbb{E}[Z_{t+1}^{1-\gamma}]/\mathbb{E}[Z_{t+1}^{-\gamma}]$ when $K = \tilde{K}^*$, where $\tilde{K}^*$ is defined as in (3.5.4), $\tilde{r}_{f,\mathrm{LA}}$ is decreasing in K, and the monotonicity becomes strict when $b > 0$. Finally, $\tilde{r}_{f,\mathrm{LA}}$ is strictly decreasing (increasing) in b when $K > \tilde{K}^*$ ($K < \tilde{K}^*$).

3.5.2 Agents with Unitary RRAD and EIS

Next, we study the market equilibrium when every agent has unitary RRAD and EIS, i.e., when $\gamma = \rho = 1$. Similar to the case of no probability weighting, a typical agent in the market with unitary RRAD and EIS solves the following single-period portfolio choice problem:

$$\begin{aligned} \max_{\theta} \quad & \exp\big[\mathbb{E}(\ln(a + \theta(X - a)))\big] + b\left[\mathcal{E}_+(\theta(X - a)) - K\mathcal{E}_-(\theta(X - a))\right] \\ \text{subject to} \quad & a + \theta(X - a) \geq 0, \end{aligned} \quad (3.5.6)$$

where a and X stand for the returns of a risk-free asset and a stock, respectively.

3.5.2.1 Portfolio Choice

Corollary 3 *Assume $\underline{x} := \mathrm{essinf} X > 0$, $\mathbb{E}[X] < \infty$, $\mathcal{E}_+(X) < +\infty$, and Assumptions 6–8 hold. Denote $\overline{x} := \mathrm{esssup} X \in (\underline{x}, +\infty]$. Suppose $a \in (\underline{x}, \overline{x})$. Then (3.5.6) admits unique optimal solution $\tilde{\varphi}(a; b, K)$. Furthermore,* *(i) There exists* $\overline{a}_l \in (\underline{x}, \overline{x})$ *and* $\underline{a}_s \in [\overline{a}_l, \overline{x}]$ *such that* $\tilde{\varphi}(a; b, K) > 0$ *if and only if* $a \in (\underline{x}, \overline{a}_l)$ *and* $\tilde{\varphi}(a; b, K) < 0$ *if and only if* $a \in (\underline{a}_s, \overline{x})$*. Moreover, there exists* $\underline{a}_l \in [\underline{x}, \overline{a}_l)$ *such that* $\tilde{\varphi}(a; b, K) = \overline{\theta}(a) := a/(a - \underline{x})$ *if and only if* $a \in (\underline{x}, \underline{a}_l]$*; when* $\overline{x} < +\infty$*, there exists* $\overline{a}_s \in (\underline{a}_s, \overline{x}]$ *such that* $\tilde{\varphi}(a; b, K) = \underline{\theta}(a) := a/(a - \overline{x})$ *if and only if* $a \in [\overline{a}_s, \overline{x})$*.*

(ii) For fixed $b \geq 0$ *and* $K \geq 1$*,* $\tilde{\varphi}(a; b, K)$ *is decreasing and continuous in* $a \in (\underline{x}, \overline{x})$ *and is strictly decreasing in* a *when* $\tilde{\varphi}(a; b, K) \neq 0$*. Moreover,* $\lim_{a \downarrow \underline{x}} \tilde{\varphi}(a; b, K) = +\infty$*,* $\lim_{a \uparrow \overline{x}} \tilde{\varphi}(a; b, K) = -\infty$ *if* $\overline{x} < +\infty$*, and* $\lim_{a \uparrow \overline{x}} \tilde{\varphi}(a; b, K) = 0$ *if* $\overline{x} = +\infty$*.*

(iii) For fixed $b > 0$ *and* $a \in (\underline{x}, \overline{x})$*,* $|\tilde{\varphi}(a; b, K)|$ *is decreasing in* K *and the monotonicity becomes strict when* $\tilde{\varphi}(a; b, K) \in (0, \overline{\theta}(a))$ *or when* $\tilde{\varphi}(a; b, K) \in (\underline{\theta}(a), 0)$*.*

(iv) For fixed $K \geq 1$ *and* $a \in (\underline{x}, \overline{x})$*,* $\max(\tilde{\varphi}(a; b, K), 0)$ *is increasing (decreasing) in* b *if* $\mathcal{E}_+(X - a) - K\mathcal{E}_-(X - a) \geq 0$ *(*≤ 0*); moreover, the monotonicity becomes strict when* $\mathcal{E}_+(X - a) - K\mathcal{E}_-(X - a) \neq 0$ *and* $\tilde{\varphi}(a; b, K) \in (0, \overline{\theta}(a))$*. Similarly, for fixed* $K \geq 1$ *and* $a \in (\underline{x}, \overline{x})$*,* $\max(-\tilde{\varphi}(a; b, K), 0)$ *is increasing (decreasing) in* b *if* $\mathcal{E}_+(a - X) - K\mathcal{E}_-(a - X) \geq 0$ *(*≤ 0*); moreover, the monotonicity becomes strict when* $\mathcal{E}_+(a - X) - K\mathcal{E}_-(a - X) \neq 0$ *and* $\tilde{\varphi}(a; b, K) \in (\underline{\theta}(a), 0)$*.*

(v) When short-selling is not allowed, $\max(\tilde{\varphi}(a; b, K), 0)$ *is the unique optimal solution to* (3.4.1)*.*

Corollary 3 shows that in the presence of probability weighting, the optimal allocation to the stock has the same properties as that in the absence of probability weighting except that with probability weighting, the agent may buy the stock when the equity premium is negative or short-sell the stock when the equity premium is positive. The intuition is as follows: with probability weighting, the agent can be risk seeking if she significantly overweighs large returns of the stock that occur with a small probability; in consequence, she may be willing to buy the stock even if the equity premium is negative.

3.5.2.2 Existence and Uniqueness of Equilibrium

Corollary 4 *Suppose $\rho = \gamma = 1$, Assumptions 6–8 hold, and $\mathcal{E}_{+,i}(Z_{t+1}) < +\infty$. Then, the competitive equilibrium uniquely exists with stock return $R_{t+1} = Z_{t+1}/\beta$ and risk-free return $R_{f,t+1}$ uniquely determined by $\sum_{i=1}^{m} \tilde{\varphi}(R_{f,t+1}; b_i, K_i)Y_{i,t} = 1$, where $Y_{i,t} \in [0, 1]$ is agent i's post-consumption wealth share at time t and $\tilde{\varphi}$ is the optimal solution to (3.5.6) with $X = R_{t+1}$. Moreover,* $\operatorname{essinf} R_{t+1} < R_{f,t+1} < \operatorname{esssup} R_{t+1}$.

Furthermore, if short-selling is not allowed, all the above results still hold with the equation determining $R_{f,t+1}$ replaced by $\sum_{i=1}^{m} \max(\tilde{\varphi}(R_{f,t+1}; b_i, K_i), 0)Y_{i,t} = 1$.

Corollary 4 is parallel to Theorem 4. However, we note that in the presence of probability weighting, the equilibrium risk-free rate is not necessarily strictly lower than the expected stock return, i.e., the equity premium is not necessarily strictly positive. Moreover, some agents in the market may short-sell the stock. This is because LA-agents can be risk seeking due to probability weighting. Consequently, the equity premium in equilibrium can be negative. This observation is consistent with the asset pricing model for lottery-type stocks in Barberis and Huang (2008b). We also observe that the prohibition of short-selling can change the market equilibrium because otherwise some agents may short-sell the stock.

3.5.2.3 Equilibrium with One EZ- and One LA-agent

In this subsection, we consider the case in which there are only two agents in the market: one is an EZ-agent and the other is an LA-agent. We index the EZ-agent by $i = 0$ and set $b_0 = 0$, and we index the LA-agent by $i = 1$ and denote her NFD, LAD, and probability weighting functions to be $b > 0$, $K \geq 1$, and $T_{\pm}$, respectively. Assume both of them have unitary RRAD and EIS, i.e., $\rho = \gamma = 1$.

Recall the equilibrium gain-loss ratio $\tilde{K}^*$ as defined in (3.5.4). We denote $R_{f,\text{EZ}}$ and $\tilde{R}_{f,\text{LA}}$ as the risk-free returns when the market is populated with the EZ-agent only and with the LA-agent only, respectively. Suppose Assumptions 6–8 hold and $\mathcal{E}_+(Z_{t+1}) < +\infty$. The following corollaries are parallel to Propositions 15–17, respectively.

Corollary 5 *Suppose $\rho = \gamma = 1$. Let $\theta^*_{0,t}$ and $\theta^*_{1,t}$ be the optimal percentage allocation to the stock of the EZ- and LA-agents, respectively.*

(i) If $K = \tilde{K}^*$, then $\theta^*_{0,t} = \theta^*_{1,t} = 1$.

(ii) If $K < \tilde{K}^*$, then $\theta^*_{0,t} < 1 < \theta^*_{1,t}$.

(iii) If $K > \tilde{K}^*$, then $\theta^*_{0,t} > 1 > \theta^*_{1,t}$.

Moreover, the above results still hold if short-selling is not allowed.

Corollary 6 *Suppose $\rho = \gamma = 1$. Denote Y_t as the wealth share of the EZ-agent in the market. Then, the following are true:*

(i) If $K = \tilde{K}^*$, then $R_{f,t+1} \equiv R_{f,\text{EZ}}$ and $\text{EP}_t = \mathbb{E}[R_{t+1}] - R_{f,\text{EZ}}$.

(ii) If $K < \tilde{K}^*$, then $R_{f,t+1}$ is continuous and strictly decreasing with respect to $Y_t \in [0, 1]$. Consequently, EP_t *is continuous and strictly increasing with respect to* $Y_t \in [0, 1]$.

(iii) If $K > \tilde{K}^*$, then $R_{f,t+1}$ is continuous and strictly increasing with respect to $Y_t \in [0, 1]$. Consequently, EP_t *is continuous and strictly decreasing with respect to* $Y_t \in [0, 1]$.

Corollary 7 *Suppose $\rho = \gamma = 1$. Suppose $K \neq \tilde{K}^*$. Then, the EZ-agent dominates the market whether short-selling is allowed or not.*

3.6 Conclusions

In this chapter, we considered a multi-period equilibrium asset pricing model with EZ- and LA-agents. The EZ-agents' preferences for consumption are represented by a recursive utility. The LA-agents derive the utility from investment gains and losses in addition to the consumption utility and their preferences are represented by the GH model.

We defined an equilibrium gain-loss ratio and showed that the LA-agents behave the same as the EZ-agents in equilibrium if their LAD is equal to this ratio. We further proposed a measure to quantify the risk aversion of the LA-agents.

With unitary RRAD and EIS, we proved the existence and uniqueness of the equilibrium. When the market is populated with one EZ- and one LA-agent, we found that the equity premium is increasing (decreasing) with respect to the wealth share of the LA-agent in the market when the LAD of this agent is higher (lower) than the equilibrium gain-loss ratio. We also proved that the EZ-agent dominates the market in the long run when the LA-agent's LAD is not equal to the equilibrium gain-loss ratio.

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Appendix

Proofs for Chapter 1

Proof of Proposition 1 We first consider (i), i.e., the case in which $\gamma \neq 1$. Because of Assumption 9-(i), we have

$$\mathbb{U}h(x) := \mathbb{E}_t \left[h(X_{t+1}) \mathbb{E}_t \left[u \left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} \right) | X_t, X_{t+1} \right] | X_t = x \right] = \sum_{y \in \mathbb{X}} \tilde{\mathbf{P}}_{x,y} h(y).$$

Because $\mathbf{P}$ is irreducible and e^{κ} is positive, we conclude that $\tilde{\mathbf{P}}$ is also irreducible. Thus, we have (1.4.1), where η and v are the Perron-Frobenius eigenvalue and eigenvector of $\tilde{\mathbf{P}}$, respectively, and $\eta > 0$, $v \in \mathcal{X}_{++}$; see for instance Meyer (2000, p. 673).

Next, we consider (ii), i.e., the case in which $\gamma = 1$. It is straightforward to see that (1.4.2) is equivalent to

$$\mathbf{P}v = v + \eta \mathbf{1} - w, \tag{0.1}$$

where w denotes the vector of $\mathbb{E}_t[\kappa(X_t, X_{t+1}, Y_{t+1}) | X_t = x]$, $x \in \mathbb{X}$ and $\mathbf{1}$ denotes the vector of all ones. Because $\mathbf{P}$ is an irreducible stochastic matrix, the kernel of $\mathbf{I} - \mathbf{P}^\top$, where $\mathbf{I}$ is the identity mapping, is the linear space spanned by the left-Perron-Frobenius eigenvector of $\mathbf{P}$, i.e., by the stationary distribution

π of $\{X_t\}$. As a result, the range of $\mathbf{I}-\mathbf{P}$ is the space of all vectors that are orthogonal to π. By the definition of η, $\eta\mathbf{1}-w$ is orthogonal to π and thus is in the range of $\mathbf{I}-\mathbf{P}$. As a result, there exists v such that (1.4.2) holds. Moreover, by multiplying the stationary distribution π on both sides of (0.1), we can see that η is uniquely determined.

Finally, we prove (iii). We first consider the case in which $\gamma \neq 1$. Because η is the Perron-Frobenius eigenvalue of $\tilde{\mathbf{P}}$, according to max-min version of the Collatz-Wielandt formula (Meyer, 2000, p. 673), we have

$$\begin{aligned}
\eta &= \max_{g\in\mathcal{X}_+^o} \min_{x\in\mathbb{X}, g(x)\neq 0} \frac{\sum_{y\in\mathbb{X}} \tilde{\mathbf{P}}_{x,y} g(y)}{g(x)} \\
&= \max_{g\in\mathcal{X}_+^o} \min_{x\in\mathbb{X}, g(x)\neq 0} \frac{\sum_{y\in\mathbb{X}} \mathbf{P}_{x,y} \mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}\right) g(X_{t+1}) | X_t = x, X_{t+1} = y\right]}{g(x)} \\
&= \max_{g\in\mathcal{X}_+^o} \min_{x\in\mathbb{X}, g(x)\neq 0} \frac{\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}\right) g(X_{t+1}) | X_t = x\right]}{g(x)}.
\end{aligned}$$

Moreover, it is straightforward to see that the maximum in the above formula is attained when g is chosen to be the Perron-Frobenius eigenvector of $\tilde{\mathbf{P}}$. Because the eigenvector lies in $\mathcal{X}_{++}$, we conclude that $\mathcal{X}_+^o$ can be replaced with $\mathcal{X}_{++}$ in the above formula. Now, recalling $\delta = \eta^{1/(1-\gamma)}$ and setting $f = g^{1/(1-\gamma)}$, we conclude that when $\gamma < 1$,

$$\delta = \max_{f\in\mathcal{X}_{++}} \min_{x\in\mathbb{X}} \frac{u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})} f(X_{t+1})\right) | X_t = x\right]\right)}{f(x)}. \tag{0.2}$$

Similarly, according to the min-max version of the Collatz-Wielandt formula (Meyer, 2000, p. 669),[5] we

[5] The formula therein is presented for postive matrices, but it also holds for irreducible nonnegative matrices because the Perron-Frobenius eigenvectors for these matrices are positive; see for instance Meyer (2000, p. 673).

have

$$\eta = \min_{g \in \mathcal{X}_{++}} \max_{x \in \mathbb{X}} \frac{\mathbb{E}_t \left[u \left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} \right) g(X_{t+1}) | X_t = x \right]}{g(x)}.$$

Recalling $\delta = \eta^{1/(1-\gamma)}$ and setting $f = g^{1/(1-\gamma)}$, we conclude that (0.2) also holds when $\gamma > 1$.

Finally, we show that (0.2) also holds when $\gamma = 1$. For each $f \in \mathcal{X}_{++}$, denote

$$\begin{aligned}
\xi_f :&= \min_{x \in \mathbb{X}} \frac{u^{-1} \Big(\mathbb{E}_t \left[u \left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} f(X_{t+1}) \right) | X_t = x \right] \Big)}{f(x)} \\
&= \exp \left\{ \min_{x \in \mathbb{X}} \Big(\mathbb{E}_t \left[\kappa(X_t, X_{t+1}, Y_{t+1}) + \ln f(X_{t+1}) | X_t = x \right] - \ln f(x) \Big) \right\}.
\end{aligned}$$

As a result, $\ln(\xi_f) + \ln f(X_t) \le \mathbb{E}_t \left[\kappa(X_t, X_{t+1}, Y_{t+1}) + \ln f(X_{t+1}) | X_t \right]$. Taking expectation on both sides under the stationary distribution of $\{X_t\}$ and recalling η that is derived in part (ii) of the proof, we conclude that $\ln(\xi_f) + \mathbb{E}[\ln f(X_t)] \le \eta + \mathbb{E}[\ln f(X_{t+1})]$, which implies $\ln(\xi_f) \le \eta$. Therefore, we conclude

$$\eta \ge \max_{f \in \mathcal{X}_{++}} \min_{x \in \mathbb{X}} \Big(\mathbb{E}_t \left[\kappa(X_t, X_{t+1}, Y_{t+1}) + \ln f(X_{t+1}) | X_t = x \right] - \ln f(x) \Big). \tag{0.3}$$

On the other hand, recall v defined in part (ii) of the proof. Then, $e^v \in \mathcal{X}_{++}$ and (0.1) can be written as

$$\mathbb{E}_t \left[\ln e^{v(X_{t+1})} | X_t = x \right] = \ln e^{v(x)} + \eta - \mathbb{E}_t \left[\kappa(X_t, X_{t+1}, Y_{t+1}) | X_t = x \right], \quad x \in \mathbb{X}.$$

Combining with (0.3), we immediately conclude that

$$\eta = \max_{f \in \mathcal{X}_{++}} \min_{x \in \mathbb{X}} \Big(\mathbb{E}_t \left[\kappa(X_t, X_{t+1}, Y_{t+1}) + \ln f(X_{t+1}) | X_t = x \right] - \ln f(x) \Big).$$

Therefore, (0.2) holds. ☐

Proof of Theorem 1 For ease of exposition, the proof is divided into two parts.

Part One: existence and uniqueness of the fixed point

In the first part of the proof, we show the existence and uniqueness of the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$. Observe that $\mathbb{T}f$ is well-defined for any $f \in \mathcal{X}_+$ and $\mathbb{T}$ is increasing.

We first note from Proposition 1-(i) that when $\gamma \neq 1$, with η, δ, and v as defined in Proposition 1, we can define $M_{t+1} := \eta^{-1} e^{(1-\gamma)\kappa(X_{t+1},Y_{t+1},X_t)} v(X_{t+1})/v(X_t)$ and show that $M_{t+1} > 0$ and $\mathbb{E}_t[M_{t+1}] = 1$. As a result, we can define a new measure $\tilde{\mathbb{P}}$ by using M_{t+1} as the Radon-Nikodym density. Note that $\{X_t\}$ is still an irreducible Markov chain under $\tilde{\mathbb{P}}$. Denote the corresponding expectation as $\tilde{\mathbb{E}}$. Then,

$$\begin{aligned}
&\mathbb{E}_t \left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})} f(X_{t+1}) \right) | X_t = x \right] \\
=& \mathbb{E}_t \left[M_{t+1} v(X_{t+1})^{-1} v(X_t) \eta f(X_{t+1})^{1-\gamma} | X_t = x \right] \\
=& \tilde{\mathbb{E}}_t \left[v(X_{t+1})^{-1} v(X_t) \eta f(X_{t+1})^{1-\gamma} | X_t = x \right].
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
&u^{-1}\left(\tilde{\mathbb{E}}_t \left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})} f(X_{t+1}) \right) | X_t = x \right] \right) \\
&= \delta u^{-1}(v(x)) u^{-1}\left(\tilde{\mathbb{E}}_t \left[u\left(f(X_{t+1})/u^{-1}(v(X_{t+1})) \right) | X_t = x \right] \right).
\end{aligned} \tag{0.4}$$

After a careful calculation, one can conclude from Proposition 1-(ii) that (0.4) holds for the case $\gamma = 1$ as well with $\tilde{\mathbb{E}}$ replaced by $\mathbb{E}$. Therefore, in the following, we will use (0.4) regardless of the value of γ, and

$\tilde{\mathbb{E}}$ stands for $\mathbb{E}$ when $\gamma = 1$. Using (0.4) and the homogeneity of $H(c, z)$, we obtain

$$\frac{\mathbb{T}f(x)}{u^{-1}\big(v(x)\big)} = H\left(\frac{1}{u^{-1}\big(v(x)\big)}, \delta\left[u^{-1}\left(\tilde{\mathbb{E}}_t\left[u\left(\frac{f(X_{t+1})}{u^{-1}\big(v(X_{t+1})\big)}\right)|X_t = x\right]\right) + \frac{\delta^{-1}\varpi(x)}{u^{-1}\big(v(x)\big)}\right]\right). \quad (0.5)$$

We first consider the case in which $\rho \neq 1$. In this case, denoting $g(x) := \big(f(x)/u^{-1}(v(x))\big)^{1-\rho}$, we conclude that f is a fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$ if and only if g is a fixed point of $\mathbb{S}$ in $\mathcal{X}_{++}$, where $\mathbb{S}$ is an operator on $\mathcal{X}_+$ defined as

$$\mathbb{S}g(x) := \frac{1-\beta}{\tilde{u}^{-1}(v(x))} + \beta\delta^{1-\rho}\left\{\left[\tilde{u}^{-1}\left(\tilde{\mathbb{E}}_t\left[\tilde{u}\big(g(X_{t+1})\big)|X_t = x\right]\right)\right]^{\frac{1}{1-\rho}} + \frac{\delta^{-1}\varpi(x)}{u^{-1}\big(v(x)\big)}\right\}^{1-\rho} \quad (0.6)$$

with $\tilde{u}(x) := u(x^{1/(1-\rho)}), x \geq 0$.

