Viscosity Characterizations of Explosions and Arbitrage

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ABSTRACT

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This thesis analyzes the viscosity characterizations of the explosion time distribution for diffusions and of the arbitrage function in an equity market model with uncertainty.

In the first part, we show that the tail distribution U of the explosion time for a multidimensional diffusion – and more generally, a suitable function $\mathscr{U}$ of the Feynman-Kac type involving the explosion time – is a viscosity solution of an associated parabolic partial differential equation (PDE), provided that the dispersion and drift coefficients of the diffusion are continuous. This generalizes a result of Karatzas and Ruf (2015), who characterize U as a classical solution of a Cauchy problem for the PDE in the one-dimensional case, under the stronger condition of local Hölder continuity on the coefficients. We also extend their result to $\mathscr{U}$ in the one-dimensional case by establishing the joint continuity of $\mathscr{U}$. Furthermore, we show that $\mathscr{U}$ is dominated by any nonnegative classical supersolution of this Cauchy problem. Finally, we consider another notion of weak solvability, that of the distributional (sub/super)solution, and show that $\mathscr{U}$ is no greater than any nonnegative distributional supersolution of the relevant PDE.

In the second part, a more elaborate mathematical finance setting is taken. We show that, in an equity market model with Knightian uncertainty regarding the relative risk and covariance structure of its assets, the *arbitrage function* – defined as the reciprocal of the highest return on investment that can be achieved relative to the market using nonanticipative strategies, and under any admissible market model con-

figuration – is a viscosity solution of an associated Hamilton-Jacobi-Bellman (HJB) equation under appropriate boundedness, continuity and Markovian assumptions on the uncertainty structure. This result generalizes that of Fernholz and Karatzas (2011), who characterized this arbitrage function as a classical solution of a Cauchy problem for this HJB equation under much stronger conditions than those needed here. Our approach and results also extend to the Markovian Market Weight model introduced in Fernholz and Karatzas (2010b).

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Chapter 1

Introduction

This thesis consists of two main chapters, both developing viscosity characterizations for a function of interest – the tail distribution function of the explosion time for diffusions (Chapter 2), and the arbitrage function for an equity market model in Stochastic Portfolio Theory (Chapter 3), respectively.

The work presented in this thesis was primarily motivated by Karatzas and Ruf [32] and Fernholz and Karatzas [15], and extends their results that each of the above functions is the smallest nonnegative classical (super)solution of the Cauchy problem for an associated second-order parabolic partial differential equation (PDE) under certain assumptions. Unlike their approaches, we attack the problems through stochastic control and dynamic programming techniques inspired by Bouchard, Soner and Touzi [4, 56, 60], among others.

Chapter 2 investigates the tail distribution function U of the time to explosion for multidimensional diffusions. For the one-dimensional case, Karatzas and Ruf [32] showed that if the dispersion and drift coefficients of the diffusion are locally Hölder continuous, then the function U solves the Cauchy problem for an associated second-order parabolic PDE determined by these coefficients, and is in fact the smallest nonnegative classical (super)solution of this Cauchy problem. We weaken the Hölder condition to continuity only, and show that the function U is a viscosity solution

of this PDE. Further, we generalize this result to multidimensional settings and to functions $\mathscr{U}$ given by multidimensional Feynman-Kac-type expressions involving the explosion time.

We also extend their minimality result to $\mathscr{U}$ in the one-dimensional case by establishing the joint continuity of $\mathscr{U}$. Moreover, we show that in multidimensional settings, the function $\mathscr{U}$ is dominated by any nonnegative classical supersolution of the associated Cauchy problem.

Finally, we consider another notion of weak solvability, that of the distributional (sub/super)solution, and show that $\mathscr{U}$ is no greater than any nonnegative distributional supersolution of the relevant PDE.

Chapter 2 contains the results of the paper

> Wang, Y. (2015) Viscosity characterization of the explosion time distribution for diffusions. *Bernoulli*, forthcoming.

Chapter 3 studies the viscosity characterization of another function – the arbitrage function in Stochastic Portfolio Theory.

The pioneering work of Fernholz in Stochastic Portfolio Theory [16] discovered a systematic strategy that outperforms the market portfolio over sufficiently long time horizons under appropriate conditions. Many efforts have been made to understand various aspects of such "relative arbitrage"; see the survey paper [18] by Fernholz and Karatzas, a major contributor to this field, for an overview of the recent developments.

The arbitrage function in a Markovian equity market model is defined as the reciprocal of the highest return on investment that can be achieved relative to the market using nonanticipative strategies, in other words, the best arbitrage with respect to the market portfolio. In the presence of uncertainty regarding the local relative risk and covariance structure of the assets in the model, the arbitrage function is defined as the reciprocal of the highest return on investment that can be achieved relative to the market using nonanticipative strategies, and under any admissible market model configuration, in other words, the best robust arbitrage.

Under appropriate assumptions on the uncertainty structure of the model, the arbitrage function was characterized by Fernholz and Karatzas [15] as a classical solution of the Cauchy problem for an associated Hamilton-Jacobi-Bellman (HJB) fully nonlinear PDE determined by the uncertainty structure, and in fact, the smallest nonnegative classical (super)solution of this Cauchy problem. Under much weaker conditions, we establish the characterization of the arbitrage function as a viscosity solution to this HJB equation.

Replacing capitalizations with market weights, our approach and results also extend to the Markovian Market Weight (MMW) model introduced in [14]. In an n-asset MMW model without uncertainty, the connection with explosions exists in a subtle way: the arbitrage function in this model is equal to the probability under an auxiliary probability measure – the so-called Föllmer exit measure [21, 22] – that the process of relative market weights, with the given initial market weight configuration, stays in $(0, 1)^n$ throughout the given time interval [13]. In the presence of uncertainty about the local characteristics of the MMW model, the arbitrage function becomes a viscosity solution to a fully nonlinear HJB-Pucci type equation.

Chapter 3 is based on the paper

> Wang, Y. (2015) Viscosity characterization of the arbitrage function under model uncertainty.

The following two chapters are self-contained and therefore exhibit some redundancies. Notational inconsistencies in the two chapters have been minimized, but not entirely eliminated.

Chapter 2

Distribution of the Explosion Time for Diffusions

2.1 Introduction

Let $\mathcal{O}$ be a fixed domain of $\mathbb{R}^n$, not necessarily bounded, and $b = (b_1, \ldots, b_n) : \mathcal{O} \to \mathbb{R}^n$ and $\sigma = (\sigma_{ik})_{n\times n} : \mathcal{O} \to \mathbb{M}(n)$ two measurable functions. We assume throughout this paper that for every $x \in \mathcal{O}$, the stochastic differential equation

$$\mathrm{d}X_i(t) = \sum_k \sigma_{ik}(X(t))\,\mathrm{d}W_k(t) + b_i(X(t))\,\mathrm{d}t\,, \quad i = 1, \ldots, n\,, \quad X(0) = x \qquad (2.1.1)$$

admits a weak solution $(X(\cdot), W(\cdot))$, $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}(t)\}_{0\le t<\infty}$ which is *unique* in law and defined up until the *explosion time*

$$S := \inf\left\{t \ge 0 : X(t) \notin \mathcal{O}\right\}.$$

Here $\mathbb{M}(n)$ is the set of $n \times n$ real matrices, $W(\cdot) = (W_1(\cdot), \ldots, W_n(\cdot))$ is an $n-$dimensional Brownian motion, and the state process $X(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))$ is strongly Markovian as a consequence of the uniqueness in law. In (2.1.1) and throughout this paper, summations extend from 1 to n. We shall denote the distribution of the process $X(\cdot)$ starting at x by $\mathbb{P}_x$; the expectation $\mathbb{E}^{\mathbb{P}_x}$ corresponding

to this probability measure by $\mathbb{E}_x$; and by U the *tail distribution function* of the explosion time S:

$$[0,\infty)\times\mathcal{O}\ni(t,x)\longmapsto U(t,x)\;:=\;\mathbb{P}_x[S>t]\in[0,1]\,. \tag{2.1.2}$$

2.1.1 Previous results: the one-dimensional case

In [32], the authors studied the one-dimensional case $n=1$, when $\mathcal{O}=(\ell,r)$ is a fixed open interval with $-\infty\le\ell<r\le\infty$ and $b:\mathcal{O}\to\mathbb{R}$ and $\sigma:\mathcal{O}\to\mathbb{R}\setminus\{0\}$ are two measurable functions satisfying

$$\int_K\left(\frac{1}{\sigma^2(y)}+\frac{|b(y)|}{\sigma^2(y)}\right)\mathrm{d}y<\infty$$

for every compact set $K\subset\mathcal{O}$. According to the same arguments as in [10, 11] or [33, Theorem 5.5.15], the stochastic differential equation (2.1.1) admits then a weak solution defined up until the explosion time S, which is unique in law. As a result, the state process $X(\cdot)$ has the strong Markov property. It is then shown in [32] that, in the one-dimensional case and under the above conditions, the function U of (2.1.2) is jointly continuous, and furthermore, that if σ and b are locally Hölder continuous, then U solves the Cauchy problem for the parabolic partial differential equation (PDE)

$$u_t(t,x)-\frac{1}{2}\sigma^2(x)u_{xx}(t,x)-b(x)u_x(t,x)=0\,,\quad(t,x)\in(0,\infty)\times\mathcal{O} \tag{2.1.3}$$

with the initial condition

$$u(0,x)=1\,,\qquad x\in\mathcal{O}, \tag{2.1.4}$$

and is in fact the smallest nonnegative (super)solution of this Cauchy problem.

Remark: In the one-dimensional case, the Feller test provides necessary and sufficient conditions under which explosion occurs with positive probability, i.e., $\mathbb{P}_x[S<\infty]>0$; in this case the Cauchy problem (2.1.3), (2.1.4) has lots of classical solutions in addition to its trivial solution $u\equiv1$, of which the function U in (2.1.2) is the smallest. A host of examples is provided in [32, Section 6]. □

2.1.2 Generalizations

A natural question to ask then, is whether U is still a solution to (2.1.3), probably in some weak or generalized sense. The answer is affirmative. We weak the Hölder condition on σ and b to *continuity only*. Moreover, we generalize this result to multidimensional settings and to functions $\mathscr{U}$ given by multidimensional *Feynman-Kac-type* expressions involving the explosion time, namely

$$\mathscr{U}(t,x) := \mathbb{E}_x\left[\mathbf{1}_{\{S>t\}} f(X(t)) \exp\left(-\int_0^t h(X(s))\,\mathrm{d}s\right)\right]. \tag{2.1.5}$$

Here f and $h : \mathcal{O} \to \mathbb{R}$ are measurable functions, such that $\mathscr{U}(t,x)$ is well-defined by (2.1.5) and finite for all $(t,x) \in (0,\infty)\times\mathcal{O}$; when $f \equiv 1$ and $h \equiv 0$, the function $\mathscr{U}$ coincides with the tail distribution U of the explosion time, as defined in (2.1.2).

Remark: A sufficient condition for the right-hand side of (2.1.5) to be well-defined and finite, is that f be bounded and h be bounded from below (e.g., $h \geq 0$) on $\mathcal{O}$. □

For ease of notation we introduce the continuous, adapted, strictly positive process

$$Y(t) := \exp\left(-\int_0^t h(X(s))\,\mathrm{d}s\right), \tag{2.1.6}$$

and thus $\mathscr{U}(t,x) = \mathbb{E}_x\left[\mathbf{1}_{\{S>t\}} f(X(t))Y(t)\right]$. Note the dynamics

$$\mathrm{d}Y(t) = -h(X(t))Y(t)\,\mathrm{d}t\,.$$

2.1.3 Preview

Section 2.2 introduces the definition of *viscosity solution* and characterizes the function $\mathscr{U}$ as a viscosity solution to the Feynman-Kac-type version of (2.1.3), (2.1.4), namely

$$(u_t - \mathcal{L}'u)(t,x) = 0\,, \quad (t,x) \in (0,\infty)\times\mathcal{O} \tag{2.1.7}$$

with

$$u(0,x) = f(x)\,, \quad x \in \mathcal{O}\,, \tag{2.1.8}$$

where we have set

$$\mathcal{L}'u(t,x) := \frac{1}{2}\sum_{i,j} a_{ij}(x)u_{x_ix_j}(t,x) + \sum_i b_i(x)u_{x_i}(t,x) - h(x)u(t,x)\,, \qquad (2.1.9)$$

$(a_{ij})_{n\times n} = a := \sigma\sigma^T$.

Section 2.3 establishes the joint continuity of $\mathscr{U}$ in its arguments (t,x) in the one-dimensional case, and therefore characterizes $\mathscr{U}$ as a classical solution of the Cauchy problem (2.1.7), (2.1.8). Section 2.4 shows that $\mathscr{U}$ is dominated by *any* nonnegative classical supersolution of this Cauchy problem. Finally, Section 2.5 defines another kind of weak solution, the *distributional (sub/super) solution*, and shows that $\mathscr{U}$ is no greater than *any* nonnegative distributional supersolution of (2.1.7).

2.1.4 The special case U

We need the following global assumption for this chapter.

Assumption 2.1.1. For every $x \in \mathcal{O}$, the stochastic differential equation (2.1.1) admits a weak solution $(X(\cdot), W(\cdot))$, $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}(t)\}_{0\le t<\infty}$ which is unique in law and defined up until the explosion time $S := \inf\{t \ge 0 : X(t) \notin \mathcal{O}\}$. Denote the distribution of the process $X(\cdot)$ starting at x by $\mathbb{P}_x$.

Remark: In fact, the uniqueness in law in Assumption 2.1.1 is more than we need; our proofs only use the strong Markov property of the process $X(\cdot)$ in the solution. □

Remark: A sufficient condition for Assumption 2.1.1 to hold is that σ and b are bounded and measurable and σ is continuous and strictly elliptic (cf. Stroock and Varadhan [57, 58]). This assumption still holds under much weaker conditions than boundedness; in fact, it is this possibility that opens the door to explosions. See examples illustrated in [32, Section 6]; for instance, we have $\mathbb{P}_x[S = \infty] = 0$ in the model with $n = 1$, $\mathcal{O} = (0,\infty)$, $\sigma(x) = -x^2$ and $b(x) = x^3$. □

In the case of $\mathscr{U} \equiv U$, our results can be summarized as follows.

Theorem 2.1.2. *Suppose that Assumption 2.1.1 holds. If the functions $a := \sigma\sigma^T$ and b are continuous, then the function $U(t,x) := \mathbb{P}_x[S > t]$, $(t,x) \in [0,\infty) \times \mathcal{O}$ is a viscosity solution of the following parabolic equation:*

$$\big(u_t - \mathcal{L}u\big)(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathcal{O} \tag{2.1.10}$$

with

$$\mathcal{L}u(t,x) := \frac{1}{2} \sum_{i,j} a_{ij}(x)\, u_{x_i x_j}(t,x) + \sum_i b_i(x)\, u_{x_i}(t,x)\,, \tag{2.1.11}$$

and thus a viscosity solution of the Cauchy problem (2.1.10), (2.1.4), *since* $U(0,\cdot) = 1$.

Theorem 2.1.3. *Suppose that Assumption 2.1.1 holds. The function U is dominated by any nonnegative, classical supersolution $\mathcal{U} \in C([0,\infty) \times \mathcal{O}) \cap C^{1,2}((0,\infty) \times \mathcal{O})$ of the Cauchy problem* (2.1.10), (2.1.4).

Theorem 2.1.4. *Suppose that Assumption 2.1.1 holds, and that σ and b are locally bounded and a is locally strictly elliptic. If $V \in W^{1,2}_{loc}((0,\infty) \times \mathcal{O})$ is a nonnegative distributional supersolution (see Definition 2.5.3) of* (2.1.10) *with $V(0,\cdot) \geq 1$ on $\mathcal{O}$, then $V \geq U$ on $[0,\infty) \times \mathcal{O}$.*

Consequently, if the function U belongs to the space $W^{1,2}_{loc}((0,\infty)\times\mathcal{O})$ and is a distributional supersolution of (2.1.10), *then it is the smallest nonnegative distributional supersolution of* (2.1.10) *that satisfies $u(0,\cdot) \geq 1$ on $\mathcal{O}$.*

2.2 Viscosity characterization of $\mathscr{U}(t,x)$

This section develops a Feynman-Kac-type result involving the explosion time, which characterizes the function $(t,x) \mapsto \mathscr{U}(t,x)$ as a viscosity solution of the associated parabolic equation (2.1.7).

2.2.1 Definition of viscosity solutions

We first recall from [6] the definition of viscosity (sub/super)solutions of a second-order parabolic PDE. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and $(t,x,y,p,q) \mapsto F(t,x,y,p,q)$

a continuous, real-valued mapping defined on $(0,\infty)\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}(n)$ and satisfying the ellipticity condition

$$F(t,x,y,p,q_1) \le F(t,x,y,p,q_2) \quad \text{whenever} \quad q_1 \ge q_2$$

for all $(t,x,y,p) \in (0,\infty)\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^n$. Here $\mathbb{S}(n)$ is the set of $n\times n$ real symmetric matrices.

Consider the second-order parabolic PDE

$$u_t(t,x) + F\left(t,x,u(t,x),Du(t,x),D^2u(t,x)\right) = 0\,, \quad (t,x)\in(0,\infty)\times\mathcal{O} \tag{2.2.1}$$

with $Du = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})'$ and $D^2u = (u_{x_ix_j})_{n\times n}$.

Definition 2.2.1. (i) We say that a function $u : (0,\infty)\times\mathcal{O}\to\mathbb{R}$ is a *viscosity supersolution* of the equation (2.2.1), if

$$\varphi_t(t_0,x_0) + F\left(t_0,x_0,u_*(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)\right) \ge 0 \tag{2.2.2}$$

holds for all $(t_0,x_0)\in(0,\infty)\times\mathcal{O}$ and test functions $\varphi\in C^{1,2}((0,\infty)\times\mathcal{O})$ such that (t_0,x_0) is a minimum of $u_*-\varphi$ on $(0,\infty)\times\mathcal{O}$. We have denoted by

$$u_*(t,x) := \liminf_{(s,y)\to(t,x)} u(s,y), \qquad (t,x)\in(0,\infty)\times\mathcal{O}$$

the *lower-semicontinuous envelope* of u, i.e., the largest lower-semicontinuous function dominated pointwise by the function u.

(ii) Similarly, a function $u(t,x) : (0,\infty)\times\mathcal{O}\to\mathbb{R}$ is a *viscosity subsolution* of (2.2.1), if

$$\varphi_t(t_0,x_0) + F\left(t_0,x_0,u^*(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)\right) \le 0 \tag{2.2.3}$$

holds for all $(t_0,x_0)\in(0,\infty)\times\mathcal{O}$ and test functions $\varphi\in C^{1,2}((0,\infty)\times\mathcal{O})$ such that (t_0,x_0) is a maximum of $u^*-\varphi$ on $(0,\infty)\times\mathcal{O}$. We have denoted by

$$u^*(t,x) := \limsup_{(s,y)\to(t,x)} u(s,y), \qquad (t,x)\in(0,\infty)\times\mathcal{O} \tag{2.2.4}$$

the *upper-semicontinuous envelope* of u, i.e., the smallest upper-semicontinuous function that dominates pointwise the function u.

(iii) Finally, we say that u is a *viscosity solution* of (2.2.1), if it is both a viscosity supersolution and a viscosity subsolution of this equation.

In addition, this definition is not changed if the minimum and maximum are *strict and/or local*.

In our setting we have

$$F(t,x,y,p,q) \;=\; -\frac{1}{2}\sum_{i,j} a_{ij}(x)\, q_{ij} - \sum_{i} b_i(x)\, p_i + h(x)\, y\,,$$

and the left-hand sides of (2.2.2) and (2.2.3) simplify to $\big(\varphi_t - \mathcal{L}'\varphi\big)(t_0,x_0)$ with $\mathcal{L}'$ defined in (2.1.9). Since $a = \sigma\sigma^T$ is positive-semidefinite, the function F satisfies the ellipticity condition. We also need F to be continuous, which means that the functions a_{ij}, b_i and h must be continuous for all indices i, j.

2.2.2 Viscosity characterization

Theorem 2.2.2. *Suppose that Assumption 2.1.1 holds, and that the functions a, b and h are continuous. Then the function $\mathscr{U}$ of* (2.1.5) *is a viscosity solution of the parabolic equation* (2.1.7)*, and thus a viscosity solution of the Cauchy problem* (2.1.7)*,* (2.1.8)*, since $\mathscr{U}(0,x) = f(x)$.*

We first highlight the main idea for the proof of viscosity subsolution property without many of the technicalities. Similar arguments will lead to the viscosity supersolution property.

We prove by contradiction, assuming the contrary of (2.2.3) in Definition 2.2.1 that there exist $\varphi \in C^{1,2}\left((0,\infty)\times\mathcal{O}\right)$ and $(t_0,x_0) \in (0,\infty)\times\mathcal{O}$ such that (t_0,x_0) is a strict maximum of $\mathscr{U}^* - \varphi$, that the maximal value equals 0, and that

$$(\varphi_t - \mathcal{L}'\varphi)(t_0,x_0) > 0\,. \tag{2.2.5}$$

It then follows from the definition (2.2.4) of $\mathscr{U}^*$ that we can take a pair (t^*, x^*) close to (t_0, x_0) such that $(\varphi - \mathscr{U})(t^*, x^*)\,(\geq 0)$ is sufficiently small, say less than a small constant C_2. Under $\mathbb{P}_{x^*}$, we have

$$\varphi(t^*, x^*) - \mathbb{E}_{x^*}\left[\varphi\big(t^* - \nu, X(\nu)\big)Y(\nu)\right] = \mathbb{E}_{x^*}\left[\int_0^\nu \mathscr{G}\big(t^* - t, X(t)\big)Y(t)\,\mathrm{d}t\right] \geq 0\,, \tag{2.2.6}$$

for any sufficiently small stopping time ν, where the process $Y(\cdot)$ is defined (2.1.6), and $\mathscr{G}(t,x) := (\varphi_t - \mathcal{L}'\varphi)(t,x)$, which is positive for any (t,x) sufficiently close to (t^*, x^*) by assumption (2.2.5).

On the other hand, in (2.2.6), we estimate $\varphi(t^*, x^*)$ from above by $C_2 + \mathscr{U}(t^*, x^*)$, and $\varphi\big(t^* - \nu, X(\nu)\big)$ from below by $C_1 + \mathscr{U}\big(t^* - \nu, X(\nu)\big)$ with C_1 a small positive constant, and deduce that

$$\mathscr{U}(t^*, x^*) > \mathbb{E}_{x^*}\left[\mathscr{U}(t^* - \nu, X(\nu))Y(\nu)\right].$$

This inequality contradicts the martingale property of the process $\mathscr{U}(t^*\!-\nu, X(\nu))Y(\nu)$, which is a consequence of the strong Markov property of $X(\cdot)$.

When implementing this idea, the stopping time ν needs to be not only small, but also satisfy that on $[0, \nu]$, the process $X(\cdot)$ is bounded and close to x^*; however, the ν cannot be too small, in order to ensure that $\varphi\big(t^* - \nu, X(\nu)\big) \geq C_1 + \mathscr{U}\big(t^* - \nu, X(\nu)\big)$ holds. These considerations inspire us to design ν as in (2.2.12) below. □

Proof of Theorem 2.2.2. Let us show that $\mathscr{U}$ is a viscosity subsolution to (2.1.7). The proof of the viscosity supersolution property is similar.

According to the definition of viscosity subsolution, if suffices to verify that for any test function $\varphi \in C^{1,2}((0,\infty)\times\mathcal{O})$, and for any $(t_0, x_0) \in (0,\infty)\times\mathcal{O}$ with

$$(\mathscr{U}^* - \varphi)(t_0, x_0) = 0 > (\mathscr{U}^* - \varphi)(t, x)\,, \quad \forall\ (t,x) \in (0,\infty)\times\mathcal{O}\,, \tag{2.2.7}$$

i.e., such that (t_0, x_0) is a strict maximum of $\mathscr{U}^* - \varphi$, we have the inequality

$$\mathscr{G}(t_0, x_0) \leq 0 \qquad \text{for the function} \qquad \mathscr{G}(t,x) := (\varphi_t - \mathcal{L}'\varphi)(t,x)\,, \tag{2.2.8}$$

with $\mathscr{L}'$ and $\mathscr{U}^*$ defined in (2.1.9) and (2.2.4). We shall argue this by contradiction, assuming that

$$\mathscr{G}(t_0, x_0) > 0\,.$$

Since the function $\mathscr{G}$ just introduced in (2.2.8) is continuous, there exists a neighborhood

$$\mathcal{N}_\delta := (t_0 - \delta, t_0 + \delta) \times B_\delta(x_0) \subset\subset (0, \infty) \times \mathcal{O} \tag{2.2.9}$$

of (t_0, x_0), on which $\mathscr{G} > 0$ holds ("$\subset\subset$" means *compactly contained in*). Recalling (2.2.7) and the condition that h is continuous and thus locally bounded, we then introduce the constants

$$C_1 := -\max_{\partial \mathcal{N}_\delta} \big(\mathscr{U}^* - \varphi\big)(t, x) > 0\,, \quad C_2 := C_1 \exp\left(-2\delta \sup_{x \in B_\delta(x_0)} |h(x)|\right). \tag{2.2.10}$$

We observe that

$$\limsup_{(t,x)\to(t_0,x_0)} (\mathscr{U} - \varphi)(t, x) = (\mathscr{U}^* - \varphi)(t_0, x_0) = 0$$

holds by the definition (2.2.4) of $\mathscr{U}^*$ and the continuity of φ, and hence there exists $(t^*, x^*) \in \mathcal{N}_\delta$ such that

$$(\mathscr{U} - \varphi)(t^*, x^*) > -C_2\,. \tag{2.2.11}$$

Let us consider the stopping time

$$\nu \big(= \nu(\omega)\big) := \inf\big\{s \in (0, t^*] : \big(t^* - s, X(s)\big) \notin \mathcal{N}_\delta\big\} \wedge t^*, \tag{2.2.12}$$

and note that the definition (2.2.9) of $\mathcal{N}_\delta$ implies

$$\nu < S \qquad \text{and} \qquad \nu \le t^* - (t_0 - \delta) = (t^* - t_0) + \delta < t^* \wedge (2\delta)\,. \tag{2.2.13}$$

Now thanks to the assumption $\varphi \in C^{1,2}((0, \infty) \times \mathcal{O})$, we can apply Itô's change of variable rule to $\varphi(t^* - t, X(t))Y(t)$ for $t \in [0, \nu] \subset [0, t^*)$ and plug in (2.1.1) to derive the semimartingale decomposition

$$\begin{aligned} \mathrm{d}\big[\varphi(t^* - t, X(t))Y(t)\big] = &- \mathscr{G}(t^* - t, X(t))Y(t)\,\mathrm{d}t \\ &+ \sum_{i,k} \varphi_{x_i}(t^* - t, X(t))\,\sigma_{ik}(X(t))Y(t)\,\mathrm{d}W_k(t)\,, \end{aligned} \tag{2.2.14}$$

with the process $Y(\cdot)$ defined in (2.1.6) and the function $\mathscr{G}$ in (2.2.8). (see Appendix A.2 for a proof).

Integrating (2.2.14) with respect to t over $[0,\nu]$ and taking the expectation under $\mathbb{P}_{x^*}$ yields

$$\varphi(t^*,x^*) - \mathbb{E}_{x^*}\left[\varphi\big(t^*-\nu, X(\nu)\big)Y(\nu)\right] = \mathbb{E}_{x^*}\left[\int_0^\nu \mathscr{G}(t^*-t, X(t))\,Y(t)\,\mathrm{d}t\right] \geq 0\,. \tag{2.2.15}$$

Here the last inequality comes from the assumption $\mathscr{G} > 0$ on $\mathcal{N}_\delta$, whereas the equality holds because the expectations of the integrals with respect to $\mathrm{d}W_k(t)$ have all vanished, due to:

(1) the uniform boundedness of $Y(t)$ on $[0,\nu]$; in fact, we have

$$Y(t) \leq \exp\left(t \sup_{x\in B_\delta(x_0)} |h(x)|\right) \leq \exp\left(2\delta \sup_{x\in B_\delta(x_0)} |h(x)|\right)$$

for all $0 \leq t \leq \nu < 2\delta$; and

(2) the boundedness of φ_{x_i} on $\mathcal{N}_\delta$, and a_{ij} and thus σ_{ik} on $B_\delta(x_0)$ (recalling that $a = \sigma\sigma^T$).

