Enhanced Continuum Damage Modeling of Mechanical Failure in Ice and Rocks

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Modeling fracture in geomaterials is essential to the understanding of many physical phenomenon which may posses natural hazards e.g. landslides, faults and iceberg calving or man-made processes e.g. hydraulic fracture and excavations. Continuum Damage Mechanics (CDM) models the crack as a solid region with a degraded stiffness. This continuum definition of cracks in CDM allows more feasible coupling with other forms of material non-linearity and eliminates the need to track complicated crack geometry. Using CDM to analyze fracture for the modeling of fracture in geomaterials encounters several challenges e.g.: 1) the need to model the multiple physical processes occurring in geomaterials, typically: coupled fluid flow and solid deformation, 2) the need to consider non-local damage and transport in order to capture the underlying long-range interactions and achieve mesh-independent finite element solutions and 3) the elevated computational cost associated with non-linear mixed finite element formulations.

The research presented in this thesis aims at improving the CDM formulations for modeling fracture geomaterials. This research can be divided into three main parts. The first is the introduction of a novel non-local damage transport formulation for modeling fracture in poroelastic media. The mathematical basis of the formulation are derived from thermodynamic equilibrium that considers non-local processes and homogenization principles. The non-local damage transport model leads to two additional regularization equations, one for non-local damage and the other for non-local transport which is reduced to non-local permeability. We consider two options for the implementation of the derived non-local transport damage model. The first option is the four-field formulation which extends the \(u/P\) formulation widely used in poroelasticity to include the non-local damage and transport phenomena. The second option is the three-field formulation, which is based on the coupling of the regularization equations under the assumptions of similar damage and permeability length scales and similar driving local stress/strain for the evolution of the damage and permeability. The three-field formulation is computationally cheaper but it degrades the physical modeling capabilities of the model. For each of these formulations, a non-linear mixed-finite
element solution is developed and the Jacobian matrix is derived analytically. The developed formulations are used in the analysis of hydraulic fracture and consolidation examples.

In the second part, a novel approach for CDM modeling of hydraulic fracture of glaciers is pretended. The presence of water-filled crevasses is known to increase the penetration depth of crevasses and this has been hypothesized to play an important role controlling iceberg calving rate. Here, we develop a continuum-damage-based poro-mechanics formulation that enables the simulation of water-filled basal and/or surface crevasse propagation. The formulation incorporates a scalar isotropic damage variable into a Maxwell-type viscoelastic constitutive model for glacial ice and the effect of the water pressure on fracture propagation using the concept of effective solid stress. We illustrate the model by simulating quasi-static hydro-fracture in idealized rectangular slabs of ice in contact with the ocean. Our results indicate that water-filled basal crevasses only propagate when the water pressure is sufficiently large and that the interaction between simultaneously propagating water-filled surface and basal crevasses can have a mutually positive influence leading to deeper crevasse propagation which can critically affect glacial stability.

In the third part, we propose a coupled Boundary Element Method (BEM) and Finite Element Method (FEM) for modeling localized damage growth in structures. BEM offers the flexibility of modeling large domains efficiently while the nonlinear damage growth is accurately accounted by a local FEM mesh. An integral-type nonlocal continuum damage mechanics with adapting FEM mesh is used to model multiple damage zones and follow their propagation in the structure. Strong form coupling, BEM hosted, is achieved using Lagrange multipliers. Since the non-linearity is isolated in the FEM part of the system of equations, the system size is reduced using Schur complement approach, then, the solution is obtained by a monolithic Newton method that is used to solve both domains simultaneously. The method is applied to multiple fractures growth benchmark problems and shows good agreement with the literature.
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Chapter 1

Introduction

1.1 Problem statement

The science that concerns the mechanical behavior of porous fluid-filled solids, “poroelasticity”, is currently a prominent research topic. The study of poroelasticity is essential to many applications, including: rock mechanics [1, 2], geotechnical engineering[3, 4], hydraulic fracture [5–8], hydrology [9–11], biomedical engineering [12, 13], glaciology [14, 15] and others.

In standard linear poroelasticity [16], the bulk stiffness, permeability and other physical properties of the solid-fluid mixture are assumed to be constant. While this assumption may be valid in limited cases where the mechanical deformation and fluid flow processes do not lead to changes in the properties of the mixture [17], it is not the case when fracture processes play a role. In reality, upon extreme loading conditions, mechanical behavior of the solid skeleton becomes nonlinear. A successful and more general model of poroelastic media should be capable of modeling, at least, the following physical processes:

1. variation of the porous solid material properties: under larger stresses/strains, the solid grains mechanical behaviour becomes non-linear and may experience softening and/or irreversible deformation
CHAPTER 1. INTRODUCTION

2. variation of fluid flow properties due to damage of the porous solid: degradation of the solid material properties may lead to geometric changes in the mixture i.e. a space that was filled by the intact solid becomes available for fluid flow; thus leading to permeability increase. In the cases of hydraulically driven cracks, the permeability increase becomes of extreme importance because it provides the mechanism by which fluid displaces solid material in the mixture.

3. the effect of solid degradation on the solid-fluid interaction mechanisms: the changes in solid stiffness affect the solid compressibility which in turn affects the solid-fluid interaction described by Biot’s coefficient and modulus.

Iceberg calving from marine-terminating glaciers accounts for nearly 50% of the mass lost from both the Greenland and Antarctic ice sheets [18–22]. However, the mechanical failure of glacier ice is a complex process owing to the multiscale and multiphysics nature, involving a bewildering variety of deformation and damage mechanisms at various length scales ranging from localized micro-scale (or milli-scale) failure to rifts that exceed hundreds of kilometers. Moreover, because ice remains brittle up to the melting point, it is necessary to simultaneously model the slow ductile flow (creep deformation) and fast brittle fracture of glacier ice. A consequence of this complexity is that researchers have not yet agreed on a versatile mathematical model that can be universally implemented in large-scale ice sheet and glacier models to describe fracture and eventual calving behavior [23–28]. An efficient and applicable mathematical model should be able to reproduce the observed glacier behavior and to easily amalgamate into traditional continuum ice-flow models used to simulate decadal to millennial-scale variations in ice dynamics.

In the research presented in this thesis, we aim to improve the current state of the continuum non-linear models of geomaterials, especially rocks and ice. The research in this thesis is focused on two major points: 1) improving the coupled damage-fluid infiltration formulations in porous media, and 2) minimizing the computational cost of the non-linear damage mechanics modeling.
1.2 Literature review

1.2.1 Mechanics of porous media

Poroelasticity deals with a multi-physics problem, involving the mechanical behavior of a porous solid skeleton and the fluid flow within the solid, approaching the porous medium as the superimposition of two continua [29]. The earliest efforts which coined the theory of poroelasticity are attributed to Terzaghi [30] and Biot [16], more than half a century ago. These theories combined mechanical equilibrium equations with fluid flow continuity equations, employing linear stress-strain relationships and Darcy’s flow law to provide the constitutive laws for solid and fluid behaviors, respectively. The finite element method (FEM) for modeling poroelasticity has been investigated by many researchers starting from the seminal work of Zienkiewicz and his co-workers [31–34] in the early 1980s, who formulated the basics of FEM for poroelasticity. Due to the complexity of the underlying physics, the FEM modeling of poroelasticity requires additional degrees of freedom to model the coupled solid behaviour and fluid flow. Mixed finite elements were first proposed for poroelasticity by Taylor and Zienkiewicz [32] and discussed in details in [33].

1.2.2 Modeling non-linear mechanical behaviour in porous media

In standard linear poroelasticity [16], the bulk stiffness, permeability and other physical properties of the solid-fluid mixture are assumed to be constant. While this assumption may be valid in limited cases where the mechanical deformation and fluid flow processes do not lead to changes in the properties of the mixture [17], it is not the case when fracture processes play a role. In reality, upon extreme loading conditions, mechanical behavior of the solid skeleton becomes nonlinear. A successful and more general non-linear model of poroelastic media should be capable of modeling, at least, the following physical processes:

1. **variation of the porous solid material properties**: under larger stresses/strains, the solid
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3. the effect of solid degradation on the solid-fluid interaction mechanisms: the changes in solid stiffness affect the solid compressibility which in turn affects the solid-fluid interaction described by Biot’s coefficient and modulus.

Poro-inelasticity has also been an active field of research. For example, poro-visco-elastic constitutive laws were proposed in [35–37] to describe time dependent properties of the solid skeleton e.g. clay, magma rocks and hydrogels. Poro-plastic constitutive laws and associated numerical implementations were proposed in [38–42] to describe ductile mechanical behaviour of rocks and geomaterials. Fracture is another extremely important process in geomaterials such as rocks and ice. Linear Elastic Fracture Mechanics (LEFM) theories of poroelastic media were investigated in [5, 43]. Research efforts have also explored the use of generalized/Extended Finite Element (G/XFEM) [44, 45] and cohesive zone elements [46, 47] to describe hydraulic fracture of porous media.

Continuum Damage Mechanics (CDM) is another well-established technique that has been used to model failure in solids. Unlike fracture mechanics approaches, the material failure in CDM is described by degradation of the stiffness of the solid. The amount of damage accumulated at a material point represents the amount of micro-cracks and voids initiated by material failure [48]. A “damage” variable is inserted into the material constitutive law to quantify the amount
1.2. LITERATURE REVIEW

of stiffness degradation occurring at a certain elastic or inelastic state [49]. Damage variables are calculated using constitutive damage laws that have been developed using experimental data, micro-mechanical analysis and thermodynamic derivations.

Moving from CDM modeling of solid behavior to CDM modeling of porous media is not straightforward. The fluid and solid compressibility factors become dependent on accumulated damage values [50]. In addition, the permeability of a porous medium is dependent on the fluid viscosity and the solid porosity and experiments show that permeability changes with deformation [51–53]. Also, we note that the damage length scale has to be much larger than the porous media scale in order to preserve the continuum description [29] and apply CDM principles to Biot's poroelasticity equations. Many phenomenological damage laws have been proposed in the literature for different classes of poroelastic materials e.g. [54–59] for geo-materials and [60–63] for bio-materials. Application of CDM to poroelasticity has been investigated in the context of enhanced consolidation [4, 64], damage-enhanced Mandel-Cryer effects [65], hydraulic fracture [66–68] and others. Furthermore, all of the previous studies concerned with poroelastic damage dealt with local damage only, which is known to yield mesh dependent results.

1.2.3 Non-local damage and transport

The meso-scale processes (occurring at grain-scale), mentioned in Section 1.2.2, affect the macroscopic response of the porous media. For example, see Figure 1.1, the mechanical failure of solid grains and consequently fluid infiltration in the fracture process zone at the meso-scale leads to the formation of macroscopic hydraulically driven fracture. The introduction of a non-local length scale to the macro-scale model has been proposed in damage mechanics [69] and transport in porous media [70] to account for the long-range meso-scale processes occurring in the fracture process zone.

The LEFM approaches typically neglect the meso-scale crack formation processes around the macroscopic crack and ignore the effect of crack propagation on the solid-fluid interaction mecha-
Figure 1.1: Transition from meso-scale to macro-scale using non-local formulations. The physical processes occurring in the fracture process zone at the meso-scale (grain-scale) e.g. fracture and fluid infiltration leads to macroscopic crack growth. The transition between the processes occurring at meso- and macro-scales is accounted for by introducing a non-local length scale to the macroscopic behaviour [69, 70].

Figure 1.1: Transition from meso-scale to macro-scale using non-local formulations. The physical processes occurring in the fracture process zone at the meso-scale (grain-scale) e.g. fracture and fluid infiltration leads to macroscopic crack growth. The transition between the processes occurring at meso- and macro-scales is accounted for by introducing a non-local length scale to the macroscopic behaviour [69, 70].

nisms in the crack vicinity. Cohesive zone models generalize LEFM based methods by introducing damage-like effects at the crack tip but fail to account for gradual failure in other regions of the domain [71, 72] their capability of modeling gradual mechanical failure away from the crack tip zone. In addition, another drawback of discrete crack modeling methods is the difficulty of tracking complicated crack topology e.g. curved cracks, crack coalescence and 3D crack surfaces [45, 73].

Non-local damage

Considering discretized numerical solutions, if damage effects are calculated and applied "locally" at each material point i.e. local damage, the equations exhibit loss of ellipticity leading to lack of uniqueness and the numerical solution suffers from mesh dependence [74]. In solid mechanics, different solutions were proposed to overcome the problems introduced by local damage, including: integral [74, 75] and gradient-type [76–79] non-local damage models. One aspect of gradient non-local damage is the continuous broadening of the damage zone due to the non-vanishing degradation driving thermodynamic force [80]. Nevertheless, the implicit non-local gradient approach proposed by Geers et al. [81] overcomes this drawback by introducing an internal length scale that varies with the evolution of damage. The original formulation in [81] provides
a description of a non-local length scale that increases with deformation to activate the non-local behaviour in the damage zone only. The increasing interaction zone hypothesis is backed up by micro mechanical analysis and acoustic emission results [82]. More recently, a localizing gradient approach was introduced by assuming that the interaction length scale decreases as damage evolves [83, 84].

The Phase-field method is another technique that is closely related to gradient-nonlocal damage mechanics in the sense that it provides gradual damage growth within a specific length scale of the material [85, 86]. Phase field modeling of hydraulic fracture problems has been proposed in [87–91]. The similarities and differences between phase-field and gradient non-local damage approaches have been reported in [80, 92], where these studies conclude that phase field may be considered as a special variant of gradient damage model.

Non-local transport

Since the early works on poroelasticity by Terzaghi [30] and Biot [16], Darcy’s law has always been the most widely used transport law to describe the fluid flow inside porous media. Darcy’s law is believed to be an efficient simplification of the Navier-Stokes equations that is valid for laminar flow cases and low Reynolds number [93, 94], which is the characteristic of many geomechanics applications. Poiseuille’s flow, which is an approximation of Navier-Stokes for laminar flow between two close plates, is also widely used in fracture applications in geomechanics to model hydraulic fracture effects. This model requires an explicit definition of a crack width, which is easily defined in LEFM models but would require additional assumptions in continuum fracture models e.g. damage and phase-field.

More recently, the idea of “non-local transport” started to get more recognition. In transport laws e.g. Fick’s law and Darcy’s law, the flux is a function of local pressure gradient and conductivity or permeability, respectively; however, experimental and theoretical analyses prove that flow is affected by the medium properties within a neighboring length scale [95, 96]. To illustrate this
concept, consider the fluid flow in porous media in Figure 1.2. The calculation of local flow parameters, as shown in Figure 1.2a, lead to a flow path defined by local material point flow parameters. The introduction of a non-local length scale, as in Figure 1.2b, leads to the incorporation of flow parameters from neighboring material points in the calculations. As shown in Figure 1.2b, the flow path may be altered because non-local flow calculations may uncover more probable flow paths. To this end, different non-local transport formulations have been proposed in the literature. An integral formulation approach for non-local transport was proposed in [70]. This formulation describes the flow as a function of the integral of the product of pressure gradients and “conductivity kernel” over a given domain, calculated at the material point. Estimates of equivalent permeability for heterogeneous media were proposed by [97, 98]. However, these models are formulated from a fluid mechanics point of view and transferring them to solid/fracture mechanics applications is a non-trivial process [95]. Peridynamic models for non-local flow have also emerged [99, 100], however, coupling the peridynamics approach with continuum models may not be straightforward.

Figure 1.2: Fluid flow through porous media. With the local flow calculation in (a), the fluid follows the path dictated by the highest pressure gradient and highest possible permeability calculated at a material point. The introduction of a length scale in (b) couples the local behavior of the flow to the behavior of the flow at neighboring points, leading the flow to move along the path that has higher permeability in a non-local sense at each material point.
1.2.4 Fracture in glaciers

Historically, researchers first sought empirical relations that parameterized the iceberg calving process in terms of a spectrum of internal and external variables that included water-depth [101–103], ice-front thickness [104], lateral stretching [105, 106] or a critical height-above-buoyancy [23, 107]. The validity and applicability of these models are limited to a few specific cases. For instance, the water-depth and height-above-buoyancy models are limited to grounded termini only [e.g., 108, 109]. Moreover, these parameterizations ignore the physical factors that contribute to the calving process, such as strain rate and hydrofracture.

It is also possible to attempt to predict the penetration depth (height) of surface (basal) crevasses using various formulations of fracture mechanics [25, 110–113]. Alternatively, the Nye zero-stress model [114–116] may be used to predict crevasse penetration depths based on the assumption that a crevasse penetrates to the point where horizontal stress vanishes. Calving in these crevasse depth or height-based models is then assumed to occur when the combination of surface and basal crevasses exceeds a critical threshold [e.g., 25, 27, 116]. However, these models ignore the viscoelastic effects on failure and assume that introducing fractures, no matter how large, has a negligible effect on the macro-scale strain rate or stress field within the glacier or ice sheet. Recently, a viscoelastic Maxwell model was used to compute surface displacements in the Jelbart Ice Shelf in Antarctica and the results matched reasonably well with observations, which emphasizes the need to include viscoelastic effects [117].

Recently, researchers have begun to incorporate the bulk effect of fractures on the deformation of ice using continuum damage mechanics, which describes the gradual time dependent failure of the material around pre-existing defects, rather than an abrupt failure described by fracture mechanics approaches. [118] and [119] proposed the first damage-based models and used them to analyze the detachment of a hanging glacier, assuming that ice behaves as an incompressible viscous fluid. [120–122] extended these models to include viscoelastic effects and to ensure thermodynamic con-
sistency using the nonlocal damage formulation in a Lagrangian finite element framework. [123] sought to combine damage and fracture mechanics approaches to model the calving behavior of ice sheets using damage mechanics to predict the ‘starter crack’ depth needed to initiate brittle failure.

Studies have also attempted to apply the formalism of damage mechanics to simplified thin-film formulations of ice shelf dynamics. For example, [124] proposed a two-dimensional ‘fracture density field’ that is in the same spirit as damage mechanics, but uses a phenomenological/empirical parameterization of rifts and fractures in ice shelves. Similarly, [125] introduced a conceptual model of damage in ice shelves, pointing that even in thin film approximations, damage remains three-dimensional. These so-called “forward” approaches sought to predict the evolution of damage using numerical models. Following an inverse approach, [126] used satellite observed surface velocities to estimate damage in ice shelves. Because this study was diagnostic, [127] were unable to describe the physical mechanisms behind damage evolution, but they found that, in contrast to the assumptions of fracture-mechanics-based studies, the flow of ice was substantially influenced by the presence of damaged regions of ice.

1.2.5 Modeling hydraulic fracture using damage mechanics

In the last few decades, fluid driven fracture in porous media has become a prominent research topic due to its wide range of applications, such as: geomechanics [2, 5], glaciology [15] and biomechanics [12, 13]. In Glaciology, several studies have hypothesized surface run-off or melt-water in crevasses as driving force in iceberg calving [e.g., 111, 112, 128] and ocean water intrusion as an important mechanism for basal crevasse propagation [110]. Different approaches were presented in the literature to model hydraulic fracture of porous media. Current continuum models may be classified into discrete-crack methods and continuum damage based methods. Discrete-crack methods are mainly based on LEFM [6, 43] or cohesive zone methods [7, 47] which extend LEFM to include traction-separation laws at the crack surfaces. These approaches were enhanced using XFEM to model more complicated crack paths in 2d and 3d [5, 45, 47, 73]. In general,
1.2. LITERATURE REVIEW

fracture mechanics based methods assume the domain to be elastic or poroelastic, and use the fluid filled macroscopic crack to model hydraulic fracture. A Poiseuille’s flow is typically assumed inside the crack, while Darcy flow is employed away from the crack.

A key advantage of damage mechanics is its compatibility with the finite element method for simulating fracture propagation in three-dimensions without having to explicitly track the fracture surface. However, a key challenge of this approach is that it is not possible to specify water pressure as a boundary condition on the fracture surface to incorporate the effects of hydrofracture, an effect that observations indicate is crucial. The previous continuum approaches for modeling hydraulic fracture [87–90] encountered a challenge when they tried to model the fluid flow inside the fluid-driven crack. These studies assumed a Poiseuille’s flow or Stokes flow inside the crack zone and estimated fracture-width like quantities from their continuum models. The estimation of the fracture width leads to several problems e.g. non-physical estimated quantities and discontinuity in fluid flow description due to the use of Poiseuille’s flow inside the crack and Darcy’s law in the intact domain.

1.2.6 Coupling FEM and BEM to minimize the computational cost of damage mechanics models

Problems involving damage mechanics, or non-linear behaviour in general, are known to require extensive computational effort [129]. Domain decomposition approaches have been proposed to accelerate the solution of computational models [130–133]. The Boundary Element Method (BEM) has been developed through the past few decades for solving different engineering problems. Each method of BEM and FEM has its advantages and disadvantages depending on the problem solved. Finite element modeling may be more suitable for the solution of non-linear problems. However, the boundary element method provides efficient and accurate modeling of linear elastic domains with a reduced dimensional order and hence a reduced number of degrees of freedom.
The earliest recorded implementations of the coupled BEM-FEM systems were presented by Zienkiewicz[134] and Brebbia [135]. Since then, many contributions have been recorded on the deployment of BEM-FEM for the solution of linear elasticity [136–140] and other potential-type problems e.g. heat conduction and fluid flow [141]. The coupled solution is based on a domain decomposition approach. The coupling can be either FEM hosted or BEM hosted. In the FEM hosted solution e.g. [142], a stiffness matrix of the BEM domain is calculated from the BEM influence matrices; then, a standard assembly procedure is followed to combine the domains. In the BEM hosted solution e.g. [143], the finite element matrices are arranged in boundary element like form and coupled with boundary element matrices in their typical format.

While FEM formulations are based on force-displacement relations, classic BEM formulations are based on traction-displacement relations. Also, FEM formulations produce system of equations with one influence matrix, the sparse and symmetric stiffness matrix. On the other hand, classic BEM formulations based on “collocation” result in influence matrices that are fully populated and non-symmetric. The incompatibility of basic variables, along with the difference in the system of equations setup have always been an issue to consider in BEM-FEM coupled formulations. In addition, the final coupled system of equations is non-symmetric. Forcing the symmetry on the coupled system of equations was proved to be mechanically inconsistent [144]. Variational BEM formulations [145, 146] have been developed in the last few decades, leading to symmetric Galerkin BEM [147–150]. The symmetric Galerkin BEM is often referred to as the “weak” BEM formulation [151]; it leads to BEM influence matrices that are symmetric, sparse and positive definite [152] and hence more suitable to be coupled with FEM. However, the drawback of that system is the computational cost which is higher than that of the collocation BEM [151].

Parallel computing was found to be effective in the domain decomposition problems because it allows faster processing of individual domains. For finite element problems, Farhat first proposed Finite Element Tearing and Interconnecting (FETI) [130, 131]. Later, the idea was extended to
include BEM and FEM domains in works on FETI/BETI [153–155]. Domain decomposition techniques improvements also include: iterative coupling [156–158] and overlapping domain decomposition [159]. The stability of the coupled numerical systems was investigated in [144, 160, 161].

The research significance of coupling BEM and FEM can be demonstrated through non-linear applications where the BEM domain is a linear elastic sub-domain and the FEM is a sub-domain susceptible to a localized non-linearity. For illustration, consider a large elastic domain with localized points of stress concentrations introducing localized non-linear behaviour such as fracture process zones [162, 163]. Modeling the entire problem using non-linear FEM would come at a huge computational cost but can be significantly reduced by using BEM to model the majority of the linear elastic domain. This can be realized by coupling FEM with BEM. Coupled BEM-FEM for the solution of non-linear problems has been investigated by many researchers. Examples include elasto-plasticity [164–168], fracture mechanics [142, 169], contact problems [143, 170, 171], among other. Yazdchi et. al [172] coupled damage susceptible FEM with linear BEM for the non-linear analysis of a gravity dam; the FEM domain was used to model the dam and surrounding soil and BEM was used to model the semi-infinite far field.

1.3 Goals and structure

In this thesis, we aim to improve the CDM modeling of fracture in geomaterials. We achieve this by introducing:

1. A thermodynamically consistent framework for non-local damage-transport model to describe failure in porous media [173–175]
3. An adaptive modeling of damage growth using a coupled FEM/BEM approach [176]

The thesis is organized as follows. In Chapter 2, the mathematical basis of the non-local transport-damage formulation is derived. The derivation is based on a thermodynamics formulation featuring the non-local damage and transport effects. Using the Clausius-Duhem inequality which sums up
CHAPTER 1. INTRODUCTION

first and second laws of thermodynamics, the non-local state laws and regularization equations are defined. An alternative derivation of the model based on volumetric homogenization is also presented. The initial boundary value problem is formulated for the possible implementation options: the 3-field and 4-field formulations. The 3-field formulation provides an simplification of the non-local damage transport model that assumes that the damage and transport length scales are similar and that the driving local stress/strain measures for the evolution of damage and permeability are the same.

