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Abstract

This paper studies optimal nonlinear pricing for a monopolist when consumers' preferences exhibit temptation and self-control as in Gul and Pesendorfer (2001a). Consumers are subject to temptation inside the store but exercise self-control, and those foreseeing large self-control costs do not enter the store. Consumers differ in their preferences under temptation. When all consumers are tempted by more expensive, higher quality choices, the optimal menu is a singleton, which saves consumers from self-control and extracts consumers' commitment surplus. When some consumers are tempted by cheaper, lower quality choices, the optimal menu may contain a continuum of choices.

JEL Classification: D42, D82, L12, L15.

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1 Introduction

Experimental studies often observe "preference reversals" whereby agents' observed actions appear to deviate systematically from stated intentions even when the external environment is closely controlled (for surveys of this evidence, see Ainslie (1992) and Frederick, Loewenstein, and O'Donoghue (2002)). In a series of papers, Gul and Pesendorfer (2000, 2001a, 2001b; hereafter GP) introduce *temptation preferences with self-control*, or *self-control preferences* for short, to rationalize these behavioral anomalies within a time-consistent decision-theoretic framework.

If consumers exhibit preference reversals in market situations, a firm may take advantage of the reversals when it determines the selection of goods to sell and their prices, perhaps by leading consumers into temptation or by lowering costs that consumers incur from self-control. To address the issue, this paper studies the optimal pricing problem for a monopolist that faces a population of heterogeneous consumers with self-control preferences. As such, we build on the literature of nonlinear pricing, and the work of GP, to study the supply-side of markets in which consumers' preferences exhibit reversals caused by temptation and self-control.[1]

Our question has a relevance for the design and pricing of product-lines. For instance, consider car dealerships. Some consumers know that once they enter the car dealership and see (and test-drive) a variety of models, they may be tempted to purchase more expensive, higher quality models than what they originally intended to purchase. Some of these consumers wholly cave-in to temptation and purchase the car they are tempted by; some others stick to their original intention but have to exercise self-control in the process. There are also consumers who succumb to temptation to some degree, but not completely, and choose a compromise between what they are tempted by and their original intentions, perhaps by purchasing the model they initially intended to purchase but adding a few optional features. If a consumer anticipates large self-control costs, he may even decide not to go to the dealership at all. Furthermore, there may be consumers who are tempted in the other direction. For example, once these consumers enter the store, they may become overly frugal, feel guilty about spending so much on a car, and end up either not purchasing any car at all or purchasing a less expensive, lower quality car.

[1] There are several papers that consider GP's self-control preferences to address different questions. Krusell, Kuruşcu, and Smith (2000) consider a competitive neoclassical growth model with GP preferences to study the problem of taxation. Krusell, Kuruşcu, and Smith (2002) analyze a general-equilibrium asset pricing model where consumers are tempted to save in the short run. DeJong and Ripoll (2003) analyze whether GP preferences in an asset pricing model can explain the behavior of stock prices. Ameriks, Caplin, Leahy, and Tyler (2003) use survey data to test for GP self-control preferences within a two-period allocation model.

Given these preferences of consumers, how should the car dealership price its products? Should it only stock tempting cars and exploit the consumers' preference reversals? Should it only offer a small number of models to reduce consumers' self-control costs? The results that we derive in this paper can give some insight into these questions.

To study preferences with temptation and self-control, GP axiomatically derive a class of preferences for which a subset of alternatives is strictly preferred to the entire set. Whether the consumer chooses a tempting alternative or not depends on whether he exercises self-control or succumbs to temptation. A consumer who exercises self-control avoids the tempting alternative, but incurs a utility cost from self-control. A consumer who succumbs to temptation chooses the tempting alternative. A consumer who partially succumbs to temptation chooses a compromise between the tempting alternative and his original intentions, and incurs self-control costs.

GP show that this type of preferences can be represented by a utility function W that is defined over sets of alternatives and given by

$$W(A) \equiv \max_{x \in A} \big[U(x) + V(x)\big] - \max_{y \in A} V(y), \tag{1}$$

where U and V are two (von Neumann–Morgenstern) utility functions defined over alternatives, and A is the choice set faced by consumers. The interpretation is that U represents the preferences of the "committed self" inside the consumer, while V represents those of the "tempted self." The alternative that the consumer actually chooses is one that maximizes $U + V$ and is considered as a compromise between the preferred alternatives of the committed and the tempted selves. The disutility from self-control is valued by the best forgone alternative, and therefore $\max_{y \in A} V(y) - V(x)$ quantifies the "self-control cost" (measured in utility terms).

We study a monopolist's optimal nonlinear price schedule when it faces a population of heterogeneous consumers who have preferences of the type described above. We consider the standard framework of Mussa and Rosen (1978) and Maskin and Riley (1984), where the monopolist sells goods that are indexed by a single-dimensional quality (or quantity) level $q \in \mathbb{R}$. The monopolist does not observe consumers' preferences directly, and therefore the monopolist can set prices only via indirect price discrimination schemes that rely on consumers' self-selection. The monopolist's problem is to choose a set of goods $Q \subseteq \mathbb{R}$ to sell and a price function $p\colon Q \to \mathbb{R}$ that specifies the price $p(q)$ for each quality level to maximize the expected profits.

Consumers differ in their temptation preferences and therefore in their self-control costs. Consumers also differ in the "direction" of their preference reversals. Some consumers are "tempted upwards" towards more expensive, higher quality choices, and others are “tempted downwards” towards less expensive, lower quality choices.[2]

For a consumer to buy a good from the monopolist, the menu (Q, p) has to satisfy two conditions of individual rationality. One is a standard condition and states that the consumer does not prefer to exit the store without buying the good. The other is introduced by the consumers' self-control costs and states that, for a consumer to buy a good, he has to have an incentive to *enter the store*. This condition is distinct from the first one since, even if the consumer prefers to buy a good once inside the store, if he anticipates large self-control costs, he may prefer not to enter the store at all. We refer to this condition as the *ex-ante individual rationality* condition, and it turns out that this condition gives an upper bound on the quality-price pairs that the monopolist includes in its menu.

One particularly simple result that we obtain states that, if all consumers have preferences with upward temptation, then the optimal menu contains a single quality level. The singleton menu saves consumers from temptation and self-control, and extracts the entire commitment surplus from each consumer. The intuition is that, when all consumers have preferences with upward temptation, the monopolist can offer a singleton menu that works as a perfect commitment device. Since all consumers are tempted upwards and no higher quality good is offered, consumers do not incur any self-control costs, and the monopolist extracts the entire commitment surplus.

On the other hand, if some consumers exhibit downward temptation, then the monopolist might offer a continuum of choices and lead some consumers to incur self-control costs. The idea is that, for those consumers who have downward temptation, the singleton menu no longer works as a perfect commitment device since these consumers prefer not purchasing any bundle to purchasing the bundle in the singleton menu. As a result, the monopolist does not earn any profits from sales to these consumers. For the monopolist to obtain profits from these consumers, it must lower the price of the bundle in the singleton menu, which can then be improved upon by price discrimination. In general, since the ex-ante individual rationality condition imposes an upper bound on the quantity-price pairs in the menu, the optimal menu differs qualitatively from the optimal menu for the standard problem (where consumers' utility functions are given simply by $U + V$). We compare and contrast the optimal tariff under self-control preferences with that obtained in the standard problem.

There are several papers that study pricing problems when consumers' preference reversals are caused by hyperbolic discounting (an instance of time-varying discount

[2]Ameriks, Caplin, Leahy, and Tyler (2003) test for the presence of GP self-control preferences in survey data using a two-period allocation problem. They find heterogeneity in the way in which respondents are tempted. Some respondents exhibit temptation towards over-consumption in the first period, while a significant number of respondents exhibit temptation towards under-consumption.

rate), which creates time-inconsistency because of a conflict between the different selves of a consumer in different periods.[3] Gilpatric (2001) examines the design of incentive contracts when agents with hyperbolic discounting preferences exert unobservable effort. Della Vigna and Malmendier (2001) examine the optimal two-part price schedule in markets for goods with delayed benefits ("investment goods") or costs ("leisure goods") when consumers discount future payoffs hyperbolically.

There is also an empirical literature that tests for the presence of preference reversals using pricing data. Della Vigna and Malmendier (2002) look for this evidence in health club membership data and find that the data can be better explained with the behavior of time-inconsistent consumers. Miravete (2003) tests for the presence of irrational behavior in calling plan choices (where the popularity of flat rate plans has been viewed as evidence of irrational choices). He finds the data to be consistent with a model of learning and rational behavior. Wertenbroch (1998) finds evidence of lower price-elasticity for vice goods (goods that have delayed negative effects; e.g., cigarettes), which suggests that consumers ration the purchase of these goods to control their impulse consumption.

Section 2 presents the basic model, including the characterization of self-control preferences. Sections 3 and 4 examine the cases where (respectively) the firm faces only upwardly-tempted consumers and where it faces only downwardly-tempted consumers. Section 5 examines the general case where both types of consumers exist. Section 5 also includes results from some illustrative simulations.

2 The Model

We consider a store that sells a collection of goods. The goods that the store sells are indexed by $q \in \mathbb{R}_+$, which may represent either quantity or quality. A *bundle* in the menu is represented by a vector $(q, t) \in \mathbb{R}_+ \times \mathbb{R}$, which means that a consumer can buy good q for price t. A *menu* is a non-empty subset $M \subseteq \mathbb{R}_+ \times \mathbb{R}$ of bundles. Since a consumer can always choose not to buy any good at all and consume $(0, 0)$, it will be convenient to denote by $M_0 = M \cup \{(0, 0)\}$ all the consumption choices available to consumers. The seller is assumed to be a monopolist, and we study its profit-maximizing menu of bundles when consumers have Gul and Pesendorfer's temptation preferences with self-control.

[3] For a review of the literature on hyperbolic discounting, see, e.g., Laibson (1997) and O'Donoghue and Rabin (1999).

2.1 Consumers' Problem

Consumers have complete information about the menu that the monopolist offers, and have preferences over menus. Their preferences differ in the way in which they are tempted and are parameterized by a number $\gamma \in \mathbb{R}_+$. We assume that γ is private information and is distributed in the population according to a distribution function F. The associated density function is denoted by f, and its support is given by $\Gamma \equiv [a, b]$, where $b = +\infty$ is admissible.

Given a menu M, the utility of type γ is given by

$$W^{EA}(M, \gamma) \equiv \sup_{(q,t)\in M_0} \big[U(q,t) + V_\gamma(q,t)\big] - \sup_{(q,t)\in M_0} V_\gamma(q,t), \tag{2}$$

where U and V_γ are functions from $\mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$.

The two maximization problems in (2) are called the $U+V$ (or $U+V_\gamma$) problem and the V (or V_γ) problem, respectively. When these problems have maxima, we denote by x_γ and z_γ bundles that solve the $U + V_\gamma$ and V_γ problems, respectively. We also refer to (2) as the *ex-ante problem* and to the $U + V$ problem as the *ex-post problem.* The maximum (or supremum) utility in the ex-post problem is denoted by $W^{EP}(M, \gamma)$. Thus $W^{EA}(M, \gamma) = W^{EP}(M, \gamma) - \sup_{(q,t)\in M_0} V_\gamma(q, t)$.

While sup is used in (2) to accommodate all menus, we will examine profit-maximizing menus within the class of menus for which at least the $U + V$ problem has a maximum. The monopolist's problem is defined formally in the next subsection.

Functions U and V_γ are interpreted as the utility functions of the two different "selves" inside consumer γ. Function U represents the preferences of the committed self, while V_γ represents those of the tempted self. The alternative that the consumer chooses in the store is x_γ, which is a bundle that maximizes $U + V_\gamma$ and considered as a compromise between the preferred alternatives of the committed and the tempted selves. The alternatives that maximize V_γ are the most tempting alternatives. The difference $\sup_{z\in M_0} V_\gamma(z) - V_\gamma(x_\gamma)$ measures the consumer's disutility from exercising self-control and is referred to as the self-control cost. Thus, the consumer's utility is equal to the utility in terms of U evaluated at the bundle that he chooses in the store (i.e., $U(x_\gamma)$) minus the self-control cost.

We assume that U and V_γ are continuous, strictly increasing in q, strictly decreasing in t, quasi-concave, and satisfy $U(0, 0) = V_\gamma(0, 0) = 0$. We also assume that $V_\gamma(q, \cdot)$ is unbounded for any q, which is trivially satisfied if V_γ is quasi-linear in t. Furthermore, we assume that the process $\{V_\gamma\}_{\gamma\in\mathbb{R}_+}$, viewed as a function of (q, t, γ), is continuous.

Given two functions V_γ and $\hat{V}_\gamma$, we write $\hat{V}_\gamma \succsim V_\gamma$ if, at any point $(q, t) \in \mathbb{R}_+ \times \mathbb{R}$, the indifference curve of $\hat{V}_\gamma$ is at least as steep as that of V_γ when we measure the first (respectively, second) argument on the horizontal (respectively, vertical) axis. Formally, $\hat{V}_\gamma \succsim V_\gamma$ if and only if, for all $(q,t),(q',t') \in \mathbb{R}_+ \times \mathbb{R}$ such that $q' > q$,

$$V_\gamma(q',t') \geq V_\gamma(q,t) \text{ implies } \hat{V}_\gamma(q',t') \geq \hat{V}_\gamma(q,t), \text{ and}$$
$$V_\gamma(q',t') > V_\gamma(q,t) \text{ implies } \hat{V}_\gamma(q',t') > \hat{V}_\gamma(q,t).$$

If $\hat{V}_\gamma \succsim V_\gamma$ and $V_\gamma \succsim \hat{V}_\gamma$, then the two functions are ordinally equivalent in the sense that they induce the same indifference map. This is denoted as $\hat{V}_\gamma \sim V_\gamma$.

We also write $\hat{V}_\gamma \succ V_\gamma$ if the indifference curve of $\hat{V}_\gamma$ is strictly steeper than that of V_γ at any point $(q,t) \in \mathbb{R}_+ \times \mathbb{R}$. Formally, $\hat{V}_\gamma \succ V_\gamma$ if and only if, for all $(q,t),(q',t') \in \mathbb{R}_+ \times \mathbb{R}$ such that $q' > q$,

$$V_\gamma(q',t') \geq V_\gamma(q,t) \text{ implies } \hat{V}_\gamma(q',t') > \hat{V}_\gamma(q,t).$$

We assume the following on U and V_γ.

A1. For all $\gamma, \gamma' \in \mathbb{R}_+$, if $\gamma' \geq \gamma$, then $V_{\gamma'} \succ V_\gamma$.

