can be approximated by the conductance value from classical radiative transfer theory when diffraction effects are negligible. $G_c(d,T)$ for two unequal spheres of equal emissivity ε and temperature T is given by [54]:

$$G_c(d,T) = \frac{4\sigma T^3(4\pi R_1^2)}{[(1-\varepsilon)/\varepsilon](1+R_1^2/R_2^2)+1/F_{12}(d)}, \tag{42}$$

where F_{12} is the gap dependent view factor between the two spheres and can be approximated (to an accuracy of 1%) by [55]:

$$F_{12}(d) = \frac{1}{2}\left(1-\sqrt{1-\frac{1}{(d/R_2+R_1/R_2+1)^2}}\right). \tag{43}$$

The emissivity for a silica half-space has been computed and plotted as a function of frequency in eV in Fig. 4(b). The conductance values have been computed by integrating over the frequency range 0.041 eV to 0.164 eV, and the value of $G_c(d,T)$ has been appropriately adjusted to reflect this [43, see supplemental information]. For the radius of the spheres that we have considered in our study (R_1 = 2.5 μm, 13.7 μm) the value of $G(d)$ at $d \approx 40\,\mu$m (when near-field effects are negligible) is taken to be the value G_c at that gap and the effect of gap-dependence of view factor for smaller gaps is included by using:

$$\frac{G_c(d_1,T)}{G_c(d_2,T)} = \frac{[(1-\varepsilon)/\varepsilon](1+R_1^2/R_2^2)+1/F_{12}(d_2)}{[(1-\varepsilon)/\varepsilon](1+R_1^2/R_2^2)+1/F_{12}(d_1)}, \tag{44}$$

The comparison between G and G^{MPA} is shown in Fig. 5(a) for two spheres with $R_1 = 13.7\,\mu$m and $2.5\,\mu$m and $R_2 = 40R_1$. The error between G and G^{MPA} is plotted in Fig. 5(b). For gaps $d/R_1 < 0.1$, MPA is observed to be able to predict the exact computed values of the conductance with errors less than 1%.

Since the total radiative conductance between the two spheres has contributions from frequencies where there is resonant enhancement from surface phonon polaritons as well as contributions from non-resonant frequencies where surface phonon polaritons are not active (see Fig. 2 in [43]), it would be of interest to observe if the MPA accurately predicts the contribution from both these regions. Computed values (by DGF formalism) and MPA predictions of the spectral conductance G_ω at a resonant (0.061 eV) and a non-resonant (0.081 eV) frequency are plotted in Fig. 6(a) and Fig. 7(a). The contribution from the far-field radiative conductance $G_c(d,T)$ in Eq. (41) has to be appropriately modified to reflect spectral conductance. From proximity approximation theory, as $d/R_1 \to 0$, the spectral conductance $G_\omega(d)$ for two spheres of unequal radii R_1 and R_2 is expected to scale as [56] :

$$G_\omega(d) \sim \frac{R_1R_2}{R_1+R_2}\frac{1}{d} \approx \frac{R_1}{d}\ (\text{for } R_2 \gg R_1). \tag{45}$$

This characteristic R_1/d behavior is observed for the resonant frequency contributions shown in Fig. 6(a) for $d/R_1 \lesssim 0.02$ where a slope of -1 is indicated. However such behavior is not observed for the non-resonant frequency contributions shown in Fig. 7(a).

The error between G_ω and the spectral conductance predicted by MPA, G_ω^{MPA}, for the resonant and non-resonant frequency contributions are shown in Fig. 6(b) and Fig. 7(b) respectively. In the far-field region ($d/R_1 \gtrsim 2$ for $R_1 = 13.7\,\mu$m, and $d/R_1 \gtrsim 10$ for $R_1 = 2.5\,\mu$m) where the enhancement due to tunneling of waves (surface waves at the resonant frequency and evanescent waves at non-resonant frequency) is negligible, the form of MPA in Eq. (41) predicts that the variation in G_ω with gap is primarily due to the changing view factor between the spheres