Chapters 3 and 4 discuss the computational implementation and numerical results of the three-field model. Chapter 3 presents a strain-based constitutive laws model while Chapter 4 presents a stress-based constitutive laws model. For each model, a non-linear mixed finite element framework is developed and the Jacobian matrix is analytically derived. In Chapter 3, numerical examples are used to model 1d and 2d hydraulic fracture, and 2d consolidation examples. The results Chapter 3 show that the proposed non-local damage model achieves mesh independence and provides and non-local continuum way to model hydraulic fracture and consolidation. The results in 4 focus on the analysis of energy dissipation mechanisms in hydraulic fracture under different loading and material configurations. Also, the results in 4 show the significance of non-local transport in modeling hydraulic fracture.

Chapter 5 discusses the computational implementation of the 4-field formulation. The non-linear mixed finite element framework is developed and a pressure projection stabilization technique is used in order to allow the use of equal order finite elements which reduces the computational cost. The Jacobian matrix is analytically derived. The numerical examples show the added value of the 4-field formulation First, hydraulic fracture modeling is attempted with increasing transport length scale while the damage length scale is fixed. Increasing transport length scale can be interpreted as increased size of inter-granular fluid-transport network. The results imply that increasing the transport length scale leads to the creation of a fluid-pool that dissipates more
fluid through fracture walls and hinders hydraulic fracture propagation. Second, 2d consolidation example is modeled using different configurations of local stresses driving damage and permeability evolution. The analysis concludes that if permeability increases inside the shear crack, a hydraulic-fracture-like behaviour that accelerated damage growth will be observed because the crack becomes water-filled.

Chapter 6 discusses the developed continuum damage formulation for hydraulic fracture of glaciers. Assuming hydrostatic water-pressure inside water-filled crevasses, a poro-visco-elastic damage model is developed. The model requires the solid mechanics equations only because of the hydrostatic water-pressure assumption. The developed model is used to analyze the mutual propagation of water-filled surface and basal crevasses in ice sheets. The numerical results suggest that mutual propagation of water-filled surface and basal crevasses increases the chances of hydraulic fracture of glaciers.

Chapter 7 discusses the development of an adaptive FEM/BEM domain decomposition approach to model damage propagation. In this approach, the finite element subdomain is used to model the non-linear damage zone and boundary elements is used to model the linear elastic subdomain. Non-local integral damage is implemented to avoid mesh dependent results. Adaptive remeshing is performed to model the increase of the size of the damage zone. A schur complement approach is used to decrease the size of the system of equations used in every time step, making use of the fact that BEM equations remain constant for every mesh state. Numerical examples show that the well comparison of the model compared to numerical and experimental results of benchmark fracture mechanics examples.
Chapter 2

Non-local damage transport model: theoretical basis

2.1 Modeling poroelastic damage

Recalling Biot’s theory of poroelasticity [16], the relationship between the total Cauchy stress $\sigma_{ij}$ and homogenized solid stress $\sigma_{ij}^s$ and fluid stress $\sigma_{ij}^f$, under saturated conditions, can be defined as

$$\sigma_{ij} = \sigma_{ij}^s + \sigma_{ij}^f$$  \hspace{1cm} (2.1)

$$= (1 - \phi)\tilde{\sigma}_{ij}^s + \phi\tilde{\sigma}_{ij}^f$$  \hspace{1cm} (2.2)

where $\phi$ is the average porosity, and $\tilde{\sigma}_{ij}^s$ and $\tilde{\sigma}_{ij}^f$ denote the effective solid and fluid stresses, respectively.

Consider the saturated porous media schematic in Figure 2.1, the solid grains are packed inside the fluid media. Upon various loading conditions, e.g. mechanical loads or fluid flux entering the porous media, a damage zone may develop and lead to a localized failure behaviour.
2.1. MODELING POROELASTIC DAMAGE

One of the popular interpretations of damage defines it as the growth of micro-pores and micro-cracks that lead to material degradation [48, 49]. Hence, damage can be viewed as an additional porosity. However, such interpretation may not always be appropriate due to the presence of pores even before any loading has occurred, which has led to the introduction of new constitutive laws that were proved beneficial in the analyses of damage enhanced consolidation [4], Mandel-Cryer effects [65] and hydraulic fracture of glaciers [15]. Another approach for the growth of damage and porosity as two distinct material degradation parameters was proposed in [177, 178]. In this approach porosity growth accounts for the dilation of pores that could be fluid filled and the damage growth accounts for the growth of micro-cracks that lead to weakening of the material without providing additional accessible space for fluid to percolate inside.

![Figure 2.1: Schematic showing the macroscale modeling of the microscale phenomena happening at a grain level in porous media.](image)

In the formulation proposed in this study, the porosity growth effects are incorporated in the model through the damage and permeability variables growth and the macroscopic failure behaviour is achieved by introducing the following:

- damage mechanics approach to model the degraded stiffness of the solid material
- permeability-strain relationship to model the decrease in fluid flow resistance upon solid material failure and porosity growth
2.1.1 Governing equations

Poroelasticity as introduced by Biot [16] provides an understanding for the coupled behaviour of the mechanical deformation and the fluid flow in porous media. The governing partial differential equations are:

- **Equilibrium equation:**
  \[ \sigma_{ij,j} + b_i = 0 \]  
  (2.3)

- **Fluid flow continuity equation:**
  \[ \frac{\partial \zeta}{\partial t} + v_{i,i} = 0 \]  
  (2.4)

where \( \sigma_{ij} \) is the Cauchy stress tensor, \( b_i \) is the mechanical body load vector, \( \zeta \) is the increment in fluid volume content, \( t \) is time and \( v_i \) is the fluid velocity. In this study, Einstein’s indicial notation is used with index repetition denoting summation.

The relationship between the fluid velocity \( v_i \) and the fluid pressure \( P \) needs to be specified by another constitutive law such as Darcy’s law, which provides a linear relationship between the fluid velocity and the pressure gradient. By using Darcy’s law we assume that the flow is laminar with a low Reynolds number, which is a valid assumption for most geomechanics applications because of the narrow inter-granular spaces in such cases [93, 94]. Nonetheless, non-linear, higher order laws may also be useful to describe flows at higher speeds and/or wider inter-granular channels, however, in this study we complete the model using Darcy’s law, which is written in the isotropic case as:

\[ v_i = -\kappa_{ij}P_{,j} \]  
(2.5)

where \( P \) is fluid pressure and \( \kappa_{ij} \) is the anisotropic permeability tensor. In the isotropic case, permeability is defined as \( \kappa = k/\mu^f \) where \( k \) is the solid intrinsic hydraulic conductivity and \( \mu^f \) is the fluid viscosity.
2.2 Thermodynamically consistent non-local damage transport model

In this section, we introduce a new thermodynamic framework for modeling non-local transport and damage in porous media. Non-local transport is aimed to model the long-range intergranular fluid flow that is not readily captured by the macroscopic fluid flow law [70]. Experimental results and theoretical models confirm that the flow in porous media is affected by flow parameters within a neighboring length scale [95, 96]. Previous non-local transport models described in the literature were formulated from fluid mechanics point of view, and are difficult to incorporate within solid mechanics or poroelasticity frameworks [70, 95].

Non-local damage aims at incorporating the size of the fracture process zone within a damage framework [69] to provide a more physical definition of the damage evolution. The introduction of a material length scale regularizes the governing set of partial differential equations and has been shown to alleviate the mesh-dependence associated with local damage models [74].

In this section, we propose a novel thermodynamically consistent derivation of a model exhibiting non-local damage and transport effects within a poroelasticity framework. We start by introducing a new Helmholtz free energy expression \( \Psi_{nl} \), which is the sum of local free energy, nonlocal solid deformation energy and nonlocal transport energy, as:

\[
\Psi_{nl}(\epsilon_{ij}, \tilde{\epsilon}, D, \zeta, \tilde{\zeta}) = \Psi(\epsilon_{ij}, \zeta, D) + \Psi_{nl,\epsilon}(\epsilon_{ij}, \tilde{\epsilon}) + \Psi_{nl,\zeta}(\epsilon_{ij}, \zeta, \tilde{\zeta}) \quad (2.6)
\]

where \( \epsilon_{ij} = \frac{1}{2} [u_{i,j} + u_{j,i}] \) is the total strain (assuming small deformations), \( u_i \) is the solid displacement, \( D \) is the damage variable, \( \zeta \) is the increment volume of fluid content. The strain \( \epsilon_{ij} \) and the fluid increment \( \zeta \) are the fundamental variables used to describe solid and fluid kinematics in poroelastic derivations [17, 179]. Herein, we introduce a non-local effect adding the non-local perturbed variables, that is, \( \tilde{\epsilon} \) which denotes the non-local strain and \( \tilde{\zeta} \) which denotes the non-local
CHAPTER 2. NON-LOCAL DAMAGE TRANSPORT MODEL: THEORETICAL BASIS

transport variable.

The first component of free energy, $\Psi$, is the local free energy of the poroelastic media derived in [17, 50, 179, 180]. The second and third terms in Equation (2.6) represent the non-local effects introduced by the solid deformation ($\Psi^{nl, \varepsilon}$) and fluid transport ($\Psi^{nl, \zeta}$).

Following [50], the local free energy $\Psi$ can be decomposed as:

$$\Psi(\varepsilon_{ij}, \zeta, D) = \Psi^{dry}(\varepsilon_{ij}, D) + \Psi^{wet}(\varepsilon_{ij}, \zeta, D)$$

(2.7)

in which the free energy component of the dry solid skeleton material is $\Psi^{dry}$ and the free energy component of the saturated wet material is $\Psi^{wet}$. For a dry material, the absence of fluid content is obtained by setting $\zeta = 0$, hence $\Psi^{dry}$ can be defined as the strain energy of the damaged solid:

$$\Psi^{dry}(\varepsilon_{ij}, D) = \frac{1}{2} C_{ijkl}^{\varepsilon}(D) \varepsilon_{kl} \varepsilon_{ij}$$

(2.8)

where $C_{ijkl}(D)$ is the damage dependent stiffness tensor. The superscript $u$ denotes the undrained properties of the material, i.e. when the fluid is trapped and not allowed to move inside or outside the material under the assumption of $\zeta = 0$. The wet component of the potential $\Psi^{wet}$ represents the energy accumulated by the dilation caused by the fluid content gain. $\Psi^{wet}$ is defined as [50]:

$$\Psi^{wet}(\varepsilon_{ij}, \zeta, D) = M(D) \zeta \left[ \frac{1}{2} \zeta - \alpha(D) \varepsilon_{ii} \right]$$

(2.9)

where the scalars $M$ and $\alpha$ are commonly known as Biot’s Modulus (which is a stiffness-like parameter) and Biot’s coefficient respectively, which provide measures of solid and fluid compressibility parameters. The energy computation in Equation (2.9) defines the stored energy due to fluid intake as the product of the force $M(D) \zeta$ and the net dilation due to fluid content variation $\left[ \frac{1}{2} \zeta - \alpha(D) \varepsilon_{ii} \right]$, where the force $M(D) \zeta$ is the product of the stiffness-like parameter $M$ and the
2.2. THERMODYNAMICALLY CONSISTENT NON-LOCAL DAMAGE TRANSPORT MODEL

kinematic quantity $\zeta$. Note that $M$ and $\alpha$ are dependent on the extent of damage sustained by the solid skeleton.

The non-local strain free energy contribution $\Psi_{nl,e}$, following the derivation in [181] (for solids without fluid) may be expressed as:

$$
\Psi_{nl,e}(\varepsilon_{ij}, \bar{e}) = \frac{1}{2} h(\bar{e} - \varepsilon)^2 + \frac{1}{2} (l^d)^2 h \bar{e}_{i,i} \varepsilon_{ii}
$$

where $h$ is a stiffness parameter and $l^d$ describes the length scale of the non-local strain interactions. The first term in Equation (2.10) quantifies the amount of energy stored in the system due to the difference between the local scalar measure of the strain $\bar{e}$ field ($\bar{e} = \bar{e}(\varepsilon_{ij})$) and the non-local scalar measure of the strain field $\tilde{e}$.

The second term in Equation (2.10) represents the energy stored by the gradient of the non-local equivalent strain $\tilde{e}$. Similarly, we introduce the energy stored due to non-local transport, $\Psi_{nl,\zeta}$, as:

$$
\Psi_{nl,\zeta}(\varepsilon_{ij}, \zeta, \tilde{\zeta}) = - (\zeta - \tilde{\zeta}) \left[ c^M \alpha \varepsilon_{kk} + \frac{1}{2} c^M (\zeta - \tilde{\zeta}) \right]
$$

$$
- (l^k)^2 (\tilde{\zeta}_{,i}) \left[ c_l \varepsilon_{kk} + \frac{1}{2} c^M \tilde{\zeta}_{,i} \right] (2.11)
$$

where $c^M$ and $c^\alpha$ are parameters similar to $M$ and $\alpha$ respectively, the vector $c_l$ is the energy conjugate of the non-local fluid increment gradient $\tilde{\zeta}_{,i}$ and $l^k$ is the length scale of the non-local transport phenomenon. Here for simplicity we assume the length scale associated with non-local transport to be constant. The first two terms in Equation (2.11) denote the energy stored by the difference between the non-local fluid increment $\tilde{\zeta}$ and the local fluid increment $\zeta$. The third and fourth terms in Equation (2.11) introduce the energy stored by the gradient of the non-local fluid increment $\tilde{\zeta}$.

In order to develop the constitutive relationships and analyze energy dissipation mechanisms, we consider the Clausius-Duhem inequality which incorporates the first and second laws of ther-
modynamics. Following the derivations in [17], in classical local isothermal poroelasticity, the Clausius-Duhem inequality can be written as:

$$\Phi = \Phi_s + \Phi_f \geq 0 \quad (2.12)$$

where $\Phi$ is the total local dissipation rate, $\Phi_s$ is the solid local dissipation rate, and $\Phi_f$ is the fluid local dissipation rate due to viscous fluid flow. In the presence of non-local phenomenon, the local Clausius-Duhem cannot be enforced directly following Equation (2.12) due to the presence of diffusive non-local processes at each material point. Alternatively, the dissipation inequality may be evaluated globally by defining the global dissipation function $\mathcal{D}$, defined as the integral of $\Phi$ in space and time as follows: $\mathcal{D} = \int_t \int_V \Phi dV dt$. The global dissipation can be decomposed additively, similar to Equation (2.12), into solid and fluid contributions as follows:

$$\mathcal{D} = \mathcal{D}_s + \mathcal{D}_f \geq 0 \quad (2.13)$$

where the solid global accumulated dissipation $\mathcal{D}_s$ is defined as $\mathcal{D}_s = \int_t \int_V \Phi_s dV dt$ and the fluid global accumulated dissipation $\mathcal{D}_f$ is defined as $\mathcal{D}_f = \int_t \int_V \Phi_f dV dt$. From the additive decomposition in Equation 2.13 we conclude the conditions: $\mathcal{D}_s \geq 0$ and $\mathcal{D}_f \geq 0$ to be sufficient for thermodynamic consistency. Following the derivation procedure of dissipation functions in isothermal poroelasticity [17, 179, 180], the following rate inequalities can be defined:

$$\dot{\mathcal{D}}_s := \int_V \sigma_{ij} \dot{e}_{ij} + P \dot{\zeta} - \Psi^r dV \geq 0 \quad (2.14)$$

$$\dot{\mathcal{D}}_f := \int_V -P_{il} v_i dV \geq 0 \quad (2.15)$$

where the dot decoration denotes derivative in time and $V$ is the domain volume. The physical
interpretation of the global solid dissipation rate in Equation (2.14) is that it is equal to the difference between the power provided by reversible processes and the power loss from the Helmholtz free energy decay rate $\dot{\Psi}_{nl}$. The reversible process in Equation (2.14) is associated with the reversible mechanical deformation presented by the first term and the work done by fluid pressure to change the fluid-filled pore volume presented by the second term. The global fluid dissipation rate in Equation (2.15) describes the power (rate of work) done by the fluid velocity $v_i$ on the change of the fluid pressure $P$ gradient which denotes the energy lost by friction due to the viscous fluid flow [17, 179, 180]. Using the chain rule, the expansion of the decay rate of the non-local Helmholtz free energy in Equation (2.6) leads to:

$$\dot{\Psi}_{nl} = \frac{\partial \Psi_{nl}}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} + \frac{\partial \Psi_{nl}}{\partial \zeta} \dot{\zeta} + \frac{\partial \Psi_{nl}}{\partial D} \dot{D} + \frac{\partial \Psi_{nl}}{\partial \tilde{\varepsilon}} \dot{\tilde{\varepsilon}} + \frac{\partial \Psi_{nl}}{\partial \tilde{\zeta}} \dot{\tilde{\zeta}} \quad (2.16)$$

The substitution of the $\dot{\Psi}_{nl}$ expression in Equation (2.16) in the global solid dissipation inequality in (2.14) leads to:

$$\dot{\mathcal{D}}_s = \int_V \left[ \sigma_{ij} - \frac{\partial \Psi_{nl}}{\partial \varepsilon_{ij}} \right] \dot{\varepsilon}_{ij} + \left[ P - \frac{\partial \Psi_{nl}}{\partial \zeta} \right] \dot{\zeta} - \frac{\partial \Psi_{nl}}{\partial D} \dot{D} - \frac{\partial \Psi_{nl}}{\partial \tilde{\varepsilon}} \dot{\tilde{\varepsilon}} - \frac{\partial \Psi_{nl}}{\partial \tilde{\zeta}} \dot{\tilde{\zeta}} dV \geq 0 \quad (2.17)$$

The first and second terms in the above inequality are non-dissipative; therefore, they are equal to zero and will be used to derive the constitutive relationships defining $\sigma_{ij}$ and $P$ following the approach in [17, 179]. The third term describes energy lost due to damage evolution in the system. The damage evolution thermodynamic driving force is defined as $Y = \frac{\partial \Psi_{nl}}{\partial D}$. In the absence of damage growth, no solid dissipation should be observed ($\dot{\mathcal{D}}_s = 0$); which provides a condition for the forth and fifth term in Inequality (2.17) to be equal to zero. Hence, the following relations can be defined:
• Stress-strain relationship: consider the first term in the Inequality (2.17) as:

\[
\int_V \left[ \sigma_{ij} - \frac{\partial \Psi^{ml}}{\partial \epsilon_{ij}} \right] \dot{\epsilon}_{ij} dV = 0 \tag{2.18}
\]

the non-trivial solution of Equation (2.18) is:

\[
\sigma_{ij} - \frac{\partial \Psi^{ml}}{\partial \epsilon_{ij}} = 0 \tag{2.19}
\]

which leads to the following definition of the total stress:

\[
\sigma_{ij} = \frac{1}{2} C_{ijkl}^{u} (D) \epsilon_{kl} - \zeta M(D) \alpha(D) \delta_{ij} + h(\bar{\epsilon} - \tilde{\epsilon}) \frac{\partial \bar{\epsilon}}{\partial \epsilon_{ij}}
\]

\[
- (\zeta - \tilde{\zeta}) c M c \alpha \frac{\partial \epsilon_{kk}}{\partial \epsilon_{ij}} - (i^k)^2 (\tilde{\zeta}, i^l) c_i \frac{\partial \epsilon_{kk}}{\partial \epsilon_{ij}}
\]

The first and second term in Equation (2.20) present the classical stress-strain relationship in poroelasticity with damage effects [50], the third term represents the contribution of the non-local strain; the fourth and fifth terms represent the effect of the non-local fluid increment.

• \( P - \zeta \) relationship: the constitutive law defining fluid pressure in terms of fluid content may be found from the second term in Inequality (2.17) as:

\[
\int_V \left[ P - \frac{\partial \Psi^{ml}}{\partial \zeta} \right] \dot{\zeta} dV = 0 \tag{2.21}
\]

the non-trivial solution of Equation (2.21) is:

\[
P - \frac{\partial \Psi^{ml}}{\partial \zeta} = 0 \tag{2.22}
\]
which leads to the following definition of the fluid pressure $P$:

$$
P = M(D) \left[ \zeta - \alpha(D) e_{ii} \right] + c^M e^{\alpha} e_{ii} - c^M (\zeta - \tilde{\zeta}) 
$$

(2.23)

The first term in Equation (2.23) is the classical definition of fluid pressure in poroelastic damage mechanics[50] while the second and third terms introduce the effect of the non-local transport to the definition of $P$.

- Non-local strain: by expanding the fourth term in Inequality (2.17), we get:

$$
\int_V \left[ h(\tilde{\epsilon} - \bar{\epsilon}) \dot{\tilde{\epsilon}} - (l^d)^2 h \tilde{\epsilon}_{,i} \dot{\tilde{\epsilon}}_{,i} \right] dV = 0
$$

(2.24)

Integrating by parts the second term in Equation (2.24) leads to:

$$
\int_V \left[ h(\tilde{\epsilon} - \bar{\epsilon}) + (l^d)^2 h \tilde{\epsilon}_{,ii} \right] \dot{\tilde{\epsilon}} dV - \int_S \left[ (l^d)^2 h \tilde{\epsilon}_{,in} \right] \dot{\tilde{\epsilon}} dS = 0
$$

(2.25)

where $S$ is the domain boundary. The additive relationship in Equation (2.25) leads to the following conditions:

$$
h(\tilde{\epsilon} - \bar{\epsilon}) + (l^d)^2 h \tilde{\epsilon}_{,ii} = 0 \text{ in } V
$$

(2.26)

which defines the differential equation governing the behaviour of the non-local equivalent strain $\tilde{\epsilon}$ with respect to the local equivalent strain $\bar{\epsilon}$, and:

$$
(l^d)^2 h \tilde{\epsilon}_{,in} = 0 \text{ on } S
$$

(2.27)

Equations (2.26) and (2.27) can be simplified by dividing them by $h$ and $(l^d)^2 h$ respectively.
The resulting simplified Equations are:

\[(\ddot{\varepsilon} - \dot{\varepsilon}) + (l^d)^2 \ddot{\varepsilon}_{,ii} = 0 \text{ in } V \]  
(2.28)

\[\ddot{\varepsilon}_{,in_i} = 0 \text{ on } S \]  
(2.29)

The non-trivial solution of Equation (2.29) leads to the boundary condition: \(\ddot{\varepsilon}_{,n_i} = 0\). Equations (2.28) and (2.29) are the non-local gradient strain equations that were first derived from volumetric homogenization principles in [79] and from thermodynamic principles for solid mechanics in [181].

- Non-local transport: by expanding the last term in Inequality (2.17), we get:

\[
\int_V \left( \left[ c^M c^{\alpha} \varepsilon_{ii} - c^M (\zeta - \bar{\zeta}) + (l^k)^2 \frac{(\bar{\zeta}_{,i})}{\zeta} c_{i j} \varepsilon_{ii} \right] \dot{\zeta} - (l^k)^2 c^M \bar{\zeta}_{,ii} \dot{\zeta}_{,ii} \right) dV = 0
\]  
(2.30)

Integrating the last term in Equation (2.30) by parts leads to:

\[
\int_V \left[ c^M c^{\alpha} \varepsilon_{ii} + c^M (\zeta - \bar{\zeta}) + (l^k)^2 \frac{(\bar{\zeta}_{,i})}{\zeta} c_{i j} \varepsilon_{ii} + (l^k)^2 c^M \bar{\zeta}_{,ii} \right] \dot{\zeta} dV
\]

\[
- \int_S \left[(l^k)^2 c^M (\bar{\zeta}_{,i}) n_i \dot{\zeta}\right] dS = 0
\]  
(2.31)

where \(n_i\) is the normal direction to the boundary \(S\). The additive relationship in Equation (2.31) leads to the following conditions:

\[c^M c^{\alpha} \varepsilon_{ii} + c^M (\zeta - \bar{\zeta}) + (l^k)^2 c^M \bar{\zeta}_{,ii} = 0 \text{ in } V\]  
(2.32)

which defines the differential equation governing the behaviour of the non-local fluid incre-
2.2. THERMODYNAMICALLY CONSISTENT NON-LOCAL DAMAGE TRANSPORT MODEL

ment $\tilde{\zeta}$ with respect to the local fluid increment $\zeta$, and:

$$(l^k)^2c^M(\tilde{\zeta},i)n_i = 0 \text{ on } S \quad (2.33)$$

Equations (2.32) and (2.33) can be simplified by dividing them by $c^M$ and $(l^k)^2c^M$ respectively. The resulting simplified Equations are:

$$c^\alpha \varepsilon_{ii} + (\zeta - \tilde{\zeta}) + (l^k)^2\tilde{\zeta}_{,ii} = 0 \text{ in } V \quad (2.34)$$

$$(\tilde{\zeta},i)n_i = 0 \text{ on } S \quad (2.35)$$

The non-trivial solution of Equation (2.35) describes the boundary condition: $(\tilde{\zeta},i)n_i = 0$.

Equations (2.34) and (2.35) provide a novel thermodynamically based definition of non-local transport that can be incorporated within a damage mechanics framework.