A2. There exists a type $\gamma^* \in \mathbb{R}_+$ such that

$$\begin{aligned} U + V_\gamma &\succsim V_\gamma && \text{if } \gamma \leq \gamma^*, \\ V_\gamma &\succsim U + V_\gamma && \text{if } \gamma \geq \gamma^*. \end{aligned}$$

A3. For any pair $\gamma, \gamma' \in \mathbb{R}_+$ and any pair of bundles $x, x' \in \mathbb{R}_+ \times \mathbb{R}$, there exists another bundle $y \in \mathbb{R}_+ \times \mathbb{R}$ such that $U(x) + V_\gamma(x) = U(y) + V_\gamma(y)$ and $U(x') + V_{\gamma'}(x') = U(y) + V_{\gamma'}(y)$.

A1 is a standard single-crossing property and says that the indifference curves of V_γ are steeper (strictly) for higher types. Given a menu, this assumption implies that the most preferred quantity level is (weakly) larger for higher types.

A2 says that there exists a critical type γ^* such that any consumer $\gamma < \gamma^*$ behaves as a lower type in the V_γ problem than in the $U + V_\gamma$ problem, while any consumer $\gamma > \gamma^*$ behaves as a higher type in V_γ than in $U + V_\gamma$. In other words, for consumers $\gamma < \gamma^*$, the tempted self has lower marginal willingness to pay for additional quality than the committed self, while for consumers $\gamma > \gamma^*$, the tempted self has higher marginal willingness to pay than the committed self. This implies that, given a menu, consumers $\gamma < \gamma^*$ are tempted by less expensive, lower quality bundles, while consumers $\gamma > \gamma^*$ are tempted by more expensive, higher quality ones. Consumer γ^* does not have temptation problems since his tempted self agrees with his committed self.

We say that consumers $\gamma < \gamma^*$ have *downward temptation* preferences, while consumers $\gamma > \gamma^*$ have *upward temptation* preferences.

A3 says that an indifference curve of $U + V_\gamma$ crosses an indifference curve of $U + V_{\gamma'}$ somewhere.

A2 implies that, for $\gamma = \gamma^*$, $U + V_\gamma$ and V_γ are ordinally equivalent. This implies the following.

Lemma 1. *V_{γ^*} is ordinally equivalent to U; i.e., $V_{\gamma^*} \sim U$.*

Proof. Take two bundles $x, x' \in \mathbb{R}_+ \times \mathbb{R}$ such that $V_{\gamma^*}(x) = V_{\gamma^*}(x')$. Since V_{γ^*} is ordinally equivalent to $U + V_{\gamma^*}$, it follows that $U(x) + V_{\gamma^*}(x) = U(x') + V_{\gamma^*}(x')$, which implies $U(x) = U(x')$.

Now, take two bundles such that $V_{\gamma^*}(q, t) > V_{\gamma^*}(q', t')$. Since $V_{\gamma^*}(q, \cdot)$ is continuous and unbounded, there exists $\alpha > 0$ such that $V_{\gamma^*}(q, t+\alpha) = V_{\gamma^*}(q', t')$. By the previous paragraph, we obtain $U(q, t + \alpha) = U(q', t')$, which implies $U(q, t) > U(q', t')$. Q.E.D.

We sometimes consider the case in which U and V_γ are quasi-linear in the price paid; i.e., there exist functions $u\colon \mathbb{R}_+ \to \mathbb{R}$ and $v\colon \mathbb{R}_+ \times \Gamma \to \mathbb{R}$ such that, for all $(q, t) \in \mathbb{R}_+ \times \mathbb{R}$ and all $\gamma \in \Gamma$,

$$U(q, t) = u(q) - t, \tag{3}$$

$$V_\gamma(q, t) = v(q, \gamma) - t, \tag{4}$$

and $v(\cdot, \gamma^*) = u(\cdot)$. A particular example of quasi-linear utility functions satisfying all of our assumptions is

$$U(q, t) = q - t, \tag{5}$$

$$V_\gamma(q, t) = \gamma q - t, \tag{6}$$

where $\gamma > 0$ and $\gamma^* = 1$.

Before we close this section, we return to general preferences and prove the following basic fact.

Proposition 1. *For any menu M and any type $\gamma \in \Gamma$, if the ex-post problem has a maximizer x_γ, then*

$$W^{EA}(M, \gamma) \leq U(x_\gamma) \leq \sup_{x \in M_0} U(x), \tag{7}$$

$$W^{EA}(M, \gamma^*) = \max_{x \in M_0} U(x) \quad \textit{if } \gamma = \gamma^*. \tag{8}$$

Proof. Fix $\gamma \in \Gamma$, and let x_γ be a bundle that solves the $U + V_\gamma$ problem. Then

$$\begin{aligned} W^{EA}(M,\gamma) &= U(x_\gamma) + V_\gamma(x_\gamma) - \sup_{x \in M_0} V_\gamma(x) \\ &\leq U(x_\gamma) \leq \sup_{x \in M_0} U(x). \end{aligned} \tag{9}$$

To prove the second part of the lemma, note that, by Lemma 1, $U + V_{\gamma^*} \sim V_{\gamma^*} \sim U$. This implies $V_{\gamma^*}(x_{\gamma^*}) = \max_{x \in M_0} V_{\gamma^*}(x)$ and $U(x_{\gamma^*}) = \max_{x \in M_0} U(x)$. Thus, for γ^*, both inequalities in (9) hold with equality. Q.E.D.

The first inequality in (7) means that self-control is costly. The second inequality means that the bundle chosen by the consumer does not necessarily maximize his commitment utility. Equation (8) means that, for type γ^*, the commitment utility is maximized.

2.2 Monopolist's Problem

Let $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the per-consumer cost function; i.e., $C(q)$ is the cost of offering good q to a consumer. We assume that C is strictly increasing, convex, and satisfies $C(0) = 0$.

An *assignment function* for a given menu M is a function $x(\cdot) = (q(\cdot), t(\cdot)) : \Gamma \to M_0$ that assigns a bundle in the menu, or $(0,0)$, to each $\gamma \in \Gamma$. Here $x(\gamma)$ denotes the bundle that the monopolist plans to sell to type γ. We will use $x(\gamma)$ and x_γ interchangeably. We allow for $x(\gamma) = (0,0)$, which means that the monopolist does not plan to sell any good to type γ.

Given a menu, consumers may prefer not to enter the store if they expect large self-control costs. Consumers γ prefer to enter the store only if $W^{EA}(M,\gamma) \geq 0$, in which case we say that the menu M satisfies *ex-ante IR* for type γ. In what follows, we let $E \subseteq \Gamma$ denote the set of types that choose to enter the store.[4] Since consumers $\gamma \notin E$ do not enter the store, the assignment function has to be such that $x(\gamma) = (0,0)$ for all $\gamma \notin E$.

The monopolist's problem is to choose a menu M, an assignment function $x(\cdot) = (q(\cdot), t(\cdot))\colon \Gamma \to M_0$, and a subset $E \subseteq \Gamma$ that maximize

$$\int [t(\gamma) - C(q(\gamma))]\, dF(\gamma)$$

[4] But some of the types in E may choose $(0,0)$ in the store.

subject to that, for all $\gamma \in E$,

$$W^{EA}(M,\gamma) \geq 0, \tag{ex-ante IR}$$

$$U(x(\gamma)) + V_\gamma(x(\gamma)) \geq 0, \tag{ex-post IR}$$

$$U(x(\gamma)) + V_\gamma(x(\gamma)) \geq U(y) + V_\gamma(y) \quad \text{for all } y \in M, \tag{ex-post IC}$$

and, for all $\gamma \notin E$,

$$W^{EA}(M,\gamma) \leq 0,$$
$$x(\gamma) = (0,0).$$

If a list (M, x, E) satisfies all the constraints of this problem, we say that (M, x, E) is a *feasible schedule*.[5] If (M, x, E) solves the monopolist's problem, then we say that (M, x, E) is an *optimal schedule* and M is an *optimal menu*. We only consider the problem for which the monopolist can earn strictly positive profits.

The ex-post IR and IC conditions together imply that, for each type that enters the store, $x(\gamma)$ is a solution to the ex-post problem (hence the ex-post problem has a maximum). On the other hand, for all types that do not enter the store, $x(\gamma)$ is $(0,0)$ and is not required to be a solution to the ex-post problem.

The ex-post IR and IC conditions are independent of consumers' V problem (their second maximization problem). The V problem affects the monopolist's problem only because it affects the consumers' ex-ante utility levels and whether they enter the store (i.e., whether ex-ante IR is satisfied for them). If the consumers' entry decisions are ignored and all consumers are assumed to enter the store, then the monopolist's problem reduces to the standard problem of Mussa and Rosen with utility functions $U + V_\gamma$.[6]

Accordingly, we denote by the *standard problem* the monopolist's profit maximization problem subject to the constraint that all types satisfy ex-post IR and ex-post IC. An optimal menu in the standard problem satisfies the following properties: it consists of a continuum of bundles, leaves no surplus for the lowest type (i.e., ex-post IR holds with equality for type a), and generates no distortion at the highest type (the indifference curve of $U + V_\gamma$ for type b is tangent to a vertically translated cost curve).[7]

Since ex-ante IR distinguishes our problem from the standard one, we begin by

[5]Consumers for whom $W^{EA}(M,\gamma) = 0$ are indifferent about the entry decision. Our formulation of the problem assumes that these consumers' entry decisions can be resolved in the way that the monopolist prefers.

[6]We would also ignore consumers' entry decisions if consumers incurred self-control costs regardless of whether they enter the store or not.

[7]See, for example, Fudenberg and Tirole (1992, Chapter 7).

characterizing this condition.

The first property to be proved states that, under any tariff, all consumers for whom ex-ante IR is satisfied receive non-negative U surplus.

Lemma 2. *For any menu M, any type $\gamma \in \Gamma$ for whom M satisfies ex-ante IR, and any bundle $x \in M$ that solves the $U + V_\gamma$ problem, we have $U(x) \geq 0$.*

Proof. Let M be a menu, $\gamma \in \Gamma$ be a type for whom M satisfies ex-ante IR, and $x \in M$ be a bundle that solves the $U + V_\gamma$ problem. Then $V_\gamma(x) \leq \sup_{z \in M_0} V_\gamma(z)$. Thus ex-ante IR for γ implies

$$0 \leq U(x) + V_\gamma(x) - \sup_{z \in M_0} V_\gamma(z) \leq U(x). \qquad \text{Q.E.D.}$$

In words, the proof of Lemma 2 goes as follows. The ex-ante utility is given by the utility in terms of U minus self-control costs. Since self-control costs are non-negative, it follows that a necessary condition for the ex-ante utility to be non-negative is that the utility in terms of U is non-negative as well.

On the other hand, for consumers for whom ex-ante IR is violated, $x(\gamma) = 0$ and hence $U(x(\gamma)) = 0$. Thus, we obtain the following corollary.

Corollary 1. *Let (M, x, E) be a feasible schedule. Then $U(x(\gamma)) \geq 0$ for all $\gamma \in \Gamma$.*

This implies that the monopolist cannot sell bundles x such that $U(x) < 0$. Thus, for the monopolist to be able to earn positive profits, it has to be the case that there exists a bundle (q, t) such that $U(q, t) \geq 0$ and $t > C(q)$, and this is assumed in what follows. Formally, define a set B by

$$B = \{(q, t) \in \mathbb{R}_+ \times \mathbb{R} : U(q, t) \geq 0 \text{ and } t - C(q) \geq 0\}, \tag{10}$$

which is illustrated in Figure 1. Then we assume

A4. The set B has a non-empty interior.

We also assume

A5. There exists a quantity level $Q > 0$ such that $U(Q, C(Q)) = 0$.

This means that the curve of $U = 0$ and the cost curve intersect somewhere (other than at the origin), and implies that B is bounded. The point of intersection, $(Q, C(Q))$, gives an upper bound to the bundles that the monopolist may want to offer.

The following lemma is a converse of Lemma 2 and Corollary 1, and states that if all bundles in a menu leave non-negative U surplus, then the menu necessarily satisfies ex-ante IR for all types.

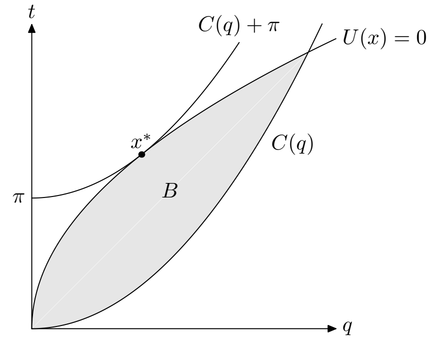

Figure 1: Set B and a most profitable bundle x^*

Lemma 3. *If a menu M satisfies $U(x) \geq 0$ for all $x \in M$, then the menu satisfies ex-ante IR for all types.*

Proof. Let M be a menu such that $U(x) \geq 0$ for all $x \in M$. To show that M satisfies ex-ante IR for all types, choose any $\gamma \in \Gamma$. Then for any bundle $x \in M_0$,

$$W^{EA}(M,\gamma) \geq U(x) + V_\gamma(x) - \sup_{z \in M_0} V_\gamma(z). \tag{11}$$

Now, choose any $\varepsilon > 0$. Then there exists a bundle $x \in M_0$ such that $V_\gamma(x) > \sup_{z \in M_0} V_\gamma(z) - \varepsilon$. Substituting this x into (11) gives

$$\begin{aligned} W^{EA}(M,\gamma) &> U(x) - \varepsilon \\ &\geq -\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $W^{EA}(M,\gamma) \geq 0$. Q.E.D.

To obtain the intuition, let z_γ be a bundle that solves the V_γ problem; i.e., a most tempting bundle in the menu for consumer γ. Then, when the consumer enters the store, one option for him is to succumb to the temptation and choose z_γ, in which case his ex-ante utility is given by $U(z_\gamma)$. By ex-post IC, the bundle that he actually chooses in the store gives him at least this level of ex-ante utility. Therefore, if all bundles in the menu give non-negative utilities in terms of U, then the ex-ante IR is satisfied.

Corollary 1 says that all the bundles that are chosen by consumers in the store give non-negative U surplus, while Lemma 3 says that if all bundles in the menu give non-negative surplus, then all consumers have an incentive to enter the store. These results together imply that if a feasible schedule (M, x, E) is not "decorated," in that the menu does not contain any bundle that is not consumed by any consumer, then ex-ante IR is satisfied for all types. Formally,

Definition. A feasible schedule (M, x, E) is *undecorated* if

$$M = \{x(\gamma) : \gamma \in \Gamma\}.$$

That is, M does not contain any bundle y such that $y \neq x(\gamma)$ for all $\gamma \in \Gamma$. Then Corollary 1 and Lemma 3 imply the following.

