The Existence of an Optimal Path in a Growth Model with Endogenous Technical Change

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1 Introduction

Consider an infinite horizon growth model where the production function changes through time in response to investment in R & D (Gruenwald, [5]). At the end of each period the sole good can be either consumed, saved as next period's capital, or invested in a research sector to increase next period's productivity. Since each unit of capital becomes more productive with R & D, the overall "production possibilities frontier" across time may exhibit increasing returns to scale. The problem is to find conditions which assure the existence of an optimal consumption-accumulation-R & D investment path. Mathematically, the problem is one of maximizing a (non-linear) functional over a function space, the space of feasible paths, which is in general a non-convex set. The feasible paths satisfy a difference equation representing capital accumulation, R&D investment and technological change through time. The problem is of interest because technology is endogenously determined, and because of its possible non-convexity. The standard conditions for existence, which rely on convexity (e.g. Von Weisacker [8]), do not apply. Classical pieces on growth without convexities (Dixit, Mirrlees and Stern [4], Weitzman [7]) do not allow a choice of R&D investment which endogenously determines technological change. Furthermore, standard compactness-continuity arguments do not work *a priori* because the logarithmic utility function which we use is ill-defined on some regions of the space which include compactness-continuity arguments do not work *a priori* because the logarithmic utility function which we use is ill-defined on some regions of the space which include paths with some zero components, and the feasible set is unbounded, as it may grow through time in response to technological change.

Here is a summary of the paper. We find a growth path B which is not feasible, and which grows exponentially. This path B bounds all feasible paths. Using this bound we define a finite measure on the set of integers, λ, and consider a "weighted" Banach space H_λ of all sequences which are summable with respect to this measure λ. This space includes all bounded sequences, and many exponentially growing sequences; in particular, it contains all the feasible growth paths in our economy. Therefore, without loss of generality, we consider the problem of maximizing a welfare function in the space H_λ, restricted to the set of all feasible growth paths. For this it would suffice to prove that the set of all feasible paths is compact, and the welfare function is continuous on it in the norm of the space H_λ. However, these conditions are not satified: the logarithmic utility function is undefined over certain paths, and the closed bounded subset of H_λ which includes all the feasible growth paths, is not closed and therefore not compact. However, we find a closed subset of the space H_λ which contains all feasible paths yielding utility values which exceed a minimal level. We prove that this set is compact and that, when restricted to this set, the welfare function is norm continuous. This set contains an optimum of our problem, thus establishing existence of an optimal growth path with endogenous technical change. The techniques utilized here rely on the use of weighted L_p spaces introduced in Chichilnisky [1], [2] and Chichilnisky and Kalman [3].

2 The Model

The problem of a planner is to maximize the discounted logarithm of consumption over an infinite time horizon:

$$\max_{\{C_t, Z_t, K_{t+1}\}_{t=0}^{\infty}} W(C_t) = \sum_{t=0}^{\infty} \beta^{-t} \ln C_t \tag{1}$$

subject to

$$\eta_t \, K_t = C_t + Z_t + K_{t+1} \tag{2}$$

$$\eta_{t+1} = \eta_t \, g[\gamma \frac{Z_t}{\eta_t K_t}] \tag{3}$$

$$\eta(0) = \eta_0, \;\; K(0) = K_0$$

$$0 < \eta_0, \, K_0 < \infty, \;\; \beta > 1 \tag{4}$$

$$\eta_0 \, g(\gamma) > 1 \tag{5}$$

where

$C_t =$ consumption in period t
$K_t =$ capital stock in period t (which fully depreciates)
$\eta_t =$ capital productivity in period t
$Z_t =$ research allocation in period t (used to increase η_{t+1})
$g(.) =$ the deterministic research sector function
$\gamma =$ research productivity parameter
$\beta =$ discount factor.