Combining (2.2.15) with (2.2.10) and (2.2.11) leads to

$$\begin{aligned} 0 \leq\ & \varphi(t^*,x^*) - \mathbb{E}_{x^*}\left[\varphi\big(t^*-\nu, X(\nu)\big)Y(\nu)\right] \\ & < C_2 + \mathscr{U}(t^*,x^*) - \mathbb{E}_{x^*}\left[\big(C_1 + \mathscr{U}\big(t^*-\nu, X(\nu)\big)\big)Y(\nu)\right] \\ & = \{C_2 - C_1\,\mathbb{E}_{x^*}[Y(\nu)]\} + \left\{\mathscr{U}(t^*,x^*) - \mathbb{E}_{x^*}\left[\mathscr{U}\big(t^*-\nu, X(\nu)\big)Y(\nu)\right]\right\}. \end{aligned} \tag{2.2.16}$$

We see that the first term on the right-hand side is at most zero by the definition (2.2.10) of C_2 and the definition (2.1.6) of $Y(\cdot)$, along with the fact $\nu < 2\delta$ from (2.2.13). Thus we will arrive at a contradiction as soon as we have shown that the second term on the right-hand side of (2.2.16) equals zero, namely

$$\mathbb{E}_{x^*}\left[\mathscr{U}(t^*-\nu, X(\nu))Y(\nu)\right] = \mathscr{U}(t^*,x^*)\,. \tag{2.2.17}$$

In fact, it follows from the strong Markov property of $X(\cdot)$ that

$$\begin{aligned}\mathscr{U}(t^*-\nu, X(\nu)) &= \mathbb{E}_\xi\left[\mathbf{1}_{\{S>t^*-\nu\}} f(X(t^*-\nu))\exp\left(-\int_0^{t^*-\nu} h(X(s))\,\mathrm{d}s\right)\right]\Big|_{\xi=X(\nu)} \\ &= \mathbb{E}_{x^*}\left[\mathbf{1}_{\{S>t^*\}} f(X(t^*))\exp\left(-\int_\nu^{t^*} h(X(s))\,\mathrm{d}s\right)\,\middle|\,\mathscr{F}(\nu)\right],\quad \mathbb{P}_{x^*}\text{-a.s..}\end{aligned}$$

Thus plugging into the left-hand side of (2.2.17) along with the definition (2.1.6) of $Y(\cdot)$ yields

$$\begin{aligned}&\mathbb{E}_{x^*}\left[\mathbb{E}_{x^*}\left[\mathbf{1}_{\{S>t^*\}} f(X(t^*))\exp\left(-\int_\nu^{t^*} h(X(s))\,\mathrm{d}s\right)\exp\left(-\int_0^{\nu} h(X(s))\,\mathrm{d}s\right)\,\middle|\,\mathscr{F}(\nu)\right]\right] \\ &= \mathbb{E}_{x^*}\left[\mathbf{1}_{\{S>t^*\}} f(X(t^*))\exp\left(-\int_0^{t^*} h(X(s))\,\mathrm{d}s\right)\right] = \mathscr{U}(t^*, x^*)\,. \qquad \square\end{aligned}$$

2.3 Joint continuity of $\mathscr{U}$ in the one-dimensional case

In this section, we fix $n = 1$. The function $U(t, x)$ is shown to be jointly continuous in (t, x) on $[0, \infty) \times \mathcal{O}$ [32, Proposition 4.3]. This section generalizes this result for $\mathscr{U}(t, x)$ by starting with the continuity in t and then using a "coupling" argument in conjunction with [32, (4.4)].

Thanks to the joint continuity of $\mathscr{U}(t, x)$, arguments similar to those in [32, Lemma 5.1 and Proposition 5.2] show that $\mathscr{U}$ is actually a classical solution of the Cauchy problem (2.1.7), (2.1.8), whenever the function f is continuous and functions $a > 0$, b and $h \geq 0$ are locally Hölder continuous. This result extends [32, Proposition 5.2] to the more general Feynman-Kac context (see Corollary 2.3.3).

Theorem 2.3.1. *When $n = 1$, the function $\mathscr{U}(t, x)$ is jointly continuous in (t, x) on $[0, \infty) \times \mathcal{O}$ if f and h are bounded, $a > 0$ and either of the following conditions holds:*

(i) *f is Hölder continuous and a and b are bounded;*

(ii) *$f \in C^2(\mathcal{O})$ and the functions $f'\sigma$ and $f'b + \frac{1}{2}f''a$ are bounded on $\mathcal{O}$.*

Remark 2.3.2. The tail distribution function $U(t,x)$ with $f \equiv 1$ and $h \equiv 0$ is a special case that satisfies Condition (ii). □

Corollary 2.3.3. *Under the assumptions of Theorem 2.3.1, if the functions a, b and $h \geq 0$ are locally Hölder continuous, then $\mathscr{U}(t,x)$ is a classical solution of the Cauchy problem* (2.1.7), (2.1.8).

Proof of Theorem 2.3.1. Let $C_0 > 0$ be an upper bound of $|f|$ and $|h|$ on $\mathcal{O}$, then

$$0 < Y(t) \leq e^{t\,C_0}\,, \quad \forall\, t \in [0,\infty)\,. \tag{2.3.1}$$

We first prove the continuity of $\mathscr{U}(t,x)$ in t.

Lemma 2.3.4. *The function $t \mapsto \mathscr{U}(t,x)$ is continuous for any given $x \in \mathcal{O}$.*

Proof. Fix $(t,x) \in [0,\infty) \times \mathcal{O}$. Let us show the left-continuity first. Assume that $t \in (0,\infty)$. For any $t' \in [0,t)$, by definition we have

$$\begin{aligned}
\mathscr{U}(t',x) - \mathscr{U}(t,x) &= \mathbb{E}_x\left[\mathbf{1}_{\{S>t'\}} f(X(t'))Y(t')\right] - \mathbb{E}_x\left[\mathbf{1}_{\{S>t\}} f(X(t))Y(t)\right] \\
&= \mathbb{E}_x\left[\left(\mathbf{1}_{\{S>t'\}} - \mathbf{1}_{\{S>t\}}\right) f(X(t'))Y(t')\right] \\
&\quad + \mathbb{E}_x\left[\mathbf{1}_{\{S>t\}} f(X(t'))(Y(t') - Y(t))\right] \\
&\quad + \mathbb{E}_x\left[\mathbf{1}_{\{S>t\}} (f(X(t')) - f(X(t)))Y(t)\right] \\
&=: \Delta_1 + \Delta_2 + \Delta_3\,.
\end{aligned}$$

It then suffices to show that each $\Delta_j \to 0$ as $t' \uparrow t$.

Case 1. $j = 1$. Since $\mathbf{1}_{\{S>t\}}$ is decreasing in t and U is continuous, we obtain

$$|\Delta_1| \leq C_0\, e^{t\,C_0}\, \mathbb{E}_x\left[\mathbf{1}_{\{S>t'\}} - \mathbf{1}_{\{S>t\}}\right] = C_0\, e^{t\,C_0}[(U(t',x) - U(t,x)] \to 0\,, \quad \text{as} \;\; t' \uparrow t\,.$$

Case 2. $j = 2$. For any $t < S$, we have

$$|Y(t') - Y(t)| = \left|Y(t)\left[\exp\left(\int_{t'}^{t} h(X(s))\,\mathrm{d}s\right) - 1\right]\right| \leq e^{t\,C_0}\left(e^{(t-t')\,C_0} - 1\right),$$

since

$$e^{(t-t')\,C_0} - 1 \geq \exp\left(\int_{t'}^{t} h(X(s))\,\mathrm{d}s\right) - 1 \geq e^{-(t-t')\,C_0} - 1 \geq -\left(e^{(t-t')\,C_0} - 1\right)$$

(as $z + 1/z \geq 2$ for all $z > 0$). Hence

$$|\Delta_2| \leq C_0\, \mathbb{E}_x\big[\mathbf{1}_{\{S>t\}}|Y(t') - Y(t)|\big] \leq C_0\, e^{t\,C_0}\left(e^{(t-t')\,C_0} - 1\right) \to 0\,, \quad \text{as} \;\; t' \uparrow t\,.$$

Case 3. $j = 3$. **(i)** If Condition (i) holds, there exist constants $\kappa \in (0, 1]$ and $C_\kappa, C > 0$ such that $|f(x) - f(y)| \leq C_\kappa |x - y|^\kappa$ and $a(x), |b(x)| \leq C$ for all $x, y \in \mathcal{O}$. Thus Hölder's inequality gives

$$\begin{aligned} |\Delta_3| &\leq e^{t\,C_0}\, \mathbb{E}_x\big[\mathbf{1}_{\{S>t\}}|f(X(t)) - f(X(t'))|\big] \leq e^{t\,C_0} C_\kappa\, \mathbb{E}_x\big[|I(t')|^\kappa\big] \\ &\leq e^{t\,C_0} C_\kappa \left(\mathbb{E}_x\left[|I(t')|^2\right]\right)^{\kappa/2}, \end{aligned} \tag{2.3.2}$$

where $I(s) := \mathbf{1}_{\{S>t\}}(X(t) - X(s \wedge t))$, $s \in [0, \infty)$. Let us recall from (2.1.1) that

$$I(s) = \mathbf{1}_{\{S>t\}} \int_s^t \sigma(X(\varsigma))\, \mathrm{d}W(\varsigma) + \mathbf{1}_{\{S>t\}} \int_s^t b(X(\varsigma))\, \mathrm{d}\varsigma =: M(s) + A(s)\,, \quad \forall\; s \leq t\,.$$

Therefore Cauchy-Schwarz inequality and Burkholder-Davis-Gundy inequalities give

$$\begin{aligned} \mathbb{E}_x\left[|I(t')|^2\right] &\leq 2\, \mathbb{E}_x\big[|M(t')|^2\big] + 2\, \mathbb{E}_x\big[|A(t')|^2\big] \\ &\leq 8\, \mathbb{E}_x\big[\langle M\rangle(t')\big] + 2\, \mathbb{E}_x\big[|A(t')|^2\big]. \end{aligned} \tag{2.3.3}$$

Since $\sigma^2 = a \leq C$ and $|b| \leq C$ on $\mathcal{O}$, we obtain

$$\langle M\rangle(t') = \mathbf{1}_{\{S>t\}} \int_{t'}^t \sigma^2(X(\varsigma))\, \mathrm{d}\varsigma \leq (t - t')\, C \quad \text{and} \quad |A(t')| \leq (t - t')\, C\,. \tag{2.3.4}$$

Substituting into (2.3.3) and then into (2.3.2) yields

$$|\Delta_3| \leq e^{t\,C_0}\, C_\kappa \left[8(t - t')\, C + 2(t - t')^2\, C^2\right]^{\kappa/2} \to 0\,, \quad \text{as} \;\; t' \uparrow t\,. \tag{2.3.5}$$

(ii) If Condition (ii) holds, let $C' > 0$ be an upper bound of $(f'\sigma)^2$ and $\left|f'b + \frac{1}{2} f''a\right|$ on $\mathcal{O}$. Our idea is similar to that of (i) with the process $I(s)$ replaced by $I'(s) := \mathbf{1}_{\{S>t\}}(f(X(t)) - f(X(s \wedge t)))$, $s \in [0, \infty)$, the exponent κ set to 1 and the constant C_κ removed. More precisely, we have

$$|\Delta_3| \leq e^{t\,C_0} \mathbb{E}_x\big[\mathbf{1}_{\{S>t\}}|f(X(t)) - f(X(t'))|\big] = e^{t\,C_0} \mathbb{E}_x\big[|I'(t')|\big] \leq e^{t\,C_0} \big(\mathbb{E}_x\big[|I'(t')|^2\big]\big)^{1/2}.$$

For any $0 \le s \le t < S$, we apply Itô's change of variable rule to $f(X(\varsigma))$, $\varsigma \in [s,t]$ and plug in (2.1.1) to get

$$\begin{aligned} I'(s) &= \mathbf{1}_{\{S>t\}} \int_s^t (f'\sigma)(X(\varsigma))\,\mathrm{d}W(\varsigma) + \mathbf{1}_{\{S>t\}} \int_s^t \left(f'b + \frac{1}{2}f''a\right)(X(\varsigma))\,\mathrm{d}\varsigma \\ &=: M'(s) + A'(s)\,. \end{aligned}$$

Thus (2.3.3)–(2.3.5) hold for I', M', A' and C', with σ replaced by $f'\sigma$ in (2.3.4), C_κ removed and $\kappa = 1$.

Switching t and t' in the above argument proves the right-continuity (see Remark 2.3.5 for a simpler proof). More precisely, we summarize the above estimates as the following inequality that holds for all $t, t' \in [0,\infty)$ to complete the proof:

$$\begin{aligned} &|\mathscr{U}(t',x) - \mathscr{U}(t,x)| \qquad (2.3.6)\\ &\le e^{(t\vee t')\,C_0}\Big\{C_0\,|U(t',x) - U(t,x)| + C_0\big(e^{|t-t'|\,C_0} - 1\big) \\ &\quad + \Big[C_\kappa\left(8|t-t'|\,C + 2(t-t')^2C^2\right)^{\kappa/2}\Big] \vee \Big[\left(8|t-t'|\,C' + 2(t-t')^2C'^2\right)^{1/2}\Big]\Big\} \\ &=: e^{(t\vee t')\,C_0}\,C_0\,|U(t',x') - U(0,x')| + d(t,t') \to 0\,, \quad \text{as } \ t' \to t\,. \end{aligned}$$ □

Now let us go back to the joint continuity. Define the stopping time

$$H_x := \inf\{t \ge 0 : X(t) = x\}\,, \quad x \in \mathcal{O}\,.$$

Fix $(t,x) \in [0,\infty)\times\mathcal{O}$. If $t = 0$, then for any $(t',x') \in [0,\infty)\times\mathcal{O}$, it follows from (2.3.6) that

$$\begin{aligned} |\mathscr{U}(t',x') - \mathscr{U}(0,x)| &\le |\mathscr{U}(t',x') - \mathscr{U}(0,x')| + |\mathscr{U}(0,x') - \mathscr{U}(0,x)| \\ &\le e^{t'C_0}\,C_0\,|U(t',x') - U(0,x')| + d(0,t') + |f(x') - f(x)| \end{aligned}$$

Since f is continuous under either Condition (i) or (ii), $d(0,t') \to 0$ as $t' \to 0$, and

$$|U(t',x') - U(0,x')| \le |U(t',x') - U(0,x)| + |U(0,x) - U(0,x')| \to 0\,, \ \text{ as } (t',x') \to (0,x)$$

by the joint continuity of U, we conclude that $|\mathscr{U}(t',x') - \mathscr{U}(0,x)| \to 0$, as $(t',x') \to (0,x)$.

Now assume $t > 0$. For any $\varepsilon > 0$, according to Lemma 2.3.4, there exists $\delta \in (0, t/2)$ such that

$$|\mathscr{U}(\tilde{t}, x) - \mathscr{U}(t, x)| < \varepsilon\,, \quad \forall\ \tilde{t} \in (t - 2\delta, t + 2\delta)\,. \tag{2.3.7}$$

For any $t' \in (t - \delta, t + \delta)$ (thus $t' > \delta$), we decompose $\mathscr{U}(t, x)$ and $\mathscr{U}(t', x')$ into two terms, respectively:

$$\mathscr{U}(t, x) = \mathbb{E}_{x'}\left[\mathbf{1}_{\{H_x < \delta\}}\mathscr{U}(t, x)\right] + \mathbb{E}_{x'}\left[\mathbf{1}_{\{H_x \geq \delta\}}\mathscr{U}(t, x)\right] =: E_1 + E_2\,,$$

$$\begin{aligned}\mathscr{U}(t', x') &= \mathbb{E}_{x'}\left[\mathbf{1}_{\{S > t'\}}\mathbf{1}_{\{H_x < \delta\}} f(X(t'))Y(t')\right] + \mathbb{E}_{x'}\left[\mathbf{1}_{\{S > t'\}}\mathbf{1}_{\{H_x \geq \delta\}} f(X(t'))Y(t')\right] \\ &=: E_1' + E_2'\,.\end{aligned} \tag{2.3.8}$$

Then using the well-known property that $\lim_{x' \to x} \mathbb{P}_{x'}[H_x \geq \delta] = 0$ (a simple proof of this property can be found in [32, (4.4)]; see also [29, Section 3.3]), we obtain

$$|E_2| = |\mathscr{U}(t, x)|\,\mathbb{P}_{x'}[H_x \geq \delta] \to 0\,, \quad |E_2'| \leq C_0\, e^{(t+\delta)\,C_0}\,\mathbb{P}_{x'}[H_x \geq \delta] \to 0\,, \quad \text{as } x' \to x\,.$$

Finally, the strong Markov property of $X(\cdot)$ implies that

$$\begin{aligned}E_1' &= \mathbb{E}_{x'}\left[\mathbf{1}_{\{H_x < \delta\}}\,\mathbb{E}_{x'}\left[\mathbf{1}_{\{S > t'\}} f(X(t'))Y(t') \,\middle|\, \mathscr{F}(H_x)\right]\right] \\ &= \mathbb{E}_{x'}\left[\mathbf{1}_{\{H_x < \delta\}}\,\mathbb{E}_{x}\left[\mathbf{1}_{\{S > t' - H_x\}} f(X(t' - H_x))Y(t' - H_x)\right]\right] \\ &= \mathbb{E}_{x'}\left[\mathbf{1}_{\{H_x < \delta\}}\,\mathscr{U}(t' - H_x, x)\right].\end{aligned}$$

Since $t' - H_x \in (t - 2\delta, t + 2\delta)$ holds on $\{H_x < \delta\}$, on the strength of (2.3.7) we arrive at

$$\begin{aligned}|E_1' - E_1| &= \mathbb{E}_{x'}\left[\mathbf{1}_{\{H_x < \delta\}}\left(\mathscr{U}(t' - H_x, x) - \mathscr{U}(t, x)\right)\right] \\ &\leq \mathbb{E}_{x'}\left[\mathbf{1}_{\{H_x < \delta\}}|\mathscr{U}(t' - H_x, x) - \mathscr{U}(t, x)|\right] < \varepsilon\,.\end{aligned}$$

Hence $\mathscr{U}(\cdot\,, \cdot)$ is jointly continuous at (t, x). □

Remark 2.3.5. We have a simpler proof of the right-continuity of $\mathscr{U}(t,x)$ in t, by taking advantage of the right-continuity of $\mathbf{1}_{\{S>\cdot\}}$ in conjunction with the continuity and boundedness of the paths of $f(X(\cdot))$ and $Y(\cdot)$ (see (2.3.1)). It follows from the dominated convergence theorem that

$$\begin{aligned}\lim_{t'\downarrow t}\mathscr{U}(t',x) &= \lim_{t'\downarrow t}\mathbb{E}_x\left[\mathbf{1}_{\{S>t'\}}f(X(t'))Y(t')\right]\\ &= \mathbb{E}_x\left[\lim_{t'\downarrow t}\mathbf{1}_{\{S>t'\}}f(X(t'))Y(t')\right] = \mathscr{U}(t,x)\,.\end{aligned}$$

□

2.4 Minimality

For the one-dimensional case, the function U of (2.1.2) is dominated by every nonnegative classical supersolution of the Cauchy problem (2.1.10), (2.1.4) [32, Proposition 5.3]. Therefore, whenever the function U is known in advance to be a classical solution of this problem (for instance, when the functions a and b are Hölder continuous), it is also the smallest nonnegative classical supersolution of this problem. This section proves these results in the more general Feynman-Kac case. Our minimality results (Theorem 2.4.1 and Corollary 2.4.2 below) have similarities – at least in spirit – to those in Proposition 5.3 of [32], Problem 3.5.1 of [45], in Exercise 4.4.7 of [33], as well as to [13, 15, 55].

Theorem 2.4.1. *Assume that the function f is nonnegative on $\mathcal{O}$. Then the function $\mathscr{U}$ defined in* (2.1.5) *is dominated by every nonnegative, classical supersolution $\mathcal{U}(t,x)\in C([0,\infty)\times\mathcal{O})\cap C^{1,2}((0,\infty)\times\mathcal{O})$ of the Cauchy problem* (2.1.7), (2.1.8).

Combining Theorem 2.4.1 with Corollary 2.3.3 leads directly to the following corollary.

Corollary 2.4.2. *If the assumptions of Theorem 2.4.1 and Corollary 2.3.3 hold, then $\mathscr{U}$ is the smallest nonnegative classical (super)solution of the Cauchy problem* (2.1.7), (2.1.8).

Proof of Theorem 2.4.1. It follows from definition (2.1.5) and the supersolution property that $\mathscr{U}(0,x) = f(x) \le \mathcal{U}(0,x)$ holds on $\mathcal{O}$. Let $\mathcal{O}_m \subset\subset \mathcal{O}$ (i.e., $\overline{\mathcal{O}_m} \subset \mathcal{O}$) be a sequence of bounded domains in $\mathbb{R}^n$ such that $\mathcal{O}_m \uparrow \mathcal{O}$ and define the stopping times

$$S_m := \inf\{t \ge 0 : X(t) \notin \mathcal{O}_m\}, \quad m \in \mathbb{N}\,. \tag{2.4.1}$$

Then $S_m \uparrow S$.

Let us fix a positive integer m. For any given $(T,x) \in (0,\infty) \times \mathcal{O}_m$, thanks to the assumption $\mathcal{U} \in C^{1,2}((0,\infty)\times\mathcal{O})$, we also have (2.2.14) with φ replaced by $\mathcal{U}$, t^* by T, and t by $t \wedge S_m$, $0 \le t \le T$. This way we deduce from the supersolution property that $\mathcal{U}_t - \mathcal{L}'\mathcal{U} \ge 0$ holds on $(0,\infty)\times\mathcal{O}$. Therefore the process $\mathcal{U}(T - t\wedge S_m\,,\, X(t\wedge S_m))\,Y(t\wedge S_m)$, $0 \le t \le T$ is a local $\mathbb{P}_x$-supermartingale.

This local supermartingale is nonnegative and hence a true supermartingale. Therefore we obtain

$$\begin{aligned}\mathcal{U}(T,x) &\ge \mathbb{E}_x\big[\mathcal{U}(T - T\wedge S_m, X(T\wedge S_m))Y(T\wedge S_m)\big] \\ &\ge \mathbb{E}_x\big[\mathbf{1}_{\{S_m>T\}}\,\mathcal{U}(0,X(T))Y(T)\big] \;\ge\; \mathbb{E}_x\big[\mathbf{1}_{\{S_m>T\}} f(X(T))Y(T)\big]\end{aligned}$$

by optional sampling. The last expression converges to $\mathbb{E}_x\left[\mathbf{1}_{\{S>T\}} f(X(T))Y(T)\right] = \mathscr{U}(T,x)$ as $m\to\infty$ by the monotone convergence theorem, so we conclude $\mathcal{U}(T,x) \ge \mathscr{U}(T,x)$. □

2.5 The distributional solution

In this section we define and study yet another kind of weak solution, the *distributional solution.* We shall adopt the following definition from [37]. Set

$$\begin{aligned}||f||_{C(\overline{Q})} &:= \sup_{y\in\overline{Q}} |f(y)|\,, \\ ||f||_{W^{1,2}(\overline{Q})} &:= ||f_t||_{L^{n+1}(\overline{Q})} + \sum_i ||f_{x_i}||_{L^{n+1}(\overline{Q})} + \sum_{i,j} ||f_{x_ix_j}||_{L^{n+1}(\overline{Q})} + ||f||_{C(\overline{Q})}\,.\end{aligned}$$

Definition 2.5.1. *([37, Definition 2.1.1])* Let Q be a bounded domain of $\mathbb{R}^{n+1}$. We denote by $W^{1,2}(Q)$ the space of functions $u : \overline{Q} \to \mathbb{R}$, for each of which there exists a sequence of functions $u^{(m)} \in C^{1,2}(\overline{Q})$ such that $||u - u^{(m)}||_{C(\overline{Q})} \to 0$ and $||u^{(m)} - u^{(m')}||_{W^{1,2}(\overline{Q})} \to 0$ as $m, m' \to \infty$.

The continuity of the functions $u^{(m)}$ implies that $u \in C(\overline{Q})$. In addition, every function $u \in W^{1,2}(Q)$ possesses generalized (a.k.a. "weak") derivatives u_t, u_{x_i} and $u_{x_i x_j}$ on Q, which are unique almost everywhere ([37, Definition 2.1.2]).

We generalize Definition 2.5.1 for unbounded domains as follows.

Definition 2.5.2. Let Q' be a domain of $\mathbb{R}^{n+1}$, not necessarily bounded. Define

$$W^{1,2}_{loc}(Q') := \left\{ u : Q' \to \mathbb{R} \mid u \in W^{1,2}(Q) \text{ for all bounded domain } Q \subset Q' \right\}.$$

For $u \in W^{1,2}_{loc}(Q')$, we have $u \in C(\overline{Q})$ for all bounded domain $Q \subset Q'$, and hence $u \in C(\overline{Q'})$ as well. Further, the generalized derivatives u_t, u_{x_i} and $u_{x_i x_j}$ can be shown to be well-defined and unique a.e. To see this, take a sequence of bounded domains $Q_m \subset Q'$ such that $Q_m \uparrow Q'$ and use the fact that the generalized derivatives are uniquely defined a.e. on each Q_m.

Now we are able to define the distributional (sub/super)solution. Let Q' be a domain of $\mathbb{R}^{n+1}$ and $F(z, y, p, q)$ a continuous map from $Q' \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n)$ to $\mathbb{R}$ satisfying the ellipticity condition:

$$F(z, y, p, q_1) \leq F(z, y, p, q_2) \quad \text{whenever} \quad q_1 \geq q_2,$$

for all $(z, y, p) \in Q' \times \mathbb{R} \times \mathbb{R}^n$ (by analogy with the F in Section 2.2.1).

Consider a second-order parabolic PDE

$$u_t(t,x) + F\left((t,x), u(t,x), Du(t,x), D^2u(t,x)\right) = 0, \quad (t,x) \in Q'. \tag{2.5.1}$$

Definition 2.5.3. We say that $u \in W^{1,2}_{loc}(Q')$ is a *distributional sub(super)-solution* of (2.5.1) if the generalized derivatives u_t, Du and D^2u can be chosen such that

$$u_t(t,x) + F\left((t,x), u(t,x), Du(t,x), D^2u(t,x)\right) \leq 0 \ \ (\geq 0), \quad \forall\ (t,x) \in \Gamma \tag{2.5.2}$$

holds for some $\Gamma \subset Q'$ with $\text{meas}\,(Q' \setminus \Gamma) = 0$.

A *distributional solution* is a function that is both a distributional subsolution and a distributional supersolution.

In our setting, we have $Q' = (0,\infty) \times \mathcal{O}$ and

$$F((t,x),y,p,q) = -\frac{1}{2}\sum_{i,j} a_{ij}(x)\, q_{ij} - \sum_i b_i(x)\, p_i + h(x)\, y\,,$$

so the left-hand side of (2.5.2) simplifies to $(u_t - \mathcal{L}'u)(t,x)$ (recall the operator $\mathcal{L}'$ from (2.1.9)). We see that F satisfies the ellipticity condition since $a = \sigma\sigma^T$ is positive-semidefinite.

2.5.1 Domination by nonnegative distributional supersolutions

We have the following minimality result for $\mathscr{U}(t,x)$.

Theorem 2.5.4. *Assume that σ, b and h are locally bounded and a is locally strictly elliptic on $\mathcal{O}$. If $v \in W^{1,2}_{loc}((0,\infty)\times\mathcal{O})$ is a nonnegative distributional supersolution of* (2.1.7) *with*

$$V(0,x) \geq f(x)\,, \quad x \in \mathcal{O}\,, \tag{2.5.3}$$

then $v \geq \mathscr{U}$ on $[0,\infty)\times\mathcal{O}$.

Thus, if the function $\mathscr{U}$ belongs to the space $W^{1,2}_{loc}((0,\infty)\times\mathcal{O})$ and is a distributional supersolution of (2.1.7)*, then it is the smallest nonnegative distributional supersolution of* (2.1.7) *that satisfies* (2.5.3).

Proof. Set the generalized derivatives v_t, v_{x_i} and $v_{x_ix_j}$ to be identically equal to zero on $((0,\infty)\times\mathcal{O}) \setminus \Gamma$ (with Γ as defined in Definition 2.5.3). Fix an arbitrary point $x_0 \in \mathcal{O}$ and real number $T > 0$. Denote $Q_m = (0,T)\times\mathcal{O}_m$, with $\mathcal{O}_m$ as in the proof of Theorem 2.4.1.

Let $u(t,x) = V(T-t,x)$, $(t,x) \in [0,T]\times\mathcal{O}$, then $u \in W^{1,2}(Q_m)$ is a distributional subsolution of $(u_t + \mathcal{L}'u)(t,x) = 0$ on Q_m. The following result is well known.

Lemma 2.5.5. ([37, Theorem 2.10.1]) *Let Q be a bounded domain of $\mathbb{R}^{n+1}$ and $u \in W^{1,2}(Q)$. For any $(s_0, x_0) \in Q$ and any stopping time*

$$\tau \le \tau(Q, s_0) := \inf\{t > 0 : (s_0 + t, X(t)) \notin Q\},$$

we have

$$u(s_0, x_0) = \mathbb{E}_{x_0}\left[Y(\tau)u(s_0 + \tau, X(\tau)) - \int_0^\tau Y(t)(u_t + \mathcal{L}'u)(s_0 + t, X(t))\,\mathrm{d}t\right],$$

where $X(\cdot)$ is the diffusion of (2.1.1) starting at $X(0) = x_0$.