It is easy to see that $\mathbb{S}$ is an increasing mapping from $\mathcal{X}_+$ into $\mathcal{X}_{++}$. Consider function $\varphi(z) := \left(z^{\frac{1}{1-\rho}} + a\right)^{1-\rho}, z \geq 0$ for some $a \geq 0$. It is straightforward to see that $\varphi'(z) = \left(z^{\frac{1}{1-\rho}}/(z^{\frac{1}{1-\rho}} + a)\right)^{\rho} \leq 1$. Consequently, $|\varphi(z_1) - \varphi(z_2)| \leq |z_1 - z_2|$ for any $z_1, z_2 \geq 0$. As a result, for any g_1 and g_2, we have

$$|\mathbb{S}g_1(x) - \mathbb{S}g_2(x)| \leq \beta\delta^{1-\rho}\left|\tilde{u}^{-1}\left(\tilde{\mathbb{E}}_t\left[\tilde{u}\big(g_1(X_{t+1})\big)|X_t = x\right]\right) - \tilde{u}^{-1}\left(\tilde{\mathbb{E}}_t\left[\tilde{u}\big(g_2(X_{t+1})\big)|X_t = x\right]\right)\right|.$$

When $\alpha := (1-\gamma)/(1-\rho) \geq 1$, $\tilde{u}(x) = x^{\alpha}$ is a convex, power function, so we conclude that

$$|\mathbb{S}g_1(x) - \mathbb{S}g_2(x)| \leq \beta\delta^{1-\rho}\left[\tilde{\mathbb{E}}_t\left(|g_1(X_{t+1}) - g_2(X_{t+1})|^{\alpha}|X_t = x\right)\right]^{\frac{1}{\alpha}}, \quad x \in \mathbb{X},$$

which implies

$$|\mathbb{S}g_1(X_t) - \mathbb{S}g_2(X_t)|^{\alpha} \leq \left(\beta\delta^{1-\rho}\right)^{\alpha}\tilde{\mathbb{E}}_t\left(|g_1(X_{t+1}) - g_2(X_{t+1})|^{\alpha}|X_t\right).$$

Recall that $\{X_t\}$ is an irreducible Markov chain under measure $\tilde{\mathbb{P}}$, so it has a unique stationary distribution. Taking expectation on both sides of the inequality under this stationary distribution, and noting that the marginal distributions of X_t and X_{t+1} are the same, we conclude

$$\left[\tilde{\mathbb{E}}\left(|\mathbb{S}g_1(X_t) - \mathbb{S}g_2(X_t)|^\alpha\right)\right]^{1/\alpha} \leq \beta\delta^{1-\rho}\left[\tilde{\mathbb{E}}\left(|g_1(X_t) - g_2(X_t)|^\alpha\right)\right]^{1/\alpha}. \tag{0.7}$$

Because $\beta\delta^{1-\rho} < 1$, $\mathbb{S}$ is a contraction mapping on $\mathcal{X}_+$ with norm $\left[\tilde{\mathbb{E}}|g(X_t)|^\alpha\right]^{1/\alpha}$. Consequently, for any $g \in \mathcal{X}_+$, the limit of $\{\mathbb{S}^n g\}_{n\geq 0}$ exists and is the unique fixed point of $\mathbb{S}$ in $\mathcal{X}_+$. Moreover, because $\mathbb{S}g(x) \geq (1-\beta)/\tilde{u}^{-1}(v(x)) > 0, x \in \mathbb{X}$, the fixed point must lie in $\mathcal{X}_{++}$. As a result, $\mathbb{T}$ has a unique fixed point in $\mathcal{X}_{++}$.

When $\alpha < 1$, we consider the following operator:

$$\tilde{\mathbb{S}}g(x) := \frac{1-\beta}{\tilde{u}^{-1}(v(x))} + \beta\delta^{1-\rho}\left\{\left(\tilde{\mathbb{E}}_t\left[g(X_{t+1})|X_t = x\right]\right)^{\frac{1}{1-\rho}} + \frac{\delta^{-1}\varpi(x)}{u^{-1}\big(v(x)\big)}\right\}^{1-\rho}. \tag{0.8}$$

Because $\tilde{u}$ is either a concave power or a logarithmic function when $\alpha < 1$, we have $\tilde{u}^{-1}\left(\mathbb{E}_t[\tilde{u}(Z)]\right) \leq \mathbb{E}_t[Z]$ for any nonnegative random variable Z. As a result, $\mathbb{S}g(x) \leq \tilde{\mathbb{S}}g(x), x \in \mathbb{X}$ for any g and in particular for $g_0(x) := (1-\beta)/\tilde{u}^{-1}(v(x)) > 0, x \in \mathbb{X}$. One can see that both $\{\mathbb{S}^n g_0\}_{n\geq 0}$ and $\{\tilde{\mathbb{S}}^n g_0\}_{n\geq 0}$ are increasing sequences and the former is dominated by the latter. On the other hand, following the same proof as in the case in which $\alpha \geq 1$, we can show that $\tilde{\mathbb{S}}$ is a contraction mapping from $\mathcal{X}_+$ into $\mathcal{X}_{++}$. As a result, $\{\tilde{\mathbb{S}}^n g_0\}_{n\geq 0}$ converges and so does $\{\mathbb{S}^n g_0\}_{n\geq 0}$. Consequently, the limit of $\{\mathbb{S}^n g_0\}_{n\geq 0}$ is a fixed point of $\mathbb{S}$ and lies in $\mathcal{X}_{++}$, and thus the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$ exists. We then show the uniqueness of the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$ when $\alpha < 1$. For the sake of contradiction, suppose there are two distinct fixed points f_1 and f_2 in $\mathcal{X}_{++}$. Without loss of generality, we assume $f_1(x) < f_2(x)$ for some

$x \in \mathbb{X}$. Define $x^* := \underset{x\in\mathbb{X}}{\text{argmin}}\ f_1(x)/f_2(x)$ and denote the corresponding minimum value as r^*. Because $\mathbb{X}$ is finite and f_i's are positive, x^* is well defined and $r^* \in (0,1)$. Define $f(x) := r^* f_2(x), x \in \mathbb{X}$. Then, $f(x) \le f_1(x), x \in \mathbb{X}$ and $f(x^*) = f_1(x^*)$. Denote $\mathbb{I}$ as the identity mapping. Then, for each $x \in \mathbb{X}$, we have

$$\begin{aligned}(\mathbb{T}-\mathbb{I})f(x) &= H\left(1, u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})} r^* f_2(X_{t+1})\right)|X_t = x\right]\right) + \varpi(x)\right) - r^* f_2(x) \\ &> r^* H\left(1, u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})} f_2(X_{t+1})\right)|X_t = x\right]\right) + \varpi(x)\right) - r^* f_2(x) \\ &= r^*(\mathbb{T}-\mathbb{I})f_2(x) = 0,\end{aligned}$$

where the inequality is the case because $\beta < 1$ and $\varpi(x) \ge 0$ and the last equality is the case because f_2 is a fixed point of $\mathbb{T}$. In particular, we have $\mathbb{T}f(x^*) > f(x^*) = f_1(x^*)$. On the other hand, because $\mathbb{T}$ is increasing and $f \le f_1$, we have $\mathbb{T}f \le \mathbb{T}f_1 = f_1$, where the equality is the case because f_1 is a fixed point of $\mathbb{T}$. In particular, $\mathbb{T}f(x^*) \le f_1(x^*)$. Thus, we have a contradiction, so the fixed point of $\mathbb{T}$ must be unique.

Next, we consider the case in which $\rho = 1$. In this case,

$$\mathbb{T}f(x) = \exp\left\{\beta \ln\left[u^{-1}\left(\mathbb{E}_t\left(u(e^{\kappa(X_t,X_{t+1},Y_{t+1})} f(X_{t+1}))|X_t = x\right)\right) + \varpi(x)\right]\right\}, \quad x \in \mathbb{X}.$$

Because $\mathbb{X}$ is finite, there exists $\epsilon > 0$ such that

$$\epsilon \le \min_{x\in\mathbb{X}}\left[u^{-1}\left(\mathbb{E}_t\left(u(e^{\kappa(X_t,X_{t+1},Y_{t+1})})|X_t = x\right)\right)\right]^{\frac{\beta}{1-\beta}}. \tag{0.9}$$

It is straightforward to verify that for such $\epsilon > 0$, $\mathbb{T}\epsilon \ge \epsilon > 0$. Because $\mathbb{T}$ is increasing, $\{\mathbb{T}^n \epsilon\}_{n\ge 0}$ is an increasing sequence. On the other hand, because $\beta < 1$ and $\mathbb{X}$ is finite, there exists $N > \epsilon$ such that

$\mathbb{T}N \le N$. Consequently, $\mathbb{T}^n\epsilon \le \mathbb{T}^n N \le N, n \ge 0$, so the limit of $\{\mathbb{T}^n\epsilon\}_{n\ge 0}$ exists and is a fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$. Using the same proof as in the case in which $\rho \ne 1$ and $\alpha < 1$, we can show that the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$ is unique.

Part Two: computation of the fixed point.

In the second part of the proof, we show that $\{\mathbb{T}^n f\}_{n\ge 0}$ converges to the fixed point of $\mathbb{T}$ for any $f \in \mathcal{X}_{++}$. Denote the fixed point as f^*.

Because $f \in \mathcal{X}_{++}$, $f^* \in \mathcal{X}_{++}$, and $\mathbb{X}$ is finite, there exists $r \ge 1$ such that $f \le rf^*$ and $f \ge (1/r)f^*$. Then,

$$\mathbb{T}(rf^*) \le r\mathbb{T}(f^*) = rf^*,$$

where the inequality is the case because $\beta \le 1$ and $\varpi \ge 0$ and the equality is the case because f^* is the fixed point of $\mathbb{T}$. Consequently, $\{\mathbb{T}^n(rf^*)\}_{n\ge 0}$ is a decreasing sequence. Similarly, $\{\mathbb{T}^n((1/r)f^*)\}_{n\ge 0}$ is an increasing sequence. Moreover, $\mathbb{T}^n((1/r)f^*) \le \mathbb{T}^n f \le \mathbb{T}^n(rf^*)$ because $\mathbb{T}$ is increasing. As a result, both $\{\mathbb{T}^n(rf^*)\}_{n\ge 0}$ and $\mathbb{T}^n((1/r)f^*)$ converge in $\mathcal{X}_{++}$ and the convergent points are fixed points of $\mathbb{T}$. Because the fixed point of $\mathbb{T}$ is unique, both $\{\mathbb{T}^n(rf^*)\}_{n\ge 0}$ and $\mathbb{T}^n((1/r)f^*)$ converge to this fixed point, i.e,. to f^*. By the squeeze theorem, $\{\mathbb{T}^n f\}_{n\ge 0}$ also converges to f^*. $\square$

Proof of Theorem 2. Define operator $\mathbb{T}_+$ on $\mathcal{X}_+$ by

$$\mathbb{T}_+ f(x) := H\left(1, u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}f(X_{t+1})\right)|X_t = x\right]\right) + \varpi^+(x)\right), \quad x \in \mathbb{X}.$$

It is obvious that $\mathbb{T}f \le \mathbb{T}_+ f$ for any f. According to Assumption 3, sequence $\{\mathbb{T}^n f_0\}_{n\ge 0}$ is increasing. Consequently, $\mathbb{T}_+ f_0 \ge \mathbb{T}f_0 \ge f_0$ and thus $\{\mathbb{T}_+^n f_0\}_{n\ge 0}$ is also an increasing sequence and dominates

$\{\mathbb{T}^n f_0\}_{n\geq 0}$. By Assumption 3, $\mathbb{T}^m f_0(x) > f_0(x) \geq 0, x \in \mathbb{X}$ for some $m \geq 0$. As a result, $\mathbb{T}_+^n f_0 \in \mathcal{X}_{++}$ for sufficiently large n and thus $\{\mathbb{T}_+^n f_0\}_{n\geq 0}$ converges to the fixed point of $\mathbb{T}_+$ in $\mathcal{X}_{++}$ according to Theorem 1. Consequently, the limit of $\{\mathbb{T}^n f_0\}_{n\geq 0}$ exists, is a fixed point of $\mathbb{T}$, and is strictly larger than f_0 point-wisely.

Next, we show the uniqueness of the fixed point of $\mathbb{T}$. We first note that for any fixed point f^* of $\mathbb{T}$, we have $f^* = \mathbb{T}f^* \geq f_0$. Because $\mathbb{T}^m f_0 > f_0$ for some $m \geq 0$, f^* must be strictly larger than f_0 point-wisely. Now, for the sake of contradiction, suppose we have two distinctive fixed points f_1 and f_2. We already showed that $f_i(x) > f_0(x), x \in \mathbb{X}$, $i = 1, 2$. Without loss of generality, we assume $f_1(x) < f_2(x)$ for some $x \in \mathbb{X}$. Define

$$x^* := \text{argmin}_{x\in\mathbb{X}} \frac{f_1(x) - f_0(x)}{f_2(x) - f_0(x)}$$

and denote the corresponding minimum value as r^*. Because $\mathbb{X}$ is finite, x^* must exist and $r^* \in (0, 1)$. Define $f(x) := r^* f_2(x) + (1 - r^*) f_0(x), x \in \mathbb{X}$. Then, one can verify that $f_0(x) < f(x) \leq f_1(x), x \in \mathbb{X}$ and $f(x^*) = f_1(x^*)$. Because $\mathbb{T}f_0$ is well defined, so is $\mathbb{T}f$. Recall that $\mathbb{T}^m f_0(x) > f_0(x), x \in \mathbb{X}$ for some $m \geq 1$. Because $\mathbb{T}$ is increasing and concave, so is $\mathbb{T}^m$. Denote $\mathbb{I}$ as the identity mapping. Then, for any $x \in \mathbb{X}$,

$$(\mathbb{T}^m - \mathbb{I})f(x) \geq r^*(\mathbb{T}^m - \mathbb{I})f_2(x) + (1 - r^*)(\mathbb{T}^m - \mathbb{I})f_0(x) = (1 - r^*)(\mathbb{T}^m - \mathbb{I})f_0(x) > 0,$$

where the first inequality is the case due to the concavity of $\mathbb{T}^m$ and the equality is the case because f_2 is a fixed point of $\mathbb{T}$. Thus, $\mathbb{T}^m f(x^*) > f(x^*) = f_1(x^*)$. On the other hand, $\mathbb{T}^m f(x) \leq \mathbb{T}^m f_1(x) = f_1(x), x \in \mathbb{X}$ because $\mathbb{T}$ is increasing and f_1 is a fixed point of $\mathbb{T}$. In particular, $\mathbb{T}^m f(x^*) \leq f_1(x^*)$, which is a contradiction.

Finally, we show that for any f such that $\mathbb{T}f$ is well defined, $\{\mathbb{T}^n f\}_{n\geq 0}$ converges to the fixed point of $\mathbb{T}$. We first note that $\mathbb{T}f \geq f_0$ and $\mathbb{T}f_0$ is well defined according to Assumption 3. As a result, $\mathbb{T}^n f$ is well defined for any $n \geq 0$. Recall that $\mathbb{T}^m f_0(x) > f_0(x), x \in \mathbb{X}$ for some $m \geq 1$, so $\mathbb{T}^{m+1} f(x) \geq \mathbb{T}^m f_0(x) > f_0(x), x \in \mathbb{X}$. Thus, in the following, we assume $f(x) > f_0(x), x \in \mathbb{X}$ without loss of generality.

Denote f^* as the unique fixed point of $\mathbb{T}$, and we already showed that $f^*(x) > f_0(x), x \in \mathbb{X}$. Because $\mathbb{X}$ is finite, there must exist $r \in (0,1]$ such that $f - f_0 \leq (f^* - f_0)/r$, i.e., $rf + (1-r)f_0 \leq f^*$. Then,

$$f^* = \mathbb{T}f^* \geq \mathbb{T}(rf + (1-r)f_0) \geq r\mathbb{T}f + (1-r)\mathbb{T}f_0,$$

where the equality is the case because f^* is the fixed point, the first inequality is the case because $\mathbb{T}$ is increasing, and the second inequality is the case because $\mathbb{T}$ is concave. Applying $\mathbb{T}$ on both sides of the above inequality, we conclude that $f^* \geq r\mathbb{T}^n f + (1-r)\mathbb{T}^n f_0$ for any $n \geq 0$, which implies

$$\mathbb{T}^n f \leq \left[f^* - (1-r)\mathbb{T}^n f_0\right]/r.$$

On the other hand, we have $\mathbb{T}^n f \geq \mathbb{T}^n f_0$. Because $\{\mathbb{T}^n f_0\}_{n\geq 0}$ converges to f^*, the squeeze theorem shows that $\{\mathbb{T}^n f\}_{n\geq 0}$ converges to f^* as well. □

Proof of Proposition 2. One can observe from (1.4.9) that $\Phi \geq \mathbb{V}_{c,\theta}\Phi$ for any $(c,\theta) \in \mathcal{A}$. Consequently, $\{\mathbb{V}^n_{c,\theta}\Phi\}_{n\geq 0}$ is a decreasing sequence and so is $\{\mathbb{T}^n_{c,\theta}(\Phi/c)\}$. By Theorem 1 its limit is the fixed point of $\mathbb{V}_{c,\theta}$, i.e., is $F_{c,\theta}$. Thus, $\Phi(x) \geq F_{c,\theta}(x), x \in \mathbb{X}$.

If there exists $(c^*, \theta^*) \in \mathcal{A}$ such that $(c^*(x), \theta^*(x))$ is a maximizer of (1.4.10) for each $x \in \mathbb{X}$, then

$\Phi = \mathbb{V}_{c^*,\theta^*}\Phi$. By the uniqueness of the fixed point of $\mathbb{V}_{c^*,\theta^*}$, we conclude that $\Phi = F_{c^*,\theta^*}$. As a result, (c^*,θ^*) and Φ are a maximizer and the optimal value, respectively, of (1.4.8). $\square$

Proof of Proposition 3. It is straightforward to see from (1.4.10) that $\mathbb{W}$ is continuous and increasing, and

$$\mathbb{W}\Phi(x) = \max_{(c,\theta)\in\mathcal{A}} H\left(c(x),(1-c(x))D_\Phi(x,\theta(x))\right) = \max_{(c,\theta)\in\mathcal{A}} c(x)\mathbb{T}_{c,\theta}(\Phi/c)(x), \quad x\in\mathbb{X}. \tag{0.10}$$

Moreover, because I_x and J_x are compact for each $x \in \mathbb{X}$, we conclude from (1.4.10) and (0.10) that for each Φ such that $\mathbb{W}\Phi$ is well defined, there exists $(c,\theta)\in\mathcal{A}$ such that $\mathbb{W}\Phi = c\mathbb{T}_{c,\theta}(\Phi/c)$, i.e., the maximum in (0.10) is attained by (c,θ).

In the following, we divided the proof the theorem into three parts.

Part One: existence of the fixed point

We prove the existence of the fixed point of $\mathbb{W}$ in this part. We first consider the case in which there exists $(c,\theta)\in\mathcal{A}$ such that $\varpi_{c,\theta}(x) < 0$ for some $x\in\mathbb{X}$.

Recall $f_{0,c,\theta}$ as defined in Assumption 3, i.e., $f_{0,c,\theta} = H(1,\varpi^+_{c,\theta})$ for each (c,θ). Define

$$\Phi_0(x) := \max_{(c,\theta)\in\mathcal{A}} c(x)f_{0,c,\theta}(x) = \max_{\bar{c}\in I_x} H\left(\bar{c},(1-\bar{c})\max_{\bar{\theta}\in J_x}\Big(\sum_{i=1}^n \bar{\theta}_i b_i g_i(x)\Big)^+\right), \quad x\in\mathbb{X}.$$

By Assumption 4, for each $(c,\theta)\in\mathcal{A}$, $\mathbb{T}_{c,\theta}f_{0,c,\theta}$ is well defined. Because $\Phi_0 \geq cf_{0,c,\theta}$, i.e., $\Phi_0/c \geq f_{0,c,\theta}$, for each $(c,\theta)\in\mathcal{A}$, we conclude that $\mathbb{T}_{c,\theta}(\Phi_0/c)$ is well defined for any $(c,\theta)\in\mathcal{A}$, so $\mathbb{W}\Phi_0$ is well defined. Moreover, because $\mathbb{T}_{c,\theta}(f)\geq f_{0,c,\theta}$ for any $(c,\theta)\in\mathcal{A}$ and f such that $\mathbb{T}_{c,\theta}(f)$ is well defined, we conclude that $\mathbb{T}_{c,\theta}(\Phi_0/c)\geq f_{0,c,\theta}$ for any $(c,\theta)\in\mathcal{A}$. As a result, $\mathbb{W}\Phi_0\geq\Phi_0$. Now, define $\Phi_n := \mathbb{W}^n\Phi_0$, $n\geq 1$. Becuase $\mathbb{W}$ is increasing, $\{\Phi_n\}_{n\geq 0}$ is an increasing sequence.

Because $\varpi_{c_0,\theta_0}(x) < 0$ for some $x\in\mathbb{X}$ and some $(c_0,\theta_0)\in\mathcal{A}$, Assumption 4 yields that

$\mathbb{T}^m_{c_0,\theta_0} f_{c_0,\theta_0} > f_{c_0,\theta_0}$ for some $m \geq 1$. On the other hand,

$$\mathbb{W}\Phi_0 \geq c_0 \mathbb{T}_{c_0,\theta_0}(\Phi_0/c_0) \geq c_0 \mathbb{T}_{c_0,\theta_0}(f_{c_0,\theta_0}),$$

which implies $\Phi_1/c_0 = (\mathbb{W}\Phi_0)/c_0 \geq \mathbb{T}_{c_0,\theta_0}(f_{c_0,\theta_0})$. Repeating the the above calculation, we conclude that $\Phi_m/c_0 \geq \mathbb{T}^m_{c_0,\theta_0}(f_{c_0,\theta_0}) > f_{c_0,\theta_0}$. As a result, $\Phi_m > c_0 f_{c_0,\theta_0}$, showing that $\Phi_m \in \mathcal{X}_{++}$. Therefore, in the following we assume the sequence $\{\Phi_n\}_{n\geq 0}$ is in $\mathcal{X}_{++}$ without loss of generality.

Now, we show that $\{\Phi_n\}_{n\geq 0}$ is bounded from above and thus its limit exists. Note that for each $n \geq 1$, there exists $(c_n, \theta_n) \in \mathcal{A}$ such that

$$\Phi_n \leq \Phi_{n+1} = \mathbb{W}\Phi_n = c_n \mathbb{T}_{c_n,\theta_n}(\Phi_n/c_n),$$

which implies $\Phi_n/c_n \leq \mathbb{T}_{c_n,\theta_n}(\Phi_n/c_n)$. Because $\Phi_n \in \mathcal{X}_{++}$, according to Theorems 1 and 2, $\{\mathbb{T}^m_{c_n,\theta_n}(\Phi_n/c_n)\}_{m\geq 0}$ converges to f_{c_n,θ_n}, the fixed point of $\mathbb{T}_{c_n,\theta_n}$, as m goes to infinity. Consequently, $\Phi_n/c_n \leq f_{c_n,\theta_n}$, i.e., $\Phi_n \leq c_n f_{c_n,\theta_n}$. Thus, we only need to show that $\{c_n f_{c_n,\theta_n}\}_{n\geq 1}$ is bounded for any sequence $(c_n, \theta_n) \in \mathcal{A}$. Because $c_n \leq 1$, we only need to show that $\{f_{c_n,\theta_n}\}_{n\geq 1}$ is bounded. Moreover, when $\rho > 1$, we have $H(1, z) \leq (1-\beta)^{1/(1-\rho)}$, so $f_{c_n,\theta_n} = \mathbb{T}_{c_n,\theta_n} f_{c_n,\theta_n} \leq (1-\beta)^{1/(1-\rho)}$. Thus, in the following, we only need to consider the case in which $\rho \leq 1$.