The following assumptions are made on the function $g(.)$:
(A1) $g(0) = 1$
(A2) $g'(.) > 0$
(A3) g is C^1 on the interval (o, γ).
The *feasible consumption set* for an initial $\eta_0 K_0$ is

$$F(\eta_0 K_0) = \left\{ \{C_t\}_{t=0}^{\infty},\, C_t \in R^+ \text{ s.t. there exists } \{\eta_t, \eta_{t+1}, K_t, K_{t+1}, Z_t\} \text{ satisfying (2) and (3)} \right\}.$$

The problem is to find a sequence $\{C\} = \{C_t\}_{t=0}^{\infty}$ in $F(\eta_0 K_0)$ which maximizes W.

3 Feasible Paths

Our next task will be to show that the space of feasible paths $F(\eta_0 K_0)$ is naturally included in a Banach space, consisting of sequences which are summable in absolute value, with respect to a finite measure on the space of integers. First we define our Banach space H_λ and then we show in Lemmas 1 and 2 that all feasible paths are included in this space.

Let Z denote the set of integers and R the real line. Define the space of sequences $\mathrm{H}_\lambda \equiv \left\{ f : Z \to R : \sum f_t \, \lambda^{-t^2} < \infty, \text{ for } \lambda > \eta_0 \mathrm{g}(\gamma) > 1 \right\}$. H_λ is a Banach space with the norm

$$\|f\| = \sum |f_t| \, \lambda^{-t^2}.$$

This is a weighted L_1 space with the bounded measure $\lambda(t) = \lambda^{-t^2}$ for $t \in Z$.

Lemma 1 *For all sequences $\{K_t\}_{t=0}^{\infty}$ and $\{\eta_t\}_{t=0}^{\infty}$ which satisfy (2) and (3):*

$$\begin{aligned} &a) \quad K_t \le \eta_0 K_0 \, g(\gamma)^{\frac{t(t-1)}{2}} \\ &b) \quad \eta_t \le \eta_0 \, g(\gamma)^t \\ &c) \quad \eta_t K_t \le \eta_0^{t+1} K_0 \, g(\gamma)^{\frac{t(t+1)}{2}} \end{aligned} \tag{6}$$

for $t = 0, 1, ..., \infty$. In particular, all feasible consumption paths are in H_λ, i.e. $F(\eta_0 K_0) \subset H_\lambda$.

Proof. Consider the (infeasible) path $\{\bar{\eta}_t, \bar{K}_t\}_{t=0}^{\infty}$ where

$$\bar{\eta}_0 = \eta_0, \ \bar{K}_0 = K_0, \ \bar{\eta}_{t+1} = \bar{\eta}_t \ g(\gamma) \text{ and } \bar{K}_{t+1} = \bar{\eta}_t \bar{K}_t$$

for all t. Note that, from direct calculation:

$$\bar{K}_1 = \eta_0 K_0$$

$$\bar{K}_2 = \eta_1 K_1 = \eta_0^2 K_0 g(\gamma)$$
$$\bar{K}_3 = \eta_2 K_2 = \eta_0^3 K_0 g(\gamma)^3$$
$$\bar{K}_4 = \eta_3 K_3 = \eta_0^4 K_0 g(\gamma)^6.$$

In fact, induction yields

$$\bar{K}_t = \eta_0^t K_0 g(\gamma)^{\frac{t(t-1)}{2}}$$

which proves a). Part b) is obvious. From a) and b) we obtain

$$\begin{aligned}
\bar{\eta}_1 \bar{K}_1 &= \eta_0^2 K_0 g(\gamma) \\
\bar{\eta}_2 \bar{K}_2 &= \eta_0^3 K_0 g(\gamma)^3 \\
\bar{\eta}_3 \bar{K}_3 &= \eta_0^4 K_0 g(\gamma)^6 \\
\bar{\eta}_4 \bar{K}_4 &= \eta_0^5 K_0 g(\gamma)^{10}.
\end{aligned}$$

