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On the Uniqueness of Convex-ranged Probabilities

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On the uniqueness of convex-ranged probabilities

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Abstract

We provide an alternative proof of a theorem of Marinacci [2] regarding the equality of two convex-ranged measures. Specifically, we show that, if P and Q are two nonatomic, countably additive probabilities on a measurable space (S, Σ), the condition [$\exists A^* \in \Sigma$ with $0 < P(A^*) < 1$ such that $P(A^*) = P(B) \Longrightarrow Q(A^*) = Q(B)$ whenever $B \in \Sigma$] is equivalent to the condition [$\forall A, B \in \Sigma$ $P(A) > P(B) \Longrightarrow Q(A) \geq Q(B)$]. Moreover, either one is equivalent to $P = Q$.

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In [2], Marinacci proved the following theorem.

Theorem (Marinacci [2]) Let P and Q be two finitely additive probabilities on a λ-system Σ. Suppose that P is convex-ranged and that Q is countably additive. If $\exists A^* \in \Sigma$ with $0 < P(A^*) < 1$ such that

$$P(A^*) = P(B) \Longrightarrow Q(A^*) = Q(B)$$

whenever $B \in \Sigma$, then $P = Q$.

The motivation for studying this kind of questions comes from Bayesian decision theory. This is discussed in [2] to which we refer the reader.

In this note, we provide an alternative proof to Marinacci's theorem by showing that the condition above is equivalent to the condition that for any $A, B \in \Sigma$, $P(A) > P(B) \Longrightarrow Q(A) \geq Q(B)$. Proposition 1 below shows that this latter is equivalent to $P = Q$, and Proposition 2 proves the equivalence to Marinacci's condition. Our condition is probably easier to check in applications. It is in this form, for instance, that the result about the equality of convex-ranged

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probabilities is repeatedly used in [1] for studying the class of unambiguous events in the sense of Epstein and Zhang [3].

Given the limited scope of this note, we make two assumptions which render the proof entirely elementary. We assume at the outset that both P and Q be nonatomic, countably additive probabilities, and that the class Σ of measurable sets be a σ-algebra.

Let P and Q be nonatomic, countably additive probabilities on the measurable space (S, Σ).

Proposition 1 *If $P \neq Q$, then there exist $A, B \in \Sigma$ such that*

$$P(A) > P(B) \qquad \textit{and} \qquad Q(A) < Q(B)$$

Proof. We have $P, Q \ll \mu = \frac{1}{2}P + \frac{1}{2}Q$. Let (Radon-Nikodym) f_P and f_Q be the two densities. Define

$$\begin{aligned} X &= \{s \in S \mid f_P > f_Q\} \\ Y &= \{s \in S \mid f_P < f_Q\} \end{aligned}$$

CLAIM 1: X and Y are measurable with $\mu(X) > 0$ and $\mu(Y) > 0$.

Measurability is immediate. Let $P \neq Q$. Then, $\exists Z \in \Sigma : P(Z) \neq Q(Z)$. Without loss, assume $P(Z) > Q(Z)$ (and $P(Z^c) < Q(Z^c)$). Now, suppose $\mu(X) = 0$. Then,

$$\begin{aligned} P(Z) - Q(Z) &= \int_Z (f_P - f_Q) d\mu \\ &= \int_{Z \cap X} (f_P - f_Q) d\mu + \int_{Z \cap X^c} (f_P - f_Q) d\mu \\ &= (\text{by } \mu(X) = 0) \int_{Z \cap X^c} (f_P - f_Q) d\mu \\ &\leq \int_{Z \cap X^c} (f_Q - f_Q) d\mu = 0 \end{aligned}$$

which implies $P(Z) \leq Q(Z)$, a contradiction. Similarly, for $\mu(Y) > 0$.

Now, observe that $\forall C \subseteq X$ $[D \subseteq Y]$ such that $\mu(C) > 0$ $[\mu(D) > 0]$, we have $P(C) > Q(C)$ $[Q(D) > P(D)]$. This is immediate as for $C \subseteq X$ with $\mu(C) > 0$,

$$P(C) - Q(C) = \int_C (f_P - f_Q) d\mu > 0$$

Similarly for the other case. In particular, $P(X) > 0$ and $Q(Y) > 0$.

Since P and Q are nonatomic, so is μ. Hence, there exists $\tilde{A} \subset X : Q(Y) > \mu(\tilde{A}) > 0$.

Pick one such an $\tilde{A}$. By the preceding, $P(\tilde{A}) > Q(\tilde{A}) \geq 0$. We distinguish between two cases.

(i) $Q(\tilde{A}) = 0$. By the nonatomicity of Q, $\exists B \subset Y : P(\tilde{A}) > Q(B) > 0$. By the preceding, $Q(B) > P(B)$. By setting $A = \tilde{A}$, we have

$$P(A) > P(B) \qquad \text{and} \qquad Q(B) > 0 = Q(\tilde{A}) = Q(A)$$

(ii) $Q(\tilde{A}) > 0$. By the nonatomicity of Q, $\exists B \subset Y : Q(B) = Q(\tilde{A})$. Hence, $P(\tilde{A}) > Q(\tilde{A}) = Q(B) > P(B)$. By the nonatomicity of Q, $\exists A \subset \tilde{A}$: $Q(B) > Q(A) > P(B)$. Since $A \subset X$ implies $P(A) > Q(A)$, we have

$$P(A) > P(B) \qquad \textit{and} \qquad Q(A) < Q(B)$$

■

It is immediately checked that the converse to the above statement is true as well, and that it does not require nonatomicity of the measures. On the other hand, nonatomicity is essential to the above result. To see this, consider two measures, P and Q, both having two atoms, s_1 and s_2, with $P(s_1) = 2/3$, $P(s_2) = 1/3$, $Q(s_1) = Q(s_2) = 1/2$.