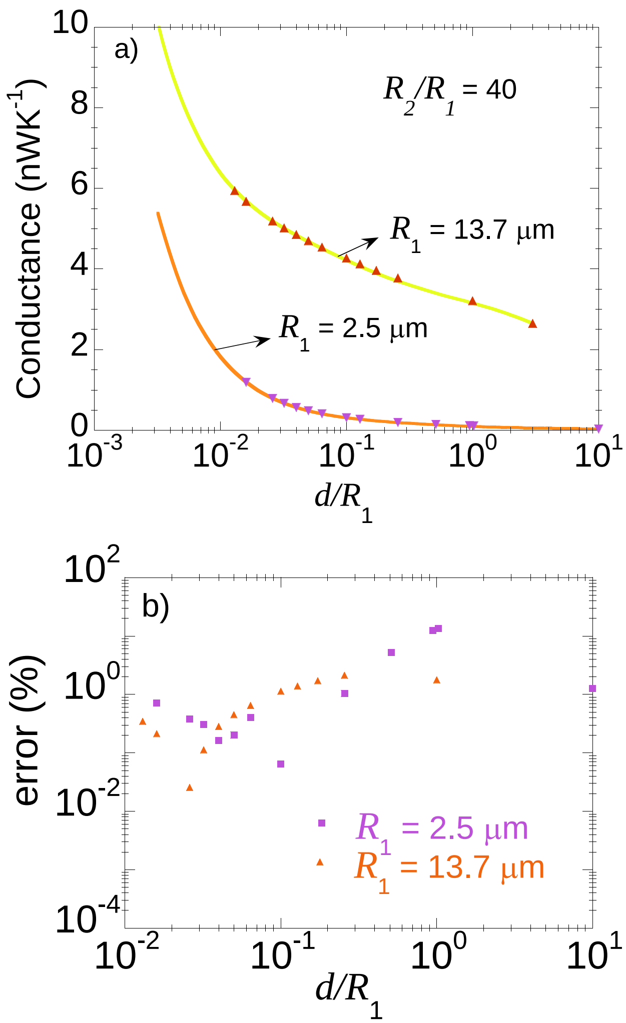

Fig. 5. (a) Plot of computed values of the total conductance (dotted) and the MPA (solid line)as a function of the non-dimensional gap d/R_1 for two spheres with $R_2/R_1 = 40$. The study has been performed for $R_1 = 13.7\mu$m and 2.5μm (b) The % error between the computed values and values from the MPA are plotted as a function of d/R_1

with gap. As observed from Fig. 6(b) and Fig. 7(b) there is good agreement with the exact computed values of the conductance at such gaps. For intermediate gaps ($2 \lesssim d/R_1 \lesssim 0.07$, for $R_1 = 13.7\,\mu$m) the variation of G_ω with gap is dependent on both the changing view factor with gap as well as increased tunneling of waves. For gaps $d/R_1 \lesssim 0.07$ (below which the view factor increases by less than 1%) the enhanced radiative transfer with decreasing gap can be attributed entirely due to increased tunneling of waves. At such small gaps MPA is able to model the enhancement at the resonant frequency within $\approx 5\%$ errors irrespective of the value of R_1, whereas at the non-resonant frequency the error is observed to be greater than 10% when $R_1 = 2.5\,\mu$m. Despite such high errors at the non-resonant frequencies, MPA is successful in predicting the overall conductance when $R_1 = 2.5\,\mu$m with error less than 1% for $d/R_1 \lesssim 0.1$

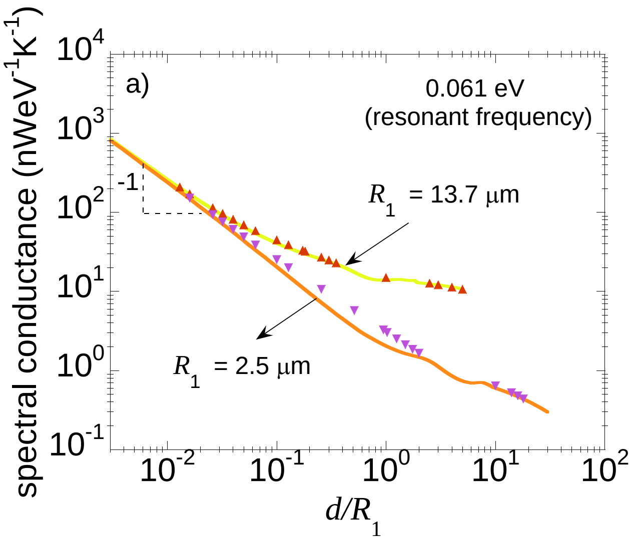

Fig. 6. (a) Comparison between the computed values of G_ω at a surface phonon-polariton frequency (0.061 eV) (dotted) and G_ω^{MPA} (solid line) as a function of the non-dimensional gap d/R_1 for two spheres of radius $R_1 = 2.5\,\mu$m, 13.7 μm and $R_2 = 40R_1$ (b) The % error between G_ω and G_ω^{MPA} as a function of d/R_1

as observed in Fig. 5. This apparent discrepancy has been analyzed in detail in Ref. [57] and has been attributed to decreased contribution from non-resonant frequencies as R_1 decreases.