Equations (2.20), (2.23), (2.28), (2.29), (2.34) and (2.35) provide a complete thermodynamic framework for a poroelastic damage model featuring non-local transport and non-local deformation effects. However, the practical implementation of this model is difficult due to several reasons. First, there is no clear approach to quantify the additional constitutive parameters: $c^\alpha$, $c^M$, $c_l$ and $h$.

This problem could have been avoided by formulating the non-local thermodynamic potentials in Equations (2.10) and (2.11) in another approach e.g. [182, 183]; however, these approaches are model specific and lead to non-local models that are non-standard and hard to implement in a computational framework. The second difficulty is the computational modeling of the non-local transport parameter $\tilde{\zeta}$. The finite element approximation of Equation (2.34), would require the projection of the values of $\tilde{\zeta}$ to FEM nodes. The fluid mass balance Equation (2.4) (will be discussed later in Section ???) implies that $\zeta$ is a function of the velocity gradient. The fluid velocity
is a function of the fluid pressure gradient for most fluid flow constitutive relationships e.g. Darcy’s law and Forchheimer law [184]. Therefore, the finite element approximation of Equation (2.32) is not trivial and would require the development of stable mixed finite element formulations. To this end, we introduce simplifications in the next section that result in a more computationally feasible model.

### 2.3 Reduced non-local damage-transport model

In this section we aim at reducing the model developed in Section 2.2 to a more computationally-favorable form. First, we systematically eliminate the terms featuring uncallibrated additional constitutive parameters from the state laws introduced in Section 2.2 (following the approach in [181]). Then, we aim at lumping Equations (2.38) and (2.39) into a single regularization equation that provides the effects of non-local strain and transport. In Section 2.3.2, we show how Equation (2.39) may be reduced to a non-local gradient permeability relationship. Later, in Section 2.3.3, we detail the assumptions required to combine the non-local gradient permeability equation with the non-local strain Equation (2.38). Finally, we show the analogy of the simplified approach with the Darcy-Brinkman fluid flow in Section 2.3.4.

#### 2.3.1 Model reduction

Following the approach presented in [181], we simplify the model by taking the limit of the model Equations (2.20), (2.23) and (2.34) as the additional constitutive parameters go to zero \((c^M \rightarrow 0, \ c^\alpha \rightarrow 0, \ c_l \rightarrow 0 \text{ and } h \rightarrow 0)\) This assumption implies that we are "switching off" the energy storage caused by the introduction of the non-local fields. The derivation of the regularization Equations (2.28) and (2.34) requires the additional constitutive parameters to have finite values. The non-local model can directly reduce to a local model if the additional constitutive parameters are all set directly to be equal to zero \((c^M = 0, \ c^\alpha = 0, \ c_l = 0 \text{ and } h = 0)\). Therefore, taking the limit of the constitutive Equations (2.20), (2.23) and (2.34) as the additional parameters tend to zero, rather than simply ignoring them, allows us to preserve the regularization relation-
2.3. REDUCED NON-LOCAL DAMAGE-TRANSPORT MODEL

ships derived in Equations (2.28) and (2.34). This simplification is necessary due to the lack of a clear methodology to calibrate the additional constitutive parameters. Hence, the model equations reduce to:

\[ \sigma_{ij} = \frac{1}{2} C_{ijkl}^u e_{kl} - \zeta M(D) \alpha(D) \delta_{ij} \]  

\[ P = M(D) [\zeta - \alpha(D) e_{kk}] \]

\[ (\tilde{\varepsilon} - \bar{\varepsilon}) - (t^d)^2 \tilde{\varepsilon}_{,ii} = 0 \quad \text{with} \quad (\tilde{\varepsilon}_{,i}) n_i = 0 \text{ on } S \]  

\[ (\tilde{\zeta} - \bar{\zeta}) - (t^k)^2 \tilde{\zeta}_{,ii} = 0 \quad \text{with} \quad (\tilde{\zeta}_{,i}) n_i = 0 \text{ on } S \]

The increment in fluid content \( \zeta \) is controlled by the fluid mass balance equation which defines the rate of fluid content increment as the fluid velocity gradient. Note that two regularization equations are introduced, one for gradient-damage in Eq. 2.38, and one for nonlocal transport in eq. 2.39.

In this paper, Darcy law is employed to describe the fluid flow constitutive law, which can be expressed as:

\[ v_i = -\kappa P_{,i} \]

where \( \kappa \) is an isotropic permeability defined as \( \kappa = k/\mu_f \) where \( k \) is the solid intrinsic hydraulic conductivity and \( \mu_f \) is the fluid viscosity.

2.3.2 Non-local permeability

The differential increment of the fluid content can be additively decomposed as follows:

\[ d\zeta = F^\kappa d\kappa + \sum_i F^i d\xi^i \]
where $\xi^i$ is the $i^{th}$ field variable contributing to $\zeta$, $F^i = \frac{\partial \zeta}{\partial \xi^i}$ and $F^\kappa = \frac{\partial \zeta}{\partial \kappa}$. The first term in Equation (2.41) is the variation in fluid content due to a permeability increment, the second term is the variation of the fluid content due to other variations. The second assumption that we introduce in this section is that the non-locality in the fluid flow is caused only by non-local permeability. Hence, we can write the differential of the non-local increment of fluid content as follows:

$$d\tilde{\zeta} = F^\kappa d\tilde{\kappa} + \sum F^i d\xi^i \quad (2.42)$$

where $F^\kappa = \frac{\partial \tilde{\zeta}}{\partial \tilde{\kappa}}$. This definition assumes that $F^k$ is the same for the local and non-local transport processes. Hence, by taking the derivative of Equation (2.39), we arrive at:

$$(\tilde{\kappa} - \kappa) - (l^k)^2 \kappa_{,ii} = 0 \quad (2.43)$$

$$(\tilde{\kappa},_{i})n_i = 0 \text{ on } S \quad (2.44)$$

Equations (2.43) and (2.44) provide a definition for non-local transport via non-local permeability only. A similar gradient non-local permeability relationship was derived previously by the authors in [173] based on volumetric averaging.

Nonetheless, this formulation is general and following the approach in Equations (2.41) - (2.44), different forms of non-local transport laws may be derived. Section 2.3.4 details the derivation of Darcy-Brinkman flow law from Equation (2.39) by modifying Equations (2.41) and (2.42).

### 2.3.3 Lumped non-local permeability-damage

Upon material failure, it is expected that the hydraulic conductivity will increase and hence its permeability. However, it is hard to quantify the damage-permeability relationship directly because only part of the damage will contribute to pores dilation and permeability increase. For
example a purely shear driven crack in a solid subjected to confining loads may not facilitate fluid flow. Providing a relationship between damage and permeability is a research direction that requires more experimental analyses. The available data in the literature provides correlations between axial strain and permeability increase in soil specimens under different confining pressure values [51–53]. Based on the available experimental data, researchers have proposed empirical permeability-strain [65, 89] and permeability-stress [185] type relationships through polynomial or exponential laws.

The choice of the equivalent local strain measure is generally dependent on the type of the material and loading applied. Different strain measures were previously proposed in the damage mechanics literature [49, 64]. In some cases, the use of an equivalent stress has been proven to be useful to model complex crack propagation scenarios [15, 121, 182].

In order to lump Equations (2.43) and (2.38), we assume that the local stress/strain measure driving damage and permeability are similar and that the local equivalent strain and local permeability evolution are driven by the same evolution law i.e. \( \tilde{\varepsilon} = \bar{\varepsilon} \). Hence, Equations (2.38) and (2.43) can be rewritten as:

\[
\tilde{\kappa} - (l^d)^2 \tilde{\kappa}_{,ii} = \kappa(\sigma_{eq}(\varepsilon_{ij}), \bar{\varepsilon}(\varepsilon_{ij}))
\]  

(2.45)

where \( \sigma_{eq} \) is a local scalar stress measure. The non-local stress-dependent gradient permeability definition in Equation (2.45) is consistent with the derivation in Section 2.2 because the equivalent stress is a direct function of the strain i.e. \( \sigma_{eq} = \sigma_{eq}(\varepsilon_{ij}) \). In summary, with the assumptions and simplifications made in this section, the two nonlocal equations (2.38)-(2.39) are combined into a single nonlocal equation corresponding to a nonlocal fluid permeability, which is demonstrated to capture both processes of damage and transport. Furthermore, from a computational point of view, this model requires a finite element formulations with addition of one scalar degree of freedom (\( \tilde{\kappa} \)) which is more computationally feasible.
2.3.4 Analogy to Darcy-Brinkman flow

Darcy-Brinkman [186] flow law is a modified version of Darcy’s flow that introduces an additional viscous diffusion term which becomes significant in low permeability porous media. Also, the additional Brinkman correction term allows for applying better boundary conditions to describe the no-slip condition for fluid flow between two plates [187]. Darcy-Brinkman [186] fluid flow law is usually written as follows:

\[
q_j - \beta q_{j,ii} = -k \frac{\mu}{\mu_f} P_j \tag{2.46}
\]

where \( q_j \) is the Darcy-Brinkman velocity vector. The correction term introduced in the Brinkman Equation \( (\beta q_{j,ii}) \) provides a better physical description of fluid flow in cases of: 1) high permeability porous materials and 2) porous material in which the solid particles are porous (e.g. double porosity materials) [187]. Another advantage of the Darcy-Brinkman relation in Equation (2.46) over Darcy’s law is that it allows for the application of the no-slip condition at the flow boundary [187]. The Darcy-Brinkman equation is not popularly used in computational models due to the difficulty and additional computational cost associated with its implementation and the lack of calibration data for the effective viscosity parameter \( \beta \) [188].

Here we show an alternative derivation of Darcy-Brinkman fluid flow law [186] that follows directly from Equation (2.39). The derivation from the thermodynamic point of view is more general and provides a way to calibrate the additional effective viscosity parameter \( \beta \). To this end, we first rewrite the expansion of the differential increment of fluid content in Equation (2.41).

Hence, the differential increment of the fluid content is rewritten as follows:

\[
d\zeta = F^v_j dv_j + \sum_i F^i d\xi^i \tag{2.47}
\]

where \( \xi^i \) is the \( i^{th} \) field variable contributing to \( \zeta \), \( F^i = \frac{\partial \zeta}{\partial \xi^i} \) and \( F^v_j = \frac{\partial \zeta}{\partial v_j} \). Then, the differential
2.3. REDUCED NON-LOCAL DAMAGE-TRANSPORT MODEL

of the non-local increment of fluid content is as follows:

\[ d\tilde{\xi} = F^v_j d\tilde{v}_j + \sum_i F^i d\tilde{\xi}^i \]  

(2.48)

where \( \tilde{v}_j \) is the non-local fluid velocity vector and \( F^v_j = \frac{\partial \tilde{\xi}}{\partial \tilde{v}_j} \). This definition assumes that \( F^v_j \) is the same for the local and non-local transport processes. Hence, by taking the derivative of Equation (2.39) with respect to \( F^v_j \), we arrive at:

\[ \tilde{v}_j - (l^k)^2 \tilde{v}_{j,ii} = v_j \]  

(2.49)

\[ (\tilde{v}_{j,i})n_i = 0 \text{ on } S \]  

(2.50)

Equations (2.49) and (2.50) provide a definition for non-local transport via a non-local velocity field only. By replacing the local velocity vector on the right hand-side of Equation (2.49) by Darcy’s law \( v_j = -\frac{k}{\mu_f}P_j \) in Equation (2.40), we get:

\[ \tilde{v}_j - (l^k)^2 \tilde{v}_{j,ii} = -\frac{k}{\mu_f}P_j \]  

(2.51)

which resembles Darcy-Brinkman flow law [186], where the Darcy-Brinkman velocity is equal to the non-local velocity vector \( q_j = \tilde{v}_j \) and \( \beta = (l^k)^2 \) is the effective viscosity term introduced by the Brinkman correction of Darcy’s law. This derivation of Darcy-Brinkman law provides a way to define the Brinkman effective viscosity \( \beta \) in terms of the non-local transport length scale.

This derivation of the Darcy-Brinkman fluid flow suggests that the Brinkman correction can be viewed as non-local correction to Darcy’s law that leads to a form of non-local transport. In order to model non-local transport, the scalar non-local permeability relationship in Equation (2.45)
provides a computationally cheaper alternative to the Darcy-Brinkman vector Equation. In addition, subject to the assumption in Section 2.3.3, Equation (2.45) can be used to model non-local deformation and non-local permeability in one scalar equation.

### 2.4 Gradient non-local permeability model

One of the approaches for non-local transport that resembles homogenization is the volumetric averaging of flow properties [96]. In a finite element context, as in Figure 2.2, the volumetric averaging of permeability may be computed by the following operation:

\[
\kappa_{nl}(X_p) = \frac{\int_{\Omega} \Phi(X_p - X) \kappa(X) d\Omega}{\int_{\Omega} \Phi(X_p - X) d\Omega}
\]  

(2.52)

where \(\kappa(X)\) is the local permeability calculated at point \(X\), \(X_p\) is the material point at which the non-local permeability is calculated, \(\Phi\) is a weighting function and \(\Omega\) is averaging domain which has a radius equal to the material characteristic length \(l^k\). The relation in Equation (2.52) is similar to the definition of the integral non-local damage proposed in [74, 75].

Following the derivation in [76], the Taylor series expansion of the local equivalent strain measure \(\kappa(X)\) is performed about the point \(X_p = X + \xi\) as follows:

\[
\kappa(X + \xi) = \kappa(X) + \nabla \kappa(X) \cdot \xi + \frac{1}{2!} \nabla^{(2)} \kappa(X) \cdot \xi^2 + \frac{1}{3!} \nabla^{(3)} \kappa(X) \cdot \xi^3 + \ldots
\]  

(2.53)

where \(\nabla^{(n)}\) is the n-th order gradient operator and \(\xi^n\) is the n factor dyadic product. By substituting Equation (2.53) into Equation (2.52) and neglecting higher order terms [76], we get:

\[
\tilde{\kappa} = \kappa(X) + c\nabla^2 \kappa(X)
\]  

(2.54)

where \(\tilde{\kappa}\) is the regularized permeability, \(\nabla^2\) is the Laplacian operator and \(c\) is a coefficient that is calculated from the size of the averaging domain. The coefficient \(c\) has the order of characteristic
2.5 Variable length scale

Quantifying the size of the length scale that accompanies the transport phenomena is an open question. The transport length scale does not only depend on the medium properties e.g. grain size and mechanical properties, but it also changes with the flow properties. The stochastic study in [192] suggests that longer flow paths would have larger characteristic length scales that may even become of regional scale. Therefore, introducing a variable length scale to the non-local permeability behaviour would enhance the transport model.

The study in [81] shows that the gradient non-local formulation in Equation (2.56) allows damage to keep widening even after damage reaches a value \( (D = 1) \). This lack of localization may lead to non-physical damage propagation as the damage-crack length becomes longer. This
Figure 2.2: A non-local length proposed in Figure 1.2 may be implemented here into a finite element context by introducing a non-local permeability. The squares present finite elements and crosses represent gauss points.

A phenomenon has been analyzed in [92] and the reason for the continuous damage widening is that the thermodynamic driving degradation force does not vanish upon failure (i.e. when $D = 1$). A remedy to this behavior, first proposed in [81], is the use of transient-gradient approach in which the constant $c$ is substituted by a function of the local equivalent strain. The physical justification of this approach is that the material behaves in a local manner before damage growth, once the material failure is initiated, the non-local activity starts until the non-local length scale is completely recovered in the fully damaged zone. The analysis in [92] shows that the transient-gradient approach leads to the vanishing of the driving degradation force upon complete material failure.

2.6 Energy dissipation

In this section we derive the expressions for the energy dissipation functions discussed earlier in Equations (2.14) and (2.15). These dissipation functions were introduced for local damage-
poroelasticity in [193] and are extended here to nonlocal damage and transport proposed in this paper. We start by calculating the total solid dissipation \( D_s \), and in this case the only process leading to energy dissipation is the damage evolution. Hence, after defining the constitutive relationships in Sections 2.2 and 2.3, Equation (2.17) reduces to:

\[
\dot{D}_s = \int_V -Y\dot{D}dV \geq 0
\]

(2.57)

By expanding \( \Psi^{nl} \) using the definition in Equation (2.6), we get:

\[
Y = \frac{\partial \Psi^{dry}}{\partial D} + \frac{\partial \Psi^{wet}}{\partial D}
\]

(2.58)

where \( \frac{\partial \Psi^{dry}}{\partial D} \) and \( \frac{\partial \Psi^{wet}}{\partial D} \) are the dry and wet components of the thermodynamic damage driving forces, respectively. Using the definition in Equation (2.8), \( \frac{\partial \Psi^{dry}}{\partial D} \) can be written as follows:

\[
\frac{\partial \Psi^{dry}}{\partial D} = \frac{1}{2} \frac{\partial C_{ijkl}^u(D)}{\partial D} \varepsilon_{kl} \varepsilon_{ij}
\]

(2.59)

where the damage derivative of the undrained damaged stiffness tensor can be written as:

\[
\frac{\partial C_{ijkl}^u(D)}{\partial D} = \frac{\partial}{\partial D} \left[ C_{ijkl}(D) + M(D) \alpha^2(D) \delta_{ij} \delta_{kl} \right] = -C_{ijkl} + \delta_{ij} \delta_{kl} \left[ \alpha^2(D) \frac{\partial M(D)}{\partial D} + 2M(D) \alpha(D) \frac{\partial \alpha(D)}{\partial D} \right]
\]

(2.60)

after the substitution of the definition of \( C_{ijkl}^u(D) \) in Equation (2.68). Using the definition in Equation (2.9), \( \frac{\partial \Psi^{wet}}{\partial D} \) can be written as follows:

\[
\frac{\partial \Psi^{wet}}{\partial D} = -\zeta \varepsilon_{ii} \left[ M(D) \frac{\partial \alpha(D)}{\partial D} + \alpha(D) \frac{\partial M(D)}{\partial D} \right] + \frac{1}{2} \zeta^2 \frac{\partial M(D)}{\partial D}
\]

(2.61)
Following the definitions above, the expression of \( \dot{\mathcal{D}}_s \) in Equation (2.57) can be additively decomposed as:

\[
\dot{\mathcal{D}}_s = \dot{\mathcal{D}}_s^d + \dot{\mathcal{D}}_s^w
\]

where the expressions \( \dot{\mathcal{D}}_s^d = \int_V \frac{\partial \Psi_{\text{dry}}}{\partial D} D dV \) and \( \dot{\mathcal{D}}_s^w = \int_V \frac{\partial \Psi_{\text{wet}}}{\partial D} D dV \) denote the dry and wet components contributing to the solid dissipation. The dry component of the dissipation is the energy lost through the damage of the solid skeleton and the wet dissipation is the energy lost due to the change in the fluid-structure interaction mechanism represented by the change of Biot’s coefficient and modulus.

In order to write the expression of the fluid dissipation described by Equation (2.15) after substituting the anisotropic Darcy’s law in Equation (2.5), we write the pressure gradient as:

\[
P_{ij} = -H_{ij} v_i
\]

where \( H_{ij} \) is the inverse permeability tensor defined by \( H_{ij} \kappa_{jk} = \delta_{ik} \), and \( \kappa_{jk} \) is the non-local anisotropic permeability tensor defined in Equation (4.4). Hence, the dissipation due to the fluid flow given in Equation (2.15) can be rewritten as:

\[
\dot{\mathcal{D}}_f := \int_V H_{ij} v_i v_j dV \geq 0
\]

In the case of isotropic permeability, Equation (2.64) is reduced to:

\[
\dot{\mathcal{D}}_f := \int_V \frac{v_i v_j}{\kappa} dV \geq 0
\]

which is similar to the expression introduced in [17], with the non-local \( \tilde{\kappa} \) replacing the local permeability \( \kappa \).
2.7 Constitutive relations of poroelasticity and isotropic damage

Reorganizing Equation (2.37), the fluid content $\zeta$ may be written as:

$$\zeta = \frac{p}{M(D)} + \alpha(D)\varepsilon_{kk} \quad (2.66)$$

then, by substituting $\zeta$ in the second term in Equation (2.36) and simplifying we get:

$$\sigma_{ij} = C_{ijkl}^u(D)\varepsilon_{kl} - \left[ \frac{p}{M(D)} + \alpha(D)\varepsilon_{kk} \right] M(D)\alpha(D)\delta_{ij}$$

$$= \left[ C_{ijkl}^u(D) - M(D)\alpha^2(D)\delta_{ij}\delta_{kl} \right] \varepsilon_{kl} - \alpha(D)\delta_{ij}P \quad (2.67)$$

$$= C_{ijkl}(D)\varepsilon_{kl} - \alpha(D)\delta_{ij}P$$

where the drained stiffness tensor $C_{ijkl}(D)$ is related to the undrained stiffness tensor $C_{ijkl}^u(D)$ by the relation [50]:

$$C_{ijkl}(D) = C_{ijkl}^u(D) - M(D)\alpha^2(D)\delta_{ij}\delta_{kl} \quad (2.68)$$

Following the definition of isotropic damage given by [49], the damaged drained stiffness tensor may be defined as:

$$C_{ijkl}(D) = (1 - D) \left[ \lambda \delta_{ij}\delta_{kl} + \mu \left[ \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right] \right] \quad (2.69)$$

where $D$ is the damage parameter varying from $(0 \rightarrow 1)$, 0 for intact material and 1 for completely damaged material. The constants $\lambda$ and $\mu$ are elastic constants for the undamaged porous domain, $\lambda = \frac{2\mu\nu}{(1-2\nu)}$ is Lame constant, $\mu$ is the shear modulus and $\nu$ is Poisson’s ratio.

Following the derivations presented in [50, 194, 195], the damage dependent Biot’s coefficient and modulus can be defined as:

$$\alpha(D) = 1 - \frac{K(D)}{K_s} \quad (2.70)$$
and

\[ M(D) = \frac{K^u - K(D)}{\alpha(D)^2} \]  

(2.71)

where the damaged bulk modulus is \( K(D) = (1 - D)K \) and the bulk modulus is \( K = \lambda + \frac{2\mu}{3} \), \( K_s \) is the solid grain bulk modulus and \( K^u \) is the bulk modulus defined as:

\[ K^u = \frac{2\mu(1 + v^u)}{3(1 - 2v^u)} \]  

(2.72)

where \( v^u \) is the undrained Poisson’s ratio. The stiffness parameters above are governed by the following bounds in order to ensure a positive definite strain energy potential [196]:

\[ \mu > 0; \quad -1 < v < v^u < 0.5 \]  

(2.73)

The relationship in Equation (2.70) indicates that as damage increases, the damaged bulk modulus decreases while the solid grains preserve their compressibility. Hence, upon material failure \( (D = 1) \), the full incompressible solid condition \( \alpha = 1 \) is retrieved. Similarly, in Equation (2.71), upon material failure \( (D = 1) \), the value of the Biot’s modulus approaches the undrained bulk modulus \( (M = K^u) \) which is infinite in fully incompressible media \( (v^u = 0.5, \ K^u = \infty) \). Therefore, the relations in Equations (2.70) and (2.71) suggest that the material becomes more incompressible as damage grows. This behaviour can be justified by the fact that, in our model, damage only affects the solid skeleton stiffness; therefore, upon material failure \( (D = 1) \) the material retain the compressibility properties of the fluid.

### 2.8 Possible implementation approaches

The computational implementation of the non-local damage transport model presented in this Chapter is not straightforward. In the following Chapters, we will introduce two possible implementation frameworks for the derived non-local transport damage model. In Chapters 3 and 4,
a three-field formulation is proposed. The three-field formulation couples Equations (2.38) and (2.45) into a single equation that provides the effects of non-local transport and damage. This is done under the assumptions of equal length scales for permeability and damage and assuming the similarity of the local strain/stress measure driving the evolution of damage and permeability. Chapter 5 presents a four-field approach that preserves the regularization Equations derived in this Chapter; however, the four-field formulation comes with an additional computational cost.

2.9 Summary

In this chapter, we derived a non-local transport damage model from thermodynamic principles. First, we introduced a new expression for Helmholtz Free Energy that takes into account the energy stores by the non-local damage and transport behaviour. Using Clasius-Duhem inequality, the new state laws and regularization equations are derived. Due to the difficulty of the implementation of the non-local transport relationship, we assume that non-local transport can be modeled as a non-local permeability relationship. An equivalent alternative derivation is presented based on volumetric homogenization. The constitutive laws for poroelastic damage are provided. The energy dissipation functions for the non-local transport damage formulation are derived analytically.
Chapter 3

Three-field modeling of non-local damage transport model: strain-based formulation

3.1 Introduction

In this chapter we present the development of a strain-based version of the three-field non-local damage transport model discussed in Section 2.8. First, the strain-based constitutive evolution laws are presented. Then, the computational implementation featuring the non-linear mixed finite element formulation is discussed in details, the analytical derivation of the Jacobian matrix and the solution algorithm are discussed. The numerical examples in this chapter show the use of the developed model to analyze 1d and 2d hydraulic fracture and 2d consolidation examples. The results show the capability of the developed non-local transport damage model in capturing the essential physical features of consolidation and hydraulic fracture. The results of the presented non-local transport damage model prove to be mesh-independent.
3.2 Strain-based evolution laws

In this section, we define a set of strain-based constitutive definitions that control the evolution of damage, permeability and variable length scale.

