The complexity of two-point boundary-value problems with piecewise analytic data

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Abstract. Previous work on the ε-complexity of elliptic boundary-value problems $Lu = f$ assumed that the class F of problem elements f was the unit ball of a Sobolev space. In a recent paper, we considered the case of a model two-point boundary-value problem, with F being a class of analytic functions. In this paper, we ask what happens if F is a class of piecewise analytic functions. We find that the complexity depends strongly on how much a priori information we have about the breakpoints. If the location of the breakpoints is known, then the ε-complexity is proportional to $\ln(\varepsilon^{-1})$, and there is a finite element p-method (in the sense of Babuška) whose cost is optimal to within a constant factor. If we know neither the location nor the number of breakpoints, then the problem is unsolvable for $\varepsilon < \sqrt{2}$. If we know only that there are $b \geq 2$ breakpoints, but we don't know their location, then the ε-complexity is proportional to $b\varepsilon^{-1}$, and a finite element h-method is nearly optimal. In short, knowing the location of the breakpoints is as good as knowing that the problem elements are analytic, whereas only knowing the number of breakpoints is no better than knowing that the problem elements have a bounded derivative in the L_2 sense.

1. INTRODUCTION

Most work on the ε-complexity of elliptic boundary-value problems $Lu = f$ has assumed that the class F of problem elements f consisted of functions whose smoothness was fixed and known, see, e.g., [6]. In particular, if F is the unit ball of a Sobolev space, then $\text{comp}(\varepsilon)$ is a power of ε^{-1}; moreover, we found conditions that are necessary and sufficient for a finite element h-method[1] to be (almost) optimal.

Unfortunately, assuming that F is the unit ball of a Sobolev space of fixed smoothness means that we must know the smoothness in advance. In practice, this may often be difficult. One possible way around this problem is to note that problem elements are often either analytic or piecewise analytic. If we restrict ourselves to such f, then we don't have to worry so much about quantifying the exact smoothness of f. Moreover, any lack of smoothness can be confined to a small set of points.

In an earlier paper [7], we looked at the case of analytic F for a simple model two-point boundary-value problem. These results were encouraging. Rather than depending on a power of ε^{-1}, we found that the ε-complexity was proportional to $\ln(\varepsilon^{-1})$ or to $\ln^2(\varepsilon^{-1})$, depending on whether or not there was "breathing room" between the domain on which the problem was defined and the interior of the domain of analyticity of

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[1] Here we use the widely-used classification of finite element methods that was introduced by Babuška and his colleagues:

(1) h-methods, in which the degree of the finite element method is held fixed and the partition varies (these are the usual finite element methods),
(2) p-methods, in which the partition is fixed and the degree is allowed to vary,
(3) (h, p)-methods, in which the partition and degree are both allowed to vary.

See [1] for further discussion.

the problem elements. Moreover, we found finite element methods (FEMs) for computing ε-approximations, whose costs were within a constant factor of the ε-complexity. In the case where there was breathing room between the domains, this FEM was a p-method; in the case where there was no breathing room, this FEM was an (h, p)-method.

In this paper, we consider the case where F consists of piecewise analytic functions. We will assume that the pieces of a piecewise analytic function belong to a common class of analytic functions with "breathing room." Our piecewise analytic classes may then be defined in several ways, mainly differing in how much we know about their breakpoints. We will analyze three such classes.

(1) Suppose we know the locations of the breakpoints. In this case, the complexity is roughly the same as when the problem elements are analytic, i.e., the ε-complexity is proportional to $\ln(\varepsilon^{-1})$, and a finite element p-method is nearly optimal.
(2) We next assume that the location of the breakpoints is unknown, in which case we either know or don't know how many breakpoints there are.
    (a) If we don't know the number of breakpoints, then the problem is unsolvable, i.e., we cannot find an ε-approximation for any $\varepsilon < \sqrt{2}$.
    (b) If we know that there are (at most) $b \geq 2$ breakpoints, then the ε-complexity is proportional to $b\varepsilon^{-1}$, and a finite element h-method is nearly optimal.

We briefly comment on this last subcase. It tells us that if we know how many breakpoints there are (but not their location) and that there are at least two of them, then the assumption that the problem elements are piecewise analytic is not much better than the assumption that they have a bounded derivative in the L_2 sense (see [6, Section 5.5]. In short, piecewise analyticity buys us very little if there are more than two pieces. Note that the case of one breakpoint whose location is unknown is still open.