We prove the boundedness of $\{f_{c_n,\theta_n}\}_{n\geq 1}$ for the case in which $\rho < 1$ and $\gamma \neq 1$ as other cases can be dealt with similarly. When $\gamma \neq 1$, recall the Perron-Frobenius eigenvalue η_{c_n,θ_n} and eigenvector v_{c_n,θ_n} as defined in Proposition 1 corresponding to (c_n, θ_n). Here and hereafter, we always choose the Perron-Frobenius eigenvector such that its L^1 norm is one. Recalling that f_{c_n,θ_n} is the fixed point of

$\mathbb{T}_{c_n,\theta_n}$, we conclude from (0.5) that

$$f_{c_n,\theta_n}(x)v_{c_n,\theta_n}(x)^{-\frac{1}{1-\gamma}} = H\Bigg(v_{c_n,\theta_n}(x)^{-\frac{1}{1-\gamma}}, \varpi_{c_n,\theta_n}(x)v_{c_n,\theta_n}(x)^{-\frac{1}{1-\gamma}} + \delta_{c_n,\theta_n}u^{-1}\left(\tilde{\mathbb{E}}_t\left[u\big(f_{c_n,\theta_n}(X_{t+1})v_{c_n,\theta_n}(X_{t+1})^{-\frac{1}{1-\gamma}}\big)|X_t=x\right]\right)\Bigg)$$
$$\leq H\Bigg(\|v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty, \|\varpi^+_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty + \delta_{c_n,\theta_n}\|f_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty\Bigg),$$

where $\|\cdot\|_\infty$ stands for the L^∞ norms of functions on $\mathbb{X}$. Consequently,

$$\|f_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty \leq H\Bigg(\|v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty, \|\varpi^+_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty + \delta_{c_n,\theta_n}\|f_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty\Bigg).$$

With $\rho < 1$, we have $(y_1+y_2)^{1-\rho} \leq y_1^{1-\rho} + y_2^{1-\rho}$ for any $y_1, y_2 \geq 0$, so

$$\|f_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho} \leq (1-\beta)\|v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho} + \beta\left(\|\varpi^+_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty + \delta_{c_n,\theta_n}\|f_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty\right)^{1-\rho}$$
$$\leq (1-\beta)\|v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho} + \beta\|\varpi^+_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho} + \beta\delta_{c_n,\theta_n}^{1-\rho}\|f_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho}.$$

As a result,

$$\|f_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho} \leq \frac{(1-\beta)\|v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho} + \beta\|\varpi^+_{c_n,\theta_n}v_{c_n,\theta_n}^{-\frac{1}{1-\gamma}}\|_\infty^{1-\rho}}{1-\beta\delta_{c_n,\theta_n}^{1-\rho}}. \tag{0.11}$$

Note that the Perron-Frobenius eigenvalue and eigenvector of an irreducible matrix A is continuous in A.[6] As a result, $\delta_{c,\theta}$ and $v_{c,\theta}$ are continuous in (c,θ). Because I_x and J_x are compact for each $x \in \mathbb{X}$, $\mathcal{A}$ is

[6] To prove the continuity, we only need to show that for any sequence of irreducible matrices A_n with Perron-Frobenius eigenvalue r_n and eigenvector p_n (normalized under L^1 norm) such that it converges to an irreducible matrix A, there exists a subsequence, indexed by n_k, such that $\{r_{n_k}\}$ and $\{p_{n_k}\}$ converge to the Perron-Frobenius eigenvalue and eigenvector of A, respectively. Because A_n converges to A and A is an irreducible matrix, by the

also compact. Because $\beta\delta_{c,\theta}^{1-\rho} < 1$ and $v_{c,\theta} \in \mathcal{X}_{++}$ for each $(c,\theta) \in \mathcal{A}$, we conclude that $\sup_n \beta\delta_{c_n,\theta_n}^{1-\rho} < 1$ and v_{c_n,θ_n}'s are uniformly bounded from above and bounded away from zero. Moreover, because I_x and J_x are compact, ϖ_n^+'s are also uniformly bounded. Consequently, because $\rho < 1$, we conclude from (0.11) that f_{c_n,θ_n}'s are uniformly bounded.

We have proved that the limit of $\{\Phi_n\}$ exists and must be in $\mathcal{X}_{++}$. Then, by the continuity of $\mathbb{W}$, the limit must be a fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$.

Next, we consider the case in which $\varpi_{c,\theta} \geq 0$ for any $(c,\theta) \in \mathcal{A}$. In this case, for any $(c_0,\theta_0) \in \mathcal{A}$, $\mathbb{W}F_{c_0,\theta_0}$ is well defined. Moreover, we have $\mathbb{W}F_{c_0,\theta_0} \geq c_0\mathbb{T}_{c_0,\theta_0}(F_{c_0,\theta_0}/c_0) = F_{c_0,\theta_0}$, so $\{\mathbb{W}^n F_{c_0,\theta_0}\}_{n\geq 0}$ is an increasing sequence. On the one hand, the sequence $\{\mathbb{W}^n F_{c_0,\theta_0}\}_{n\geq 0}$ is in $\mathcal{X}_{++}$ because $F_{c_0,\theta_0} \in \mathcal{X}_{++}$. On the other hand, following the same proof as in the previous case, we can show that this sequence is bounded from above. As a result, this sequence converges and the convergent point is a fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$.

Part Two: uniqueness of the fixed point

Because I_x and J_x are compact for each $x \in \mathbb{X}$, according to Proposition 2, any fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ is the optimal value of (1.4.8), so the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ is unique.

Part Three: computing the fixed point

Denote the unique fixed point of $\mathbb{W}$ as Φ^*. We first consider the case in which $\varpi_{c,\theta} \geq 0$ for any $(c,\theta) \in \mathcal{A}$. Note that for any $\Phi \in \mathcal{X}_{++}$, because $\mathbb{X}$ is finite, there exists $r > 1$ such that $(1/r)\Phi^*(x) \leq$

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min-max and max-min version of the Collatz-Wielandt formula, we conclude that r_n's are uniformly bounded from above and uniformly bounded away from zero. On the other hand, p_n's are uniformly bounded because they are normalized. As a result, there exists a subsequence, indexed by n_k, such that $\{r_{n_k}\}$ and $\{p_{n_k}\}$ converge. Denote the convergent points as r and p, respectively. Then, $r > 0$, $p \geq 0$, and p is normalized, i.e., the L^1 norm of p is one. Moreover, because $A_{n_k}p_{n_k} = r_{n_k}p_{n_k}$, we conclude $Ap = rp$, i.e., p is an eigenvector of A with eigenvalue r. Because there is no nonnegative eigenvector for A except for positive multiples of the Perron-Frobenius eigenvector regardless of the eigenvalue (Meyer, 2000, p. 673), we immediately conclude that r and p are the Perron-Frobenius eigenvalue and eigenvector of A, respectively.

$\Phi(x) \le r\Phi^*(x), x \in \mathbb{X}$. Because $\varpi_{c,\theta} \ge 0$ for all $(c,\theta) \in \mathcal{A}$ and $\beta < 1$,

$$\mathbb{W}(r\Phi^*)(x) = \max_{(c,\theta)\in\mathcal{A}} c(x)\mathbb{T}_{c,\theta}(r\Phi^*/c)(x) \le \max_{(c,\theta)\in\mathcal{A}} c(x)r\mathbb{T}_{c,\theta}(\Phi^*/c)(x) = r\mathbb{W}\Phi^*(x) = r\Phi^*(x).$$

Therefore, $\{\mathbb{W}^n(r\Phi^*)\}_{n\ge 0}$ is a decreasing sequence. Similarly, $\{\mathbb{W}^n((1/r)\Phi^*)\}_{n\ge 0}$ is an increasing sequence. Moreover, $\mathbb{W}^n((1/r)\Phi^*) \le \mathbb{W}^n\Phi \le \mathbb{W}^n(r\Phi^*)$ because $\mathbb{W}$ is increasing. As a result, both $\{\mathbb{W}^n(r\Phi^*)\}_{n\ge 0}$ and $\{\mathbb{W}^n((1/r)\Phi^*)\}_{n\ge 0}$ converge in $\mathcal{X}_{++}$ and the convergent points are fixed points of $\mathbb{W}$ in $\mathcal{X}_{++}$. Because the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ is unique, both $\{\mathbb{W}^n(r\Phi^*)\}_{n\ge 0}$ and $\{\mathbb{W}^n((1/r)\Phi^*)\}_{n\ge 0}$ converge to this fixed point, i.e., to Φ^*. By the squeeze theorem, $\{\mathbb{W}^n\Phi\}_{n\ge 0}$ converges to Φ^* as well.

Next, we consider the case in which $\varpi_{c,\theta}(x) < 0$ for some $x \in \mathbb{X}$ and some $(c,\theta) \in \mathcal{A}$. In this case, for any Φ such that $\mathbb{W}\Phi$ is well defined, we have

$$\mathbb{W}\Phi = \max_{(c,\theta)\in\mathcal{A}} c(x)\mathbb{T}_{c,\theta}(\Phi/c) \ge \max_{(c,\theta)\in\mathcal{A}} c(x)f_{0,c,\theta} = \Phi_0.$$

As a result, $\mathbb{W}^n\Phi \ge \mathbb{W}^{n-1}\Phi_0$, $n \ge 1$. On the other hand, consider the following operator:

$$\mathbb{W}_+\Phi := \max_{(c,\theta)\in\mathcal{A}} c(x)\mathbb{T}_{+,c,\theta}(\Phi/c),$$

where $\mathbb{T}_{+,c,\theta}$ is defined by replacing $\varpi_{c,\theta}$ in $\mathbb{T}_{c,\theta}$ with $\varpi^+_{c,\theta}$. We already showed that $\mathbb{W}_+\Phi$ has a unique fixed point in $\mathcal{X}_{++}$, and denote this fixed point as Φ^*_+. Because $\mathbb{X}$ is finite, there exists $r \ge 1$ such that $\Phi \le r\Phi^*_+$. Then,

$$\mathbb{W}(\Phi) \le \mathbb{W}(r\Phi^*_+) \le \mathbb{W}_+(r\Phi^*_+) \le r\mathbb{W}_+\Phi^*_+ = r\Phi^*_+,$$

where the first inequality is the case because $\mathbb{W}$ is increasing, the second inequality is the case because $\mathbb{W}_+$ dominates $\mathbb{W}$, the third inequality is the case because $\beta < 1$ and $\varpi^+_{c,\theta} \geq 0$ for all $(c, \theta) \in \mathcal{A}$, and the equality is the case because Φ^*_+ is the fixed point of $\mathbb{W}_+$. As a result, $\{\mathbb{W}^n(r\Phi^*_+)\}_{n\geq 1}$ is a decreasing sequence and dominates $\{\mathbb{W}^n(\Phi)\}_{n\geq 1}$ and thus dominates $\{\mathbb{W}^{n-1}(\Phi_0)\}_{n\geq 1}$. We already showed that $\{\mathbb{W}^{n-1}(\Phi_0)\}_{n\geq 1}$ converges to Φ^*, so $\{\mathbb{W}^n(r\Phi^*_+)\}_{n\geq 1}$ must converge in $\mathcal{X}_{++}$ and the convergent point is a fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$. Because the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ is unique, we conclude that $\{\mathbb{W}^n(r\Phi^*_+)\}_{n\geq 1}$ converges to Φ^* as well. By the squeeze theorem, we conclude that $\{\mathbb{W}^n\Phi\}_{n\geq 0}$ converges to Φ^*. $\square$

Portfolio Selection with the GH Model

In this appendix, we provide the general results of the existence and uniqueness of the total utility process in the GH model and portfolio selection results in this model.

Existence and Uniqueness of the Total Utility Process

Recall the GH model (2.2.9). Denote

$$c_t := C_t/W_t, \quad R_{t+1} := W_{t+1}/(W_t - C_t), \tag{0.12}$$

which stand for the fraction of the agent's wealth used for consumption and the gross return of the agent's investment, respectively. Denote

$$g_{i,t} := G_{i,t}/(W_t - C_t), \quad A_{t+1} := (1 - c_t)\left(1 + \sum_{i=1}^{n} \frac{b_i}{\mathcal{M}_t(R_{t+1})} g_{i,t}\right) R_{t+1}. \tag{0.13}$$

We impose the following assumption:

Assumption 9 *(i)* $\{(X_t, Y_t)\}$ *is a Markov process and the joint distribution of* (X_{t+1}, Y_{t+1}) *conditioned on* (X_t, Y_t) *depends only on* X_t*. Moreover, the state space of* $\{X_t\}$*, denoted as* $\mathbb{X}$*, is finite.*

(ii) $c_t > 0$ *and* $A_{t+1} > 0$*,* $t = 0, 1, \dots$

(iii) $\log(c_{t+1}) - \log(c_t) + \log A_{t+1} = \kappa(X_t, X_{t+1}, Y_{t+1}), \quad t = 0, 1, \dots$ *for some real-valued function* κ*. Moreover, for any state* x*,* $\mathbb{E}_t\left[u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})}\right)|X_t = x\right]$ *exists.*

Assumption 9-(i) posits that given $\{X_t\}$, $\{Y_t\}$ is an independent sequence. This assumption also implies that $\{X_t\}$ is a Markov process. Thus, we can consider $\{X_t\}$ to be the market state process and $\{Y_t\}$ to be an innovation process. Assumption 9-(ii) stipulates that A_{t+1} needs to be positive, which is equivalent to assuming

$$-\sum_{i=1}^{n} b_i g_{i,t} < \mathcal{M}_t(R_{t+1}). \tag{0.14}$$

This inequality means that the disutility of gains and losses experienced by the agent as a consequence of narrow framing cannot be too large, i.e., cannot exceed the certainty equivalent of the gross return on the agent's investment. Assumption-(iii) ensures a Markovian structure for the agent's consumption and investment.

In view of Assumption 9 and using the homogeneity of H, we can write (2.2.9) as

$$\frac{U_t}{C_t} = H\left(1, \mathcal{M}_t\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})}\frac{U_{t+1}}{C_{t+1}}\right)\right). \tag{0.15}$$

Because of the Markovian structure imposed in Assumption 9, we can focus on Markovian solutions to

(0.15), i.e., $U_t/C_t = f(X_t)$ for some function f. It is straightforward to see that f is a fixed point of

$$\mathbb{T}f(x) := H\left(1, u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}f(X_{t+1})\right)|X_t = x\right]\right)\right), \quad x \in \mathbb{X}. \tag{0.16}$$

Thus, to find the Markovian solution to (0.15), we only need to find the fixed point of $\mathbb{T}$. We consider the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$, the space of positive functions on $\mathbb{X}$, i.e., $\mathcal{X}_{++} := \{f|f(x) > 0, x \in \mathbb{X}\}$.[7]

The following quantity is crucial for the existence and uniqueness of the total utility process:

$$\delta = \max_{f\in\mathcal{X}_{++}} \min_{x\in\mathbb{X}} \frac{u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}f(X_{t+1})\right)|X_t = x\right]\right)}{f(x)}. \tag{0.17}$$

This quantity is related to the Perron-Frobenius eigenvalue of certain operator; see Proposition 1. We can see that when the state space $\mathbb{X}$ is singleton,

$$\delta = u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}\right)\right]\right),$$

which is nothing but the certainty equivalent of $\exp[\kappa(X_t, X_{t+1}, Y_{t+1})]$. Thus, in general δ can be regarded as a variant of the certainty equivalent of $\exp[\kappa(X_t, X_{t+1}, Y_{t+1})]$.

Theorem 5 *Suppose Assumption 9 holds and $\beta\delta^{1-\rho} < 1$. Then, the fixed point of $\mathbb{T}$ in $\mathcal{X}_{++}$ uniquely exists. Moreover, for any $f \in \mathcal{X}_{++}$, $\{\mathbb{T}^n f\}_{n\geq 0}$ converges to the fixed point.*

Proof of Theorem 5. This theorem is a direct consequence of Theorem 1.

[7]Note that $\mathbb{T}f$ is well defined for any nonnegative function f. However, as shown in Chapter 1, $\mathbb{T}$ can have multiple fixed points on the space of nonnegative functions. Moreover, because we assume the consumption to be positive (i.e., $c_t > 0$) and the disutility of losses cannot be too large (i.e., $A_{t+1} > 0$), it is reasonable for us to expect a positive total utility process.

Theorem 5 proves the existence and uniqueness of the fixed point of $\mathbb{T}$ and thus those of the total utility process in the GH model. Moreover, it provides a simple algorithm to compute the fixed point: one can start from any positive function, e.g., a positive constant function, to do iteration, and one can obtain a sequence that eventually converges to the fixed point.

Dynamic Programming

Suppose an agent can invest in a risk-free asset with gross return $R_{f,t+1}$ in period t to $t+1$ and n risky assets, indexed by $i = 1, \ldots, n$, with gross returns $R_{i,t+1}$, $i = 1, \ldots, n$, respectively, in period t to $t+1$. Suppose at time t, the agent has wealth W_t prior to any consumption at that time. The agent decides to consume $c_t W_t$, invests $\theta_{i,t}(1-c_{i,t})W_t$ in asset i, $i = 1, \ldots, n$, and thus the remaining amount $(1 - \sum_{i=1}^{n} \theta_{i,t})(1 - c_t)W_t$ in the risk-free asset. For this strategy, we assume the agent's total utility process is modeled by

$$U_t = H\left(c_t W_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} \frac{\mathcal{M}_t(U_{t+1})}{\mathcal{M}_t(W_{t+1})} b_i G_{i,t}\right) \tag{0.18}$$

where b_i's are nonnegative constants and

$$G_{i,t} := \mathbb{E}_t \left[\nu\left(\theta_{i,t}(1-c_t)W_t R_{i,t+1} - \theta_{i,t}(1-c_t)W_t R_{f,t+1}\right)\right].$$

In other words, the agent may frame the investment in asset i separately from other risks and evaluate it using the GH model.

With Assumption 9-(i) in force, we assume $R_{i,t+1} = r_i(X_t, X_{t+1}, Y_{t+1})$ for some function r_i, $i = 1, \ldots, n$, and $R_{f,t+1} = r_0(X_t)$ for some function r_0. In this Markovian setting, it is reasonable that the agent only considers Markovian strategies, i.e., $c_t = c(X_t)$ and $\theta_{i,t} = \theta_i(X_t)$, $i = 1, \ldots, n$, for some functions c and θ_i's. Denote $\boldsymbol{\theta}(x) := (\theta_1(x), \ldots, \theta_n(x))'$. Suppose the feasible set of strategies of the agent is

$$\mathcal{A} := \{(c, \boldsymbol{\theta}) | c(x) \in I_x, \boldsymbol{\theta}(x) \in J_x, x \in \mathbb{X}\},$$

where, for each $x \in \mathbb{X}$, I_x is a nonempty compact subset of $(0, 1)$ and J_x is a nonempty compact subset of $\mathbb{R}^n$. To highlight the dependence on the agent's strategy, we denote the total utility process in (0.18) as $\{U_t^{c,\boldsymbol{\theta}}\}$. The agent faces the following consumption-investment problem:

$$\max_{(c,\boldsymbol{\theta}) \in \mathcal{A}} U_t^{c,\boldsymbol{\theta}}. \tag{0.19}$$

We need to impose certain assumption on the feasible set $\mathcal{A}$ so that the total utility process for each feasible strategy uniquely exists. Denote

$$R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1}) := r_0(X_t) + \sum_{i=1}^{n} \theta_i(X_t)(r_i(X_t, X_{t+1}, Y_{t+1}) - r_0(X_t)) \tag{0.20}$$

as the gross return of the agent's portfolio and denote

$$g_i(x, \bar{\theta}_i) := \mathbb{E}_t \left[\nu \left(\bar{\theta}_i \big(r_i(x, X_{t+1}, Y_{t+1}) - r_0(x) \big) \right) | X_t = x \right], \quad \bar{\theta}_i \in \mathbb{R}, \quad i = 1, \ldots, n. \tag{0.21}$$

In view of Theorem 5, we impose the following:

Assumption 10 *(i) $\{(X_t, Y_t)\}$ is a Markov process and the joint distribution of (X_{t+1}, Y_{t+1}) conditioned on (X_t, Y_t) depends only on X_t. Moreover, the state space of $\{X_t\}$, denoted as $\mathbb{X}$, is finite.*

(ii) For each $(c, \boldsymbol{\theta}) \in \mathcal{A}$, $R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1})) > 0$ *and*

$$-\sum_{i=1}^{n} b_i g_i(X_t, \theta_i(X_t)) < \mathcal{M}_t(R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1})).$$

(iii) For each $(c, \boldsymbol{\theta}) \in \mathcal{A}$ *and any state* x, $\mathbb{E}_t\left[u\left(e^{\kappa_{c,\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1})}\right)|X_t = x\right]$ *exists, where*

$$\begin{aligned}
\kappa_{c,\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1}) &:= \log\left(c(X_{t+1})\right) - \log\left(c(X_t)\right) + \log\left(1 - c(X_t)\right) \\
&+ \log\left(\mathcal{M}_t(R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1})) + \sum_{i=1}^{n} b_i g_i(X_t, \theta_i(X_t))\right) \\
&+ \log\left(R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1})\right) - \log\left(\mathcal{M}_t(R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1}))\right).
\end{aligned} \tag{0.22}$$

(iv) For each $(c, \boldsymbol{\theta}) \in \mathcal{A}$, $\beta\left(\delta_{c,\boldsymbol{\theta}}\right)^{1-\rho} < 1$, *where*

$$\delta_{c,\boldsymbol{\theta}} := \max_{f \in \mathcal{X}_{++}} \min_{x \in \mathbb{X}} \frac{u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa_{c,\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1})} f(X_{t+1})\right)|X_t = x\right]\right)}{f(x)}. \tag{0.23}$$

With Assumption 10, for each $(c, \boldsymbol{\theta}) \in \mathcal{A}$, the total utility process $\{U_t^{c,\boldsymbol{\theta}}\}$ uniquely exists. Denote $\Phi(x)$ as the agent's optimal utility per unit capital, i.e.,

$$\Phi(x) = \max_{(c,\boldsymbol{\theta}) \in \mathcal{A}} \left(U_t^{c,\boldsymbol{\theta}} / W_t\right), \quad \text{given } X_t = x.$$

Then, we expect the following dynamic programming equation:

$$\Phi(x) = \mathbb{W}\Phi(x), \quad x \in \mathbb{X}, \tag{0.24}$$

where

$$\mathbb{W}\Phi(x) := \max_{\bar{c}\in I_x} H\left(\bar{c}, (1-\bar{c}) \max_{\bar{\boldsymbol{\theta}}\in J_x} D_\Phi(x,\bar{\boldsymbol{\theta}})\right), \quad x \in \mathbb{X}, \tag{0.25}$$

$$\begin{aligned} D_\Phi(x,\bar{\boldsymbol{\theta}}) &:= u^{-1}\Bigg(\mathbb{E}_t\Big[u\Big(\big(r_0(x) + \sum_{i=1}^{n} \bar{\theta}_i \left(r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)\right)\big)\Phi(X_{t+1})\Big)|X_t = x\Big]\Bigg) \\ &\quad \times \left(1 + \frac{\sum_{i=1}^n b_i g_i(x,\bar{\theta}_i)}{u^{-1}\Big(\mathbb{E}_t\big[u\big(r_0(x) + \sum_{i=1}^n \bar{\theta}_i \left(r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)\right)\big)|X_t = x\big]\Big)}\right). \end{aligned} \tag{0.26}$$

Theorem 6 *Suppose Assumption 10 holds. Then, the fixed point of $\mathbb{W}$ in $\mathcal{X}_{++}$ uniquely exists and $\{\mathbb{W}^n\Phi\}_{n\geq 0}$ converges to this fixed point for any $\Phi \in \mathcal{X}_{++}$. Moreover, if $(c^*(x), \boldsymbol{\theta}^*(x)) \in I_x \times J_x$ is a maximizer of the maximization in the dynamic programming equation (0.24), which must exist due to the compactness of I_x and J_x, $x \in \mathbb{X}$, then $(c^*, \boldsymbol{\theta}^*)$ is an optimal solution to (0.19).*

Proof of Theorem 6. The proof is exactly the same as for Propositions 2 and 3 in Chapter 1 in the case in which $\varpi_{c,\theta}$ therein is nonnegative for any (c,θ). $\square$

Theorem 6 shows the existence and uniqueness of the solution to the dynamic programming equation. Moreover, it shows that starting from any Φ that is positive and applying the dynamic programming equation repeatedly, one eventually obtains the solution to the equation.