Corollary 2. *Let (M, x, E) be a feasible schedule that is undecorated. Then $U(x) \geq 0$ for all $x \in M$, and M satisfies ex-ante IR for all types.*

For the study of optimal menus, focusing on undecorated menus is without loss of generality. The reason is that, for any optimal menu that is decorated, removing the bundles that are not consumed by any types generates an optimal menu that is undecorated. This is because the removal of the unconsumed bundles can only make the menu less tempting and decrease the consumers' self-control costs. Thus, the removal of unconsumed bundles can only increase the ex-ante utilities of consumers and make the ex-ante IR easier to satisfy.

Formally, we can prove the following.

Proposition 2. *If there exists an optimal schedule, then there exists an optimal schedule (M, x, E) such that*

1. *$M = \{x(\gamma) : \gamma \in \Gamma\}$ (i.e., the schedule is not decorated),*
2. *$U(y) \geq 0$ for all $y \in M$ (i.e., all bundles in the menu generate non-negative U surplus),*
3. *$E = \Gamma$ (i.e., all types enter the store),*
4. *$t - C(q) \geq 0$ for all $(q, t) \in M$ (i.e., all bundles in the menu generate non-negative profits).*

Proof. Let $(\hat{M}, \hat{x}, \hat{E})$ be an optimal schedule. Define a menu M by

$$M = \{\hat{x}(\gamma) : \gamma \in \Gamma\} \cap B,$$

where B is defined by (10). By A5, M is bounded. Let $\bar{M}$ be the closure of M. Then, for each type $\gamma \notin \hat{E}$, there exists a bundle, which we denote by $x(\gamma)$, that solves the $U + V_\gamma$ problem for the menu $\bar{M}$. Lemma 2 implies that, for all $\gamma \in \hat{E}$, $U(\hat{x}(\gamma)) \geq 0$, but it is possible that $\hat{x}(\gamma) \notin B$ for some $\gamma \in \hat{E}$. For all $\gamma \in \hat{E}$ such that $\hat{x}(\gamma) \notin B$, let $x(\gamma)$ be a bundle that solves the $U + V_\gamma$ problem for the menu $\bar{M}$. On the other hand, for all $\gamma \in \hat{E}$ such that $\hat{x}(\gamma) \in B$, let $x(\gamma) = \hat{x}(\gamma)$; for these types, $\hat{x}(\gamma)$ is in M and remains a solution to the $U + V_\gamma$ problem for $\bar{M}$. Then let

$$M^* = \{x(\gamma) : \gamma \in \Gamma\},$$

and consider a schedule (M^*, x, Γ). Since $M^* \subseteq B$, Lemma 3 implies that the ex-ante IR condition is satisfied for all types. By the construction of $x(\cdot)$, the ex-post IC and IR conditions are also satisfied for all types. Hence (M^*, x, Γ) is feasible. The schedule is optimal since for all types such that $x(\gamma) \neq \hat{x}(\gamma)$, $\hat{x}(\gamma)$ generates non-positive profits while $x(\gamma)$ generates non-negative profits. Q.E.D.

Given this result, we introduce the following definition.

Definition. An optimal schedule (M, x, E) is *regular* if it satisfies 1–4 in Proposition 2.

3 Upward Temptation

We begin our derivation of optimal menus with the case where all consumers exhibit upward temptation; i.e., $\gamma \geq \gamma^*$ for all consumers, or $a \geq \gamma^*$.

We denote by x^* a most profitable bundle in B (see Figure 1). This bundle is identified as a point where the indifference curve of U through $(0, 0)$ and a translated cost curve (i.e., an isoprofit curve) are tangent. Formally, $x^* = (q^*, t^*)$ is a pair that solves

$$\max_{(q,t)\in\mathbb{R}_+\times\mathbb{R}} \; t - C(q) \quad \text{s.t. } U(q, t) \geq 0. \tag{12}$$

By Corollary 1, x^* is a most profitable bundle that the monopolist can sell to consumers.

When U is quasi-linear, q^* is a solution to

$$\max_{q\in\mathbb{R}_+} \; u(q) - C(q). \tag{13}$$

We refer to q^* as a *socially optimal quantity* since it maximizes social surplus when the committed utility U is used for consumers.

The following proposition characterizes optimal menus when all consumers have preferences with upward temptation.

Proposition 3. *Assume that* $a \geq \gamma^*$.[8] *Then an optimal menu is* $M = \{x^*\}$ *where*

[8] The proposition also holds when the number of types is finite.

x^* is a solution of (12).[9] *Conversely, if a feasible schedule* (M, x, E) *is optimal, then, for some* x^* *that solves* (12), $x(\gamma) = x^*$ *for all* $\gamma > a$.

Proof. To prove the first statement, suppose that the monopolist offers $\{x^*\}$ such that x^* solves (12). By Corollary 1, x^* is a most profitable bundle that the monopolist can sell to consumers. Since $x^* \in B$, Lemma 3 implies that all consumers have an incentive to enter the store. It remains to show that all consumers do choose x^*, rather than $(0,0)$, in the $U+V$ problem. This follows from the fact that, by Lemma 1, $U+V_\gamma$ is at least as "steep" as U; i.e., $U+V_\gamma \succsim U$ for all $\gamma \in \Gamma$. Since x^* is at least as good as $(0,0)$ for the U utility, it follows that x^* is also at least as good as $(0,0)$ for the $U+V_\gamma$ utility.

To prove the second statement in the proposition, note that the first statement implies that, for a schedule (M, x, E) to be optimal, $x(\gamma)$ has to be a solution of (12) for almost all $\gamma \in \Gamma$. Furthermore, ex-post IC implies that, if $x(\gamma)$ solves (12) for some $\gamma > \gamma^*$, $x(\gamma') = x(\gamma)$ for all $\gamma' > \gamma$. This implies that $x(\gamma) = x(\gamma')$ for all $\gamma, \gamma' > a$. Q.E.D.

The intuition behind the first statement in the proposition is as follows. Since all consumers have preferences with upward temptation, all consumers are tempted by more expensive, higher quality bundles. However, only one bundle, x^*, is offered. Thus all consumers are tempted by the same bundle they consume, which makes their self-control costs equal to zero and allows the monopolist to price the bundle to extract the entire ex-ante surplus. Ex post, however, each consumer $\gamma > \gamma^*$ receives a surplus equal to $U + V_\gamma = V_\gamma$, but this surplus cannot be extracted by the monopolist since the ex-ante IR condition is binding.[10]

4 Downward Temptation

We now consider the case in which all consumers have preferences with downward temptation; i.e., $\gamma \leq \gamma^*$ for all consumers, or $b \leq \gamma^*$. The following proposition shows that, in this case, ex-ante IR is implied by ex-post IR for all types.

Proposition 4. *For all* $\gamma \leq \gamma^*$ *and all menus* M*, if the* $U + V_\gamma$ *problem has a maximum and* $W^{EP}(M, \gamma) \geq 0$*, then* $W^{EA}(M, \gamma) \geq 0$.[11]

[9] The result can be generalized to the case when the (ex-ante) reservation utility level for each type is given by $\underline{W} \in \mathbb{R}_+$. In this case an optimal menu consists of (q^*, t^*) such that $U(q^*, t^*) = \underline{W}$.

[10] The finding that the firm provides a contract that works as a perfect commitment device also appears in Della Vigna and Malmendier (2001) (Proposition 7(i)), who analyze optimal two-part tariffs when consumers discount future payoffs hyperbolically.

[11] This proposition itself does not require $b \leq \gamma^*$.

Proof. Let $\gamma \leq \gamma^*$ and M be such that $U + V_\gamma$ problem has a maximum and $W^{EP}(M, \gamma) \geq 0$. Let x_γ be a bundle that solves the $U + V_\gamma$ problem, and let $t' \in \mathbb{R}$ be such that bundle $(0, t')$ is indifferent to x_γ for the $U + V_\gamma$ utility. Since x_γ gives a non-negative ex-post utility, $t' \leq 0$. Since $V_\gamma \precsim U + V_\gamma$, it follows that $(0, t')$ is at least as good as any bundle in the menu for the V_γ utility; i.e., $V_\gamma(0, t') \geq \sup_{z \in M_0} V_\gamma(z)$. Thus

$$\begin{aligned} W^{EA}(M, \gamma) &= U(0, t') + V_\gamma(0, t') - \sup_{z \in M_0} V_\gamma(z) \\ &\geq U(0, t') \geq 0. \end{aligned}$$ Q.E.D.

A corollary to this proposition is that, when $b \leq \gamma^*$, consumers' entry decisions can be ignored and the monopolist's problem reduces to the standard problem. Hence, when $b \leq \gamma^*$, the standard analysis of nonlinear pricing can be used to characterize optimal menus.

Corollary 3. *If $b \leq \gamma^*$, a menu is optimal if and only if it is an optimal menu for the standard problem.*

5 Mixed Case

This section studies the general case where some consumers have preferences with downward temptation and others with upward temptation; i.e., $a < \gamma^* < b$. We divide our analysis into three sections. The first section presents properties that optimal menus satisfy in general environments. Section 5.2 examines the case when there are only two types. Finally, Section 5.3 returns to the continuum-type case and obtains stronger characterizations with quasi-linear and differentiable preferences.

5.1 General Properties of Optimal Menus

This section presents properties that optimal menus satisfy in general environments. The first property states that the ex-post IR condition binds for the lowest consumer type.

Proposition 5. *For any regular optimal schedule, $U(x(a)) + V_a(x(a)) = 0$.*

Proof. See Appendix A.1.

This means that consumer a is assigned by the monopolist a bundle that yields zero ex-post surplus. Otherwise, if the bundle derives a positive ex-post surplus, the monopolist can find a new tariff that satisfies ex-ante and ex-post participation of all types and yields higher profits. It should be noted that this proposition does not hold when $a > \gamma^*$ since the optimal menu described in Proposition 3 gives a strictly positive ex-post utility to type a.[12]

The next result to be proved states that, for any regular optimal schedule, the bundle at the right end-point gives no surplus in terms of U provided that the following assumption is satisfied. The assumption guarantees that, for any bundle x in the set B (which is a bounded set by A5), there exists a type $\gamma \in \Gamma$ whose indifference curve at x is steeper than that of the cost curve.

A6. For all $(q, t) \in B$, there exist a type $\gamma \in \Gamma$ and a bundle $(q', t') \gg (q, t)$ such that[13]

$$U(q', t') + V_\gamma(q', t') > U(q, t) + V_\gamma(q, t), \text{ and} \tag{14}$$

$$t' - C(q') > t - C(q). \tag{15}$$

Given this additional assumption, the right end of the optimal tariff is on the $U = 0$ curve.

Proposition 6. *For any regular optimal schedule, the right-end of the menu, defined as*

$$\bar{x} \equiv (\sup_{\gamma \in \Gamma} q(\gamma), \sup_{\gamma \in \Gamma} t(\gamma)), \tag{16}$$

is finite (i.e., $\bar{x} \in \mathbb{R}^2$) and satisfies $U(\bar{x}) = 0$.

Proof. See Appendix A.2.

While a complete proof is provided in the Appendix, Figure 2 illustrates its workings. Suppose, by way of contradiction, that the optimal tariff ends before intersecting the $U(x) = 0$ curve, as illustrated by $\bar{x}$ in the figure, and translate the cost curve vertically onto $\bar{x}$ to identify all the bundles that are more profitable than $\bar{x}$. Assume that the highest type, b, is finite, and thus $\bar{x} = x(b)$ (although the result also holds when $b = +\infty$). Then draw the indifference curve of type b through $\bar{x}$. A6 implies that type b's indifference curve is steeper than the translated cost curve at $\bar{x}$ (the isoprofit curve) and therefore creates a lens-shaped area of bundles that are preferred by type b and are more profitable. Suppose that we add to the menu any bundle, such as y in the figure, in the interior of the lens-shape area and below the $U(x) = 0$ curve. Then a

[12]Proposition 5 also holds when $a = \gamma^*$, which follows from the second statement in Proposition 3.

[13]We use the following notation for vector inequalities: $(q', t') \gg (q, t)$ if $q' > q$ and $t' > t$; and, $(q', t') \geq (q, t)$ if $q' \geq q$ and $t' \geq t$.

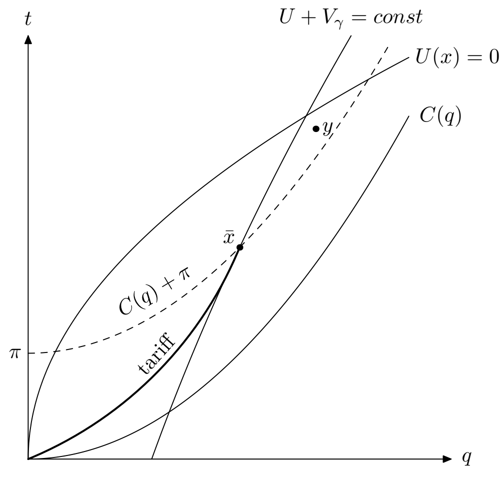

Figure 2: No U surplus at the top

positive mass of consumers, consumers close to type b, strictly prefer y to what they are assigned in the initial menu. Since y lies above the isoprofit curve, it is more profitable than what these consumers are initially assigned, and the modified menu generates more profits.

A6, which is required for Proposition 6, restricts b to be large enough so that the lens-shaped area discussed in the previous paragraph exists for every bundle in B. If this assumption is not satisfied, then the optimal tariff may have $U > 0$ for type b, while still being the case that $b > \gamma^*$.

We can also show that $U(x(\gamma)) = 0$ may hold only at the end-points of the menu (see Proposition 9 in Appendix A.5). The idea is that, by ex-post IC, all other bundles must leave positive U surplus.

We can also show that the tariff intersects the $U = 0$ curve to the right of x^*. For the quasi-linear case, this means that the largest quantity that the firm sells, $\bar{q}$, is not smaller than the socially optimal quantity.

Proposition 7. *Let* (M, x, Γ) *be a regular optimal schedule, and* $\bar{x}$ *be defined by* (16)*. Then, for all bundles* $x \geq \bar{x}$ *such that* $U(x) \geq 0$*, we have* $t - C(q) \leq \bar{t} - C(\bar{q})$*. If* U *is quasi-linear,* $\bar{q} \geq q^*$ *for some* q^* *that solves* (13)*. If* u *and* C *are differentiable, then* $C'(\bar{q}) \geq u'(\bar{q})$*.*

Proof. See Appendix A.3.

The proof is similar to that of Proposition 6. The idea is that if the tariff intersected the $U = 0$ curve to the left of x^*, then the monopolist could increase its profits by adding x^* to the menu.