We summarize the main contributions from this work:

(a) The exponential behavior of the spherical Hankel function $z_n^{(3)}(k_f r'')$ when $n \geq k_f r''$ has been utilized to propose a simplified form, referred to in this work as the one-term approximation, for the coefficients in the translation addition theorem. The one-term approximation is valid when $n \geq Ck_f r''$ where C is a constant which is chosen depending on the desired accuracy. The recursion relations for these simplified translation forms are also given. They are simpler

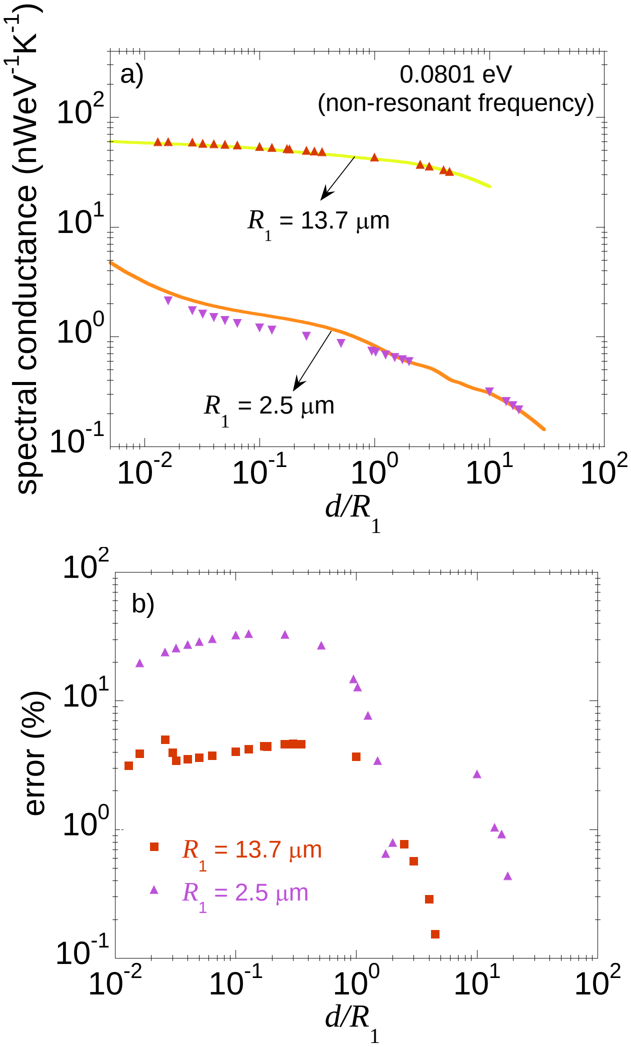

Fig. 7. (a) Comparison between the computed values of G_ω at frequency 0.0801 eV (dotted) and G_ω^{MPA} (solid line) as a function of the non-dimensional gap d/R_1 for two spheres of radius $R_1 = 2.5\mu$m, 13.7μm and $R_2 = 40R_1$ (b) The % error between G_ω and G_ω^{MPA} as a function of d/R_1

and computationally less expensive compared to those for the exact forms of the translation coefficients.

(b) A method to normalize the translation coefficients has been proposed and the behavior of the normalized translation coefficients on the relative size of the two spheres has been exploited to simplify the computations of near-field radiative transfer between two spheres.

(c) These simplifications are utilized to compute near-field radiative transfer between a mesoscopic and a macroscopic sphere with size ratio of 40 and we show that there is agreement with error less than 1% between the predictions of the modified proximity approximation and the exact computations for $d/R_1 < 0.1$. Thus the modified proximity approximation, which was initially proposed for the near-field radiative transfer between two equal sized spheres, has been shown to be valid for a more general configuration of unequal sized spheres as well.

Appendix: Normalizing factors for the vector translation coefficients $A_{\nu m,nm}$ and $B_{\nu m,nm}$