Applying the above lemma with $Q = Q_m$, $s_0 = 1/i$ and $\tau = \tau_{i,m} := \big(T - (1/i)\big) \wedge S_m = \tau(Q, s_0)$ with S_m defined in (2.4.1), and combining with the distributional subsolution property of u leads to

$$u\left(\frac{1}{i}, x_0\right) \ge \mathbb{E}_{x_0}\left[Y(\tau_{i,m})\, u\left(\frac{1}{i} + \tau_{i,m}, X(\tau_{i,m})\right)\right]. \tag{2.5.4}$$

Recall that v is continuous (see the explanation right below Definition 2.5.2), and therefore so is u. Thus u is bounded on $[0,T] \times \mathcal{O}$. Moreover, given m, for all $t \in [0, S_m \wedge T]$ we have

$$Y(t) = \exp\left(-\int_0^t h(X(s))\,\mathrm{d}s\right) \le \exp\left(t \sup_{x \in \mathcal{O}_m} |h(x)|\right) \le \exp\left(T \sup_{x \in \mathcal{O}_m} h(x)\right) < \infty,$$

since h is bounded on $\mathcal{O}_m$. Therefore, the family of random variables $\{Y(t),\ t \in [0, S_m \wedge T]\}$ is uniformly bounded. Notice further that the paths of the processes $X(\cdot)$ and $Y(\cdot)$ are continuous, and that $\lim_{i\to\infty} \tau_{i,m} = S_m \wedge T$. Taking $i \to \infty$ in (2.5.4) yields

$$\begin{aligned}
V(T, x_0) \;=\; u(0, x_0) \;&=\; \lim_{i\to\infty} u\big(1/i\,, x_0\big) \;\ge\; \lim_{i\to\infty} \mathbb{E}_{x_0}\left[Y(\tau_{i,m})\, u\left(\frac{1}{i} + \tau_{i,m}, X(\tau_{i,m})\right)\right] \\
&=\; \mathbb{E}_{x_0}\left[\lim_{i\to\infty} Y(\tau_{i,m})\, u\left(\frac{1}{i} + \tau_{i,m}, X(\tau_{i,m})\right)\right] \\
&=\; \mathbb{E}_{x_0}\big[Y(S_m \wedge T)\, v\,(T - S_m \wedge T, X(S_m \wedge T))\,\big] \\
&\ge\; \mathbb{E}_{x_0}\left[\mathbf{1}_{\{S_m > T\}} Y(T) f^+(X(T))\right] \\
&\to\; \mathbb{E}_{x_0}\left[\mathbf{1}_{\{S > T\}} Y(T) f^+(X(T))\right] \;\ge\; \mathscr{U}(T, x_0), \quad \text{as } m \to \infty,
\end{aligned}$$

by the monotone convergence theorem, as claimed, where $f^+(\cdot) = f(\cdot) \vee 0$. □

Remark 2.5.6. In the case of $\mathcal{O}$ bounded with $\partial\mathcal{O}$ piecewise smooth, besides the condition of Corollary 2.3.3, we have another sufficient condition for $\mathscr{U}$ to be a classical solution of (2.1.7), (2.1.8) whenever it can be checked directly that $\mathscr{U} \equiv 0$ on $(0,\infty) \times \partial\mathcal{O}$: *the functions a, b, f and h are bounded and a is continuously differentiable and uniformly elliptic on $\mathcal{O}$.* Under these conditions, a classical PDE result (see e.g. [39, Theorem III.12.1]) states that the Cauchy problem (2.1.7), (2.1.8) has a unique weak solution $\mathscr{V}$ (in the distributional sense) that vanishes on $(0,\infty) \times \partial\mathcal{O}$; further, we have $\mathscr{V} \in C([0,T] \times \mathcal{O}) \cap C^{1,2}((0,T) \times \mathcal{O})$ for any $T \in (0,\infty)$. One can then show that $\mathscr{V}$ coincides with $\mathscr{U}$ on $(0,\infty) \times \mathcal{O}$ by integrating (2.2.14) with respect to t over $[0, S \wedge t^*]$, for any $t^* \in (0,\infty)$, and the conclusion follows. The result of $\mathscr{V} = \mathscr{U}$ is similar to Theorem II.2.3 and the discussion in [23, pp. 133–135]. The special case U was dealt with in [50, Theorem 2.7]. □

2.5.2 Connections between distributional and viscosity solutions

In [28], H. Ishii showed that the two kinds of weak solutions of (2.1.7) we have been discussing are equivalent, under very nice regularity conditions on the coefficients as well as on the weak solution u. We state his results in our setting:

Theorem 2.5.7. ([28, Theorems 1 and 2]) *Let $\mathcal{Q} := (0,\infty) \times \mathcal{O}$. Assume that $a_{ij} \in C^{1,1}(\mathcal{Q})$ and $b_i \in C^{0,1}(\mathcal{Q})$ with $\mathcal{Q}$ regarded as an open subset of $(0,\infty) \times \mathbb{R}^n$, and h and u are continuous. Then*

(i) *if u is a viscosity solution of* (2.1.7)*, it is also a distributional solution of* (2.1.7)*; whereas*

(ii) *if $\sigma \in C^1(\mathcal{Q})$ and u is a distributional solution of* (2.1.7)*, then u is also a viscosity solution of* (2.1.7)*.*

We have shown in Theorem 2.2.2 that $\mathscr{U}$ is a viscosity solution of (2.1.7). In order to apply Theorem 2.5.7 to prove that $\mathscr{U}$ is also a distributional solution, we need

$a_{ij} \in C^{1,1}(\mathcal{Q})$, $b_i \in C^{0,1}(\mathcal{Q})$, and $\mathscr{U}$ to be continuous. The difficulty in our setting comes from the fact that we want to assume the coefficients to be only continuous, and that the continuity of $\mathscr{U}$ is not known *a priori*. However, if all of the conditions in Theorem 2.5.7 hold, we have shown that $\mathscr{U}$ is then a classical solution (see the discussion in the second paragraph of Section 2.3).

2.6 Conclusions

We have characterized the function $\mathscr{U}$ as a viscosity solution of of the Cauchy problem (2.1.7), (2.1.8), and shown that $\mathscr{U}$ is dominated by any nonnegative classical supersolution of this Cauchy problem. We have also shown that $\mathscr{U}$ is jointly continuous under appropriate conditions when $n = 1$, and therefore the smallest nonnegative classical (super)solution of this Cauchy problem. It would be interesting to investigate whether those conditions can be relaxed.

It would be also interesting to study whether $\mathscr{U}$ is a distributional (super)solution, a requirement which consists of two parts: that $\mathscr{U} \in W^{1,2}_{loc}((0,\infty) \times \mathcal{O})$ and that (2.5.2) holds for some Γ. The analysis in Section 2.5.2 shows that one cannot apply the results or the approach in [28] directly.

Finally, it would be interesting to explore the joint continuity of $\mathscr{U}$ in the multidimensional case; as was pointed out in the second paragraph of Section 2.3, with jointly continuous $\mathscr{U}$, continuous f, and locally Hölder $a\,(>0)$, b and $h\,(\geq 0)$, one can conclude that $\mathscr{U}$ is a classical solution to (2.1.7).

Chapter 3

Optimal Arbitrage under Model Uncertainty

3.1 Introduction

We consider an equity market with asset capitalizations $\mathfrak{X}(t) = (X_1(t), ..., X_n(t))' \in (0,\infty)^n$ at time $t \in [0,\infty)$, and with local covariation rates $\alpha(t,\mathfrak{X}) = (\alpha_{ij}(t,\mathfrak{X}))_{1\leq i,j\leq n}$ and local relative risk rates $\vartheta(t,\mathfrak{X}) = (\vartheta_1(t,\mathfrak{X}), ..., \vartheta_n(t,\mathfrak{X}))'$, which are nonanticipative functionals of (i.e., are determined by) the past and present capitalizations for any given time t. We denote by $\mathbb{S}_+(n)$ is the space of real, symmetric and positive-definite $n \times n$ matrices, fix a collection $\{\mathcal{K}(y)\}_{y\in(0,\infty)\}}$ of nonempty, compact and convex subsets on $\mathbb{R}^n \times \mathbb{S}_+(n)$, and pose the following question:

If the pair $(\vartheta(t,\mathfrak{X}), \alpha(t,\mathfrak{X}))$ is restricted to take values in a given nonempty subset $\mathcal{K}(\mathfrak{X}(t))$ of $\mathbb{R}^n \times \mathbb{S}_+(n)$, what is the highest return on investment relative to the market portfolio over the given time horizon $[0,T]$, that can be achieved using nonanticipative investment rules, when starting with initial capitalizations $x = (x_1, \ldots, x_n)' \in (0,\infty)^n$, and with probability one under all possible market model configurations with the above covariance and relative risk structure?

Equivalently, if the initial configuration of asset capitalizations is $x = (x_1, \ldots, x_n)$,

what is the smallest proportion of the initial total market capitalization $x_1 + \cdots + x_n$, starting with which one can match or outperform the market capitalization over a given time horizon $[0, T]$, by using nonanticipative investment rules, and with probability one under all possible market model configurations with the above covariance and relative risk structure?

Our main result offers the following answers to these two questions: $1/\mathfrak{u}(T, x)$ and $\mathfrak{u}(T, x)$, respectively. Here the function $\mathfrak{u} : [0, \infty) \times \mathbb{R}^n_+ \to (0, 1]$ is, subject to appropriate conditions that will be specified as we progress, a *viscosity solution* to the Cauchy problem for the *Hamilton-Jacobi-Bellman (HJB)* fully nonlinear partial differential equation

$$\big(u_t - \widehat{\mathcal{L}} u\big)(t, x) = 0\,, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n_+ \tag{3.1.1}$$

of parabolic type, subject to the initial condition

$$u(0, x) = 1\,, \quad x \in \mathbb{R}^n_+\,. \tag{3.1.2}$$

Here we are using the notation

$$\begin{aligned} \widehat{\mathcal{L}} u(t, x) &:= \sup_{a \in \mathcal{A}(x)} \mathcal{L}_a u(t, x)\,, \\ \mathcal{L}_a u(t, x) &:= \sum_{i,j} x_i x_j a_{ij} \left(\frac{D^2_{ij}}{2} + \frac{D_i}{||x||_1} \right) u(t, x)\,, \end{aligned} \tag{3.1.3}$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^n_+$ with $a = (a_{ij})_{1 \le i,j \le n}$; we are also using the ℓ^1-norm

$$||x||_1 := \sum_i x_i\,,$$

$$\mathcal{A}(x) := \big\{ a \in \mathbb{S}_+(n) : \exists\ \theta \in \mathbb{R}^n \text{ s.t. } (\theta, a) \in \mathcal{K}(x) \big\} \tag{3.1.4}$$

and employing the notation $D_i u = u_{x_i}$, $D^2_{ij} u = u_{x_i x_j}$, and $\mathbb{R}^n_+ := (0, \infty)^n$. Furthermore, the above function $\mathfrak{u}$ is dominated by any nonnegative classical supersolution of this Cauchy problem; thus, it is the smallest nonnegative classical supersolution of this Cauchy problem, whenever it is of class $C\left([0, \infty) \times \mathbb{R}^n_+\right) \cap C^{1,2}\left((0, \infty) \times \mathbb{R}^n_+\right)$.

The function $\mathfrak{u}$ is called the *arbitrage function* for a model with uncertainty, in the terminology of [15, Sections 1 and 4]; this extends the arbitrage function $\mathfrak{u}_{\mathcal{M}}$ for a specified model $\mathcal{M}$ in the terminology of [13, Section 6]. In [15] the authors characterized the arbitrage function $\mathfrak{u}$ as a classical solution of the HJB equation (3.1.1), subject to the initial condition of (3.1.2), but under much stronger assumptions on the uncertainty structure; see Theorem 3.3.3 below.

Under much weaker conditions than in [15], we develop here a different characterization of the arbitrage function $\mathfrak{u}$, as a viscosity solution to the Cauchy problem of (3.1.1), (3.1.2). We first prove in Theorems 3.4.5 and 3.4.6 that the function $\widehat{\Phi}$ – defined as the supremum of $\mathfrak{u}_{\mathcal{M}}$ over all possible market models $\mathcal{M}$ that satisfies certain strong Markov property (*strongly Markovian admissible systems* in Definition 3.2.2) – and the function Φ – defined as the supremum of $\mathfrak{u}_{\mathcal{M}}$ over all possible market models $\mathcal{M}$ – are viscosity subsolution and viscosity supersolution of this Cauchy problem, respectively.

Moreover, we show in Theorem 3.7.2 that the function $\mathfrak{u}$ coincides with Φ, if this latter function is continuous. As a consequence, the function $\mathfrak{u}$ is shown to be a viscosity supersolution of (3.1.1), and further, a viscosity solution of (3.1.1) if $\Phi \equiv \widehat{\Phi}^*$ (the *upper-semicontinuous envelope* of Φ ; see (3.4.4)).

3.1.1 Preview

Section 3.2 sets up the model for an equity market with model uncertainty regarding its covariance and relative risk characteristics, and Section 3.3 interprets the variables in this model, introduces the concepts of investment rules and portfolios as well as the notion of arbitrage function, and reviews the results of [15].

Section 3.4 recalls the definition of viscosity solutions, states our main results and discusses related work. Section 3.5 characterizes the function $\widehat{\Phi}$ as a viscosity subsolution – and further, in Section 3.6, the function Φ as a viscosity supersolution – to the Cauchy problem of (3.1.1), (3.1.2).

Section 3.7 provides conditions, under which the arbitrage function $\mathfrak{u}$ coincides with the function Φ (Theorems 3.7.1 and 3.7.2), and thus becomes a viscosity solution to the Cauchy problem (3.1.1), (3.1.2). Furthermore, these conditions imply that, if $\mathfrak{u}$ is of class $C\left([0,\infty)\times\mathbb{R}^n_+\right)\cap C^{1,2}\left((0,\infty)\times\mathbb{R}^n_+\right)$, it is a classical solution and in fact the smallest nonnegative (super)solution of this Cauchy problem (Corollary 3.7.3). Additional results, namely, Propositions 3.7.5 and 3.7.10, provide conditions on the covariance and relative risk structure, under which $\mathfrak{u}\equiv\Phi\equiv\widehat{\Phi}$ and it is indeed the smallest nonnegative (super)solution of this Cauchy problem.

Section 3.8 develops the proof of Theorem 3.7.1. Section 3.9 concludes with examples from the (generalized) volatility-stabilized model of [17, 52]. Finally, Appendix B.2 presents an alternative proof of the viscosity characterizations of the functions $\widehat{\Phi}$ and Φ.

3.2 Notation and terminology

We shall fix the dimension n, let $\Omega:=C([0,\infty);\mathbb{R}^n_+)$ be the canonical space of continuous paths $\omega:[0,\infty)\to\mathbb{R}^n_+$ equipped with the topology of locally uniform convergence. We shall also denote by $\mathcal{F}$ the Borel σ-field of Ω, and $\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$ the raw filtration generated by the canonical process $\mathfrak{B}(t,\omega):=\omega(t)$.

We shall let $\mathbf{0}=(0,\cdots,0)'$ denote the origin in $\mathbb{R}^n$, and

$$\mathbb{K}=\{\mathcal{K}(y)\}_{y\in[0,\infty)^n\setminus\{\mathbf{0}\}} \tag{3.2.1}$$

be a collection of nonempty, compact and convex subsets on $\mathbb{R}^n\times\mathbb{S}_+(n)$ (recall that $\mathbb{S}_+(n)$ is the space of real, symmetric, positive-definite $n\times n$ matrices). We denote by $\mathfrak{K}$ the collection of pairs (σ,ϑ) consisting of progressively measurable functionals $\sigma=(\sigma_{ik})_{n\times n}:[0,\infty)\times\Omega\to\mathrm{GL}(n)$ and $\vartheta=(\vartheta_1,\ldots,\vartheta_n)':[0,\infty)\times\Omega\to\mathbb{R}^n$, such that

$$\big(\vartheta(T,\omega),\alpha(T,\omega)\big)\in\mathcal{K}(\omega(T))\quad\text{and}\quad\int_0^T\big(||\vartheta(t,\omega)||^2+\mathrm{Tr}(\alpha(t,\omega))\big)\,\mathrm{d}t<\infty \tag{3.2.2}$$

hold for all $\omega \in \Omega$, $T \in (0, \infty)$, where

$$\alpha := \sigma\sigma' \,. \tag{3.2.3}$$

Here and throughout the paper, $'$ denotes transposition and $\mathrm{GL}(n)$ the space of $n \times n$ invertible real matrices.

Definition 3.2.1. Admissible Systems [15, Sections 1 and 2]**:** For a given $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n_+$, we shall call *admissible system*, subject to the *Knightian uncertainty* $\mathbb{K}$ with initial configuration x, a quintuple $\mathcal{M} = (\sigma, \vartheta, \mathbb{P}, W, \mathfrak{X})$ consisting of

(i) a pair $(\sigma, \vartheta) \in \mathfrak{K}$; of

(ii) a probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$; of

(iii) an n-dimensional $\mathbb{F}-$Brownian motion $W(\cdot) = (W_1(\cdot), \ldots, W_n(\cdot))'$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F}$; and of

(iv) a continuous, $\mathbb{F}-$adapted process $\mathfrak{X}(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'$ with values in $\mathbb{R}^n_+$ and

$$\mathrm{d}X_i(t) = X_i(t) \sum_k \sigma_{ik}(t, \mathfrak{X}) \big(\vartheta_k(t, \mathfrak{X})\, \mathrm{d}t + \mathrm{d}W_k(t)\big) \,, \;\; i = 1, \ldots, n \,, \;\; \mathfrak{X}(0) = x \,. \tag{3.2.4}$$

The integrability condition (3.2.2) guarantees that the process $\mathfrak{X}(\cdot)$ indeed takes values in $\mathbb{R}^n_+$, $\mathbb{P}-$a.s.

We shall write $\sigma^{\mathcal{M}}, \vartheta^{\mathcal{M}}, \mathbb{P}^{\mathcal{M}}, W^{\mathcal{M}}$ and $\mathfrak{X}^{\mathcal{M}}$ for the elements $\sigma, \vartheta, \mathbb{P}, W$ and $\mathfrak{X}$ of the quintuple $\mathcal{M}$, respectively, and $\mathfrak{M}(x)$ for the collection of admissible systems with initial configuration $x \in \mathbb{R}^n_+$. □

In Definition 3.2.1 and throughout this paper, all vectors are assumed to be column vectors, and summations to extend from 1 to n.

Definition 3.2.2. Strongly Markovian Admissible Systems: For a given initial configuration $x \in \mathbb{R}^n_+$, we shall call *strongly Markovian admissible system*, subject

to the Knightian uncertainty $\mathbb{K}$ with initial configuration x, an admissible system $\mathcal{M} = (\sigma, \vartheta, \mathbb{P}, W, \mathfrak{X}) \in \mathfrak{M}(x)$ satisfying:

(i) the functionals σ and ϑ are *Markovian* and *time-homogeneous*, i.e.,

$$\sigma(t,\omega) = \mathbf{s}(\omega(t)) = (\mathbf{s}_{ij}(\omega(t)))_{1\le i,j\le n}\,,$$

and

$$\vartheta(t,\omega) = \boldsymbol{\theta}(\omega(t)) = (\boldsymbol{\theta}_1(\omega(t)), \ldots, \boldsymbol{\theta}_n(\omega(t)))' \tag{3.2.5}$$

for some measurable functions $\mathbf{s} : \mathbb{R}^n_+ \to \mathrm{GL}(n)$ and $\boldsymbol{\theta} : \mathbb{R}^n_+ \to \mathbb{R}^n$; and

(ii) for every $y \in \mathbb{R}^n_+$, there exists an admissible system $\mathcal{M}^y \in \mathfrak{M}(y)$ with the same $\mathbf{s}(\cdot)$ and $\boldsymbol{\theta}(\cdot)$ as in $\mathcal{M}$, and a strongly Markovian state process $\mathfrak{X}(\cdot)$.

We shall denote by $\widehat{\mathfrak{M}}(x)$ the subcollection of $\mathfrak{M}(x)$ consisting of all strongly Markovian admissible systems with initial configuration x. □

Remark 3.2.3. It follows from the Markovian selection results of Krylov (see [36], [58, Chapter 12] and [12, Theorem 5.4]) that, if the collection of subsets $\mathbb{K}$ satisfies the linear growth condition

$$\sup_{\substack{(\theta,a)\in\mathcal{K}(y)\\ b=\varsigma\theta,\ \varsigma\varsigma'=a}} \left[\sum_{i,j} y_i y_j a_{ij} + \sum_i (y_i b_i)^2\right] \le C(1+||y||)^2\,, \quad \forall\ y \in [0,\infty)^n\backslash\{\mathbf{0}\}\,, \tag{3.2.6}$$

for some constant $C > 0$, then the state process $\mathfrak{X}(\cdot)$ can be chosen to be strongly Markovian under $\mathbb{P}^{\mathcal{M}}$ for any admissible system $\mathcal{M}$ with Markovian and time-homogeneous σ and ϑ as in (3.2.5). □

Remark 3.2.4. **(i)** A sufficient condition for $\widehat{\mathfrak{M}}(x) \neq \emptyset$ to hold for all $x \in \mathbb{R}^n_+$, is that there exist locally Lipschitz functions $\mathbf{s}(\cdot)$ and $\boldsymbol{\theta}(\cdot)$ satisfying Condition (i) of Definition 3.2.2, that $\big(\boldsymbol{\theta}(y), \mathbf{s}(y)\mathbf{s}'(y)\big) \in \mathcal{K}(y)$ for all $y \in \mathbb{R}^n_+$, and that $\mathbf{s}(\cdot)$ and $\mathbf{b}(\cdot) := \mathbf{s}(\cdot)\boldsymbol{\theta}(\cdot)$ are linearly growing, i.e.,

$$||\mathbf{s}(y)|| + ||\mathbf{b}(y)|| \le C(1+||y||) \quad \text{for all } y \in \mathbb{R}^n_+\,, \tag{3.2.7}$$

for some real constant $C > 0$. Under this condition and for any $x \in \mathbb{R}^n_+$, the SDE (3.2.4) with the σ and ϑ as in (3.2.5), always has a pathwise unique, strong solution starting at x ([24, Theorem 5.2.2]; [60, p. 8]).

(ii) In particular, if $\mathcal{K}(y) = \{(\boldsymbol{\theta}(y), \mathbf{s}(y)\mathbf{s}'(y))\}$ for all $y \in \mathbb{R}^n_+$ with such $\mathbf{s}$ and $\boldsymbol{\theta}$, then we have $\mathfrak{M}(x) = \widehat{\mathfrak{M}}(x) \neq \emptyset$ for all $x \in \mathbb{R}^n_+$. □

3.3 Interpretation and previous results

The above variables can be interpreted in a model for an equity market with n assets, say stocks, as follows:

(i) $\mathfrak{X}(t)$ as the vector of capitalizations for the various assets $i = 1, \cdots, n$ at time t, and

$$X(t) := \sum_i X_i(t) \tag{3.3.1}$$

as the total capitalization at that time;

(ii) $W(\cdot)$ as the vector of independent factors (sources of randomness) in the resulting model;

(iii) $\sigma_{ik}(t, \mathfrak{X})$, $k = 1, \cdots, n$ as the local volatilities for the ith asset at time t;

(iv) $\alpha_{ij}(t, \mathfrak{X})$ as the local covariation rate between assets i and j at time t;

(v) $\vartheta(t, \mathfrak{X})$ as the vector of local market prices of risk at time t; and

(vi) $\beta(t, \mathfrak{X}) := (\sigma\vartheta)(t, \mathfrak{X})$ as the vector of local rates of return at time t.

3.3.1 Investment rules and portfolios

Consider now an investor who is "small", in the sense that his actions have no effect on market prices. Starting with initial fortune $v > 0$, he uses a rule that invests a proportion $\Pi_i(t, \mathfrak{X})$ of current wealth in the i-th asset of the equity market at time $t \in [0, \infty)$ $(i = 1, \ldots, n)$, and holds the remaining proportion in cash – or equivalently in a zero-interest money market.

We shall call *investment rule* a progressively measurable functional

$$\Pi = (\Pi_1, \cdots, \Pi_n)' : [0, \infty) \times \Omega \to \mathbb{R}^n$$

satisfying

$$\int_0^T \left(|\Pi'(t,\omega)\sigma(t,\omega)\vartheta(t,\omega)| + \Pi'(t,\omega)\alpha(t,\omega)\Pi(t,\omega) \right) \mathrm{d}t < \infty \tag{3.3.2}$$

for all $T \in (0,\infty)$ and $\omega \in \Omega$, and denote by $\mathfrak{P}$ the set of all such (nonanticipative) investment rules.

We shall call an investment rule Π *bounded*, if Π is bounded uniformly on $[0,\infty) \times \Omega$; for a bounded investment rule, the requirement (3.3.2) is satisfied automatically, on the strength of (3.2.2).

We shall call an investment rule Π a *portfolio,* if $\sum_i \Pi_i = 1$ on $[0,\infty) \times \Omega$; in other words, if it never invests, in or borrows from, the money market. We shall call a portfolio Π *long-only* if $\Pi_i \geq 0$, $i = 1, \ldots, n$ also holds on this domain, that is, it never sells any stock short. A long-only portfolio is also bounded, since it satisfies $0 \leq \Pi_i \leq 1$, $i = 1, 2, \ldots, n$.

• Given an initial wealth v, an investment rule Π and an admissible model $\mathcal{M} \in \mathfrak{M}(x)$, the resulting wealth process $Z(\cdot) := Z^{v,\Pi}(\cdot)$ satisfies the initial condition $Z(0) = v$ and, by (3.2.4),

$$\frac{\mathrm{d}Z(t)}{Z(t)} = \sum_i \Pi_i(t,\mathfrak{X}) \frac{\mathrm{d}X_i(t)}{X_i(t)} = \Pi'(t,\mathfrak{X})\sigma(t,\mathfrak{X}) \left[\vartheta(t,\mathfrak{X})\,\mathrm{d}t + \mathrm{d}W(t) \right]. \tag{3.3.3}$$

3.3.2 The market portfolio

In the special case with

$$\Pi_i(t,\omega) \equiv \mu_i(t,\omega) := \frac{\omega_i(t)}{\omega_1(t) + \cdots + \omega_n(t)}, \quad \forall\, i = 1, \cdots, n, \;\; 0 \leq t < \infty, \tag{3.3.4}$$

we have $\mu_i(\cdot\,,\mathfrak{X}) = X_i(\cdot)/X(\cdot)$: the resulting strategy μ invests in all stocks in proportion to their relative market weights. We call the resulting strategy $\Pi \equiv \mu$ the (long-only) *market portfolio.* It follows from the first equality in the dynamics (3.3.3) that investing according to the market portfolio amounts to owning the entire market, in proportion of course to the initial wealth: $Z^{v,\mu}(\cdot) = vX(\cdot)/X(0)$.

3.3.3 The arbitrage function

With these ingredients in place, we define the *arbitrage function*

$$\mathfrak{u} : [0,\infty) \times \mathbb{R}^n_+ \to (0,1]\,,$$

as

$$\mathfrak{u}(T,x) := \inf\Big\{r > 0 : \exists\, \Pi \in \mathfrak{P} \;\text{ s.t. }\; \mathbb{P}^{\mathcal{M}}\left[Z^{rX^{\mathcal{M}}(0),\Pi}(T) \geq X^{\mathcal{M}}(T)\right] = 1\,, \quad \forall\, \mathcal{M} \in \mathfrak{M}(x)\Big\}\,. \tag{3.3.5}$$

For the strict positivity of this quantity, see (3.3.11) below. □

We call the function $\mathfrak{u}(\cdot,\cdot)$ the arbitrage function because, for the initial configuration $x = (x_1,\ldots,x_n)' \in \mathbb{R}^n_+$ of asset capitalizations, the quantity $\mathfrak{u}(T,x)$ can be thought of as the smallest proportion of the initial total market capitalization $x_1 + \cdots + x_n$, starting with which one can find a nonanticipative investment rule, whose performance matches or outperforms that of the market portfolio over the time horizon $[0,T]$, with probability one under all admissible systems. Equivalently, $\mathfrak{u}(T,x)$ can be thought of as the reciprocal of the highest return on investment relative to the market portfolio over the time horizon $[0,T]$, that can be achieved using nonanticipative investment rules when starting with the vector $x = (x_1,\ldots,x_n)' \in \mathbb{R}^n_+$ of initial capitalizations, and with probability one under all admissible systems.

Given an admissible system $\mathcal{M} = (\sigma,\vartheta,\mathbb{P},W,\mathfrak{X}) \in \mathfrak{M}(x)$, we define the stochastic discount factor $L(\cdot)$ as the associated exponential $\mathbb{P}-$local martingale

$$L(t) := \exp\left(-\int_0^t \vartheta'(s,\mathfrak{X})\,\mathrm{d}W(s) - \int_0^t \frac{1}{2}\left|\left|\vartheta(s,\mathfrak{X})\right|\right|^2 \mathrm{d}s\right), \quad 0 \leq t < \infty\,. \tag{3.3.6}$$

This process is well-defined and a strictly positive $\mathbb{P}-$local martingale (thus a $\mathbb{P}-$supermartingale), on the strength of the integrability condition (3.2.2); *but is not necessarily a* $\mathbb{P}-$*martingale*. It plays the rôle of a state-price-density or "deflator" in the present context. We also write $L^{\mathcal{M}}(\cdot)$ for this $L(\cdot)$ under $\mathcal{M}$ when needed.