We now outline the contents of this paper. In Section 2, we precisely describe the problem to be solved, including a definition of the three classes of piecewise analytic functions. In Section 3, we consider the case where the breakpoints are known in advance. Finally, in Section 4, we consider the cases where the location of the breakpoints is not known in advance, considering the two subcases of whether or not the number of breakpoints is known.

2. Problem description

Let $I = (-1, 1)$. In what follows, we use the standard notations and definitions for Sobolev spaces of functions defined on I, as well as Sobolev norms, seminorms, and inner products. See the appendix of [6] and the references found therein for further details. We use one slightly nonstandard notation; namely, we define

$$\|f\|_{H^{-1}(I)} = \sup_{v \in H^1(I)} \frac{\langle f, v\rangle_{L_2(I)}}{\|v\|_{H^1(I)}}.$$

That is, we consider $H^{-1}(I)$ to be the dual space of $H^1(I)$, rather than of $H_0^1(I)$. See [6, pg. 127] for further discussion.

Define a bilinear form B on $H^1(I)$ by

$$B(u, v) = \int_I u'v' + uv \qquad \forall\, u, v \in H^1(I).$$

We let F be one of several classes of piecewise analytic functions defined on I, which will be defined in the sequel. Then for $f \in F$, we seek $Sf \in H^1(I)$ satisfying

$$B(Sf, v) = \langle f, v\rangle_{L_2(I)} \qquad \forall\, v \in H^1(I).$$

It is a standard result that $u = Sf$ is the variational solution of the two-point boundary-value problem

$$\begin{aligned} -u'' + u &= f \qquad \text{in } I \\ u'(-1) &= u'(1) = 0 \end{aligned} \tag{2.1}$$

with natural boundary conditions.

We now describe F, the set of piecewise analytic functions that will be our class of problem elements. Given $\rho > 1$, we let D_ρ denote the open disk in the complex plane with radius ρ centered at the origin, and let $F(D_\rho)$ be the set of real-valued functions on I having an analytic extension to D_ρ, this extension being bounded by 1 on D_ρ. Let us say that $\mathrm{b} = \{\beta_1, \dots, \beta_b\}$ is a set of *breakpoints* for a piecewise analytic function $f : D_\rho \to \mathbb{C}$ if there exist functions $f_1, \dots, f_b \in F(D_\rho)$ such that $f|_{(\beta_{s-1}, \beta_s)} = f_s$ for $1 \le s \le b+1$, with $\beta_0 = -1$ and $\beta_{b+1} = 1$.

Then we will let F be any of the following function classes:

(1) The class $F_{\rho,\mathrm{b}}$ of functions with a *known* set $\mathrm{b} = \{\beta_1, \dots, \beta_b\}$ of breakpoints. That is, b is the set of breakpoints for all $f \in F_{\rho,\mathrm{b}}$.

(2) The class $F_{\rho,*}$ of functions whose breakpoints are *unknown*. This means that for any $f \in F_{\rho,*}$, there exists $b = b(f)$ and a set $\mathrm{b} = \{\beta_1, \dots, \beta_b\}$ of breakpoints for f.

(3) The class $F_{\rho,b}$ of functions having at most b breakpoints whose locations are unknown. This means that for any $f \in F_{\rho,b}$, there exists a set $\mathrm{b} = \{\beta_1, \dots, \beta_b\}$ of b breakpoints for f. Note that $F_{\rho,0} \subset F_{\rho,1} \subset \dots$.

We assume that only standard information is available for solving our problem. Thus, information has the form

$$Nf = [f(x_1), \dots, f(x_{n(f)})],$$

where the number $n(f)$ and choice $x_1, \dots, x_{n(f)}$ of sample points may be determined adaptively. (See [5, Chapter 3] for further discussion.)

Our model of computation is the standard one given in [5]. The evaluation of any function f from F at any point in I has cost c, and the cost of basic combinatory operations is 1. Typically, $c \gg 1$.

In this paper, we consider the worst case setting. Hence, the error of any algorithm ϕ using information N is given by

$$e(\phi, N, F) = \sup_{f \in F} \|Sf - \phi(Nf)\|_{H^1(I)}.$$

The radius of information N is

$$r(N, F) = \inf_{\phi} e(\phi, N, F),$$

and the nth minimal radius is

$$r(n, F) = \inf\{\, r(N, F) : \operatorname{card} N \le n \,\}.$$

The cost of an algorithm ϕ using N is given by

$$\operatorname{cost}(\phi, N, F) = \sup_{f \in F} \operatorname{cost}(\phi, N, f),$$

with $\operatorname{cost}(\phi, N, f)$ denoting the cost of computing ϕ for a particular problem element f. As always, the ε-complexity

$$\operatorname{comp}(\varepsilon, F) = \inf\{\, \operatorname{cost}(\phi, N, F) : e(\phi, N, F) \le \varepsilon \,\}$$

of our problem is the minimal cost of computing an ε-approximation, for $\varepsilon \ge 0$.