The coupled linear equations are given by (Eq. 20 in Ref. [20]):

$$C_{nm}^{lM} + \overline{u}_n(R_1)\frac{z_n^{(1)}(k_f R_1)}{z_n^{(3)}(k_f R_1)} \sum_{\nu=(m,1)}^{N_{max}} \left[D_{\nu m}^{lM} A_{\nu m,nm}(-k_f r'') + D_{\nu m}^{lN} B_{\nu m,nm}(-k_f r'') \right] = p_N^M \delta_{Nl}, \quad (46)$$

$$D_{nm}^{lM} + \overline{u}_n(R_2)\frac{z_n^{(1)}(k_f R_2)}{z_n^{(3)}(k_f R_2)} \sum_{\nu=(m,1)}^{N_{max}} \left[C_{\nu m}^{lM} A_{\nu m,nm}(+k_f r'') + C_{\nu m}^{lN} B_{\nu m,nm}(+k_f r'') \right] = 0, \quad (47)$$

$$C_{nm}^{lN} + \overline{v}_n(R_1)\frac{z_n^{(1)}(k_f R_1)}{z_n^{(3)}(k_f R_1)} \sum_{\nu=(m,1)}^{N_{max}} \left[D_{\nu m}^{lM} B_{\nu m,nm}(-k_f r'') + D_{\nu m}^{lN} A_{\nu m,nm}(-k_f r'') \right] = 0, \quad (48)$$

$$D_{nm}^{lN} + \overline{v}_n(R_2)\frac{z_n^{(1)}(k_f R_2)}{z_n^{(3)}(k_f R_2)} \sum_{\nu=(m,1)}^{N_{max}} \left[C_{\nu m}^{lM} B_{\nu m,nm}(+k_f r'') + C_{\nu m}^{lN} A_{\nu m,nm}(+k_f r'') \right] = 0, \quad (49)$$

where the symbol $(m,1)$ denotes the maximum of m and 1, $\overline{u}_n(R_1)$ and $\overline{v}_n(R_1)$ are given by:

$$\overline{u}_n(R_1) = \left(\frac{k_b \dfrac{z_{n+1}^{(1)}(k_b R_1)}{z_n^{(1)}(k_b R_1)} - k_f \dfrac{z_{n+1}^{(1)}(k_f R_1)}{z_n^{(1)}(k_f R_1)}}{k_b \dfrac{z_{n+1}^{(1)}(k_b R_1)}{z_n^{(1)}(k_b R_1)} - k_f \dfrac{z_{n+1}^{(3)}(k_f R_1)}{z_n^{(3)}(k_f R_1)}} \right), \quad (50)$$

and

$$\overline{v}_n(R_1) = \left(\frac{n_b \frac{z_{n+1}^{(1)}(k_f R_1)}{z_n^{(1)}(k_f R_1)} - \frac{z_{n+1}^{(1)}(k_b R_1)}{z_n^{(1)}(k_b R_1)} + \frac{(n+1)}{k_f R_1}\frac{(1-n_b^2)}{n_b}}{n_b \frac{z_{n+1}^{(1)}(k_f R_1)}{z_n^{(1)}(k_f R_1)} - \frac{z_{n+1}^{(3)}(k_b R_1)}{z_n^{(3)}(k_b R_1)} + \frac{(n+1)}{k_f R_1}\frac{(1-n_b^2)}{n_b}} \right). \quad (51)$$

$\overline{u}_n(R_2)$ and $\overline{v}_n(R_2)$ have similar expressions. The normalizing factors for the coefficients $D_{\nu m}^{lM}$ and $D_{\nu m}^{lN}$ are $\left(z_l^{(1)}(k_a R_1)/z_\nu^{(1)}(k_f R_2)\right)$ and $\left(z_l^{(1)}(k_a R_1)/\zeta_\nu^{(1)}(k_f R_2)\right)$ respectively. The equivalent normalizing factors for C_{nm}^{lM} and C_{nm}^{lN} are $\left(z_l^{(1)}(k_b R_2)/z_n^{(1)}(k_f R_1)\right)$ and $\left(z_l^{(1)}(k_b R_2)/\zeta_n^{(1)}(k_f R_1)\right)$ respectively. Using these in Eqs. (46)–(49) and rearranging, the coupled linear equations reduce to:

$$\left(\frac{z_l^{(1)}(k_b R_2)}{z_n^{(1)}(k_f R_1)} C_{nm}^{lM}\right) + \overline{u}_n(a)\frac{z_l^{(1)}(k_b R_2)}{z_l^{(1)}(k_a R_1)} \times \sum_{\nu=(m,1)}^{N_{max}} \left[\begin{array}{l} \left(\frac{z_l^{(1)}(k_a R_1)}{z_\nu^{(1)}(k_f R_2)} D_{\nu m}^{lM}\right)\left(\frac{z_\nu^{(1)}(k_f R_2)}{z_n^{(3)}(k_f R_1)} A_{\nu m,nm}(-k_f D)\right) \\ + \left(\frac{z_l^{(1)}(k_a R_1)}{\zeta_\nu^{(1)}(k_f R_2)} D_{\nu m}^{lN}\right)\left(\frac{\zeta_\nu^{(1)}(k_f R_2)}{z_n^{(3)}(k_f R_1)} B_{\nu m,nm}(-k_f D)\right) \end{array} \right] = \frac{z_l^{(1)}(k_b R_2)}{z_n^{(1)}(k_f R_1)} p_N^M \delta_{Nl}, \quad (52a)$$