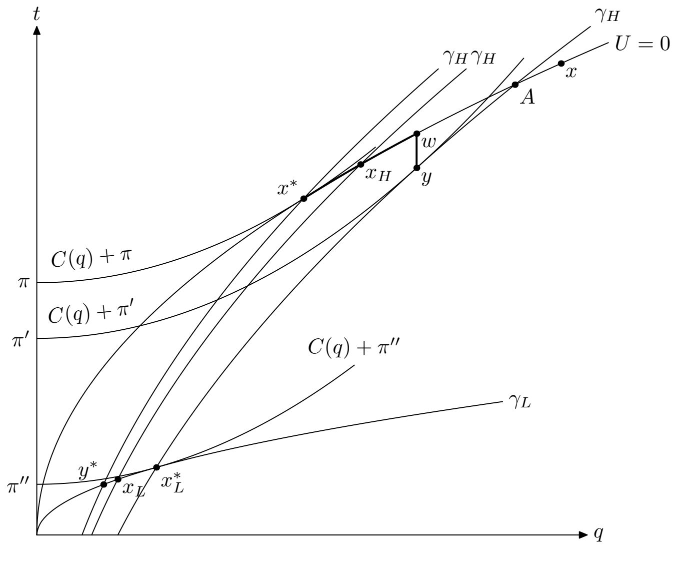

Figure 3: Two-type case

5.2 Two-Type Case

This section departs from our basic setup and examines the case in which only two types of consumers exist. Let these types be denoted by γ_L and γ_H where $\gamma_L < \gamma^* < \gamma_H$, and be distributed in proportions n_L and n_H, respectively. We also assume that preferences are quasi-linear.

See Figure 3. We denote by x_L^* the most profitable bundle (which we assume to be unique) under the ex-post IR condition of the low type consumers. If this bundle is offered to the low type consumers, then the set of bundles that can be offered to the high type consumers is given by the lower envelope of the $U = 0$ curve and the high type's indifference curve through x_L^*; i.e., the kinked curve that connects $x_L^* yAxU$ (and bundles below the curve). It is useful to distinguish two cases.

Case 1. The first case is when the most profitable bundle along the kinked curve $x_L^* yAxU$ is on the right side of A; e.g., suppose that it is x. In this case, the $U = 0$ curve is tangent with the isoprofit curve at x. (Note that the isoprofit curves in the figure are *not* drawn for this case.) Then offering $\{x_L^*, x\}$ is optimal for the seller. Indeed, increasing the quantity level offered to γ_L can only contract the set of bundles that can be offered to γ_H. Decreasing the quantity level for γ_L, on the other hand, does expand the set of bundles that can be offered to γ_H, but x remains the most profitable bundle in the set.

Case 2. The other case to consider is when the most profitable bundle along $x_L^* yAxU$ is on the left side of A; e.g., suppose that it is y. (The isoprofit curves in the figure are drawn for this case.) Offering $\{x_L^*, y\}$ is not necessarily optimal. Indeed, as in the standard problem, decreasing the quantity level for γ_L enlarges the set of bundles that can be offered to γ_H, which increases profits that can be extracted from γ_H. The most profitable bundle that can be offered to γ_H moves up along the vertical segment yw. At point w, the $U \geq 0$ constraint comes into effect, from which point on, the bundle offered to γ_H moves along the $U = 0$ curve. It is easy to see that it is not optimal to offer to γ_H a bundle to the left of x^*. Thus the monopolist offers to γ_L some bundle x_L between y^* and x_L^* on the indifference curve of γ_L, and to γ_H some bundle on the kinked curve x^*wy such that ex-post IC binds for γ_H (e.g., offering $\{x_L, x_H\}$ in the figure).

From the figure above, it can be seen that, if $\{x_L, x_H\}$ is the optimal menu, then

$$q_L \geq q_L^S \text{ and } q_H \leq q_H^S, \tag{17}$$

where $\{q_L^S, q_H^S\}$ denote the quantities in the optimal menu for the standard problem.

To see (17), first consider Case 1; thus suppose that $\{x_L^*, x\}$ in the figure is the optimal menu. Let x_H^* be the most profitable bundle on γ_H's indifference curve through x_L^*. Since $\gamma_H > \gamma^*$, we have $q_H^* \geq q$, where q_H^* and q denote the quantity levels of x_H^* and x, respectively. Since $q_L^S \leq q_L^*$ and $q_H^S = q_H^*$, the desired result follows.

For Case 2, recall that by decreasing the quantity level for the low type consumer, the monopolist can increase the surplus that can be extracted from the high type one. As the bundle offered to γ_L moves from x_L^* to y^*, the bundle offered to γ_H moves along ywx^*. We obtain (17) since the marginal increase in profits that can be extracted from γ_H is higher in the standard problem. The reason is that, in the standard problem, the bundles that can be offered to γ_H are not bounded by the $U = 0$ curve. In our problem with self-control, on the other hand, the profits from sales to the high type consumers are bounded because of the ex-ante IR condition, and hence there is less incentive for the monopolist to decrease the quantity level for the low type consumers.

The optimal menu for Case 2 can be characterized analytically. Since ex-post IR binds for γ_L, we have $u(q_L) + v_L(q_L) = 2t_L$. Since ex-post IC binds for γ_H, we have $2t_H = u(q_H) + v_H(q_H) - v_H(q_L) + v_L(q_L)$. Then ex-ante IR (or non-negative U surplus) can be written as

$$u(q_H) - v_H(q_H) + v_H(q_L) - v_L(q_L) \geq 0. \tag{18}$$

Therefore, the monopolist's problem is to choose (q_L, q_H) that maximizes

$$\begin{aligned} &n_H\big[u(q_H) + v_H(q_H) - v_H(q_L) + v_L(q_L) - 2C(q_H)\big] \\ &+ n_L\big[u(q_L) + v_L(q_L) - 2C(q_L)\big] \end{aligned}$$

subject to (18). Letting λ denote the Lagrange multiplier for (18), and assuming that all functions are differentiable, we obtain the following first-order conditions:

$$n_H\left[u'(q_H) + v'_H(q_H) - 2C'(q_H)\right] + \lambda\left[u'(q_H) - v'_H(q_H)\right] = 0, \tag{19}$$

$$\begin{aligned} n_H\left[v'_L(q_L) - v'_H(q_L)\right] + n_L\left[u'(q_L) + v'_L(q_L) - 2C'(q_L)\right] \\ + \lambda\left[v'_H(q_L) - v'_L(q_L)\right] = 0. \end{aligned} \tag{20}$$

In the second term of (19), we have $u'(q_H) - v'_H(q_H) < 0$ since $\gamma_H > \gamma^*$. Thus the first term of the equation is non-negative, which implies $q_H \le q_H^S$ (assuming that the optimal menu is unique in either problem). Similarly, in the last term of (20), we have $v'_H(q_L) - v'_L(q_L) > 0$, which implies that the reversed argument follows and yields $q_L \ge q_L^S$.

5.3 Quasi-Linear and Differentiable Preferences

Here we return to the case in which there is a continuum of types. In what follows, we assume that the utility functions U and V are quasi-linear (see (3) and (4)) in order to obtain stronger characterizations of the optimal tariffs. Additionally, we assume that u and v are C^2 and C^3, respectively, and satisfy the following assumptions.

A7. $v_{12}(q,\gamma) > 0$.

A8. $v_2(q,\gamma) > 0$ if $q > 0$.

A9. $v_{122}(q,\gamma) \le 0$ and $v_{112}(q,\gamma) \ge 0$.[14]

A10. f is log-concave; i.e., $f'(\gamma)/f(\gamma)$ is non-increasing in γ.

Functions f and C are C^1 and C^2, respectively, and C satisfies $C' > 0$ and $C'' > 0$. We also assume that the maximum type in the support is finite; i.e., $b < +\infty$.

The monopolist maximizes its profits, which are

$$\int_a^b \Big[u(q(\gamma)) + v(q(\gamma),\gamma) - W(\gamma) - 2C(q(\gamma))\Big] f(\gamma)\, d\gamma \tag{21}$$

[14]See A8 in Fudenberg and Tirole (1992, page 263).

subject to, for all $\gamma \in [a, b]$,

$$W'(\gamma) = v_2(q(\gamma), \gamma), \tag{22}$$

$$W(\gamma) \geq v(q(\gamma), \gamma) - u(q(\gamma)), \tag{23}$$

$$W(a) = 0, \tag{24}$$

$$q(\gamma) \geq 0, \tag{25}$$

$$q \text{ is non-decreasing}, \tag{26}$$

where $W(\gamma)$ is the ex-post utility of type γ (i.e., $W(\gamma) = W^{EP}(M, \gamma)$) and (23) is the ex-ante IR condition (by $W = u + v - 2t$). Since the values of $q(\cdot)$ at the points of discontinuity are immaterial, we consider only $q(\cdot)$ that is continuous from the left.

For comparison, the necessary condition characterizing the optimal tariff in the standard problem (i.e., without the ex-ante IR constraint (23)) is:

$$u'(q^S(\gamma)) + v_1(q^S(\gamma), \gamma) - v_{21}(q^S(\gamma), \gamma)\frac{1 - F(\gamma)}{f(\gamma))} = 2C'(q^S(\gamma)). \tag{27}$$

The superscript S denotes optimal variables for the standard problem.

As usual, we first characterize the solution to the "relaxed" problem, which is the above problem ignoring the monotonicity condition in (26). The relaxed problem can be formulated as an optimal control problem in which W is the state variable and q is the control. The associated Hamiltonian is given by

$$\begin{aligned} H(W, q, \mu, \lambda, \gamma) = &\big[u(q) + v(q, \gamma) - W - 2C(q)\big] f(\gamma) \\ &+ \mu v_2(q, \gamma) + \lambda[u(q) - v(q, \gamma) + W]. \end{aligned}$$

As in the standard case, the log-concavity assumption A10 on the type distribution f guarantees that the quantity function $q(\cdot)$, characterized by the necessary conditions, is non-decreasing; this fact will be proved for our problem as part of Proposition 8 below.

The ex-ante IR condition (23) places restrictions on the values that the state $W(\gamma)$ can take. As is well-known, when such state restrictions are present, a first-order approach (using the necessary conditions for the optimal control problem given by the Maximum Principle) may fail to isolate the optimal solution due to a constraint qualification (see Seierstad and Sydsæter (1987, page 278, Note 4)). The constraint qualification (CQ) associated with the inequality constraint $u - v + W \geq 0$ states that the derivative of the constraint with respect to q (i.e., $u' - v_1$) must not be equal to zero at all types at which the constraint binds. This implies that the CQ is violated if and only if $U = 0$ (or equivalently $u - v + W = 0$) for type γ^*. Hence, we consider two families of feasible tariffs wherein the optimal tariff may lie: (1) those that violate the

CQ, and (2) those that satisfy the CQ.

5.3.1 Case 1: Optimal Tariff Violating Constraint Qualification

For tariffs such that $U = 0$ for γ^*, ex-post and ex-ante IR conditions imply that all types $\gamma < \gamma^*$ are excluded and all types $\gamma > \gamma^*$ consume bundles such that $U = 0$. Hence, if an optimal tariff violates CQ, then the tariff takes the form described in Proposition 3. Specifically, if an optimal tariff violates the constraint qualification, then the tariff offers a bundle (q^*, t^*) such that $u'(q^*) = C'(q^*)$ and $t^* = u(q^*)$ to all types $\gamma > \gamma^*$, and excludes all types $\gamma < \gamma^*$.

Moreover, since $\gamma^* < b$, (27) implies

$$u'(q^S(\gamma^*)) - C'(q^S(\gamma^*)) = \frac{1}{2}v_{21}(q^S(\gamma^*), \gamma^*)\frac{1 - F(\gamma^*)}{f(\gamma^*)} > 0,$$

and hence $q^S(\gamma^*) < q^*$. In words, the monopolist sells more to type γ^* than in the standard problem.

5.3.2 Case 2: Optimal Tariff Satisfying Constraint Qualification

Let (q, W) be a pair that solves the relaxed problem and satisfies CQ. Then it satisfies the following necessary conditions: for all $\gamma \in [a, b]$,

$$\begin{aligned}\big[u'(q(\gamma)) + v_1(q(\gamma), \gamma) - 2C'(q(\gamma))\big] f(\gamma) + \mu(\gamma)v_{21}(q(\gamma), \gamma)&\\ + \lambda(\gamma)(u'(q(\gamma)) - v_1(q(\gamma), \gamma)) + \delta(\gamma) = 0,&\end{aligned} \tag{28}$$

$$\mu'(\gamma) = f(\gamma) - \lambda(\gamma), \tag{29}$$

$$\lambda(\gamma) \cdot (u(q(\gamma) - v(q(\gamma), \gamma) + W(\gamma)) = 0, \quad \lambda(\gamma) \geq 0, \tag{30}$$

$$\mu(b) = 0, \tag{31}$$

$$q(\gamma)\delta(\gamma) = 0, \quad \delta(\gamma) \geq 0 \tag{32}$$

The following proposition characterizes the solution to the relaxed problem.

Proposition 8. *Let (q, W) be a pair that solves the relaxed problem and satisfies CQ. Then q is non-decreasing. Moreover, let*

$$\begin{aligned}&\underline{\gamma} \equiv \max\{\gamma \in [a, b] : W(\gamma) = 0\} \text{ and}\\ &\bar{\gamma} \equiv \sup\{\gamma \in [a, b] : W(\gamma) > v(q(\gamma), \gamma) - u(q(\gamma))\}.\end{aligned} \tag{33}$$

Then $\underline{\gamma} < \gamma^* < \bar{\gamma}$, and

1. *if $\bar{\gamma} < b$, then q is constant over $(\bar{\gamma}, b]$;*

2. for all $\gamma \in (\underline{\gamma}, \bar{\gamma})$*,*

$$u'(q(\gamma)) + v_1(q(\gamma), \gamma) - v_{12}(q(\gamma), \gamma)\frac{\beta(\bar{\gamma}) - F(\gamma)}{f(\gamma)} = 2C'(q(\gamma)), \tag{34}$$

where $\beta(\bar{\gamma}) \equiv F(\bar{\gamma}) - \mu(\bar{\gamma}) \in [0, 1]$*;*

3. if $\underline{\gamma} > a$*, then* $q(\gamma) = W(\gamma) = 0$ *for all* $\gamma \in [a, \underline{\gamma}]$ *and*

$$u'(0) + v_1(0, \underline{\gamma}) - v_{12}(0, \underline{\gamma})\frac{\beta(\bar{\gamma}) - F(\underline{\gamma})}{f(\underline{\gamma})} = 2C'(0). \tag{35}$$

Proof. See Appendix A.4.