The maximization of $H(\bar{c}, (1-\bar{c})z)$ in $\bar{c}$ for each fixed $z > 0$ is easy, but the maximization of $D_\Phi(x,\bar{\boldsymbol{\theta}})$ in $\bar{\boldsymbol{\theta}}$ for each fixed $\Phi > 0$ is not straightforward in general. When the state space of $\{X_t\}$ is singleton or when $\gamma = 1$, however, it is easy to solve the maximization of $D_\Phi(x,\bar{\boldsymbol{\theta}})$. Indeed, in these two cases, we have

$$D_\Phi(x,\bar{\boldsymbol{\theta}}) = \mathcal{M}_t(\Phi(X_{t+1}))\Big[\sum_{i=1}^n b_i g_i(x,\bar{\theta}_i) \\ + u^{-1}\Big(\mathbb{E}_t\big[u\big(r_0(x) + \sum_{i=1}^n \bar{\theta}_i\left(r_i(x,X_{t+1},Y_{t+1}) - r_0(x)\right)\big)|X_t = x\big]\Big)\Big]. \tag{0.27}$$

Because the certainty equivalent $\mathcal{M}_t$ is a concave functional and $g_i(x,\bar{\theta}_i)$ is concave in θ_i for each i, $D_\Phi(x,\bar{\boldsymbol{\theta}})$ is concave in $\bar{\boldsymbol{\theta}}$. Thus, it is straightforward to find the maximizer of $D_\Phi(x,\bar{\boldsymbol{\theta}})$ in $\bar{\boldsymbol{\theta}}$.

Finally, because in many applications of the GH model, the state space is assumed to be a singleton, for convenience we present the dynamic programming equation in this case. Note that in this case, any function f on $\mathbb{X}$ is actually a scalar. Moreover, the gross return of the risk-free asset is a constant R_f, the gross returns of the risky assets are i.i.d. over time, and the agent's consumption strategy c and portfolio strategy $\boldsymbol{\theta}$ are constants.

Corollary 8 *Suppose Assumption 10 holds and the state space $\mathbb{X}$ is a singleton. Then, the dynamic programming equation becomes*

$$\Phi = \max_{c\in I} H\left(c,(1-c)\Phi\Theta\right), \tag{0.28}$$

$$\Theta = \max_{\boldsymbol{\theta}\in J}\left(u^{-1}\Big(\mathbb{E}\big[u\big(R_f + \sum_{i=1}^n \theta_i\left(R_{i,t+1} - R_f\right)\big)\big]\Big) + \sum_{i=1}^n b_i g_i(\theta_i)\right), \tag{0.29}$$

where I and J are the feasible sets of c and $\boldsymbol{\theta}$, respectively, and $g_i(\theta_i) := \mathbb{E}\big[\nu\big(\theta_i(R_{t+1} - R_f)\big)\big]$.

Dynamic Programming in the GH-I Model

We consider the GH-I model (2.2.16). Let us set aside the issue of the existence and uniqueness of the total utility process in this model. Recall the market setting in the portfolio selection problem considered in Section 3.6 and suppose the agent's preferences are represented by the GH-I model. We can write down the dynamic programming equation heuristically:

$$\Phi(x) = \tilde{\mathbb{W}}\Phi(x), \quad x \in \mathbb{X}, \tag{0.30}$$

where

$$\tilde{\mathbb{W}}\Phi(x) := \max_{\bar{c}\in I_x} H\left(\bar{c}, (1-\bar{c}) \max_{\bar{\boldsymbol{\theta}}\in J_x} \tilde{D}_\Phi(x, \bar{\boldsymbol{\theta}})\right), \quad x \in \mathbb{X}, \tag{0.31}$$

$$\begin{aligned}\tilde{D}_\Phi(x, \bar{\boldsymbol{\theta}}) &:= u^{-1}\Bigg(\mathbb{E}_t\Big[u\Big(\big(r_0(x) + \sum_{i=1}^n \bar{\theta}_i\left(r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)\right)\big)\Phi(X_{t+1})\Big)|X_t = x\Big]\Bigg) \\ &\quad + u^{-1}\Bigg(\mathbb{E}_t\Big[u\Big(\Phi(X_{t+1})\Big)|X_t = x\Big]\Bigg)\sum_{i=1}^n b_i g_i(x, \bar{\theta}_i).\end{aligned} \tag{0.32}$$

Clearly, for each $x \in \mathbb{X}$ and Φ, $\tilde{D}_\Phi(x, \bar{\boldsymbol{\theta}})$ is concave in $\bar{\boldsymbol{\theta}}$, so in general it is easier to maximize $\tilde{D}_\Phi(x, \bar{\boldsymbol{\theta}})$ than to maximize $D_\Phi(x, \bar{\boldsymbol{\theta}})$ in $\bar{\boldsymbol{\theta}}$. However, neither the existence and uniqueness of the total utility process in the GH-I model nor those of the solution to the dynamic programming equation (0.30) are known.

Analysis of the GH-II Model

We consider the GH-II model (2.2.17). We first establish the existence and uniqueness of the total utility process in this model. To this end, recall c_t, R_{t+1}, and $g_{i,t}$ as defined in (0.12) and (0.13), and define

$$\hat{A}_{t+1} := (1 - c_t) \left(1 + \sum_{i=1}^{n} b_i g_{i,t}\right) R_{t+1}. \tag{0.33}$$

Suppose Assumption 9 holds with A_{t+1} replaced by $\hat{A}_{t+1}$, denote κ therein as $\hat{\kappa}$, i.e., $\hat{\kappa}(X_t, X_{t+1}, Y_{t+1}) = \log(c_{t+1}) - \log(c_t) + \log \hat{A}_{t+1}$, denote $\mathbb{T}$ in (0.16) with κ replaced by $\hat{\kappa}$ as $\hat{\mathbb{T}}$, and denote δ defined in (0.17) with κ replaced by $\hat{\kappa}$ as $\hat{\delta}$. Following the same argument as in Section 3.6, we can focus on Markovian solutions to (2.2.17), i.e., $U_t/C_t = f(X_t)$ for some function f, and U_t/C_t is a solution to (2.2.17) if and only if f is a fixed point of $\hat{\mathbb{T}}$. The existence and uniqueness of the fixed point of $\hat{\mathbb{T}}$ can be established as follows:

Corollary 9 *Suppose Assumption 9 holds with A_{t+1} replaced by $\hat{A}_{t+1}$ and $\beta\hat{\delta}^{1-\rho} < 1$. Then, the fixed point of $\hat{\mathbb{T}}$ in $\mathcal{X}_{++}$ uniquely exists. Moreover, for any $f \in \mathcal{X}_{++}$, $\{\hat{\mathbb{T}}^n f\}_{n\geq 0}$ converges to the fixed point.*

Consider the same portfolio selection setting as in Section 3.6. Define

$$\begin{aligned}\hat{\kappa}_{c,\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1}) &:= \log(c(X_{t+1})) - \log(c(X_t)) + \log(1 - c(X_t)) \\ &\quad + \log\left(1 + \sum_{i=1}^{n} b_i g_i(X_t, \theta_i(X_t))\right) + \log R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1}).\end{aligned} \tag{0.34}$$

Consider the following dynamic programming equation:

$$\Phi(x) = \hat{\mathbb{W}}\Phi(x), \quad x \in \mathbb{X}, \tag{0.35}$$

where

$$\hat{\mathbb{W}}\Phi(x) := \max_{\bar{c}\in I_x} H\left(\bar{c}, (1-\bar{c})\max_{\bar{\boldsymbol{\theta}}\in J_x} \hat{D}_\Phi(x,\bar{\boldsymbol{\theta}})\right), \quad x \in \mathbb{X}, \tag{0.36}$$

$$\hat{D}_\Phi(x,\bar{\boldsymbol{\theta}}) := u^{-1}\left(\mathbb{E}_t\left[u\left(\left(r_0(x) + \sum_{i=1}^n \bar{\theta}_i\left(r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)\right)\right)\Phi(X_{t+1})\right)|X_t = x\right]\right)$$
$$\times\left(1 + \sum_{i=1}^n b_i g_i(x, \bar{\theta}_i)\right). \tag{0.37}$$

Corollary 10 *Consider the same portfolio selection setting as in Section 3.6 and assume the agent's preferences are represented by the GH-II model (2.2.17). Suppose Assumption 10 holds with $\kappa_{c,\boldsymbol{\theta}}$ replaced by $\hat{\kappa}_{c,\boldsymbol{\theta}}$. Then, the fixed point of $\hat{\mathbb{W}}$ in $\mathcal{X}_{++}$ uniquely exists and $\{\hat{\mathbb{W}}^n\Phi\}_{n\geq 0}$ converges to this fixed point for any $\Phi \in \mathcal{X}_{++}$. Moreover, if $(c^*(x), \boldsymbol{\theta}^*(x)) \in I_x \times J_x$ is a maximizer of the maximization in the dynamic programming equation (0.35), which must exist due to the compactness of I_x and J_x, $x \in \mathbb{X}$, then $(c^*, \boldsymbol{\theta}^*)$ is an optimal solution of the agent's portfolio selection problem.*

Note that the maximization of $\hat{D}_\Phi(x,\bar{\boldsymbol{\theta}})$ in $\boldsymbol{\theta}$ is more complicated than that of $D_\Phi(x,\bar{\boldsymbol{\theta}})$ taking a special form in (0.27), showing that the optimal portfolio in the GH-II model is more complicated to solve than in the GH-model when the state space of $\{X_t\}$ is a singleton or when $\gamma = 1$. In general, however, $D_\Phi(x,\bar{\boldsymbol{\theta}})$ takes the general form (0.26), so it is unclear which of $D_\Phi(x,\bar{\boldsymbol{\theta}})$ and $\hat{D}_\Phi(x,\bar{\boldsymbol{\theta}})$ is easier to maximize.

Probability Weighting, Reference Points, and S-Shaped Utility Functions

In the main text and the previous sections, we followed Barberis and Huang (2009) to assume in the GH model that the agent evaluates $\xi_{i,t+1}$, the gain and loss experienced by the agent in risk i in period from time t to $t+1$, according to CPT without probability weighting. In the portfolio selection context, we also assumed that the agent uses the risk-free gross return of his initial wealth as the reference point to calculate the gain and loss experienced in the investment in each individual stock. These two assumptions, however, are for ease of expositions only and can be removed easily. Moreover, we can incorporate diminishing sensitivity with respect to gains and losses, another important feature of CPT, in the GH model as well.

Consider the following piece-wise power utility function

$$\bar{v}(x) = x^{\alpha}\mathbf{1}_{\{x\geq 0\}} - \lambda(-x)^{\alpha}\mathbf{1}_{\{x<0\}}, \tag{0.38}$$

where $\alpha \in (0,1]$. Consider, $T_{\pm}$, two increasing and continuous mappings from $[0,1]$ onto $[0,1]$, that represent the probability weighting functions with respect to gains and losses, respectively. Denote

$$\mathcal{E}_t(Z) := \int_0^{\infty} zd\left[-T_+\big(\mathbb{P}(Z > z|\mathcal{F}_t)\big)\right] + \int_{-\infty}^{0} zd\left[T_-\big(\mathbb{P}(Z \leq z|\mathcal{F}_t)\big)\right]. \tag{0.39}$$

We model the utility of the gain and loss $\xi_{i,t+1}$ as

$$G_{i,t} = \text{sign}\left(\mathcal{E}_t\big(\bar{\nu}(\xi_{i,t+1})\big)\right)\left|\mathcal{E}_t\big(\bar{\nu}(\xi_{i,t+1})\big)\right|^{1/\alpha}, \tag{0.40}$$

where $\text{sign}(x)$ stands for the sign of x. It is straightforward to verify that $G_{i,t}$ is positively homogeneous in $\xi_{i,t+1}$; i.e., for any positive quantity a_t that is known at time t, if $\xi_{i,t+1}$ is scaled by a_t, so is $G_{i,t}$.

In the portfolio selection context, we consider a general reference gross return rate $R_{\text{rp},i,t+1}$, which can even be random from time t's perspective, in the evaluation of the investment in stock i. Then,

$$\xi_{i,t+1} = (W_t - C_t)\theta_{i,t}(R_{i,t+1} - R_{\text{rp},i,t+1}). \tag{0.41}$$

Recalling $g_{i,t}$ as defined in (0.13) and thanks to the positive homogeneity of $G_{i,t}$ in $\xi_{i,t+1}$, we have

$$g_{i,t} = \text{sign}\Big(\mathcal{E}_t\big(\bar{\nu}\big(\theta_{i,t}(R_{i,t+1} - R_{\text{rp},i,t+1})\big)\big)\Big)\Big|\mathcal{E}_t\big(\bar{\nu}\big(\theta_{i,t}(R_{i,t+1} - R_{\text{rp},i,t+1})\big)\big)\Big|^{1/\alpha}. \tag{0.42}$$

With Assumption 9-(i) in place and assuming $R_{i,t+1} = r_i(X_t, X_{t+1}, Y_{t+1})$ and $\theta_{i,t} = \theta_i(X_t)$ as in Section 3.6 and $R_{\text{rp},i,t+1} = r_{\text{rp},i}(X_t, X_{t+1}, Y_{t+1})$ for some deterministic function $r_{\text{rp},i}$, we can see that $g_{i,t}$ is a function of X_t. Thus, Theorem 5 can still be applied to show the existence and uniqueness of the total utility process in the case of probability weighting, piece-wise power utility functions, and general reference points. Moreover, Theorem 6 can also be applied in this case with $g_i(x, \bar{\theta}_i)$ in (0.21) replaced by

$$\begin{aligned} g_i(x,\bar{\theta}_i) := \text{sign}\,\Big(\mathcal{E}_t\Big(\bar{\nu}\big(\bar{\theta}_i\big(r_i(x, X_{t+1}, Y_{t+1}) - r_{\text{rp},i}(x, X_{t+1}, Y_{t+1})\big)\big)\Big)\Big) \\ \times\Big|\mathcal{E}_t\Big(\bar{\nu}\big(\bar{\theta}_i\big(r_i(x, X_{t+1}, Y_{t+1}) - r_{\text{rp},i}(x, X_{t+1}, Y_{t+1})\big)\big)\Big)\Big|^{1/\alpha}, \\ \bar{\theta}_i \in \mathbb{R},\ \text{given } X_t = x. \end{aligned} \tag{0.43}$$

Note that $g_i(x, \bar{\theta}_i)$ is linear in $\theta_i \geq 0$ and in $\theta_i \leq 0$, so the inclusion of probability weighting, piece-wise power utility functions, and general reference points actually does not increase the complexity of solving the dynamic programming equation in Theorem 6. Consequently, the GH model is still tractable in this case.

The above idea of incorporating probability weighting arises from De Giorgi and Legg (2012). Note that the positive homogeneity of $G_{i,t}$ in $\xi_{i,t+1}$ is crucial in the above argument of the validity of Theorems 5 and 6 in the case of probability weighting, piece-wise power utility functions, and general reference points. For this reason, we model $G_{i,t}$ as in (0.40), which is different from De Giorgi and Legg (2012) wherein $G_{i,t}$ is modeled as $\mathcal{E}_t\big(\bar{\nu}(\xi_{i,t+1})\big)$.

He and Zhou (2014) consider a portfolio selection problem in which an agent derives utility from consumption and from the gain and loss experienced in her *investment portfolio*. The GH model can be applied to such problems as well. Recall the portfolio gross return $R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1})$ as in (0.20). Suppose the reference gross return for the agent's portfolio is $r_{\mathrm{rp}}(X_t, X_{t+1}, Y_{t+1})$. Then, the gain and loss experienced by the agent is

$$\xi_{t+1} = (W_t - C_t)\big(R_{\boldsymbol{\theta}}(X_t, X_{t+1}, Y_{t+1}) - r_{\mathrm{rp}}(X_t, X_{t+1}, Y_{t+1})\big),$$

and denote the corresponding utility that is modeled in (0.40) as G_t. Because G_t is positively homogeneous in ξ_{t+1}, it is not difficult to see that Theorems 5 and 6 are still valid in this case.

Proofs for Chapter 2

Proof of Proposition 7. Straightforward calculation yields

$$\mathbb{V}'(\Psi) = \delta\beta\big(H(c/(\delta\Psi + c\bar{b}\zeta), 1)\big)^{\rho} = \delta\beta\big(\mathbb{V}(\Psi)/(\delta\Psi + c\bar{b}\zeta)\big)^{\rho}, \tag{0.44}$$

so one can easily see that $\mathbb{V}$ is strictly increasing and concave. Moreover,

$$\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta} \mathbb{V}(\Psi) = \begin{cases} c(1-\beta)^{1/(1-\rho)}, & \rho < 1, \\ 0, & \rho \geq 1, \end{cases} \quad \lim_{\Psi\uparrow +\infty} \mathbb{V}(\Psi) = \begin{cases} +\infty, & \rho \leq 1, \\ c(1-\beta)^{1/(1-\rho)}, & \rho > 1, \end{cases}$$

$$\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta} \mathbb{V}'(\Psi) = \begin{cases} +\infty, & \rho \leq 1, \\ \left(\beta\delta^{1-\rho}\right)^{1/(1-\rho)}, & \rho > 1, \end{cases} \quad \lim_{\Psi\uparrow +\infty} \mathbb{V}'(\Psi) = \begin{cases} \left(\beta\delta^{1-\rho}\right)^{1/(1-\rho)}, & \rho < 1, \\ 0, & \rho \geq 1. \end{cases}$$

If $\rho < 1$ and $-\bar{b}\zeta/\delta < (1-\beta)^{1/(1-\rho)}$, we have $\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta} \mathbb{V}(\Psi) > -c\bar{b}\zeta/\delta$, $\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta} \mathbb{V}'(\Psi) = +\infty$, and $\lim_{\Psi\uparrow +\infty} \mathbb{V}'(\Psi) < 1$, so by the concavity of $\mathbb{V}$, we can see that the solution to (2.2.8) uniquely exists.

If $\rho < 1$ and $-\bar{b}\zeta/\delta > (1-\beta)^{1/(1-\rho)}$, we have $\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta} \mathbb{V}(\Psi) < -c\bar{b}\zeta/\delta$, $\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta} \mathbb{V}'(\Psi) = +\infty$, and $\lim_{\Psi\uparrow +\infty} \mathbb{V}'(\Psi) < 1$. In this case, the line starting from $(0,0)$ and tangent to $\mathbb{V}(\Psi)$ uniquely exists and the tangent point Ψ_0 solves $\mathbb{V}(\Psi_0) = \mathbb{V}(\Psi_0)'\Psi_0$. Straightforward yields that

$$\mathbb{V}'(\Psi_0) = \delta(1-\beta)^{1/(1-\rho)} \left[(-\bar{b}\zeta)^{-(1-\rho)/\rho} + \left(\frac{\beta}{1-\beta}\right)^{1/\rho} \right]^{\rho/(1-\rho)}.$$

Because $\delta\beta^{1/(1-\rho)} < 1$, we conclude that $\mathbb{V}'(\Psi_0) > 1$ if and only if

$$-\bar{b}\zeta/\delta < (1-\beta)^{1/(1-\rho)} \left[1 - \left(\beta\delta^{1-\rho}\right)^{1/\rho} \right]^{-\rho/(1-\rho)}.$$

In this case, because $\mathbb{V}$ is strictly concave, $\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta} \mathbb{V}(\Psi) < -c\bar{b}\zeta/\delta$, and $\lim_{\Psi\uparrow +\infty} \mathbb{V}'(\Psi) < 1$, we conclude that there exists two fixed points of $\mathbb{V}$ on $[-c\bar{b}\zeta/\delta, +\infty)$, and we denote them as $\Psi_1^* < \Psi_2^*$. Moreover, $\Psi_1^* > -c\bar{b}\zeta/\delta$, $\mathbb{V}'(\Psi_1^*) > 1$, and $\mathbb{V}(\Psi) < \Psi$ for any $\Psi < \Psi_1^*$. In consequence, for such Ψ,

$\{\mathbb{V}^n(\Psi)\}$ is a strictly decreasing sequence. We claim that there exists n_0 such that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$. Otherwise, the limit of $\{\mathbb{V}^n(\Psi)\}$ exists in $[-c\bar{b}\zeta/\delta, +\infty)$ and thus must be a fixed point of $\mathbb{V}$ by the continuity of $\mathbb{V}$, and this contradicts with the fact that Ψ_1^* and Ψ_2^* are the only two fixed points of $\mathbb{V}$. On the other hand, $\mathbb{V}'(\Psi_2^*) < 1$, $\mathbb{V}(\Psi) > \Psi$ for $\Psi \in (\Psi_1^*, \Psi_2^*)$, and $\mathbb{V}(\Psi) < \Psi$ for $\Psi > \Psi_2^*$. Thus, $\{\mathbb{V}^n(\Psi)\}_{n\geq 1}$ converges to Ψ_2^* for any $\Psi > \Psi_1^*$.