3.2.1 Equivalent strain measure

In this study we use an equivalent strain that is defined using a deviatoric-volumetric split as:

\[
\varepsilon^{eq} = \begin{cases} 
\chi^e & \text{if } \chi^e \geq 0 \\
0 & \text{otherwise}
\end{cases} \tag{3.1}
\]

where \( \chi \) is the strain measure:

\[
\chi^e = a_1 s^d - (1 - a_1) s^v \tag{3.2}
\]

where \( s^d \) is the second invariant of the deviatoric strain which describes shear component of the deformation, \( s^d = \sqrt{e_{ij} e_{ij}} \), where \( e_{ij} \) is the deviatoric strain: \( e_{ij} = \varepsilon_{ij} - \frac{1}{3} \delta_{ij} \varepsilon_{kk} \) and \( s^v = \frac{1}{3} \varepsilon_{kk} \) is the volumetric strain which describes pore dilation. The constant \( a_1 \) controls the failure behaviour and should be calibrated against experimental data.

3.2.2 Deformation dependent permeability

In this study we follow the permeability-strain relationship proposed by Selvadurai and Shirazi [65]:

\[
\kappa(\varepsilon^{eq}) = \left[ b_1 + b_2 (\varepsilon^{eq})^{b_3} \right] \kappa_0 \tag{3.3}
\]

where \( b_1, b_2 \) and \( b_3 \) are constants that need to be calibrated from experimental data, \( \kappa_0 \) is the permeability of the unloaded material and \( \varepsilon^{eq} \) is an equivalent strain measure that will be defined in Section 3.2.1.
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3.2.3 Damage evolution law

After the calculation of the non-local permeability $\tilde{\kappa}$, the non-local strain measure $\tilde{\varepsilon}^{eq}$ may be calculated using the inverse of the local permeability-strain relationship in Equation (3.3). The non-local equivalent strain $\tilde{\varepsilon}^{eq}$ can be calculated using:

$$\tilde{\varepsilon}^{eq} = \left[ \frac{\tilde{\kappa}/\kappa_0 - b_1}{b_2} \right]^{\frac{1}{b_3}}$$

(3.4)

The damage law adopted in this study is the bilinear damage law used previously in [79] and other studies. The damage law is defined as:

$$D(\tilde{\varepsilon}^{eq}) = \begin{cases} 
0 & \text{if } \tilde{\varepsilon}^{eq} \leq \varepsilon_i^{eq} \\
\frac{\varepsilon_f^{eq}(\tilde{\varepsilon}^{eq} - \varepsilon_i^{eq})}{\tilde{\varepsilon}^{eq}(\varepsilon_f^{eq} - \varepsilon_i^{eq})} & \text{if } \varepsilon_i^{eq} \leq \tilde{\varepsilon}^{eq} \leq \tilde{\varepsilon}^{eq} \\
D_{max} & \text{if } \tilde{\varepsilon}^{eq} \leq \varepsilon_f^{eq} 
\end{cases}$$

(3.5)

where $\varepsilon_i^{eq}$ and $\varepsilon_f^{eq}$ represent the damage initiation and failure strains respectively. Their values should be calibrated from experimental data. The maximum damage is $D_{max}$ and the strain measure $\tilde{\varepsilon}^{eq}$ is defined as:

$$\tilde{\varepsilon}^{eq} = -\frac{\varepsilon_f^{eq}\varepsilon_i^{eq}}{(\varepsilon_f^{eq} - \varepsilon_i^{eq})} \left[ D_{max} - \frac{\varepsilon_f^{eq}}{(\varepsilon_f^{eq} - \varepsilon_i^{eq})} \right]^{-1}$$

(3.6)

3.2.4 Variable length scale

Substituting the gradient Equation (2.56) by the transient-gradient approach leads to:

$$\tilde{\kappa} - g(\varepsilon^{eq})\nabla^2 \tilde{\kappa} = \kappa(\varepsilon^{eq})$$

(3.7)
where $g(\varepsilon^{eq})$ is the variable length scale that replaces the constant length scale. In this study we use the following exponential relationship for the definition of $g(\varepsilon^{eq})$:

$$g(\varepsilon^{eq}) = (l)^2 [1 - k_1 \exp(-k_2 \varepsilon^{eq})]$$  \hspace{1cm} (3.8)

where $k_1$ and $k_2$ are constants that need to be calibrated from experimental data. The diagram in Figure 3.1 shows the variation of $g(\varepsilon^{eq})$ with $\varepsilon^{eq}$. Figure 3.1 also shows that $k_1$ controls the length scale in the undamaged zone, taking $k_1 = 1$ means that the length scale in the undamaged zone is zero while taking $k_1 = 0$ recovers the fully constant length scale and leads to recovery of Equation (2.56). The constant $k_2$ controls the range of strains in which the transition from local to non-local behavior occurs.

![Figure 3.1: The variation of the gradient variable $g(\varepsilon^{eq})$ with $\varepsilon^{eq}$.](image)


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3.3 Computational implementation

3.3.1 Boundary value problem

This section summarizes the statement of the boundary value problem of non-local damage-transport model proposed in this paper:

\[
\left[C_{ijkl}(D)e_{kl} - \alpha(D)\delta_{ij}P\right]_{,j} + b_i = 0 \text{ in } \Omega \]  
\[\frac{\partial}{\partial t}\left[\frac{P}{M(D)} + \alpha(D)e_{ii}\right] - [\tilde{\kappa}P,]_{,j} = 0 \text{ in } \Omega \]  
\[
\tilde{\kappa} - g(e^{eq})\nabla^2 \tilde{\kappa} = \kappa(e^{eq}) \text{ in } \Omega
\]  

with the boundary conditions:

\[
u_i = \bar{u}_i \text{ on } \Gamma_u \text{ & } \sigma_{ij}n_j = t_i \text{ on } \Gamma_t
\]
\[P = \bar{P} \text{ on } \Gamma_P \text{ & } v_i n_i = s \text{ on } \Gamma_s
\]
\[
\tilde{\kappa},n_i = 0 \text{ on } \Gamma_\kappa
\]
\[D|_{t=0} = 0 \text{ in } \Omega
\]

where Equation (3.9) is derived by substituting the definition of the total stress from Equation (2.67) in the balance of momentum Equation (2.3), Equation (3.10) is derived by substituting the definition of \(\zeta\) from Equation (2.66) and the anisotropic Darcy’s law from Equation (2.5) into the continuity Equation (2.4). Equation (3.11) is the non-local permeability Equation (3.7). The domain space is denoted by \(\Omega\). The boundary conditions \(\bar{u}_i, t_i, \bar{P}\) and \(s\) are the boundary conditions resembling displacements, tractions, pressure and normal flow flux respectively. The boundary conditions \(\bar{u}_i, t_i, \bar{P}\) and \(s\) are applied on the boundary segments \(\Gamma_u, \Gamma_t, \Gamma_P\) and \(\Gamma_s\) respectively as shown in Figure 3.2. The boundary condition in Equation (3.14) is the Neumann boundary con-
3.3. COMPUTATIONAL IMPLEMENTATION

dition on Equation (3.11), which was proposed for the gradient non-local strain in [79]. This
boundary condition does not contribute to the physics of the problem, however, it is required to
complete the definition of the boundary value problem. The damage-dependent material variables
$C_{ijkl}(D), \alpha(D)$ and $M(D)$ are calculated according to Equations (2.69), (2.70) and (2.71). Note
that these terms are also implicit functions of time since damage changes with strain and time. The
non-local equivalent strain $\tilde{\varepsilon}^{eq}$ is calculated according to Equation (3.4) and the damage variable $D$
is calculated according to (3.5). The variable length scale $g(\varepsilon^{eq})$ is calculated according to Equa-
tion (3.8). It has to be noted that in the case of incompressible solid and fluid constituents ($\alpha = 1$
and $M = \infty$), the first, second and fourth terms of Equation (3.10) drop.

![Figure 3.2: Schematic illustrating the non-local damage-transport boundary value problem. The contours on the left figure show the damage increase in the crack zone and the contours in the right figure show the permeability increase in the crack zone. The arrows on the right figure indicate fluid flow directions.](image)

3.3.2 Mixed finite element formulation

The finite element solution of Equations (3.9) to (3.15) involves the development of a mixed
finite element formulation with the main variables of interest $(u, P, \tilde{\kappa})$. The finite element implementa-
tion in [81] of Equation (3.11) requires the introduction of additional nodal degrees of free-
domain for the length scale $g(\varepsilon^{eq})$ due to the presence of its derivative in the weak form. The simple modification proposed in [197] leads to an alternative formulation that does not require additional degrees of freedom. However, it still requires additional computational cost due to the introduction of the additional non-linear relationship describing the evolution of $g(\varepsilon^{eq})$. In this paper, we follow an explicit time update of the variable length scale function $g(\varepsilon^{eq})$, i.e. $g^{n+1} \approx g((\varepsilon^{eq})^n)$ where the superscript $n$ indicates the last converged time step and the superscript $n+1$ indicates the current time step. This lagged update approach leads to a piece-wise constant definition of $g(\varepsilon^{eq})$ and hence avoids the modeling complications in [81] and [197]. To this end, a mixed finite element formulation $(u, P, \tilde{\kappa})$ is proposed to solve the boundary value problem described in Section 3.3.1. Equations (3.9) to (3.11) can be written in weak form as the following residual functions:

\begin{align}
    R^u_i(u, P, \tilde{\kappa}) &= \int_{\Omega} w^u \left[ C_{ijkl}(D) \varepsilon_{kl} - \alpha(D) \delta_{ij} P \right]_{,j} + b_i \right] d\Omega \\
    R^P(u, P, \tilde{\kappa}) &= \int_{\Omega} w^P \left[ \frac{1}{M(D)} \frac{\partial P}{\partial t} + P \frac{\partial (1/M(D))}{\partial t} + \alpha(D) \frac{\partial \varepsilon_{ii}}{\partial t} + \varepsilon_{ii} \frac{\partial \alpha(D)}{\partial t} - [\tilde{\kappa} P_{,i}]_{,i} \right] d\Omega \\
    R^\kappa(u, P, \tilde{\kappa}) &= \int_{\Omega} w^\kappa \left[ \tilde{\kappa} - g((\varepsilon^{eq})^n) \nabla^2 \tilde{\kappa} - \kappa(\varepsilon^{eq}) \right] d\Omega
\end{align}

where $w^u, w^P$ and $w^\kappa$ are the test functions for displacement, fluid pressure and non-local permeability fields, respectively. The following discretization functions are adopted for the three field variables $u, P$ and $\tilde{\kappa}$:

$u = N^u u^h; \quad P = N^P P^h; \quad \tilde{\kappa} = N^\kappa \tilde{\kappa}^h$

where $N^u, N^P$ and $N^\kappa$ are the shape functions for the displacement ($u_i$), fluid pressure ($P$) and non-local permeability ($\tilde{\kappa}$) fields, respectively. The superscript $h$ in $u^h, P^h$ and $\tilde{\kappa}^h$ indicates nodal values of each field. The discretization in Equation (3.17) extends the formulation proposed in [33] to include damage and non-local permeability effects. In order to satisfy the Babuška-Brezzi condition [198–201] the displacement shape function is taken to be quadratic 8-node element known as the serendipity element, while the pressure and non-local permeability shape functions are taken to be
bilinear 4-node element. A schematic of the mixed element formulation is shown in Figure 3.3.

Figure 3.3: Schematic showing the \((u - P - \tilde{\kappa})\) mixed finite element degrees of freedom and Gauss integration points. 8 nodes serendipity element is used for displacements while 4 nodes bilinear elements are for pressure and nonlocal permeability.

In order to continue with the linearizion of the weak form presented in Equation (3.16a) to (3.16c), the solution vector \(x\) and the residual vector \(R\) are defined as follows:

\[
x = \begin{bmatrix} u_i \\ P \\ \tilde{\kappa} \end{bmatrix} ; \quad R = \begin{bmatrix} R_u \\ R^p \\ R^\kappa \end{bmatrix}
\] (3.18)

The residual statement in Equation 3.16 may be written as:

\[
R^{n+1} = M\dot{x}^{n+1} + Kx^{n+1} = 0
\] (3.19)

where \(M\) and \(K\) are square matrices and the superscript \((n + 1)\) denotes the solution at the next time step. Using a difference scheme [202], we may write the velocity vector \(\dot{x}^{n+1}\) as:

\[
\dot{x}^{n+1} = \frac{1}{\beta dt} [x^{n+1} - x^n] - \frac{1 - \beta}{\beta} \dot{x}^n
\] (3.20)
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where $\beta$ is the parameter that defines the time integration scheme, $\beta = 1$ for backward Euler, $\beta = 0.5$ for a Crank-Nicolson method and $\beta = 0$ for forward Euler. The final residual statement, after temporal discretization, is defined as:

$$R^{n+1} = \left[ \frac{1}{\beta \Delta t} M + K \right] x^{n+1} - \frac{1}{\beta \Delta t} M x^n - \frac{1 - \beta}{\beta} M \dot{x}^n = 0 \quad (3.21)$$

In this paper we assume an implicit scheme, which means that at every time step a Newton-Raphson method is used to solve the resulting non-linear system of equations leading to the following linearized system:

$$J^{n+1} \delta x^{n+1} = -R^{n+1} \quad (3.22)$$

where $\delta x$ is the incremental solution vector computed at each Newton iteration and the Jacobian (tangent stiffness) matrix is:

$$J^{n+1} = \frac{\partial R^{n+1}}{\partial x^{n+1}} = \left[ \frac{1}{\beta \Delta t} M + K \right] = \begin{bmatrix} J_{uu} & J_{uP} & J_{uK} \\ J_{Pu} & J_{PP} & J_{PK} \\ J_{Ku} & J_{KP} & J_{KK} \end{bmatrix} \quad (3.23)$$

where the components of the Jacobian matrix $J^{n+1}$ are calculated as:

$$J_{uu} = \frac{\partial R^{n+1}}{\partial u} = \int_{\Omega} B^T \bar{C} B u \, d\Omega \quad (3.24a)$$

$$J_{uP} = \frac{\partial R^{n+1}}{\partial P} = -\int_{\Omega} B^T \alpha(D) I N^P \, d\Omega \quad (3.24b)$$

$$J_{uK} = \frac{\partial R^{n+1}}{\partial \kappa} = -\int_{\Omega} B^T C \varepsilon \frac{\partial D}{\partial \kappa} N^\kappa + B^T P I \frac{\partial \alpha(D)}{\partial \kappa} N^\kappa \, d\Omega \quad (3.24c)$$

$$J_{Pu} = \frac{\partial R^{n+1}}{\partial u} = \frac{1}{\beta \Delta t} \int_{\Omega} N^P T \alpha(D) B^{u,\text{vol}} \, d\Omega + \int_{\Omega} N^P T \frac{\partial \alpha(D)}{\partial t} B^{u,\text{vol}} \, d\Omega \quad (3.24d)$$

$$J_{PP} = \frac{\partial R^{n+1}}{\partial P} = \frac{1}{\beta \Delta t} \int_{\Omega} N^P T \frac{1}{M(D)} N^P \, d\Omega + \int_{\Omega} N^P T \frac{\partial (1/M(D))}{\partial t} N^P \, d\Omega + \int_{\Omega} B^{P T} \bar{K} B^P \, d\Omega \quad (3.24e)$$
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\[ J_{\kappa P} = \frac{\partial R^P}{\partial \tilde{\kappa}} = \int_{\Omega} N^{P^T} \left[ \frac{\partial P}{\partial t} \frac{\partial [1/M(D)]}{\partial \tilde{\kappa}} + P \frac{\partial \frac{1}{M(D)}}{\partial \tilde{\kappa}} + \varepsilon_{il} \frac{\partial \alpha(D)}{\partial \tilde{\kappa}} + \varepsilon_{ii} \frac{\partial \alpha(D)}{\partial \tilde{\kappa}} \right] N^{\kappa} + B^P B^P N^{\kappa} d\Omega \] (3.24f)

\[ J_{\kappa u} = \frac{\partial R^\kappa}{\partial u} = - \int_{\Omega} N^{\kappa^T} \left[ \frac{\partial \kappa'(\varepsilon_{eq})}{\partial \varepsilon} \right] B^u d\Omega \] (3.24g)

\[ J_{\kappa P} = \frac{\partial R^\kappa}{\partial P} = 0 \] (3.24h)

\[ J_{\kappa \kappa} = \frac{\partial R^\kappa}{\partial \tilde{\kappa}} = \int_{\Omega} N^{\kappa^T} N^{\kappa} d\Omega + g((\varepsilon_{eq})^n) \int B^{\kappa^T} B^{\kappa} d\Omega \] (3.24i)

where \( B^u \) is the shape function derivative of \( N^u \) used in the calculation of strains as in: \( \varepsilon_{ij} = B^u u^h; B^{u,\text{vol}} \) is the shape function derivative of \( N^u \) used in the calculation of volumetric strain as in: \( \varepsilon_{ii} = B^{u,\text{vol}} u^h; B^P \) is the shape function derivative of \( N^P \) corresponding to \( P_i = B^P P^h; B^{\kappa} \) is the shape function derivative of \( N^{\kappa} \) corresponding to \( \tilde{\kappa}_i = B^{\kappa} \tilde{\kappa}^h \) and the superscript \( T \) indicates matrix transpose. The matrix forms of the stiffness tensor \( C_{ijkl} \) and the damaged stiffness tensor \( (1 - D)C_{ijkl} \) are denoted by \( C \) and \( \bar{C} \), respectively. The kronecker \( \delta_{ij} \) operator is the identity matrix \( I \). In this paper, we employ a backward difference scheme with \( \beta = 1 \). Detailed expressions of the derivatives used in the calculation of the residual vector and Jacobian matrix are provided in Appendix A. Note that the Jacobian matrix is sparse and nonsymmetric, which would require appropriate solvers.

### 3.3.3 Solution algorithm

This section details the non-linear solution algorithm used to solve the system of equations in Equation (3.23). The solution is programmed within the FEAP software [203] as a user-defined element model. The time-step is calculated adaptively using FEAP built-in functions as:

\[ \Delta T^{n+1} = \begin{cases} \min(\Delta t_{\text{max}}, 10^{\log(\Delta t^n) + 0.2}) & \text{if } I^n < I_{\text{min}} \\ \max(\Delta t_{\text{min}}, 10^{\log(\Delta t^n) - 0.2}) & \text{if } I^n > I_{\text{max}} \end{cases} \] (3.25)
The symbol $I^n$ is the number of Newton iterations at the previous time step $n$ and the parameters $I^{\text{min}}$ and $I^{\text{max}}$ denote the user-defined target minimum and maximum number of iterations, respectively, used to estimate the next time step size at $n+1$. We emphasize that these parameters are only used as criterion for the adaptive time stepping estimation. $\Delta t^{\text{min}}$ and $\Delta t^{\text{max}}$ are user-defined minimum and maximum time step values, respectively. The complete solution scheme is summarized in Algorithm 1. In the solution scheme in Algorithm 1, the element computations are performed between steps 5 and 14 and the solution of equations is done in step 17. Once convergence is reached in step 4, the time step for the next time increment is adjusted adaptively following the formula in Eq. (3.25) (step 20). This adjustment aims at keeping the time step small enough to obtain good Newton convergence while allowing it to grow when the nonlinear equations are easier to solve. Note that once the time step size is set, Newton iterations proceed until the desired tolerance is met and are not directly affected by $I^{\text{min}}$ and $I^{\text{max}}$ parameters.

Finite element solutions of Biot’s poroelasticity equations are well-known to experience spurious non-physical pressure oscillations when time marching is done using small time steps [204–206]. Different researchers attribute these oscillations to the time integration scheme[204, 205, 207, 208], violation of finite element compatibility [209–211], ill-conditioning of poroelasticity matrices [212, 213] or other reasons[214, 215]. These studies were based on linear poroelasticity models, where the need to use small time steps is only to get high resolution output or analyze specific loading mechanisms. In non-linear models like the damage model in this paper, small time steps are essential for convergence of the non-linear solver but too small steps may also lead back to spurious pressure oscillations. The solution to these numerical artifacts is still an open question and so far no robust way to eliminate these numerical artifacts was identified. Therefore, in the following numerical examples, the adaptive time step size in Eq. (3.25) is bounded below to prevent spurious pressure oscillations and bounded above to ensure reasonable convergence of the Newton scheme. Forcing a large time step degrades the convergence of the Newton solver and
3.4. EXAMPLES

Algorithm 1 Solution Algorithm

1: Initialize all variables \((\mathbf{u}^h, \mathbf{P}^h, \tilde{\kappa}^h)\)
2: while \(t < T\) do
3: Calculate \(g((\tilde{\varepsilon}^{eq})^n)\)  
4: while \(\|\mathbf{R}\| < tol\|\mathbf{R}_0\|\) do
5: for each finite element do
6: for each material point do
7: Interpolate \(\tilde{\kappa}\) value at material point from \(\tilde{\kappa}^h\)
8: Calculate the value of the non-local equivalent strain \(\tilde{\varepsilon}^{eq}\)  
9: Calculate non-local damage \(D(\tilde{\varepsilon}^{eq})\)  
10: Interpolate local strain tensor \(\varepsilon_{ij}\) from \(\mathbf{u}^h\)
11: Calculate local equivalent strain \(\varepsilon^{eq}\)  
12: Calculate local permeability \(\kappa(\varepsilon^{eq})\)  
13: Calculate Jacobian matrix \(\mathbf{J}\) and residual vector \(\mathbf{R}\)  
14: end for
15: end for
16: Assemble Jacobian matrix \(\mathbf{J}\) and residual vector \(\mathbf{R}\) for all elements
17: Solve for \(\delta\mathbf{x}\)  
18: \(\mathbf{x}^{n+1} \rightarrow \mathbf{x}^{n+1} + \delta\mathbf{x}\)  
19: end while
20: Calculate adaptive time step: \(dt = \Delta T^{n+1}\) using \(I^{min}, I^{max}, \Delta t^{min}\) and \(\Delta t^{max}\)  
21: \(t \rightarrow t + dt\)  
22: end while

Quadratic convergence may not be guaranteed [216, 217].

3.4 Examples

3.4.1 Fluid driven failure of a poroelastic column

In this section, we consider a fluid driven failure example of a poroelastic column. The column, shown in Figure 3.4, is subjected to compressive loading \(E\varepsilon_{zz}\) at the top and fixed at the bottom \(u_z|_{z=0} = 0\), while the pressure at the top is zero \(P|_{z=H} = 0\) and a fluid flux is applied at the bottom \(\kappa \frac{\partial P}{\partial z}|_{z=0} = q\). In order to achieve a stable steady state equilibrium, we limit the damage growth to \(D^{max} = 0.75\). We consider the behaviour of the column in the following cases:

- linear poroelastic model, no damage and constant permeability (LM)
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- local damage and constant permeability model (LDCP)
- local damage and variable permeability model (LDVP)
- non-local damage-transport model (NDT)

Figure 3.4: A poroelastic column subjected to compressive loading at the top and fluid flux at the bottom. The expected strain $\varepsilon_z(z)$ and fluid pressure $P(z)$ profiles are demonstrated for the linear poroelastic (LM) and non-local damage transport (NDT) models.

The analytical solution of the fluid pressure and vertical strain at the steady state is given in B for the cases of LM and and LDCP. In the local damage model with constant permeability (LDCP), the analytical derivations conclude that strain at steady state will always be higher than $\varepsilon^{eq}_f$.

Nonetheless, these analytical solutions are only derived for steady state and could be used to verify the numerical method needed in order to obtain the time dependent response of the column. The problem is modeled using the finite element formulation presented in Section 3.3. In order to model the 1d problem using the developed multi-dimension formulation, the constitutive relation
defining stress in Equation (2.67) is modified to be:

\[ \sigma_{ij} = C_{ijkl}(D)\varepsilon_{kl} - \alpha(D)\delta_{ij}\delta_{zz} \]  

(3.26)

and the value of the Poisson’s ratio is kept as \( \nu = 0 \). The multiplication of the fluid pressure term by \( \delta_{zz} \) assures that fluid pressure is added to the stress in \( z \)-direction only. Three different mesh densities of sizes \( h = \{1.00 \text{ m}, 0.50 \text{ m}, 0.25 \text{ m}\} \) are considered in the numerical testing. The numerical simulation is run until \( t = 2 \times 10^6 \text{s} \) (approximately 555.5 hours) in order to get to the steady state behaviour. The model parameters used in the simulations are listed in Table 3.1.