3. Breakpoints Known

In this section, we consider the case $F = F_{\rho,\mathbf{b}}$. That is, we know the breakpoints in advance. We will show that the nth minimal radius decreases exponentially with n, and that a there is a finite element method (FEM) using n evaluations whose error does decrease exponentially with n. Since we will also be using FEMs in the next section of this paper when we discuss the case $F = F_{\rho,b}$, we first describe the FEM in terms of an unspecified partition Δ and degree k. Then, we later give a specific choice of Δ and k for the classes $F = F_{\rho,\mathbf{b}}$ and $F = F_{\rho,b}$.

Recall that the FEM is described as follows. Choose a partition $\Delta = \{x_0, \dots, x_m\}$ of I, with $-1 = x_0 < x_1 < \cdots < x_m = 1$. For $1 \le i \le m$, we write $\Delta_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$. Choose $k \in \mathbb{Z}$. The spline space $\mathscr{S}_{k,\Delta}$ is defined to be the set of all $v \in C(I)$ such that $v\big|_{\Delta_i} \in \mathscr{P}_k$ for $1 \le i \le m$, where $\mathscr{P}_k$ is the space of polynomials of degree at most k. Letting $n = \dim \mathscr{S}_{k,\Delta}$, we choose a basis $\{s_1, \dots, s_n\}$ for $\mathscr{S}_{k,\Delta}$. There exist points $t_1, \dots, t_n$ such that $s_j(t_i) = \delta_{i,j}$ for $1 \le i, j \le n$. Then the $\mathscr{S}_{k,\Delta}$-interpolation operator $\Pi_{k,\Delta}$ is defined by

$$\Pi_{k,\Delta} f = \sum_{j=1}^{n} f(t_j) s_j.$$

For $f \in F$, we find $u_n \in \mathscr{S}_{k,\Delta}$ for which

$$B(u_n, s) = \langle \Pi_{k,\Delta} f, s \rangle \qquad \forall s \in \mathscr{S}_{k,\Delta}.$$

It is easy to check that u_n is well-defined, and that we can write

$$u_n = \phi_{n,k,\Delta}(N_{n,k,\Delta} f),$$

where

$$N_{n,k,\Delta} f = [f(t_1), \dots, f(t_n)].$$

The algorithm $\phi_{n,k,\Delta}$ is the *finite element method* (FEM) of degree k over Δ, and $N_{n,k,\Delta}$ is the *finite element information* (FEI) that $\phi_{n,k,\Delta}$ uses. (For further information on FEMs, see [2] and [4], as well as the references cited in [6].)

A standard error bound is given by

$$\|Sf - u_n\|_{H_1(I)} \le \inf_{s \in \mathscr{S}_{k,\Delta}} \|Sf - s\|_{H^1(I)} + \|f - \Pi_{k,\Delta} f\|_{H^{-1}(I)}. \tag{3.1}$$

See the proof of [6, Theorem 5.7.4] for details.

We now define the sample points $t_1, \dots, t_n$. Let $y_1, \dots, y_{k+1}$ be the zeros of the Legendre polynomial P_{k+1}. Set

$$\tau_{i,j} = \tfrac{1}{2} h_i (1 + y_j) + x_{i-1} \qquad (1 \le j \le k+1).$$

We then let

$$t_{(i-1)(k+1)+j} = \tau_{i,j} \qquad (1 \le j \le k+1, 1 \le i \le m).$$

so that $\tau_{i,1}, \dots, \tau_{i,k+1}$ are the sample points belonging to Δ_i. The dimension n of our spline space $\mathscr{S}_{k,\Delta}$ is

$$n = (k+1)m - (m-1) = km + 1,$$

because functions in $\mathscr{S}_{k,\Delta}$ must be continuous.