Again, induction yields

$$\bar{\eta}_t \bar{K}_t = \eta_0^{t+1} K_0 g(\gamma)^{\frac{t(t+1)}{2}}$$

which proves c), since $\bar{\eta}_t \bar{K}_t$ exceeds any path satisfying (2) and (3).□

Lemma 2 *The set $F = F(\eta_0 K_0)$ is a compact subset of H_λ.*

Proof. Let $B = \{B_t\}_{t=0}^{\infty}$ denote the sequence $\{\eta_0^{t+1} K_0 g(\gamma)^{\frac{t(t+1)}{2}}\}$ from Lemma 1. Since $\forall \ \{C\} \in F(\eta_0 K_0), C_t \leq \eta_t K_t$, it follows that $C_t \leq B_t$, $\forall t$. Note that $B \in H_\lambda$. This is because for $t \geq 2$

$$\eta_0^{t+1} K_0 g(\gamma)^{\frac{t(t+1)}{2}} < [\eta_0 g(\gamma)]^{\frac{t(t+1)}{2}} \eta_0 K_0 < \lambda^{\frac{t(t+1)}{2}} \eta_0 K_0$$

since $\lambda > \eta_0 g(\gamma)$ by assumption. This implies that $\sum_{t=0}^{\infty} B_t \lambda^{-t^2} < \infty$.

The set $F(\eta_0 K_0) \subset H_\lambda$ is therefore bounded above by the sequence $B \in H_\lambda$ and is also bounded below by the zero sequence because consumption $\widetilde{C}$ is positive. Since the function g is continuous, F is a closed subset of H_λ. Since F is a norm bounded subset of H_λ, by Banach Alaoglu's theorem, F is weak* compact. By definition, this means that every sequence $\{f^n\}$ in H_λ has a subsequence which converges pointwise $f^n \rightarrow_w f^*$. Since f^* is also bounded by $B \in H_\lambda$, by Lebesgue's bounded convergence theorem the convergence is also in the norm, i.e., $f^n \rightarrow_{\|\cdot\|} f^*$ It follows that $F(\eta_0 K_0)$ is compact in the norm of H_λ. □

4 Existence of an Optimal Growth Path

Our next task is to prove that, although the welfare function W is not continuous on all of H_λ, and indeed it is not defined for any sequence which contains zero consumption, if we restrict the set of feasible paths to those which attain at least a given utility level, then the set of feasible paths which yield at least this level is compact and the welfare function is well defined and continuous on this set. Thus we prove the existence of an optimal growth path.

Lemma 3 *There exists a consumption sequence* $\{\tilde{C}\} \in F(\eta_0 K_0)$ *with* $W\{\tilde{C}\} = \sum_0^\infty \beta^{-t} \ln \tilde{C}_t > -\infty$.

Proof. Consider the feasible policy $\{\tilde{C}_t\} = 1/3\tilde{\eta}_t\tilde{K}_t, \tilde{Z}_t = 1/3\tilde{\eta}_t\tilde{K}_t$, and $\tilde{K}_{t+1} = 1/3\tilde{\eta}_t\tilde{K}_t$ for all t with $\tilde{\eta}_0 = \eta_0, \tilde{K}_0 = K_0$. As in Lemma 1, straightforward calculation yields

$$\tilde{\eta}_1\tilde{K}_1 = 1/3\eta_0^2 K_0 g(\gamma/3)$$
$$\tilde{\eta}_2\tilde{K}_2 = (1/3)^2\eta_0^3 K_0 g(\gamma/3)^3$$
$$\tilde{\eta}_3\tilde{K}_3 = (1/3)^3\eta_0^4 K_0 g(\gamma/3)^6$$