Assuming that $\widehat{\mathfrak{M}}(x) \neq \emptyset$ holds for all $x \in \mathbb{R}^n_+$, we consider the functions

$$\Phi(T,x) := \sup_{\mathcal{M}\in\mathfrak{M}(x)} \mathfrak{u}_{\mathcal{M}}(T,x) \qquad \text{and} \qquad \widehat{\Phi}(T,x) := \sup_{\mathcal{M}\in\widehat{\mathfrak{M}}(x)} \mathfrak{u}_{\mathcal{M}}(T,x) \tag{3.3.7}$$

for $(T,x) \in [0,\infty) \times \mathbb{R}^n_+$, where

$$\mathfrak{u}_{\mathcal{M}}(T,x) := \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}\left[L^{\mathcal{M}}(T)X^{\mathcal{M}}(T)\right] / ||x||_1 \tag{3.3.8}$$

(recall the ℓ^1-norm $||x||_1 = \sum_i x_i$) and the total capitalization

$$X^{\mathcal{M}}(T) := ||\mathfrak{X}^{\mathcal{M}}(T)||_1 = \sum_i X_i^{\mathcal{M}}(T)\,. \tag{3.3.9}$$

As was shown in [18, Section 10, pp. 127–129], [34], or [54], the quantity $\mathfrak{u}_{\mathcal{M}}(T,x)$ in (3.3.8) is obtained by fixing an admissible system $\mathcal{M}$ in the definition (3.3.5) of $\mathfrak{u}$, namely,

$$\begin{aligned}\mathfrak{u}_{\mathcal{M}}(T,x) = &\inf\left\{r>0 : \exists\ \Pi \in \mathfrak{P} \ \text{ s.t. } \ \mathbb{P}^{\mathcal{M}}\left[Z^{rX^{\mathcal{M}}(0),\Pi}(T) \geq X^{\mathcal{M}}(T)\right] = 1\right\} \\ &\in (0,1]\,.\end{aligned} \tag{3.3.10}$$

This can be interpreted as the reciprocal of the highest return on investment over the time horizon $[0,T]$, that can be achieved relative to the market portfolio in the context of the model $\mathcal{M}$, by using nonanticipative strategies and starting with the vector x of initial capitalizations. It can also can be interpreted as the *arbitrage function for* $\mathcal{M}$ in the terminology of [13, Section 6], at least when $(\mathbb{P},\mathbb{F})$-martingales can be represented as stochastic integrals with respect to the $W(\cdot)$ in (3.2.4).

Since the processes $L(\cdot)$ and $X(\cdot)$ are strictly positive, so is the function $\mathfrak{u}_{\mathcal{M}}(\cdot\,,\cdot)$ for all admissible system $\mathcal{M}$. It then follows from the definitions (3.3.5)–(3.3.10) that

$$1 \geq \mathfrak{u}(T,x) \geq \Phi(T,x) \geq \widehat{\Phi}(T,x) > 0\,, \qquad \forall\ (T,x) \in [0,\infty)\times\mathbb{R}^n_+\,. \tag{3.3.11}$$

Remark 3.3.1. **Strong Arbitrage:** If $\mathfrak{u}(T,x) < 1$, then a *strong arbitrage* relative to the market portfolio in the terminology of [18, Definition 6.1] exists on $[0,T]$ with the

initial capitalizations x. Such strong arbitrage is *robust*, that is, holds under every possible admissible system or model that might materialize.

Instances of $\mathfrak{u}(T,x) < 1$ with $T \in (0,\infty)$ occur when there exists a real constant $C > 0$ such that either

$$\inf_{a\in\mathcal{A}(y)} \left(\sum_i \frac{y_i a_{ii}}{y_1 + \cdots + y_n} - \sum_{i,j} \frac{y_i y_j a_{ij}}{(y_1 + \cdots + y_n)^2} \right) \geq C$$

or

$$\frac{(y_1 \cdots y_n)^{1/n}}{y_1 + \cdots + y_n} \cdot \inf_{a\in\mathcal{A}(y)} \left(\sum_i a_{ii} - \frac{1}{n} \sum_{i,j} a_{ij} \right) \geq C$$

holds for every $y \in \mathbb{R}^n_+$ (recall $\mathcal{A}(\cdot)$ from (3.1.4) and see [18, Examples 11.1, 11.2], [17] and [19]). □

Remark 3.3.2. **No Unbounded Profits with Bounded Risk:** The inequality $\mathfrak{u}(T,x) > 0$ in (3.3.11) rules out scalable arbitrage opportunities, also known as *Unbounded Profits with Bounded Risk (UPBR)*. We refer the reader to [7] for the origin of the resulting "No Unbounded Profit with Bounded Risk" (NUPBR) concept, and to [31] for an elaboration of this point in a different context, namely, the existence and properties of the so-called "numéraire" portfolio. □

3.3.4 Previous results

The Knightian uncertainty in the above model shares a lot with the uncertainty regarding the underlying volatility structure of assets in [44]. The approach in [15] is reminiscent of the Dubins-Savage [8] and Sudderth [27, 48, 51, 59] approaches to stochastic optimization.

The arbitrage function $\mathfrak{u}$ of (3.3.5) was characterized in [15] as a classical solution and in fact, the smallest nonnegative classical (super)solution of the Cauchy problem (3.1.1), (3.1.2), but under rather strong assumptions on the uncertainty structure (see Theorem 3.3.3 below), which amount to: $\Phi \equiv \mathfrak{u}_{\mathcal{M}}$ for some strongly Markovian admissible system $\mathcal{M}$, and $\mathfrak{u}_{\mathcal{M}}$ solves (3.1.1).

Theorem 3.3.3. *[15, Proposition 3, Remark 2] We have* $\mathfrak{u} \equiv \Phi$ *on* $[0,\infty)\times\mathbb{R}^n_+$, *and in fact, this function is the smallest nonnegative (super)solution of* (3.1.1), (3.1.2), *if there exists a strongly Markovian admissible system* $\mathcal{M}_o$ *under which EITHER:*

(i) *the functions* $\mathbf{s}$ *and* $\boldsymbol{\theta}$ *of* (3.2.5) *are locally Lipschitz, and*

(ii) *the function* $u(t,x) := \mathfrak{u}_{\mathcal{M}_o^x}(t,x)$, *which, by [55, Theorem 4.7], is of class* $C^{1,2}$ *and solves*

$$\left(u_t - \mathcal{L}_{\mathbf{a}(x)}u\right)(t,x) = 0 \quad \text{with} \quad \mathbf{a}(x) := \mathbf{s}(x)\mathbf{s}'(x)\,, \quad (t,x)\in(0,\infty)\times\mathbb{R}^n_+ \tag{3.3.12}$$

($\mathcal{M}_o^x := (\mathcal{M}_o)^x$; recall Definition 3.2.2 for $\mathcal{M}^y$ and (3.1.3) *for $\mathcal{L}_a$), is a classical supersolution of* (3.1.1)*;*

OR both of the following conditions hold:

(i)′ *the functions* $\mathbf{s}$ *and* $\boldsymbol{\theta}$ *of* (3.2.5) *are continuous,*

(ii)′ *there exists a positive constant* C *such that*

$$\sum_{i,k} y_i|\mathbf{s}_{ik}(y)\boldsymbol{\theta}_k(y)| \;\le\; C(1+||y||)$$

holds for all $y\in\mathbb{R}^n_+$,

(iii)′ *there exists a* C^2*-function* $\mathbf{h}:\mathbb{R}^n_+\to\mathbb{R}$ *such that* $\boldsymbol{\theta}_k(y) = \sum_i y_i\mathbf{s}_{ik}(y)D_i\mathbf{h}(y)$, $k=1,\dots,n$,

(iv)′ *the function*

$$\begin{aligned}\mathscr{G}(t,x) &:= \mathbb{E}^{\mathbb{P}^{\mathcal{M}_o^x}}\left[\mathscr{F}(\mathfrak{X}(t))\exp\left(\int_0^t \mathscr{K}\big(\mathfrak{X}(t)\big)\right)\right]\\ &\in C\left([0,\infty)\times\mathbb{R}^n_+\right)\cap C^{1,2}\left((0,\infty)\times\mathbb{R}^n_+\right),\end{aligned}$$

where

$$\mathscr{F}(y) \;:=\; \frac{1}{2}\sum_{i,j}\mathbf{a}_{ij}(y)\left[D^2_{ij}\mathbf{h} + D_i\mathbf{h}\cdot D_j\mathbf{h}\right](y) \quad \text{and} \quad \mathscr{K}(y) \;:=\; ||y||_1\exp\big(-\mathbf{h}(y)\big),$$

and

(v)′ *the function* $\mathscr{U}(t,x) := \mathscr{G}(t,x)/\mathscr{F}(x)$ *is a classical supersolution of* (3.1.1).

A natural question to ask then, is whether the arbitrage function $\mathfrak{u}$ of (3.3.5) is still a solution to (3.1.1), perhaps in some weak or generalized sense, when regularity and other conditions are weakened. The answer turns out to be affirmative, though it is somewhat indirect; it is provided in Theorems 3.4.5, 3.4.6 and Corollary 3.7.3 below.

3.4 Viscosity characterizations of the functions Φ and $\widehat{\Phi}$

We first recall from [6] the definition of viscosity (sub/super)solutions for a second-order parabolic partial differential equation, and then state our main results with a discussion of related results.

3.4.1 Viscosity subsolution and supersolution of a second-order parabolic PDE

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, let $\mathbb{S}(n)$ be the set of $n \times n$ real symmetric matrices, and consider a continuous, real-valued mapping $(t, x, r, p, q) \mapsto F(t, x, r, p, q)$ defined on $(0, \infty) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n)$ and satisfying the *ellipticity condition*:

$$F(t, x, r, p, q_1) \leq F(t, x, r, p, q_2) \quad \text{whenever} \quad q_1 \geq q_2 \tag{3.4.1}$$

for all $(t, x, r, p) \in (0, \infty) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n$.

Consider the second-order parabolic partial differential equation

$$u_t + F\left(t, x, u(t, x), Du(t, x), D^2u(t, x)\right) = 0\,, \quad (t, x) \in (0, \infty) \times \mathcal{O} \tag{3.4.2}$$

with the gradient $Du = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})'$ and the Hessian $D^2u = (u_{x_ix_j})_{n\times n}$.

Definition 3.4.1. Viscosity Solution: (i) We say that a function $u : (0, \infty) \times \mathcal{O} \to \mathbb{R}$ is a *viscosity subsolution* of the equation (3.4.2), if

$$\varphi_t + F\left(t_0, x_0, u^*(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)\right) \leq 0 \tag{3.4.3}$$

holds for all $(t_0, x_0) \in (0,\infty) \times \mathcal{O}$ and test functions $\varphi \in C^{1,2}((0,\infty) \times \mathcal{O})$ such that (t_0, x_0) is a (strict) (local) maximum of $u^* - \varphi$ on $(0,\infty) \times \mathcal{O}$. We have denoted here by

$$u^*(t,x) := \limsup_{(s,y)\to(t,x)} u(s,y), \qquad (t,x) \in (0,\infty) \times \mathcal{O} \tag{3.4.4}$$

the *upper-semicontinuous envelope* of u, i.e., the smallest upper-semicontinuous function that dominates pointwise the function u.

(ii) Similarly, we say that $u : (0,\infty) \times \mathcal{O} \to \mathbb{R}$ is a *viscosity supersolution* of (3.4.2), if

$$\varphi_t + F\left(t_0, x_0, u_*(t_0,x_0), D\varphi(t_0,x_0), D^2\varphi(t_0,x_0)\right) \geq 0 \tag{3.4.5}$$

holds for all $(t_0, x_0) \in (0,\infty) \times \mathcal{O}$ and test functions $\varphi \in C^{1,2}((0,\infty) \times \mathcal{O})$ such that (t_0, x_0) is a (strict) (local) minimum of $u_* - \varphi$ on $(0,\infty) \times \mathcal{O}$. We have denoted here by

$$u_*(t,x) := \liminf_{(s,y)\to(t,x)} u(s,y), \qquad (t,x) \in (0,\infty) \times \mathcal{O} \tag{3.4.6}$$

the *lower-semicontinuous envelope* of u, i.e., the largest lower-semicontinuous function dominated pointwise by the function u.

(iii) Finally, we say that $u : (0,\infty) \times \mathcal{O} \to \mathbb{R}$ is a *viscosity solution* of (3.4.2), if it is both a viscosity subsolution and a viscosity supersolution of this equation.

Remark 3.4.2. The above definition implies that u is a viscosity subsolution (supersolution) of (3.4.2) if and only if u^* (u_*) is a viscosity subsolution (supersolution) of this equation. □

3.4.2 Main results

In our setting we have $\mathcal{O} = \mathbb{R}^n_+$ and

$$F(t,x,r,p,q) = -\sup_{a\in\mathcal{A}(x)} \left(\sum_{i,j} x_i x_j a_{ij} \left(\frac{q_{ij}}{2} + \frac{p_i}{||x||_1} \right) \right) \tag{3.4.7}$$

for $q = (q_{ij})_{1\leq i,j\leq n}$, $p = (p_1, \ldots, p_n)'$, and thus the left-hand sides of (3.4.3) and (3.4.5) simplify to $\left(\varphi_t - \widehat{\mathcal{L}}\varphi\right)(t_0, x_0)$ in the notation of (3.1.3).

Since each matrix a in the collection $\mathcal{A}(x)$ of (3.1.4) is positive-definite, we deduce that the matrix $(x_i x_j a_{ij})_{n\times n} = x'ax$ is always positive-definite, and hence F satisfies the ellipticity condition (3.4.1).

In the results that follows, we shall also need F to be a continuous mapping, as well as the following conditions:

Assumption 3.4.3. Local Boundedness: The collection $\mathbb{K}$ of (3.2.1) is locally bounded on $\mathbb{R}^n_+$; that is, for any $x \in \mathbb{R}^n_+$, there exists a neighborhood $\mathcal{D}(x) \subset \mathbb{R}^n_+$ of x such that $\bigcup_{y\in\mathcal{D}(x)} \mathcal{K}(y)$ is bounded.

Assumption 3.4.4. Continuity: For any $\iota > 0$, $x \in \mathbb{R}^n_+$ and $a = (a_{ij})_{1\le i,j\le n} \in \mathcal{A}(x)$, there exist a positive number $\delta < \iota$ and locally Lipschitz functions $\mathbf{s} : \mathbb{R}^n_+ \to \mathrm{GL}(n)$ and $\boldsymbol{\theta} : \mathbb{R}^n_+ \to \mathbb{R}^n$ such that $\mathbf{s}(\cdot)$ and $\mathbf{b}(\cdot) := \mathbf{s}(\cdot)\boldsymbol{\theta}(\cdot)$ are linearly growing (i.e., satisfy the condition (3.2.7)), and that for $\mathbf{a}(\cdot) = (\mathbf{a}_{ij}(\cdot))_{1\le i,j\le n} := \mathbf{s}(\cdot)\mathbf{s}'(\cdot)$, we have $\big(\boldsymbol{\theta}(y), \mathbf{a}(y)\big) \in \mathcal{K}(y)$ for all $y \in \mathbb{R}^n_+$ and

$$|\mathbf{a}_{ij}(y) - a_{ij}| < \iota\,, \quad 1 \le i,j \le n\,, \quad \text{for all } y \in B_\delta(x)\,. \tag{3.4.8}$$

Remark: All of the conditions in Assumption 3.4.4, except for (3.4.8), are inspired by Remark 3.2.4. The aim is to guarantee the existence of an admissible system with the functional $\sigma(t,\omega) = \mathbf{s}(\omega(t))$, as in (3.2.5). □

We have then the following results.

Theorem 3.4.5. Viscosity Subsolution: *Suppose that the real-valued function F of* (3.4.7) *is continuous on $(0,\infty)\times\mathbb{R}^n_+\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}(n)$, and that Assumption 3.4.3 holds.*

The function $\widehat{\Phi}$ of (3.3.7) *is then a viscosity subsolution of the HJB equation* (3.1.1)*, and thus a viscosity subsolution of the Cauchy problem* (3.1.1), (3.1.2)*, since it satisfies $\widehat{\Phi}(0,\cdot) = 1$.*

Theorem 3.4.6. Viscosity Supersolution: *Suppose that the real-valued function F of* (3.4.7) *is continuous on $(0,\infty)\times\mathbb{R}^n_+\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}(n)$, and that Assumptions 3.4.3 and 3.4.4 hold.*

The function Φ *of* (3.3.7) *is then a viscosity supersolution of the HJB equation* (3.1.1)*, and thus a viscosity supersolution of the Cauchy problem* (3.1.1), (3.1.2)*, since it satisfies* $\Phi(0,\cdot) = 1$.

3.4.3 Discussion of related work

These results echo similar themes from the literature on models with an analogous type of uncertainty, under which the functionals σ and ϑ are *fixed*; instead, the uncertainty comes from a *control* process $\mathcal{C}(\cdot)$. At any time t, the values of σ and ϑ are determined not only by the present capitalizations $\mathfrak{X}(t)$, but also by the present value $\mathcal{C}(t)$ of the control process $\mathcal{C}$, i.e., the local volatility matrix and the relative risk vector at time t are $\sigma(\mathfrak{X}(t), \mathcal{C}(t))$ and $\vartheta(\mathfrak{X}(t), \mathcal{C}(t))$, respectively. A control process is a progressively measurable process that takes values in a given subset Γ of some Euclidean space and satisfies certain integrability condition.

Among those papers in the literature are the ground-breaking works [40]–[43] by P.L. Lions, specifically, [43, Theorem III.1] (or [41, Theorem I.1]). These impose much stronger assumptions on the volatility and drift structure: namely, the continuity of $h(x,\cdot)$ for all x and $\sup_{\gamma\in\Gamma} ||h(\cdot,\gamma)||_{W^{2,\infty}(\mathbb{R}^n)} < \infty$, where $h = \sigma_{ik}, \beta_i$, $1 \leq i,k \leq n$.

A similar result was proved in [56, Theorem 4.1], but under the stronger assumptions that both functions σ and β be bounded and Lipschitz, that the analogue in their formulation of the function F of (3.4.7) be locally Lipschitz, and that the set Γ be compact.

If the functions $\alpha_{ij}(\cdot,\gamma)$ ($\gamma \in \Gamma, 1 \leq i,j \leq n$) are all of class $C^{1,\eta}_{\text{loc}}\left(\mathbb{R}^n_+\right)$ for some constant $\eta \in (0,1]$, then in [2, Theorem 3.3], and more generally, in [35, Theorem 2.1], the asymptotic-growth-optimal trading strategy is characterized in terms of a generalized version of the principal eigenvalue of the following fully nonlinear elliptic operator and its associated eigenfunction:

$$\widetilde{\mathcal{L}}u(t,x) := \frac{1}{2}\sum_{i,j} x_i x_j a_{ij} D^2_{ij} u(t,x)\,.$$

For a model with *no* uncertainty and with local volatility matrix $\sigma(\mathfrak{X}(t))$ and relative risk vector $\vartheta(\mathfrak{X}(t))$ at time t, the viscosity characterization was obtained in [3, Proposition 4.5] but with additional local Lipschitz condition on σ and ϑ. This (local) Lipschitz condition is also a typical assumption in previous literature on stochastic control and dynamic programming, e.g., [4, 26, 60] (it is even assumed in [20] that $\sigma(y,\gamma)$ and $\vartheta(y,\gamma)$ are continuous and twice differentiable in y).

In the one-dimensional case ($n = 1$) with *zero* drift ($\beta \equiv 0$) but *no* uncertainty, the authors of [5] removed the local Lipschitz condition and hence chose not to pursue a viscosity characterization; instead, provided that the function σ is continuous and satisfies $\int_1^\infty x\sigma^{-2}(x)\,\mathrm{d}x = \infty$, they approximated the arbitrage function by classical solutions to Cauchy problems [5, Theorem 5.3].

3.5 The proof of Theorem 3.4.5: viscosity subsolution

We first highlight the main idea without many of the technicalities. We argue by contradiction, assuming the negation of (3.4.3) in Definition 3.4.1 with the function F as in (3.4.7): namely, that there exist $\varphi \in C^{1,2}\big((0,\infty)\times\mathbb{R}^n_+\big)$ and $(t_0,x_0) \in (0,\infty)\times\mathbb{R}^n_+$, such that (t_0,x_0) is a strict maximum of $\widehat{\Phi}^* - \varphi$; that the maximal value is equal to zero; and that

$$\big(\varphi_t - \widehat{\mathcal{L}}\varphi\big)(t_0,x_0) > 0\,. \tag{3.5.1}$$

It follows from the definition (3.4.4) of $\widehat{\Phi}^*$ that we can take a pair (t^*,x^*) close to (t_0,x_0) such that the nonnegative difference $\big(\varphi - \widehat{\Phi}\big)(t^*,x^*)$ is sufficiently small, say less than a small positive constant C_3; further, by the definition (3.3.7) of $\widehat{\Phi}$, we can take an admissible system $\mathcal{M}^{x^*} \in \widehat{\mathfrak{M}}(x^*)$ such that $0 \le \big(\widehat{\Phi} - \mathfrak{u}_{\mathcal{M}^{x^*}}\big)(t^*,x^*) < C_3\,.$

Therefore $0 \leq (\varphi - \mathfrak{u}_{\mathcal{M}^{x^*}})(t^*, x^*) < 2\,C_3$. Under this system, we have

$$||x^*||_1\,\varphi(t^*,x^*) - \mathbb{E}\left[L(\rho)X(\rho)\varphi\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right] = \mathbb{E}\left[\int_0^\rho L(s)X(s)\,g\big(t^*-s,s,\mathfrak{X}\big)\,\mathrm{d}s\right] > 0\,, \tag{3.5.2}$$

for any sufficiently small positive stopping time ρ, where

$$g(t,s,\mathfrak{X}) := (\varphi_t - \mathcal{L}_{\alpha(s,\mathfrak{X})}\varphi)\,(t,\mathfrak{X}(s)) \geq \big(\varphi_t - \widehat{\mathcal{L}}\varphi\big)\,(t,\mathfrak{X}(s))$$

for $(t,s) \in (0,\infty)\times[0,\infty)$ and with $\mathcal{L}_a$ and $\widehat{\mathcal{L}}$ as in (3.1.3). This displayed quantity is positive for any sufficiently small s, and t sufficiently close to t^*, by virtue of (3.5.1) and the continuity of the function F in (3.4.7).

On the other hand, on the left-hand side of (3.5.2) we can estimate $\varphi(t^*,x^*)$ from above by $\mathfrak{u}_{\mathcal{M}^{x^*}}(t^*,x^*) + 2\,C_3$, and $\varphi\big(t^*-\rho,\mathfrak{X}(\rho)\big)$ from below by $\widehat{\Phi}\big(t^*-\rho,\mathfrak{X}(\rho)\big) + C_2$ (for some ω's in Ω) or by $\widehat{\Phi}\big(t^*-\rho,\mathfrak{X}(\rho)\big)$ (for other ω's in Ω) with C_2 a small positive constant; this allows us to deduce

$$||x^*||_1\,\mathfrak{u}_{\mathcal{M}^{x^*}}(t^*,x^*) > \mathbb{E}\left[L(\rho)X(\rho)\,\widehat{\Phi}\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right]. \tag{3.5.3}$$

But the inequality (3.5.3) turns out to contradict the martingale property of the process

$$L(\cdot)X(\cdot)\,\mathfrak{u}_{\mathcal{M}^{\mathfrak{X}(\cdot)}}\big(T-\cdot\,,\mathfrak{X}(\cdot)\big)\,;$$

see Proposition 3.5.3, and recall $\mathcal{M}^x,\ x \in \mathbb{R}^n_+$ from Definition 3.2.2.

When implementing this program, the stopping time ρ needs to be not only small, but also such that on $[0,\rho]$ the processes $L(\cdot)$ and $\mathfrak{X}(\cdot)$ are bounded, and $\mathfrak{X}(\cdot)$ is close to x^*; however, ρ cannot be too small, in order to ensure that $\varphi\big(t^*-\rho,\mathfrak{X}(\rho)\big) \geq \widehat{\Phi}\big(t^*-\rho,\mathfrak{X}(\rho)\big) + C_2$ holds with a probability greater than some positive constant independent of C_2 (1/2 in the following proof, see Lemma 3.5.2). These considerations inspire us to construct ρ as in (3.5.10)–(3.5.12) below.

Proof of Theorem 3.4.5: According to Definition 3.4.1 (i) of viscosity subsolution with the function F as in (3.4.7), it suffices to show that for any test function $\varphi \in C^{1,2}\big((0,\infty)\times\mathbb{R}^n_+\big)$ and $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n_+$ with

$$\big(\widehat{\Phi}^*-\varphi\big)(t_0,x_0)=0>\big(\widehat{\Phi}^*-\varphi\big)(t,x)\,,\quad\forall\,(t,x)\in(0,\infty)\times\mathbb{R}^n_+\,,\tag{3.5.4}$$

(i.e., such that (t_0,x_0) is a strict maximum of $\widehat{\Phi}^*-\varphi$), we have

$$\big(\varphi_t-\widehat{\mathcal{L}}\varphi\big)(t_0,x_0)\le 0\,.$$

Here $\widehat{\mathcal{L}}$ is defined in (3.1.3), and $\widehat{\Phi}^*$ is the upper-semicontinuous envelope of $\widehat{\Phi}$ as in the definition (3.4.4). *We shall argue this by contradiction, assuming that*

$$\widehat{\mathcal{G}}(t_0,x_0)>0\qquad\text{holds for the function}\qquad\widehat{\mathcal{G}}(t,x):=\big(\varphi_t-\widehat{\mathcal{L}}\varphi\big)(t,x)\,.\tag{3.5.5}$$

Since the function F of (3.4.7) is continuous, so is the function $\widehat{\mathcal{G}}$ just introduced in (3.5.5). There will exist then, under this hypothesis and Assumption 3.4.3, a neighborhood $\mathcal{D}_\delta:=(t_0-\delta,t_0+\delta)\times B_\delta(x_0)$ of (t_0,x_0) in $(0,\infty)\times\mathbb{R}^n_+$ with $0<\delta<||x_0||_1/n$, on which $\mathcal{K}(\cdot)$ is bounded and $\widehat{\mathcal{G}}(\cdot\,,\cdot)>0$ holds.

Let C be a constant such that $||\theta||<C$ and $|a_{ij}|<C$ $(1\le i,j\le n)$ hold for all pairs $(\theta,a=(a_{ij})_{n\times n})\in\mathcal{K}(x)$ and all $x\in B_\delta(x_0)$. We notice that $|x_i-(x_0)_i|\le|x-x_0|<\delta$ holds for any $x=(x_1,\ldots,x_n)\in\mathcal{D}_\delta$, thus

$$0<||x_0||_1-n\delta<||x||_1<||x_0||_1+n\delta\,,\tag{3.5.6}$$

and introduce the strictly positive constants

$$\begin{aligned}C_1&:=\sqrt{32\,\delta C^2+4\,\delta^2C^4}\,,\\ C_2&:=-\max_{\partial\mathcal{D}_\delta}\big(\widehat{\Phi}^*-\varphi\big)(t,x)\,,\\ C_3&:=\frac{C_2\,e^{-C_1}(||x_0||_1-n\delta)}{4(||x_0||_1+n\delta)}\end{aligned}\tag{3.5.7}$$

(the positivity of C_2 and C_3 follows from (3.5.4) and (3.5.6), respectively). We observe that

$$\limsup_{(t,x)\to(t_0,x_0)}\big(\widehat{\Phi}-\varphi\big)(t,x)=\big(\widehat{\Phi}^*-\varphi\big)(t_0,x_0)=0\,,$$

and hence there exists $(t^*, x^*) \in \mathcal{D}_\delta$ such that

$$\big(\widehat{\Phi} - \varphi\big)(t^*, x^*) > -C_3\,; \tag{3.5.8}$$

and by the definition (3.3.7) of $\widehat{\Phi}$, there exists an admissible system $\mathcal{M}^{x^*} \in \widehat{\mathfrak{M}}(x^*)$ such that

$$\mathfrak{u}_{\mathcal{M}^{x^*}}(t^*, x^*) > \widehat{\Phi}(t^*, x^*) - C_3 > \varphi(t^*, x^*) - 2\,C_3\,, \quad \text{by (3.5.8)}\,. \tag{3.5.9}$$

The remaining discussion in this section (with the exception of Proposition 3.5.3) will be carried out under this admissible system.