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
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<td>Undrained Poisson’s ratio</td>
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<td>Solid grain bulk modulus</td>
<td>( K_s )</td>
<td>( 1.0 \times 10^50 \text{ Pa} )</td>
</tr>
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<td>Constant in Equation (3.3)</td>
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<td>( 1.0 \times 10^8 )</td>
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<tr>
<td>Constant in Equation (3.3)</td>
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<td>Maximum damage</td>
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<td>Damage initiation strain</td>
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<td>( 5.0 \times 10^{-5} )</td>
</tr>
<tr>
<td>Damage final strain</td>
<td>( \varepsilon_{eq}^f )</td>
<td>( 2.0 \times 10^{-4} )</td>
</tr>
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<tr>
<td>Constant in Equation (3.8)</td>
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<td>( H )</td>
<td>10 \text{ m}</td>
</tr>
<tr>
<td>Fluid flux</td>
<td>( q )</td>
<td>(-1.0 \times 10^{-8} \text{ m}^3/\text{s} )</td>
</tr>
</tbody>
</table>

The fluid pressure, equivalent strain and permeability profiles are presented in Figures 3.5, 3.6 and 3.7, respectively. The fluid pressure results in Figure 3.5 show that the LDCP model FEM
results match the analytical solution for different mesh sizes, the LDVP model results are mesh dependent while the NDT models are mesh independent. The strain results in the LDCP and LDVP show that there is a discontinuity in the values of the strain in the damage zone, in addition, the LDCP and LDVP strain results are mesh dependent. A similar discontinuity and mesh dependence is observed in the permeability field of the LDVP model. However, in the NDT model strain and permeability results, the transition between the no-damage to damage zone is smooth and the final profiles of strain and permeability are mesh independent.

Figure 3.5: The fluid pressure profile at steady state calculated from different models: LM, LDCP, LDVP and NDT. The LDCP model matches the analytical solution derived in Appendix B. The LDVP model demonstrates mesh dependence which is overcome by the non-local formulation in NDT.

In order to have a better understanding of the difference between the analysis models, we further look into the temporal evolution of the key variables. First, the temporal evolution of equivalent strain and fluid pressure $P$ results from the LDCP model are presented in Figures 3.8 and 3.9, respectively. The strain steady state solutions for all mesh sizes are higher than $\hat{\varepsilon}_c$ as concluded by the analytical derivation in B. The results in Figures 3.8 and 3.9 show that the LDCP model suffers from two undesirable phenomena, which are:
3.4. EXAMPLES

Figure 3.6: The equivalent strain profile at steady state calculated from different models: LDVP and NDT. The LDVP model demonstrates mesh dependence which is overcome by the non-local formulation in NDT.

Figure 3.7: The permeability profile at steady state calculated from different models: LDVP and NDT. The LDCP and LDVP model demonstrates mesh dependence which is overcome by the non-local formulation in NDT.

- significant mesh dependence on the damage, strain and pressure evolution paths as well as the steady state values of strain,
• spurious oscillations in strain and pressure evolution; these oscillations are caused by the nature of the local damage growth at a material point which leads to cycles of successive unloading and stress redistribution.

Figure 3.8: Strain evolution numerical results for constant permeability, local damage (LDCP) model. The final strain at steady state is higher than $\hat{\varepsilon}_z = 1.14 \times 10^{-4}$, as expected from the analytical derivation. The strain evolution and steady state values demonstrate significant mesh dependence on the finite element mesh size $h$. The local damage modeling leads to spurious oscillations during strain growth. Results are presented at the point $z = 0$.

The LDVP and NDT results are shown in Figures 3.10, 3.11 and 3.12 for strain, fluid pressure and permeability respectively. The LDVP model results exhibit the same problems that were found in the LDCP model which are: mesh dependence of the results and the spurious oscillations. However, the additional downside of using local damage is demonstrated here in the spurious oscillations in the local permeability growth. Due to the permeability-strain relationship, the permeability behaviour follows the cycles of relaxation and hardening exhibited by the local equivalent strain which are caused by local damage growth. The local permeability oscillations imply that the permeability may decrease during material failure, which is a non physical phenomena and does not appear in experimental data [51–53]. In addition, if permeability is interpreted as a damage
3.4. EXAMPLES

Figure 3.9: Fluid pressure evolution numerical results for constant permeability, local damage (LDCP) model. The fluid pressure at steady state is equal to the analytical derived value from Equation (B.11). The local damage modeling leads to spurious oscillations during fluid pressure growth. Results are presented at the point $z = 0$.

parameter that contributes to the fluid flow properties of the problem, then permeability decrease indicates a false healing effect.

The non-local model results demonstrate mesh-independence as well as smooth evolution of all variable without any oscillations. Hence, the non-local model overcomes all the discussed drawbacks of the local model. In the local model, the damage behaviour is dependent on the discretization length scale; therefore, as we decrease the element size, we reach an extremely localized crack and the numerical discretization controls the physical behaviour of the problem.

The introduction of a non-local length scale to govern the material softening behaviour provides a mathematically consistent [74] definition of damage that disassociates the numerical discretization from the physical behaviour.

By analyzing the fluid pressure evolution in the non-local model demonstrated in Figure 3.11, we find that the pressure evolution profile matches the numerical results for fluid fracture problems [88, 89]. In fluid fracture, the fluid pressure at the crack tip increases until the crack opens, then
the pressure inside the crack drops partially to a value lower than that at the tip.

![Graph showing local strain evolution](image)

Figure 3.10: Local strain $\varepsilon_z$ evolution numerical results for variable permeability, local model (denoted by L) and non-local model (denoted by NL) for different mesh sizes $h$. The final strain at steady state is higher than $\varepsilon_z = 1.14 \times 10^{-4}$, as expected from the analytical derivation. The local models demonstrate spurious oscillations upon strain growth and mesh dependence in the steady state values. Non-local model results show smooth strain growth and consistent values at steady state for different mesh sizes. Results are presented at the point $z = 0$.

### 3.4.2 Enhanced two-dimensional consolidation

Consolidation of soils has always been one of the major interests of geomechanics studies. Under poroelastic assumptions, the consolidation process is described by the loss of pore water pressure in a recently loaded soil, which leads to increased stresses and strain in the solid skeleton of the porous medium. The solid skeleton may experience high stresses during this process which may result in damage of the solid. In this example we use the proposed non-local damage-transport model to analyze the consolidation behaviour when the solid skeleton experiences damage. Damage-enhanced consolidation has been previously investigated in [4]; however, the study employed local damage only which is known to be mesh dependent and as was illustrated in the previous example may also exhibit spurious oscillations.
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Figure 3.11: Fluid pressure evolution numerical results for variable permeability, local model (denoted by \( L \)) and non-local model (denoted by \( NL \)) for different mesh sizes \( h \). The local models demonstrate spurious oscillations upon fluid pressure growth and mesh dependence in the steady state values. Non-local model results show smooth fluid pressure evolution and consistent values at steady state for different mesh sizes. Results are presented at the point \( z = 0 \).

Consider the consolidation problem in Figure 3.13, the footing base is assumed rigid. Due to symmetry, only half of the problem is analyzed with symmetric boundary conditions as shown in Figure 3.14. The problem is analyzed using local and non-local damage-transport model developed in this paper. Three mesh sizes are considered in the analysis \( h = \{0.15 \text{ m}, 0.10 \text{ m}, 0.05 \text{ m}\} \). The modeling parameters are listed in Table 3.2.

Damage and fluid pressure evolution contour results for the non-local damage-transport model are shown in Figure 3.15. Upon loading, the fluid pressure in the soil below the footing increases as expected by the traditional theory of consolidation [16]. The excess fluid pressure below the footing leads to the creation of a pressure gradient that leads to water flowing outside the domain through the permeable boundaries as dictated by the boundary conditions, demonstrated in Figure 3.14. Upon fluid flow away from the footing, the fluid pressure decreases and the solid skeleton bears more stresses which leads to damage. The damage evolution pattern is similar to expected...
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Figure 3.12: Permeability evolution numerical results for local model (denoted by \( L \)) and non-local model (denoted by \( NL \)) for different mesh sizes \( h \). The local models demonstrate spurious oscillations upon permeability growth and mesh dependence in the steady state values. Non-local model results show smooth permeability evolution and consistent values at steady state for different mesh sizes. Results are presented at the point \( z = 0 \).

Figure 3.13: Schematic diagram demonstrating the loaded concrete footing resting on the soil medium, resembling the 2d consolidation problem.

shear failure footings and the damage results presented in [4, 65] and the theoretical expectations for footing failure [218]. A fluid velocity arrow diagram is demonstrated in Figure 3.16, the diagram shows that the fluid velocity is high in the damage process zone where higher strains and permeability are present.

The fluid pressure, equivalent strain and permeability results at point A are demonstrated in Figures 3.17, 3.18 and 3.19, respectively. The results are shown for the local model with variable
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Figure 3.14: Mechanical model showing the boundary conditions of the consolidation problem including the boundary conditions for the equilibrium and mass balance equations. The rigid footing is loaded with a vertical surcharge $\sigma_0$.

Figure 3.15: Damage and fluid pressure evolution results for the consolidation problem from the non-local damage-transport model with variable characteristic length. The fluid pressure results in Figure 3.17 show that the pressure evolution profile is less affected by the mesh size in the non-local model. The local model results for equivalent strain and permeability shown in Figures
Table 3.2: Modeling parameters for Example 3.4.2

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Footing pressure</td>
<td>$\sigma_0$</td>
<td>$1.0 \times 10^5$ Pa</td>
</tr>
<tr>
<td>Bulk Modulus</td>
<td>$K$</td>
<td>$2.08 \times 10^8$ Pa</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>$\mu$</td>
<td>$9.26 \times 10^7$ Pa</td>
</tr>
<tr>
<td>Undrained Poisson’s ratio</td>
<td>$\nu^u$</td>
<td>0.49999</td>
</tr>
<tr>
<td>Solid grain bulk modulus</td>
<td>$K_s$</td>
<td>$2.0 \times 10^{20}$ Ma</td>
</tr>
<tr>
<td>Initial permeability</td>
<td>$\kappa_0$</td>
<td>$1.00 \times 10^{-11}$ m$^2$/Pa s</td>
</tr>
<tr>
<td>Constant in Equation (3.2)</td>
<td>$a_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (3.3)</td>
<td>$b_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (3.3)</td>
<td>$b_2$</td>
<td>$5.0 \times 10^2$</td>
</tr>
<tr>
<td>Constant in Equation (3.3)</td>
<td>$b_3$</td>
<td>1.0</td>
</tr>
<tr>
<td>Maximum damage</td>
<td>$D^{max}$</td>
<td>0.95</td>
</tr>
<tr>
<td>Damage initiation strain</td>
<td>$\epsilon_i^{eq}$</td>
<td>$2.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>Damage final strain</td>
<td>$\epsilon_f^{eq}$</td>
<td>$3.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>Non-local length scale</td>
<td>$l^k$</td>
<td>0.75 m</td>
</tr>
<tr>
<td>Constant in Equation (3.8)</td>
<td>$k_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (3.8)</td>
<td>$k_2$</td>
<td>$8.0 \times 10^3$</td>
</tr>
<tr>
<td>Maximum time step</td>
<td>$\Delta t^{max}$</td>
<td>$1.0 \times 10^2$ s</td>
</tr>
<tr>
<td>Minimum time step</td>
<td>$\Delta t^{min}$</td>
<td>$5.0 \times 10^{-1}$ s</td>
</tr>
<tr>
<td>Adaptive time stepping iteration target range</td>
<td>$I_{min} - I_{max}$</td>
<td>15–20</td>
</tr>
</tbody>
</table>

3.18a and 3.19a demonstrate significant mesh dependence. In the local permeability results, the permeability decreases after the initial peak which is not physical [51–53] and represents a false healing caused by the local damage growth. This behaviour is similar to the local permeability oscillations discussed in Section 3.4.1. The non-local equivalent strain and permeability results show that the model results are mesh independent and no oscillations are recorded.

The results in Figures 3.17, 3.18 and 3.19 show that we run the consolidation simulations until they approach steady state. The time taken until steady state mainly depends on the soil initial permeability. In our case ($\kappa_0 = 1.0 \times 10^{-11} m^2/Pa$) it takes a few seconds to approach a steady state behaviour, but soils with lower permeabilities may take much longer times to approach steady state. However, modeling lower permeability soils will lead to a computational challenge because the time scale of the poroelastic behaviour is much slower than the damage evolution behaviour.
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Figure 3.16: Fluid velocity distribution at 1.0 s. The arrow directions represent the fluid velocity direction and the color is the velocity magnitude.

From a geotechnical engineering point of view, the primary variable of interest in the consolidation problem is the settlement below the footing. We examine the resulting displacement evolution from the proposed non-local damage transport against the linear poroelastic results and the local displacement with variable permeability model proposed in [4, 65], the results are demonstrated in Figure 3.20. The results show that using local damage may lead to higher settlement expectation than the proposed non-local model. In addition, the settlement of the soil computed from the local damage model is mesh dependent; hence it is not suitable for footing design purposes.

3.4.3 Fluid driven fracture in porous media

Modeling hydraulic fracture in porous media is a challenge that has been approached by researchers using different methods e.g.: analytical solutions [196, 219], generalized/extended finite...
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Figure 3.17: Pressure variation with time at Point A (on Figure 3.14) for local damage-transport model (a) vs. non-local damage-transport model (b). In the non-local model, the calculated values of pressure are less influenced by mesh size than in the local model.

Figure 3.18: Equivalent strain variation with time at Point A (on Figure 3.14) for local damage-transport model (a) vs. non-local damage-transport model (b). The mesh dependence of the local equivalent strain in the local damage model is overcome in the non-local equivalent strain results.

element method (G/XFEM) [45, 47, 73, 220] and phase-field [88–90]. In this section, we tackle the hydraulic fracture problem using the non-local damage-transport model developed in this paper. Consider the schematic in Figure 3.21, the hydraulic fracture problem is modeled using a poroelastic domain of dimensions 80m × 80m, the external boundaries of the domain are mechanically
3.4. EXAMPLES

Figure 3.19: Permeability variation with time at Point A (on Figure 3.14) for local damage-transport model (a) vs. non-local damage-transport model (b). The mesh dependence of the local permeability in the local model is overcome in the non-local permeability results.

Figure 3.20: Vertical displacement variation with time at Point A (on Figure 3.14) for linear poroelastic model (LM), local damage with variable permeability (LDVP) and non-local damage transport model (NDT).

restrained and permeable. An initial void with dimensions of 1m × 1m, is used to pump fluid inside the domain with a flux of $Q$. The modeling parameters are listed in Table 3.3. In this example, we ignore the in-situ stresses to simplify the modeling; however, they can be easily included by changing the boundary conditions in Figure 3.21. The model boundary conditions and parameters
are similar to those in [90].

Figure 3.21: Schematic diagram demonstrating the hydraulic fracture domain with boundary conditions. The external boundary of the domain is mechanically restrained ($u_i = 0$) and permeable ($P = 0$). A fluid flux $Q$ is injected in the initial crack in the middle of the domain. The domain dimensions are $80m \times 80m$ and the initial void dimensions are $1m \times 1m$. Section line $A-A$ and marking points $a$, $b$ and $c$ are going to be used to demonstrate specific results in later figures.

This example demonstrates the efficiency and robustness of the solution methodology described in Algorithm 1. First, we plot the variation in time step size $dt$ and the number of iterations at each time increment $I^n$ against the simulation time in Figure 3.22. The results show the effectiveness of Algorithm 1 in keeping the number of iterations within the range prescribed in Table 3.3. Also, Figure 3.22a shows that later stages of the simulation requires smaller time steps in order to preserve the desired convergence tolerance. Second, we plot the number of Newton iterations required by the solver to reach a relative tolerance of $\frac{||R||}{||R_0||} = 1 \times 10^{-12}$ in Figure 3.23, convergence at two different stages of the simulations are shown. The first is the convergence at an early stage of the simulation ($T = 212.0$ s) before the adaptive time stepping techniques reaches the range of number of iterations prescribed in Table 4.1. The second is the convergence of the Newton solver.
Table 3.3: Modeling parameters for Example 3.4.3

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid flux</td>
<td>$Q$</td>
<td>$2.8 \times 10^{-4} \text{m}^3\text{s}^{-1}$</td>
</tr>
<tr>
<td>Bulk Modulus</td>
<td>$K$</td>
<td>$2.08 \times 10^8 \text{Pa}$</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>$\mu$</td>
<td>$9.26 \times 10^7 \text{Pa}$</td>
</tr>
<tr>
<td>Undrained Poisson’s ratio</td>
<td>$\nu_u$</td>
<td>0.49999</td>
</tr>
<tr>
<td>Solid grain bulk modulus</td>
<td>$K_s$</td>
<td>$1.1 \times 10^9 \text{Pa}$</td>
</tr>
<tr>
<td>Initial permeability</td>
<td>$\kappa_0$</td>
<td>$1.00 \times 10^{-11} \text{m}^2/\text{Pa s}$</td>
</tr>
<tr>
<td>Constant in Equation (3.2)</td>
<td>$a_1$</td>
<td>0.0</td>
</tr>
<tr>
<td>Constant in Equation (3.3)</td>
<td>$b_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (3.3)</td>
<td>$b_2$</td>
<td>$5.0 \times 10^3$</td>
</tr>
<tr>
<td>Constant in Equation (3.3)</td>
<td>$b_3$</td>
<td>1</td>
</tr>
<tr>
<td>Maximum damage</td>
<td>$D_{\text{max}}$</td>
<td>0.96</td>
</tr>
<tr>
<td>Damage initiation strain</td>
<td>$\varepsilon_i^{eq}$</td>
<td>$7.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>Damage final strain</td>
<td>$\varepsilon_f^{eq}$</td>
<td>$1.0 \times 10^{-2}$</td>
</tr>
<tr>
<td>Non-local length scale</td>
<td>$l^k$</td>
<td>0.75 m</td>
</tr>
<tr>
<td>Constant in Equation (3.8)</td>
<td>$k_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (3.8)</td>
<td>$k_2$</td>
<td>$5.0 \times 10^2$</td>
</tr>
<tr>
<td>Maximum time step</td>
<td>$\Delta t_{\text{max}}$</td>
<td>$1.0 \times 10^2 \text{s}$</td>
</tr>
<tr>
<td>Minimum time step</td>
<td>$\Delta t_{\text{min}}$</td>
<td>$5.0 \times 10^{-1} \text{s}$</td>
</tr>
<tr>
<td>Adaptive time stepping iteration target range</td>
<td>$I_{\text{min}} - I_{\text{max}}$</td>
<td>15–20</td>
</tr>
</tbody>
</table>

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at a later stage of the simulation ($T = 1.3 \times 10^4 \text{s}$) where the number of iterations is around 15.

The suboptimal convergence rate at the later stage is caused by the large number of iterations that we force in order to use a time step that is large enough to overcome the spurious fluid pressure oscillations as discussed in Section 3.3.3. The strict tolerance criteria, $\| \mathbf{R} \|_{\| \mathbf{R}_0 \|} = 1 \times 10^{-12}$, is required to reach a reasonable final absolute residual in each iteration. The convergence results in Figures 3.22 and 3.23 show the robustness of the solution Algorithm and its capability to adapt the time-stepping during the simulation to ensure the convergence of the Newton solver at each time step while minimizing the overall computational cost.

The evolution of damage, fluid pressure and fluid velocity in $x$-direction ($v_x$) in the poroelastic domain under the effect of the hydraulically driven fracture is demonstrated by the contours in Figure 3.24. We notice from the results in Figure 3.24 that fluid pressure grows in all directions.
Figure 3.22: The variation of the solution time step $dt$ (a) and the number of iterations (b) against simulation time. The plots show the effectiveness of the adaptive time stepping approach detailed in Section 3.3.3.

Figure 3.23: Convergence of the Newton solver at different stages of the simulation for a relative tolerance of $\frac{||R||}{||R_0||} = 1 \times 10^{-12}$. The plots show the robustness of the Newton’s solver developed in Section 4.3.2; even with larger time steps, the algorithm is capable of converging to the desired tolerance.

around the crack while damage and fluid velocity grow in the direction of fluid pumping shown in Figure 3.21. This difference in behaviour is attributed to the use of isotropic permeability in our
model which allows the fluid pressure to build up in all directions by fluid seeping slowly in all directions. However, the velocity contours still show that the majority of the fluid mass is moving along with the crack. The evolution of the characteristic length scale $l^k$ according to the function in Equation (3.8) is shown in Figure 3.25 at points $a, b$ and $c$ marked on Figure 3.21. The results show the time dependent evolution of $l^k$ at points $a$ and $b$ along the fracture, while the material behaviour remains almost local at point $c$ far from the fracture.

The flow of fracture fluid through the crack walls to the surrounding domain is referred to as the fluid "leak-off" [221]. This leak-off phenomenon has been previously reported in hydraulic fracture literature and the leaking fluid is believed to cause additional damage around the major fracture [222]. Previous formulations for hydraulic fracture e.g. phase-field [89] and LEFM [221] needed to add an artificial flux to the hydraulic fracture boundary in order to model the leakage. The aforementioned models could not accommodate for the fluid leak-off because they use a Poiseuille flow inside the crack and Darcy flow in the elastic domain. In addition, the extra flux term indicates the amount of fluid lost but does not lead to additional damage in these models. In the present model, leak-off is naturally present in the model because of the continuum non-local transport that we use inside and outside the crack.

In order to have a better understanding of the fluid flow, we plot the profiles of damage and fluid velocity in x-direction along a line 7m away from the flux input point, denoted by section A-A on Figure 3.21. The profiles are plotted at different time steps in Figure 3.26. The results in Figure 3.26 show the localization of hydraulic fracture in the poroelastic domain and the correlation between the damage zone and the high flow speed which demonstrates that the crack is hydraulically driven. However, we also notice that the width of the high velocity zone is larger than the damage zone which leads to fluid seeping around the crack, thus, causes the slow pressure build-up in the poroelastic domain. We investigate the model response to changing the input hydraulic flux, the fracture length is plotted against time in Figure 3.28. The results in Figure 3.28 show that using a
higher input flux leads to a higher fracture rates and that lower input flux models may be subject to approaching a steady state at a lower fracture length.

In order to have a closer look at the pressure-damage evolution at the crack tip, we plot the evolution of damage and fluid pressure at the flux input point (crack initiation) in Figure 3.27. In this simulation we let damage grow to $D_{\text{max}} = 0.99$ rather than $D_{\text{max}} = 0.96$ used in other simulations in order to model a completely open crack upon complete solid material loss. The results in Figure 3.27 show that the fluid pressure keeps increasing until the material completely breaks ($D = 0.99$), then it drops to a lower value and stabilizes in the open crack. This behaviour of fluid pressure in hydraulically driven fracture was recorded in several numerical studies [88, 89].

### 3.5 Summary and conclusion

This paper introduces a novel non-local damage-transport model for porous media. The poromechanics system consists of a solid skeleton coupled to fluid flow and a damage evolution law. The solid is modeled as a linear elastic material and the fluid flow is modeled via Darcy’s law assuming an isotropic but non constant strain-dependent permeability. Mechanical loading such as in consolidation type problems or fluid injection loadings as those in hydraulic fracture type problems may both cause damage to the solid skeleton and accelerate its failure. In such cases, if local damage schemes are used to model fracture growth, the results are mesh dependent and furthermore the pressure, strain and permeability fields experience unphysical spurious oscillations. Hence, the objective of this work is to develop a reliable nonlocal formulation for these multiphysics systems.

To this end, we propose a novel non-local permeability formulation (which can also be viewed as non-local damage-transport model) with a variable length scale and show that it indeed leads to reliable mesh independent results and smooth response of pressure, strain and permeability fields.

A displacement-pressure-permeability ($u - P - \tilde{\kappa}$) mixed finite element formulation is proposed to discretize the governing poromechanics equations. An implicit scheme with adaptive time stepping is used to evolve the system in time, where at every time step Newton’s method with analytical
3.5. SUMMARY AND CONCLUSION

Figure 3.24: Damage, fluid pressure and fluid velocity in x-direction ($v_x$) evolution results for the 2-d hydraulic fracture example.
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![Figure 3.25](image)

Figure 3.25: Evolution of the characteristic length $l^k$ at points $a$, $b$ and $c$ marked on Figure 3.21. Points $a$ and $b$ show time-dependent evolution of the characteristic length $l^k$ while the behaviour at point $c$ remains almost local.

![Figure 3.26](image)

Figure 3.26: The profiles of damage and fluid velocity in x-direction at section A-A as shown in Figure 3.21. The profiles of damage and fluid velocity are demonstrated at different time steps.

derivation of the Jacobian matrix, is employed to solve the nonlinear system.

Numerical examples of 1d and 2d hydraulic fracture problems and 2d damage-enhanced consolidation are presented and the proposed nonlocal damage-transport model is compared with var-
3.5. SUMMARY AND CONCLUSION

Figure 3.27: Damage and fluid pressure evolution with time at the flux input point. In this simulation, the maximum damage is set to be $D_{\text{max}} = 0.99$ to resemble a completely open crack upon full solid material loss.