The optimal tariff for this case resembles that derived by Besanko, Donnenfeld, and White (1987) for nonlinear pricing with minimum quality standards. In order to compare the optimal tariff obtained in our problem to that in the standard problem, we evaluate (28) at $\bar{\gamma}$ to obtain

$$\begin{aligned}\left[u'(q(\bar{\gamma})) + v_1(q(\bar{\gamma}), \bar{\gamma}) - 2C'(q(\bar{\gamma}))\right] f(\bar{\gamma}) + \mu(\bar{\gamma}) v_{21}(q(\bar{\gamma}), \bar{\gamma})& \\ + \lambda(\bar{\gamma})(u'(q(\bar{\gamma})) - v_1(q(\bar{\gamma}), \bar{\gamma})) = 0.&\end{aligned}$$

Since the second and third terms are non-positive, it follows that $u'(q(\bar{\gamma})) + v_1(q(\bar{\gamma}), \bar{\gamma}) \geq 2C'(q(\bar{\gamma}))$. By the definition of $\beta(\bar{\gamma})$ (given in item 2 above) and the expression for the multiplier $\mu(\bar{\gamma})$ given in (56) in the Appendix, we have

$$\begin{aligned}u'(q(\bar{\gamma})) + v_1(q(\bar{\gamma}), \bar{\gamma}) \geq 2C'(q(\bar{\gamma})) &\Longrightarrow \frac{2[C'(q(\bar{\gamma})) - u'(q(\bar{\gamma}))]}{v_1(q(\bar{\gamma}), \gamma) - u'(q(\bar{\gamma}))} \leq 1 \\ &\Longrightarrow \beta(\bar{\gamma}) \leq 1.\end{aligned}$$

Hence, a comparison of (27) and (34) yields $q(\gamma) \geq q^S(\gamma)$ for all $\gamma \in (\underline{\gamma}, \bar{\gamma})$: the monopolist expands production to these types. A similar comparison yields that $\underline{\gamma} \leq \underline{\gamma}^S$: the monopolist expands the range of consumers served relative to the standard case.

The intuition behind these two comparisons is the same as with two consumer types. In the standard price discrimination problem, the monopolist deteriorates the quality sold to the low type consumers, and excludes some of them, in order to increase the surplus extracted from the high type ones. With self-control preferences, the ex-ante IR condition effectively represents a bound on the feasibility of price discrimination towards higher types. As a result, the monopolist has less incentives to deteriorate the quality sold to lower types, or exclude them, than it does in the standard problem.

In solving the relaxed problem, we have maintained the assumptions that underlie Proposition 6 above, which ensure that the optimal tariff features $U = 0$ at the top. If, instead, $U > 0$ at the top, so that $\bar{\gamma} = b$, then (34), which characterizes the optimal tariff for our problem, becomes identical to the first-order condition (27) for the standard problem. In that case, our tariff would coincide with the standard tariff, and features of the standard tariff — such as no distortion at the top — would hold for our tariff as well.[15]

5.3.3 Illustrative Simulations

In this section, we present simulation results from a parameterized example to illustrate our findings in the previous sections. We use the example introduced in (5) and (6). For this example, the critical type, γ^*, equals 1. We assume that γ is drawn uniformly from $[0, 6]$ (which satisfies A6), and take the cost function $C(q) = \frac{1}{2}q^2$.

In Figure 4, we simulate the optimal tariff with self-control preferences and the optimal tariff for the standard problem and compare the properties of these two tariffs. In the top graph, where we graph the optimal quantities, we see that the tariff with self-control preferences serves a larger range of consumer types (i.e., $\underline{\gamma} < \underline{\gamma}^S$, as derived in the previous section). This can also be seen in the second graph of Figure 4, which contains the payment functions $t(\cdot)$ for the standard and self-control tariffs. The interpretation is that the standard optimal pricing problem lowers the quality offered to lower types, and excludes some of them, in order to extract more surplus from the high type consumers. The ex-ante IR condition, however, serves as a bound to the profitability of this strategy, which lowers the incentives to cut back the quality served to the lower type consumers or to exclude some of them.

The bottom graph of Figure 4 plots the tariffs as pairs of price and quality. The tariff with self-control costs has larger quantity discounts than the standard tariff, which means that increasing quality has a lower increase in price with self-control preferences.

In Figure 5, we simulate the consumption and the temptation choices for all the consumers in our example tariff; that is, the choices solving the $U + V$ and V problems, respectively. The temptation choice is illustrated with the crossed line and the consumption choice with the solid line. The top graph shows how consumers with $\gamma < \gamma^* = 1$ are tempted towards lower quality goods; thus, for a given γ, the temptation choice is of lower quality than the consumption choice. Instead, for $\gamma > \gamma^*$, the opposite follows. Notice that at the top, however, consumers are tempted by the same bundle they consume; thus, for a sufficiently high γ, their temptation and consumption

[15] Outside the quasi-linear case, however, it may not be true that, if the optimal tariff in our problem features $U > 0$ at the top, then this tariff coincides with the optimal tariff in the standard problem.

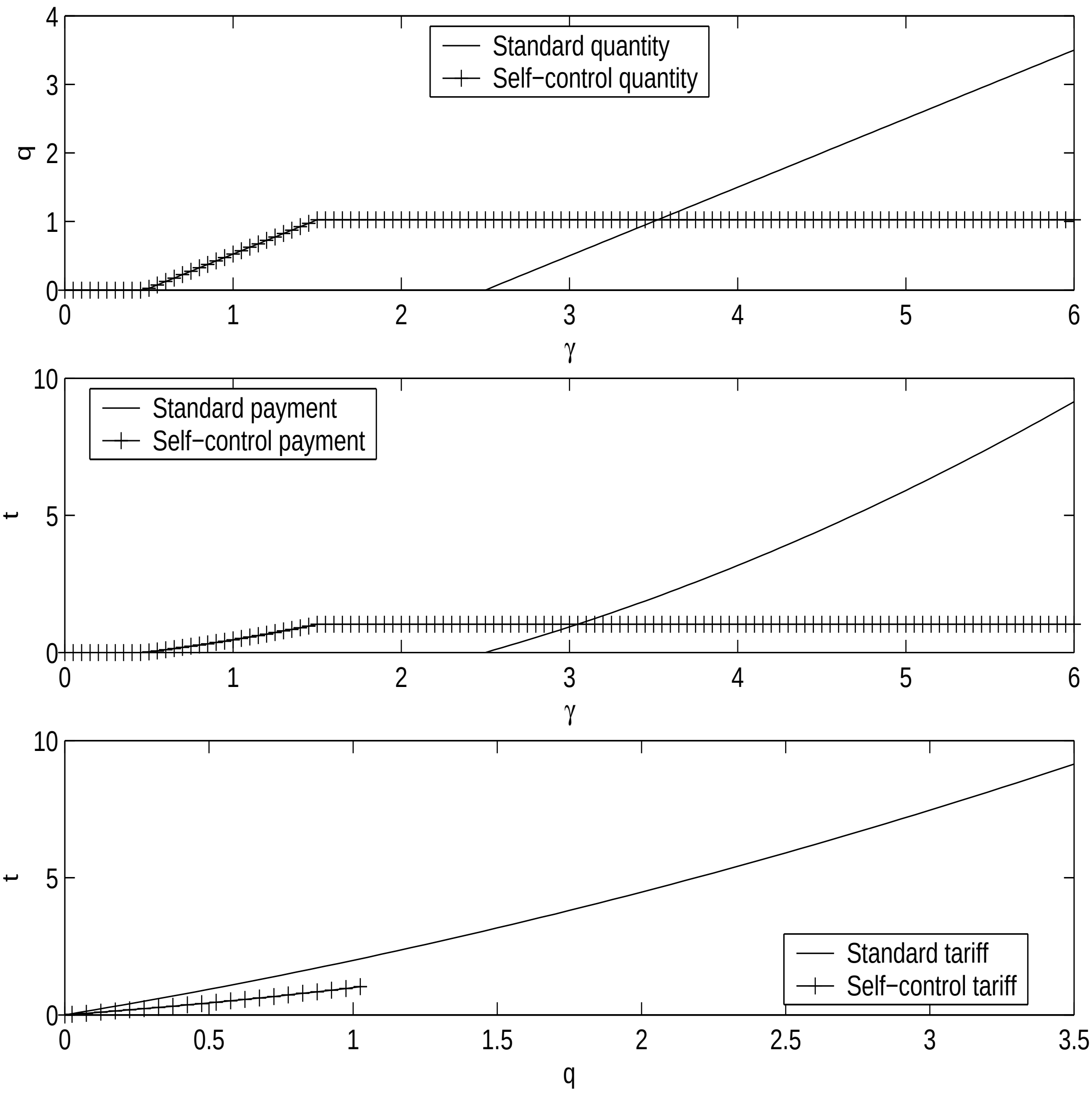

Figure 4: Comparing optimal standard and self-control tariffs.
Utility functions parameterized as in (5) and (6); cost function $C(q) = \frac{1}{2}q^2$;
$\gamma \sim \mathcal{U}[0, 6]$.

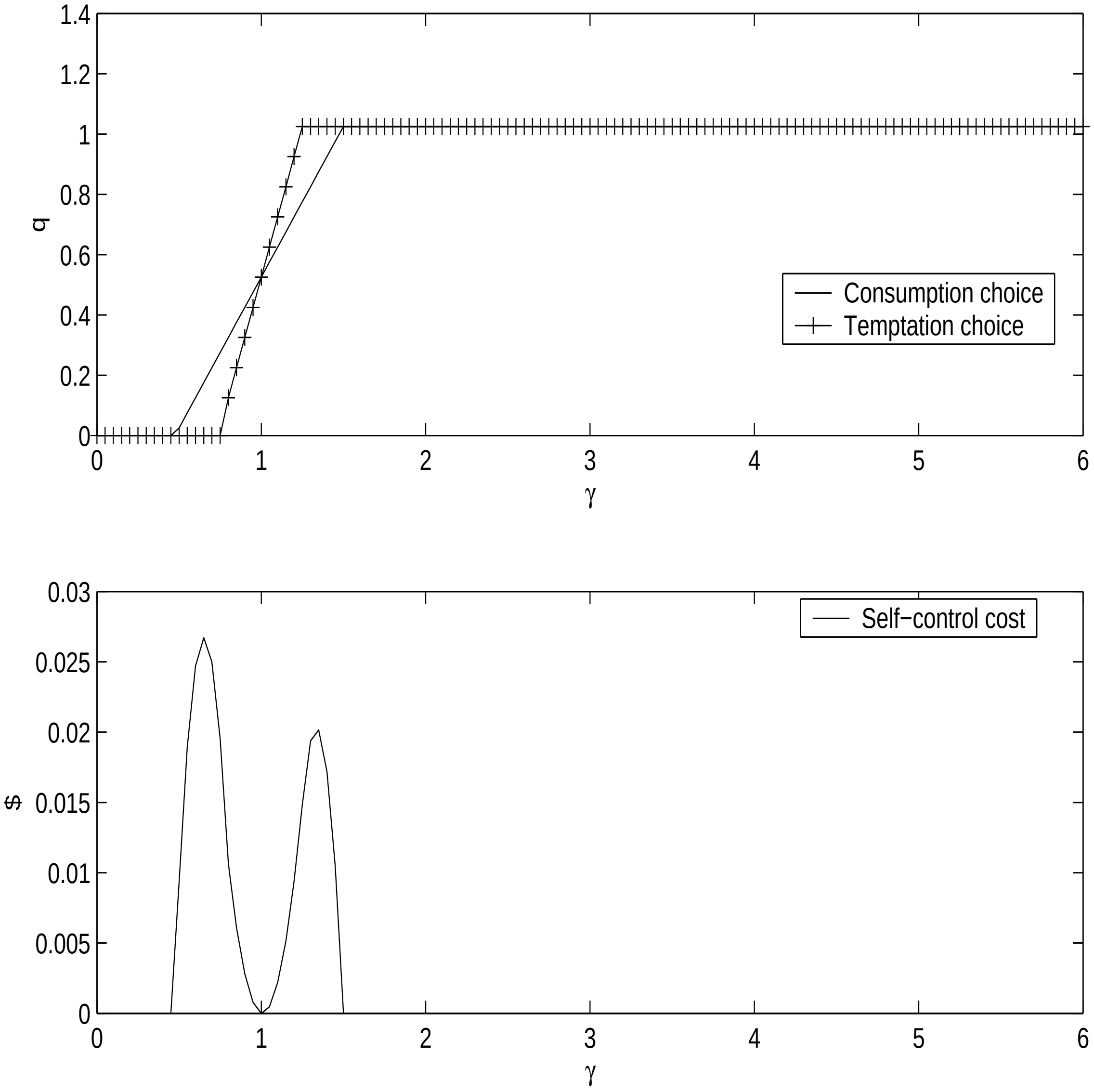

Figure 5: Consumption and temptation choices.
Utility functions parameterized as in (5) and (6); cost function $C(q) = \frac{1}{2}q^2$;
$\gamma \sim \mathcal{U}[0, 6]$.

choices are the same.

The bottom graph computes the self-control cost (i.e., the difference $V(x_\gamma) - V(z_\gamma)$) given the consumption and temptation choices in the example. Consumer γ^* and consumers above $\bar{\gamma}$ incur no self-control costs. Since consumer γ^* does not have preference reversals, he incurs zero self-control costs. On the other hand, consumers above $\bar{\gamma}$ are tempted to buy higher quality goods; however, since there are none available, they incur zero self-control costs.

6 Concluding Remarks

We have studied a monopolist's supply decision when it faces consumers with temptation and self-control preferences. We have analyzed the case where consumers are heterogeneous in their temptation preferences and shown that, if all consumers have preferences with upward temptation, the optimal menu consists of a single bundle, and consumers do not incur any self-control costs. Instead, if some consumers have preferences with downward temptation, then the optimal menu may contain a continuum of bundles and consumers incur self-control costs. Generally, the optimal tariff differs from the one for the standard nonlinear pricing problem.

A number of questions arise from this paper. For example, the role of advertising can be interesting in our setting. If consumers become aware of the monopolist's offerings only via advertisements, then the seller may choose to advertise only a subset of its products, to lead consumers to believe that self-control costs are small and lure them inside the store.[16] Indeed, the optimal advertising decision is to advertise only one bundle $(\varepsilon_q, \varepsilon_t)$, where ε_q and ε_t are both small and satisfy $U(\varepsilon_q, \varepsilon_t) \geq 0$, so that all consumers are willing, based upon the advertisement, to enter the store. Because all consumers enter the store, the ex-ante IR condition is no longer an issue, and the monopolist will be able to implement the optimal menu in the standard problem to its customers.

In ongoing work, we are considering what happens if the monopolist can open multiple stores and separate different consumer types into different locations. Since self-control costs are lower in stores with smaller menus, the monopolist may want to divide the set of bundles that it sells into multiple stores. But characterizing the monopolist's optimal decision is a non-trivial problem. Another interesting extension is to examine how optimal pricing would be affected by the presence of rival firms, in the manner of the competitive price discrimination literature, surveyed in Stole (2002).

[16] In the present paper, we assume that consumers are completely aware of the monopolist's offerings before they enter the store.

A Appendix

A.1 Proof of Proposition 5

Let (M, x, Γ) be a regular optimal schedule. Since the tariff may have "holes," which would make it difficult to handle, we first fill the holes in a way that will be convenient for the proof.