Similarly, $\mathbb{V}'(\Psi_0) < 1$ if and only if

$$-\bar{b}\zeta/\delta > (1-\beta)^{1/(1-\rho)}\left[1 - \left(\delta\beta^{1/(1-\rho)}\right)^{(1-\rho)/\rho}\right]^{-\rho/(1-\rho)}.$$

Because $\mathbb{V}$ is strictly concave, $\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta}\mathbb{V}(\Psi) < -c\bar{b}\zeta/\delta$, and $\lim_{\Psi\uparrow+\infty}\mathbb{V}'(\Psi) < 1$, we conclude that there does not exist any fixed point of $\mathbb{V}$ on $[-c\bar{b}\zeta/\delta, +\infty)$. Using the same argument as in the case in which $\mathbb{V}'(\Psi_0) > 1$, we can show that for any $\Psi > -c\bar{b}\zeta/\delta$, there exists n_0 such that $\mathbb{V}^{n_0}(\Psi) < -c\bar{b}\zeta/\delta$.

Finally, we consider the case in which $\rho \geq 1$. In this case, $\lim_{\Psi\downarrow -c\bar{b}\zeta/\delta}\mathbb{V}(\Psi) = 0 < -c\bar{b}\zeta/\delta$. Again, the line starting from $(0,0)$ and tangent to $\mathbb{V}(\Psi)$ uniquely exists and the tangent point Ψ_0 solves $\mathbb{V}(\Psi_0) = \mathbb{V}(\Psi_0)'\Psi_0$. Straightforward yields that

$$\mathbb{V}'(\Psi_0) = \begin{cases} \delta(1-\beta)^{1/(1-\rho)}\left[(-\bar{b}\zeta)^{-(1-\rho)/\rho} + \left(\frac{\beta}{1-\beta}\right)^{1/\rho}\right]^{\rho/(1-\rho)} & \rho > 1, \\ (-\bar{b}\zeta)^{-(1-\beta)}\delta(1-\beta)^{1-\beta}\beta^{\beta}, & \rho = 1. \end{cases}$$

Then, (iv) and (v) follow from the same argument as in the proof of (i)–(iii). □

Proof of Proposition 8. By definition, Ψ^* is a fixed point of $\mathbb{V}$. We then examine $\mathbb{V}'(\Psi^*)$. Straightforward calculation yields

$$\mathbb{V}'(\Psi^*) = \delta\beta\left(\mathbb{V}(\Psi^*)/(\delta\Psi^* + c\bar{b}\zeta)\right)^{\rho} = \delta\beta\left(\Psi^*/(\delta\Psi^* + cb\Psi^*\zeta)\right)^{\rho} = \frac{\delta}{\delta + cb\zeta}\beta(\delta + cb\zeta)^{1-\rho}.$$

Thus, $\mathbb{V}'(\Psi^*) < 1$ if and only if $cb\zeta < (\delta\beta)^{1/\rho} - \delta$. In this case, $\mathbb{W}$ in Proposition 7 has two fixed points $\Psi_1^* < \Psi_2^*$ and $\Psi^* = \Psi_1^*$ because $\mathbb{V}'(\Psi_1^*) < 1$ and $\mathbb{V}'(\Psi_2^*) > 1$. Similarly, $\mathbb{V}'(\Psi^*) > 1$ if and only if $cb\zeta > (\delta\beta)^{1/\rho} - \delta$, in which case $\mathbb{W}$ in Proposition 7 has two fixed points $\Psi_1^* < \Psi_2^*$ and $\Psi^* = \Psi_2^*$. □

Proof of Proposition 9. Denote the right-hand side of (2.2.14) as $\mathbb{W}(\Phi)$. First, because $0 \in J$, $\mathbb{W}(\Phi)$ is well defined for any $\Phi \geq 0$. Moreover, it is obvious that $\mathbb{W}(\Phi)$ is continuous in Φ.

Next, because $0 \in J$, we have $\mathbb{W}(\Phi) \geq H\left(c_0, (1 - c_0)\Phi R_f\right)$ for some fixed $c_0 \in I$. In consequence, by the homogeneity of H,

$$\liminf_{\Phi \downarrow 0}\left(\mathbb{W}(\Phi)/\Phi\right) \geq \liminf_{\Phi \downarrow 0} H\left(c_0/\Phi, (1 - c_0)R_f\right) = \begin{cases} +\infty, & \rho \leq 1, \\ \beta^{1/(1-\rho)}(1 - c_0)R_f, & \rho > 1. \end{cases}$$

In consequence, when $\rho \leq 1$, we have $\liminf_{\Phi\downarrow 0}\left(\mathbb{W}(\Phi)/\Phi\right) = +\infty > 1$; when $\rho > 1$, (2.2.15) implies that $\beta\big((1 - c_0)R_f\big)^{1-\rho} < 1$, so $\liminf_{\Phi\downarrow 0}\left(\mathbb{W}(\Phi)/\Phi\right) = \left[\beta\big((1 - c_0)R_f\big)^{1-\rho}\right]^{1/(1-\rho)} > 1$. Thus, we conclude that $\mathbb{W}(\Phi) > \Phi$ when Φ is sufficiently small.

Finally, we show $\mathbb{W}(\Phi) < \Phi$ when Φ is sufficiently large so that the fixed point of $\mathbb{W}$ exists. Note that because $I \subset (0, 1)$ is compact, there exists $\bar{c} \in (0, 1)$ such that $\bar{c} \geq c, \forall c \in I$. When $\rho \geq 1$, because I and J are compact, there exists $a > 0$ such that

$$(1 - c)\max_{\theta \in J}\left\{\mathcal{M}\big(R_f + \theta(R_{S,t+1} - R_f)\big) + \bar{b}\theta\mathbb{E}[\nu(R_{S,t+1} - R_f)]/\Phi\right\} < a, \quad \forall c \in I$$

for sufficiently large Φ. In consequence,

$$\limsup_{\Phi\uparrow+\infty}\big(\mathbb{W}(\Phi)/\Phi\big)\le\limsup_{\Phi\uparrow+\infty}\max_{c\in I}H\left(c/\Phi,a\right)\le\limsup_{\Phi\uparrow+\infty}H\left(\bar{c}/\Phi,a\right)=0.$$

When $\rho<1$, because of condition (2.2.15) and the compactness of J, there exists $\alpha<\beta^{-1/(1-\rho)}$ such that

$$(1-c)\max_{\theta\in J}\big\{\mathcal{M}\big(R_f+\theta(R_{S,t+1}-R_f)\big)+\bar{b}\theta\mathbb{E}[\nu(R_{S,t+1}-R_f)]/\Phi\big\}<\alpha,\quad\forall c\in I$$

when Φ is sufficiently large. In consequence,

$$\limsup_{\Phi\uparrow+\infty}\big(\mathbb{W}(\Phi)/\Phi\big)\le\limsup_{\Phi\uparrow+\infty}\max_{c\in I}H\left(c/\Phi,\alpha\right)\le\limsup_{\Phi\uparrow+\infty}H\left(\bar{c}/\Phi,\alpha\right)=\beta^{1/(1-\rho)}\alpha<1.$$

Therefore, we conclude that for any value of ρ, $\mathbb{W}(\Phi)<\Phi$ for sufficiently large Φ. □

Proof of Proposition 10. Denote $\delta:=1/W_t$. From (2.3.1), we only need to investigate when

$$f(\delta):=\mathcal{M}\left(1+\delta\xi\right)+bG\left(\delta\xi\right),\quad\delta\ge0$$

is strictly larger than one.

Because $\mathcal{M}(\cdot)$ is strictly concave and $G(\delta\xi)$ is linear in $\delta\ge0$, we conclude that f is strictly concave in δ. Moreover, $f(0)=1$ and $f'(0)=\mathbb{E}[\xi]+bG(\xi)$. In consequence, if $f'(0)\le0$, which is the case if and only if $\lambda\ge1+\big(1+1/b\big)\big(\mathbb{E}[\xi]/\mathbb{E}[\xi_-]\big)$, we have $f(\delta)<1$ for any $\delta>0$, so the agent rejects ξ at any wealth level W_t.

If $f'(0)>0$, which is the case if and only if $\lambda<1+\big(1+1/b\big)\big(\mathbb{E}[\xi]/\mathbb{E}[\xi_-]\big)$, there exists unique

$\delta^* > 0$ that solves $f(\delta^*) = 1$ such that $f(\delta) \geq 1$ for $\delta \leq \delta^*$. Note that $\delta^* = 1/W^*$, where W^* solves $\mathcal{M}(1 + \xi/W^*) + b\mathbb{E}[\nu(\xi/W^*)] = 0$, so the proof completes. $\square$

Proof of Proposition 11. Denote the constant consumption-wealth ratio as $1/c$, i.e., the agent consumes a constant fraction c of her wealth at each time. In equilibrium, the net supply of the risk-free asset is zero and the total consumption at time t is C_t, so $S_t + N_t = W_t - C_t = (1-c)W_t$, $\forall t$. On the other hand, because $S_t/(S_t + N_t) = \iota$, we have $N_t/S_t = 1/\iota - 1$ and $S_t/W_t = \iota(1-c)$.

Denote $C_{N,t}$ and $C_{S,t}$ as the consumption good payout by the non-financial asset and by the stock, respectively, at time t. Then,

$$\begin{aligned} R_{N,t+1} := \frac{N_{t+1} + C_{N,t+1}}{N_t} &= \frac{W_{t+1} - C_{t+1} - S_{t+1} + C_{t+1} - C_{S,t+1}}{N_t} = \frac{W_{t+1} - (S_{t+1} + C_{S,t+1})}{N_t} \\ &= \frac{(W_{t+1}/W_t)(W_t/S_t) - R_{S,t+1}}{N_t/S_t} = \frac{(C_{t+1}/C_t)/\big(\iota(1-c)\big) - R_{S,t+1}}{1/\iota - 1}. \end{aligned}$$

Because $\big(\log(C_{t+1}/C_t), \log(R_{S,t+1})\big)$'s are i.i.d. over time, so are $\big(\log(R_{N,t+1}), \log(R_{S,t+1})\big)$'s.

Now, consider the optimal consumption and portfolio selection problem faced by the agent. Because the gross returns of the assets are i.i.d. over time, the agent's optimal utility per unit wealth is a constant, and we denote it by Ψ. Then, we have the following dynamic programming equation:

$$\Phi = \max_{c \in I} H\left(c, (1-c) \max_{\theta_N \in J_N, \theta_S \in J_S} D_\Phi(\theta_N, \theta_S)\right),$$

where I is a subinterval of $(0,1)$ and J_N and J_S are two subintervals of $\mathbb{R}$ that specify the feasible set of the agent's strategies, and

$$D_\Phi(\theta_N, \theta_S) = \Phi\Big[\mathcal{M}\big(R_f + \theta_N\left(R_{N,t+1} - R_f\right) + \theta_S\left(R_{S,t+1} - R_f\right)\big) + bG\big(\theta_S(R_{S,t+1} - R_f)\big)\Big];$$

see Corollary 8. Because the right-hand side of the dynamic programming is strictly concave in c, θ_N and θ_S, the first-order condition is necessary and sufficient for the optimality of c^*, θ_N^* and θ_S^*. Here, we assume I, J_N, and J_S to contain c^*, θ_N^*, and θ_S^*, respectively.

Because we assume the net supplies of the non-financial asset and of the stock are positive, in equilibrium the optimal θ_N^* and θ_S^* must be positive as well. Then, taking the first order derivatives of $D_\Phi(\theta_N, \theta_S)$ with respect to θ_N and θ_S, respectively, we have

$$\frac{\mathbb{E}\left[u'\left(R_{p,t+1}\right)\left(R_{N,t+1}-R_f\right)\right]}{u'\left(\mathcal{M}\left(R_{p,t+1}\right)\right)}=0, \tag{0.45}$$

$$\frac{\mathbb{E}\left[u'\left(R_{p,t+1}\right)\left(R_{S,t+1}-R_f\right)\right]}{u'\left(\mathcal{M}\left(R_{p,t+1}\right)\right)}+bG\left(R_{S,t+1}-R_f\right)=0, \tag{0.46}$$

where $R_{p,t+1} := R_f + \theta_N^*\left(R_{N,t+1}-R_f\right) + \theta_S^*\left(R_{S,t+1}-R_f\right)$ is the optimal portfolio return. Multiplying θ_N^* on both sides of (0.45), and noting that $\theta_N^*(R_{N,t+1}-R_f) = R_{p,t+1} - R_f - \theta_S^*(R_{S,t+1}-R_f)$, we conclude

$$\frac{\mathbb{E}\left[u'\left(R_{p,t+1}\right)R_{p,t+1}\right]}{u'\left(\mathcal{M}\left(R_{p,t+1}\right)\right)}=(1-\theta_S^*)R_f\frac{\mathbb{E}\left[u'\left(R_{p,t+1}\right)\right]}{u'\left(\mathcal{M}\left(R_{p,t+1}\right)\right)}+\theta_S^*\frac{\mathbb{E}\left[u'\left(R_{p,t+1}\right)R_{S,t+1}\right]}{u'\left(\mathcal{M}\left(R_{p,t+1}\right)\right)}. \tag{0.47}$$

Note that in equilibrium, the gross return of the optimal portfolio is the same as the growth rate of the wealth, i.e.,

$$R_{p,t+1}=\frac{W_{t+1}}{W_t-C_t}=\frac{1}{1-c^*}\frac{C_{t+1}}{C_t}.$$

Moreover, $\theta_N^*+\theta_S^*=1$ and $\theta_S^*=\iota$. In consequence, we conclude (2.5.1) and (2.5.2) from (0.46) and (0.47) immediately. Moreover, we have

$$
\begin{aligned}
D_\Phi(\theta_N^*, \theta_S^*) &= \Phi\left[\mathcal{M}(R_{p,t+1}) + b\iota G(R_{S,t+1} - R_f)\right] \\
&= \Phi\Bigg[\frac{1}{1-c^*}\mathcal{M}\left(\frac{C_{t+1}}{C_t}\right) - \iota\frac{\mathbb{E}\left[u'\big(C_{t+1}/C_t\big)R_{S,t+1}\right]}{u'\left(\mathcal{M}\big(C_{t+1}/C_t\big)\right)} - \frac{\iota^2}{1-\iota}\frac{\mathbb{E}\left[u'\big(C_{t+1}/C_t\big)R_{S,t+1}\right]}{u'\left(\mathcal{M}\big(C_{t+1}/C_t\big)\right)} \\
&\quad + \frac{\iota}{1-\iota}\frac{1}{1-c^*}\frac{\mathbb{E}\left[u'\big(C_{t+1}/C_t\big)\big(C_{t+1}/C_t\big)\right]}{u'\left(\mathcal{M}\big(C_{t+1}/C_t\big)\right)}\Bigg].
\end{aligned} \tag{0.48}
$$

Note that

$$
\Phi = \max_{c\in I} H\left(c, (1-c)D_\Phi(\theta_N^*, \theta_S^*)\right), \quad c^* = \mathrm{argmax}_{c\in I} H\left(c, (1-c)D_\Phi(\theta_N^*, \theta_S^*)\right). \tag{0.49}
$$

Straightforward calculation yields that for each $z > 0$,

$$
\mathrm{argmax}_{c\in(0,1)} H\left(c, (1-c)z)\right) = 1/\left[1 + (\beta/(1-\beta))^{1/\rho} z^{(1-\rho)/\rho}\right], \tag{0.50}
$$

$$
\max_{c\in(0,1)} H\left(c, (1-c)z)\right) = (1-\beta)\left[1 + (\beta/(1-\beta))^{1/\rho} z^{(1-\rho)/\rho}\right]^{\rho}. \tag{0.51}
$$

Recalling $D_\Phi(\theta_N^*, \theta_S^*)$ in (0.48) and setting $z = D_\Phi(\theta_N^*, \theta_S^*)$ in (0.50) and (0.51), we can solve Φ and c^*, provided that I contains c^*. In particular,

$$
1 - c^* = \left[\beta\big(D_\Phi(\theta_N^*, \theta_S^*)/\Phi\big)^{1-\rho}\right]^{1/\rho},
$$

which yields (2.5.3).

Finally, according to Theorem 5, for the total utility process of the agent to exist when she takes the optimal consumption and investment strategy, we need to assume $-b\theta_S^* G(R_{S,t+1} - R_f) < \mathcal{M}(R_{p,t+1})$ and $\beta\delta^{1-\rho} < 1$, where $\delta := (1-c^*)\big[\mathcal{M}(R_{p,t+1}) + b\theta_S^* G(R_{S,t+1} - R_f)\big]$. Recalling $(1-c^*)\mathcal{M}(R_{p,t+1}) =$

$\mathcal{M}(C_{t+1}/C_t)$ and $\theta_S^* = \iota$, the proof completes. $\square$

Proofs for Chapter 3

Proof of Lemma 1. On the one hand, if $\{R_{f,t+1}, P_t\}$ and $\{c_{i,t}, \theta_{i,t}\}$, $i = 1, 2, \ldots, m$ constitute a competitive equilibrium, then $c_{i,t}$ and $\theta_{i,t}$, $t \geq 0$ are the optimal consumption propensity and percentage allocation to the stock, respectively, of agent i. Combining the clearing conditions for the stock and for the risk-free asset, we obtain $\sum_{i=1}^{m}(W_{i,t} - c_{i,t}W_{i,t}) = P_t$, and then (3.2.6) follows from the clearing condition for consumption.

On the other hand, suppose $c_{i,t}$ and $\theta_{i,t}$, $t \geq 0$ are the optimal consumption propensity and percentage allocation to the stock, respectively, of agent i and satisfy (3.2.6). One can check that $(1 - c_{i,t})W_{i,t} := Y_{i,t}P_t$ is agent i's post-consumption wealth and the clearing conditions in Definition 1 are satisfied. $\square$

Proof of Theorem 3. The equilibrium analysis when the market is populated with EZ-agents only is standard in the literature; see for instance Epstein and Zin (1989, 1991). Note that in equilibrium, the optimal percentage allocation to the stock for the EZ-agents is 1.

Now, suppose that LA-agents also exist in the market with LAD equal to K^*. We show the equilibrium does not change. Because $K_i = K^*$, we conclude that for any $\theta_{i,t} \geq 0$, $\mathbb{E}\big[\nu_i\big(\theta_{i,t}(R_{t+1} - R_{f,t+1})\big)|\mathscr{F}_t\big] = \theta_{i,t}\mathbb{E}\big[\nu_i\big(R_{t+1} - R_{f,t+1}\big)|\mathscr{F}_t\big] = 0$. On the other hand, because $K_i \geq 1$, we have $-\mathbb{E}\big[\nu_i\big(R_{f,t+1} - R_{t+1}\big)|\mathscr{F}_t\big] \geq \mathbb{E}\big[\nu_i\big(R_{t+1} - R_{f,t+1}\big)|\mathscr{F}_t\big] = 0$. As a result, for any $\theta_{i,t} < 0$, we have

$$\mathbb{E}\left[\nu_i\left(\theta_{i,t}(R_{t+1} - R_{f,t+1})\right)\right] = -\theta_{i,t}\mathbb{E}\left[\nu_i\left(R_{f,t+1} - R_{t+1}\right)\right] \leq 0.$$

In consequence,

$$M(\Psi_{i,t+1}R_{i,t+1}|\mathscr{F}_t)\Big(1+b_i\left(M(R_{i,t+1}|\mathscr{F}_t)\right)^{-1}\mathbb{E}\left[\nu_i\left(\theta_{i,t}(R_{t+1}-R_{f,t+1})\right)|\mathscr{F}_t\right]\Big)\le M(\Psi_{i,t+1}R_{i,t+1}|\mathscr{F}_t)$$

holds for any $\theta_{i,t}$ and the inequality becomes equality for any $\theta_{i,t}\ge 0$. Observe from (3.2.5) that the maximizer of $\max_{\theta_{i,t}} M(\Psi_{i,t+1}R_{i,t+1}|\mathscr{F}_t)$ is the optimal portfolio of the EZ-agents, which is 1 in equilibrium, so the optimal portfolio for the LA-agents is also 1. As a result, the optimal consumption for the LA-agents is also the same as that for the EZ-agents. Thus, the market equilibrium does not change and the consumption-investment strategies of the LA- and EZ-agents are the same. □

Proof of Proposition 12. We only need to show that $\mathbb{E}\left(Z_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma}\right)$ is strictly decreasing in $\gamma\ge 0$.

We first show that $h(t):=\ln\mathbb{E}\left(e^{tY}\right)$ is strictly convex in $t\le 1$, where $Y:=\ln Z_{t+1}$. Because essinf $Z_{t+1}>0$ and $\mathbb{E}(Z_{t+1})<\infty$, $\mathbb{E}\left(e^{tY}\right)$ is continuous and well defined for $t\le 1$ and is twice continuously differentiable in $t<1$. Furthermore, its first- and second-order derivatives can be computed by interchanging the differential and expectation operators. Then, $h(t)$ is twice continuously differentiable and

$$h''(t)=\frac{\left(\mathbb{E}[Y^2e^{tY}]\right)\left(\mathbb{E}[e^{tY}]\right)-\left(\mathbb{E}[Ye^{tY}]\right)^2}{(\mathbb{E}[e^{tY}])^2}=\tilde{\mathbb{E}}[Y^2]-(\tilde{\mathbb{E}}[Y])^2,$$

where $\tilde{\mathbb{E}}$ is the expectation associated with $\tilde{\mathbb{P}}$ defined by $d\tilde{\mathbb{P}}/d\mathbb{P}=e^{tY}/\mathbb{E}[e^{tY}]$. Jensen's inequality immediately yields $h''(t)>0$, so $h(t)$ is strictly convex in $t\le 1$.

Now, for any $0\le\gamma_1<\gamma_2$, we have

$$\mathbb{E}\left(Z_{t+1}^{1-\gamma_1}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma_1}\right)=e^{h(1-\gamma_1)-h(-\gamma_1)}>e^{h(1-\gamma_2)-h(-\gamma_2)}=\mathbb{E}\left(Z_{t+1}^{1-\gamma_2}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma_2}\right),$$

where the inequality is the case because $-\gamma_1 > -\gamma_2$ and h is strictly convex.

Finally, it is obvious that $\mathbb{E}\left(Z_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(Z_{t+1}^{-\gamma}\right) = \mathbb{E}[Z_{t+1}]$ when $\gamma = 0$, so K^* becomes 1 when $\gamma = 0$. $\square$

Proof of Proposition 13. We conjecture that the price-dividend ratio is a constant, so the stock return $R_{t+1} = \alpha_{\text{LA}}^{-1} Z_{t+1}$ for some constant $\alpha_{\text{LA}} > 0$. Furthermore, Ψ_t and $R_{f,t+1}$ are also constants, and we denote them as Ψ and $R_{f,\text{LA}}$, respectively.