Now that we have given a general description of the FEM of degree k over the partition Δ, we need to specify k and Δ for our class $F = F_{\rho,\mathbf{b}}$ of problem elements. Our space $\mathscr{S}_{k,\Delta}$ must be chosen in such a way that the breakpoints for $F_{\rho,\mathbf{b}}$ are partition points for Δ. This means that Δ must be chosen so that $\beta_s = x_{l_s}$

for indices $l_1 < \cdots < l_b$. (Since $\beta_0 = -1 = x_0$ and $\beta_{b+1} = 1 = x_m$, we also let $l_0 = 0$ and $l_{b+1} = m$.) Let $\eta_s = \beta_s - \beta_{s-1}$ denote the size of the sth breakpoint subinterval for $1 \le s \le b$. Then

$$h_i = \frac{\eta_s}{l_s - l_{s-1} + \delta_{s,0} - 1} \tag{3.2}$$

for all indices i such that $x_i \in [\beta_{s-1}, \beta_s]$. Next, we want the set of grids to be quasi-uniform in n. This means that we need to choose the indices $l_1, \ldots, l_b$ so that $h_i \approx h$ for $1 \le i \le m$. From (3.2), we see that

$$l_s = l_{s-1} + \frac{\eta_s}{h} \qquad (1 \le s \le b+1),$$

whose solution is

$$l_s = \frac{1}{h}\sum_{i=1}^{s} \eta_i = \frac{1}{h}(\beta_s + 1) \qquad (1 \le s \le b+1).$$

Letting $s = b + 1$, we get $h = 2/m$, and so

$$l_s = \tfrac{1}{2}m(\beta_s + 1) \qquad (1 \le s \le b+1). \tag{3.3}$$

We are now ready to define our information and algorithm. Choose

$$\begin{aligned} m &= \left\lceil \frac{e}{2(\rho - 1)} \right\rceil, \\ k &= \left\lceil \frac{n-1}{m} \right\rceil. \end{aligned} \tag{3.4}$$

We then have

THEOREM 3.1.

(1) *Let* $q_1 = (\rho + \sqrt{\rho^2 + 1})^2$. *Then*

$$r(n, F_{\rho,\mathrm{b}}) = \Omega(q_1^{-n}) \qquad \text{as } n \to \infty.$$

(2) *Let* $N_{n,k,\Delta}$ *and* $\phi_{n,k,\Delta}$ *be the FEI and FEM determined by the parameters (3.2), (3.3), and (3.4). Then*

$$e(\phi_{n,k,\Delta}, N_{n,k,\Delta}) = O(q_2^{-n}) \qquad \text{as } n \to \infty$$

for any $q_2 < \exp\big(2(\rho - 1)/e\big)$.

PROOF: To see that the lower bound in part (1) holds, note that $F(D_\rho) \subset F_{\rho,\mathrm{b}}$, and so

$$r(n, F_{\rho,\mathrm{b}}) \ge r(n, F(D_\rho)) = \Omega(q_1^{-n}) \qquad \text{as } n \to \infty,$$

the latter by [7, Theorem 3.1].

We now turn to the upper bound in part (2). Choose $f \in F$, and let $u = Sf$. Write $e = u - \Pi_{k,\Delta}u$ and $\tilde{e} = f - \Pi_{k,\Delta}f$. Note that f and u are analytic on each subinterval (β_{s-1}, β_s). Thus for $j = 0$ and $j = 1$, we have

$$\|e^{(j)}\|^2_{L_2(I)} = \sum_{s=1}^{b+1} \|e^{(j)}\|^2_{L_2(\beta_{s-1},\beta_s)} = \sum_{s=1}^{b+1} \sum_{i=l_{s-1}+1}^{l_s} \|e^{(i)}\|^2_{L_2(\Delta_i)}.$$

From Lemmas A.1, A.2, and A.4 of [7], we have

$$\|e\|^2_{L_2(\Delta_i)} \le 2\pi \frac{M^2}{(k+1)^2}(\rho-1)^5 \left(\frac{h_i}{4(\rho-1)}\right)^{2k+3}$$

and

$$\|e'\|^2_{L_2(\Delta_i)} \le \pi \left(\frac{h_i}{4(\rho-1)}\right)^{2k+1} \left[\frac{M^2}{(k+2)^2}(\rho-1) + \frac{M^2}{(k+1)^2}(\rho-1)^5(k+\tfrac{3}{2})^3\right].$$