Let $w(x, \gamma)$ be defined by

$$w(x, \gamma) = U(x) + V_\gamma(x),$$

which is the ex-post utility level of consumer type γ when he consumes bundle x. For all $q \in \mathbb{R}_+$ and all $\gamma \in [a, b]$, let $T(q, \gamma) \in \mathbb{R}$ be the unique number t such that

$$w(q, t, \gamma) = W^{EP}(M, \gamma) \equiv w(x(\gamma), \gamma).$$

That is, $t = T(q, \gamma)$ is the unique amount such that bundle (q, t) is indifferent to $x(\gamma)$ for type γ. There exists such a t since U and V_γ are continuous and unbounded in t, and t is unique since U and V_γ are strictly increasing in t.

Lemma 4. *Function T is continuous.*

Proof. Let $(q, \gamma) \in \mathbb{R}_+ \times [a, b]$, $t = T(q, \gamma)$, and $\delta > 0$. Then for a small number $\varepsilon > 0$,

$$w(q, t - \delta, \gamma) + 2\varepsilon < W^{EP}(M, \gamma) < w(q, t + \delta, \gamma) - 2\varepsilon.$$

Since w and $W^{EP}(M, \cdot)$ are continuous, if we take (q', γ') that is sufficiently close to (q, γ), we have

$$\begin{aligned} w(q', t - \delta, \gamma') &< w(q, t - \delta, \gamma) + \varepsilon < W^{EP}(M, \gamma) - \varepsilon < W^{EP}(M, \gamma') \\ &< W^{EP}(M, \gamma) + \varepsilon < w(q, t + \delta, \gamma) - \varepsilon < w(q', t + \delta, \gamma'). \end{aligned}$$

Thus $T(q', \gamma') \in (t - \delta, t + \delta)$. Q.E.D.

We define a function $\hat{\gamma}\colon [q(a), q(b)] \to [a, b]$ as follows. For a given $q \in [q(a), q(b)]$, if $q = q(\gamma)$ for some $\gamma \in [a, b]$, then we choose such a γ arbitrarily and let $\hat{\gamma}(q) = \gamma$. On the other hand, if q is such that $q \neq q(\gamma)$ for all $\gamma \in [a, b]$, then, since $q(\cdot)$ is non-decreasing, there exists a unique type $\gamma \in (a, b)$ such that

$$q(\gamma') < q < q(\gamma'') \quad \text{for all } \gamma' < \gamma \text{ and all } \gamma'' > \gamma.$$

We then set $\hat{\gamma}(q) = \gamma$.

The definition implies that, for all $q \in [q(a), q(b)]$,

$$q(\gamma') \leq q \leq q(\gamma'') \quad \text{for all } \gamma' < \hat{\gamma}(q) \text{ and all } \gamma'' > \hat{\gamma}(q). \tag{36}$$

Lemma 5. *Function $\hat{\gamma}$ is non-decreasing.*

Proof. If $q' > q$ and $\hat{\gamma}(q') < \hat{\gamma}(q)$, then we can pick $\gamma'' \in (\hat{\gamma}(q'), \hat{\gamma}(q))$. By (36), $q(\gamma'') \leq q$ and $q' \leq q(\gamma'')$, which is impossible. Q.E.D.

Let $\hat{t}\colon [q(a), q(b)] \to \mathbb{R}$ be a function defined by

$$\hat{t}(q) = T(q, \hat{\gamma}(q)).$$

Lemma 6. *Function $\hat{t}$ is continuous.*

Proof. Consider a sequence $\{q_k\}_{k=1}^{\infty}$ such that $q_k \to q$ as $k \to \infty$. Without loss of generality, assume that $q_k \leq q_{k+1}$ for all k. Since $\hat{\gamma}$ is non-decreasing, $\hat{\gamma}(q_k)$ converges to some γ. Since $\hat{\gamma}(q_k) \leq \hat{\gamma}(q)$ for all k, we have $\hat{\gamma}(q_k) \leq \gamma \leq \hat{\gamma}(q)$. Since T is continuous, $\hat{t}(q_k) = T(q_k, \hat{\gamma}(q_k))$ converges to $T(q, \gamma)$. We would like to show that $T(q, \gamma) = \hat{t}(q)$. If $\gamma = \hat{\gamma}(q)$, we are done since $T(q, \hat{\gamma}(q)) = \hat{t}(q)$ by definition. Thus suppose $\gamma < \hat{\gamma}(q)$.

We first show that $q(\gamma') = q$ for all $\gamma' \in (\gamma, \hat{\gamma}(q))$. Indeed, by (36), $q(\gamma') \leq q$. Suppose, by way of contradiction, that $q(\gamma') < q$ for some $\gamma' \in (\gamma, \hat{\gamma}(q))$. Since $\hat{\gamma}(q(\gamma')) = \gamma'$, the monotonicity of $\hat{\gamma}$ implies that $\hat{\gamma}(q_k) \geq \gamma'$ for a sufficiently large k (since then $q(\gamma') \leq q_k \leq q$). But since $\hat{\gamma}(q_k) \leq \gamma < \gamma'$, we obtain a contradiction.

The previous paragraph implies that (M, x, Γ) assigns $(q, \hat{t}(q))$ to all types $\gamma' \in (\gamma, \hat{\gamma}(q))$. This implies that, for all these types γ', $T(q, \gamma') = \hat{t}(q)$. Since T is continuous, $T(q, \gamma) = \hat{t}(q)$. Q.E.D.

We now show that the ex-post IC of $x(\cdot)$ is preserved in the enlarged set of bundles $\{(q, \hat{t}(q)) : q(a) \leq q \leq q(b)\}$; i.e., there exists no bundle $(q, \hat{t}(q))$ that type γ strictly prefers to $x(\gamma) = (q(\gamma), \hat{t}(q(\gamma)))$.

Lemma 7. *For all $\gamma \in [a, b]$ and all $q \in [q(a), q(b)]$,*

$$w(x(\gamma), \gamma) \geq w(q, \hat{t}(q), \gamma). \tag{37}$$

Proof. Let $\gamma \in [a, b]$ and $q \in [q(a), q(b)]$. If $q = q(\gamma')$ for some γ', then (37) follows from the ex-post IC of (M, x, Γ). Thus assume that $q \neq q(\gamma')$ for all γ'.

If $\hat{\gamma}(q) = \gamma$, then $\hat{t}(q) = T(q, \gamma)$, so (37) holds with equality. Thus, we assume, without loss of generality, that $\hat{\gamma}(q) > \gamma$; the case when $\hat{\gamma}(q) < \gamma$ can be proved similarly. Consider any sequence $\{\gamma_k\}_{k=1}^{\infty}$ such that $\gamma_k \leq \gamma_{k+1} < \hat{\gamma}(q)$ for all k and $\gamma_k \to \hat{\gamma}(q)$. Since $x(\cdot)$ is non-decreasing, $x(\gamma_k)$ converges to some point $x' = (q', t')$.

Since w is continuous, the ex-post IC of $x(\cdot)$ implies

$$w(x', \hat{\gamma}(q)) = w(x(\hat{\gamma}(q)), \hat{\gamma}(q)).$$

That is, x' is on $\hat{\gamma}(q)$'s indifference curve that passes through $x(\hat{\gamma}(q))$. Let (q^c, t^c) be the point where $\hat{\gamma}(q)$'s indifference curve on $x(\hat{\gamma}(q))$ crosses γ's indifference curve on $x(\gamma)$. Since γ does not prefer $x(\gamma_k)$ to $x(\gamma)$, it follows that $q' \geq q^c$. Since $\gamma_k < \hat{\gamma}(q)$, (36) implies $q' \leq q$. Therefore $q \geq q^c$, which implies that γ does not prefer $(q, \hat{t}(q))$ to $x(\gamma)$; i.e., (37) holds. Q.E.D.

Now, suppose, by way of contradiction, that $W^{EP}(M, a) > 0$. The argument that follows can be divided into several steps.

Step 1 (Preliminaries). We first consider the case when

$$w(q, \hat{t}(q), a) = w(q', \hat{t}(q'), a) \quad \text{for all } q, q' \in [q(a), q(b)].$$

That is, all bundles in the enlarged menu lie on a single indifference curve of type a. Then by ex-post IC, $q(\gamma) = q(b)$ for all $\gamma > a$, and hence the menu contains at most two quantities. Let

$$\gamma' \in \operatorname*{argmax}_{\gamma \in \{a,b\}} \left[t(\gamma) - C(q(\gamma))\right].$$

Then for a sufficiently small $\delta > 0$, a singleton menu $\{(q(\gamma'), t(\gamma') + \delta)\}$ satisfies both conditions of IR and generates more profits than (M, x, Γ), in contradiction with the optimality of (M, x, Γ).

Thus we assume, in what follows, that there exists $k \in (q(a), q(b))$ such that

$$w(q(a), t(a), a) > w(k, \hat{t}(k), a) > 0. \tag{38}$$

Since w and $\hat{t}$ are continuous, we can choose $k > q(a)$ sufficiently small to ensure that

$$w(q, \hat{t}(q), a) > 0 \quad \text{for all } q \in [q(a), k]. \tag{39}$$

Let $\pi \colon [q(a), q(b)] \to \mathbb{R}$ be a function defined by

$$\pi(q) = \hat{t}(q) - C(q),$$

which denotes the profit level of bundle $(q, \hat{t}(q))$. Since this function is continuous, there exists a quantity level $q^m \in [q(a), k]$ that maximizes π over $[q(a), k]$. By (39), there exists a small number $\varepsilon > 0$ such that

$$w(q^m, \hat{t}(q^m) + \varepsilon, a) > 0. \tag{40}$$

Step 2 (Construction of a tariff). We now construct a tariff that generates more profits than (M, x, Γ). We basically consider the upper envelope of $\hat{t}$ and the isoprofit curve that passes through $(q^m, \hat{t}(q^m)+\varepsilon)$. Formally, let $\tau\colon [q(a), q(b)] \to \mathbb{R}$ be a function defined by

$$\tau(q) = \max\{\hat{t}(q), C(q) + \pi(q^m) + \varepsilon\}. \tag{41}$$

Note that τ is continuous. It is possible that a part of the upper envelope generates negative U utility, in which case we remove the part. Formally, let

$$D = \{q \in [q(a), q(b)] : U(q, \tau(q)) \geq 0\}.$$

Note that D contains q^m and is compact by A5. Then the menu that we consider is given by

$$M' = \{(q, \tau(q)) : q \in D\}.$$

Let $x'\colon [a, b] \to M' \cup \{(0, 0)\}$ be an assignment function for M' such that, for all $\gamma \in [a, b]$,

$$x'(\gamma) \in \operatorname*{argmax}_{x \in M'} w(x, \gamma) \tag{42}$$

and, if $x(\gamma)$ belongs to M' and solves (42), then $x'(\gamma) = x(\gamma)$.

Step 3 (IR and IC). We observe that (M', x', Γ) is a feasible schedule. The ex-post IC condition follows from (42). The ex-ante IR condition is satisfied for all types because of the definition of D. The ex-post IR condition is also satisfied for all types since (40) implies that type a can obtain a positive ex-post utility from M'.

Step 4. We observe that in terms of the ex-post utilities, all types are weakly worse off in M' than M. Indeed, Lemma 7 and (41) imply that, for all $\gamma \in [a, b]$,

$$W^{EP}(M, \gamma) \geq w(q'(\gamma), \hat{t}(q'(\gamma)), \gamma) \geq w(q'(\gamma), \tau(q'(\gamma)), \gamma) = W^{EP}(M', \gamma).$$

Step 5. We prove that (M', x', Γ) is at least as profitable as (M, x, Γ). We actually prove that each type generates weakly more profits in (M', x', Γ) than in (M, x, Γ); i.e.,

$$\tau(q'(\gamma)) - C(q'(\gamma)) \geq \pi(q(\gamma)) \quad \text{for all } \gamma. \tag{43}$$

To prove this, we first consider types γ such that

$$\pi(q(\gamma)) \geq \pi(q^m) + \varepsilon.$$

Then, by (41), $\tau(q(\gamma)) = \hat{t}(q(\gamma))$. This implies that $x(\gamma)$ is available in M'. Then, by Step 4 and the tie-breaking rule for x', we have $x'(\gamma) = x(\gamma)$, and (43) holds with equality.

We now consider types γ such that

$$\pi(q(\gamma)) < \pi(q^m) + \varepsilon. \tag{44}$$

Then by (41),

$$\tau(q(\gamma)) = C(q(\gamma)) + \pi(q^m) + \varepsilon > \hat{t}(q(\gamma)).$$

Using (41) again, we obtain

$$\tau(q'(\gamma)) - C(q'(\gamma)) \geq \pi(q^m) + \varepsilon > \pi(q(\gamma)). \tag{45}$$

Step 6 (Final). We prove that (M', x', Γ) generates strictly more profits than (M, x, Γ). By (45), it suffices to show that there exists a positive measure of types that satisfy (44). First, note that (44) holds for type a by the definition of q^m. Furthermore, the first inequality in (38) implies $T(k, a) < \hat{t}(k)$ and hence $\hat{\gamma}(k) > a$. By $T(k, \hat{\gamma}(k)) = \hat{t}(k)$ and Lemma 7, we have

$$w(k, \hat{t}(k), \hat{\gamma}(k)) \geq w(q, \hat{t}(q), \hat{\gamma}(k)) \quad \text{for all } q \in [q(a), q(b)].$$

Then by single-crossing and Lemma 7, it follows that, for all types $\gamma \in (a, \hat{\gamma}(k))$, $q(\gamma) \leq k$. Hence, by the definition of q^m, all these types satisfy (44). Q.E.D.

A.2 Proof of Proposition 6

Let (M, x, Γ) be a regular optimal schedule. By regularity, $x(\gamma) \in B$ for all $\gamma \in \Gamma$, which implies that $\bar{x} \in B$ and hence $\bar{x}$ is finite. Since the assignment function x is monotonic, it follows that $\bar{x} = x(b)$; if $b = +\infty$, then $\bar{x} = \lim_{\gamma \to +\infty} x(\gamma)$. Since $U(x(\gamma)) \geq 0$ for all γ, we have $U(\bar{x}) \geq 0$. We assume, by way of contradiction, that $U(\bar{x}) > 0$.

By A6, there exist a type $\gamma' \in \Gamma$ and a bundle $x' = (q', t') \gg \bar{x}$ such that

$$U(x') + V_{\gamma'}(x') > U(\bar{x}) + V_{\gamma'}(\bar{x}), \tag{46}$$

$$t' - C(q') > \bar{t} - C(\bar{q}). \tag{47}$$

Since $U(\bar{x}) > 0$, we can choose x' close to $\bar{x}$ so that $U(x') \geq 0$.