• Let us start by recalling the definitions of $\mathcal{D}_\delta$ and t^*, and by constructing the positive stopping times

$$\begin{aligned}
\nu\,(=\nu(\omega)) &:= \inf\big\{s \in (0, t^*] : \big(t^* - s, \mathfrak{X}(s)\big) \notin \mathcal{D}_\delta\big\} \\
&\leq t^* - (t_0 - \delta) = (t^* - t_0) + \delta < t^* \wedge 2\delta\,, \qquad (3.5.10)\\
\lambda\,(=\lambda\,(\omega)) &:= \inf\{s > 0 : |\log L(s)| > C_1\}\,, \qquad (3.5.11)\\
\rho\,(=\rho(\omega)) &:= \nu \wedge \lambda \qquad (3.5.12)
\end{aligned}$$

with the usual convention $\inf\emptyset = \infty$. From the definitions (3.5.5) and (3.1.3), we see that

$$\begin{aligned}
g(t, s, \mathfrak{X}) &:= \big(\varphi_t - \mathcal{L}_{\alpha(s,\mathfrak{X})}\varphi\big)\big(t, \mathfrak{X}(s)\big) \\
&\geq \widehat{\mathcal{G}}\big(t, \mathfrak{X}(s)\big)\,, \quad \forall\ (t, s) \in (0, \infty) \times [0, \infty)\,. \qquad (3.5.13)
\end{aligned}$$

Recall that $\widehat{\mathcal{G}}(\cdot\,,\cdot) > 0$ holds on $\mathcal{D}_\delta$, from the discussion right below (3.5.5). Combining with (3.5.13), this observation leads to

$$g(t^* - s, s, \mathfrak{X}) > 0\,, \quad \forall\ s \in [0, \rho)\,. \tag{3.5.14}$$

Thanks to the assumption $\varphi \in C^{1,2}\big((0, \infty) \times \mathbb{R}^n_+\big)$, we can apply Itô's change of variable rule to $X(t)L(t)\varphi(T - t, \mathfrak{X}(t))$, $0 \leq t \leq T$ and derive the following decomposition (see Appendix B.1 for a detailed proof).

Lemma 3.5.1. *For any* $0 \le t < T < \infty$, $x \in \mathbb{R}^n_+$, $\varphi \in C^{1,2}\big((0,\infty)\times\mathbb{R}^n_+\big)$, *and diffusion* $\mathfrak{X}(\cdot)$ *satisfying* (3.2.4), *we have*

$$\begin{aligned} & \mathrm{d}\big(L(t)X(t)\varphi\big(T-t,\mathfrak{X}(t)\big)\big) \qquad (3.5.15)\\ = \; & -L(t)X(t)g\big(T-t,t,\mathfrak{X}\big)\,\mathrm{d}t - X(t)\varphi\big(T-t,\mathfrak{X}(t)\big)L(t)\,\vartheta'(t,\mathfrak{X})\,\mathrm{d}W(t)\\ & + L(t)\sum_{i,k} X_i(t)\big[\varphi + X(t)D_i\varphi\big]\big(T-t,\mathfrak{X}(t)\big)\sigma_{ik}(t,\mathfrak{X})\,\mathrm{d}W_k(t)\,. \end{aligned}$$

Let us apply now Lemma 3.5.1 with $T = t^*$, integrating (3.5.15) with respect to t over $[0,\rho]$ and taking the expectation under $\mathbb{P}$, to obtain

$$\begin{aligned} ||x^*||_1\,\varphi(t^*,x^*) - \mathbb{E}\left[L(\rho)X(\rho)\varphi\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right] &= \mathbb{E}\left[\int_0^\rho L(s)X(s)\,g\big(t^*-s,s,\mathfrak{X}\big)\,\mathrm{d}s\right]\\ &> 0\,. \qquad (3.5.16) \end{aligned}$$

Here, the strict inequality comes from (3.5.14) and the positivity of ρ; whereas, in the equality, the expectations of the integrals with respect to $\mathrm{d}W(t)$ or $\mathrm{d}W_k(t)$ have all vanished. This is due to the the boundedness of the processes $\mathfrak{X}(\cdot)$ and $L(\cdot)$ on $[0,\rho]$, of the functions φ and $D_i\varphi$ on $\overline{\mathcal{D}_\delta}$, and of the functionals $\vartheta(\cdot,\mathfrak{X})$, $\alpha_{ij}(\cdot,\mathfrak{X})$ (by Assumption 3.4.3) and thus $\sigma_{ik}(\cdot,\mathfrak{X})$ on $[0,\rho]$.

(We have made use here of the following facts. The eigenvalues e_i of α are the nonnegative roots of the characteristic polynomial of α, which is determined by the entries α_{ij}; since the $\alpha_{ij}(\cdot,\mathfrak{X})$'s are bounded on $[0,\rho]$, so are the e_i's. Thus σ, which can be written as $\mathbf{QD}$ for some $n\times n$ orthonormal matrix $\mathbf{Q}$ and diagonal matrix $\mathbf{D}$ with diagonal entries $\sqrt{e_i}$, is also bounded.)

Notice that $\big(t^*-\nu,\mathfrak{X}(\nu)\big) \in \partial\mathcal{D}_\delta$ holds by the definition (3.5.10) of ν, so we have

$$\varphi\big(t^*-\nu,\mathfrak{X}(\nu)\big) \ge \widehat{\Phi}^*\big(t^*-\nu,\mathfrak{X}(\nu)\big) + C_2 \ge \widehat{\Phi}\big(t^*-\nu,\mathfrak{X}(\nu)\big) + C_2\,. \qquad (3.5.17)$$

Plugging (3.5.4), (3.5.9) and (3.5.17) into (3.5.16) yields

$$\begin{aligned} 0 < ||x^*||_1 \left[\mathfrak{u}_{\mathcal{M}^{x^*}}(t^*, x^*) + 2\, C_3 \right] - \mathbb{E}\left[\mathbf{1}_{\{\rho=\nu\}} L(\rho) X(\rho) \left(\widehat{\Phi}\big(t^* - \rho, \mathfrak{X}(\rho)\big) + C_2 \right) \right. \\ \left. + \, \mathbf{1}_{\{\rho\neq\nu\}} L(\rho) X(\rho)\, \widehat{\Phi}\big(t^* - \rho, \mathfrak{X}(\rho)\big) \right] \\ = ||x^*||_1\, \mathfrak{u}_{\mathcal{M}^{x^*}}(t^*, x^*) - \mathbb{E}\left[L(\rho) X(\rho)\, \widehat{\Phi}\big(t^* - \rho, \mathfrak{X}(\rho)\big) \right] + 2\, C_3\, ||x^*||_1 \\ - C_2\, \mathbb{E}\big[\mathbf{1}_{\{\rho=\nu\}} L(\rho) X(\rho) \big] . \end{aligned} \tag{3.5.18}$$

We start by estimating the last term on the right-hand side of (3.5.18). Recalling the definition (3.5.12) of ρ and the second inequality in (3.5.6), we see that

$$L(\rho) \geq e^{-C_1} \quad \text{and} \quad X(\rho) > ||x_0||_1 - n\delta\ > 0\,, \tag{3.5.19}$$

and hence

$$\mathbb{E}\big[\mathbf{1}_{\{\rho=\nu\}} L(\rho) X(\rho) \big] \geq e^{-C_1} \big(||x_0||_1 - n\delta \big)\ \mathbb{P}\big(\rho = \nu\big)\,. \tag{3.5.20}$$

Lemma 3.5.2. *We have*

$$\mathbb{P}\big(\rho = \nu\big)\ =\ \mathbb{P}\big(\lambda \geq \nu\big) \geq \frac{1}{2}\,. \tag{3.5.21}$$

Proof. For any $t \in (0, \nu]$, we have

$$\begin{aligned} \big(\log L(t) \big)^2 \ &= \ \left| - \int_0^t \vartheta'(s, \mathfrak{X})\, \mathrm{d}W(s) - \int_0^t \frac{1}{2} \left|\left| \vartheta(s, \mathfrak{X}) \right|\right|^2 \mathrm{d}s \right|^2 \\ &\leq \ 2 \left| \int_0^t \vartheta'(s, \mathfrak{X})\, \mathrm{d}W(s) \right|^2 + 2 \left| \int_0^t \frac{1}{2} \left|\left| \vartheta(s, \mathfrak{X}) \right|\right|^2 \mathrm{d}s \right|^2 . \end{aligned}$$

It follows from $t \leq \nu < 2\delta$ that

$$\int_0^t \frac{1}{2} \left|\left| \vartheta(s, \mathfrak{X}) \right|\right|^2 \mathrm{d}s\ \leq\ \frac{t}{2}\, C^2\ \leq\ \delta C^2,$$

and therefore

$$\mathbb{E}\left[\sup_{0\leq t\leq \nu} \big(\log L(t) \big)^2 \right] \leq 2\, \mathbb{E}\left[\sup_{0\leq t\leq \nu} \left| \int_0^t \vartheta'(s, \mathfrak{X})\, \mathrm{d}W(s) \right|^2 \right] + 2\, \delta^2 C^4.$$

Further, the Burkholder-Davis-Gundy Inequality gives

$$2\, \mathbb{E}\left[\sup_{0\leq t\leq \nu} \left| \int_0^t \vartheta'(s, \mathfrak{X})\, \mathrm{d}W(s) \right|^2 \right] \leq 8\, \mathbb{E}\left[\int_0^\nu \left|\left| \vartheta'(s, \mathfrak{X}) \right|\right|^2 \mathrm{d}s \right] \leq 8\, \mathbb{E}\left[\nu C^2 \right] \leq 16\, \delta C^2,$$

and therefore

$$\mathbb{E}\left[\sup_{0\le t\le\nu}\big(\log L(t)\big)^2\right]\le 16\,\delta C^2+2\,\delta^2C^4.$$

Finally, appealing to Markov's Inequality yields

$$\mathbb{P}\big(\lambda<\nu\big)=\mathbb{P}\left[\sup_{0\le t\le\nu}|\log L(t)|>C_1\right]\le\frac{16\,\delta C^2+2\,\delta^2C^4}{C_1^2}=\frac{1}{2}$$

(this is why we defined C_1 as in (3.5.7); in fact, setting C_1 to be any value greater than the right-hand side of the first equation in (3.5.7) would also work), and the claim (3.5.21) follows. □

Substituting the estimate of Lemma 3.5.2 into (3.5.20), we obtain

$$C_2\,\mathbb{E}\big[\mathbf{1}_{\{\rho=\nu\}}L(\rho)X(\rho)\big]\ge\frac{1}{2}C_2\,e^{-C_1}\big(||x_0||_1-n\delta\big)=2\,C_3\big(||x_0||_1+n\delta\big)>2\,C_3\,||x^*||_1\,,$$

where we used the definition (3.5.7) of C_3 and the last inequality in (3.5.6). Plugging into (3.5.18) yields the inequality (3.5.3); however, this inequality contradicts Proposition 3.5.3 (ii) right below with $T=t^*$ and $\tau=\rho$. (This explains why we constructed C_3 as we did in (3.5.7); in fact, setting C_3 to be any value less than the right-hand side of (3.5.7) would also work.)

The proof of Theorem 4.5 is complete. □

Proposition 3.5.3. Martingale Property: *Recall the strongly Markovian admissible systems $\mathcal{M}^y\in\widehat{\mathfrak{M}}(y)$ $(y\in\mathbb{R}^n_+)$ from Definition 3.2.2.*

(i) *For $0\le t\le T<\infty$, we have*

$$L(t)X(t)\,\mathfrak{u}_{\mathcal{M}^{\mathfrak{X}(t)}}\big(T-t,\mathfrak{X}(t)\big)=\mathbb{E}\big[\,L(T)X(T)\,\big|\,\mathcal{F}(t)\,\big]\,,\ \mathbb{P}-\text{a.s.}$$

In particular, the process on the left-hand side is a martingale.

(ii) *For any stopping time $\tau\le T<\infty$, we have*

$$\mathbb{E}\left[L(\tau)X(\tau)\,\mathfrak{u}_{\mathcal{M}^{\mathfrak{X}(\tau)}}\big(T-\tau,\mathfrak{X}(\tau)\big)\right]=||x^*||_1\,\mathfrak{u}_{\mathcal{M}^{x^*}}(T,x^*)\,.$$

Proof. (i) To alleviate notation somewhat, we write $\mathbb{P}^y$, $W^y(\cdot)$, $X^y(\cdot)$ and $L^y(\cdot)$ for $\mathbb{P}^{\mathcal{M}^y}$, $W^{\mathcal{M}^y}(\cdot)$ $X^{\mathcal{M}^y}(\cdot)$ and $L^{\mathcal{M}^y}(\cdot)$ ($y \in \mathbb{R}^n_+$), respectively. The definitions (3.3.8) of $\mathfrak{u}_{\mathcal{M}}$ and (3.3.6) of $L^{\mathcal{M}}$ give

$$\begin{aligned}
\text{LHS} &= L(t)\,\mathbb{E}^{\mathbb{P}^{\mathfrak{X}(t)}}\left[X^{\mathfrak{X}(t)}(T-t)L^{\mathfrak{X}(t)}(T-t)\right] \\
&= L(t)\,\mathbb{E}^{\mathbb{P}^{\mathfrak{X}(t)}}\left[X^{\mathfrak{X}(t)}(T-t)\exp\left(-\int_0^{T-t}\vartheta'\left(s,\mathfrak{X}^{\mathfrak{X}(t)}\right)\,\mathrm{d}W^{\mathfrak{X}(t)}(s)\right.\right. \\
&\qquad\qquad \left.\left.-\int_0^{T-t}\frac{1}{2}\left|\left|\vartheta\left(s,\mathfrak{X}^{\mathfrak{X}(t)}\right)\right|\right|^2\mathrm{d}s\right)\right] \\
&= L(t)\,\mathbb{E}\left[X(T)\exp\left(-\int_t^T\vartheta'(s,\mathfrak{X})\,\mathrm{d}W(s)-\int_t^T\frac{1}{2}\left|\left|\vartheta(s,\mathfrak{X})\right|\right|^2\mathrm{d}s\right)\middle|\,\mathcal{F}(t)\right] \\
&= L(t)\,\mathbb{E}\big[X(T)L(T)\,/\,L(t)\,|\,\mathcal{F}(t)\big] = \text{RHS}\,,\ \mathbb{P}-\text{a.s.}
\end{aligned}$$

We note that in the third equality we took advantage of (3.2.5) and of the strong Markov property for the process $\mathfrak{X}(\cdot)$.

(ii) On the strength of the martingale property from (i), the Optional Sampling Theorem gives $\text{LHS} = L(0)X(0)\mathfrak{u}_{\mathcal{M}^{x^*}}(T,\mathfrak{X}(0)) = \text{RHS}\,.$ □

Remark 3.5.4. In the above proof of Theorem 3.4.5, the special structure of strongly Markovian admissible systems that we selected in Definition 3.2.2, is indispensable in the context of Proposition 3.5.3. On the other hand, the Assumption 3.4.3 is important for the existence of the neighborhood $\mathcal{D}_\delta$ with the stated properties; see the discussion right below (3.5.5). □

3.6 The proof of Theorem 3.4.6: viscosity supersolution

The proof that follows shares many similarities with that in Section 3.5 for Theorem 3.4.5, the counterpart of Theorem 3.4.6, but also requires the additional Assumption 3.4.4 and a much stronger result – the *Dynamic Programming Principle* (or DPP,

Proposition 6.1 below) – than the martingale property of Proposition 3.5.3. Before outlining and presenting the proof, we explain the reasons for such differences.

• We begin with an idea similar to that in Section 3.5 (with corresponding inequalities in opposite directions, and with $\widehat{\Phi}$ replaced by Φ); however, we cannot proceed in the same way for two reasons:

(i) The reverse inequality to (3.5.13), namely, $g(t,s,\mathfrak{X}) \leq \widehat{\mathcal{G}}\big(t,\mathfrak{X}(s)\big)$ does not hold in general, by the definition (3.1.3) of $\widehat{\mathcal{L}}$ (recall g from (3.5.13) and $\widehat{\mathcal{G}}$ from (3.5.5)). Therefore, with g as in (3.5.13) and ρ as in (3.5.12), we cannot obtain

$$g(t^*-s,s,\mathfrak{X}) < 0 \quad \text{for all} \quad s \in [0,\rho)\,, \tag{3.6.1}$$

the reverse inequality to (3.5.14), as we did in Section 3.5. Instead, we need to find an admissible system in $\mathfrak{M}(x^*)$ under which (3.6.1) holds.

If we still want to argue by contradiction, assuming the reverse inequality to (3.5.5), then according to the definitions (3.1.3) of $\mathcal{L}$ and (3.5.5) of $\widehat{\mathcal{G}}$, there exists $a_0 \in \mathcal{A}(x_0)$ such that $(\varphi_t - \mathcal{L}_{a_0}\varphi)(t_0,x_0) < 0$. Plugging in the definition (3.1.3) of $\mathcal{L}_{a_0}$ and comparing the left-hand side of this inequality with the $g(t^*-s,s,\mathfrak{X})$ of (3.5.13), we see that (3.6.1) holds if the α in (3.5.13) is very close to a_0 when s is sufficiently small. This accounts for the requirement (3.4.8) of Assumption 3.4.4. Other conditions in Assumption 3.4.4 are inspired by Remark 3.2.4 aimed for the existence of an admissible system with such α.

(ii) The reverse inequality of (3.5.3) with $\widehat{\Phi}$ replaced by Φ, namely

$$||x^*||_1\, \mathfrak{u}_{\mathcal{M}^{x^*}}(t^*,x^*) < \mathbb{E}\left[L(\rho)X(\rho)\,\Phi\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right], \tag{3.6.2}$$

actually holds in general, on the strength of Proposition 3.5.3 and the definition (3.3.7) of Φ. Therefore we need to estimate more accurately the value of φ on the left-hand side of the counterpart of (3.5.2), by using Φ instead of $\mathfrak{u}_{\mathcal{M}^{x^*}}$, so that we arrive at

$$||x^*||_1\, \Phi(t^*,x^*) < \mathbb{E}\left[L(\rho)X(\rho)\,\Phi\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right], \tag{3.6.3}$$

instead of (3.6.2). We then need the DPP of Proposition 3.6.2, to obtain a contradiction to (3.6.3).

3.6.0.1 Informal outline

Now we outline the main steps of the proof. We prove by contradiction, assuming the negation of (3.4.5) in Definition 3.4.1 with the function F as in (3.4.7), that there exist $\varphi \in C^{1,2}\big((0,\infty)\times\mathbb{R}^n_+\big)$ and $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n_+$ such that: (t_0,x_0) is a strict minimum of $\Phi_* - \varphi$; the minimal value is equal to zero; and $\big(\varphi_t - \widehat{\mathcal{L}}\varphi\big)(t_0,x_0) < 0$.

Since $\widehat{\mathcal{L}}\varphi = \sup_{a\in\mathcal{A}(x)} \mathcal{L}_a\varphi$ (definition (3.1.3)), there exists $a_0 \in \mathcal{A}(x_0)$ such that

$$(\varphi_t - \mathcal{L}_{a_0}\varphi)(t_0,x_0) < 0\,. \tag{3.6.4}$$

We take $(x,a) = (x_0,a_0)$ and a sufficiently small ι in Assumption 3.4.4, and let δ, $\mathbf{s}$ and $\boldsymbol{\theta}$ be the corresponding elements. Further, by the definition (3.4.6) of Φ_*, we can take a pair (t^*,x^*) close to (t_0,x_0) such that the nonnegative difference $(\Phi-\varphi)(t^*,x^*)$ is sufficiently small, say less than a small positive constant C_3^* (depending on δ; defined similarly to the C_3 of (3.5.7)).

Thanks to Assumption 3.4.4, there exists an admissible system $\mathcal{M}^{x^*} \in \mathfrak{M}(x^*)$ with the functionals σ and ϑ defined by (3.2.5). Under this admissible system, we derive (3.6.1) from (3.6.4), and thus

$$\begin{aligned}||x^*||_1\,\varphi(t^*,x^*) - \mathbb{E}\left[L(\rho)X(\rho)\varphi\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right] &= \mathbb{E}\left[\int_0^\rho L(s)X(s)\,g\big(t^*-s,s,\mathfrak{X}\big)\,\mathrm{d}s\right]\\ &< 0\,.\end{aligned} \tag{3.6.5}$$

On the other hand, on the left-hand side of (3.6.5) we estimate the real number $\varphi(t^*,x^*)$ from below by $\Phi(t^*,x^*)-C_3^*$, and the random quantity $\varphi\big(t^*-\rho,\mathfrak{X}(\rho)\big)$ from above by $\Phi\big(t^*-\rho,\mathfrak{X}(\rho)\big)-C_2^*$ (for some ω's in Ω) or $\Phi\big(t^*-\rho,\mathfrak{X}(\rho)\big)$ (for other ω's in Ω) with C_2^* a small positive constant similar to the C_2 of (3.5.7), and then deduce (3.6.3), which contradicts the Dynamic Programming Principle of Proposition 3.6.2. □

3.6.1 The supersolution property

We are ready now to present the argument proper.

Proof of Theorem 3.4.6: According to Definition 3.4.1 (ii) of viscosity supersolution with the function F as in (3.4.7), it suffices to show that for any test function $\varphi \in C^{1,2}\left((0,\infty)\times\mathbb{R}^n_+\right)$ and $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n_+$ with

$$(\Phi_*-\varphi)(t_0,x_0)=0<(\Phi_*-\varphi)(t,x)\,,\quad\forall\,(t,x)\in(0,\infty)\times\mathbb{R}^n_+ \tag{3.6.6}$$

(i.e., such that (t_0,x_0) is a strict minimum of $\Phi_*-\varphi$), and with Φ_* the lower-semicontinuous envelope of Φ as in the definition (3.4.6), we have

$$\big(\varphi_t-\widehat{\mathcal{L}}\varphi\big)(t_0,x_0)\geq 0\,.$$

Recalling $\widehat{\mathcal{L}}$ from (3.1.3), it suffices to establish $(\varphi_t-\mathcal{L}_a\varphi)(t_0,x_0)\geq 0$ for every fixed $a\in\mathcal{A}(x)$.

We shall argue this by contradiction, assuming that for some $a_0\in\mathcal{A}(x_0)$ we have

$$g_0:=-\,\mathcal{G}_{a_0}(t_0,x_0)>0\,, \tag{3.6.7}$$

where

$$\mathcal{G}_a(t,x):=(\varphi_t-\mathcal{L}_a\varphi)(t,x)\,,\ (a,t,x)\in\mathcal{A}(x)\times(0,\infty)\times\mathbb{R}^n_+\,. \tag{3.6.8}$$

Under Assumption 3.4.3, there exists a positive number $\delta_1<t_0\wedge(||x_0||_1/n)$ such that $\mathcal{K}(\cdot)$ is bounded on $\mathcal{D}_{\delta_1}:=(t_0-\delta_1,t_0+\delta_1)\times B_{\delta_1}(x_0)$. Let $C>1$ be a constant such that $||\theta||,|a_{ij}|<C$ $(1\leq i,j\leq n)$ hold for all pairs $(\theta,a=(a_{ij})_{n\times n})\in\mathcal{K}(x)$ and all $x\in B_{\delta_1}(x_0)$.

Since the functions $\mathcal{G}_{a_0}(\cdot,\cdot)$, $\varphi_t(\cdot,\cdot)$ and

$$H_{ij}(s,y):=D_i\varphi(s,y)\,/\,||y||_1+y_iy_jD^2_{ij}\varphi(s,y)\,/\,2\,,\quad(s,y)\in(0,\infty)\times\mathbb{R}^n_+\,, \tag{3.6.9}$$

$1\leq i,j\leq n$, are continuous, there exists under the hypothesis (3.6.7), a positive number $\delta_2<\delta_1$ such that for all $H\in\{\varphi_t,H_{ij}\,(1\leq i,j\leq n)\}$, we have

$$|H(t,x)-H(t_0,x_0)|<g_0\,/\,3n^2C<g_0\,/\,3\,, \tag{3.6.10}$$

holds for all

$$(t,x) \in \mathcal{D}_{\delta_2} := (t_0 - \delta_2, t_0 + \delta_2) \times B_{\delta_2}(x_0)\,.$$

Lemma 3.6.1. *With $\mathcal{G}_a(\cdot,\cdot)$ defined in* (3.6.8)*, the inequality*

$$|\mathcal{G}_a(t,x) - \mathcal{G}_{a_0}(t_0,x_0)| < g_0 \tag{3.6.11}$$

holds for all $(t,x) \in \mathcal{D}_{\delta_2}$, $a \in \mathcal{A}(x)$ with

$$\max_{1\le i,j\le n} |a_{ij} - (a_0)_{ij}| < \iota := \delta_2 \wedge g_0 \left(1 + 3n^2 \max_{i,j} |H_{ij}(t_0,x_0)|\right)^{-1}. \tag{3.6.12}$$

Recalling the number g_0 from the definition (3.6.7)*, we have also*

$$\mathcal{G}_a(t,x) < 0 \quad \text{for all } (a,t,x) \text{ in (3.6.12).} \tag{3.6.13}$$

Proof. Plugging the definition (3.6.8) of $\mathcal{G}_a$ into the left-hand side of (3.6.11) yields

$$\begin{aligned}
&\text{LHS of (3.6.11)}\\
=\ & |(\varphi_t - \mathcal{L}_a\varphi)(t,x) - (\varphi_t - \mathcal{L}_{a_0}\varphi)(t_0,x_0)|\\
\le\ & |\varphi_t(t,x) - \varphi_t(t_0,x_0)| + |\mathcal{L}_a\varphi(t_0,x_0) - \mathcal{L}_a\varphi(t,x)| + |\mathcal{L}_{a_0}\varphi(t_0,x_0) - \mathcal{L}_a\varphi(t_0,x_0)|\\
=:\ & \Lambda_1 + \Lambda_2 + \Lambda_3\,,
\end{aligned} \tag{3.6.14}$$

i.e., Λ_j $(j = 1,2,3)$ denotes the j-th term in (3.6.14). It suffices to show that $\Lambda_j < g_0\,/\,3$ for all j.

Since $(t,x) \in \mathcal{D}_{\delta_2}$, we can take advantage of the property (3.6.10) and get $\Lambda_1 < g_0\,/\,3$. Moreover, we notice that $\mathcal{L}_a\varphi(t,x) = \sum_{i,j} a_{ij}H_{ij}(t,x)$ (from the definitions (3.1.3) of $\mathcal{L}_a$ and (3.6.9) of H_{ij}) and obtain

$$\Lambda_2 = \left|\sum_{i,j} a_{ij}\big[H_{ij}(t_0,x_0) - H_{ij}(t,x)\big]\right| < n^2 \cdot C \cdot g_0\,/\,(3n^2C) = g_0\,/\,3 \quad \text{by (3.6.10)}\,,$$

$$\Lambda_3 = \left|\sum_{i,j} \big[(a_0)_{ij} - a_{ij}\big] H_{ij}(t_0,x_0)\right| < n^2 \cdot \iota \cdot \left(\max_{i,j} |H_{ij}(t_0,x_0)|\right) < g_0\,/\,3 \quad \text{by (3.6.12)}\,.$$

This completes the proof. □

Take $x = x_0$ and $a = a_0$ in Assumption 3.4.4 with ι defined in (3.6.12). Let δ, $\mathbf{s}$, $\boldsymbol{\theta}$ and $\mathbf{a}$ be the corresponding elements described in Assumption 3.4.4. We shall now adopt the definitions of C_1 from (3.5.7) and introduce the strictly positive constants

$$C_2^* := \min_{\partial \mathcal{D}_\delta} (\Phi_* - \varphi)(t,x) > 0 \quad \text{(by (3.6.6))}\,, \tag{3.6.15}$$
$$C_3^* := \frac{C_2^*\, e^{-C_1}(||x_0||_1 - n\delta)}{2(||x_0||_1 + n\delta)} > 0 \quad \text{(by (3.5.6))}$$

by analogy with C_2 and C_3 in (3.5.7). We observe from the definition (3.4.6) of Φ_* that

$$\liminf_{(t,x)\to(t_0,x_0)} (\Phi - \varphi)(t,x) = (\Phi_* - \varphi)(t_0,x_0) = 0\,,$$

and hence there exists $(t^*, x^*) \in \mathcal{D}_\delta$ such that

$$(\Phi - \varphi)(t^*, x^*) < C_3^*\,; \tag{3.6.16}$$

and thanks to Assumption 3.4.4, there exists an admissible system $\mathcal{M}^{x^*} \in \mathfrak{M}(x^*)$ with the functionals σ and ϑ defined by (3.2.5). The remaining discussion in this section (with the exception of Proposition 3.6.2) will be carried out under this admissible system.

Now we shall adopt the definitions of ν, λ and ρ from (3.5.10)– (3.5.12). For any $0 \le s \le \rho$, we have $\big(t^* - s, \mathfrak{X}(s)\big) \in \overline{\mathcal{D}_\delta} \subset \mathcal{D}_{\delta_2}$ and therefore (3.6.12) holds for $(a,t,x) = (\alpha(s,\mathfrak{X}), t^* - s, \mathfrak{X}(s))$ by virtue of (3.4.8) (recall from (3.2.5) that $\alpha(s,\mathfrak{X}) = \mathbf{a}(\mathfrak{X}(s))$). Therefore, we can apply (3.6.13) and obtain

$$\mathcal{G}_{\alpha(s,\mathfrak{X})}\big(t^* - s, \mathfrak{X}(s)\big) < 0\,. \tag{3.6.17}$$

Let us apply now Lemma 3.5.1 with $T = t^*$, integrating (3.5.15) with respect to t over $[0,\rho]$ and taking the expectation under $\mathbb{P}$, to obtain

$$||x^*||_1\, \varphi(t^*,x^*) - \mathbb{E}\left[L(\rho)X(\rho)\varphi\big(t^* - \rho, \mathfrak{X}(\rho)\big)\right] = \mathbb{E}\left[\int_0^\rho L(s)X(s)\, g\big(t^* - s, s, \mathfrak{X}\big)\, \mathrm{d}s\right]$$
$$< 0\,, \tag{3.6.18}$$

by (3.6.17) and the same reasoning as right below (3.5.16). Here g is defined in (3.5.13), and thus the quantity $g\big(t^*-s,s,\mathfrak{X}\big)$ is the left-hand side of (3.6.17).