Similarly, we have

$$\|\tilde{e}\|^2_{L_2(I)} = \sum_{s=1}^{b+1} \sum_{i=l_{s-1}+1}^{l_s} \|\tilde{e}\|^2_{L_2(\Delta_i)},$$

with

$$\|\tilde{e}\|^2_{L_2(\Delta_i)} \le 2\pi(\rho-1)\left(\frac{h_i}{4(\rho-1)}\right)^{2k+3}$$

by Lemmas A.1 and A.3 of [7]. Combining these results and using (3.1), we find

$$\|u-u_n\|_{H^1(I)} = O(\sqrt{km})\left(\frac{h}{4(\rho-1)}\right)^{k+1/2} = O(\sqrt{k})\left(\frac{h}{4(\rho-1)}\right)^{k},$$

the latter since $m = 2/h$. Using (3.4), we see that

$$\|u-u_n\|_{H^1(I)} = O(\sqrt{n})\exp(-2rn/e).$$

Since $f \in F$ is arbitrary, the desired conclusion follows. □

We remark that since $\exp(-2/e) \doteq 0.479$, we find that $e(\phi_{n,k,\Delta}, N_{n,k,\Delta}) = O(2^{-(\rho-1)n})$.

Note that since $q_1 > q_2$, the ratio of the upper bound in part (2) of Theorem 3.1 to the lower bound in part (1) is not bounded by a constant. Hence there is a gap between the estimates provided by these bounds. Despite this, we can determine the ε-complexity to within a constant factor.

Suppose that F is a class of problem elements such that the following hold:

(1) For any information N, there exists a *linear optimal error algorithm* using N. That is, if

$$Nf = [f(x_1), \dots, f(x_n)],$$

then there exist functions $v_1, \dots, v_n \in H^1(I)$, which may be computed in advance, such that the *linear algorithm* ϕ^{L} given by

$$\phi^{\mathrm{L}}(Nf) = \sum_{j=1}^{n} f(x_j)v_j \tag{3.5}$$

is an *optimal error algorithm* using N, i.e.,

$$e(\phi^{\mathrm{L}}, N, F) = r(N, F).$$

(2) We do not charge for precomputation, i.e., calculations that may be done in advance, independent of any $f \in F$. In particular, this means that we do not charge for determining the functions $v_1, \dots, v_n$ in (3.5) that characterize the linear optimal error algorithm using N.

Then

$$\text{comp}(\varepsilon, F) = \Theta\big(c\, m(\varepsilon, F)\big) \qquad \text{as } \varepsilon \to 0.$$

where the ε-cardinality number is given by

$$m(\varepsilon, F) = \inf\{\, n \in \mathbb{Z} : r(n) \le \varepsilon \,\}.$$

For further discussion and details, see [5, Chapter 4].

Let

$$\text{cost}^{\text{FE}}(\varepsilon, F) = \inf\{\, \text{cost}(\phi_{n,k,\Delta}, N_{n,k,\Delta}) : e(\phi_{n,k,\Delta}, N_{n,k,\Delta}) \le \varepsilon \,\}$$

denote the minimal cost of using an FEM to compute an ε-approximation. From the discussion in the previous paragraph and Theorem 3.1, we immediately have

COROLLARY 3.1.

(1) *The ε-complexity satisfies*

$$\text{comp}(\varepsilon, F_{\rho,\text{b}}) = \Theta\big(c \ln(\varepsilon^{-1})\big) \qquad \text{as } \varepsilon \to 0.$$

(2) *Let $\phi_{n,k,\Delta}$ be the FEM using FEI $N_{n,k,\Delta}$ of cardinality*

$$n \sim \frac{\ln \varepsilon^{-1}}{\ln q_2}$$

and whose degree k and partition Δ are determined by (3.2), (3.3), and (3.4). Then

$$e(\phi_{n,k,\Delta}, N_{n,k,\Delta}, F_{\rho,\text{b}}) \le \varepsilon$$

and

$$\text{cost}^{\text{FE}}(\phi_{n,k,\Delta}, N_{n,k,\Delta}, F_{\rho,\text{b}}) = \Theta\big(c \ln(\varepsilon^{-1})\big) \qquad \text{as } \varepsilon \to 0$$

as $\varepsilon \to 0$.

The Θ-constants that appear here are independent of the number and the locations of the (known) breakpoints. □

Hence, the finite element p-method described in Corollary 3.1 is *quasi-optimal*, i.e., its cost is within a constant factor of being optimal.

4. BREAKPOINTS UNKNOWN

In the previous section, we showed that when we know the breakpoints, the complexity for piecewise analytic functions is roughly the same as that for analytic functions. We now look at what happens when the breakpoints are unknown.

We first suppose that $F_{\rho,*}$ is our class of problem elements. That is, for any problem element f, there is an unknown set of breakpoints. For any information N, we define the *zero algorithm* ϕ_0 as

$$\phi_0(Nf) \equiv 0 \qquad \forall f \in F_{\rho,*}.$$

THEOREM 4.1. *For any $n \in \mathbb{Z}$,*

$$r(n, F_{\rho,*}) = \sqrt{2},$$

and the zero algorithm is an nth minimal error algorithm.