Given the stock return $R_{t+1} = \alpha_{\text{LA}}^{-1} Z_{t+1}$ and risk-free return $R_{f,\text{LA}}$, (3.2.5) yields that the LA-agent's investment problem is

$$\Theta := \max_{\theta}\Big\{M\big(R_{f,\text{LA}} + \theta(R_{t+1} - R_{f,\text{LA}})\big) + b\mathbb{E}\big[\nu\big(\theta(R_{t+1} - R_{f,\text{LA}})\big)\big]\Big\}. \tag{0.52}$$

Because $K \geq 1$, the objective function in (0.52) is concave. As a result, in equilibrium, the optimal percentage to the stock $\theta^* = 1$ if and only if the following first-order condition holds:

$$M(R_{t+1}) - M(R_{t+1})^{\gamma}\mathbb{E}[R_{t+1}^{-\gamma}]R_{f,\text{LA}} + b\mathbb{E}[\nu(R_{t+1} - R_{f,\text{LA}})] = 0. \tag{0.53}$$

The equilibrium risk-free return $R_{f,\text{LA}}$ is then solved from (0.53). Because $R_{t+1} = \alpha_{\text{LA}}^{-1} Z_{t+1}$, we conclude $R_{f,\text{LA}} = \alpha_{\text{LA}}^{-1} r_{f,\text{LA}}$, where $r_{f,\text{LA}}$ is the solution to (3.3.2). The left-hand side of (3.3.2) is strictly decreasing in $r_{f,\text{LA}}$, is strictly positive when $r_{f,\text{LA}} = 0$, and becomes $-\infty$ when $r_{f,\text{LA}} \to +\infty$, so (3.3.2) admits a unique solution. Moreover, it is straightforward to check that $r_{f,\text{LA}} = \mathbb{E}[Z_{t+1}^{1-\gamma}]/\mathbb{E}[Z_{t+1}^{-\gamma}]$ when $K = K^*$, $r_{f,\text{LA}}$ is decreasing in K, and the monotonicity becomes strict when $b > 0$. Finally, $r_{f,\text{LA}}$ is strictly decreasing (increasing) in b when $K > K^*$ ($K < K^*$).

Next, we compute α_{LA}. Recall the optimal consumption problem that is obtained from the dynamic programming equation (3.2.5):

$$\Psi = \max_{c\in(0,1)} \quad H\big(c, (1-c)\Psi\Theta)\big). \tag{0.54}$$

We immediately conclude that the optimal consumption propensity is

$$c^* = 1\Big/ \left[1 + (\beta/(1-\beta))^{1/\rho} (\Psi\Theta)^{(1-\rho)/\rho}\right]. \tag{0.55}$$

Plugging (0.55) into (0.54), we solve $\Psi = \left[(1-\beta)^{1/\rho}/\left(1-(\beta\Theta^{1-\rho})^{1/\rho}\right)\right]^{\rho/(1-\rho)}$ when $\rho \neq 1$ and $\Psi = (1-\beta)\beta^{\beta/(1-\beta)}\Theta^{\beta/1-\beta}$ when $\rho = 1$. Here, we implicitly assume that $\beta\Theta^{1-\rho} < 1$ and will show that this is true later on. In consequence, plugging Ψ into (0.55), we obtain

$$1/c^* = 1 + (\beta/(1-\beta))^{1/\rho}\,\Theta^{(1-\rho)/\rho}\left[(1-\beta)^{1/\rho}/\left(1-(\beta\Theta^{1-\rho})^{1/\rho}\right)\right] = 1/\left(1-(\beta\Theta^{1-\rho})^{1/\rho}\right).$$

Now, plugging the first-order condition (0.53) into (0.52), we conclude

$$\Theta = M(R_{t+1})^{\gamma}\mathbb{E}[R_{t+1}^{-\gamma}]R_{f,\text{LA}} = \alpha_{\text{LA}}^{-1}M(Z_{t+1})^{\gamma}\mathbb{E}[Z_{t+1}^{-\gamma}]r_{f,\text{LA}}.$$

On the other hand, from the market clearing condition (3.2.6), we obtain that $P_t/D_t = (1/c^*) - 1$, so we have $R_{t+1} = \big((P_{t+1}/D_{t+1}+1)/(P_t/D_t)\big)(D_{t+1}/D_t) = Z_{t+1}/(1-c^*)$. In consequence, we have

$$\alpha_{\text{LA}} = 1 - c^* = (\beta\Theta^{1-\rho})^{1/\rho} = \beta^{1/\rho}\left(\alpha_{\text{LA}}^{-1}M(Z_{t+1})^{\gamma}\mathbb{E}[Z_{t+1}^{-\gamma}]r_{f,\text{LA}}\right)^{(1-\rho)/\rho},$$

which yields $\alpha_{\text{LA}} = \beta\left(M(Z_{t+1})^{\gamma}\mathbb{E}[Z_{t+1}^{-\gamma}]r_{f,\text{LA}}\right)^{1-\rho}$.

Finally, note that $\Theta = \beta^{-1}\left(M(Z_{t+1})^{\gamma}\mathbb{E}[Z_{t+1}^{-\gamma}]r_{f,\text{LA}}\right)^{\rho}$. In consequence, $\beta\Theta^{1-\rho} = \alpha_{\text{LA}}^{\rho} < 1$ because it is assumed that $\alpha_{\text{LA}} < 1$. Thus, Ψ is uniquely determined by (0.54); i.e., the LA-agents' total utility defined by (3.2.1) with the optimal consumption-investment strategy uniquely exists. $\square$

Proof of Proposition 14. For fixed $b \geq 0$ and $K \geq 1$, denote

$$h(\theta, a) : = \mathbb{E}[\ln(a + \theta(X - a))], \quad g_+(a) := \mathbb{E}[\nu(X - a)], \quad g_-(a) := -\mathbb{E}[\nu(a - X)] \tag{0.56}$$

$$f_0(\theta, a) : = e^{h(\theta,a)}, \quad f(\theta, a) := f_0(\theta, a) + \theta b \left[g_+(a)\mathbf{1}_{\theta\geq 0} + g_-(a)\mathbf{1}_{\theta<0}\right]. \tag{0.57}$$

Then, f is the objective function of (3.4.1). Moreover, we must have $\theta \in [\underline{\theta}(a), \overline{\theta}(a)]$ so that $a+\theta(X-a) \geq 0$. Here and hereafter, we set $1/-\infty = 0$ so that $\underline{\theta}$ is well defined, i.e., $\underline{\theta} = 0$, when $\overline{x} = +\infty$. We also set $\ln(0) = -\infty$ and $1/0 = +\infty$, so h, f_0, and f are well-defined at $\theta = \overline{\theta}(a)$ and at $\theta = \underline{\theta}(a)$. In consequence, the optimal solution to (3.4.1) is the maximizer of $f(\theta, a)$ in $\theta \in [\underline{\theta}(a), \overline{\theta}(a)]$.

We first prove the continuity and compute the derivatives of f. Recalling the assumption $\mathbb{E}[X] < \infty$ and applying the dominated convergence theorem, one can prove that $\frac{\partial h}{\partial \theta}$, $\frac{\partial h}{\partial a}$, $\frac{\partial^2 h}{\partial \theta^2}$, and $\frac{\partial^2 h}{\partial \theta \partial a}$ exist for $\theta \in (\underline{\theta}(a), \overline{\theta}(a)), a \in (\underline{x}, \overline{x})$, and that these derivatives, together with h, are continuous in (θ, a) in the same region. Furthermore, these derivatives can be computed by interchanging the expectation and differential operators; i.e.,

$$\frac{\partial h}{\partial \theta}(\theta, a) = \mathbb{E}\left[\frac{X - a}{a + \theta(X - a)}\right], \quad \frac{\partial h}{\partial a}(\theta, a) = \mathbb{E}\left(\frac{1 - \theta}{a + \theta(X - a)}\right), \tag{0.58}$$

$$\frac{\partial^2 h}{\partial \theta^2}(\theta, a) = -\mathbb{E}\left[\frac{(X - a)^2}{(a + \theta(X - a))^2}\right], \quad \frac{\partial^2 h}{\partial \theta \partial a}(\theta, a) = -\mathbb{E}\left[\frac{X}{(a + \theta(X - a))^2}\right]. \tag{0.59}$$

In consequence, f is continuous in (θ, a) for $\theta \in (\underline{\theta}(a), \overline{\theta}(a)), a \in (\underline{x}, \overline{x})$. Furthermore, noting that

$$h(\theta, a) = \mathbb{E}[\ln(a + \theta(X - a))\mathbf{1}_{X\geq a}] + \mathbb{E}[\ln(a + \theta(X - a))\mathbf{1}_{X<a}]$$

and applying the monotone convergence theorem, we conclude that $\lim_{\theta\uparrow\underline{\theta}(a)} h(\theta,a) = h(\overline{\theta}(a),a)$ and $\lim_{\theta\downarrow\underline{\theta}(a)} h(\theta,a) = h(\underline{\theta}(a),a)$. Consequently, f is continuous in $\theta \in [\underline{\theta}(a),\overline{\theta}(a)]$.

We already showed that h is twice continuously differentiable in (θ,a) for $\theta \in (\underline{\theta}(a),\overline{\theta}(a)), a \in (\underline{x},\overline{x})$. In consequence, f is twice continuously differentiable in (θ,a) for $\theta \in (\underline{\theta}(a),0)\cup(0,\overline{\theta}(a)), a \in (\underline{x},\overline{x})$. Moreover, the right-partial derivative of f in θ at $\theta = 0$, denoted as $\frac{\partial f}{\partial\theta}(0+,a)$, exists and $\lim_{\theta\downarrow 0}\frac{\partial f}{\partial\theta}(\theta,a) = \frac{\partial f}{\partial\theta}(0+,a)$; similarly, when $\overline{x} < +\infty$ so that $\underline{\theta}(a) < 0$, the left-partial derivative of f in θ at $\theta = 0$, denoted as $\frac{\partial f}{\partial\theta}(0-,a)$, exists and $\lim_{\theta\uparrow 0}\frac{\partial f}{\partial\theta}(\theta,a) = \frac{\partial f}{\partial\theta}(0-,a)$. Furthermore,

$$\frac{\partial f}{\partial\theta}(\theta,a) = f_0(\theta,a)\mathbb{E}\left[\frac{X-a}{a+\theta(X-a)}\right] + b\left[g_+(a)\mathbf{1}_{\theta>0} + g_-(a)\mathbf{1}_{\theta<0}\right], \tag{0.60}$$

$$\frac{\partial^2 f}{\partial\theta^2}(\theta,a) = f_0(\theta,a)\left[-\mathbb{E}\left(\frac{X-a}{a+\theta(X-a)}\right)^2 + \left(\mathbb{E}\left(\frac{X-a}{a+\theta(X-a)}\right)\right)^2\right] < 0, \tag{0.61}$$

where the inequality follows from Jensen's inequality. On the other hand, $g_+(0) \le g_-(0)$ because $K \ge 1$. Consequently, f is strictly concave in $\theta \in [\underline{\theta}(a),\overline{\theta}(a)]$, and this, together with the continuity of f in $\theta \in [\underline{\theta}(a),\overline{\theta}(a)]$, implies that the maximizer of f in $\theta \in [\underline{\theta}(a),\overline{\theta}(a)]$ uniquely exists. As a result, the optimal portfolio $\varphi(a;b,K)$ of problem (3.4.1) is well defined.

Next, we compute $\varphi(a;b,K)$. Because f is strictly concave in $\theta \in [\underline{\theta}(a),\overline{\theta}(a)]$, we only need to find the maximizer of f in $\theta \in [0,\overline{\theta}(a)]$, denoted as $\phi_+(a)$, and the maximizer of f in $\theta \in [\underline{\theta}(a),0]$, denoted as $\phi_-(a)$. Indeed, we have $\varphi(a;b,K) = \phi_+(a)\mathbf{1}_{\phi_+(a)>0} + \phi_-(a)\mathbf{1}_{\phi_-(a)<0}$ and $\phi_+(a)\phi_-(a) = 0$.

Because f is strictly concave in θ, we conclude that $\phi_+(a) = 0$ if and only if $\frac{\partial f}{\partial\theta}(0+,a) \le 0$. Straightforward computation yields $\frac{\partial f}{\partial\theta}(0+,a) = \mathbb{E}(X) - a + bg_+(a)$, which is continuous and strictly decreasing in a because $g_+(a)$ is decreasing in a. One can see that the zero of $\frac{\partial f}{\partial\theta}(0+,a)$, denoted as $\overline{a}_l$, uniquely exists and $\phi_+(a) = 0$ if and only if $a \ge \overline{a}_l$. Moreover, $\frac{\partial f}{\partial\theta}(0+,\underline{x}) > 0$ and $\frac{\partial f}{\partial\theta}(0+,\mathbb{E}(X)) \le 0$ because $K \ge 1$, so $\overline{a}_l \in (\underline{x},\mathbb{E}(X)]$.

On the other hand, $\phi_+(a) = \overline{\theta}(a)$ if and only if $\lim_{\theta\uparrow\overline{\theta}(a)} \frac{\partial f}{\partial \theta}(\theta, a) \geq 0$. Note that this limit must exist and take values in $[-\infty, +\infty)$ because f is strictly concave in θ. Straightforward calculation yields

$$\frac{\partial f}{\partial \theta}(\theta, a) = e^{\mathbb{E}(\ln(c+X-\underline{x}))} - (c + a - \underline{x})e^{\mathbb{E}(\ln(c+X-\underline{x}))}\mathbb{E}[1/(c + X - \underline{x})] + bg_+(a), \tag{0.62}$$

where $c := (a/\theta) - (a - \underline{x})$. Note that θ increasingly goes to $\overline{\theta}(a)$ if and only if c decreasingly goes to zero, so

$$\lim_{\theta\uparrow\overline{\theta}(a)} \frac{\partial f}{\partial \theta}(\theta, a) = e^{\mathbb{E}(\ln(X-\underline{x}))} - (a - \underline{x})\xi + bg_+(a)$$

where $\xi := \lim_{c\downarrow 0} \{(\exp[\mathbb{E}(\ln(c + X - \underline{x}))]) (\mathbb{E}[1/(c + X - \underline{x})])\} \in [0, +\infty]$. Here, the limit defining ξ exists because $\lim_{\theta\uparrow\overline{\theta}(a)} \frac{\partial f}{\partial \theta}(\theta, a)$ exists. Furthermore, by Jensen's inequality, we conclude that $\xi \in [1, +\infty]$. If $\xi < +\infty$, $\lim_{\theta\uparrow\overline{\theta}(a)} \frac{\partial f}{\partial \theta}(\theta, a)$ is continuous and strictly decreasing in a, its value is nonnegative when $a = \underline{x}$, and

$$\lim_{\theta\uparrow\overline{\theta}(a)} \frac{\partial f}{\partial \theta}(\theta, \overline{a}_l) = e^{\mathbb{E}(\ln(X-\underline{x}))} - (\overline{a}_l - \underline{x})\xi + bg_+(\overline{a}_l) < \mathbb{E}[X] - \underline{x} - (\overline{a}_l - \underline{x})\xi + bg_+(\overline{a}_l)$$
$$= -(\xi - 1)(\overline{a}_l - \underline{x}) \leq 0,$$

where the first inequality is the case due to Jensen's inequality and the second equality is the case because $\frac{\partial f}{\partial \theta}(0+, \overline{a}_l) = 0$. Thus, there exists $\underline{a}_l \in [\underline{x}, \overline{a}_l)$ such that $\lim_{\theta\uparrow\overline{\theta}(a)} \frac{\partial f}{\partial \theta}(\theta, a) \geq 0$ and thus $\phi_+(a) = \overline{\theta}(a)$ if and only if $a \leq \underline{a}_l$. If $\xi = +\infty$, we have $\lim_{\theta\uparrow\overline{\theta}(a)} \frac{\partial f}{\partial \theta}(\theta, a) = -\infty$, so by setting $\underline{a}_l := \underline{x}$ in this case, we also conclude that $\phi_+(a) = \overline{\theta}(a)$ if and only if $a \in (\underline{x}, \underline{a}_l]$.

We have shown that $\phi_+(a) = 0$ if and only if $a \in [\overline{a}_l, \overline{x})$ and $\phi_+(a) = \overline{\theta}(a)$ if and only if $a \in$

$(\underline{x}, \underline{a}_l]$. Therefore, for $a \in (\underline{a}_l, \overline{a}_l)$, we have $\phi_+(a) \in (0, \overline{\theta}(a))$ and thus $\phi_+(a)$ is uniquely determined by $\frac{\partial f}{\partial \theta}(\phi_+(a), a) = 0$. Because $\frac{\partial f}{\partial \theta}(\theta, a)$ is strictly increasing in θ and continuous in (θ, a) for $\theta \in (0, \overline{\theta}(a)), a \in (\underline{x}, \overline{x})$, we conclude that $\phi_+(a)$ is continuous in $a \in (\underline{a}_l, \overline{a}_l)$.

Next, we show that $\phi_+(a)$ is strictly decreasing in $a \in (\underline{a}_l, \overline{a}_l)$. Note that fixing $\theta \in [1, \overline{\theta}(a))$ and defining $Y = \frac{1}{a+\theta(X-a)} > 0$, we calculate from (0.58) and (0.59) that

$$\begin{aligned}\frac{\partial^2 f_0}{\partial \theta \partial a}(\theta, a) &= f_0(\theta, a) \left\{ -\mathbb{E}\left[\frac{X}{(a+\theta(X-a))^2}\right] + \mathbb{E}\left[\frac{X-a}{a+\theta(X-a)}\right] \cdot \mathbb{E}\left[\frac{1-\theta}{a+\theta(X-a)}\right] \right\} \\ &= f_0(\theta, a) \left\{ \frac{1-\theta}{\theta} a[\mathbb{E}(Y)^2 - (\mathbb{E}(Y))^2] - \mathbb{E}[Y] \right\} \leq -f_0(\theta, a)\mathbb{E}[Y] < 0, \end{aligned} \tag{0.63}$$

where the first inequality is due to Jensen's inequality. In consequence, $\frac{\partial f_0}{\partial \theta}(\theta, a)$ is strictly decreasing in a for $\theta \in [1, \overline{\theta}(a))$ and $a \in (\underline{x}, \overline{x})$. Now, consider any a_i, $i = 1, 2$ such that $\underline{a}_l < a_1 < a_2 < \overline{a}_l$ and at least one of $\phi_+(a_i)$, $i = 1, 2$ is larger than or equal to 1. We claim that $\phi_+(a_1) > \phi_+(a_2)$. Indeed, because $a_i \in (\underline{a}_l, \overline{a}_l)$, we conclude $\frac{\partial f}{\partial \theta}(\phi_+(a_i), a_i) = \frac{\partial f_0}{\partial \theta}(\phi_+(a_i), a_i) + bg_+(a_i) = 0$, $i = 1, 2$. Suppose $\phi_+(a_1) \geq 1$. If $\phi_+(a_1) \geq \overline{\theta}(a_2)$, then $\phi_+(a_1) > \phi_+(a_2)$ because $a_2 > \underline{a}_l$ and thus $\phi_+(a_2) < \overline{\theta}(a_2)$. If $\phi_+(a_1) < \overline{\theta}(a_2)$, then

$$0 = \frac{\partial f}{\partial \theta}(\phi_+(a_1), a_1) > \frac{\partial f_0}{\partial \theta}(\phi_+(a_1), a_2) + bg_+(a_1) \geq \frac{\partial f_0}{\partial \theta}(\phi_+(a_1), a_2) + bg_+(a_2) = \frac{\partial f}{\partial \theta}(\phi_+(a_1), a_2),$$

where the first inequality is the case because $\frac{\partial f_0}{\partial \theta}(\theta, a)$ is strictly decreasing in a for $\theta \geq 1$ and $a \in (\underline{x}, \overline{x})$ and the second inequality is the case because g_+ is strictly decreasing in $a \in (\underline{x}, \overline{x})$. Because $\frac{\partial f}{\partial \theta}(\theta, a)$ is strictly decreasing in $\theta \in (0, \overline{\theta}(a))$, we immediately conclude that $\phi_+(a_2) < \phi_+(a_1)$. A similar argument yields that $\phi_+(a_2) < \phi_+(a_1)$ when $\phi_+(a_2) \geq 1$.

To complete the proof that $\phi_+(a)$ is strictly decreasing in $a \in (\underline{a}_l, \overline{a}_l)$, we only need to show that for any $a_i \in (\underline{a}_l, \overline{a}_l)$ such that $a_1 < a_2$ and $\phi_+(a_i) \in (0,1)$, $i = 1, 2$, $\phi_+(a_1) > \phi_+(a_2)$. We first note that $\phi_+(a) \in (0,1)$ for any $a \in (a_1, a_2)$; otherwise, if $\phi_+(a_0) \geq 1$ for some $a_0 \in (a_1, a_2)$, we conclude that $\phi_+(a_1) > \phi_+(a_0) \geq 1$, which is a contradiction. Now, note that g_+ is concave and thus absolutely continuous in a. Then, a non-standard implicit function theorem (Ettlinger, 1928, Theorem II) yields that $\phi_+(a)$ is absolutely continuous in $a \in (a_1, a_2)$ and

$$\frac{\partial^2 f}{\partial \theta^2}(\phi_+(a), a)\phi_+'(a) + \frac{\partial^2 f}{\partial \theta \partial a}(\phi_+(a), a) = 0.$$

Because, $\frac{\partial^2 f}{\partial \theta^2}(\theta, a) < 0$ for any $\theta \in (0, \overline{\theta}(a))$, $a \in (\underline{x}, \overline{x})$, we only need to show that $\frac{\partial^2 f}{\partial \theta \partial a}(\phi_+(a), a) < 0$, $a \in (a_1, a_2)$. To this end, by taking the derivative with respect to θ, we can verify that $\mathbb{E}\left[\frac{1-\theta}{a+\theta(X-a)}\right]$ is decreasing in θ. Therefore, for any $\theta \in [0,1)$, we have $0 < \mathbb{E}\left[\frac{1-\theta}{a+\theta(X-a)}\right] \leq \frac{1}{a}$. On the other hand, because of the concavity of g_+, we conclude that $g_+(a) \geq g_+(0) + g_+'(a)a > g_+'(a)a$, where the second equality is the case because $g_+(0) > 0$. Because g_+ is concave and strictly decreasing, we must have $g_+'(a) < 0$. Consequently, $g_+(a)/g_+'(a) < a$. Therefore, we conclude that for any $\theta < 1$,

$$-g_+(a)\mathbb{E}\left[\frac{1-\theta}{a+\theta(X-a)}\right] + g_+'(a) = -g_+'(a)\left\{\frac{g_+(a)}{g_+'(a)}\mathbb{E}\left[\frac{1-\theta}{a+\theta(X-a)}\right] - 1\right\} < 0.$$

Now, for any $a \in (a_1, a_2)$, because $0 < \phi_+(a) < 1$ and thus $\frac{\partial f}{\partial \theta}(\phi_+(a), a) = 0$, straightforward calculation yields

$$\begin{aligned}\frac{\partial^2 f}{\partial \theta \partial a}(\phi_+(a), a) = &-bg_+(a)\mathbb{E}\left[\frac{1-\phi_+(a)}{a+\phi_+(a)(X-a)}\right] + bg_+'(a)\\ &-f_0(\phi_+(a), a)\mathbb{E}\left[\frac{X}{(a+\phi_+(a)(X-a))^2}\right] < 0.\end{aligned}$$

We have shown that $\phi_+(a)$ is continuous, strictly decreasing, and taking values in $(0, \overline{\theta}(a))$ on

$(\underline{a}_l, \overline{a}_l)$. Moreover, $\phi_+(a)$ is equal to 0 on $[\overline{a}_l, \overline{x})$ and equal to $\overline{\theta}(a)$ on $(\underline{x}, \underline{a}_l]$. Because $\overline{\theta}(a)$ is strictly decreasing in a, we conclude that $\phi_+(a)$ is decreasing in $a \in (\underline{x}, \overline{x})$ and the monotonicity becomes strict when $\phi_+(a) > 0$.