Notice that $\big(t^*-\nu,\mathfrak{X}(\nu)\big)\in\partial\mathcal{D}_\delta$ holds by the definition (3.5.10) of ν. Thus

$$\varphi\big(t^*-\nu,\mathfrak{X}(\nu)\big)\leq\Phi_*\big(t^*-\nu,\mathfrak{X}(\nu)\big)-C_2^*\,. \tag{3.6.19}$$

Plugging (3.6.6), (B.2.6) and (3.6.19) into (3.6.18) yields

$$\begin{aligned}0\;>\;&||x^*||_1\big[-C_3^*+\Phi(t^*,x^*)\big]-\mathbb{E}\left[\mathbf{1}_{\{\rho=\nu\}}L(\rho)X(\rho)\left(\Phi_*\big(t^*-\rho,\mathfrak{X}(\rho)\big)-C_2^*\right)\right.\\&\left.+\,\mathbf{1}_{\{\rho\neq\nu\}}L(\rho)X(\rho)\Phi_*\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right]\\=\;&-C_3^*\,||x^*||_1+||x^*||_1\,\Phi(t^*,x^*)-\mathbb{E}\left[L(\rho)X(\rho)\Phi_*\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right]\\&+C_2^*\,\mathbb{E}\big[\mathbf{1}_{\{\rho=\nu\}}L(\rho)X(\rho)\big].\end{aligned} \tag{3.6.20}$$

On the strength of (3.5.19), Lemma 3.5.2, the definition (3.6.15) of C_3^*, and (3.5.6), we obtain now

$$\begin{aligned}C_2^*\,\mathbb{E}\big[\mathbf{1}_{\{\rho=\nu\}}L(\rho)X(\rho)\big]\;&\geq\;C_2^*\,e^{-C_1}(||x_0||_1-n\delta)\,\mathbb{P}\big(\rho=\nu\big)\\&\geq\;C_2^*\,e^{-C_1}(||x_0||_1-n\delta)\,/\,2=C_3^*(||x_0||_1+n\delta)>C_3^*\,||x^*||_1\,.\end{aligned}$$

(This explains why we constructed C_3^* as we did in (3.6.15); in fact, setting C_3^* to be any value less than the right-hand side of (3.6.15) would also work.)

Substituting this inequality into (3.6.20), and recalling $\Phi_*(\cdot,\cdot)\leq\Phi(\cdot,\cdot)$ from the definition (3.4.6) of Φ_*, leads now to the inequality

$$||x^*||_1\,\Phi(t^*,x^*)<\mathbb{E}\left[L(\rho)X(\rho)\,\Phi\big(t^*-\rho,\mathfrak{X}(\rho)\big)\right]$$

of (3.6.3). However, this inequality contradicts the Dynamic Programming Principle of Proposition 3.6.2 right below, so the proof of Theorem 3.4.6 is complete. □

Proposition 3.6.2. Dynamic Programming Principle ([46]–[47])**:** *For any given* $(T,x)\in(0,\infty)\times\mathbb{R}^n_+$ *and any stopping time* $\tau\leq T<\infty$*, we have*

$$||x||_1\,\Phi(T,x)=\sup_{\mathcal{M}\in\mathfrak{M}(x)}\mathbb{E}^{\mathbb{P}^{\mathcal{M}}}\left[L^{\mathcal{M}}(\tau)X^{\mathcal{M}}(\tau)\,\Phi\big(T-\tau,\mathfrak{X}^{\mathcal{M}}(\tau)\big)\right].$$

Proof. We refer to [46, Proposition 2.2, Theorem 2.4, Remark 2.7] and [47, Theorem 2.3]. □

3.7 Viscosity characterization of the arbitrage function

Let us go back to the arbitrage function $\mathfrak{u}$ of (3.3.5). As a consequence of the minimality result Theorem 3.7.1 below, if Φ of (3.3.7) is a classical supersolution of (3.1.1), then the function $\mathfrak{u}$ coincides with Φ and hence is the smallest nonnegative classical supersolution of the Cauchy problem of (3.1.1), (3.1.2); in fact, we have $\mathfrak{u} \equiv \Phi$ if Φ is only continuous (see Theorem 3.7.2 below).

Theorem 3.7.1. ((3.3.11) and [15, Proposition 2]) *For any nonnegative classical supersolution U of the Cauchy problem* (3.1.1), (3.1.2), *we have*

$$U(T,x) \geq \mathfrak{u}(T,x) \geq \Phi(T,x) \geq \widehat{\Phi}(T,x) > 0\,, \quad \forall\ (T,x) \in [0,\infty) \times \mathbb{R}^n_+\,.$$

Proof. We adopt the idea from the proof in [15, Proposition 2, (5.3)–(5.15)]; the detailed proof is provided in Section 3.8. □

Theorem 3.7.2. *The arbitrage function $\mathfrak{u}$ coincides with the function Φ of* (3.3.7) *if Φ is continuous.*

This theorem is proved right below. Combining it with Theorems 3.4.5, 3.4.6 and 3.7.1, and recalling Remark 3.4.2 and $\widehat{\Phi}^*$ from (3.4.4), gives the following characterizations of the arbitrage function $\mathfrak{u}$.

Corollary 3.7.3. *Suppose that the conditions in Theorem 3.4.6 are in force and the function Φ is continuous.*

Then the arbitrage function $\mathfrak{u}$ is a viscosity supersolution of the HJB equation (3.1.1) *subject to the initial condition* (3.1.2). *If furthermore $\Phi \equiv \widehat{\Phi}^*$, then $\mathfrak{u}$ is a viscosity solution of* (3.1.1) *subject to* (3.1.2).

If in addition $\mathfrak{u}$ is of class $C\left([0,\infty) \times \mathbb{R}^n_+\right) \cap C^{1,2}\left((0,\infty) \times \mathbb{R}^n_+\right)$, then it is the smallest nonnegative classical (super)solution of the Cauchy problem (3.1.1), (3.1.2).

Remark 3.7.4. If a robust strong arbitrage relative to the market exists on some time horizon $[0,T]$ for some initial capitalization x (see Remark 3.3.1), then $\mathfrak{u}(T,x) < 1$. This amounts to a failure of uniqueness of classical/viscosity solutions for the Cauchy problem of (3.1.1), (3.1.2), since the constant $u \equiv 1$ is always a (trivial) solution to this problem.

We refer the reader to [15, p. 2205] or to [44], for an interpretation of Theorem 3.7.2. □

Proof of Theorem 3.7.2: Let $\mathcal{U}$ be the collection of positive classical supersolutions of the Cauchy problem (3.1.1), (3.1.2), and $\breve{\mathcal{U}}$ the collection of continuous functions $\breve{U} : [0,\infty)\times\mathbb{R}^n_+ \to \mathbb{R}_+$ that satisfies (3.1.2) and that the process $L(t)X(t)\Phi(T-t,\mathfrak{X}(t))$ is a supermartingale under every admissible system. Note that $\Phi \in \breve{\mathcal{U}}$ by virtue of [47, Theorem 2.3].

Following the idea in [15, Theorem 1], we have for $T = 0$ the identities $\mathfrak{u}(0,x) = 1 = \Phi(0,x)$ for all $x \in \mathbb{R}^n_+$ by the initial condition (3.1.2). Now we fix an arbitrary pair $(T,x) \in (0,\infty)\times\mathbb{R}^n_+$. For every $\varepsilon > 0$, there exists a mollification $U_\varepsilon \in \mathcal{U}$ of the function Φ with $0 < U_\varepsilon(T,x) \leq \Phi(T,x) + \varepsilon$. Combining with Theorem 3.7.1 gives

$$\mathfrak{u}(T,x) \leq U_\varepsilon(T,x) \leq \Phi(T,x) + \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, this leads to $\mathfrak{u}(T,x) \leq \Phi(T,x)$. On the other hand, the reverse inequality $\mathfrak{u}(T,x) \geq \Phi(T,x)$ holds on the strength of (3.3.11). Hence, $\mathfrak{u}(T,x) = \Phi(T,x)$ on $[0,\infty)\times\mathbb{R}^n_+$. □

3.7.1 The Markovian Market Weight model

With slight modifications our approach can also show that, under appropriate conditions described in Theorems 3.4.5, 3.4.6 and Corollary 3.7.3 but now with

$$F(t,x,r,p,q) \;=\; -\frac{1}{2}\sup_{a\in\mathcal{A}(x)}\left(\sum_{i,j} x_i x_j a_{ij} q_{ij}\right),\;\text{ for }\; q=(q_{ij})_{1\leq i,j\leq n}\,,\;\; p=(p_1,\ldots,p_n)',$$

and $a = (a_{ij})_{1 \leq i,j \leq n}$, the functions

$$||x||_1 \, \Phi(T,x) \, , \qquad ||x||_1 \, \widehat{\Phi}(T,x) \, , \qquad \text{and} \qquad ||x||_1 \, \mathfrak{u}(T,x) \, , \quad (T,x) \in [0,\infty) \times \mathbb{R}^n_+$$

are classical/viscosity (super/sub)solutions of an HJB equation simpler than (3.1.1) – namely, the *Pucci-maximal type equation*

$$u_t(t,x) - \frac{1}{2} \sup_{a \in \mathcal{A}(x)} \left(\sum_{i,j} x_i x_j a_{ij} D^2_{ij} u(t,x) \right) = 0 \, , \quad (t,x) \in (0,\infty) \times \mathbb{R}^n_+$$

subject to the initial condition $u(0,x) = ||x||_1$, and are dominated by any nonnegative classical supersolution of this Cauchy problem.

Similarly, we can extend the above results to the *Markovian Market Weight (MMW)* model introduced in [14], in which the volatilities and market prices of risk are functions in terms of the relative market weights of individual assets instead of their capitalizations.

In this model, we define the *relative arbitrage function* $\mathfrak{v}(T,z)$ – by analogy with the arbitrage function $\mathfrak{u}(T,x)$ of (3.3.5) – as, for the initial market weight configuration $z \in \Delta^o$, the smallest relative initial wealth, starting with which one can find a nonanticipative investment rule, to attain relative wealth of at least 1 with respect to the market almost surely at time T, under all admissible market model configurations. Here Δ^o is the interior of the unit simplex $\{z = (z_1, \ldots, z_n)' \in [0,1]^n : \sum_i z_i = 1\}$ in $n-1$ dimensions. Equivalently, $\mathfrak{v}(T,z)$ can be thought of as the reciprocal of the maximal relative amount by which the market portfolio can be outperformed over the time horizon $[0,T]$ almost surely, by using nonanticipative investment rules when starting with the vector z of initial market weights, under all admissible market model configurations.

Then we see that, under appropriate conditions analogous to those described in Theorems 3.4.5, 3.4.6 and Corollary 3.7.3, the relative arbitrage function $\mathfrak{v}$ is a classical/viscosity (super/sub)solution of the Pucci-maximal type equation

$$u_t(t,z) - \frac{1}{2} \sup_{\mathrm{a} \in \mathrm{A}(z)} \left(\sum_{i,j} z_i z_j \mathrm{a}_{ij} D^2_{ij} u(t,z) \right) = 0 \, , \quad (t,z) \in (0,\infty) \times \Delta^o$$

subject to the initial condition $u(0,z) = 1$ for all $z \in \Delta^o$, and are dominated by any nonnegative classical supersolution of this Cauchy problem.

3.7.2 Sufficient conditions for $\mathfrak{u} \equiv \Phi \equiv \widehat{\Phi}$ to be a classical supersolution of (3.1.1)

Now let us provide some sufficient conditions under which we have $\mathfrak{u} \equiv \Phi \equiv \widehat{\Phi}$, and this function is a classical solution of (3.1.1) – thus also the smallest nonnegative classical (super)solution of the Cauchy problem (3.1.1), (3.1.2) by virtue of Theorem 3.7.1.

In particular, via the discussions below, we will see that one sufficient condition is the following specific requirements on the Knightian uncertainty $\mathbb{K}$.

Proposition 3.7.5. *Suppose that there exist locally Lipschitz functions* $\mathbf{s} : \mathbb{R}^n_+ \to \mathrm{GL}(n)$ *and* $\boldsymbol{\theta} : \mathbb{R}^n_+ \to \mathbb{R}^n$*, and subsets* $\mathcal{R}(y)$ $(y \in \mathbb{R}^n_+)$ *of* $\mathbb{R}$ *such that the functions* $\mathbf{s}(\cdot)$ *and* $\mathbf{b}(\cdot) := \mathbf{s}(\cdot)\boldsymbol{\theta}(\cdot)$ *are linearly growing (i.e., satisfy* (3.2.7)*) and with* $\mathbf{a}(y) := \mathbf{s}(y)\mathbf{s}'(y)$ $(y \in \mathbb{R}^n_+)$ *we have*

$$\big(\boldsymbol{\theta}(y), \mathbf{a}(y)\big) \in \mathcal{K}(y)\,, \quad \mathcal{A}(y) = \{r \cdot \mathbf{a}(y) : r \in \mathcal{R}(y)\}\,, \quad \min \mathcal{R}(y) = 1 \tag{3.7.1}$$

holds for all $y \in \mathbb{R}^n_+$*. Then* $\mathfrak{u} \equiv \Phi \equiv \widehat{\Phi}$ *is the smallest nonnegative classical (super)solution of the Cauchy problem* (3.1.1), (3.1.2)*, and the smallest nonnegative classical (super)solution of the Cauchy problem* (3.3.12), (3.1.2)*.*

Proof. This result follows directly from Remark 3.7.8 and Theorem 3.7.10 below. □

We start with the following observation.

Proposition 3.7.6. *If there exist admissible systems* $\mathcal{M}^y \in \widehat{\mathfrak{M}}(y)$ $(y \in \mathbb{R}^n_+)$ *such that*

$$V(t,y) := \mathfrak{u}_{\mathcal{M}^y}(t,y)\,, \quad (t,y) \in [0,\infty) \times \mathbb{R}^n_+ \tag{3.7.2}$$

is a classical supersolution of (3.1.1)*, then* $\mathfrak{u} \equiv \Phi \equiv \widehat{\Phi} \equiv V$ *is the smallest nonnegative classical (super)solution of the Cauchy problem* (3.1.1), (3.1.2)*.*

Proof. Theorem 3.7.1 and the definition (3.3.7) of $\widehat{\Phi}$ give $V(t,y) \geq \mathfrak{u}(t,y) \geq \Phi(t,y) \geq \widehat{\Phi}(t,y) \geq \mathfrak{u}_{\mathcal{M}^y}(t,y) = V(t,y)$. Hence $\mathfrak{u} \equiv \Phi \equiv \widehat{\Phi} \equiv V$ is a classical supersolution of (3.1.1). □

To proceed further, we need the following assumption.

Assumption 3.7.7. There exist admissible systems $\mathcal{M}^x \in \widehat{\mathfrak{M}}(x)$, $x \in \mathbb{R}^n_+$ such that
(i) they share the same functionals $\sigma(t,\mathfrak{X}) = \mathbf{s}(\mathfrak{X}(t))$ and $\vartheta(t,\mathfrak{X}) = \boldsymbol{\theta}(\mathfrak{X}(t))$ as in (3.2.5); and
(ii) for every $x \in \mathbb{R}^n_+$, the process $\mathfrak{X}$ in $\mathcal{M}^x$ is unique in distribution in the following sense (and thus strongly Markovian): for any admissible system $\widetilde{\mathcal{M}} \in \mathfrak{M}(x)$ with the same functionals σ and ϑ as in $\mathcal{M}^x$, the two processes $\mathfrak{X}^{\mathcal{M}^x}$ and $\mathfrak{X}^{\widetilde{\mathcal{M}}}$ have the same law. □

Remark 3.7.8. Assumption 3.7.7 holds when the conditions in Remark 3.2.4 (i) are satisfied. □

Proposition 3.7.9. *Under Assumption 3.7.7, the function V of* (3.7.2) *is*
(i) *dominated by any nonnegative classical supersolution of the Cauchy problem* (3.3.12), (3.1.2)*;*
(ii) *a viscosity solution of* (3.3.12)*, if $\boldsymbol{\theta}(\cdot)$ and $\mathbf{a}(\cdot)$ are locally bounded and $\mathbf{a}(\cdot)$ is continuous; and*
(iii) *a classical solution of the Cauchy problem* (3.3.12), (3.1.2) *(and thus its smallest nonnegative (super)solution), if $\mathbf{s}(\cdot)$ and $\boldsymbol{\theta}(\cdot)$ are locally Lipschitz.*

Proof. We will see that *(i)* and *(ii)* are special cases of Theorems 3.7.1 and Theorems 3.4.5–3.4.6, respectively, with $\mathcal{K}(y) = \{(\boldsymbol{\theta}(y), \mathbf{a}(y))\}$ $(y \in \mathbb{R}^n_+)$ via the following observations. First, in this case we have $\widehat{\mathcal{L}}(t,y) = \mathcal{L}_{\mathbf{a}(y)}(t,y)$ (recall the definition (3.1.3) for $\widehat{\mathcal{L}}$ and $\mathcal{L}_a$). Moreover, by virtue of Assumption 3.7.7 and definition (3.3.8), we have $\mathfrak{u}_{\mathcal{M}^y}(t,y) = \mathfrak{u}_{\mathcal{M}}(t,y)$ for all $\mathcal{M} \in \mathfrak{M}(y)$, and by the definition (3.3.7) of Φ and $\widehat{\Phi}$ gives

$$\Phi(t,y) = \widehat{\Phi}(t,y) = \mathfrak{u}_{\mathcal{M}^y}(t,y) = V(t,y)\,.$$

(iii) Under these conditions, we have $V(\cdot\,,\cdot) \in C^{1,2}((0,\infty)\times\mathbb{R}^n_+)$ (see [55, Theorem 4.7] for a proof that uses results from the theory of stochastic flows [38, 53] and from parabolic partial differential equations [9, 30], and conclude by invoking (ii) since the local Lipschitz condition on $\mathbf{s}$ and $\boldsymbol{\theta}$ implies the condition in *(ii)*. □

Proposition 3.7.10. *If Assumption 3.7.7 holds with locally Lipschitz functions* $\mathbf{s}(\cdot)$ *and* $\boldsymbol{\theta}(\cdot)$*, and there exist subsets* $\mathcal{R}(y)$ $(y\in\mathbb{R}^n_+)$ *of* $\mathbb{R}$ *such that* (3.7.1) *holds, then, with* $\mathcal{M}^y \in \widehat{\mathfrak{M}}(y)$ *as in Assumption 3.7.7, the function*

$$\mathfrak{u}(t,y) \;\equiv\; \Phi(t,y) \;\equiv\; \widehat{\Phi}(t,y) \;\equiv\; \mathfrak{u}_{\mathcal{M}^y}(t,y) \tag{3.7.3}$$

is the smallest nonnegative classical (super)solution of the Cauchy problem (3.1.1), (3.1.2)*, as well as the smallest nonnegative classical (super)solution of the Cauchy problem* (3.3.12), (3.1.2)*.*

Proof. By Proposition 3.7.9 (iii), the right-hand side of (3.7.3), i.e., the function V of (3.7.2) solves (3.3.12):

$$V_t(t,y) = \mathcal{L}_{\mathbf{a}(y)}V(t,y)\,,\quad (t,y)\in(0,\infty)\times\mathbb{R}^n_+\,.$$

Thus

$$\mathcal{L}_{r\cdot\mathbf{a}(y)}V(t,y) \;=\; r\cdot\mathcal{L}_{\mathbf{a}(y)}V(t,y) \;=\; r\cdot V_t(t,y)\,,\quad (t,y)\in(0,\infty)\times\mathbb{R}^n_+\,.$$

Once we have shown that $V_t(t,y)\le 0$, i.e., that V is nonincreasing in t on $(0,\infty)$, for all $y\in\mathbb{R}^n_+$, then V is a classical supersolution of (3.1.1) on the strength of (3.7.1), and the proof will be complete by Proposition 3.7.6. In fact, under any given admissible system, the positive process $L(\cdot)X(\cdot)$ is a local martingale, and hence a supermartingale (one can derive the formula $\mathrm{d}(L(t)X(t)) = L(t)X(t)\,(\pi'\sigma-\vartheta')\,(t,\mathfrak{X})\,\mathrm{d}W(t)$ with π the market portfolio, via Itô's Rule; see (3.8.8) for details). Therefore $V(t,y) = \mathbb{E}^{\mathbb{P}^{\mathcal{M}^y}}\left[L^{\mathcal{M}^y}(t)X^{\mathcal{M}^y}(t)\right]/\,||y||_1$ is indeed nonincreasing in t. □

Remark 3.7.11. This result is in agreement with general regularity theory for fully nonlinear parabolic equations, as in [42, Theorem II.4].

Remark 3.7.12. We have tried to find weaker conditions for Theorem 3.7.2 to hold, or for the function Φ to be continuous, but did not succeed. Even if all the functions $\mathfrak{u}_{\mathcal{M}}$ are of class $C^{1,2}$, their supremum Φ might still fail to be continuous. □

3.8 The proof of Theorem 3.7.1: minimality

The proof consists of two parts, Theorems 3.8.1 and 3.8.2. Theorem 3.8.1 shows that any nonnegative classical supersolution U of the Cauchy problem (3.1.1), (3.1.2) is strictly positive, by proving that $U(T,x) \geq \Phi(T,x)$ and then applying the fact $\Phi(T,x) > 0$ from (3.3.11).

In Theorem 3.8.2, the positivity of U from Theorem 3.8.1 enables us to construct an investment rule from U (see (3.8.5) below) that matches or outperforms the market portfolio over the time horizon $[0,T]$, with probability one under all admissible systems. We then conclude that $U(T,x) \geq \mathfrak{u}(T,x)$ from the definition (3.3.5) of $\mathfrak{u}(T,x)$.

The following proofs of Theorems 3.8.1 and 3.8.2 adopt the idea from [15, Proposition 2, (5.3)–(5.15)] and provide details for completeness.

Theorem 3.8.1. *For any nonnegative classical supersolution U of the Cauchy problem* (3.1.1), (3.1.2), *we have*

$$U(T,x) \geq \Phi(T,x) > 0\,, \quad \forall\ (T,x) \in [0,\infty) \times \mathbb{R}^n_+\,. \tag{3.8.1}$$

Proof. The second inequality was shown in (3.3.11). For the first inequality, let us fix an admissible system $\mathcal{M} \in \mathfrak{M}(x)$; the remaining discussion in this proof will be carried out under this system. The key point, is to show that the process

$$\Xi(t) := X(t)L(t)\,U\big(T-t, \mathfrak{X}(t)\big) \tag{3.8.2}$$

is a supermartingale. Once this is proved, with the initial condition $U(0,\cdot) \geq 1$, we

obtain

$$\begin{aligned}||x||_1\, U(T,x) &= \mathbb{E}\big[\Xi(0)\big] \geq \mathbb{E}\big[\Xi(T)\big] = \mathbb{E}\big[X(T)L(T)\,U(0,\mathfrak{X}(T))\big] \\ &\geq \mathbb{E}\big[X(T)L(T)\big] = ||x||_1\,\mathfrak{u}_{\mathcal{M}}(T,x)\,, \quad \text{by the definition (3.3.8)}\,.\end{aligned}$$

Since $||x||_1 > 0$, we deduce $U(T,x) \geq \mathfrak{u}_{\mathcal{M}}(T,x)$, which leads to (3.8.1) by the definition (3.3.7).

To show the supermartingale property of $\Xi(\cdot)$, we apply Lemma 3.5.1 with $\varphi = U$ and get

$$\begin{aligned}&\mathrm{d}\,(\Xi(t)) \qquad\qquad (3.8.3)\\ =& -L(t)X(t)\,\big(U_t - \mathcal{L}_{\alpha(t,\mathfrak{X})}U\big)\big(T-t,\mathfrak{X}(t)\big)\mathrm{d}t - X(t)U\big(T-t,\mathfrak{X}(t)\big)L(t)\,\vartheta'(t,\mathfrak{X})\,\mathrm{d}W(t)\\ &+ L(t)\sum_{i,k} X_i(t)\,\big[U\big(T-t,\mathfrak{X}(t)\big) + X(t)D_iU\big(T-t,\mathfrak{X}(t)\big)\big]\,\sigma_{ik}(t,\mathfrak{X})\,\mathrm{d}W_k(t)\,.\end{aligned}$$

Thanks to the supersolution property of U that

$$\big(U_t - \mathcal{L}_{\alpha(t,\mathfrak{X})}U\big)\,(s,y) \geq \big(U_t - \widehat{\mathcal{L}}U\big)(s,y) \geq 0\,, \quad \forall\ (s,y) \in [0,\infty)\times\mathbb{R}^n_+ \qquad (3.8.4)$$

(recall $\mathcal{L}_a$ and $\widehat{\mathcal{L}}$ from (3.1.3)) and the nonnegativity of the processes $L(\cdot)$, $X(\cdot)$ and the function $U(\cdot,\cdot)$, we conclude that $\Xi(t)$ is a nonnegative local martingale, and hence a supermartingale. □

Theorem 3.8.2. *For any nonnegative classical supersolution U of the Cauchy problem* (3.1.1), (3.1.2)*, the investment rule $\pi^U \in \mathfrak{P}$ generated by this function U through*

$$\pi^U_i(t,\omega) := \omega_i(t)D_i\log U(T-t,\omega(t)) + \frac{\omega_i(t)}{||\omega(t)||_1}\,, \quad i=1,\ldots,n\,,\ \ t\in[0,T] \qquad (3.8.5)$$

for continuous function $\omega : [0,\infty) \to \mathbb{R}^n_+$, satisfies the inequality

$$Z^{U(T,x)X^{\mathcal{M}}(0),\pi^U}(T) \geq X^{\mathcal{M}}(T)\,, \quad \mathbb{P}-\text{a.s.}, \quad \forall\ \mathcal{M}\in\mathfrak{M}(x)\,. \qquad (3.8.6)$$

It then follows from the definition (3.3.5) *of $\mathfrak{u}(T,x)$ that*

$$U(T,x) \geq \mathfrak{u}(T,x)\,, \quad \forall\ (T,x)\in[0,\infty)\times\mathbb{R}^n_+\,.$$

Proof. The investment rule π^U is well-defined since U is positive by Theorem 3.8.1. Let us fix $\mathcal{M} \in \mathfrak{M}(x)$; the remaining discussion in this proof will be carried out under this system.

We shall set $v := U(T,x)X(0)$ and $\pi := \pi^U$. The main goal is to show that the growth rate of the process $\log\left(L(t)Z^{v,\pi}(t)\right)$ is no less than that of $\log \Xi(t)$ with $\Xi(t)$ defined in (3.8.2). Once this is proved, noticing that these two processes start at the same initial value v, we obtain

$$L(T)Z^{v,\pi}(T) \geq \Xi(T) = X(T)L(T)U(0,\mathfrak{X}(T)) \geq X(T)L(T)\,,$$

as $U(0,\cdot) \geq 1$ by the initial condition. This leads to (3.8.6) as $L(T) > 0$.

To start, we observe from (3.3.3) with $\pi = \mathit{\Pi}$ that the wealth process $Z^{v,\pi}(\cdot)$ satisfies the dynamics

$$\mathrm{d}Z^{v,\pi}(t) = Z^{v,\pi}(t)\pi'(t,\mathfrak{X})\sigma(t,\mathfrak{X})\left[\vartheta(t,\mathfrak{X})\,\mathrm{d}t + \mathrm{d}W(t)\right] \quad \text{with} \quad Z^{v,\pi}(0) = v\,. \tag{3.8.7}$$

We apply Itô's Rule for the product function $f_1(r_1,r_2) := r_1 r_2$ with (B.1.4) and (3.8.7) yields

$$\begin{aligned}
\mathrm{d}\left(L(t)Z^{v,\pi}(t)\right) &= L(t)\,\mathrm{d}Z^{v,\pi}(t) + Z^{v,\pi}(t)\,\mathrm{d}L(t) + \mathrm{d}\langle L, Z^{v,\pi}\rangle(t) \qquad (3.8.8)\\
&= L(t)Z^{v,\pi}(t)\left[\pi'\sigma\vartheta\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}W(t) - \vartheta'\,\mathrm{d}W(t) - \pi'\sigma\vartheta\,\mathrm{d}t\right](t,\mathfrak{X})\\
&= L(t)Z^{v,\pi}(t)\mathcal{H}(t,\mathfrak{X})\,\mathrm{d}W(t)\,,
\end{aligned}$$

where

$$\mathcal{H}(t,\mathfrak{X}) := (\pi'\sigma - \vartheta')(t,\mathfrak{X})\,, \tag{3.8.9}$$

whose k-th component

$$\mathcal{H}_k(t,\mathfrak{X}) = \sum_i \left[X_i(t)D_i \log U\big(T-t,\mathfrak{X}(t)\big) + \frac{X_i(t)}{X(t)}\right]\sigma_{ik}(t,\mathfrak{X}) - \vartheta_k(t,\mathfrak{X})\,, \tag{3.8.10}$$

by (3.8.5).