Figure 3.28: The hydraulic driven crack length evolution against time for different values of input fluid flux.

ious poromechanics formulations: without damage, with a local damage scheme and constant permeability and a local damage scheme and nonconstant permeability. While both local damage schemes are shown to be mesh dependent and exhibit spurious oscillations in the strain, fluid pres-
sure and permeability, the proposed model is mesh independent and smooth response is observed for all field variables.
Chapter 4

Three-field modeling of non-local damage transport model: stress-based formulation

4.1 Introduction

In this chapter we present the development of a stress-based version of the three-field non-local damage transport model discussed in Section 2.8. First, the stress-based constitutive evolution laws are presented. Then, the computational implementation featuring the non-linear mixed finite element formulation is discussed in details, the analytical derivation of the Jacobian matrix and the solution algorithm are discussed. The numerical examples in this chapter show how energy dissipation mechanisms change for different loading and material parameter configurations for hydraulic fracture problems. The numerical results also show the significance of incorporating non-local transport in hydraulic fracture modeling.
4.2 Stress-based evolution laws

4.2.1 Equivalent stress

In this study, we define the damage evolution law and the local permeability evolution as functions of an equivalent stress measure. First, we define the effective solid stress tensor $\bar{\sigma}_{ij}$, which describes the amount stress felt by the solid particles. By reorganizing Equation (2.67), the effective solid stress tensor $\bar{\sigma}_{ij}$ can be written as [15, 49]:

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij} + \alpha(D)P}{1 - D}$$  \hspace{1cm} (4.1)

Experiments [223, 224] and numerical studies [225, 226] analyzing failure in porous media have concluded that two main processes that occur at meso-scale fracture process zone lead to damage and permeability increase. The first is the micro-crack aperture size increase, or pore dilation due to direct volumetric expansion. The second is the shear dilation which occurs when the shear stress applied at a material point is higher than its shear strength. In order to model these mechanical processes we employ a Hayhurst type equivalent stress measure [227] which can be written as follows:

$$\sigma^{eq} = \begin{cases} \chi^s & \text{if } \chi^s \geq 0 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (4.2)

$$\chi^s = a_1 \bar{\sigma}^{(1)} + a_2 \bar{\tau}^e - (1 - a_1 - a_2) \bar{\sigma}_{kk}$$  \hspace{1cm} (4.3)

where $a_1$ and $a_2$ are material parameters that control the damage growth mechanism. The Hayhurst stress provides a measure of pore dilation through the first and third terms in Equation (6.17), the first component $\bar{\sigma}^{(1)}$ is the highest eigen-value of stress which controls mode-I crack opening mechanism, and the third component $\bar{\sigma}_{kk}$ is the volumetric stress controlling pore dilation. In addi-
4.2. STRESS-BASED EVOLUTION LAWS

The Hayhurst stress accounts for the shear dilation by the second component \( \tau_e = \sqrt{\frac{3}{2} \sigma_{ij}^{\text{dev}} \sigma_{ij}^{\text{dev}}} \) which is the effective von Mises stress that provides a measure of the shear stress at a material point. The von Mises stress is defined as: 

\[
\bar{\tau}_e = \sqrt{\frac{3}{2} \sigma_{ij}^{\text{dev}} \sigma_{ij}^{\text{dev}}} \]

where \( \sigma_{ij}^{\text{dev}} = \bar{\sigma}_{ij} - \frac{1}{3} \bar{\sigma}_{kk} \delta_{kl} \) is the effective deviatoric stress. The second term \( \bar{\tau}_e \) in its current form does not account for the shear strength of the material, however a threshold can be introduced to incorporate the effect of shear strength by modifying the second term to be: 

\[
< (\bar{\tau}_e - \tau^{th}) >
\]

where \( \tau^{th} \) is the material shear strength and the Macaulay brackets \( < x > = (x + |x|)/2 \). Although the Hayhurst stress was proposed as a measure for damage in creeping material [227], it is also useful to analyze damage and permeability evolution in porous media because its components provide measures for the above mentioned processes.

4.2.2 Anisotropic Darcy’s law

We define the anisotropic permeability tensor \( \kappa_{ij} \) in 2d as follows:

\[
\kappa_{ij} = \begin{bmatrix}
\kappa_0 + (\tilde{\kappa} - \kappa_0) \cos \theta & 0 \\
0 & \kappa_0 + (\tilde{\kappa} - \kappa_0) \sin \theta
\end{bmatrix}
\]  

(4.4)

where \( \kappa_0 \) is the initial permeability of the medium before loading and \( \tilde{\kappa} \) is the scalar non-local permeability calculated from Equation (2.45). \( \theta \) is the angle corresponding to the direction of the principal stress that is defined by the relationship: 

\[
\theta = \frac{1}{2} \tan^{-1} \left( \frac{2\bar{\sigma}_{12}}{\bar{\sigma}_{11} - \bar{\sigma}_{22}} \right)
\]

We note that such approach can easily be generalized to 3D.

Following the available permeability-strain and permeability-stress empirical relationships previously proposed in [51–53, 65, 89, 185], a permeability-stress polynomial functions are chosen in this study, as follows:

\[
\kappa(\sigma^{eq}) = \left[ b_1 + b_2 (\sigma^{eq})^{b_3} \right] \kappa_0
\]

(4.5)

where \( b_1, b_2 \) and \( b_3 \) are constants that need to be calibrated from experimental data and \( \kappa(\sigma^{eq}) \) is the local permeability on the right hand side of Equation (2.45).
4.2.3 Damage evolution law

After the evaluation of the non-local permeability \( \tilde{\kappa} \), the non-local stress measure \( \tilde{\sigma}^{eq} \) may be calculated using the inverse of the local permeability-stress relationship in Equation (4.5). The non-local equivalent strain \( \tilde{\sigma}^{eq} \) can be expressed as:

\[
\tilde{\sigma}^{eq} = \left[ \frac{[\tilde{\kappa}/\kappa_0] - b_1}{b_2} \right]^{\frac{1}{\nu_3}}
\]  

(4.6)

The damage law adopted in this study is the bilinear damage law defined as:

\[
D(\tilde{\sigma}^{eq}) = \begin{cases} 
0 & \text{if } \tilde{\sigma}^{eq} \leq S^i \\
\frac{S^i(\tilde{\sigma}^{eq} - S^i)}{\sigma^{eq}(S^f - S^i)} & \text{if } S^i \leq \tilde{\sigma}^{eq} \leq S^th \\
D_{max} & \text{if } S^th \leq \tilde{\sigma}^{eq}
\end{cases}
\]  

(4.7)

where \( S^i \) and \( S^f \) represent the damage initiation and failure stresses, respectively. Their values should be calibrated from experimental data. The maximum damage is \( D_{max} \) and the corresponding stress value (to \( D_{max} \)) is \( S^th \) which is defined as:

\[
S^th = -\frac{S^i S^f}{(S^f - S^i)} \left[ D_{max} - \frac{S^f}{(S^f - S^i)} \right]^{-1}
\]  

(4.8)

4.2.4 Variable length scale

In this study we employ the following linear relationship to define \( g(\sigma^{eq}) \):

\[
g(\sigma^{eq}) = \begin{cases} 
k_1(l^k)^2 & \text{if } \sigma^{eq} \leq g^i \\
1 - (1 - k_1) \frac{\sigma^{eq} - \sigma^{eq}}{g^f - g^i} (l^k)^2 & \text{if } g^i \leq \sigma^{eq} \leq g^f \\
(l^k)^2 & \text{if } \sigma^{eq} \geq g^f
\end{cases}
\]  

(4.9)
where \( k_1 \) is a scaling parameter, \( 0 < k_1 < 1 \), and \( \sigma^{eq} \) is the equivalent stress calculated from Equation (4.2). The length scale is varying linearly between equivalent stress values of \( g^i \) and \( g^f \). A schematic diagram for the definition of \( g(\sigma^{eq}) \) is given is Figure 4.1.

Figure 4.1: Definition of the variable length scale \( g(\sigma^{eq}) \) from Equation (4.9). \( g(\sigma^{eq}) \) varies linearly between \((1 - k_1)(l^k)^2\) and \((l^k)^2\) for values of equivalent stress \( \sigma^{eq} \) between \( g^i \) and \( g^f \).

### 4.3 Computational implementation

#### 4.3.1 Boundary value problem

This section summarizes the statement of the boundary value problem of non-local damage-transport model proposed in this paper:

\[
\left[ C_{ijkl}(D)\varepsilon_{kl} - \alpha(D)\delta_{ij}P \right]_{,j} + b_i = 0 \text{ in } \Omega \tag{4.10}
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{M(D)}P + \alpha(D)\varepsilon_{ii} \right) - \left[ \kappa_{ij}P_{,j} \right]_{,i} = 0 \text{ in } \Omega \tag{4.11}
\]

\[
\tilde{\kappa} - g(\sigma^{eq})\tilde{\kappa}_{ii} = \kappa(\sigma^{eq}) \text{ in } \Omega \tag{4.12}
\]
with the boundary conditions:

\[
\begin{align*}
    u_i &= \bar{u}_i \text{ on } \Gamma_u & \sigma_{ij} n_j &= t_i \text{ on } \Gamma_t \\
    P &= \bar{P} \text{ on } \Gamma_P & v_i n_i &= s \text{ on } \Gamma_s \\
    \tilde{\kappa}_i n_i &= 0 \text{ on } \Gamma_k \\
    D|_{t=0} &= 0 \text{ in } \Omega
\end{align*}
\]  

(4.13)  
(4.14)  
(4.15)  
(4.16)

where Equation (4.10) is derived by substituting the definition of the total stress from Equation (2.67) in Equation (2.3), Equation (4.11) is derived by substituting the definition of \( \zeta \) from Equation (2.66) and the non-local Darcy’s law from Equation (2.5) into the continuity Equation (2.4). Equation (4.12) is the non-local permeability Equation (2.45). The domain space is denoted by \( \Omega \). The boundary conditions \( \bar{u}_i, t_i, \bar{P} \) and \( s \) are the boundary conditions resembling displacements, tractions, pressure and normal flow flux respectively. The boundary conditions \( \bar{u}_i, t_i, \bar{P} \) and \( s \) are applied on the boundary segments \( S_u, S_t, S_P \) and \( S_s \) respectively. The boundary condition in Equation (4.15) is the Neumann boundary condition on Equation (4.12), which was proposed for the gradient non-local strain in [79]. This boundary condition is required to complete the definition of the boundary value problem. The damage-dependent material variables \( C_{ijkl}(D), \alpha(D) \) and \( M(D) \) are calculated according to Equations (2.69), (2.70) and (2.71). Note that these terms are also implicit functions of time since damage changes with strain and time. The non-local equivalent strain \( \tilde{\sigma}_{eq} \) is calculated according to Equation (4.6) and the damage variable \( D \) is calculated according to (4.7). The variable length scale \( g(\sigma_{eq}) \) is calculated according to Equation (4.9). It has to be noted that by setting \( D = 0 \) and \( l^k = 0 \), the PDE system in Equations (4.10) to (4.16) reduces to the linear poroelastic model originally proposed by Biot [16]. Also, in the case of incompressible solid and fluid constituents \( (\alpha = 1 \text{ and } M = \infty) \), the first, second and fourth terms of Equation (4.11) drop.
4.3. COMPUTATIONAL IMPLEMENTATION

4.3.2 Mixed finite element formulation

The finite element solution of Equations (4.10) to (4.16) involves the development of a mixed finite element formulation with the main variables of interest \((u, P, \tilde{\kappa})\). Following the approach discussed in Section 3.3.2, we use explicit time update of the variable length scale function \(g(e^{eq})\), i.e. \(g^{n+1} \approx g((e^{eq})^n)\) where the superscript \(n\) indicates the last converged time step and the superscript \(n + 1\) indicates the current time step. This lagged update approach leads to a piece-wise constant definition of \(g(e^{eq})\) and hence avoids the modeling complications in [81] and [197]. Similarly, we use explicit update for the calculation of the anisotropy angle \(\theta\) in Equation (4.4); i.e. \(\theta^{n+1} \approx \theta^n\) in order to simplify the calculation of the Jacobian matrix. To this end, a mixed finite element formulation \((u, P, \tilde{\kappa})\) is proposed to solve the boundary value problem described in Section 3.3.1.
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Equations (4.10) to (4.12) can be written in weak form as the following residual functions:

\[ R_u^i(u, P, \tilde{\kappa}) = \int_{\Omega} w_u^i \left[ C_{ijkl}(D)\varepsilon_{kl} - \alpha(D)\delta_{ij}P \right]_{,j} + b_i \, d\Omega \] (4.17a)

\[ R_P^i(u, P, \tilde{\kappa}) = \int_{\Omega} w_P^i \left[ \frac{1}{M(D)} \frac{\partial P}{\partial t} + \frac{P}{M(D)} \frac{\partial (1/M(D))}{\partial t} + \alpha(D)\varepsilon_{ii} \frac{\partial \alpha(D)}{\partial t} + \varepsilon_{ii} \frac{\partial \alpha(D)}{\partial t} - \left[ \kappa_{ij}P \right]_{,i} \right] \, d\Omega \] (4.17b)

\[ R^K(u, P, \tilde{\kappa}) = \int_{\Omega} w^K \left[ \tilde{\kappa} - g((\sigma_{eq})^n)\nabla^2 \tilde{\kappa} - \kappa(\sigma_{eq}) \right] \, d\Omega \] (4.17c)

where \( w_u^i, w_P^i \) and \( w^K \) are the test functions for displacement, fluid pressure and non-local permeability fields, respectively. The following discretization functions are adopted for the three field variables \( u, P \) and \( \tilde{\kappa} \):

\[ u = N^u u^h; \quad P = N^P P^h; \quad \tilde{\kappa} = N^K \tilde{\kappa}^h \] (4.18)

where \( N^u, N^P \) and \( N^K \) are the shape functions for the displacement \( (u_i) \), fluid pressure \( (P) \) and non-local permeability \( (\tilde{\kappa}) \) fields, respectively. The superscript \( h \) in \( u^h, P^h \) and \( \tilde{\kappa}^h \) indicates nodal values of each field. The discretization in Equation (3.17) extends the formulation proposed in [33] to include damage and non-local permeability effects. In order to satisfy the Babuška-Brezzi condition [198–201] the displacement shape function is taken to be quadratic 8-node element known as the serendipity element, while the pressure and non-local permeability shape functions are taken to be bilinear 4-node element. A schematic of the mixed element formulation is shown in Figure 3.3.

In order to continue with the linearization of the weak form presented in Equation (4.17a) to (4.17c), the solution vector \( \mathbf{x} \) and the residual vector \( \mathbf{R} \) are defined as follows:

\[ \mathbf{x} = \begin{bmatrix} u_i \\ P \\ \tilde{\kappa} \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} R_u^i \\ R_P^i \\ R^K \end{bmatrix} \] (4.19)
4.3. COMPUTATIONAL IMPLEMENTATION

The residual statement in Equation 4.17 may be written as:

\[ R^{n+1} = M\dot{x}^{n+1} + Kx^{n+1} = 0 \] (4.20)

where \( M \) and \( K \) are square matrices and the superscript \( (n + 1) \) denotes the solution at the next time step. Using a difference scheme [202], we may write the velocity vector \( \dot{x}^{n+1} \) as:

\[ \dot{x}^{n+1} = \frac{1}{\beta dt} [x^{n+1} - x^n] - \frac{1 - \beta}{\beta} \dot{x}^n \] (4.21)

where \( \beta \) is the parameter that defines the time integration scheme, \( \beta = 1 \) for backward Euler, \( \beta = 0.5 \) for a Crank-Nicolson method and \( \beta = 0 \) for forward Euler. The final residual statement, after temporal discretization, is defined as:

\[ R^{n+1} = \left[ \frac{1}{\beta dt} M + K \right] x^{n+1} - \frac{1}{\beta dt} Mx^n - \frac{1 - \beta}{\beta} \dot{M}x^n = 0 \] (4.22)

In this paper we assume an implicit scheme, which means that at every time step a Newton-Raphson method is used to solve the resulting non-linear system of equations leading to the following linearized system:

\[ J^{n+1} \delta x^{n+1} = -R^{n+1} \] (4.23)

where \( \delta x \) is the incremental solution vector computed at each Newton iteration and the Jacobian (tangent stiffness) matrix is:

\[ J^{n+1} = \frac{\partial R^{n+1}}{\partial x^{n+1}} = \left[ \frac{1}{\beta dt} M + K \right] = \begin{bmatrix} J_{uu} & J_{uP} & J_{uK} \\ J_{Pu} & J_{PP} & J_{PK} \\ J_{Ku} & J_{KP} & J_{KK} \end{bmatrix} \] (4.24)
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where the components of the Jacobian matrix $J^{n+1}$ are calculated as:

\[
J_{uu} = \frac{\partial R_u}{\partial u} = \int_\Omega B^u T \tilde{C} B^u d\Omega \tag{4.25a}
\]

\[
J_{uP} = \frac{\partial R_u}{\partial P} = -\int_\Omega B^u T \alpha(D) I N^P d\Omega \tag{4.25b}
\]

\[
J_{uk} = \frac{\partial R_u}{\partial \tilde{\kappa}} = -\int_\Omega B^u T \epsilon \frac{\partial D}{\partial \tilde{\kappa}} N^\kappa + B^u T P I \frac{\partial \alpha(D)}{\partial \tilde{\kappa}} N^\kappa d\Omega \tag{4.25c}
\]

\[
J_{Pu} = \frac{\partial R_P}{\partial u} = \frac{1}{\beta} dt \int_\Omega N^P T \frac{1}{M(D)} N^P d\Omega + \int_\Omega N^P T \frac{\partial (1/M(D))}{\partial t} N^P d\Omega + \int_\Omega B^P T \kappa_i B^P d\Omega \tag{4.25d}
\]

\[
J_{PP} = \frac{\partial R_P}{\partial P} = \frac{1}{\beta} dt \int_\Omega N^P T \frac{1}{M(D)} N^P d\Omega + \int_\Omega N^P T \frac{\partial (1/M(D))}{\partial t} N^P d\Omega + \int_\Omega B^P T \kappa_i B^P d\Omega \tag{4.25e}
\]

\[
J_{k\kappa} = \frac{\partial R_k}{\partial \tilde{\kappa}} = \int_\Omega N^\kappa T \left[ \frac{\partial \kappa (\sigma^{eq})}{\partial \epsilon} \right] B^\kappa d\Omega \tag{4.25f}
\]

\[
J_{ku} = \frac{\partial R_k}{\partial u} = -\int_\Omega N^\kappa T \left[ \frac{\partial \kappa (\sigma^{eq})}{\partial \epsilon} \right] B^u d\Omega \tag{4.25g}
\]

\[
J_{\kappa P} = \frac{\partial R_k}{\partial P} = 0 \tag{4.25h}
\]

\[
J_{k\kappa} = \frac{\partial R_k}{\partial \tilde{\kappa}} = \int_\Omega N^\kappa T N^\kappa d\Omega + g((\sigma^{eq})^n) \int_\Omega B^\kappa T B^\kappa d\Omega \tag{4.25i}
\]

where $B^u$ is the shape function derivative of $N^u$ used in the calculation of strains as in: $\epsilon_{ij} = B^u u^h$; $B^u, \text{vol}$ is the shape function derivative of $N^u$ used in the calculation of volumetric strain as in: $\epsilon_{ii} = B^u, \text{vol} u^h$; $B^P$ is the shape function derivative of $N^P$ corresponding to $P_j = B^P p^h$; $B^\kappa$ is the shape function derivative of $N^\kappa$ corresponding to $\kappa_i = B^\kappa \tilde{\kappa}^h$ and the superscript $T$ indicates matrix transpose. The matrix forms of the stiffness tensor $C_{ijkl}$ and the damaged stiffness tensor $(1-D)C_{ijkl}$ are denoted by $C$ and $\tilde{C}$, respectively. The kronecker $\delta_{ij}$ operator is the identity matrix $I$. In this paper, we employ a backward difference scheme with $\beta = 1$. Detailed expressions of the derivatives used in the calculation of the residual vector and Jacobian matrix are provided in Appendix A. The adaptive time stepping scheme in Equation (3.25) is used to control the time step.
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in this solution scheme. The complete solution scheme is summarized in Algorithm 2.

Algorithm 2 Solution Algorithm

1: Initialize all variables \((u^h, P^h, \tilde{\kappa}^h)\)
2: while \(t < T\) do
3:   Calculate \(g((\sigma^{eq})^n)\)  
4:   while \(||R|| < tol||R_0||\) do
5:     for each finite element do
6:       for each material point do
7:         Interpolate \(\tilde{\kappa}\) value at material point from \(\tilde{\kappa}^h\)
8:       Calculate the value of the non-local equivalent stress \(\bar{\sigma}^{eq}\)  
9:     Calculate non-local damage \(D(\bar{\sigma}^{eq})\)  
10:    Interpolate local strain tensor \(\varepsilon_{ij}\) from \(u^h\)
11:    Calculate local equivalent stress \(\sigma^{eq}\)  
12:    Calculate local permeability \(\kappa'(\sigma^{eq})\)  
13:    Calculate Jacobian matrix \(J\) and residual vector \(R\)  
14: end for
15: for each material point do
16:   Interpolate \(\tilde{\kappa}\) value at material point from \(\tilde{\kappa}^h\)
17: end for
18: Solve for \(\delta x\)  
19: \(x^{n+1} \rightarrow x^{n+1} + \delta x\)  
20: end while
21: \(dt = \Delta T^{n+1}\)
22: \(t \rightarrow t + dt\)

4.4 Numerical examples of hydraulic fracture

The model described in the previous sections is used to simulate hydraulic fracture of porous media. Consider the schematic in Figure 4.3, the hydraulic fracture problem is modeled using a poro-elastic domain of dimensions \(2L \times L\). The left edge is the symmetry line where a zero-flux condition and a horizontal translation constraint is applied. The right, top and bottom edges are the external boundaries of the domain which are mechanically restrained from moving but are permeable to fluid flow. A notch is used to pump fluid inside the domain with a flux of \(Q\). The finite element mesh size used in the crack propagation zone is 0.05 m. The modeling parameters are listed in Table 4.1.
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First, we present a qualitative example that showcases the capabilities of the proposed formulation to capture the hydraulic fracture mechanism in porous media. The hydraulic fracture is represented in the example by a damage field and leads to an elevated permeability. Then, we thoroughly investigate the energy dissipation mechanisms under different cases of model parameterizations. We investigate the effects of varying the fluid injection flux $Q$ and the characteristic length $l^k$. In order to facilitate this comparison, we introduce the following measures:

- the fracture length $L_F$, which is the distance along the fracture center line from the left edge to the farthest point experiencing damage ($D > 0$)
- the total fractured volume $V_F$, computed as the integral of the damage variable $V_F = \int_V D dV$
- the average fracture width $w_F$, which is approximated as: $w_F = \frac{V_F}{L_F}$

Finally, we simulate a special case where high permeability zones pre-exist close to the hydraulic fracture expected path. We show that the proposed non-local transport approach successfully models the fluid leakage through the preexisting high permeability zones.

### 4.4.1 Hydraulic fracture evolution

In this example, the domain in Figure 4.3 with the material properties in Table 4.1 is subjected to a fluid flux $Q = q_0$. The characteristic length scale for the nonlocal permeability is chosen as $l^k = 0.1$ m. The damage ($D$), anisotropic permeability in x-direction $\kappa_{xx}$ and fluid pressure ($P$) contours are shown in Figure 4.4. The damage contour results show that the damage indeed propagates along the expected crack direction. The anisotropic permeability $\kappa_{xx}$ plots follow damage evolution which provides the preferential direction of fluid flow in the direction of crack propagation. The fluid pressure contours show that the high pressure is confined in the region inside and around the crack and that the fluid pressure inside the crack is almost constant.

The difference in fluid pressure results between the presented model and the results in Section (3.4.3) is attributed to the anisotropic permeability; which provides a strong preferential direction
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Figure 4.3: Schematic diagram demonstrating the hydraulic fracture domain with boundary conditions and FEM mesh. The external boundary (right, top and bottom edges) of the domain is mechanically restrained ($u_i = 0$) and permeable ($P = 0$). Symmetry boundary conditions are applied on the left side of the domain.

For the fluid flow and hence leads to confinement of the fluid within the damage zone. In order to have a better understanding of the fluid pressure evolution, we plot the fluid pressure and damage evolution with time at the fluid flux input point. The plots are shown in Figure 4.6. The results show that fluid pressure approaches a constant value inside the crack which is the same conclusion recorded in several numerical studies [88, 89]. The results in Figures 4.4 and 4.6 confirm the ability of the proposed non-local damage transport model to capture the fundamental features of hydraulic fracture modeling.