PROOF: Let

$$Nf = [f(x_1), \dots, f(x_{n(f)})] \qquad \forall f \in F_{\rho,*}$$

be information (possibly adaptive) of cardinality n. Let $t_1, \dots, t_k$ be the N-evaluation points for the zero function $f \equiv 0$, so that $k = n(0) \le n$. Choosing small $\alpha > 0$, we let

$$h_\alpha(x) = \begin{cases} 0 & \text{if } |x - t_i| < \alpha \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

Then $h_\alpha \in F_{\rho,*}$ and $Nh_\alpha = 0$. Since $S1 = 1$, we may use [5, Chapter 4], to see that

$$r(N, F_{\rho,*}) \ge \|Sh_\alpha\|_{H^1(I)} \ge \|1\|_{H^1(I)} - \|S(1 - h_\alpha)\|_{H^1(I)}.$$

Since

$$\|S(1 - h_\alpha)\|_{H^1(I)} \le \|1 - h_\alpha\|_{H^{-1}(I)} \le \|1 - h_\alpha\|_{L_2(I)} \le \sqrt{2\alpha k}$$

and

$$\|1\|_{H^1(I)} = \sqrt{2},$$

we have

$$r(N, F_{\rho,*}) \ge \sqrt{2} - \sqrt{2\alpha k} \qquad \forall \alpha > 0.$$

Since α can be chosen arbitrarily small and $k \le n$, we have

$$r(N, F_{\rho,*}) \ge \sqrt{2}.$$

Since N is arbitrary adaptive information of cardinality n, we have

$$r(n, F_{\rho,*}) \ge \sqrt{2}. \tag{4.1}$$

Now consider the zero algorithm ϕ_0. We have

$$\|Sf - \phi_0(Nf)\|_{H^1(I)} = \|Sf\|_{H^1(I)} = \|f\|_{H^{-1}(I)} \le \|f\|_{L_2(I)} \le \|f\|_{C(I)}\sqrt{|I|} \le \sqrt{2}.$$

Since $f \in F_{\rho,*}$ is arbitrary we have

$$e(\phi_0, N, F_{\rho,*}) \le \sqrt{2}. \tag{4.2}$$

The theorem now follows from (4.1) and (4.2). □

From Theorem 4.1, we immediately find

COROLLARY 4.1. *The ε-complexity satisfies*

$$\text{comp}(\varepsilon, F_{\rho,*}) = \begin{cases} 0 & \text{for } \varepsilon \ge \sqrt{2}, \\ \infty & \text{for } \varepsilon < \sqrt{2}. \end{cases}$$ □

Hence if our class of problem elements is a family of piecewise analytic functions with a set of unknown breakpoints, we do not have enough knowledge about our problem to compute an ε-approximation unless $\varepsilon \ge \sqrt{2}$. This means that we need to have additional knowledge about our problem class. So, we now consider the case where our problem class is $F_{\rho,b}$. That is, we know that there are b breakpoints, but we don't know where they are.

THEOREM 4.2. *For any $b \geq 2$,*

$$r(n, F_{\rho,b}) = \Omega(bn^{-1}).$$

The constant is independent of b and n.

PROOF: Let

$$Nf = [f(x_1), \ldots, f(x_{n(f)})] \qquad \forall f \in F_{\rho,*}$$

be information (possibly adaptive) of cardinality n. Let $t_1, \ldots, t_k$ be the resulting N-evaluation points for the function $f \equiv 1$, so that $k = n(1) \leq n$. Without loss of generality, we assume that $t_0 := -1 \leq t_1 < \cdots < t_k \leq t_{k+1} := 1$. Write $h_i = t_{i+1} - t_i$ for $0 \leq i \leq k$. Choose indices $i_1, \ldots, i_{k+1}$ so that $h_{i_1} \geq h_{i_2} \geq \cdots \geq h_{i_{k+1}} > 0$. For small $\delta > 0$, let

$$\tilde{f}_\delta = \begin{cases} -1 & \text{in } \bigcup_{s=1}^{\lfloor b/2 \rfloor}[t_{i_s} + \delta, t_{i_{s+1}} - \delta], \\ 1 & \text{otherwise.} \end{cases}$$