We prove that the monopolist can increase profits by adding x' to its menu and offering $M' = M \cup \{x'\}$. The associated assignment function to consider is $\hat{x}$ defined by: $\hat{x}(\gamma) = x'$ if consumer γ strictly prefers x' to $x(\gamma)$ in terms of $U + V$ utility, and $\hat{x}(\gamma) = x(\gamma)$ otherwise.

We first show that $(M', \hat{x}, \Gamma)$ is a feasible schedule. Ex-ante IR remains to hold for all types since the added bundle satisfies $U(x') \geq 0$. Ex-post IC is also satisfied since x' is assigned to consumer γ only if the consumer prefers x' to $x(\gamma)$. Ex-post IR is also satisfied for all types since the addition of x' does not make any consumer worse off in terms of $U + V$ utility.

We now show that, for all types γ that prefer x' to $x(\gamma)$, x' generates more profits than $x(\gamma)$. Indeed, for these types, we have

$$\begin{aligned} U(x') + V_\gamma(x') &> U(x(\gamma)) + V_\gamma(x(\gamma)) \\ &\geq U(x(\hat{\gamma})) + V_\gamma(x(\hat{\gamma})) \quad \text{for all } \hat{\gamma} \in \Gamma. \end{aligned}$$

Taking the limit of $\hat{\gamma} \to b$, we obtain

$$U(x') + V_\gamma(x') > U(x(\gamma)) + V_\gamma(x(\gamma)) \geq U(\bar{x}) + V_\gamma(\bar{x}). \tag{48}$$

Now, consider the vertical translation of the cost curve that passes through $\bar{x}$. Then, (47) implies that x' is strictly above this curve, and (48) implies that the curve crosses γ's indifference curve at $\bar{x}$ from above. Then the second inequality in (48) and $x(\gamma) \leq \bar{x}$ imply that $x(\gamma)$ lies below the translation of the cost curve, which implies that $x(\gamma)$ generates strictly less profits than x'.

Finally, we show that a strictly positive measure of types γ prefer x' to $x(\gamma)$. To do so, we distinguish two cases.

Case 1: $b < +\infty$. Then $\bar{x} = x(b)$. By single-crossing, (46) also holds for type b, which can be written as

$$W^{EP}(M, b) < U(x') + V_b(x').$$

By a standard argument, $W^{EP}(M, \cdot)$ is continuous in γ. Since $V_\gamma(x')$ is also continuous in γ, there exists $\varepsilon > 0$ such that for all $\gamma > b - \varepsilon$, $W^{EP}(M, \gamma) < U(x') + V_\gamma(x')$. Thus all these types choose x' in $M \cup \{x'\}$.

Case 2: $b = +\infty$. By (46), there exists $\varepsilon > 0$ such that

$$U(x') + V_{\gamma'}(x') > U(\bar{q}, \bar{t} - \varepsilon) + V_{\gamma'}(\bar{q}, \bar{t} - \varepsilon).$$

By single-crossing, the same inequality also holds for all $\gamma \geq \gamma'$. Since $\bar{x}$ is the limit of $x(\gamma)$, there exists $\gamma'' \geq \gamma'$ such that, for all $\gamma \geq \gamma''$, $\|x(\gamma) - \bar{x}\| < \varepsilon$. Then all types $\gamma \geq \gamma''$ prefer x' to $x(\gamma)$; indeed, by $\gamma \geq \gamma'' \geq \gamma'$ and single-crossing,

$$\begin{aligned} U(x') + V_\gamma(x') &> U(\bar{q}, \bar{t} - \varepsilon) + V_\gamma(\bar{q}, \bar{t} - \varepsilon) \\ &\geq U(\bar{q}, t(\gamma)) + V_\gamma(\bar{q}, t(\gamma)) \\ &\geq U(x(\gamma)) + V_{\gamma''}(x(\gamma)), \end{aligned}$$

where the second inequality uses $\|x(\gamma) - \bar{x}\| < \varepsilon$ and the last inequality uses $\bar{q} \geq q(\gamma)$. Q.E.D.

A.3 Proof of Proposition 7

Suppose, by way of contradiction, that there exists a bundle $x' \geq \bar{x}$ such that $U(x') \geq 0$ and $t' - C(q') > \bar{t} - C(\bar{q})$. We claim that the monopolist can earn more profits by offering $M' = M \cup \{x'\}$. The argument is identical to that in the proof of Proposition 6; just note that (46) holds for all $\gamma' > \gamma^*$ since $\bar{x}$ and x' are on the curve of $U = 0$. Q.E.D.

A.4 Proof of Proposition 8

For $\underline{\gamma}$ defined as in (33), since W is non-decreasing, $W(\gamma) = 0$ for all $\gamma \in [a, \underline{\gamma}]$. Thus, if $\underline{\gamma} > a$, then for all $\gamma \in [a, \underline{\gamma}]$, $v_2(q(\gamma), \gamma) = q(\gamma) = 0$ and hence the ex-ante IR condition (23) holds with equality. By CQ, $\underline{\gamma} < \gamma^*$.

We partition $[a, b]$ into two sets on the basis of whether the ex-ante IR condition (23) binds or not:

$$\Gamma_+ = \{\gamma \in [a,b] : W(\gamma) > v(q(\gamma),\gamma) - u(q(\gamma))\},$$
$$\Gamma_0 = \{\gamma \in [a,b] : W(\gamma) = v(q(\gamma),\gamma) - u(q(\gamma))\}.$$

Claim 1. For all $\gamma \in \Gamma_0$ such that $\gamma > \underline{\gamma}$, $q(\gamma) > 0$.

Proof. Indeed, if $q(\gamma) = 0$, then

$$W(\gamma) = v(0, \gamma) - u(0) = 0,$$

which implies $\gamma \leq \underline{\gamma}$, a contradiction. Q.E.D.

Claim 2. $\bar{\gamma} \geq \gamma^*$.

Proof. By CQ, $\gamma^* \in \Gamma_+$, implying $\bar{\gamma} = \sup \Gamma_+ \geq \gamma^*$. Q.E.D.

Claim 3. If $\bar{\gamma} < b$, then q is constant over $\bar{I} \equiv [\bar{\gamma}, b] \cap \Gamma_0$. (Note that $(\bar{\gamma}, b] \subseteq \Gamma_0$ but $\bar{\gamma}$ may not be in Γ_0.)

Proof. Let $\gamma_1, \gamma_2 \in \bar{I}$ be such that $\gamma_2 > \gamma_1$. By (22),

$$W(\gamma_2) - W(\gamma_1) = \int_{\gamma_1}^{\gamma_2} v_2(q(\gamma), \gamma)\, d\gamma. \tag{49}$$

Since ex-ante IR holds with equality at γ_1 and γ_2, the left-hand side of (49) is equal to

$$\begin{aligned} v(q(\gamma_2),\gamma_2) - u(q(\gamma_2)) &- \big[v(q(\gamma_1),\gamma_1) - u(q(\gamma_1))\big] \\ &= \int_{\gamma^*}^{\gamma_2} v_2(q(\gamma_2),\gamma)\, d\gamma - \int_{\gamma^*}^{\gamma_1} v_2(q(\gamma_1),\gamma)\, d\gamma \\ &= \int_{\gamma^*}^{\gamma_2} \int_{q(\gamma_1)}^{q(\gamma_2)} v_{12}(q,\gamma)\, dq\, d\gamma + \int_{\gamma_1}^{\gamma_2} v_2(q(\gamma_1),\gamma)\, d\gamma. \end{aligned}$$

Thus (49) reduces to

$$\int_{\gamma^*}^{\gamma_2}\int_{q(\gamma_1)}^{q(\gamma_2)} v_{12}(q,\gamma)\,dq\,d\gamma = \int_{\gamma_1}^{\gamma_2}\int_{q(\gamma_1)}^{q(\gamma)} v_{12}(q,\gamma)\,dq\,d\gamma,$$

which is equivalent to

$$\int_{\gamma^*}^{\gamma_1}\int_{q(\gamma_1)}^{q(\gamma_2)} v_{12}(q,\gamma)\,dq\,d\gamma + \int_{\gamma_1}^{\gamma_2}\int_{q(\gamma)}^{q(\gamma_2)} v_{12}(q,\gamma)\,dq\,d\gamma = 0. \tag{50}$$

This equation holds for any $\gamma_1, \gamma_2 \in \bar{I}$ such that $\gamma_1 < \gamma_2$.

To prove that q is constant over $\bar{I}$, suppose, by way of contradiction, that there exist $\gamma_3, \gamma_4 \in \bar{I}$ such that $\gamma_3 < \gamma_4$ and $q(\gamma_3) \neq q(\gamma_4)$.

We first consider the case when $q(\gamma_3) < q(\gamma_4)$. Let $q' = \sup\{q(\gamma) : \gamma_3 \leq \gamma \leq \gamma_4\}$. Then there exists $\gamma' \in [\gamma_3, \gamma_4]$ such that $q(\gamma') \geq q(\gamma_4)$ and

$$\int_{\gamma_3}^{\gamma_4}\int_{q(\gamma')}^{q'} v_{12}(q,\gamma)\,dq\,d\gamma < \int_{\gamma^*}^{\gamma_3}\int_{q(\gamma_3)}^{q(\gamma_4)} v_{12}(q,\gamma)\,dq\,d\gamma, \tag{51}$$

since the right-hand side is positive[17] and the left-hand side can be set arbitrarily close to zero. By applying (50) to $\gamma_1 = \gamma_3$ and $\gamma_2 = \gamma'$, we obtain

$$\begin{aligned} 0 &= \int_{\gamma^*}^{\gamma_3}\int_{q(\gamma_3)}^{q(\gamma')} v_{12}(q,\gamma)\,dq\,d\gamma + \int_{\gamma_3}^{\gamma'}\int_{q(\gamma)}^{q(\gamma')} v_{12}(q,\gamma)\,dq\,d\gamma \\ &\geq \int_{\gamma^*}^{\gamma_3}\int_{q(\gamma_3)}^{q(\gamma_4)} v_{12}(q,\gamma)\,dq\,d\gamma - \int_{\gamma_3}^{\gamma_4}\int_{q(\gamma')}^{q'} v_{12}(q,\gamma)\,dq\,d\gamma > 0, \end{aligned} \tag{52}$$

which is a contradiction.

If $q(\gamma_3) > q(\gamma_4)$, a symmetric argument applies. Specifically, we set $q' = \inf\{q(\gamma) : \gamma_3 \leq \gamma \leq \gamma_4\}$. Then there exists $\gamma' \in [\gamma_3, \gamma_4]$ such that $q(\gamma') \leq q(\gamma_4)$ and (51) holds with the reverse inequality. We then obtain (52) with reverse inequalities. Q.E.D.

Let $\bar{q} = q(b)$.

Claim 4. $\bar{\gamma} > \gamma^*$.

Proof. Suppose, by way of contradiction, that $\bar{\gamma} = \gamma^*$. For all $\gamma > \bar{\gamma}$, the ex-ante IR condition (23) holds with equality and $q(\gamma) = \bar{q}$. Thus

$$W(\gamma^*) = v(\bar{q}, \gamma^*) - u(\bar{q}) = 0, \tag{53}$$

where the last equality follows from $v(\,\cdot\,, \gamma^*) = u(\cdot)$. On the other hand, since $\gamma^* \in \Gamma_+$

[17] We have $\gamma_3 > \gamma^*$ since $\gamma_3 \in \bar{I} \subseteq \Gamma_0$ and $\gamma^* \in \Gamma_+$.

(by CQ), we have

$$W(\gamma^*) > v(q(\gamma^*), \gamma^*) - u(q(\gamma^*)) = 0,$$

which is a contradiction with (53). Q.E.D.

Claim 5. If $\bar{\gamma} < b$, then $\mu(\bar{\gamma}) < 0$.

Proof. Since ex-ante IR binds for all $\gamma > \bar{\gamma}$,

$$v(\bar{q}, \bar{\gamma}) - u(\bar{q}) = W(\bar{\gamma}) \geq v(q(\bar{\gamma}), \bar{\gamma}) - u(q(\bar{\gamma})). \tag{54}$$

This inequality, together with $\bar{\gamma} > \gamma^*$, implies

$$q(\bar{\gamma}) \leq \bar{q}. \tag{55}$$

(28) and (29) imply that for all $\gamma > \bar{\gamma}$,

$$2[u'(\bar{q}) - C'(\bar{q})]f(\gamma) + \mu(\gamma)v_{12}(\bar{q}, \gamma) - \mu'(\gamma)[u'(\bar{q}) - v_1(\bar{q}, \gamma)] = 0.$$

This differential equation can be solved using (31) and the fact the sum of the last two terms is the derivative of $-\mu(\gamma)(u'(\bar{q}) - v_1(\bar{q}, \gamma))$. We then obtain

$$\mu(\bar{\gamma}) = \frac{2(1 - F(\bar{\gamma}))\big[u'(\bar{q}) - C'(\bar{q})\big]}{v_1(\bar{q}, \bar{\gamma}) - u'(\bar{q})}. \tag{56}$$

Noting that $q(\bar{\gamma}) \neq \bar{q}$ is possible, the argument that yields (60) yields

$$\big[u'(q(\bar{\gamma})) + v_1(q(\bar{\gamma}), \bar{\gamma}) - 2C'(q(\bar{\gamma}))\big]f(\bar{\gamma}) + \mu(\bar{\gamma})v_{12}(q(\bar{\gamma}), \bar{\gamma}) = 0. \tag{57}$$

We now suppose, by way of contradiction, that $\mu(\bar{\gamma}) \geq 0$. Then (57) implies

$$u'(q(\bar{\gamma})) + v_1(q(\bar{\gamma}), \bar{\gamma}) \leq 2C'(q(\bar{\gamma})). \tag{58}$$

On the other hand, (56) implies that $u'(\bar{q}) \geq C'(\bar{q})$. Since $\bar{\gamma} > \gamma^*$, it follows that

$$u'(\bar{q}) + v_1(\bar{q}, \bar{\gamma}) > 2C'(\bar{q}). \tag{59}$$

(59) and (58) imply $\bar{q} < \hat{q}$, in contradiction with (55). Q.E.D.

Claim 6. $\mu(\gamma) \leq 0$ for all $\gamma < \bar{\gamma}$ that is sufficiently close to $\bar{\gamma}$.