Applying Itô's Rule to the logarithm function for $L(\cdot)Z^{v,\pi}(\cdot)$, we obtain

$$\mathrm{d}\log\left(L(t)Z^{v,\pi}(t)\right) = \mathcal{H}(t,\mathfrak{X})\,\mathrm{d}W(t) - \frac{1}{2}\left(\mathcal{H}\mathcal{H}'\right)(t,\mathfrak{X})\,\mathrm{d}t\,. \tag{3.8.11}$$

To determine the growth rate for $\log \Xi(\cdot)$, we recast (3.8.3) into

$$\mathrm{d}\left(\Xi(t)\right) = \Xi(t)\left[\mathcal{I}\big(T-t, \mathfrak{X}(t)\big)\mathrm{d}t + \mathcal{H}(t,\mathfrak{X})\,\mathrm{d}W(t)\right],$$

by virtue of

$$\left(\frac{D_i U}{U}\right)(s,y) = D_i\big(\log U(s,y)\big), \quad (s,y) \in [0,\infty)\times\mathbb{R}^n_+$$

and (3.8.10), where

$$\mathcal{I}(s,y) := -\left(\frac{U_t - \mathcal{L}_{\alpha(t,\mathfrak{x})}U}{U}\right)(s,y) \leq 0\,, \quad (s,y) \in [0,\infty)\times\mathbb{R}^n_+\,,$$

by (3.8.4) and $U > 0$.

Applying Itô's Rule again to the logarithm function for $\Xi(\cdot)$ and juxtaposing with (3.8.11) leads to

$$\begin{aligned}\mathrm{d}\log\Xi(t) &= \mathcal{I}\big(T-t,\mathfrak{X}(t)\big)\mathrm{d}t + \mathcal{H}(t,\mathfrak{X})\,\mathrm{d}W(t) - \frac{1}{2}\left(\mathcal{H}\mathcal{H}'\right)(t,\mathfrak{X})\,\mathrm{d}t \\ &\leq \mathrm{d}\log\left(L(t)Z^{\nu,\pi}(t)\right),\end{aligned}$$

as desired. □

Remark 3.8.3. In the special case of a model without uncertainty, the HJB equation (3.1.1) reduces to a linear PDE. If additionally, the functions σ and ϑ have the form of (3.2.5) and are locally Lipschitz continuous, then the arbitrage function $\mathfrak{u}$ is also shown to be dominated by every nonnegative and lower-semicontinuous viscosity supersolution of the Cauchy problem for the linear PDE (3.1.1) and (3.1.2) [3, Proposition 4.7], that satisfies certain convexity and continuity conditions.

This local Lipschitz condition on σ and ϑ is indispensable in the proof of [3]. It is the subject of future research, to determine whether this result still holds with weaker assumptions and in the presence of model uncertainty. □

3.9 Examples

The volatility-stabilized model was introduced in [17] and further generalized in [52], but now we add some uncertainty regarding its local volatility and relative risk structure.

Example 3.9.1. Volatility-Stabilized Model: Take constants $c_1^* \geq c_1 \geq 1/2$ and $c_2 \geq 1$, and set

$$\mathcal{K}(y) = \left\{ \big(\gamma_2\, \mathbf{a}(y), \gamma_1\gamma_2\, \boldsymbol{\theta}(y)\big) : \gamma_1 \in [c_1, c_1^*],\ \gamma_2 \in [1, c_2] \right\},$$

where

$$\mathbf{a}(y) = \mathbf{s}(y)\mathbf{s}'(y) \ \text{ with } \ \mathbf{s}_{ij}(y) = \mathbf{1}_{\{i=j\}}(||y||_1/y_i)^{1/2}, \quad \boldsymbol{\theta}_i(y) = (||y||_1/y_i)^{1/2}, \tag{3.9.1}$$

$1 \leq i, j \leq n$. Then the system of Stochastic Differential Equations (3.2.4) becomes

$$\begin{aligned} \mathrm{d}X_i(t) = \gamma_1\gamma_2^2 \big(X_1(t) + \cdots + X_n(t)\big)\, \mathrm{d}t \\ + \gamma_2 \sqrt{X_i(t)\big(X_1(t) + \cdots + X_n(t)\big)}\, \mathrm{d}W_i(t), \quad i = 1, \ldots, n, \end{aligned}$$

or equivalently, and a bit more succinctly,

$$\mathrm{d}\, \log(X_i(t)) \ = \ \left(\gamma_1 - \frac{1}{2}\right) \frac{\gamma_2^2}{\mu_i(t, \mathfrak{X})}\, \mathrm{d}t + \frac{\gamma_2}{\mu_i(t, \mathfrak{X})}\, \mathrm{d}W_i(t), \quad i = 1, \ldots, n,$$

with $\mu(t, \mathfrak{X})$ the market portfolio defined in (3.3.4).

For every $x \in \mathbb{R}^n_+$, $\gamma_1 \in [c_1, c_1^*]$ and $\gamma_2 \in [1, c_2]$, this system of SDEs has a unique-in-distribution solution $\mathfrak{X}(\cdot)$ starting at $\mathfrak{X}(0) = x$ whose $X_i(\cdot)$'s are time-changed versions of independent squared-Bessel processes (see [1, 17, 25] for more details). In particular, we have $\mathfrak{X}(\cdot) \in \mathbb{R}^n_+$.

Moreover, this uncertainty structure satisfies the conditions in Remark 3.3.1 and Proposition 3.7.10 with the $\mathbf{s}$, $\boldsymbol{\theta}$ as in (3.9.1) and $\mathcal{R}(y) = [1, c_2]$. Hence

$$\mathfrak{u}(t, y) \equiv \Phi(t, y) \equiv \widehat{\Phi}(t, y) \equiv \mathfrak{u}_{\mathcal{M}^y}(t, y) \begin{cases} < 1, & \text{if } t > 0 \\ = 1, & \text{if } t = 0 \end{cases} \tag{3.9.2}$$

is the smallest nonnegative classical (super)solution of the Cauchy problem (3.1.1), (3.1.2), as well as the smallest nonnegative classical (super)solution of the Cauchy problem (3.3.12), (3.1.2) (recall $\mathcal{M}^y$ from Assumption 3.7.7; see [25] and [49] for a computation of the joint density of $X_1(\cdot), \ldots, X_n(\cdot)$, which leads to an explicit formula for

$$\mathfrak{u}_{\mathcal{M}^y}(t,y) \;=\; \frac{\Pi_{i=1}^n y_i}{||y||_1} \;\cdot\; \mathbb{E}^{\mathbb{P}^{\mathcal{M}^y}}\left[\frac{\Pi_{i=1}^n X_i^{\mathcal{M}^y}(t)}{||X^{\mathcal{M}^y}(t)||_1}\right]$$

and shows that this function is indeed of class $C^{1,2}$).

Example 3.9.2. Generalized Volatility-Stabilized Model: Take constants $c_i^* \geq c_i \geq 0$, $i = 1, 2, \ldots, n$ and $c_{n+1} \geq 1$, and set

$$\mathcal{K}(y) = \left\{(a,\theta) : a = \gamma_{n+1}\,\mathbf{a}(y),\ \theta_i = \frac{\gamma_i + \gamma_{n+1}^2}{2\gamma_{n+1}}\,\boldsymbol{\theta}_i(y),\ \gamma_i \in [c_i, c_i^*],\ \gamma_{n+1} \in [1, c_{n+1}],\right.$$
$$\left. 1 \leq i \leq n\right\},$$

where $\mathbf{a}(y) = \mathbf{s}(y)\mathbf{s}'(y)$ with

$$\mathbf{s}_{ij}(y) = \mathbf{1}_{\{i=j\}}\left(\frac{||y||_1}{y_i}\right)^{\kappa} G(y)\,, \quad \boldsymbol{\theta}_i(y) = \left(\frac{||y||_1}{y_i}\right)^{\kappa} G(y)\,, \quad 1 \leq i,j \leq n \tag{3.9.3}$$

where κ is a positive constant and $G : \mathbb{R}_+^n \to \mathbb{R}_+$ is a bounded and locally Lipschitz function (Example 3.9.1 is a special case of this model with $\kappa = 1/2$ and $G \equiv 1$).

Then the system of Stochastic Differential Equations (3.2.4) becomes

$$\mathrm{d}X_i(t) = X_i(t)\left[\frac{\gamma_i + \gamma_{n+1}^2}{2\left(\mu_i(t,\mathfrak{X})\right)^{2\kappa}}\,G^2(\mathfrak{X}(t))\,\mathrm{d}t + \frac{\gamma_{n+1}}{\left(\mu_i(t,\mathfrak{X})\right)^{\kappa}}\,G(\mathfrak{X}(t))\,\mathrm{d}W_i(t)\right],\ \ i = 1, \ldots, n$$

with $\mu(t,\mathfrak{X})$ the market portfolio defined in (3.3.4), or equivalently,

$$\mathrm{d}\log(X_i(t)) = \frac{\gamma_i}{2\left(\mu_i(t,\mathfrak{X})\right)^{2\kappa}}\,G^2(\mathfrak{X}(t))\,\mathrm{d}t + \frac{\gamma_{n+1}}{\left(\mu_i(t,\mathfrak{X})\right)^{\kappa}}\,G(\mathfrak{X}(t))\,\mathrm{d}W_i(t)\,, \quad i = 1, \ldots, n\,,$$

For every $x \in \mathbb{R}_+^n$, $\gamma_i \in [c_i, c_i^*]$, $i = 1, 2, \ldots, n$ and $\gamma_{n+1} \in [1, c_{n+1}]$, this system of SDEs has a unique-in-distribution solution $\mathfrak{X}(\cdot)$ starting at $\mathfrak{X}(0) = x$, the components $X_i(\cdot)$ of this solution are time-changed versions of independent squared-Bessel processes (see [1] and [52, Sections 2 and 4] for more details). In particular, we have $\mathfrak{X}(\cdot) \in \mathbb{R}_+^n$.

This uncertainty structure also satisfies the conditions in Proposition 3.7.10 with the $\mathbf{s}$, $\boldsymbol{\theta}$ as in (3.9.3) and $\mathcal{R}(y) = [1, c_{n+1}]$, and therefore

$$\mathfrak{u}(t,y) \equiv \Phi(t,y) \equiv \widehat{\Phi}(t,y) \equiv \mathfrak{u}_{\mathcal{M}^y}(t,y)$$

is the smallest nonnegative classical (super)solution of the Cauchy problem (3.1.1), (3.1.2), as well as the smallest nonnegative classical (super)solution of the Cauchy problem (3.3.12), (3.1.2) (recall $\mathcal{M}^y$ from Assumption 3.7.7). If in addition $G(\cdot)$ is bounded away from zero, then the condition in Remark 3.3.1 is satisfied as well and (3.9.2) follows.

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Appendix A

Proofs of Theorems in Chapter 2

A.1 Another proof of Theorem 2.1.2 in the one-dimensional case

This section presents a proof of Theorem 2.1.2 in the one-dimensional case which is direct, that is, does not proceed by contradiction. This proof also works in several dimensions, whenever it is known in advance that the function U of (2.1.2) is continuous.

Suppose that $\mathcal{O} = (\ell, r)$ is a fixed open interval with $-\infty \leq \ell < r \leq \infty$.

Theorem A.1.1. *Assume that σ and b are continuous. Then the function U of* (2.1.2) *is a viscosity solution of the parabolic equation* (2.1.3).

Proof. Let us show that $U(t,x)$ is a viscosity subsolution to (2.1.3). The proof of the viscosity supersolution property is similar.

According to the definition of viscosity subsolution and the joint continuity of U ([32, Proposition 4.3]), if suffices to verify that for any $(t_0, x_0) \in (0,\infty) \times \mathcal{O}$ and any $\varphi(t,x) \in C^{1,2}((0,\infty) \times \mathcal{O})$ with

$$(U - \varphi)(t_0, x_0) = \max_{(t,x)\in(0,\infty)\times\mathcal{O}} (U - \varphi)(t,x) = 0\,, \tag{A.1.1}$$

we have the inequality

$$G(t_0, x_0) \leq 0 \qquad \text{for the function} \qquad G(t,x) := (\varphi_t - \mathcal{L}\varphi)(t,x)$$

with the operator $\mathcal{L}$ defined in (2.1.11).

We start the diffusion process $X(\cdot)$ at $X(0) = x_0$, consider $\alpha > 0$ such that $[x_0 - \alpha, x_0 + \alpha] \subset \mathcal{O}$, and define $T_\alpha := \inf\{t > 0 : |X(t) - x_0| > \alpha\}$. Then we have $S > T_\alpha$ if $T_\alpha < \infty$, and $S = \infty$ if $T_\alpha = \infty$. In either case, the bounded stopping time

$$\theta_m := \frac{1}{m} \wedge T_\alpha < S\,.$$

Further, for every fixed $\omega \in \Omega$, we notice that $\theta_m(\omega) = 1/m$ when $m > 1/T_\alpha(\omega)$.

Take $m > \frac{2}{t_0}$, then we have $\theta_m \leq \frac{1}{m} \wedge S \leq \frac{t_0}{2} \wedge S < t_0$. The strong Markov property of $X(\cdot)$ gives

$$\begin{aligned}\mathbb{E}_{x_0}\left[U(t_0 - \theta_m, X(\theta_m))\right] &= \mathbb{E}_{x_0}\left[\mathbb{P}_{X(\theta_m)}[S > t_0 - \theta_m]\right] = \mathbb{E}_{x_0}\left[\mathbb{P}_{x_0}\left[S > t_0 \,|\, \mathscr{F}(\theta_m)\right]\right] \\ &= \mathbb{P}_{x_0}[S > t_0] = U(t_0, x_0)\,.\end{aligned}$$

On the other hand, we recall (A.1.1) and obtain

$$\varphi(t_0, x_0) - \mathbb{E}_{x_0}[\varphi(t_0 - \theta_m, X(\theta_m))] \leq U(t_0, x_0) - \mathbb{E}_{x_0}[U(t_0 - \theta_m, X(\theta_m))] = 0\,. \quad \text{(A.1.2)}$$

Thanks to the assumption $\varphi \in C^{1,2}((0,\infty) \times \mathcal{O})$, we can apply Itô's change of variable rule to $\varphi(t_0 - s, X(s))$, $s \in [0, \theta_m] \subset [0, t_0)$ and plug in (2.1.1) to derive the semimartingale decomposition

$$\mathrm{d}\varphi(t_0 - s, X(s)) = -G(t_0 - s, X(s))\,\mathrm{d}s + \varphi_x(t_0 - s, X(s))\,\sigma(X(s))\,\mathrm{d}W(s)\,. \quad \text{(A.1.3)}$$

Integrating (A.1.3) with respect to s over $[0, \theta_m] \subset [0, t_0)$ and taking the expectation under $\mathbb{P}_{x_0}$ yields

$$\begin{aligned}0 &\geq m \cdot \text{LHS of (A.1.2)} \\ &= m \cdot \mathbb{E}_{x_0}\left[\int_0^{\theta_m} G(t_0 - s, X(s))\,\mathrm{d}s - \int_0^{\theta_m} \sigma(X(s))\varphi_x(t_0 - s, X(s))\,\mathrm{d}W(s)\right] \\ &= \mathbb{E}_{x_0}\left[m\int_0^{\theta_m} G(t_0 - s, X(s))\,\mathrm{d}s\right] := G_m(t_0, x_0)\,.\end{aligned}$$

Here in the second equality, the expectations of the integrals with respect to $\mathrm{d}W_k(s)$ have all vanished, due to the local boundedness of functions σ and φ_x and the fact that for all $s \in [0, \theta_m]$, we have $t_0 - s \in [t_0/2, t_0]$ and $X(s) \in [x_0 - \alpha, x_0 + \alpha] \subset \mathcal{O}$ since $\theta_m < \frac{t_0}{2} \wedge S$.

It then suffices to show that $G_m(t_0, x_0) \to G(t_0, x_0)$ as $m \to \infty$. In fact, recalling that $\theta_m \leq 1/m$, as well as the boundedness of the terms in $G(t_0 - s, X(s))$, we can take the limit and appeal to the dominated convergence theorem to obtain

$$\lim_{m\to\infty} G_m(t_0, x_0) = \mathbb{E}_{x_0}\left[\lim_{m\to\infty} m \int_0^{\theta_m} G(t_0 - s, X(s))\,\mathrm{d}s\right].$$

Finally, since $\theta_m = 1/m$ $(\to 0)$ for sufficiently large m, the mean value theorem and the continuity in s give $\lim_{m\to\infty} G_m(t_0, x_0) = G(t_0, x_0)$, as desired. □

A.2 The proof of (2.2.14)

Proof. We apply Itô's change of variable rule to $\varphi(t^* - t, X(t))Y(t)$ for $t \in [0, \nu] \subset [0, t^*)$ and plug in (2.1.1) to obtain:

$$\begin{aligned}
&\mathrm{LHS} \cdot (Y(t))^{-1} \\
=\ & -\varphi_t(t^* - t, X(t))\,\mathrm{d}t + \sum_i \varphi_{x_i}(t^* - t, X(t))\,\mathrm{d}X_i(t) \\
&+ \sum_{i,j} \frac{1}{2}\,\varphi_{x_i x_j}(t^* - t, X(t))\,\mathrm{d}\langle X_i, X_j\rangle(t) - h(X(t))\varphi(t^* - t, X(t))\,\mathrm{d}t \\
=\ & -\varphi_t(t^* - t, X(t))\,\mathrm{d}t + \sum_{i,k} \varphi_{x_i}(t^* - t, X(t))\big[\sigma_{ik}(X(t))\,\mathrm{d}W_k(t) + b_i(X(t))\,\mathrm{d}t\big] \\
&+ \sum_{i,j,k} \frac{1}{2}\varphi_{x_i x_j}(t^* - t, X(t))\sigma_{ik}(X(t))\sigma_{jk}(X(t))\,\mathrm{d}t - h(X(t))\varphi(t^* - t, X(t))\,\mathrm{d}t \\
=\ & \left[-\varphi_t + \sum_i b_i(X(t))\varphi_{x_i} + \sum_{i,j} \frac{1}{2}\,a_{ij}(X(t))\varphi_{x_i x_j} - h(X(t))\varphi\right](t^* - t, X(t))\,\mathrm{d}t \\
&+ \sum_{i,k} \varphi_{x_i}(t^* - t, X(t))\sigma_{ik}(X(t))\,\mathrm{d}W_k(t) \\
=\ & \mathrm{RHS} \cdot (Y(t))^{-1}.
\end{aligned}$$

□

Appendix B

Proofs of Theorems in Chapter 3

B.1 The proof of Lemma 3.5.1

Proof. Let $\phi(t) := \varphi(T-t, \mathfrak{X}(t))$,

$$\mathfrak{s}_{ik}(t,\mathfrak{X}) := X_i(t)\sigma_{ik}(t,\mathfrak{X})\,, \quad \mathfrak{s}(t,\mathfrak{X}) := (\mathfrak{s}_{ik}(t,\mathfrak{X}))_{n\times n}\,,$$

and

$$\mathfrak{b}(t,\mathfrak{X}) = (\mathfrak{b}_1,\ldots,\mathfrak{b}_n)(t,\mathfrak{X}) := (\mathfrak{s}\vartheta)(t,\mathfrak{X})\,.$$

Then the SDE (3.2.4) can be rewritten as

$$\mathrm{d}X_i(t) = \mathfrak{b}_i(t,\mathfrak{X})\,\mathrm{d}t + \sum_k \mathfrak{s}_{ik}(t,\mathfrak{X})\,\mathrm{d}W_k(t)\,, \quad i=1,2,\ldots,n\,, \quad \mathfrak{X}(0)=x\,. \tag{B.1.1}$$

Apply Itô's Rule to $f_2(x,y_1,\ldots,y_n) := \varphi\big(T-x,(y_1,\ldots,y_n)\big)$ with (B.1.1):

$$\begin{aligned}\mathrm{d}\phi(t) = \Bigg[&-\varphi_t\,\mathrm{d}t + \sum_i D_i\varphi\left(\mathfrak{b}_i\,\mathrm{d}t + \sum_k \mathfrak{s}_{ik}\,\mathrm{d}W_k(t)\right)\\ &+\frac{1}{2}\sum_{i,j} D^2_{ij}\varphi \sum_k \mathfrak{s}_{ik}\mathfrak{s}_{jk}\,\mathrm{d}t\Bigg](T-t,t,\mathfrak{X})\,,\end{aligned} \tag{B.1.2}$$

where for convenience, throughout the paper the values of L, ϕ, $\mathfrak{b}_i$, $\mathfrak{s}_{ik}$, ϑ, φ_t, $D_i\varphi$ and $D^2_{ij}\varphi$ at $(T-t,t,\mathfrak{X})$ stand for $L(t)$, $\phi(t)$, $\mathfrak{b}_i(t,\mathfrak{X})$, $\mathfrak{s}_{ik}(t,\mathfrak{X})$, $\vartheta(t,\mathfrak{X})$, $\varphi_t(T-t,\mathfrak{X}(t))$, $D_i\varphi(T-t,\mathfrak{X}(t))$ and $D^2_{ij}\varphi(T-t,\mathfrak{X}(t))$, respectively.

Summing (B.1.1) over i from 1 to n yields

$$\mathrm{d}X(t) = \left(\sum_i \mathfrak{b}_i \,\mathrm{d}t + \sum_{i,k} \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) \right)(t, \mathfrak{X}) \,. \tag{B.1.3}$$

Finally, apply Itô's Rule to the exponential function for $L(\cdot)$:

$$\mathrm{d}L(t) = -L(t)\, \vartheta'(t, \mathfrak{X}) \,\mathrm{d}W(t) \,. \tag{B.1.4}$$

Plugging (B.1.2) – (B.1.4) into Itô's Rule for $f_3(r_1, r_2, r_3) := r_1 r_2 r_3$ gives

$$\begin{aligned}
&\mathrm{d}(XL\phi)(t) \\
&= \big[L\phi \,\mathrm{d}X(t) + X\phi \,\mathrm{d}L(t) + XL \,\mathrm{d}\phi(t) + X \,\mathrm{d}\langle L, \phi \rangle_t + L \,\mathrm{d}\langle X, \phi \rangle_t + \phi \,\mathrm{d}\langle X, L \rangle_t \big](t) \\
&= L\phi \left[\sum_i \mathfrak{b}_i \,\mathrm{d}t + \sum_{i,k} \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) \right] - X\phi L\vartheta' \mathrm{d}W(t) - XL\varphi_t \,\mathrm{d}t \qquad \text{(B.1.5)} \\
&\quad + XL \sum_i D_i\varphi \left[\mathfrak{b}_i \,\mathrm{d}t + \sum_k \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) \right] + \frac{1}{2} XL \sum_{i,j} D^2_{ij}\varphi \sum_k \mathfrak{s}_{ik}\mathfrak{s}_{jk} \,\mathrm{d}t \\
&\quad - XL \sum_k \vartheta_k \sum_i D_i\varphi \mathfrak{s}_{ik} \,\mathrm{d}t + L \sum_{k,j} \mathfrak{s}_{jk} \sum_i D_i\varphi \mathfrak{s}_{ik} \,\mathrm{d}t - \phi L \sum_{k,i} \mathfrak{s}_{ik}\vartheta_k \,\mathrm{d}t \Big|_{(T-t,t,\mathfrak{X})} \,.
\end{aligned}$$

Rearranging (B.1.5), we obtain

$$\begin{aligned}
&\mathrm{d}(XL\phi)(t) \\
&= - XL \left(\phi_t - \frac{1}{2} \sum_{i,j} D^2_{ij}\varphi \sum_k \mathfrak{s}_{ik}\mathfrak{s}_{jk} - \frac{1}{X} \sum_{k,j} \mathfrak{s}_{jk} \sum_i D_i\varphi \mathfrak{s}_{ik} \right) \mathrm{d}t + L\phi \sum_{i,k} \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) \\
&\quad - X\phi L\vartheta' \mathrm{d}W(t) + XL \sum_i D_i\varphi \sum_k \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) + L\phi \left(\sum_i \mathfrak{b}_i - \sum_{k,i} \mathfrak{s}_{ik}\vartheta_k \right) \mathrm{d}t \\
&\quad + XL \left(\sum_i D_i\varphi \mathfrak{b}_i - \sum_k \vartheta_k \sum_i D_i\varphi \mathfrak{s}_{ik} \right) \mathrm{d}t \Big|_{(T-t,t,\mathfrak{X})} \\
&= -XLg \,\mathrm{d}t + L\phi \sum_{i,k} \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) - X\phi L\vartheta' \mathrm{d}W(t) + XL \sum_i D_i\varphi \sum_k \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) \Big|_{(T-t,t,\mathfrak{X})} \\
&= -XLg \,\mathrm{d}t - X\phi L\vartheta' \mathrm{d}W(t) + L \sum_{i,k} \mathfrak{s}_{ik} \,\mathrm{d}W_k(t) \,(\phi + XD_i\varphi) \Big|_{(T-t,t,\mathfrak{X})} \,,
\end{aligned}$$

where we used the definition (3.5.5) of g and the fact that $\mathfrak{b} = \mathfrak{s}\,\vartheta$. □

B.2 An alternative proof of Theorem 3.4.5

We present here an alternative proof of Theorem 3.4.5. We still argue by contradiction, but avoid introducing the stopping time λ of (3.5.11) and thus also the stopping time ρ and the constant C_1. We also avoid using Lemma 3.5.2; instead, we provide a lower bound for $\mathbb{E}[L(\nu)]$ in (B.2.15) below. The goal is to prove (3.5.16) for ν instead of ρ. We shall approximate ν by a sequence of stopping times ν_m for which (3.5.16) holds, then apply Fatou's Lemma. This approach can also be applied to the proof in Section 3.6 for the supersolution property.

Proof. According to Definition 3.4.1 (i) of viscosity subsolution with the F in (3.4.7), it suffices to show that for any test function $\varphi \in C^{1,2}\big((0,\infty)\times\mathbb{R}^n_+\big)$ and $(t_0,x_0) \in (0,\infty)\times\mathbb{R}^n_+$ with

$$\big(\widehat{\Phi}^* - \varphi\big)(t_0,x_0) = 0 > \big(\widehat{\Phi}^* - \varphi\big)(t,x)\,, \quad \forall\ (t,x) \in (0,\infty)\times\mathbb{R}^n_+\,, \tag{B.2.1}$$

(i.e., such that (t_0,x_0) is a strict maximum of $\widehat{\Phi}^* - \varphi$), we have

$$\big(\varphi_t - \widehat{\mathcal{L}}\varphi\big)(t_0,x_0) \le 0\,.$$

Here $\widehat{\mathcal{L}}$ is defined in (3.1.3), and $\widehat{\Phi}^*$ is the upper-semicontinuous envelope of $\widehat{\Phi}$ as in the definition (3.4.4). *We shall argue this by contradiction, assuming that*

$$\widehat{\mathcal{G}}(t_0,x_0) > 0 \qquad \text{for the function} \qquad \widehat{\mathcal{G}}(t,x) := \big(\varphi_t - \widehat{\mathcal{L}}\varphi\big)(t,x)\,. \tag{B.2.2}$$

Since the function F of (3.4.2) is continuous, so is the function $\widehat{\mathcal{G}}$ just introduced in (B.2.2). There will exist then, under this hypothesis and Assumption 3.4.3, a neighborhood $\mathcal{D}_\delta := (t_0-\delta, t_0+\delta)\times B_\delta(x_0)$ of (t_0,x_0) in $(0,\infty)\times\mathbb{R}^n_+$ with $0<\delta<||x_0||_1/n$, on which $\mathcal{K}(\cdot)$ is bounded and $\widehat{\mathcal{G}}(\cdot,\cdot) > 0$ holds.

Let C be a constant such that $\varphi(t,x)$, $||\theta||$, $|a_{ij}| < C$ $(1 \le i,j \le n)$ hold for all pairs $(\theta, a = (a_{ij})_{n\times n}) \in \mathcal{K}(x)$ and all $(t,x) \in \mathcal{D}_\delta$. We can assume that

$$16\,\delta C^2 + 2\,\delta^2 C^4 < 1/2 \tag{B.2.3}$$

by selecting a sufficiently small $\delta > 0$. We notice that $|x_i - (x_0)_i| \le |x - x_0| < \delta$ holds for any $x = (x_1, \dots, x_n) \in \mathcal{D}_\delta$, and thus

$$0 < ||x_0||_1 - n\delta < ||x||_1 < ||x_0||_1 + n\delta\,, \tag{B.2.4}$$

and introduce the constants

$$C_2 := -\max_{\partial\mathcal{D}_\delta} \big(\widehat{\Phi}^* - \varphi\big)(t,x)$$

and

$$C_3^\star := \frac{C_2(||x_0||_1 - n\delta)}{2(||x_0||_1 + n\delta)} \left(\frac{1}{2} - 16\,\delta C^2 - 2\,\delta^2 C^4\right), \tag{B.2.5}$$

which are strictly positive by (B.2.1) and (B.2.3), respectively. We observe that

$$\limsup_{(t,x)\to(t_0,x_0)} (\widehat{\Phi} - \varphi)(t,x) = (\widehat{\Phi}^* - \varphi)(t_0,x_0) = 0\,,$$

and hence there exists $(t^*, x^*) \in \mathcal{D}_\delta$ such that

$$(\widehat{\Phi} - \varphi)(t^*, x^*) > -C_3^\star\,; \tag{B.2.6}$$

and by the definition (3.3.7) of $\widehat{\Phi}$, there exists an admissible system $\mathcal{M}^{x^*} \in \widehat{\mathfrak{M}}(x^*)$ such that

$$\mathfrak{u}_{\mathcal{M}^{x^*}}(t^*, x^*) > \widehat{\Phi}(t^*, x^*) - C_3^\star > \varphi(t^*, x^*) - 2\,C_3^\star\,, \quad \text{by (B.2.6).} \tag{B.2.7}$$

The remaining discussion in this section will be carried out under this admissible system, unless otherwise specified.