Since $f, \tilde{f}_\delta \in F_{\rho,b}$ with $Nf = N\tilde{f}_\delta$, we may use [5, Chapter 4] to find

$$2r(N, F_{\rho,b}) \geq d(N, F_{\rho,b}) \geq \|Sf - S\tilde{f}_\delta\|_{H^1(I)} = \|f - \tilde{f}_\delta\|_{H^{-1}(I)} \geq \int_I (f - \tilde{f}_\delta). \tag{4.3}$$

Since

$$f - \tilde{f}_\delta = \begin{cases} 2 & \text{in } \bigcup_{s=1}^{\lfloor b/2 \rfloor}[t_{i_s} + \delta, t_{i_{s+1}} - \delta], \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\int_I (f - \tilde{f}_\delta) = \sum_{s=1}^{\lfloor b/2 \rfloor} \int_{t_{i_s}+\delta}^{t_{i_s+1}-\delta} 2\,dt = 2\sum_{s=1}^{\lfloor b/2 \rfloor}(h_{i_s} - 2\delta). \tag{4.4}$$

Since $\delta > 0$ can be chosen arbitrarily small, we may combine (4.3) and (4.4) to find that

$$r(N, F_{\rho,b}) \geq \sum_{s=1}^{\lfloor b/2 \rfloor} h_{i_s}.$$

Since $h_{i_1} \geq h_{i_2} \geq \cdots \geq h_{i_{k+1}} > 0$ and $\sum_{s=1}^{k+1} h_{i_s} = 2$, it is easy to see that

$$\sum_{s=1}^{\lfloor b/2 \rfloor} h_{i_s} \geq \frac{2}{\lceil n/\lfloor b/2 \rfloor \rceil} = \Theta(bn^{-1}).$$

The desired conclusion now follows from these last two inequalities. □

We now show that this lower bound is sharp. Consider the FEM with

$$\begin{aligned} m &= n - 1, \\ h_i \equiv h &= \frac{2}{m} \qquad (1 \leq i \leq m). \\ k &= 1. \end{aligned} \tag{4.5}$$

Hence our spline space is $\mathscr{S}_{1,\Delta}$, a space of continuous piecewise linear functions on a uniform grid.

THEOREM 4.3. *Let $N_{n,1,\Delta}$ and $\phi_{n,1,\Delta}$ be the FEI and FEM determined by the parameters (4.5). Then there is a positive constant C, independent of b and n, such that*

$$e(\phi_{n,1,\Delta}, N_{n,1,\Delta}, F_{\rho,b}) \le Cbn^{-1}$$

for any positive integers b and n.

PROOF: In what follows, C_1, C_2, and C_3 will denote positive constants, independent of n, b, and any particular problem element $f \in F_{\rho,b}$.

Let $f \in F_{\rho,b}$, and let $\{\beta_1, \ldots, \beta_b\}$ be the set of breakpoints for f. Suppose that $\beta_s \in \Delta_{j_s}$ for $1 \le s \le b$. From the proof of [6, Theorem 5.7.4], we find that

$$\|Sf - \phi_{n,1,\Delta}(N_{n,1,\Delta}f)\|_{H^1(I)} \le \|\tilde{e}\|_{H^{-1}(I)} + O(n^{-1}),$$

where $\tilde{e} = f - \Pi_{n,1,\Delta}f$ and the O-constant is independent of n and b. Since $f \in F_{\rho,b}$ is arbitrary, we will be done once we show that

$$\|\tilde{e}\|_{H^{-1}(I)} = O(n^{-1}). \tag{4.6}$$

To prove (4.6), let $v \in H^1(I)$ with $\|v\|_{H^1(I)}$. Then

$$\left|\int_I \tilde{e}(x)v(x)\,dx\right| \le \sum_{s=1}^{b}\left|\int_{\Delta_{j_s}} \tilde{e}(x)v(x)\,dx\right| + \sum_{i\notin\{j_1,\ldots,j_b\}}\left|\int_{\Delta_i} \tilde{e}(x)v(x)\,dx\right|. \tag{4.7}$$

From the proof of [6, Theorem 5.7.4], we find that

$$\sum_{s=1}^{b}\left|\int_{\Delta_{j_s}} \tilde{e}(x)v(x)\,dx\right| \le b\|e\|_{L_\infty(I)} \sup_{0\le a\le 1-h}\int_a^{a+h} |v(x)|\,dx.$$

$$\|e\|_{L_\infty(I)} \le C_1\|f\|_{L_\infty(I)} \le C_1,$$

$$\sup_{0\le a\le 1-h}\int_a^{a+h} |v(x)|\,dx \le \sqrt{\frac{4}{3}}h.$$

Hence we have

$$\sum_{s=1}^{b}\left|\int_{\Delta_{j_s}} \tilde{e}(x)v(x)\,dx\right| \le \sqrt{\frac{16}{3}}\frac{C_1 b}{n+1}. \tag{4.8}$$