$$\left(\frac{z_l^{(1)}(k_a R_1)}{z_n^{(1)}(k_f R_2)} D_{nm}^{lM}\right) + \bar{u}_n(R_2) \frac{z_l^{(1)}(k_a R_1)}{z_l^{(1)}(k_b R_2)} \times \sum_{\nu=(m,1)}^{N_{max}} \left[\begin{array}{l} \left(\frac{z_l^{(1)}(k_b R_2)}{z_\nu^{(1)}(k_f R_1)} C_{\nu m}^{lM}\right) \left(\frac{z_\nu^{(1)}(k_f R_1)}{z_n^{(3)}(k_f R_2)} A_{\nu m,nm}(+k_f D)\right) \\ + \left(\frac{z_l^{(1)}(k_b R_2)}{\zeta_\nu^{(1)}(k_f R_1)} C_{\nu m}^{lN}\right) \left(\frac{\zeta_\nu^{(1)}(k_f R_1)}{z_n^{(3)}(k_f R_2)} B_{\nu m,nm}(+k_f D)\right) \end{array} \right] = 0, \tag{52b}$$

$$\left(\frac{z_l^{(1)}(k_b R_2)}{\zeta_n^{(1)}(k_f R_1)} C_{nm}^{lN}\right) + \bar{v}_n(R_1) \frac{z_n^{(1)}(k_f R_1)}{\zeta_n^{(1)}(k_f R_1)} \frac{z_l^{(1)}(k_b R_2)}{z_l^{(1)}(k_a R_1)} \times \sum_{\nu=(m,1)}^{N_{max}} \left[\begin{array}{l} \left(\frac{z_l^{(1)}(k_a R_1)}{z_\nu^{(1)}(k_f R_2)} D_{\nu m}^{lM}\right) \left(\frac{z_\nu^{(1)}(k_f R_2)}{z_n^{(3)}(k_f R_1)} B_{\nu m,nm}(-k_f D)\right) \\ + \left(\frac{z_l^{(1)}(k_a R_1)}{\zeta_\nu^{(1)}(k_f R_2)} D_{\nu m}^{lN}\right) \left(\frac{\zeta_\nu^{(1)}(k_f R_2)}{z_n^{(3)}(k_f R_1)} A_{\nu m,nm}(-k_f D)\right) \end{array} \right] = 0, \tag{52c}$$

$$\left(\frac{z_l^{(1)}(k_a R_1)}{\zeta_n^{(1)}(k_f R_2)} D_{nm}^{lN}\right) + \bar{v}_n(R_2) \frac{z_n^{(1)}(k_f R_2)}{\zeta_n^{(1)}(k_f R_2)} \frac{z_l^{(1)}(k_a R_1)}{z_l^{(1)}(k_b R_2)} \times \sum_{\nu=(m,1)}^{N_{max}} \left[\begin{array}{l} \left(\frac{z_l^{(1)}(k_b R_2)}{z_\nu^{(1)}(k_f R_1)} C_{\nu m}^{lM}\right) \left(\frac{z_\nu^{(1)}(k_f R_1)}{z_n^{(3)}(k_f R_2)} B_{\nu m,nm}(+k_f D)\right) \\ + \left(\frac{z_l^{(1)}(k_b R_2)}{\zeta_\nu^{(1)}(k_f R_1)} C_{\nu m}^{lN}\right) \left(\frac{\zeta_\nu^{(1)}(k_f R_1)}{z_n^{(3)}(k_f R_2)} A_{\nu m,nm}(+k_f D)\right) \end{array} \right] = 0. \tag{52d}$$

Thus there are two possible normalizing factors for $A_{\nu m,nm}(+k_f D)$: $\left(\zeta_\nu^{(1)}(k_f R_1)/z_n^{(3)}(k_f R_2)\right)$ and $\left(z_\nu^{(1)}(k_f R_1)/z_n^{(3)}(k_f R_2)\right)$. Since the function $\zeta_\nu^{(1)}(k_f R_1)$ behaves similar to $z_\nu^{(1)}(k_f R_1)$ for $\nu \gg k_f R_1$ only one of them (the latter) has been chosen as a representative form for discussion in Eq. (22).

Acknowledgment

This work is funded partially by ONR Grant N00014-12-1-0996.