Next, we show that the condition for the equality of the measures produced by Proposition 1 is equivalent to the condition in Marinacci's theorem.

Proposition 2 *If* $\exists A^* \in \Sigma$ *with* $0 < P(A^*) < 1$ *such that* $P(A^*) = P(B) \Longrightarrow Q(A^*) = Q(B)$, $B \in \Sigma$, *then for any* $A, B \in \Sigma$

$$P(A) > P(B) \Longrightarrow Q(A) \geq Q(B)$$

Proof. Let, $A, B \in \Sigma$ be such that $P(A) > P(B)$.

We begin by establishing the statement in some special cases. Then, we will reduce the general case to one of those.

(a) Suppose $P(B) = P(A^*)$. Then, $P(A) > P(A^*)$, and, by the nonatomicity of P, $\exists \tilde{A} \subset A$ such that $P(\tilde{A}) = P(A^*)$. We have,

$$\begin{aligned} P(\tilde{A}) &= P(A^*) \Longrightarrow Q(\tilde{A}) = Q(A^*) \\ P(B) &= P(A^*) \Longrightarrow Q(B) = Q(A^*) \end{aligned}$$

Hence, $Q(A) \geq Q(\tilde{A}) = Q(B)$.

(b) Suppose $P(A) \geq P(A^*) > P(B)$. Observe, that $1 > P(A^*)$ implies $P(B^c) > P(A^*) - P(B) > 0$. By the nonatomicity of P, there exists a $Z \subset B^c$ such that $P(Z) = P(A^*) - P(B)$. Hence, $P(B \cup Z) = P(A^*)$. Moreover, just like before, $\exists \tilde{A} \subset A$ such that $P(\tilde{A}) = P(A^*)$. Combining these two observations, we get $Q(A) \geq Q(\tilde{A}) = Q(A^*) = Q(B \cup Z) \geq Q(B)$.

(c) Observe that, trivially, A^{*c} has the same property as A^* in the statement. It follows that if we prove the statement for the case $P(A^*) > P(A) > P(B)$, then the statement is proven also for the case $P(A) > P(B) > P(A^*)$ [for if $P(A) > P(B) > P(A^*)$, then $P(A^c) < P(B^c) < P(A^{*c})$ and $Q(B^c) \geq Q(A^c) \Longrightarrow Q(A) \geq Q(B)$].

(d) So, what is left is to show that the statement is true when $P(A^*) > P(A) > P(B)$.

(d1) Suppose $\exists Z \subset A^c \cap B^c$ such that $P(Z) = P(A^*) - P(A)$. Then, $P(A \cup Z) = P(A^*) > P(B \cup Z)$, and, by (b), $Q(A \cup Z) \geq Q(B \cup Z)$ which implies $Q(A) \geq Q(B)$.

From the nonatomicity of P, we see that a necessary and sufficient condition for such a Z to exist is that $P(A^c \cap B^c) \geq P(A^*) - P(A)$. From $P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B)$, we get the sufficient condition $P(B^c) \geq P(A^*)$.

(d2) So, suppose that $P(A^*) > P(A) > P(B)$ but there exists no $Z \subset A^c \cap B^c$ such that $P(Z) = P(A^*) - P(A)$. We have $P(B^c) > P(A^c) > P(A^{*c})$. By the nonatomicity of P, there exists an integer $k \geq 1$, and two collections of pairwise disjoint sets, $\left\{\tilde{B}_i\right\}_1^k$ and $\left\{\tilde{A}_i\right\}_1^k$, such that (i) for each i, $\tilde{B}_i \subset B^c$, $\tilde{A}_i \subset A^c$; (ii) $P(\tilde{B}_i) = P(\tilde{A}_i) = P(A^{*c})$; (iii) $P(A^c \backslash \cup \tilde{A}_i) \leq P(A^{*c})$. Set $Z'' = B^c \backslash \cup \tilde{B}_i$ and $Z' = A^c \backslash \cup \tilde{A}_i$. Then, clearly, $B^c = (\cup \tilde{B}_i) \cup Z''$, $A^c = (\cup \tilde{A}_i) \cup Z'$ and $P(Z'') > P(Z')$. If we show that $Q(Z'') \geq Q(Z')$, we are done as

$$Q(B^c) = Q(\cup \tilde{B}_i) + Q(Z'') = Q(\cup \tilde{A}_i) + Q(Z'') \geq Q(\cup \tilde{A}_i) + Q(Z') = Q(A^c)$$

Hence, $Q(A) \geq Q(B)$.

Since $P(Z') \leq P(A^{*c})$, either we have $P(Z'') \geq P(A^{*c}) \geq P(Z')$ or $P(Z') < P(Z'') < P(A^{*c})$. If we are in the first case, we are done by (a) and (b). If we are in the second case, observe that

$$Z'^c = (\cup \tilde{A}_i) \cup A$$

Hence,

$$P(Z'^c) = kP(A^{*c}) + P(A) > P(A^{*c})$$

Hence, by the final observation in (d1), the sufficient condition for the existence of a $Z''' \subset Z''^c \cap Z'^c$ with $P(Z''') = P(A^{*c}) - P(Z'')$ is satisfied, and it follows that $Q(Z'') \geq Q(Z')$. $\blacksquare$

Again, the converse is immediately checked, and does not require nonatomicity. Finally, by combining Proposition 1 and Proposition 2, we get Marinacci's theorem for nonatomic, countably additive P and Q.

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