On the other hand, if $i \notin \{j_1, \ldots, j_b\}$, then

$$\left|\int_{\Delta_i} \tilde{e}(x)v(x)\,dx\right| \le \|\tilde{e}\|_{L_2(\Delta_i)}\|v\|_{L_2(\Delta_i)} \le \|\tilde{e}\|_{L_2(\Delta_i)}.$$

From [6, Lemma A.2.3.3], we have

$$\|\tilde{e}\|_{L_2(\Delta_i)} \le C_2 n^{-2}\|f''\|_{L_2(\Delta_i)},$$

while from [7, Lemma A.3], we have

$$\|f''\|_{L_2(\Delta_i)} \le \rho^{-2}.$$

Hence

$$\sum_{i\notin\{j_1,\ldots,j_b\}}\left|\int_{\Delta_i} \tilde{e}(x)v(x)\,dx\right| \le Cn^{-1}\rho^{-2}. \tag{4.9}$$

From (4.7), (4.8) and (4.9), we find that

$$\left|\int_I \tilde{e}(x)v(x)\,dx\right| \le \left(\sqrt{\frac{16}{3}}C_1 b + C_3\rho^{-2}\right)n^{-1}.$$

Since v is an arbitrary function in the unit ball of $H^1(I)$, the bound (4.6) holds as claimed, completing the proof of the theorem. □

From Theorems 4.2 and 4.3, we immediately find

COROLLARY 4.2. *Let $b \geq 2$.*

(1) *The ε-complexity satisfies*

$$\mathrm{comp}(\varepsilon, F_{\rho,b}) = \Theta(b\varepsilon^{-1}) \qquad \text{as } \varepsilon \to 0,$$

the Θ-constant being independent of b and ε.

(2) *Let $\phi_{n,1,\Delta}$ be the FEM using FEI $N_{n,1,\Delta}$ of cardinality*

$$n = \left\lceil \frac{Cb}{\varepsilon} \right\rceil$$

(where C is as in the statement of Theorem 4.3), and whose partition Δ is determined by (4.5). Then

$$e(\phi_{n,1,\Delta}, N_{n,1,\Delta}, F_{\rho,b}) \leq \varepsilon$$

and

$$\mathrm{cost}^{\mathrm{FE}}(\phi_{n,1,\Delta}, N_{n,1,\Delta}, F_{\rho,b}) = \Theta(b\varepsilon^{-1})) \qquad \text{as } \varepsilon \to 0,$$

the Θ-constant being independent of b and ε. □

Thus, we have shown that if our class of piecewise analytic functions has a fixed number b of breakpoints, the ε-complexity is proportional to $b\varepsilon^{-1}$ if $b \geq 2$. Moreover, the FEM described in Corollary 4.2 is a quasi-optimal algorithm. Note that this FEM is an h-method, i.e., we decrease its error by decreasing the mesh size.

As a final remark, we point out that the proof of Theorem 4.2 depends on the assumption that $b \geq 2$. We do not know the complexity of our problem when $b = 1$. The best upper bound known for the ε-complexity is proportional to ε^{-1}, while the best lower bound known is proportional to $\ln(\varepsilon^{-1})$. Hence there is a huge gap in our knowledge of the complexity for the case of one breakpoint.

BIBLIOGRAPHY

1. Babuška, I., *Recent progress in the p and (h, p)-versions of the finite element method*, Numerical analysis 1987, (D. F. Griffiths and G. A. Watson, eds.), Wiley, New York, 1988, pp. 1–17.
2. Becker, E. B., Carey, G. F., Oden, J. T., *Finite Elements*, Prentice-Hall, Englewood Cliffs, N.J., 1981.
3. Bojanov, B. D., *Optimal rate of integration and ε-entropy of a class of analytic functions*, Math. Notes 19 (1973), 551–556.
4. Ciarlet, P. G., *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.
5. Traub, J. F., Wasilkowski, G. W., and Woźniakowski, H., *Information-Based Complexity*, Academic Press, New York, 1988.
6. Werschulz, A. G., *The Computational Complexity of Differential and Integral Equations: An Information-Based Approach*, Oxford University Press, Oxford, 1991.
7. Werschulz, A. G., *The complexity of two-point boundary-value problems with analytic data*, Journal of Complexity 9 (1993) (to appear).