Proof. This follows immediately from Claim 5 if $\bar{\gamma} < b$ since μ is continuous. Thus assume $\bar{\gamma} = b$. Since $\mu(b) = 0$ and $\mu'(b) = f(b) - \lambda(b)$, it suffices to prove $\lambda(b) = 0$. Since this follows from (30) if $b \in \Gamma_+$, assume $b \in \Gamma_0$. Claim 1 implies $q(b) > 0$. Since q is continuous from the left, $q(\gamma) > 0$ for all γ close to b. Furthermore, since $b = \bar{\gamma}$, it follows that for all $\varepsilon > 0$, there exists $\gamma \in (b-\varepsilon, b)$ such that $\gamma \in \Gamma_+$. Therefore, taking the limit of (28) from the left at b yields

$$\big[u'(q(b)) + v_1(q(b), b) - 2C'(q(b))\big] f(b) + \mu(b) v_{12}(q(b), b) = 0. \tag{60}$$

On the other hand, evaluating (28) at b yields

$$\begin{aligned}\big[u'(q(b)) + v_1(q(b), b) - 2C'(q(b))\big] f(b) + \mu(b) v_{12}(q(b), b)& \\ + \lambda(b)\big[u'(q(b)) - v_1(q(b), b)\big] &= 0.\end{aligned}$$

Since $b > \gamma^*$, these equations imply $\lambda(b) = 0$. Q.E.D.

By Claim 6, there exists $\varepsilon > 0$ such that $\mu(\gamma) \leq 0$ for all $\gamma \in [\bar{\gamma} - \varepsilon, \bar{\gamma}]$. In what follows, let $\tilde{\gamma} \in [\bar{\gamma} - \varepsilon, \bar{\gamma}]$ be such that $\tilde{\gamma} \in \Gamma_+$. Let $\tilde{I}$ be the maximal interval such that $\tilde{\gamma} \in \tilde{I}$ and $\tilde{I} \subseteq \Gamma_+$.

Claim 7. q is non-decreasing over $\tilde{I}$.

Proof. Define a function $\beta\colon [a, b] \to \mathbb{R}$ by

$$\beta(\gamma) = F(\gamma) - \mu(\gamma).$$

Since $\beta'(\gamma) = \lambda(\gamma)$, it follows that over the interval $\tilde{I}$, $\beta(\gamma)$ is constant and equal to $\beta(\tilde{\gamma})$. Since $\mu(\tilde{\gamma}) \leq 0$, $\beta(\tilde{\gamma}) \geq 0$. Since β is non-decreasing, $\beta(\tilde{\gamma}) \leq \beta(b) = 1$. (28) implies that for all $\gamma \in \tilde{I}$,

$$u'(q(\gamma)) + v_1(q(\gamma), \gamma) - v_{12}(q(\gamma), \gamma)\frac{\beta(\tilde{\gamma}) - F(\gamma)}{f(\gamma)} + \frac{\delta(\gamma)}{f(\gamma)} = 2C'(q(\gamma)). \tag{61}$$

Define a function $\Pi\colon \mathbb{R}_+ \times [a, b] \to \mathbb{R}$ by

$$\Pi(q, \gamma) = u(q) + v(q, \gamma) - v_2(q, \gamma)\frac{\beta(\tilde{\gamma}) - F(\gamma)}{f(\gamma)} - 2C(q).$$

Then for all $\gamma \in \tilde{I}$, $q(\gamma)$ satisfies $\Pi_q(q(\gamma), \gamma) \leq 0$ and $q(\gamma)\Pi_q(q(\gamma), \gamma) = 0$. Given the way in which $\tilde{\gamma}$ is chosen, it follows that for all $\gamma \in \tilde{I}$, $\beta(\tilde{\gamma}) - F(\gamma) = -\mu(\gamma) \geq 0$. Since we assume $C'' > 0$ and $v_{112} \geq 0$, we obtain

$$\Pi_{qq} = u''(q, \gamma) + v_{11}(q, \gamma) - v_{112}(q, \gamma)\frac{\beta(\tilde{\gamma}) - F(\gamma)}{f(\gamma)} - 2C''(q) < 0, \tag{62}$$

which implies that $q(\gamma)$ is determined uniquely. Thus a sufficient condition for q to be non-decreasing is $\Pi_{q\gamma} \geq 0$. The cross derivative is given by

$$\Pi_{q\gamma} = -v_{12}(q,\gamma)\frac{\partial\left(\frac{\beta(\tilde{\gamma})-F(\gamma)}{f(\gamma)}\right)}{\partial\gamma} - v_{122}(q,\gamma)\frac{\beta(\tilde{\gamma})-F(\gamma)}{f(\gamma)}.$$

Since we assume $v_{122} \leq 0$, a sufficient condition for $\Pi_{q\gamma} \geq 0$ is that $(\beta(\tilde{\gamma}) - F(\gamma))/f(\gamma)$ is non-increasing. This condition is satisfied since $\beta(\tilde{\gamma}) \in [0,1]$ and we assume that f is log-concave (Bagnoli and Bergstrom, 1989). Q.E.D.

Define a function $h\colon \mathbb{R}_+ \times [a,b] \to \mathbb{R}$ by

$$h(q,\gamma) = W(\gamma) - v(q,\gamma) + u(q).$$

Claim 8. $h(q(\gamma),\gamma)$ is non-increasing in γ over $\tilde{I} \cap [\gamma^*, b]$.

Proof. Let $\gamma,\gamma' \in \tilde{I}$ be such that $\gamma^* \leq \gamma' < \gamma$. Since $\gamma' \geq \gamma^*$ and q is non-decreasing over $\tilde{I}$,

$$\begin{aligned} h(q(\gamma'),\gamma') &= W(\gamma') - v(q(\gamma'),\gamma') + u(q(\gamma')) \\ &\geq W(\gamma') - v(q(\gamma),\gamma') + u(q(\gamma)). \end{aligned} \tag{63}$$

Moreover, $v_{12} \geq 0$ implies that for all $\theta \in (\gamma',\gamma)$, $v_2(q(\theta),\theta) - v_2(q(\gamma),\theta) \leq 0$. This implies that the first two terms in (63) do not increase if we replace γ' by γ; i.e.,

$$\begin{aligned} W(\gamma') - v(q(\gamma),\gamma') + u(q(\gamma)) &\geq W(\gamma) - v(q(\gamma),\gamma) + u(q(\gamma)) \\ &= h(q(\gamma),\gamma). \end{aligned}$$

This and (63) prove the desired monotonicity. Q.E.D.

Let $\underset{\sim}{\gamma} = \inf \tilde{I}$ and $\underset{\sim}{q} = \inf\{q(\gamma) : \gamma \in \tilde{I}\}$.

Claim 9. $\underset{\sim}{\gamma} < \gamma^*$.

Proof. Suppose, by way of contradiction, that $\underset{\sim}{\gamma} \geq \gamma^*$. First, note that, for all $\gamma \in \tilde{I}$, ex-ante IR holds with strict inequality and hence $h(q(\gamma),\gamma) > 0$. Since $h(q(\cdot),\cdot)$ is non-increasing, we obtain $h(\underset{\sim}{q},\underset{\sim}{\gamma}) > 0$; i.e., $W(\gamma) > v(\underset{\sim}{q},\underset{\sim}{\gamma}) - u(\underset{\sim}{q})$. On the other hand, the definition of $\underset{\sim}{\gamma}$ implies that, within any neighborhood of $\underset{\sim}{\gamma}$, there exists $\gamma < \underset{\sim}{\gamma}$ such that $\gamma \in \Gamma_0$. Since q is continuous from the left, it follows that $W(\underset{\sim}{\gamma}) = v(q(\underset{\sim}{\gamma}),\underset{\sim}{\gamma}) - u(q(\underset{\sim}{\gamma}))$. Thus

$$v(q(\underset{\sim}{\gamma}),\underset{\sim}{\gamma}) - u(q(\underset{\sim}{\gamma})) > v(\underset{\sim}{q},\underset{\sim}{\gamma}) - u(\underset{\sim}{q}).$$

This, together with our assumption that $\underset{\sim}{\gamma} \geq \gamma^*$, implies

$$q(\underset{\sim}{\gamma}) > \underset{\sim}{q}. \tag{64}$$

Since (61) holds for all $\gamma \in \tilde{I}$,

$$u'(\underset{\sim}{q}) + v_1(\underset{\sim}{q}, \underset{\sim}{\gamma}) - v_{12}(\underset{\sim}{q}, \underset{\sim}{\gamma}) \frac{\beta(\tilde{\gamma}) - F(\underset{\sim}{\gamma})}{f(\underset{\sim}{\gamma})} \leq 2C'(\underset{\sim}{q}). \tag{65}$$

(64) implies $q(\underset{\sim}{\gamma}) > 0$, and evaluating (28) at $\underset{\sim}{\gamma}$ yields

$$\begin{aligned} u'(q(\underset{\sim}{\gamma})) + v_1(q(\underset{\sim}{\gamma}), \underset{\sim}{\gamma}) - v_{12}(q(\underset{\sim}{\gamma}), \underset{\sim}{\gamma}) &\frac{\beta(\tilde{\gamma}) - F(\underset{\sim}{\gamma})}{f(\underset{\sim}{\gamma})} \\ + \frac{\lambda(\underset{\sim}{\gamma})\big[u'(q(\underset{\sim}{\gamma})) - v_1(q(\underset{\sim}{\gamma}), \underset{\sim}{\gamma})\big]}{f(\underset{\sim}{\gamma})} &= 2C'(q(\underset{\sim}{\gamma})). \end{aligned} \tag{66}$$

Since $\underset{\sim}{\gamma} \geq \gamma^*$, $\lambda(\underset{\sim}{\gamma})\big[u'(q(\underset{\sim}{\gamma})) - v_1(q(\underset{\sim}{\gamma}), \underset{\sim}{\gamma})\big] \leq 0$. Therefore (66) and (65), together with (62), imply $q(\underset{\sim}{\gamma}) \leq \underset{\sim}{q}$, which is a contradiction with (64). Q.E.D.

Remark 1. Recall that Claims 7–9 are proved for any $\tilde{\gamma} < \bar{\gamma}$ that is close to $\bar{\gamma}$ (so that $\mu(\tilde{\gamma}) \leq 0$) and such that $\tilde{\gamma} \in \Gamma_+$. Thus Claim 9 implies that $\tilde{I}$ and $\underset{\sim}{\gamma}$ are the same for all these $\tilde{\gamma}$. This implies that $\sup \tilde{I} = \bar{\gamma}$. Moreover, q is non-decreasing over $\tilde{I} \cup \bar{I}$ by Claim 7, (55), and Claim 3.

Claim 10. $\underset{\sim}{\gamma} = \underline{\gamma}$.

Proof. First, $\underset{\sim}{\gamma} < \underline{\gamma}$ is not possible since ex-ante IR should be binding over $[a, \underline{\gamma}]$ and should not be at $\gamma > \underset{\sim}{\gamma}$ close to $\underset{\sim}{\gamma}$. On the other hand, if $\underline{\gamma} < \underset{\sim}{\gamma}$, then the definition of $\underset{\sim}{\gamma}$ implies that there exists $\gamma \in (\underline{\gamma}, \underset{\sim}{\gamma})$ such that $\gamma \in \Gamma_0$. But then, $0 \leq W(\gamma) = v(q(\gamma), \gamma) - u(q(\gamma)) \leq 0$, which implies $W(\gamma) = 0$ and is a contradiction with $\gamma > \underline{\gamma}$. Q.E.D.

Hence, the conclusions of Proposition 8 are proved as follows:

For the monotonicity of q, if $\tilde{I} \cup \bar{I} = [a, b]$, then the last statement in Remark 1 completes the proof. If $\underline{\gamma} > a$, then it suffices to note that $q(\gamma) = 0$ for all $\gamma \in [a, \underline{\gamma}]$. The only other case to be considered is when $\underline{\gamma} = a$ and $a \in \Gamma_0$. In this case, $0 = W(a) = v(q(a), a) - u(q(a)) \leq 0$ and hence $q(a) = 0$ and the monotonicity follows.

Statement 1 is proved in Claim 3.

Statement 2 follows from (61) and the fact $q(\gamma) > 0$ for all $\gamma > \underline{\gamma}$. Indeed, if $q(\gamma) = 0$ for some $\gamma > \underline{\gamma}$, then the monotonicity of q implies that $q(\gamma') = W(\gamma') = 0$ for all $\gamma' \in [a, \gamma]$, in contradiction with the definition of $\underline{\gamma}$.

Finally, to see (35), by taking the limit of (61) as $\gamma \to \underline{\gamma}$ from the right and noting that $q(\gamma) > 0$ for all $\gamma > \underline{\gamma}$, we obtain

$$u'(\underset{\sim}{q}) + v_1(\underset{\sim}{q}, \underline{\gamma}) - v_{12}(\underset{\sim}{q}, \underline{\gamma}) \frac{\beta(\bar{\gamma}) - F(\underline{\gamma})}{f(\underline{\gamma})} = 2C'(\underset{\sim}{q}). \tag{67}$$

On the other hand, evaluating (28) at $\underline{\gamma}$ yields

$$\begin{aligned} u'(0) + v_1(0, \underline{\gamma}) - v_{12}(0, \underline{\gamma}) &\frac{\beta(\bar{\gamma}) - F(\underline{\gamma})}{f(\underline{\gamma})} \\ &+ \frac{\lambda(\underline{\gamma})\big[u'(0) - v_1(0, \underline{\gamma})\big]}{f(\underline{\gamma})} + \frac{\delta(\underline{\gamma})}{f(\underline{\gamma})} = 2C'(0). \end{aligned} \tag{68}$$

Since the fourth and fifth terms of (68) are non-negative, (68) and (67) yield $\underset{\sim}{q} = 0$ and (35) follows. Q.E.D.

A.5 Proposition 9

Proposition 9. *Assume A1–A4. Let* (M, x, E) *be an optimal schedule and* $\bar{x}$ *be defined by* (16). *Then, if there exists* $\gamma \in \Gamma$ *such that* $0 < q(\gamma) < \bar{q}$ *and* $U(x(\gamma)) = 0$, *then* $\gamma = \gamma^*$, $\bar{x}$ *is finite,*

$$\{x(\gamma) : \gamma \in \Gamma\} = \{(0, 0), x(\gamma^*), \bar{x}\}, \tag{69}$$

and $\bar{x}$ *solves* (12).

Proof. To show $\gamma = \gamma^*$, note that all $\gamma < \gamma^*$ prefer $(0, 0)$ to $x(\gamma)$, and all $\gamma > \gamma^*$ prefer $x(\hat{\gamma})$ to $x(\gamma)$ for $\hat{\gamma}$ sufficiently close (or equal) to b. Thus $\gamma = \gamma^*$ is the only possibility. The ex-post IC of γ^* then implies that $U = 0$ is satisfied for all types, which in turn implies that all $\gamma < \gamma^*$ consume $(0, 0)$, $U(\bar{x}) = 0$, $\bar{x}$ is finite, and all $\gamma > \gamma^*$ consume $\bar{x}$. This implies (69). Furthermore, since profits from γ^* are negligible and (M, x, E) is assumed to be optimal, $\bar{x}$ has to be a solution of (12). Q.E.D.

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