By induction

$$\tilde{\eta}_t\tilde{K}_t = (1/3)^t\eta_0^{t+1}K_0.g(1/3)^{\frac{t(t+1)}{2}}$$

The utility from the consumption sequence $\{\tilde{C}\}$ is

$$W\{\tilde{C}\} = \tfrac{1}{(1-\beta^{-1})^2}\ln(1/3) + \tfrac{\beta}{(1-\beta^{-1})^2}\ln\eta_0 + \tfrac{1}{1-\beta^{-1}}\ln K_0$$
$$+\tfrac{\beta}{(1-\beta^{-1})^3}\ln\, g(\gamma/3) > -\infty. \quad \square$$

Lemma 4 *Let* $G_\epsilon = \{f \in F : W(f) \geq W(\tilde{C}) - \epsilon,$ *with* $\{\tilde{C}\}$ *as in Lemma 3*$\}$. *The function* $W(C) = \sum_0^\infty \beta^{-t}\ln C_t$ *is well defined and norm continuous on* $G_\epsilon \subset H_\lambda$, $\forall\epsilon \geq 0$.

Proof. First we show that W is well defined on G_ϵ. By Lemma 3 $G_\epsilon \neq \emptyset$. Let $x \in G_\epsilon$. Then by the definition of G_ϵ we know that $W(x) > -\infty$. We now show that $W(x) < \infty$. Since $x \in H_\lambda$, by definition $\sum_{t=0}^{\infty} x_t\lambda^{-t^2} < \infty$, which implies in particular that $\lim_{t\to\infty} x_t\lambda^{-t^2} = 0$. Therefore for t large enough, $x_t < \lambda^{t^2}$ so that $\ln\, x_t < t^2\ln\lambda$. Now $\sum_{t=0}^{\infty}\beta^{-t}t^2\ln\,\lambda < \infty$ (Knopp[6]). Therefore, W is well defined on G_ϵ. Next we prove continuity. Assume that $f^n \to f$ in the $\|.\|$ norm. Then $\lim_{n\to\infty}\sum_{t=0}^{\infty}(f_t^n - f_t)\lambda^{-t^2} = 0$. Since f^n and f are in G_ϵ, this implies that for $\overline{t}$ large enough the "cut-off" sequences $g_t^n = f_t^n$, $g_t = f_t$ for $t \leq \overline{t}$ and 0 otherwise, are also in the set G_ϵ. Therefore

$\sum_{t=0}^{\infty} \beta^{-t} |\ln g_t^n - \ln g_t| = \sum_{t=0}^{\bar{t}} \beta^{-t} |\ln f_t^n - \ln f_t| \Longrightarrow 0$ when $n \to \infty$. Furthermore, since both f^n and f are in H_λ^+, for t large enough $f_t^n < \lambda^{t^2}$ and $f_t < \lambda^{t^2}$ so that $|\ln (f_t^n)| < t^2 \ln \lambda$ and $|\ln (f_t)| < t^2 \ln \lambda$. Therefore $|\ln f_t^n - \ln f_t| < 2t^2 \ln \lambda$ and thus $\sum_{t>\bar{t}} \beta^{-t} |\ln f_t^n - \ln f_t| < \sum_{t>\bar{t}} \beta^{-t}[t^2]\ln \lambda$ which converges to 0 as $\bar{t} \to \infty$. We have therefore proven that $W(f^n) \to W(f)$, and therefore the function W is continuous on G_ϵ. □

Theorem 1 *There exists a feasible consumption path $\{C^*\}$ which maximizes $W(.)$ subject to (2) - (5).*

Proof. Let G_0 denote the set G_ϵ for $\epsilon = 0$. By Lemma 4 the set G_0 is closed because $W(.)$ is continuous on $G_\epsilon \supset G_0$ and F is closed. Since $G_0 \subset F$ and F is $\|.\|$ compact by Lemma 2, then G_0 is also $\|.\|$ compact. Finally, $W(.)$ is continuous on G_0 with the $\|.\|$ norm, so that a maximum $\{C^*\} \subset G_0$ exists. □

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