A key feature in modeling hydraulic fracture process is the ability of models to simulate the high velocity fluid flow inside the fluid-driven fracture, several orders of magnitudes faster than
Figure 4.4: Damage ($D$), fluid pressure ($P$) and permeability in x-direction ($\kappa_{xx}$) evolution results for the hydraulic fracture example.
4.4. NUMERICAL EXAMPLES OF HYDRAULIC FRACTURE

Table 4.1: Modeling parameters

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain dimension</td>
<td>$L$</td>
<td>10 m</td>
</tr>
<tr>
<td>Fluid flux</td>
<td>$q$</td>
<td>$2.8 \times 10^{-4}$ m$^3$s$^{-1}$</td>
</tr>
<tr>
<td>Bulk Modulus</td>
<td>$K$</td>
<td>$2.08 \times 10^{8}$ Pa</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>$\mu$</td>
<td>$9.26 \times 10^{7}$ Pa</td>
</tr>
<tr>
<td>Undrained Poisson’s ratio</td>
<td>$\nu^u$</td>
<td>0.49</td>
</tr>
<tr>
<td>Solid grain bulk modulus</td>
<td>$K_s$</td>
<td>$1.1 \times 10^{12}$ Pa</td>
</tr>
<tr>
<td>Initial permeability</td>
<td>$\kappa_0$</td>
<td>$1.00 \times 10^{-14}$ m$^2$/Pa s</td>
</tr>
<tr>
<td>Constant in Equation (4.3)</td>
<td>$a_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (4.3)</td>
<td>$a_2$</td>
<td>0.0</td>
</tr>
<tr>
<td>Constant in Equation (4.5)</td>
<td>$b_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (4.5)</td>
<td>$b_2$</td>
<td>$5.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>Constant in Equation (4.5)</td>
<td>$b_3$</td>
<td>2.0</td>
</tr>
<tr>
<td>Maximum damage</td>
<td>$D_{max}$</td>
<td>0.99</td>
</tr>
<tr>
<td>Damage initiation stress</td>
<td>$S^i$</td>
<td>$0.5 \times 10^{6}$ Pa</td>
</tr>
<tr>
<td>Damage final stress</td>
<td>$S^f$</td>
<td>$1.0 \times 10^{9}$ Pa</td>
</tr>
<tr>
<td>Constant in Equation (4.9)</td>
<td>$k_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (4.9)</td>
<td>$k_2$</td>
<td>$5.0 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

Figure 4.5: Isotropic vs. Anisotropic permeability results. Fluid pressure and damage along the left edge of the domain at 800s. $Z = 0$ is the crack centerline. The plots show how anisotropic permeability leads to more localized damage and fluid pressure evolution.
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Figure 4.6: Fluid pressure (a) and damage (b) evolution with time at the fluid flux input point. The results show that the pressure approaches a constant value as the material completely fractures (\( D \approx 1 \)).

Fluid seepage into intact porous media. In LEFM approaches, Poiseuille’s flow is most often used to model fluid flow inside the fracture [45, 73] which is straightforward because the crack has a well-defined geometry. Continuum approaches using phase-field, e.g.[87, 89, 90], computed quantities similar to fracture width to model the flow inside the crack Poiseuille’s flow. The computation of these fracture-width-like quantities is complicated and is often accompanied by numerical issues [87]. In addition, using Darcy’s law in the intact media and Poiseuille’s flow in the fractured media leads to a discontinuity that hinders the convergence of the non-linear solver and increases the computational cost of the model [87]. In this study, we use the permeability-stress relationship in Equation (4.5) to account for the increased fluid flow speed inside the fracture.

In Figure 4.7, we plot the evolution of damage, fluid pressure, \( \kappa_{xx} \) and \( v_x \) at a section 0.5m away from the left edge of the domain. The plots show the growth of the fluid-driven fracture in terms of increasing damage, permeability and velocity. One can observe that the proposed variable permeability approach leads to fluid velocity inside the crack that is several orders of magnitudes higher than the fluid velocity in the intact porous medium.
Figure 4.7: The evolution of damage (a), fluid pressure (b), permeability $\kappa_{xx}$ (c) and fluid velocity $v_x$ at a section 0.5 m from the left edge of the domain in Figure 4.3. The crack centerline is $z = 0$. The plots show that the proposed model successfully captures the increased damage and fluid velocity in the crack zone.

In the variable permeability approach in Equation (4.5), the permeability evolution is governed by 23 parameters: $b_1$, $b_2$ and $b_3$ that need to be calibrated from experimental data. In the numerical examples in this paper we use $b_3 = 2$ to provide a quadratic permeability-stress relationship. In Figure 4.8 we show the effect of $b_2$ on the evolution of the hydraulic fracture. We plot the damage, fluid pressure, permeability $\kappa_{xx}$ (c) and fluid velocity $v_x$ at a section 0.1 m from the left edge of the domain in Figure 4.3 at 2600 s. The plots in Figure 4.8 suggest that using higher values of $b_2$
leads to narrower hydraulic fracture and elevated permeability and fluid velocity in the hydraulic fracture. The results in Figures 4.4, 4.6, 4.7 and 4.8 confirm the ability of the proposed non-local damage transport model to capture the basic features of hydraulic fracture modeling.

Figure 4.8: Plots of damage (a), fluid pressure (b), permeability $\kappa_{xx}$ (c) and fluid velocity $v_x$ at a section 0.1 m from the left edge of the domain in Figure 4.3. The plots are recorded at 2600 s. The crack centerline is $z = 0$. The plots show that higher values of $b_2$ in Equation (4.5) lead to narrower crack zones.

The analytical expressions for the energy dissipation functions derived in Section 2.6 have to
achieve the following relationship:

\[ W^{\text{ext}} = W^{\text{int}} + D_t \] (4.26)

where \( W^{\text{ext}} \) is the external work done at the boundary, \( W^{\text{int}} \) is the internal elastic energy stored in the system and \( D_t \) is the total dissipated energy. The external work \( W^{\text{ext}} \) is calculated as:

\[ W^{\text{ext}} = \int_{\Gamma} F_id_id\Gamma + \int_T \int_{\Gamma} P v_i \cdot v_id\Gamma dt \] (4.27)

where \( F_i \) and \( d_i \) are the boundary force and displacement vectors respectively. The first term in Equation (4.27) represents the effect of the mechanical loading and the second term represents the effect of the fluid flux through the boundary. The internal energy \( W^{\text{int}} \) is calculated as:

\[ W^{\text{int}} = \int_{\Omega} \frac{1}{2} (1 - D) \bar{\sigma}_{ij} \epsilon_{ij} + P \zeta d\Omega \] (4.28)

The total dissipated energy \( D_t \) is calculated as:

\[ D_t = D_s + D_f \] (4.29)

where \( D_s \) and \( D_f \) are the solid and fluid dissipation energies defined in Equations (2.57) and (2.64) respectively. The evolution of \( W^{\text{ext}} \), \( W^{\text{int}} \) and \( D_t \) are plotted in Figure 4.9. The numerical results confirm that the computed dissipation functions are in agreement with the energy conservation statement in Equation (4.26).

### 4.4.2 Influence of the input flux \( Q \)

We investigate the effect of fluid input flux \( Q \) on the model response; three input flux values \( Q = \{0.5q_0, q_0, 2q_0\} \) are analyzed. First, we plot the hydraulic fracture length, volume and average
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Figure 4.9: Energy conservation plot. The evolution of external work $W^{\text{ext}}$, internal energy $W^{\text{int}}$ and the total dissipated energy $\mathcal{D}_t$. The results verify the expressions of $\mathcal{D}_t$ as the energy conservation condition in Equation (4.26) is achieved.

width evolution against time in Figure 4.10. The results in Figure 4.10 suggest that higher input rate $Q$ leads to higher fracture propagation rate and wider fracture zones.

The energy dissipation, derived in Section 2.6, in the three cases is plotted in Figure 4.11. The results in Figures 4.11a and 4.11b indicate that higher pumping rates lead to faster damage propagation and energy dissipation by solid material loss. However, Figures 4.11e and 4.11f suggest that the higher pumping rate will lead to more energy dissipation through viscous fluid flow. By comparing Figures 4.11a and 4.11b vs. 4.11e and 4.11f, one can see that as pumping rate increases more energy is dissipated through the viscous fluid flow than damage propagation. This observation concludes that pumping fluid at higher velocities may not be an energy efficient solution for hydraulic fracture.

4.4.3 Influence of the nonlocal length scale

We investigate the effect of the length scale parameter on the hydraulic fracture process; three values of the nonlocal length scale $l^k = \{0.1, 0.2, 0.3\}$ m are analyzed. The fracture length, volume and average width evolution with time are plotted in Figure (4.12). Figure 4.12a suggests that the fracture length is almost similar for all length scale cases. However, Figures 4.12b and 4.12c
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Figure 4.10: The hydraulic driven fracture length (a), volume (b) and average width (c) evolution against time for different values of input fluid flux $Q = \{0.5q_0, q_0, 2q_0\}$.

shows that the fracture total volume, and hence width, are significantly larger for materials with larger length scales. The results in Figure 4.12 suggest that materials having larger length scales, i.e. larger meso-scale interactions, will exhibit shorter and significantly wider cracks.

The energy dissipation, derived in Section 2.6, for the three different length scale models are plotted in Figure 4.13. Due to the significant difference in the crack width, different observations can be drawn from fracture volume vs length results in Figure 4.13. First, the solid-dry dissipation (Equation (2.62)) has an almost indifferent rate with respect to fracture volume while it varies
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Figure 4.11: Energy dissipation results for different values of input fluid flux $Q = \{q_0, 1.5q_0, 2q_0\}$. 
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significantly with respect to the fracture length. This leads to the conclusion that fracturing materials with larger length scales is not necessarily more energy consuming; it is only more energy consuming to get a longer crack to propagate. On the other hand, we find that the solid-wet dissipation vs fracture volume results are more sensitive to the changes in the characteristic length than the solid-wet dissipation vs fracture length. These results indicate that the solid-wet dissipation becomes more significant in materials with shorter length scales.

Figure 4.12: The hydraulic driven fracture length (a), volume (b) and average volume (c) evolution against time for different values of input fluid flux for different material length scales \( l^k = 0.1, 0.2, 0.3 \) m.
Figure 4.13: Energy dissipation results for different material length scales $l^k = 0.1, 0.2, 0.3$ m.
4.4.4 Pre-existing high permeability zones

In this section we aim to demonstrate the significance of non-local transport by analyzing a hydraulic fracture model where a high permeability zones pre-exist close to the fracture path. A schematic in Figure 4.14 shows zones of different permeability: 1) the majority of the domain with initial permeability $\kappa_0$, 2) the fracture path with initial permeability $\kappa_1$ and 3) the high permeability strips with initial permeability $\kappa_2$. Each strip dimensions are $h \times w$ and they are located at $d$ distances apart as shown in Figure 4.14. The dimension $a$ indicates the size of the zone with initial permeability of $\kappa_1$. The dimension $b$ defines the vertical location of the high permeability strips. These dimensions are: $a = 1.0 \text{ m}$, $b = 0.1 \text{ m}$, $h = 1.9 \text{ m}$, $w = 0.3 \text{ m}$ and $d = 1.2 \text{ m}$. We analyze the hydraulic fracture in three model configurations:

- **Model 1**: no damage is allowed to grow, non-local permeability and $\kappa_1 = \kappa_2 = 1000\kappa_0$
- **Model 2**: non-local damage, non-local permeability and $\kappa_1 = \kappa_0, \kappa_2 = 1000\kappa_0$
- **Model 3**: non-local damage, local permeability and $\kappa_1 = \kappa_0, \kappa_2 = 1000\kappa_0$

Each of these cases is analyzed using three different values of non-local length scale: $l^k = \{0.1, 0.2, 0.3\}$ m. The amount of fluid passing through points $A_1$, $A_2$ and $A_3$ (shown in Figure 4.14) are recorded for each model.

First, we consider the non-damage case in Model 1. The volume of fluid accumulated at points $A_1$, $A_2$ and $A_3$ (shown in Figure 4.14) are plotted in Figure 4.15. The plots show that as the length scale increases, the amount of fluid accumulated at points $A_1$ and $A_2$ increases while the amount of fluid accumulated at point $A_3$ decreases. This observation indicates that: as the transport length scale increases, more fluid is expected to leak-off through pre-existing high permeability zones and less fluid is expected to flow through the major cracking direction.

The volume of fluid accumulated at $A_1$, $A_2$ and $A_3$ are shown for Model 2 and Model 3 in Figure 4.16. In Model 3, more fluid is accumulated at points $A_1$ and $A_2$ as length scale increases while less fluid is accumulated at point $A_3$ as length scale increases. This observation confirms that more
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\[ u_i = 0 \]
\[ u_x = 0 \]
\[ P/n = 0 \]
\[ P = 0 \]

Figure 4.14: Schematic diagram demonstrating the hydraulic fracture domain with pre-existing high permeability zones. The majority of the domain has initial permeability of \( \kappa_0 \), the expected crack propagation zone has an initial permeability of \( \kappa_1 \) and the high permeability strips have a permeability of \( \kappa_2 \). Points \( A_1, A_2 \) and \( A_3 \) are used to measure fluid flow at different positions in the domain. The mentioned distances are: \( a = 1.0 \) m, \( b = 0.1 \) m, \( h = 1.9 \) m, \( w = 0.3 \) m and \( d = 1.2 \) m. Point \( A_3 \) is 5.5 m away from the left edge.

the hydraulic fracture experiences larger leakage volumes through pre-existing high-permeability networks as length scale increases. We also notice that the volume of fluid accumulated in Model 2 (non-local damage and permeability) is higher than the volume of fluid accumulated in Model 3 (non-local damage and local permeability). This difference is attributed to the non-local transport effect in Model 2.

In order to better understand the difference between Model 2 and Model 3, we plot the non-local permeability, velocity magnitude and damage contours at \( 3.5 \times 10^4 \)s in Figures 4.17, 4.18.
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Figure 4.15: Model 1 results: Volume of fluid accumulated at different points in the domain for different material length scales $l^k = 0.1, 0.2, 0.3$ m. The positions of points $A_1$, $A_2$ and $A_3$ are shown in marked on Figure 4.14. Points $A_1$ and $A_2$ on the pre-existing permeability zones experience more fluid accumulation for larger length scales. Point $A_3$ on the fracture path experiences less fluid accumulation as length scale increases, this is attributed to increased leakage along the fracture length.

and 4.19. The results in Figures 4.17 and 4.18 show that in Model 3 high permeability and fluid flow localize in the high permeability zones leading to narrow zones of high fluid velocities. On the other hand, Model 2 results in Figures 4.17 and 4.18 show that the flow is distributed among a larger zone of increase permeability. The results in Figure 4.19 show that in Model 2, the fluid flow forms a leakage channel in the high permeability zone in which the fluid can flow without
creating damage, while damage grows around the high permeability zone. The damage growth around the high permeability zone is attributed to the increased fluid velocity in the vicinity of the high permeability zone. In Model 3, damage propagates inside the high permeability zone due to the fluid flow localization. This major difference in the crack growth pattern proves the importance of incorporating non-local transport in hydraulic fracture in fracture geomaterials.

Figure 4.16: Model 2 and 3 results: Volume of fluid accumulated at different points in the domain for different material length scales \( l^k = 0.2, 0.3 \) m. The positions of points \( A_1, A_2 \) and \( A_3 \) are shown in marked on Figure 4.14. \textbf{NLDT} denotes Model 2 where damage and permeability are non-local, \textbf{NLD} denotes Model 3 where damage is non-local and permeability is local.

The results show that the non-local transport model presented in this paper exhibits significantly
Figure 4.17: Non-local permeability ($\bar{\kappa}$) evolution in Model 2 and Model 3 at $3.5 \times 10^4$ s. In Model 2, permeability growth is distributed along the fracture network. In Model 3, the permeability growth is localized in the high permeability zones.

larger volumes of fluid leaking-off the major hydraulic fracture through the pre-existing high permeability zone. These results indicate that the non-local damage-transport model presented in this paper can act as a platform to better understanding and simulation of hydraulic fracture through geomaterials exhibiting networks of natural fractures, which is still a persisting challenge to hydraulic fracture modeling research [228].
Figure 4.18: Velocity magnitude ($|v_i|$) evolution in Model 2 and Model 3 at $3.5 \times 10^4$ s.

4.5 Summary and conclusion

In this paper we present a novel thermodynamics based non-local damage transport approach for modeling hydraulic fracture. A new thermodynamics derivation of poroelasticity state laws is presented to account for the non-local long-range interactions in the fracture process zone. The simplification of the proposed model leads to a gradient-type non-local permeability relationship. The gradient non-local permeability formulation can readily describe non-local transport and reg-
4.5. SUMMARY AND CONCLUSION

Figure 4.19: Damage evolution in Model 2 and Model 3 at $3.5 \times 10^4$s. Damage in Model 2 passes through the pre-existing high permeability zone. In Model 3, a leakage path is created in the high permeability zone leading to damage accumulation around the high permeability zone but not inside it.

ularize regularize damage via a stress-dependent permeability evolution law. The derivation of the simplified model is shown to be analogous to Darcy-Brinkman fluid flow, leading to the conclusion that Darcy-Brinkman is a form of non-local transport and its additional viscosity parameter is a function of the flow length scale. The energy dissipation functions are derived analytically for the proposed non-local damage-transport model; energy dissipation was found to be in three forms: 1-dry-solid damage dissipation, 2-wet-solid damage dissipation and 3-fluid viscous flow dissipation.

The proposed model is applied to hydraulic fracture simulation; the model was found to be successful at capturing the fundamental features of modeling hydraulic fracture. Energy dissipation mechanisms for different configurations of hydraulic fracture loading and material parameters were analyzed. The significance of non-local transport modeling is demonstrated by modeling hydraulic fracture in a domain with pre-existing high permeability zones. The results show that adopting non-local transport is essential in modeling hydraulic fracture in fractured media where the leak-off fluid amount is significant.
Chapter 5

Four-field modeling of non-local damage transport model

5.1 Introduction

In this chapter, we present the modeling of the non-local damage transport formulation first discussed in Section 2.8 using a four field \((u, P, \bar{\sigma}, \bar{\kappa})\) formulation. This formulation allows the introduction of different length scales and evolution laws for permeability and damage; hence, it provides a more physically enriched model. In order to decrease the computational cost associated with the implementation of the four field formulation, we incorporate the stabilization scheme introduced in [229] which allows the use of equal order approximations for displacement and pressure fields. In this chapter, we use the constitutive laws presented in Section 4.2 for the evolution of damage, anisotropic permeability and variable damage length scale. The numerical results show the significance of the four-field non-local damage transport modeling. First, hydraulic fracture modeling is attempted using increasing transport length scale and fixed damage length scale. The results show that a higher transport length scale leads to the formation of a fluid pool that dissipated fluid through crack walls and hinders hydraulic fracture propagation. Second, 2d consolidation is
modeled using different configurations of driving stresses for the evolution of damage and permeability. The results show that if the permeability increases inside the shear crack, the shear crack will be water-filled leading to a hydraulic-fracture-like process that accelerates damage propagation.

5.2 Computational implementation

5.2.1 Boundary value problem

Using the set of equations derived in Chapter 2, the initial boundary value problem can be written as:

\[
[C_{ijkl}(D)\varepsilon_{kl} - \alpha(D)\delta_{ij}P]_{,j} + b_i = 0 \text{ in } \Omega \tag{5.1}
\]

\[
\frac{\partial}{\partial t} \left[ \frac{P}{M(D)} + \alpha(D)\varepsilon_{ii} \right] - [\tilde{\kappa}P_{,i}]_j = 0 \text{ in } \Omega \tag{5.2}
\]

\[
\tilde{\sigma}^{eq} - g(\sigma^{eq,D})\tilde{\sigma}^{eq}_{,ii} = \sigma^{eq,D} \text{ in } \Omega \tag{5.3}
\]

\[
\tilde{\kappa} - (l_k)^2\tilde{\kappa}_{,ii} = \kappa(\sigma^{eq,K}) \text{ in } \Omega \tag{5.4}
\]

with the boundary conditions:

\[
u_i = \bar{u}_i \text{ on } \Gamma_u \quad \sigma_{ij}n_j = t_i \text{ on } \Gamma_t \tag{5.5}
\]

\[P = \bar{P} \text{ on } \Gamma_P \quad v_i n_i = s \text{ on } \Gamma_s \tag{5.6}
\]

\[
\tilde{\sigma}^{eq}_{,i}n_i = 0 \text{ on } \Gamma_\sigma \tag{5.7}
\]

\[
\tilde{\kappa}_{,i}n_i = 0 \text{ on } \Gamma_\kappa \tag{5.8}
\]

\[D|_{t=0} = 0 \text{ in } \Omega \tag{5.9}
\]

where Equation (5.1) is derived by substituting the definition of the total stress from Equation (2.67) in the balance of momentum Equation (2.3), Equation (3.10) is derived by substituting the
CHAPTER 5. FOUR-FIELD MODELING OF NON-LOCAL DAMAGE TRANSPORT MODEL

definition of $\zeta$ from Equation (2.66) and the anisotropic Darcy’s law from Equation (2.5) into the continuity Equation (2.4). Equation (5.3) is the regularized stress relationship that can be derived from Equation (2.38). Equation (5.4) is the non-local permeability Equation (3.7). The symbols $\sigma^{eq,D}$ and $\sigma^{eq,K}$ denote the driving local stress governing the evolution of damage of permeability respectively. $\sigma^{eq,D}$ and $\sigma^{eq,K}$ are both calculated from Equation (4.3); however, different values of parameters $a_1$ and $a_2$ may be used for the calculation of each of them. This provides the flexibility of modeling the evolution of permeability and damage subject to different evolution laws. The damage-dependent material variables $C_{ijkl}(D), \alpha(D)$ and $M(D)$ are calculated according to Equations (2.69), (2.70) and (2.71). Note that these terms are also implicit functions of time since damage changes with stress and time. The non-local equivalent stress $\tilde{\sigma}^{eq}$ is calculated according to Equation (5.3) and the damage variable $D$ is calculated according to (4.7). The variable length scale $g(\sigma^{eq,D})$ is calculated according to Equation (4.9).

The domain space is denoted by $\Omega$. The boundary conditions $\bar{u}_i, t_i, \bar{P}$ and $s$ are the boundary conditions resembling displacements, tractions, pressure and normal flow flux respectively. The boundary conditions $\bar{u}_i, t_i, \bar{P}$ and $s$ are applied on the boundary segments $\Gamma_u, \Gamma_t, \Gamma_P$ and $\Gamma_s$ respectively as shown in Figure 3.2. The boundary condition in Equations (5.7) and (5.8) present the Neumann boundary condition on Equation (5.3), which was proposed for the gradient non-local strain in [79]. This boundary condition is required to complete the definition of the boundary value problem.

5.2.2 Mixed finite element formulation

The finite element solution of Equations (5.1) to (5.9) involves the development of a mixed finite element formulation with the main variables of interest $(u, P, \tilde{\sigma}^{eq}, \tilde{K})$. The finite element implementation in [81] of Equation (5.3) requires the introduction of additional nodal degrees of freedom for the length scale $g(\sigma^{eq,D})$ due to the presence of its derivative in the weak form. The simple modification proposed in [197] leads to an alternative formulation that does not require
additional degrees of freedom. However, it still requires additional computational cost due to the introduction of the additional non-linear relationship describing the evolution of $g(\sigma^{eq,D})$. In this paper, we follow an explicit time update of the variable length scale function $g(\sigma^{eq,D})$, i.e. $g^{n+1} \approx g((\sigma^{eq,D})^{n})$ where the superscript $n$ indicates the last converged time step and the superscript $n+1$ indicates the current time step. This lagged update approach leads to a piece-wise constant definition of $g(\sigma^{eq,D})$ and hence avoids the modeling complications in [81] and [197]. To this end, a mixed finite element formulation $\mathbf{u}, P, \tilde{\kappa}$ is proposed to solve the boundary value problem described in Section 5.2.1. Equations (5.1) to (5.3) can be written in weak form as the following residual functions:

$$R^u(u, P, \tilde{\sigma}^{eq}, \tilde{\kappa}) = \int_{\Omega} w^u \left[ C_{ijkl}(D) \varepsilon_{kl} - \alpha(D) \delta_{ij} P \right]_{,j} + b_j \, d\Omega$$  \hspace{1cm} (5.10a)

$$R^P(u, P, \tilde{\sigma}^{eq}, \tilde{\kappa}) = \int_{\Omega} w^P \left[ \frac{1}{M(D)} \frac{\partial P}{\partial t} + P \frac{\partial (1/M(D))}{\partial t} + \alpha(D) \frac{\partial \varepsilon_{ii}}{\partial t} + \varepsilon_{ii} \frac{\partial \alpha(D)}{\partial t} - [\tilde{\kappa} P, i]_{,i} \right] \, d\Omega$$  \hspace{1cm} (5.10b)

$$R^{\sigma}(u, P, \tilde{\sigma}^{eq}, \tilde{\kappa}) = \int_{\Omega} w^{\sigma} \left[ \tilde{\sigma}^{eq} - g((\sigma^{eq,D})^{n}) \tilde{\sigma}^{eq,ii} - \sigma^{eq,D} \right] \, d\Omega$$  \hspace{1cm} (5.10c)

$$R^{\kappa}(u, P, \tilde{\sigma}^{eq}, \tilde{\kappa}) = \int_{\Omega} w^{\kappa} \left[ \tilde{\kappa} - (l^k)^2 \tilde{\kappa}_{,ii} - \kappa(\sigma^{eq,k}) \right] \, d\Omega$$  \hspace{1cm} (5.10d)

where $w^u, w^P, w^{\sigma}$ and $w^{\kappa}$ are the test functions for displacement, fluid pressure and non-local permeability fields, respectively. The following discretization functions are adopted for the three field variables $u, P$ and $\tilde{\kappa}$:

$$\mathbf{u} = \mathbf{N}^u u^h; \quad P = \mathbf{N}^P P^h; \quad \tilde{\sigma}^{eq} = \mathbf{N}^{\sigma} \tilde{\sigma}^{eq,h}; \quad \tilde{\kappa} = \mathbf{N}^{\kappa} \tilde{\kappa}^h$$  \hspace{1cm} (5.11)

where $\mathbf{N}^u, \mathbf{N}^P, \mathbf{N}^{\sigma}$ and $\mathbf{N}^{\kappa}$ are the shape functions for the displacement ($u_i$), fluid pressure ($P$), non-local stress ($\tilde{\sigma}^{eq}$) and non-local permeability ($\tilde{\kappa}$) fields, respectively. The superscript $h$ in $\mathbf{u}^h, P^h$ and $\tilde{\kappa}^h$ indicates nodal values of each field. The discretization in Equation (3.17) extends the for-
mulation proposed in [33] to include damage and non-local permeability effects. In order to satisfy the Babuška-Brezzi condition [198–201] the displacement shape function is taken to be quadratic 8-node element known as the serendipity element, while the pressure and non-local permeability shape functions are taken to be bilinear 4-node element. Using equal order interpolation of displacement and pressure leads to lower computational effort; however, it leads to unstable numerical results featuring spurious pressure oscillations. A schematic of the interpolation functions options is shown in Figure 5.1.