Next, we show that $\phi_+(a)$ is continuous from the right at $\underline{a}_l$ if $\underline{a}_l > \underline{x}$ and continuous from the left at $\overline{a}_l$ and thus $\phi_+(a)$ is continuous in $a \in (\underline{x}, \overline{x})$. For the sake of contradiction, suppose $\underline{a}_l > \underline{x}$ and $\phi_+(a)$ is not continuous from the right at $\underline{a}_l$. Then, there exist $\epsilon_0 > 0$ and a_n's that decreasingly converge to $\underline{a}_l$ as $n \to \infty$ such that $\phi_+(a_n) \le \overline{\theta}(\underline{a}_l) - \epsilon_0 < \overline{\theta}(a_n)$. Because $f(\theta, a_n)$ is strictly concave in θ and $\phi_+(a_n)$ is the maximizer of $f(\theta, a_n)$ in $\theta \in [0, \overline{\theta}(a_n)]$, we have $\frac{\partial f}{\partial \theta}(\overline{\theta}(\underline{a}_l) - \epsilon_0, a_n) \le 0$. Sending n to infinity and recalling the continuity of $\frac{\partial f}{\partial \theta}$ in (θ, a) for $\theta \in (0, \overline{\theta}(a)), a \in (\underline{x}, \overline{x})$, we conclude that $\frac{\partial f}{\partial \theta}(\overline{\theta}(\underline{a}_l) - \epsilon_0, \underline{a}_l) \le 0$. On the other hand, by the definition of $\underline{a}_l$, we have $\lim_{\theta \uparrow \overline{\theta}(\underline{a}_l)} \frac{\partial f}{\partial \theta}(\theta, \underline{a}_l) = 0$. Because $f(\theta, a)$ is strictly concave in θ, we conclude that $\frac{\partial f}{\partial \theta}(\overline{\theta}(\underline{a}_l) - \epsilon_0, \underline{a}_l) > 0$, which is a contradiction. Thus, $\phi_+(a)$ is continuous from the right at $\underline{a}_l$. Similarly, we can show that $\phi_+(a)$ is continuous from the left at $\overline{a}_l$.

Next, we show $\lim_{a \downarrow \underline{x}} \phi_+(a) = +\infty$. Suppose that it is not case, then there exist $M \in (1, +\infty)$ and $a_n \downarrow \underline{x}$ such that $\phi_+(a_n) \le M < \overline{\theta}(a_n)$. Then, we must have $\frac{\partial f}{\partial \theta}(M, a_n) \le 0$, and thus $\limsup_{n \to \infty} \frac{\partial f}{\partial \theta}(M, a_n) \le 0$. On the other hand, from (0.60) one can conclude that $\liminf_{a \downarrow \underline{x}} \frac{\partial f}{\partial \theta}(M, a) > 0$, which is a contradiction. Therefore, we must have $\lim_{a \downarrow \underline{x}} \phi_+(a) = +\infty$.

Next, we compute $\phi_-(a)$, the maximizer of $f(\theta, a)$ in $\theta \in [\underline{\theta}(a), 0]$. We consider the case $\overline{x} < +\infty$ first. Using the same proof as for the derivation of ϕ_+, we can show that there exist $\underline{a}_s \in [\mathbb{E}[X], \overline{x})$ and $\overline{a}_s \in (\underline{a}_s, \overline{x}]$ such that $\phi_-(a) = 0$ if and only if $a \in (\underline{x}, \underline{a}_s]$, $\phi_-(a) = \underline{\theta}(a)$ if and only if $a \in [\overline{a}_s, \overline{x})$, and $\phi_-(a) \in (\underline{\theta}(a), 0)$ otherwise. Moreover, because $g_-(a)$ is decreasing in a and (0.63) holds for $\theta \in (\underline{\theta}(a), 0)$ as well, using the same proof as for $\phi_+(a)$, we conclude that $\phi_-(a)$ is continuous and decreasing in $a \in (\underline{x}, \overline{x})$ and the monotonicity becomes strict when $\phi_-(a) < 0$. In the case in which $\overline{x} = 0$, we have

$\underline{\theta}(a) = 0$, i.e., short-selling is not allowed, and we simply set $\underline{a}_s = \overline{a}_s = \overline{x} = +\infty$.

Recall that $\varphi(a; b, K) = \phi_+(a)\mathbf{1}_{\phi_+(a)>0} + \phi_-(a)\mathbf{1}_{\phi_-(a)<0}$ and $\phi_+(a)\phi_-(a) = 0$. Then, the proof of (i) and (ii) is completed.

Next, we prove (iii). Suppose $b > 0$. Note that for each $a \in (\underline{x}, \overline{x})$, $g_+(a)$ is strictly decreasing in K and $g_-(a)$ is strictly increasing in K. In consequence, we conclude from (0.60) that for any $a \in (\underline{x}, \overline{x})$, $\frac{\partial f}{\partial \theta}(\theta, a)$ is strictly decreasing in K when $\theta > 0$ and strictly increasing in K when $\theta < 0$. Because $\overline{a}_l$ is the zero of $\frac{\partial f}{\partial \theta}(0+, a)$ and $\frac{\partial f}{\partial \theta}(0+, a)$ is strictly decreasing in a, we conclude that $\overline{a}_l$ is strictly decreasing in K. A similar argument yields that $\underline{a}_l$ is decreasing in K. Moreover, for any $a \in (\underline{a}_l, \overline{a}_l)$, $\varphi(a; b, K)$ is the zero of $\frac{\partial f}{\partial \theta}(\theta, a)$ and thus is strictly decreasing in K. Similarly, $\underline{a}_s$ and $\overline{a}_s$ are increasing in K and for any $a \in (\underline{a}_s, \overline{a}_s)$, $\varphi(a; b, K)$ is strictly increasing in K. Thus, we conclude that $|\varphi(a; b, K)|$ is decreasing in K and the monotonicity becomes strict when $\varphi(a; b, K) \in (0, \overline{\theta}(a))$ or when $\varphi(a; b, K) \in (\underline{\theta}(a), 0)$.

Finally, we prove (iv). When $g_+(a) \leq 0$, we conclude from (0.60) that for any $\theta > 0$, $\frac{\partial f}{\partial \theta}(\theta, a)$ is decreasing in b and the monotonicity becomes strict when $g_+(a) < 0$. Consequently, using the same argument as for the proof of the monotonicity of $\varphi(a; b, K)$ in K, we conclude that $\underline{a}_l$ and $\overline{a}_l$ are decreasing in b. Moreover, for any $a \in (\underline{a}_l, \overline{a}_l)$, $\varphi(a; b, K)$ is decreasing in b and the monotonicity becomes strict when $g_+(a) < 0$. In other words, when $g_+(a) \leq 0$, $\max(\varphi(a; b, K), 0)$ is decreasing in b and the monotonicity becomes strict when $g_+(a) < 0$ and $\varphi(a; b, K) \in (0, \overline{\theta}(a))$. Similarly, we can show that when $g_+(a) \geq 0$, $\max(\varphi(a; b, K), 0)$ is increasing in b and the monotonicity becomes strict when $g_+(a) > 0$ and $\varphi(a; b, K) \in (0, \overline{\theta}(a))$. The case of $\max(-\varphi(a; b, K), 0)$ can be treated similarly. $\square$

Proof of Theorem 4. It is straightforward to see from (3.2.5) that the optimal consumption propensity of agent i is $c^*_{i,t} = 1 - \beta$, $i = 1, \ldots, m$. As a result, (3.2.6) leads to the unique equilibrium price dividend ratio $P_t/D_t \equiv \frac{\beta}{1-\beta}$. Consequently, the stock return in equilibrium must be

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}/D_{t+1} + 1}{P_t/D_t} \cdot \frac{D_{t+1}}{D_t} = \frac{1}{\beta} Z_{t+1}.$$

Because Z_{t+1}'s are i.i.d., so are R_{t+1}'s. Denote $\underline{x} := \operatorname{essinf} R_{t+1}$ and $\overline{x} := \operatorname{esssup} R_{t+1}$.

Next, recalling the Bellman equation (3.2.5) and noting that R_{t+1}'s are i.i.d., we immediately conclude that agent i's optimal percentage allocation to the stock at time t is $\varphi(R_{f,t+1}; b_i, K_i)$, where φ is the optimal solution to (3.4.1) with $X = R_{t+1}$. Therefore, from (3.2.6) we conclude that $R_{f,t+1}$ is the equilibrium risk-free rate if and only if it satisfies $\sum_{i=1}^m Y_{i,t}\varphi(R_{f,t+1}; b_i, K_i) = 1$. By proposition 14-(ii), $\varphi(a; b_i, K_i)$ is continuous and decreasing in $a \in (\underline{x}, \overline{x})$ and is strictly decreasing in a when $\varphi(a; b_i, K_i) \neq 0$. Moreover, $\lim_{a \downarrow \underline{x}} \varphi(a; b_i, K_i) = +\infty$ and $\lim_{a \uparrow \overline{x}} \varphi(a; b_i, K_i)$ is either 0 or $-\infty$. Thus, $R_{f,t+1}$ uniquely exists and is a continuous function of $Y_{i,t}, i = 1, \ldots, m$. Furthermore, by Proposition 14-(i), $\varphi(a; b_i, K_i) \leq 0$ for $a \geq \mathbb{E}(R_{t+1})$, so we must have $R_{f,t+1} \in (\underline{x}, \mathbb{E}(R_{t+1}))$.

Finally, we show that agent i's utility is well defined, i.e., $\{U_{i,t}\}$ defined by (3.2.1) uniquely exists, when the agent takes the optimal consumption-investment strategy $(c^*_{i,t}, \theta^*_{i,t})$ in equilibrium. Note that R_{t+1}'s are i.i.d. and $R_{f,t+1} = R_f(\mathbf{Y}_t)$ for some continuous function R_f on $\Delta := \{\mathbf{y} := (y_1, \ldots, y_m) \in \mathbb{R}^n | y_i \geq 0, i = 1, \ldots, m, \sum_{i=1}^m y_i = 1\}$, where $\mathbf{Y}_t := (Y_{1,t}, Y_{2,t}, \ldots, Y_{m,t})$ stands for the wealth share vector of the m agents in the market. Because $c^*_{i,t} = 1 - \beta$ and $\theta^*_{i,t} = \varphi(R_{f,t+1}; b_i, K_i)$, we conclude that agent i's utility $\{U_{i,t}\}$, defined by (3.2.1), with the optimal consumption-investment strategy is equivalent to the solution to the following equation

$$\begin{aligned} U_{i,t}/W_{i,t} = \exp\Big\{ &(1-\beta)\ln(1-\beta) + \beta \ln \beta + \beta \mathbb{E}[\ln(U_{i,t+1}/W_{i,t+1}) | \mathscr{F}_t] \\ &+ \beta \ln \left[f\left(\varphi(R_f(\mathbf{Y}_t); b_i, K_i), R_f(\mathbf{Y}_t)\right)\right] \Big\}, \end{aligned}$$

where $f(\theta, a)$, as defined in (0.57), is the objective function of problem (3.4.1). Because $f(\theta, a)$ is continuous in (θ, a), $\varphi(a; b, K)$ is continuous in a, and $R_f(\mathbf{y})$ is continuous in $\mathbf{y}$, we conclude that $f\big(\varphi(R_f(\mathbf{Y}_t); b_i, K_i), R_f(\mathbf{Y}_t)\big)$ is continuous in $\mathbf{Y_t}$. Moreover, $f\big(\varphi(R_f(\mathbf{Y}_t); b_i, K_i), R_f(\mathbf{Y}_t)\big) \geq f\big(0, R_f(\mathbf{Y}_t)\big) > 0$ because φ is the maximizer of $f(\theta, a)$ in θ. Thus, we conjecture that $U_{i,t}/W_{i,t} = \exp[\psi_i(\mathbf{Y}_t)]$ for some continuous function ψ_i on Δ. Furthermore, given the wealth share vector $\mathbf{Y}_t$ at time t, the wealth share vector $\mathbf{Y}_{t+1}$ at time $t+1$ is determined by Z_{t+1}, i.e., there exists a function $h(\mathbf{y}, z)$ from $\Delta \times \mathbb{R}_+$ to Δ such that $\mathbf{Y}_{t+1} = h(\mathbf{Y}_t, Z_{t+1})$. Indeed, we have

$$h(\mathbf{Y}_t, Z_{t+1}) = \left(\frac{Y_{i,t}\big(R_f(\mathbf{Y}_t) + \theta^*_{i,t}(Z_{t+1}/\beta - R_f(\mathbf{Y}_t))\big)}{\sum_{j=1}^m \big[Y_{j,t}\big(R_f(\mathbf{Y}_t) + \theta^*_{j,t}(Z_{t+1}/\beta - R_f(\mathbf{Y}_t))\big)\big]}; i = 1, \dots, m \right), \tag{0.64}$$

which is obtained by computing the return of each agent's optimal portfolio. Consequently, we have

$$\psi_i(\mathbf{y}) = (1-\beta)\ln(1-\beta) + \beta \ln \beta + \beta \ln\left[f(\varphi(R_f(\mathbf{y}); b_i, K_i), R_f(\mathbf{y}))\right] + \beta \mathbb{E}[\psi_i(h(\mathbf{y}, Z_{t+1}))], \quad \forall \mathbf{y} \in \Delta. \tag{0.65}$$

It is straightforward to see that the right-hand of (0.65) is a contraction mapping from the space of continuous functions on Δ with the maximum norm into the same space. Therefore, (0.65) admits a unique solution and, consequently, $\{U_{i,t}\}$ is well-defined. □

Proof of Proposition 15. Following the proof of Theorem 4, we conclude that $\theta^*_{0,t} = \varphi(R_{f,t+1}; 0, K)$ and $\theta^*_{1,t} = \varphi(R_{f,t+1}; b, K)$, where φ is the optimal solution to (3.4.1) with $X = R_{t+1}$. Recall f_0 and f as defined in (0.57). Then, $\varphi(R_{f,t+1}; 0, K)$ and $\varphi(R_{f,t+1}; b, K)$ are the maximizers of $f_0(\theta, a)$ and $f(\theta, a)$ in θ, respectively. According to Theorem 4, $R_{f,t+1} < \mathbb{E}[R_{t+1}]$, so Proposition 14-(i) implies that $\theta^*_{i,t} \geq 0$, $i = 0, 1$.

It is straightforward to verify that $\frac{\partial f_0}{\partial \theta}(1, R_{f,\text{EZ}}) = 0$, so $\varphi(R_{f,EZ}; 0, K) = 1$. When $K = K^*$, we have $\mathbb{E}[\nu(R_{t+1} - R_{f,\text{EZ}})] = 0$, so Proposition 14-(iv) yields that $\max(\varphi(R_{f,\text{EZ}}; b, K^*), 0) = \max(\varphi(R_{f,\text{EZ}}; 0, K^*), 0) = 1$. In consequence, the equilibrium risk-free rate $R_{f,t+1} \equiv R_{f,\text{EZ}}$ and $\theta^*_{i,t} \equiv 1, i = 0, 1$ in this case.

Next, consider the case in which $K < K^*$. We already showed that $\varphi(R_{f,\text{EZ}}; 0, K) = \varphi(R_{f,\text{EZ}}; b, K^*) = 1$, so Proposition 14-(iii) yields that for any $Y_t \in [0, 1)$,

$$Y_t\varphi(R_{f,\text{EZ}}; 0, K) + (1 - Y_t)\varphi(R_{f,\text{EZ}}; b, K) > Y_t\varphi(R_{f,\text{EZ}}; 0, K) + (1 - Y_t)\varphi(R_{f,\text{EZ}}; b, K^*) = 1.$$

In consequence, according to Proposition 14-(ii), the equilibrium risk-free rate $R_{f,t+1}$, which is determined by the equilibrium condition $Y_t\varphi(R_{f,t+1}; 0, K) + (1 - Y_t)\varphi(R_{f,t+1}; b, K) = 1$, must satisfy $R_{f,t+1} > R_{f,\text{EZ}}$. By Proposition 14-(ii) again, we conclude that $\theta^*_{0,t} = \varphi(R_{f,t+1}; 0, K) < \varphi(R_{f,\text{EZ}}; 0, K) = 1$ and thus $\theta^*_{1,t} = \varphi(R_{f,t+1}; b, K) > 1$. Consequently, Proposition 14-(iv) implies that $\mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathscr{F}_t] > 0$. Furthermore, because $R_{f,t+1} < \mathbb{E}[R_{t+1}]$ according to Theorem 4 and $\frac{\partial f_0}{\partial \theta}(0, a) > 0$ for any $a < \mathbb{E}[R_{t+1}]$, we conclude that $\theta^*_{0,t} = \varphi(R_{f,t+1}; 0, K) > 0$.

Finally, the case in which $K > K^*$ can be proved similarly. $\square$

Proof Proposition 16. Recall $\theta^*_{0,t} = \varphi(R_{f,t+1}; 0, K)$ and $\theta^*_{1,t} = \varphi(R_{f,t+1}; b, K)$, where φ is the optimal solution to (3.4.1) with $X = R_{t+1}$. We first consider the case in which $K < K^*$. In this case, $\varphi(R_{f,t+1}; b, K) > 1 > \varphi(R_{f,t+1}; 0, K)$ according to Proposition 15-(ii), so equilibrium equation (3.4.2) yields

$$Y_t = \frac{1}{1 + \big(1 - \varphi(R_{f,t+1}; 0, K)\big)/\big(\varphi(R_{f,t+1}; b, K) - 1\big)}.$$

Proposition 14-(ii) immediately yields that Y_t is strictly decreasing in $R_{f,t+1}$. Consequently, the equilibrium $R_{f,t+1}$ is strictly decreasing in Y_t. The other two cases can be proved similarly. $\square$

Proof of Proposition 17. Denote $\theta^*_{0,t}$ and $\theta^*_{1,t}$ as the optimal percentage allocations to the stock of the EZ- and LA-agent, respectively. Then, $\theta^*_{0,t} = \varphi(R_{f,t+1}; 0, K)$ and $\theta^*_{1,t} = \varphi(R_{f,t+1}; b, K)$, where φ is the optimal solution to (3.4.1) with $X = R_{t+1}$. Denote $\underline{x} := \operatorname{essinf} R_{t+1}$ and $\overline{x} := \operatorname{esssup} R_{t+1}$.

When $K < K^*$, Propositions 15 and 16 yield $R_{f,\mathrm{EZ}} < R_{f,\mathrm{LA}}$, $R_{f,t+1} \in [R_{f,\mathrm{EZ}}, R_{f,\mathrm{LA}}]$, and $\theta^*_{0,t} < 1 < \theta^*_{1,t}$ for any $Y_t \in (0,1)$. In consequence, because $R_{f,\mathrm{EZ}} > \underline{x}$ according to Theorem 4, we conclude from Proposition 14-(ii) that

$$1 < \theta^*_{1,t} \le \varphi(R_{f,\mathrm{EZ}}; b, K) < +\infty. \tag{0.66}$$

On the other hand, recall from Proposition 15-(iii) that

$$1 > \theta^*_{0,t} \ge 0. \tag{0.67}$$

When $K > K^*$, Propositions 15 and 16 yield $R_{f,\mathrm{EZ}} > R_{f,\mathrm{LA}}$, $R_{f,t+1} \in [R_{f,\mathrm{LA}}, R_{f,\mathrm{EZ}}]$, and $\theta^*_{0,t} > 1 > \theta^*_{1,t}$ for any $Y_t \in (0,1)$. On the one hand, recall that $\varphi(a; 0, K)$ is the maximizer of $f_0(\theta, a)$ in θ, where $f_0(\theta, a)$ is defined as in (0.57). Furthermore, from (0.62), we obtain

$$\frac{\partial f_0}{\partial \theta}(\theta, a) = e^{\mathbb{E}(\ln(c + R_{t+1} - \underline{x}))} \left\{1 - (c + a - \underline{x})\mathbb{E}[1/(c + R_{t+1} - \underline{x})]\right\},$$

where $c := (a/\theta) - (a - \underline{x})$. In consequence, if $\lim_{c \downarrow 0} \mathbb{E}[1/(c + R_{t+1} - \underline{x})] = +\infty$, then there exists

$\varepsilon_0 > 0$ such that for any $a \geq R_{f,\text{LA}}$,

$$\begin{aligned}\frac{\partial f_0}{\partial \theta}\left(\frac{a}{\varepsilon_0 + a - \underline{x}}, a\right) &= e^{\mathbb{E}(\ln(\varepsilon_0 + R_{t+1} - \underline{x}))}\left\{1 - (\varepsilon_0 + a - \underline{x})\mathbb{E}[1/(\varepsilon_0 + R_{t+1} - \underline{x})]\right\} \\ &< e^{\mathbb{E}(\ln(\varepsilon_0 + R_{t+1} - \underline{x}))}\left\{1 - (R_{f,\text{LA}} - \underline{x})\mathbb{E}[1/(\varepsilon_0 + R_{t+1} - \underline{x})]\right\} \\ &< 0.\end{aligned}$$

As a result, by the concavity of $f_0(\theta, a)$ in θ, we conclude that $\varphi(a; 0, K) < a/(\varepsilon_0 + a - \underline{x})$ for any $a \geq R_{f,\text{LA}}$. Consequently, we conclude that

$$1 < \theta^*_{0,t} < R_{f,t+1}/(\varepsilon_0 + R_{f,t+1} - \underline{x}) < \overline{\theta}(R_{f,t+1}) \leq \overline{\theta}(R_{f,\text{LA}}) < +\infty. \quad (0.68)$$

If $\lim_{c\downarrow 0} \mathbb{E}[1/(c + X - \underline{x})] < +\infty$, then, $\mathbb{P}(X = \underline{x}) = 0$, $a + \overline{\theta}(a)(X - a) > 0$ almost surely for any $a \in (\underline{x}, \overline{x})$, and $\mathbb{E}[1/(X - \underline{x})] < +\infty$. In consequence,

$$1 < \theta^*_{0,t} \leq \overline{\theta}(R_{f,t+1}) \leq \overline{\theta}(R_{f,\text{LA}}) < +\infty. \quad (0.69)$$

On the other hand, we conclude from Proposition 15 and Proposition 14-(ii) that

$$-\infty < \varphi(R_{f,\text{EZ}}; b, K) \leq \varphi(R_{f,t+1}; b, K) = \theta^*_{1,t} < 1. \quad (0.70)$$

Denote Γ_t as the post-consumption wealth ratio of the LA- and EZ-agents, i.e., $\Gamma_t := (1 - Y_t)/Y_t$. Recalling (0.67) when $K < K^*$, (0.68) when $K > K^*$ and $\lim_{c\downarrow 0} \mathbb{E}[1/(c + R_{t+1} - \underline{x})] = +\infty$, and (0.69) and the inequality $a + \overline{\theta}(a)(R_{t+1} - a) > 0$ almost surely for any $a \in (\underline{x}, \overline{x})$ when $K > K^*$ and $\lim_{c\downarrow 0} \mathbb{E}[1/(c + R_{t+1} - \underline{x})] < +\infty$, we conclude that $Y_t > 0$, $t \geq 0$, so $\{\Gamma_t\}$ is well defined.