- Let us start by constructing stopping times

$$\begin{aligned} \nu\,(=\nu(\omega)) :=&\ \inf\big\{s \in (0,t^*] : \big(t^* - s, \mathfrak{X}(s)\big) \notin \mathcal{D}_\delta\big\} \\ \le&\ t^* - (t_0 - \delta) = (t^* - t_0) + \delta < t^* \wedge 2\delta \end{aligned} \tag{B.2.8}$$

(by the definitions of $\mathcal{D}_\delta$ and t^*), and for $m = 1, 2, \dots,$

$$\begin{aligned} &\lambda_m\,(=\lambda_m\,(\omega)) :=\ \inf\{s > 0 : |\log L(s)| > m\} \uparrow \infty\,, \\ &\nu_m\,(=\nu_m\,(\omega)) :=\ \nu \wedge \lambda_m \uparrow \nu\,, \qquad\qquad \mathbb{P}\text{−a.s. as } m \uparrow \infty \end{aligned} \tag{B.2.9}$$

with the usual convention $\inf\emptyset = \infty$.

From definitions (B.2.2) and (3.1.3), we see that

$$g(t,s,\mathfrak{X}) := (\varphi_t - \mathcal{L}_{\alpha(s,\mathfrak{X})}\varphi)\big(t,\mathfrak{X}(s)\big) \geq \widehat{\mathcal{G}}\big(t,\mathfrak{X}(s)\big),\ \forall\,(t,s)\in(0,\infty)\times[0,\infty). \quad \text{(B.2.10)}$$

Recall that $\widehat{\mathcal{G}}(\cdot\,,\cdot) > 0$ on $\mathcal{D}_\delta$ from the discussion right below (B.2.2). Combining with (B.2.10) leads to

$$g(t^*-s,s,\mathfrak{X}) > 0\,, \quad \forall\ s\in[0,\nu)\,. \tag{B.2.11}$$

Let us apply now Lemma 3.5.1 with $T = t^*$, integrating (3.5.15) with respect to t over $[0,\nu_m]$ and taking the expectation under $\mathbb{P}$, to obtain

$$\begin{aligned} ||x^*||_1\,\varphi(t^*,x^*) \,-\, &\mathbb{E}\big[L(\nu_m)X(\nu_m)\varphi\big(t^*-\nu_m,\mathfrak{X}(\nu_m)\big)\big] \\ &= \mathbb{E}\left[\int_0^{\nu_m} L(s)X(s)g(t^*-s,s,\mathfrak{X})\,\mathrm{d}s\right] >\ 0\,. \end{aligned} \tag{B.2.12}$$

Here, the strict inequality comes from (B.2.11) and the positivity of ν_m; whereas, in the equality, the expectations of the integrals with respect to $\mathrm{d}W(t)$ or $\mathrm{d}W_k(t)$ have all vanished – due to the boundedness of the processes $\mathfrak{X}(\cdot)$ and $L(\cdot)$ on $[0,\nu_m]$, of the functions φ and $D_i\varphi$ on $\overline{\mathcal{D}_\delta}$, and of the functionals $\vartheta(\cdot,\mathfrak{X})$, $\alpha_{ij}(\cdot,\mathfrak{X})$ (by Assumption 3.4.3) and thus $\sigma_{ik}(\cdot,\mathfrak{X})$ on $[0,\nu_m]$.

(We have made use here of the following facts. The eigenvalues e_i of α are the nonnegative roots of the characteristic polynomial of α, which is determined by the entries α_{ij}; since the $\alpha_{ij}(\cdot,\mathfrak{X})$'s are bounded on $[0,\nu_m]$, so are the e_i's. Thus σ, which can be written as $\mathbf{QD}$, for some $n\times n$ orthonormal matrix $\mathbf{Q}$ and diagonal matrix $\mathbf{D}$ with diagonal entries $\sqrt{e_i}$, is also bounded.)

Since almost surely $\nu_m \uparrow \nu$ ((B.2.9)) and $L(\nu_m)X(\nu_m)\varphi\big(t^*-\nu_m,\mathfrak{X}(\nu_m)\big) > 0$ for all m (the positivity of φ follows from (B.2.1) and (3.3.11)), Fatou's Lemma along with (B.2.12) gives

$$\begin{aligned} \mathbb{E}\left[L(\nu)X(\nu)\varphi\big(t^*-\nu,\mathfrak{X}(\nu)\big)\right] \,&=\, \mathbb{E}\left[\liminf_{m\to\infty} L(\nu_m)X(\nu_m)\varphi\big(t^*-\nu_m,\mathfrak{X}(\nu_m)\big)\right] \\ \leq \liminf_{m\to\infty}\,\mathbb{E}\left[L(\nu_m)X(\nu_m)\varphi\big(t^*-\nu_m,\mathfrak{X}(\nu_m)\big)\right] \,&\leq\, ||x^*||_1\,\varphi(t^*,x^*)\,. \end{aligned} \tag{B.2.13}$$

Notice that $\big(t^* - \nu, \mathfrak{X}(\nu)\big) \in \partial\mathcal{D}_\delta$ holds by the definition (B.2.8) of ν, so we have

$$\varphi\big(t^* - \nu, \mathfrak{X}(\nu)\big) \geq \widehat{\Phi}^*\big(t^* - \nu, \mathfrak{X}(\nu)\big) + C_2 \geq \widehat{\Phi}\big(t^* - \nu, \mathfrak{X}(\nu)\big) + C_2\,. \tag{B.2.14}$$

Plugging (B.2.7) and (B.2.14) into (B.2.13) yields

$$\begin{aligned}
0 &< ||x^*||_1 \left[\, 2\, C_3^\star + \mathfrak{u}_{\mathcal{M}^{x^*}}(t^*, x^*)\right] - \mathbb{E}\left[L(\nu)X(\nu)\left(\widehat{\Phi}\big(t^* - \nu, \mathfrak{X}(\nu)\big) + C_2\right)\right] \\
&= ||x^*||_1\, \mathfrak{u}_{\mathcal{M}^{x^*}}(t^*, x^*) - \mathbb{E}\Big[L(\nu)X(\nu)\widehat{\Phi}\big(t^* - \nu, \mathfrak{X}(\nu)\big)\Big] + 2\, C_3^\star\, ||x^*||_1 - C_2\, \mathbb{E}\big[L(\nu)X(\nu)\big] \\
&\leq 2\, C_3^\star\, ||x^*||_1 - C_2\, \mathbb{E}\big[L(\nu)X(\nu)\big] \;<\; 2\, C_3^\star(||x_0||_1 + n\delta) - C_2\,(||x_0||_1 - n\delta)\, \mathbb{E}\big[L(\nu)\big]\,,
\end{aligned}$$

(we have used Proposition 3.5.3 in the third step and the last inequality of (B.2.4) at last).

Recall the definition (B.2.5) of $C_3^\star$. We will arrive at a contradiction and hence complete the argument, as soon as we have shown the following inequality:

$$\mathbb{E}\big[L(\nu)\big] \geq \frac{1}{2} - 16\,\delta C^2 - 2\,\delta^2 C^4. \tag{B.2.15}$$

(This explains why we constructed $C_3^\star$ as we did in (B.2.5); in fact, setting $C_3^\star$ to be any value less than the right-hand side of (B.2.5) would also work). First, we observe the following double inequality

$$e^r \geq \frac{3}{2e} - \frac{r^2}{2e} > \frac{1}{2} - r^2, \quad \forall\, r \in \mathbb{R}\,. \tag{B.2.16}$$

The second inequality is obvious since $2 < e < 3$. For the first inequality, we set

$$f(r) := e^r - \frac{3}{2e} + \frac{r^2}{2e}$$

and find that $f(-1) = 0$, $f'(-1) = 0$ and $f''(r) > 0$. Hence $f(r)$ achieves its minimum 0 at $r = -1$. Applying (B.2.16) to $\log L(\nu)$ yields

$$\mathbb{E}\big[L(\nu)\big] \geq \frac{1}{2} - \mathbb{E}\left[\big(\log L(\nu)\big)^2\right].$$

Therefore, it suffices to show that

$$\mathbb{E}\left[\sup_{0\leq t\leq \nu}\big(\log L(t)\big)^2\right] \leq 16\,\delta C^2 + 2\,\delta^2 C^4. \tag{B.2.17}$$

For any $t \in (0, \nu]$, we have

$$\begin{aligned}\big(\log L(t)\big)^2 &= \left|-\int_0^t \vartheta'(s,\mathfrak{X})\,\mathrm{d}W(s) - \int_0^t \frac{1}{2}\,\big|\big|\vartheta(s,\mathfrak{X})\big|\big|^2\,\mathrm{d}s\right|^2 \\ &\leq 2\left|\int_0^t \vartheta'(s,\mathfrak{X})\,\mathrm{d}W(s)\right|^2 + 2\left|\int_0^t \frac{1}{2}\,\big|\big|\vartheta(s,\mathfrak{X})\big|\big|^2\,\mathrm{d}s\right|^2\end{aligned}$$

It follows from $t \leq \nu < 2\delta$ that

$$\int_0^t \frac{1}{2}\,\big|\big|\vartheta(s,\mathfrak{X})\big|\big|^2\,\mathrm{d}s \;\leq\; \frac{t}{2}\,C^2 \;\leq\; \delta C^2,$$

and therefore

$$\mathbb{E}\left[\sup_{0\leq t\leq \nu}\big(\log L(t)\big)^2\right] \leq 2\,\mathbb{E}\left[\sup_{0\leq t\leq \nu}\left|\int_0^t \vartheta'(s,\mathfrak{X})\,\mathrm{d}W(s)\right|^2\right] + 2\,\delta^2 C^4.$$

Finally, the Burkholder-Davis-Gundy Inequality gives

$$\mathbb{E}\left[\sup_{0\leq t\leq \nu}\left|\int_0^t \vartheta'(s,\mathfrak{X})\,\mathrm{d}W(s)\right|^2\right] \leq 4\,\mathbb{E}\left[\int_0^\nu \big|\big|\vartheta'(s,\mathfrak{X})\big|\big|^2\,\mathrm{d}s\right] \leq 4\,\mathbb{E}\left[\nu C^2\right] \leq 8\,\delta C^2,$$

and (B.2.17) follows. □

Appendix C

List of Notations and Definitions in Chapter 3

We list the various notations and terminology in Chapter 3, followed by the page number and the equation number (if available) of its first occurrence or definition.

a, p. 27: a real, symmetric, positive-definite $n \times n$ matrix.

a_0, p. 52: an element of $\mathcal{A}(x_0)$ that satisfies (3.6.7).

a_{ij}, p. 27: the (i, j) entry of a.

$\mathbf{a}(\cdot)$, p. 37, Theorem 3.3.3, (3.3.12): $\mathbf{s}(\cdot)\mathbf{s}'(\cdot)$.

$\mathcal{A}(\cdot)$, p. 27, (3.1.4): subset of $\mathbb{S}_+(n)$: $\big\{a \in \mathbb{S}_+(n) : \exists\, \theta \in \mathbb{R}^n \text{ s.t. } (\theta, a) \in \mathcal{K}(x)\big\}$.

Admissible system, p. 30, Definition 3.2.1.

Arbitrage function, p. 34, (3.3.5).

Arbitrage function for $\mathcal{M}$, p. 35, (3.3.8), (3.3.10).

$\alpha(\cdot, \cdot)$, p. 30, (3.2.3): $\sigma(\cdot, \cdot)\sigma(\cdot, \cdot)'$.

$\alpha_{ij}(\cdot, \cdot)$, p. 32: the (i, j) entry of $\alpha(\cdot, \cdot)$.

$\mathfrak{b}(\cdot, \cdot)$, p. 79: $(\mathfrak{s}\vartheta)(\cdot, \cdot)$.

$\mathfrak{b}_i(\cdot, \cdot)$, p. 79: the i-th component of $\mathfrak{b}(\cdot, \cdot)$.

$B_\delta(x)$, p. 40: the ball of radius δ and center x in $\mathbb{R}^n_+$.

$\beta(\cdot, \cdot)$, p. 32: $(\sigma\vartheta)(\cdot, \cdot)$.

Bounded (investment rule): p. 33: an investment rule $\Pi(\cdot,\cdot)$ that is bounded on $[0,\infty)\times\Omega$.

C, p. 31, (3.2.6): some positive constant.

C_1, C_2, C_3, p. 44, (3.5.7): some positive constants.

C_2^*, C_3^*, p. 54, (3.6.15): some positive constants.

$C_3^\star$, p. 82, (B.2.5): some positive constant.

Du, p. 38, (3.4.2): $(u_{x_1}, u_{x_2}, \ldots, u_{x_n})'$.

$D_i u$, p. 27: u_{x_i}.

$D^2 u$, p. 38, (3.4.2): $(u_{x_i x_j})_{n\times n}$.

$D^2_{ij} u$, p. 27: $u_{x_i x_j}$.

$\mathcal{D}_\delta$, p. 44: $(t_0-\delta, t_0+\delta)\times B_\delta(x_0)$.

δ, p. 40, Assumption 3.4.4: some positive number.

Dynamic Programming Principle (DPP), p. 55, Proposition 3.6.2.

$F(\cdot,\cdot,\cdot,\cdot,\cdot)$, p. 38: a continuous function: $(0,\infty)\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}(n)\to\mathbb{R}$; in our setting: p. 39, (3.4.7).

$\mathbb{F}$, p. 29: $\{\mathcal{F}(t)\}_{0\le t<\infty}$, the filtration generated by the canonical process: $\mathfrak{B}(t,\omega)=\omega(t)$.

$\mathcal{F}, \mathcal{F}(\cdot)$, p. 29: σ-algebras.

$g(t,s,\mathfrak{X})$, p. 45, (3.5.13): $\big(\varphi_t - \mathcal{L}_{\alpha(s,\mathfrak{X})}\varphi\big)\big(t,\mathfrak{X}(s)\big)$.

g_0, p. 52, (3.6.7): $-\mathcal{G}_{a_0}(t_0,x_0)\,(>0)$.

$\mathcal{G}_a(t,x)$, p. 52, (3.6.8): $(\varphi_t-\mathcal{L}_a\varphi)(t,x)$.

$\widehat{\mathcal{G}}(t,x)$, p. 44, (3.5.5): $\big(\varphi_t-\widehat{\mathcal{L}}\varphi\big)(t,x)$.

γ, p. 41: an element of Γ.

Γ, p. 41: a given subset of some Euclidean space.

$\mathrm{GL}(n)$, p. 30: the space of $n\times n$ invertible real matrices.

$H_{ij}(s,y)$, p. 52, (3.6.9): $D_i\varphi(s,y)\,/\,||y||_1 + y_i y_j D^2_{ij}\varphi(s,y)\,/\,2$.

$\mathcal{H}(\cdot,\cdot)$, p. 64, (3.8.9): $(\pi'\sigma-\vartheta')(\cdot,\cdot)$.

$\mathcal{H}_k(\cdot,\cdot)$, p. 64, (3.8.10): the k-th component of $\mathcal{H}(\cdot,\cdot)$.

Hamilton-Jacobi-Bellman (HJB) equation, p. 27, (3.1.1).

i, p. 26: a positive integer no greater than n.

ι, p. 40, Assumption 3.4.4; p. 53, (3.6.12): a positive number.

Investment rule, p. 32.

j, p. 26: a positive integer no greater than n.

k, p. 29: a positive integer no greater than n.

$\mathbb{K}$, p. 29, (3.2.1): $\big\{\mathcal{K}(y)\big\}_{y\in\mathbb{R}^n_+}$.

$\mathcal{K}(\cdot)$, p. 29, (3.2.1): a nonempty, compact and convex subset of $\mathbb{R}^n\times\mathbb{S}_+(n)$.

$\mathfrak{K}$, p. 29: some collection of progressively measurable functional pairs.

Knightian uncertainty, p. 29, (3.2.1), Definition 3.2.1: $\mathbb{K}$.

$L(\cdot)$, p. 34, (3.3.6): the associated exponential $\mathbb{P}-$local martingale.

$L^{\mathcal{M}}(\cdot)$, p. 34: the L associated with $\mathcal{M}$.

$\mathcal{L}_a$, p. 27, (3.1.3): some linear second-order differential operator.

$\widehat{\mathcal{L}}$, p. 27, (3.1.3): some nonlinear second-order differential operator.

$\lambda(\omega)$, p. 45, (3.5.12): a stopping time: $\inf\{s>0:|\log L(s,\omega)|>C_1\}$.

$\lambda_m(\omega)$, p. 82, (B.2.9): a stopping time: $\inf\{s>0:|\log L(s,\omega)|>m\}$.

Long-only (portfolio), p. 33: a portfolio $\mathit{\Pi}(\cdot,\cdot)$ with $\mathit{\Pi}_i\geq 0$ on $[0,\infty)\times\Omega$ for all i.

Lower-semicontinuous envelope, p. 39, (3.4.6).

m, p. 82, (B.2.9): a positive integer.

$\mathcal{M}$, p. 30, Definition 3.2.1: an admissible system (a quintuple).

$\mathcal{M}^{x^*}$, p. 45: an admissible system in $\mathfrak{M}(x^*)$ that satisfies (3.5.9).

$\mathcal{M}^y$, p. 30, Definition 3.2.2: some admissible system in $\mathfrak{M}(y)$.

$\mathfrak{M}(x)$, p. 30, Definition 3.2.1: the collection of admissible systems with initial configuration $x\in\mathbb{R}^n_+$.

$\widehat{\mathfrak{M}}(x)$, p. 30, Definition 3.2.2: the collection of strongly Markovian admissible systems with initial configuration $x\in\mathbb{R}^n_+$.

Market portfolio, p. 33: $\mu(\cdot,\mathfrak{X})=\mathfrak{X}(\cdot)/X(\cdot)$.

$\mu(\cdot,\mathfrak{X})$, p. 33, (3.3.4): $\mathfrak{X}(\cdot)/X(\cdot)$, the market portfolio.

$\mu_i(\cdot, \mathfrak{X})$, p. 33, (3.3.4): $X_i(\cdot)/X(\cdot)$.

n, p. 26: the dimension.

$\nu(\omega)$, p. 45, (3.5.10): a stopping time: $\inf\left\{s \in (0, t^*] : \left(t^* - s, \mathfrak{X}(s, \omega)\right) \notin \mathcal{D}_\delta\right\}$.

No Unbounded Profits with Bounded Risk (NUPBR), p. 36, Remark 3.3.2.

$\nu_m(\omega)$, p. 82, (B.2.9): a stopping time: $\nu(\omega) \wedge \lambda_m(\omega)$.

$\mathcal{O}$, p. 38: an open subset of $\mathbb{R}^n$; in our setting: $\mathbb{R}^n_+$.

ω, p. 29: an element of Ω, a continuous path: $[0, \infty) \to \mathbb{R}^n_+$.

Ω, p. 29: $C([0, \infty); \mathbb{R}^n_+)$, the canonical space of continuous paths: $[0, \infty) \to \mathbb{R}^n_+$.

$\mathbb{P}$, p. 30, Definition 3.2.1: a probability measure on $(\Omega, \mathcal{F})$.

$\mathbb{P}^{\mathcal{M}}(\cdot, \cdot)$, p. 30, Definition 3.2.1: the $\mathbb{P}$ in $\mathcal{M}$.

$\mathfrak{P}$, p. 32: the set of investment rules.

$\phi(t)$, p. 79: $\varphi(T - t, \mathfrak{X}(t))$.

$\varphi(\cdot, \cdot)$, p. 39, (3.4.5): some function in $C^{1,2}\left((0, \infty) \times \mathcal{O}\right)$.

$\Phi(T, x)$, p. 35, (3.3.7): $\sup_{\mathcal{M} \in \mathfrak{M}(x)} \mathfrak{u}_{\mathcal{M}}(T, x)$.

$\widehat{\Phi}(T, x)$, p. 35, (3.3.7): $\sup_{\mathcal{M} \in \widehat{\mathfrak{M}}(x)} \mathfrak{u}_{\mathcal{M}}(T, x)$.

$\Phi_*(\cdot, \cdot)$, p. 52: the lower-semicontinuous envelope of Φ.

$\widehat{\Phi}^*(\cdot, \cdot)$, p. 44: the upper-semicontinuous envelope of $\widehat{\Phi}$.

$\pi(\cdot, \cdot)$, p. 64: $\pi^U(\cdot, \cdot)$.

$\pi_i(\cdot, \cdot)$, p. 64, (3.8.10): the i-th component of $\pi(\cdot, \cdot)$.

$\pi^U(\cdot, \cdot)$, p. 63, Theorem 3.8.2: $(\pi^U_1, \ldots, \pi^U_n)(\cdot, \cdot)$, the investment rule generated by (3.8.5).

$\pi^U_i(\cdot, \cdot)$, p. 63, Theorem 3.8.2, (3.8.5).

$\mathit{\Pi}(\cdot, \cdot)$, p. 33: an investment rule, a progressively measurable functional: $[0, \infty) \times \Omega \to \mathbb{R}^n$.

$\mathit{\Pi}_i(\cdot, \cdot)$, p. 32: the i-th component of $\mathit{\Pi}(\cdot, \cdot)$.

Portfolio, p. 33: an investment rule $\mathit{\Pi}(\cdot, \cdot)$ that satisfies $\sum_i \mathit{\Pi}_i = 1$ on $[0, \infty) \times \Omega$.

$\mathbb{R}^n_+$, p. 27: $(0, \infty)^n$.

$\mathcal{R}(\cdot)$, p. 59, Proposition 3.7.5: some subset of $\mathbb{R}$.

$\rho(\omega)$, p. 45, (3.5.12): a stopping time: $\nu(\omega) \wedge \lambda(\omega)$.

Robust (strong arbitrage), p. 36, Remark 3.3.1.

s, p. 40, Assumption 3.4.4, (3.4.8): some nonnegative number indicating the time.

$\mathbf{s}(\cdot)$, p. 31, (3.2.5): a measurable function: $\mathbb{R}^n_+ \to \mathrm{GL}(n)$.

$\mathbf{s}_{ik}(\cdot)$, p. 31, (3.2.5): the (i,k) entry of $\mathbf{s}(\cdot)$.

$\mathfrak{s}(\cdot,\cdot)$, p. 79: $(\mathfrak{s}_{ik}(\cdot,\cdot))_{n\times n}$.

$\mathfrak{s}_{ik}(t,\mathfrak{X})$, p. 79: $X_i(t)\sigma_{ik}(t,\mathfrak{X})$.

$\mathbb{S}(n)$, p. 38: the set of $n \times n$ real symmetric matrices.

$\mathbb{S}_+(n)$, p. 26: the set of real, symmetric, positive-definite $n \times n$ matrices.

$\sigma(\cdot,\cdot)$, p. 29: a progressively measurable functional: $[0,\infty) \times \Omega \to \mathrm{GL}(n)$.

$\sigma_{ik}(\cdot,\cdot)$, p. 29: the (i,k) entry of $\sigma(\cdot,\cdot)$.

$\sigma^{\mathcal{M}}(\cdot,\cdot)$, p. 30, Definition 3.2.1: the σ in $\mathcal{M}$.

Strong arbitrage, p. 35, Remark 3.3.1.

Strongly Markovian admissible system, p. 30, Definition 3.2.2.

t, p. 26: a nonnegative number indicating the time.

t_0, p. 39, (3.4.5): some positive number indicating the time.

t^*, p. 45, (3.5.8): some number in $(t_0 - \delta, t_0 + \delta)$.

T, p. 26: a nonnegative number indicating the time.

θ, p. 27: an element of $\mathbb{R}^n$.

$\boldsymbol{\theta}(\cdot)$, p. 31, (3.2.5): a measurable function: $\mathbb{R}^n_+ \to \mathbb{R}^n$.

$\boldsymbol{\theta}_k(\cdot)$, p. 31, (3.2.5): the k-th component of $\boldsymbol{\theta}(\cdot)$.

$\vartheta(\cdot,\cdot)$, p. 29: a progressively measurable functional: $[0,\infty) \times \Omega \to \mathbb{R}^n$.

$\vartheta_k(\cdot,\cdot)$, p. 29: the k-th component of $\vartheta(\cdot,\cdot)$.

$\vartheta^{\mathcal{M}}(\cdot,\cdot)$, p. 30, Definition 3.2.1: the ϑ in $\mathcal{M}$.

$\mathrm{Tr}(\cdot)$, p. 29, (3.2.2): the trace of a matrix.

$u(\cdot,\cdot)$, p. 27, (3.1.1): a function: $[0,\infty) \times \mathbb{R}^n_+ \to \mathbb{R}$.

$u_*(t,x)$, p. 39, (3.4.6): $\liminf_{(s,y)\to(t,x)} u(s,y)$, the lower-semicontinuous envelope of u.

$u^*(t,x)$, p. 39, (3.4.4): $\limsup_{(s,y)\to(t,x)} u(s,y)$, the upper-semicontinuous envelope of u.

$\mathfrak{u}(\cdot,\cdot)$, p. 34, (3.3.5): the arbitrage function.

$\mathfrak{u}_{\mathcal{M}}(\cdot,\cdot)$, p. 35, (3.3.8), (3.3.10): the arbitrage function for $\mathcal{M}$.

$U(\cdot,\cdot)$, p. 56, Theorem 3.7.1: a nonnegative supersolution of the Cauchy problem (3.1.1), (3.1.2).

$\mathcal{U}$, p. 57: the collection of positive supersolutions of the Cauchy problem (3.1.1), (3.1.2).

$\breve{\mathcal{U}}$, p. 57: some collection of continuous functions $\breve{U} : [0,\infty) \times \mathbb{R}^n_+ \to \mathbb{R}_+$.

Unbounded Profits with Bounded Risk (UPBR), p. 36, Remark 3.3.2.

Upper-semicontinuous envelope, p. 39, (3.4.4).

v, p. 32: the initial fortune.

$V(t,y)$, p. 59, Proposition 3.7.6, (3.7.2): $\mathfrak{u}_{\mathcal{M}^y}(t,y)$.

Viscosity (sub/super)solution, p. 38, Definition 3.4.1.

$W(\cdot)$, p. 30, Definition 3.2.1: an n-dimensional Brownian motion on $(\Omega,\mathcal{F},\mathbb{P}),\mathbb{F}$.

$W_k(\cdot)$, p. 30, Definition 3.2.1: the k-th component of $W(\cdot)$.

$W^{\mathcal{M}}(\cdot)$, p. 30: the W in $\mathcal{M}$.

x, p. 26: a vector in $\mathbb{R}^n_+$.

x_i, p. 26: the i-th component of x.

x_0, p. 38, (3.4.3): some element of $\mathcal{O}$.

$(x_0)_i$, p. 44: the i-th component of x_0.

x^*, p. 45, (3.5.8): some point in $B_\delta(x_0)$.

$||x||_1$, p. 27: $\sum_i x_i$.

$X(\cdot)$, p. 32, (3.3.1): $\sum_i X_i(\cdot)$.

$X_i(\cdot)$, p. 30, Definition 3.2.1, (3.2.4): the i-th component of $\mathfrak{X}(\cdot)$.

$X^{\mathcal{M}}(\cdot)$, p. 35, (3.3.9): $\sum_i X^{\mathcal{M}}_i(\cdot)$.

$X^{\mathcal{M}}_i(\cdot)$, p. 35: the i-th component of $\mathfrak{X}^{\mathcal{M}}(\cdot)$.

$\mathfrak{X}(\cdot)$, p. 30, Definition 3.2.1: asset capitalizations, a diffusion process.

$\mathfrak{X}^{\mathcal{M}}(\cdot)$, p. 30: the $\mathfrak{X}$ in $\mathcal{M}$.

$\Xi(t)$, p. 62, (3.8.2): $X(t)L(t)U\big(T-t,\mathfrak{X}(t)\big)$.

y, p. 30, Definition 3.2.2: a vector in $\mathbb{R}^n$.

y_i, p. 31, (3.2.6): the i-th component of y.

$Z(\cdot)$, p. 33, (3.3.3): $Z^{v,\Pi}(\cdot)$.

$Z^{v,\Pi}(\cdot)$, p. 33, (3.3.3): the wealth process starting with fortune v by using investment rule Π.

$\prime$, p. 30: transposition.

$\wedge$, p. 45, (3.5.12): min.