Straightforward calculation yields

$$\Gamma_{t+1} = \frac{(1-c^*_{1,t+1})W_{1,t+1}}{(1-c^*_{0,t+1})W_{0,t+1}} = \frac{W_{1,t+1}}{W_{0,t+1}} = \frac{[R_{f,t+1}+\theta^*_{1,t}(R_{t+1}-R_{f,t+1})](1-c^*_{1,t})W_{1,t}}{[R_{f,t+1}+\theta^*_{0,t}(R_{t+1}-R_{f,t+1})](1-c^*_{0,t})W_{0,t}}$$

$$= \left[1+(\theta^*_{1,t}-\theta^*_{0,t})A_{t+1}\right]\Gamma_t, \tag{0.71}$$

where

$$A_{t+1} := \frac{R_{t+1}-R_{f,t+1}}{R_{f,t+1}+\theta^*_{0,t}(R_{t+1}-R_{f,t+1})}. \tag{0.72}$$

Next, we show that Γ_t is integrable for any $t \geq 0$. Because Γ_0 is constant and thus integrable, we only need to show that if Γ_t is integrable, so is Γ_{t+1}. We first note from (0.66)–(0.70) that $\theta^*_{1,t}-\theta^*_{0,t}$ is uniformly bounded. For notational simplicity, we simply write $|\theta^*_{1,t}-\theta^*_{0,t}| \leq d$ for some constant $d < +\infty$. Note also that A_{t+1} is decreasing in $\theta^*_{0,t}$. In consequence, when $K < K^*$, we conclude from (0.67) that

$$-\frac{R_{f,t+1}}{R_{t+1}} \leq \frac{R_{t+1}-R_{f,t+1}}{R_{t+1}} \leq A_{t+1} \leq \frac{R_{t+1}-R_{f,t+1}}{R_{f,t+1}} \leq \frac{R_{t+1}}{R_{f,t+1}}.$$

Because $R_{f,t+1} \in [R_{f,\mathrm{EZ}}, R_{f,\mathrm{LA}}]$ and $R_{t+1} \geq \underline{x}$, we immediately conclude that

$$|A_{t+1}| \leq R_{f,\mathrm{LA}}/\underline{x} + (1/R_{f,\mathrm{EZ}})R_{t+1}. \tag{0.73}$$

As a result,

$$\Gamma_{t+1} \leq \left[1+|\theta^*_{1,t}-\theta^*_{0,t}||A_{t+1}|\right]\Gamma_t \leq \left[1+d\big(R_{f,\mathrm{LA}}/\underline{x}+(1/R_{f,\mathrm{EZ}})R_{t+1}\big)\right]\Gamma_t,$$

which implies that Γ_{t+1} is integrable because Γ_t and R_{t+1} are integrable and independent of each other. When $K > K^*$ and $\lim_{c\downarrow 0} \mathbb{E}[1/(c + R_{t+1} - \underline{x})] = +\infty$, we conclude from (0.68) that

$$\frac{R_{t+1} - R_{f,t+1}}{R_{f,t+1} + (R_{f,t+1}/(\varepsilon_0 + R_{f,t+1} - \underline{x}))\,(R_{t+1} - R_{f,t+1})} \le A_{t+1} \le \frac{R_{t+1} - R_{f,t+1}}{R_{t+1}}.$$

Because $R_{f,t+1} \le R_{f,\mathrm{EZ}}$ and $R_{t+1} \ge \underline{x}$, we conclude that

$$|A_{t+1}| \le 1 + \frac{R_{f,t+1}}{R_{f,t+1} + (R_{f,t+1}/(\varepsilon_0 + R_{f,t+1} - \underline{x}))\,(\underline{x} - R_{f,t+1})} \le 1 + \frac{\varepsilon_0 + R_{f,\mathrm{EZ}} - \underline{x}}{\varepsilon_0}, \tag{0.74}$$

showing that A_{t+1} is bounded. In consequence, Γ_{t+1} is integrable. When $K > K^*$ and $\lim_{c\downarrow 0} \mathbb{E}[1/(c + R_{t+1} - \underline{x})] < +\infty$, we already showed that $\mathbb{E}[1/(R_{t+1} - \underline{x})] < +\infty$. Moreover, (0.69) yields that

$$\frac{(R_{t+1}/R_{f,t+1} - 1)(R_{f,t+1} - \underline{x})}{R_{t+1} - \underline{x}} = \frac{R_{t+1} - R_{f,t+1}}{R_{f,t+1} + \overline{\theta}(R_{f,t+1})(R_{t+1} - R_{f,t+1})} \le A_{t+1} \le \frac{R_{t+1} - R_{f,t+1}}{R_{t+1}}.$$

Because $R_{f,t+1} \le R_{f,\mathrm{EZ}}$, we conclude

$$|A_{t+1}| \le 1 + \frac{R_{f,t+1}}{R_{t+1} - \underline{x}} \le 1 + \frac{R_{f,\mathrm{EZ}}}{R_{t+1} - \underline{x}}. \tag{0.75}$$

As a result,

$$\Gamma_{t+1} \le \left[1 + |\theta^*_{1,t} - \theta^*_{0,t}||A_{t+1}|\right]\Gamma_t \le \left[1 + d\big(1 + R_{f,\mathrm{EZ}}/(R_{t+1} - \underline{x})\big)\right]\Gamma_t,$$

which implies that Γ_{t+1} is integrable because Γ_t and $1/(R_{t+1} - \underline{x})$ are integrable and independent of each other.

The above argument also yields that A_{t+1} is integrable. Moreover, we conclude from (0.60) that

$$\mathbb{E}[A_{t+1}|\mathscr{F}_t] = \mathbb{E}[A_{t+1}|R_{f,t+1}] = \frac{\partial f_0}{\partial \theta}(\theta^*_{0,t}, R_{f,t+1})(f_0(\theta^*_{0,t}, R_{f,t+1}))^{-1}.$$

When $K < K^*$, (0.67) holds and thus $\frac{\partial f_0}{\partial \theta}(\theta^*_{0,t}, R_{f,t+1}) \leq 0$ due to the first-order condition for the optimality of $\theta^*_{0,t}$. When $K > K^*$, $\theta^*_{0,t} > 1$, so $\frac{\partial f_0}{\partial \theta}(\theta^*_{0,t}, R_{f,t+1}) \geq 0$ due to the first-order condition for the optimality of $\theta^*_{0,t}$. Consequently,

$$\mathbb{E}[\Gamma_{t+1}|\mathscr{F}_t] = \Gamma_t \left[1 + (\theta^*_{1,t} - \theta^*_{0,t})\mathbb{E}[A_{t+1}|\mathscr{F}_t]\right] \leq \Gamma_t,$$

showing that $\{\Gamma_t\}$ is a positive super-martingale. By the martingale convergence theorem, Γ_t converges almost surely and in L^1 to a nonnegative $\mathscr{F}_\infty$-measurable random variable Γ_∞. Consequently, Y_t converges almost surely to $Y_\infty := \frac{1}{1+\Gamma_\infty} \in (0, 1]$. Because $R_{f,t+1}$ is a continuous function of $Y_t \in [0, 1]$, $R_{f,t+1}$ converges almost surely, and we denote the limit as $R_{f,\infty}$. Because $R_{f,t+1}$ is bounded by a_1 and a_2, where $a_1 := \min(R_{f,\text{LA}}, R_{f,\text{EZ}}) > \underline{x}$ and $a_2 := \max(R_{f,\text{LA}}, R_{f,\text{EZ}}) < \overline{x}$, and $\varphi(a; 0, K)$ and $\varphi(a; b, K)$ are continuous in $a \in (\underline{x}, \overline{x})$, we conclude that

$$\lim_{t\to\infty} \theta^*_{0,t} = \lim_{t\to\infty} \varphi(R_{f,t+1}; 0, K) = \varphi(R_{f,\infty}; 0, K), \quad \lim_{t\to\infty} \theta^*_{1,t} = \lim_{t\to\infty} \varphi(R_{f,t+1}; b, K) = \varphi(R_{f,\infty}; b, K)$$

almost surely.

We claim that $\varphi(R_{f,\infty}; 0, K) \neq \varphi(R_{f,\infty}; b, K)$. Indeed, when $K < K^*$, $\varphi(R_{f,t+1}; 0, K) < 1 < \varphi(R_{f,t+1}; b, K)$, so we must have $\varphi(R_{f,\infty}; 0, K) \leq 1 \leq \varphi(R_{f,\infty}; b, K)$. In consequence, $\varphi(R_{f,\infty}; 0, K) = \varphi(R_{f,\infty}; b, K)$ if and only if $\varphi(R_{f,\infty}; 0, K) = \varphi(R_{f,\infty}; b, K) = 1$. A similar argument shows that this sufficient and necessary condition is also true when $K > K^*$. Now, for the sake of contradiction, suppose $\varphi(R_{f,\infty}; 0, K) = \varphi(R_{f,\infty}; b, K) = 1$. Then, Proposition 14-(iv) implies that $\mathbb{E}[\nu(R_{t+1} - R_{f,\infty})] = 0$. In addition, $\varphi(R_{f,\infty}; 0, K) = 1$ implies $R_{f,\infty} = R_{f,\text{EZ}}$. Thus, we conclude $\mathbb{E}[\nu(R_{t+1} - R_{f,\text{EZ}})] = 0$, which implies $K = K^*$. Contradiction! Therefore, we must have $\varphi(R_{f,\infty}; 0, K) \neq \varphi(R_{f,\infty}; b, K)$ and thus $\lim_{t\to\infty} \theta^*_{0,t} \neq \lim_{t\to\infty} \theta^*_{1,t}$.

Next, we show $\mathbb{P}(\lim_{t\to\infty} A_t = 0) = 0$. To this end, it is sufficient to find some $\epsilon > 0$ such that $\mathbb{P}(\cap_{n=1}^{\infty} \cup_{t=n}^{\infty} \{|A_t| > \epsilon\}) = 1$, i.e., such that $\lim_{n\to\infty} \mathbb{P}(\cap_{t=n}^{\infty}\{|A_t| \leq \epsilon\}) = 0$.

Recall that $a_1 = \min(R_{f,\text{EZ}}, R_{f,\text{LA}}) > \underline{x}$ and $a_2 = \max(R_{f,\text{EZ}}, R_{f,\text{LA}}) < \overline{x}$. There exist $\eta > 0$ and $\delta > 0$ such that $\sup_{a\in[a_1,a_2]} \mathbb{P}(R_{t+1} \in [a - \delta, a + \delta]) \leq 1 - \eta$. For any $a \in [a_1, a_2]$, consider

$$h(a) := \mathbb{P}\left(\left|\frac{R_{t+1} - a}{a + \varphi(a; 0, K)(R_{t+1} - a)}\right| \leq \epsilon\right) = \mathbb{P}\left(R_{t+1} \in \left[a - \frac{a\epsilon}{1 + \varphi(a; 0, K)\epsilon}, a + \frac{a\epsilon}{1 - \varphi(a; 0, K)\epsilon}\right]\right).$$

Choose $\epsilon > 0$ such that

$$\sup_{a\in[a_1,a_2]} \left[\frac{a\epsilon}{1 - \varphi(a; 0, K)\epsilon} + \frac{a\epsilon}{1 + \varphi(a; 0, K)\epsilon}\right] = \sup_{a\in[a_1,a_2]} \frac{2a\epsilon}{1 - (\varphi(a; 0, K)\epsilon)^2} \leq \frac{2a_2\epsilon}{1 - (\varphi(a_1; 0, K)\epsilon)^2} < \delta.$$

Then, we have $\sup_{a\in[a_1,a_2]} h(a) \leq 1 - \eta$.

For each $t \geq 0$, because $R_{f,t+1} \in [a_1, a_2]$, we have $\mathbb{P}(|A_{t+1}| \leq \epsilon | \mathscr{F}_t) = h(R_{f,t+1}) \leq 1 - \eta$. Then, for any $1 \leq n < N$,

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{t=n}^{N}\{|A_t| \leq \epsilon\}\right) &= \mathbb{E}\left[\prod_{t=n}^{N} 1_{\{|A_t|\leq\epsilon\}}\right] = \mathbb{E}\left[\mathbb{E}\left(\prod_{t=n}^{N} 1_{\{|A_t|\leq\epsilon\}} | \mathscr{F}_{N-1}\right)\right] \\
&= \mathbb{E}\left[\left(\prod_{t=n}^{N-1} 1_{\{|A_t|\leq\epsilon\}}\right) \mathbb{P}\left(|A_N| \leq \epsilon | \mathscr{F}_{N-1}\right)\right] \\
&\leq (1 - \eta)\mathbb{E}\left[\left(\prod_{t=n}^{N-1} 1_{\{|A_t|\leq\epsilon\}}\right)\right] \leq (1 - \eta)^{N-n},
\end{aligned}$$

where the last inequality is due to mathematical induction. Sending N to infinity, we obtain $\mathbb{P}\left(\bigcap_{t=n}^{\infty}\{|A_t| \le \epsilon\}\right) = 0$ for each $n \ge 1$, so we conclude $\mathbb{P}(\lim_{t\to\infty} A_t = 0) = 0$.

Now, because $\lim_{t\to\infty}\theta^*_{0,t} \neq \lim_{t\to\infty}\theta^*_{1,t}$, we conclude from (0.71) that $\lim_{t\to\infty}\Gamma_t > 0$ implies $\lim_{t\to\infty} A_t = 0$. As a result, $\mathbb{P}(\lim_{t\to\infty}\Gamma_t > 0) \le \mathbb{P}(\lim_{t\to\infty} A_t = 0) = 0$. Consequently, we conclude $\lim_{t\to\infty}\Gamma_t = 0$ almost surely, i.e., $\lim_{t\to\infty} Y_t = 1$ almost surely. □

Proof of Corollary 1. The equilibrium asset prices with EZ-agents only are the same as those in Theorem 3. We need to prove that in the presence of LA-agents satisfying (i), (ii), and (iii), the market equilibrium does not change.

Straightforward calculation yields that $\tilde{G}_{i,t} = (1-c_{i,t})W_{i,t}\theta_{i,t}\left[\tilde{g}_{i,+,t}\mathbf{1}_{\theta_{i,t}\ge 0} + \tilde{g}_{i,-,t}\mathbf{1}_{\theta_{i,t}<0}\right]$, where

$$\begin{aligned}
\tilde{g}_{i,+,t} &:= \mathcal{E}_{i,+}(R_{t+1} - R_{f,t+1}|\mathscr{F}_t) - K_i\mathcal{E}_{i,-}(R_{t+1} - R_{f,t+1}|\mathscr{F}_t),\\
\tilde{g}_{i,-,t} &:= -\big(\mathcal{E}_{i,+}(R_{f,t+1} - R_{t+1}|\mathscr{F}_t) - K_i\mathcal{E}_{i,-}(R_{f,t+1} - R_{t+1}|\mathscr{F}_t)\big)
\end{aligned}$$

and $\mathcal{E}_{i,\pm}(X|\mathscr{F}_t)$ are defined in the same way as in (3.5.2) except that the distribution of X is replaced by its conditional distribution given $\mathscr{F}_t$. Then, the following dynamic programming equation holds for the LA-agents:

$$\begin{aligned}
\Psi_{i,t} = \max_{c_{i,t},\theta_{i,t}} H\Bigg(& c_{i,t}, (1-c_{i,t})M(\Psi_{i,t+1}R_{i,t+1}|\mathscr{F}_t)\\
&\times\left(1 + b_i\left(M(R_{i,t+1}|\mathscr{F}_t)\right)^{-1}\theta_{i,t}\left[\tilde{g}_{i,+,t}\mathbf{1}_{\theta_{i,t}\ge 0} + \tilde{g}_{i,-,t}\mathbf{1}_{\theta_{i,t}<0}\right]\right)\Bigg),
\end{aligned} \tag{0.76}$$

For any X, denoting $F_{X|\mathscr{F}_t}$ and $F^{-1}_{X|\mathscr{F}_t}$ as the distribution function of X given $\mathscr{F}_t$ and its left-continuous inverse and applying change-of-variable, we obtain

$$\mathcal{E}_{i,+}(X|\mathscr{F}_t) = \int_0^{+\infty} xd[-T_{i,+}(1-F_{X|\mathscr{F}_t}(x))] = \int_0^1 F^{-1}_{X|\mathscr{F}_t}(z)\mathbf{1}_{F^{-1}_{X|\mathscr{F}_t}(z)\geq 0}T'_{i,+}(1-z)dt,$$
$$\mathcal{E}_{i,-}(-X|\mathscr{F}_t) = -\int_{-\infty}^{0} xd[T_{i,-}(F_{-X|\mathscr{F}_t}(x))] = \int_0^1 F^{-1}_{X|\mathscr{F}_t}(z)\mathbf{1}_{F^{-1}_{X|\mathscr{F}_t}(z)\geq 0}T'_{i,-}(1-z)dt.$$

In consequence, Assumption 7 yields that $\mathcal{E}_{i,+}(X|\mathscr{F}_t) \leq K_i\mathcal{E}_{i,-}(-X|\mathscr{F}_t)$ and thus $\tilde{g}_{i,+,t} \leq \tilde{g}_{i,-,t}$.

Now, recall that $\operatorname{essinf} Z_{t+1} > 0$, $\mathcal{E}_{i,+}(Z_{t+1}) < +\infty$, $R_{t+1} = Z_{t+1}/\tilde{\alpha}_{\mathrm{EZ}}$, and $R_{f,t+1}$ is constant. Then, $\mathcal{E}_{i,\pm}(\theta_{i,t}(R_{t+1} - R_{f,t+1}))$ and, consequently, $\tilde{G}_{i,t}$ are well defined for any $\theta_{i,t}$. Moreover, (3.5.3) yields that $\tilde{g}_{i,-,t} \geq \tilde{g}_{i,+,t} = 0$. Consequently, $\theta_{i,t}\left[\tilde{g}_{i,+,t}\mathbf{1}_{\theta_{i,t}\geq 0} + \tilde{g}_{i,-,t}\mathbf{1}_{\theta_{i,t}<0}\right] \leq 0$ for any $\theta_{i,t}$ and the inequality becomes an equality for any $\theta_{i,t} \geq 0$. Then, the same argument as used in the proof of Theorem 3 shows that the equilibrium does not change in the presence of the LA-agents and those agents behave the same as the EZ-agents. □

Proof of Corollary 2. The proof is the same as that for Proposition 13. □

Proof of Corollary 3. For each $a \in (\underline{x}, \overline{x})$, denote

$$\tilde{g}_+(a) := \mathcal{E}_+(X-a) - K\mathcal{E}_-(X-a), \quad \tilde{g}_-(a) := -\left(\mathcal{E}_+(a-X) - K\mathcal{E}_-(a-X)\right).$$

Then, the objective function of (3.5.6) becomes

$$\tilde{f}(\theta, a) := f_0(\theta, a) + b\theta\left[\tilde{g}_+(a)\mathbf{1}_{\theta\geq 0} + \tilde{g}_-(a)\mathbf{1}_{\theta<0}\right],$$

where f_0 is defined as in (0.57). Therefore, the analysis of $\tilde{\varphi}(a; b, K)$ should be the same as that of $\varphi(a; b, K)$ in the proof of Proposition 14 except that $g_\pm$ therein are replaced by $\tilde{g}_\pm$.

It is straightforward to see that $\tilde{g}_{\pm}$ are strictly decreasing and continuous in $a \in (\underline{x}, \overline{x})$. Moreover, $\lim_{a\downarrow\underline{x}} g_{\pm}(a) > 0$ and $\lim_{a\uparrow\overline{x}} g_{\pm}(a) < 0$. Furthermore, using the same argument as in the proof of Corollary 1, we can show that $\tilde{g}_{+}(a) \geq \tilde{g}_{-}(a)$ for any $a \in (\underline{x}, \overline{x})$, implying that $\tilde{f}(\theta, a)$ is strictly concave in θ. We further claim that $\tilde{g}_{+}(a)$ is concave in a. Indeed, denoting F_X and F_X^{-1} as the distribution function of X and its left-continuous inverse, respectively, and applying change-of-variable, we obtain

$$\tilde{g}_{+}(a) = \int_0^1 \left[\left(F_X^{-1}(z) - a\right) T_{+}'(1-z)\mathbf{1}_{F_X^{-1}(z)-a\geq 0} + K\left(F_X^{-1}(z) - a\right) T_{-}'(z)\mathbf{1}_{F_X^{-1}(z)-a<0}\right] dz. \quad (0.77)$$

Because of Assumption 8, for each $z \in (0,1)$, $T_{+}'(1-z) \leq KT_{-}'(z)$ and thus $xT_{+}'(1-z)\mathbf{1}_{x\geq 0} + KxT_{-}'(z)\mathbf{1}_{x<0}$ is concave in x. In consequence, for each $z \in (0,1)$, the integrand in (0.77) is concave in a and, consequently, $\tilde{g}_{+}(a)$ is concave in a.

Now, the same proof as of Proposition 14 yields (i)–(iv). Finally, because $\tilde{f}(\theta, a)$ is concave in θ, when short-selling is now allowed, the unique optimal solution to (3.5.6) is $\max(\tilde{\varphi}(a; b, K), 0)$. □

Proof of Corollary 4. The proof is the same as of Theorem 4. □

Proof of Corollary 5. The proof is the same as of Proposition 15. □

Proof of Corollary 6. The proof is the same as that of Proposition 16. □

Proof of Corollary 7. When short-selling is not allowed or esssup $R_{t+1} = +\infty$, the proof of Proposition 17 is still valid in the presence of probability weighting with K^* and φ replaced by $\tilde{K}^*$ and $\tilde{\varphi}$, respectively. When esssup $R_{t+1} < +\infty$ and short-selling is allowed, (0.67) and thus (0.73) no longer hold when $K < K^*$ because $\theta_{0,t}^*$ can go negative. However, using the same argument as that in the proof of Proposition 17, we can still show that Γ_{t+1} and A_{t+1} defined in (0.71) and (0.72), respectively, are integrable. The rest of the proof is the same as that of Proposition 17. □