![Figure 5.1: Schematic showing the discretization options for the \((u - P - \tilde{\sigma}^{eq} - \kappa)\) mixed finite element formulation.](image)

Over the years, several stabilization techniques were developed to stabilize the equal order mixed finite element formulations for coupled fluid flow - solid deformation problems. Formulations like Galerkin Least Squares (GLS) [230, 231] and Finite Increment Calculus (FIC) [232] and others [233] aimed to achieve non-oscillatory equal order elements. These formulations were successful in stabilizing the performance of \(u/P\) solutions; however, they suffered from certain drawbacks e.g. the need for approximation of higher order derivatives [231, 232] and/or the introduction of additional non-symmetric terms [231]. These complications lead to additional computational cost of the stabilization scheme. In this study, we adopt the polynomial projection stabilization procedure introduced in [229] for poroelasticity formulations. The pressure projection stabilization
schemes were originally proposed for Stokes [234, 235] and Darcy [236] problems. The pressure projection is introduced by modifying Equation (5.10b) to become:

\[ R^P - R^{stab} = 0 \] (5.12)

where \( R^{stab} \) is the additional stabilization term introduced to the weak term. \( R^{stab} \) can be defined as [229]:

\[ R^{stab} = \int_{\Omega} \frac{\tau}{2(1-D)} G \left( w^P - \Pi(w^P) \right) \left( P - \Pi(P) \right) d\Omega \] (5.13)

where \( G \) is the Shear Modulus, \( \tau \) is the stabilization parameter that can be calibrated to achieve the optimum stabilization. The operator \( \Pi \) is defined as:

\[ \Pi(X) = \frac{1}{V^e} \int_{\Omega^e} X d\Omega^e \] (5.14)

where \( V^e \) is the finite element volume and \( \Omega^e \) is the finite element domain. The operator \( \Pi \) provides an averaging operator that calculates the field average over the finite element domain. The addition of \( R^{stab} \) leads to stabilization by introducing a perturbation term that smoothen the effect material point pressure rate oscillations at a material point. The capability of this stabilization technique relies heavily on the finite element size and the value of the stabilization parameter \( \tau \).

In order to continue with the linearization of the weak form presented in Equation (5.10a) to (5.10c), the solution vector \( \mathbf{x} \) and the residual vector \( \mathbf{R} \) are defined as follows:

\[
\mathbf{x} = \begin{pmatrix}
    u_i \\
    P \\
    \tilde{\sigma}^{eq} \\
    \tilde{\kappa}
\end{pmatrix} \quad ; \quad \mathbf{R} = \begin{pmatrix}
    R^u_i \\
    R^P \\
    R^{\sigma} \\
    R^{\kappa}
\end{pmatrix}
\] (5.15)
The residual statement in Equation 3.16 may be written as:

\[ R^{n+1} = M\dot{x}^{n+1} + Kx^{n+1} = 0 \] (5.16)

where \( M \) and \( K \) are square matrices and the superscript \((n + 1)\) denotes the solution at the next time step. Using a difference scheme [202], we may write the velocity vector \( \dot{x}^{n+1} \) as:

\[ \dot{x}^{n+1} = \frac{1}{\beta dt} \left[ x^{n+1} - x^n \right] - \frac{1 - \beta}{\beta} x^n \] (5.17)

where \( \beta \) is the parameter that defines the time integration scheme, \( \beta = 1 \) for backward Euler, \( \beta = 0.5 \) for a Crank-Nicolson method and \( \beta = 0 \) for forward Euler. The final residual statement, after temporal discretization, is defined as:

\[ R^{n+1} = \left[ \frac{1}{\beta dt} M + K \right] x^{n+1} - \frac{1}{\beta dt} Mx^n - \frac{1 - \beta}{\beta} M\dot{x}^n = 0 \] (5.18)

In this paper we assume an implicit scheme, which means that at every time step a Newton-Raphson method is used to solve the resulting non-linear system of equations leading to the following linearized system:

\[ J^{n+1} \delta x^{n+1} = -R^{n+1} \] (5.19)

where \( \delta x \) is the incremental solution vector computed at each Newton iteration and the Jacobian (tangent stiffness) matrix is:

\[ J^{n+1} = \frac{\partial R^{n+1}}{\partial x^{n+1}} = \left[ \frac{1}{\beta dt} M + K \right] = \begin{bmatrix} J_{uu} & J_{up} & J_{u\sigma} & J_{u\kappa} \\ J_{pu} & J_{pp} & J_{p\sigma} & J_{p\kappa} \\ J_{su} & J_{sp} & J_{s\sigma} & J_{s\kappa} \\ J_{sk} & J_{kp} & J_{k\sigma} & J_{k\kappa} \end{bmatrix} \] (5.20)
where the components of the Jacobian matrix $J^{n+1}$ are calculated as:

\[ J_{uu} = \frac{\partial R_i^u}{\partial u} = \int_\Omega B_i^{uT} \bar{C} B^u d\Omega \]  
(5.21a)

\[ J_{uP} = \frac{\partial R_i^u}{\partial P} = -\int_\Omega B_i^{uT} \alpha(D) I N^P d\Omega \]  
(5.21b)

\[ J_{u\sigma} = \frac{\partial R_i^u}{\partial \sigma} = -\int_\Omega B_i^{uT} C \varepsilon \frac{\partial D}{\partial \sigma^{eq}} N^\sigma + B_i^{uT} P \frac{\partial \alpha(D)}{\partial \sigma^{eq}} N^\sigma d\Omega \]  
(5.21c)

\[ J_{uk} = \frac{\partial R_i^u}{\partial \kappa} = 0 \]  
(5.21d)

\[ J_{Pu} = \frac{\partial R_i^P}{\partial u} = \frac{1}{\beta dt} \int_\Omega N_i^{P T} \alpha(D) B_i^{u,vol} d\Omega + \int_\Omega N_i^{P T} \frac{\partial \alpha(D)}{\partial t} B_i^{u,vol} d\Omega \]  
(5.21e)

\[ J_{PP} = \frac{\partial R_i^P}{\partial P} = \frac{1}{\beta dt} \int_\Omega N_i^{P T} \frac{1}{M(D)} N_i^P d\Omega + \int_\Omega N_i^{P T} \frac{\partial (1/M(D))}{\partial P} N_i^P d\Omega + \int_\Omega B_i^{P T} \bar{k} B_i^P d\Omega \]  
(5.21f)

\[ J_{P\sigma} = \frac{\partial R_i^P}{\partial \sigma^{eq}} = \int_\Omega N_i^{P T} \left[ \frac{\partial P \partial [1/M(D)]}{\partial t} + P \frac{\partial (1/M(D))}{\partial t} \right]^{\sigma^{eq}} + \varepsilon_{ii} \frac{\partial \alpha(D)}{\partial \sigma^{eq}} + \varepsilon_{ii} \frac{\partial \alpha(D)}{\partial \sigma^{eq}} \right] N^\sigma d\Omega \]  
(5.21g)

\[ J_{P\kappa} = \frac{\partial R_i^P}{\partial \kappa} = B_i^{P T} B_i^{P} N^\kappa d\Omega \]  
(5.21h)

\[ J_{\sigma u} = \frac{\partial R_i^\sigma}{\partial u} = -\int_\Omega N_i^{\sigma T} \left[ \frac{\partial \sigma^{eq,D}}{\partial \varepsilon} \right] B_i^u d\Omega \]  
(5.21i)

\[ J_{\sigma P} = \frac{\partial R_i^\sigma}{\partial P} = 0 \]  
(5.21j)

\[ J_{\sigma\sigma} = \frac{\partial R_i^\sigma}{\partial \sigma^{eq}} = \int_\Omega N_i^{\sigma T} N_i^\sigma d\Omega + g((\sigma^{eq,D})^n) \int_\Omega B_i^{\sigma T} B_i^\sigma d\Omega \]  
(5.21k)

\[ J_{\sigma\kappa} = \frac{\partial R_i^\sigma}{\partial \kappa} = 0 \]  
(5.21l)

\[ J_{\kappa u} = \frac{\partial R_i^\kappa}{\partial u} = -\int_\Omega N_i^{\kappa T} \left[ \frac{\partial \kappa(\sigma^{eq,k})}{\partial \varepsilon} \right] B_i^u d\Omega \]  
(5.21m)

\[ J_{\kappa P} = \frac{\partial R_i^\kappa}{\partial P} = 0 \]  
(5.21n)

\[ J_{\kappa\sigma} = \frac{\partial R_i^\kappa}{\partial \sigma^{eq}} = 0 \]  
(5.21o)

\[ J_{\kappa\kappa} = \frac{\partial R_i^\kappa}{\partial \kappa} = \int_\Omega N_i^{\kappa T} N_i^\kappa d\Omega + (l^k)^2 \int_\Omega B_i^{\kappa T} B_i^\kappa d\Omega \]  
(5.21p)

where $B_i^u$ is the shape function derivative of $N_i^u$ used in the calculation of strains as in: $\varepsilon_{ij} =$
CHAPTER 5. FOUR-FIELD MODELING OF NON-LOCAL DAMAGE TRANSPORT MODEL

$B^u u^h; B^{u,\text{vol}}$ is the shape function derivative of $N^u$ used in the calculation of volumetric strain as in: $\varepsilon_{ii} = B_{\text{vol}}^{u} u^h; B^P$ is the shape function derivative of $N^P$ corresponding to $P_i = B^P P^h; B^\kappa$ is the shape function derivative of $N^\kappa$ corresponding to $\tilde{\kappa}_i = B^\kappa \tilde{\kappa}^h$ and the superscript $T$ indicates matrix transpose. The matrix forms of the stiffness tensor $C_{ijkl}$ and the damaged stiffness tensor $(1 - D)C_{ijkl}$ are denoted by $C$ and $\bar{C}$, respectively. The kronecker $\delta_{ij}$ operator is the identity matrix $I$. In this paper, we employ a backward difference scheme with $\beta = 1$. Detailed expressions of the derivatives used in the calculation of the residual vector and Jacobian matrix are provided in Appendix A. Note that the Jacobian matrix is sparse and nonsymmetric, which would require appropriate solvers. The adaptive time stepping scheme in Equation (3.25) is used to control the time step in this solution scheme. The complete solution scheme is summarized in Algorithm 3.

Algorithm 3 Solution Algorithm

1: Initialize all variables $(u^h, P^h, \bar{\sigma}^{eq}^h, \tilde{\kappa}^h)$
2: while $t < T$ do
3:   Calculate $g((\sigma^{eq})^n)$ Eq. (4.9)
4:     while $||R|| < tol ||R_0||$ do
5:       for each finite element do
6:         for each material point do
7:           Interpolate $\tilde{\kappa}$ value at material point from $\tilde{\kappa}^h$
8:           Interpolate $\bar{\sigma}^{eq}$ value at material point from $\bar{\sigma}^{eq}^h$
9:           Calculate non-local damage $D(\bar{\sigma}^{eq})$ Eq. (4.7)
10:          Interpolate local strain tensor $\varepsilon_{ij}$ from $u^h$
11:          Calculate local equivalent stresses $\sigma^{eq,D}$ and $\sigma^{eq,K}$ Eq. (4.2)
12:          Calculate local permeability $\kappa(\sigma^{eq,K})$ Eq. (4.5)
13:          Calculate Jacobian matrix $J$ and residual vector $R$ Eq. (5.20)
14:       end for
15:     end for
16:     Assemble Jacobian matrix $J$ and residual vector $R$ for all elements
17:     Solve for $\delta x$ Eq. (5.19)
18:     $x^{n+1} \rightarrow x^{n+1} + \delta x$ Update solution vector
19: end while
20: $dt = \Delta T^{n+1}$ Eq. (3.25)
21: $t \rightarrow t + dt$ Update solution time
22: end while
5.3 Numerical examples of hydraulic fracture

In this section, we present two examples that demonstrate the significance of the four-field modeling approach presented in this chapter. The first example is a hydraulic fracture problem similar to the one presented in Section 4.4.1; however, in this section we investigate the effect of varying the transport length scale while fixing the damage length scale. The increased transport length scale represents a larger size of intergranular transport network. The second example investigates of modeling the evolution of damage and permeability using different local equivalent stress parameters.

5.3.1 Investigation of different transport length scales effect

Modeling hydraulic fracture in pre-fractured geomaterials is a challenge in hydraulic fracture industry because of the difficulty of modeling the underlying wide fracture and fluid-transport networks [237–239]. In this example, we investigate the effect of varying transport length scale on the evolution of hydraulic fracture while fixing the damage length scale. The increasing transport length scale reflects and increasing size of inter-granular pore network. Consider the schematic in Figure 5.2, the hydraulic fracture problem is modeled using a poro-elastic domain of dimensions $2L \times L$. The left edge is the symmetry line where a zero-flux condition and a horizontal translation constraint is applied. The right, top and bottom edges are the external boundaries of the domain which are mechanically restrained from moving but are permeable to fluid flow. A notch is used to pump fluid inside the domain with a flux of $Q$. The finite element mesh size used in the crack propagation zone is 0.05 m. The modeling parameters are listed in Table 4.1. In this example we fix the non-local damage length scale $l^d = 0.1$ m and vary the transport length scale $l^k = \{0.1, 0.2, 0.5, 1.0\}$ m.

We first plot the damage $D$ and non-local permeability $\tilde{\kappa}$ contours for each case ($l^k = \{0.1, 0.2, 0.5, 1.0\}$ m) at $5.3 \times 10^4$ s. Damage $D$ and non-local permeability $\tilde{\kappa}$ contours are shown in Figures 5.3 and
5.4 respectively. By analyzing the trends in Figures 5.3 and 5.4, one can observe that the increase in the non-local permeability length scale leads to a wider high permeability zone which leads to higher water dissipation on the hydraulic fracture crack walls. The additional water dissipation from the crack walls leads to wider damage zones. After some time through the simulation, the wide permeability-damage zone forms a fluid pool that dissipates large amounts of fluid through the crack walls and hinders the propagation of the hydraulic fracture in the desired direction.

In order to have a better understanding of the effect of increasing the permeability length scale we plot the evolution of the fracture length $L_F$, volume $V_F$ and average width $w_F$ (defined in Section 4.4) in Figure 5.5. The results show clearly that increasing the transport length scale leads
5.3. NUMERICAL EXAMPLES OF HYDRAULIC FRACTURE

Figure 5.3: The effect of changing $l^k$ on damage evolution in hydraulic fracture example. Damage ($D$) contours are plotted at $5.3 \times 10^4$ s for different values of $l^k = \{0.1, 0.2, 0.5, 1.0\}$ m and $l^d = 0.1$ m.
Figure 5.4: The effect of changing $l^k$ on non-local permeability evolution in hydraulic fracture example. Non-local permeability ($\tilde{\kappa}$) contours are plotted at $5.3 \times 10^4$ s for different values of $l^k = \{0.1, 0.2, 0.5, 1.0\}$ m and $l^d = 0.1$ m.
5.3. NUMERICAL EXAMPLES OF HYDRAULIC FRACTURE

to shorter fracture length (lower $L_F$) and wider fracture (higher $w_F$). In the extreme case where $l^k = 1.0$ m, one can observe that the fracture average width $w_F$ keeps increasing continuously which hinders the evolution of the fracture length $L_F$ and leads to the formation of the wide fluid pool observed in Figures 5.3d and 5.4d.

![Graphs showing fracture length, volume, and average width evolution.](image)

Figure 5.5: The hydraulic driven fracture length (a), volume (b) and average width (c) evolution against time for different values of transport length scale $l^k = \{0.1, 0.2, 0.5, 1.0\}$ m.

5.3.2 Investigation of different driving local stresses effect

Shear cracks may develop in geomaterials under different loading configurations which may be natural e.g. landslides and consolidation or man-made e.g. excavation. The accurate character-
ization the coupled evolution of damage and permeability in the presence of shear cracks is a not trivial and has been the subject of extensive research in geomechanics [56, 185, 240–242]. In shear cracks, we do not expect to see fluid flow inside the crack; however, once the shear stress exceeds the shear strength of the material, the crack experiences shear dilation [243]. The 4-field formulation presented in this chapter allows us to model the evolution of damage and permeability using different driving local stresses $\sigma^{eq,D}$ and $\sigma^{eq,K}$. This capability of the 4-field formulation provides a better physical modeling of shear cracks than the 3-field formulation presented in Chapters 3 and 4 which assume similar driving local equivalent stress/strain measures for the evolution of damage and permeability.

Consider the consolidation problem in Figure 3.13, the footing base is assumed rigid. Due to symmetry, only half of the problem is analyzed with symmetric boundary conditions as shown in Figure 5.6. Points A and B shown on Figure 5.6 are measurement points which will be used later in the discussion. The modeling parameters are listed in Table 5.1. In this section, we investigate three different modeling cases summarized in Table 5.2. In all cases, the damage evolution is controlled by shear crack evolution implied by setting $a_2 = 1.0$ for the calculation of $\sigma^{eq,D}$. In Case I, the permeability evolution is controlled by shear stress evolution by setting $a_2 = 1.0$ for the calculation of $\sigma^{eq,K}$, which implies that the permeability evolution will follow the damage evolution. Case I results are expected to be similar to the three-field formulation results for the consolidation problem presented in Section 3.4.2. In Case II, the permeability evolution is controlled by local volumetric expansion which is implied by setting $a_1 = a_2 = 0.0$ for the calculation of $\sigma^{eq,K}$. In Case III, the permeability evolution is controlled by mode-I crack opening which is implied by setting $a_1 = 1.0$ for the calculation of $\sigma^{eq,K}$. Each of these cases is modeled using two valued of parameter $b_2$ in Equation (4.5): $b_2 = \{1 \times 10^{-15}, 1 \times 10^{-17}\}$; the value of the parameter $b_2$ controls how much permeability increases as the stresses increase. The lower value of $b_2 = 1 \times 10^{-17}$ indicates that permeability experiences almost insignificant upon the domain loading.
5.3. NUMERICAL EXAMPLES OF HYDRAULIC FRACTURE

Figure 5.6: Mechanical model showing the boundary conditions of the consolidation problem including the boundary conditions for the equilibrium and mass balance equations. The rigid footing is loaded with a vertical surcharge $\sigma_0$.

We first plot damage and non-local permeability contours at $3.3 \times 10^3$ s for each modeling case in Figures 5.7 and 5.8. Figure 5.7 shows contours for the $b_2 = 1 \times 10^{-15}$ model and Figure 5.8 shows contours for the $b_2 = 1 \times 10^{-17}$ model. By analyzing the results in Figure 5.7, one can observe that permeability completely follows damage path in Case I. In Case II, no significant permeability growth is observed. In Case III, the permeability is higher in the crack zone than in the rest of the domain; however, the permeability values are less than Case I and not localized in the crack zone as in Case I. It is also noticeable that in Case I and III, the damage is higher than Case II. This indicated that the absence of significant permeability increase in Case II leads to slower damage propagation. Similar conclusions about permeability evolution can be drawn by analyzing
Table 5.1: Modeling parameters

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain dimension</td>
<td>$L$</td>
<td>10 m</td>
</tr>
<tr>
<td>Fluid flux</td>
<td>$q$</td>
<td>$2.8 \times 10^{-4} \text{ m}^3\text{s}^{-1}$</td>
</tr>
<tr>
<td>Bulk Modulus</td>
<td>$K$</td>
<td>$2.08 \times 10^8 \text{ Pa}$</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>$\mu$</td>
<td>$9.26 \times 10^7 \text{ Pa}$</td>
</tr>
<tr>
<td>Undrained Poisson’s ratio</td>
<td>$\nu^u$</td>
<td>0.49</td>
</tr>
<tr>
<td>Solid grain bulk modulus</td>
<td>$K_s$</td>
<td>$1.1 \times 10^{12} \text{ Pa}$</td>
</tr>
<tr>
<td>Initial permeability</td>
<td>$k_0$</td>
<td>$1.00 \times 10^{-14} \text{ m}^2/\text{Pa s}$</td>
</tr>
<tr>
<td>Constant in Equation (4.5)</td>
<td>$b_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (4.5)</td>
<td>$b_3$</td>
<td>2</td>
</tr>
<tr>
<td>Maximum damage</td>
<td>$D_{max}$</td>
<td>0.99</td>
</tr>
<tr>
<td>Damage initiation stress</td>
<td>$S^i$</td>
<td>$0.5 \times 10^6 \text{ Pa}$</td>
</tr>
<tr>
<td>Damage final stress</td>
<td>$S^f$</td>
<td>$1.0 \times 10^9 \text{ Pa}$</td>
</tr>
<tr>
<td>Constant in Equation (4.9)</td>
<td>$k_1$</td>
<td>1.0</td>
</tr>
<tr>
<td>Constant in Equation (4.9)</td>
<td>$k_2$</td>
<td>$5.0 \times 10^{-9}$</td>
</tr>
<tr>
<td>Transport length scale</td>
<td>$l^k$</td>
<td>0.15 m</td>
</tr>
<tr>
<td>Damage length scale</td>
<td>$l^D$</td>
<td>0.15 m</td>
</tr>
<tr>
<td>Stabilization parameter in Equation (5.13)</td>
<td>$l^D$</td>
<td>0.15 m</td>
</tr>
</tbody>
</table>

Table 5.2: Consolidation modeling cases. $a_1$ and $a_2$ are parameters used in the calculation of the local equivalent stress measure in Equation (4.3).

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>$\sigma^{eq.D}$ parameters</th>
<th>$\sigma^{eq.K}$ parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>$a_1 = 0.0, a_2 = 1.0$</td>
<td>$a_1 = 0.0, a_2 = 1.0$</td>
</tr>
<tr>
<td>Case II</td>
<td>$a_1 = 0.0, a_2 = 1.0$</td>
<td>$a_1 = 0.0, a_2 = 0.0$</td>
</tr>
<tr>
<td>Case III</td>
<td>$a_1 = 0.0, a_2 = 1.0$</td>
<td>$a_1 = 1.0, a_2 = 0.0$</td>
</tr>
</tbody>
</table>

the results in Figure 5.8; however, it is important to notice that the permeability increase in Figure 5.8 is much lower than the permeability increase in Figure 5.7 which is attributed to the lower value of $b_2$. On the contrast to the observations drawn from Figure 5.7, the damage contours in Figure 5.8 are similar for the three cases which is caused by the insignificant increase in permeability in the three cases when $b_2 = 1 \times 10^{-17}$.

In order to have a closer look on the difference between the above-mentioned modeling cases, we plot temporal evolution of the main field variables damage $D$, non-local permeability $\tilde{\kappa}$ and
5.4. SUMMARY AND CONCLUSION

fluid velocity magnitude $||v_i||$ at points A and B (shown in Figure 5.6) in Figures 5.9, 5.10 and 5.11 respectively. The results in Figure 5.9 and 5.10 confirm the two main conclusions drawn from analyzing the contour results, which are:

1. in the case of too small $b_2$, the permeability increase upon loading is too small to have a significant effect on the damage evolution behaviour; the damage results for all cases are almost identical when $b_2 = 1 \times 10^{-17}$
2. the increase of permeability in the crack zone directly affects damage increase; in cases I and III where a large increase in permeability is experienced, higher damage is observed as well.

As expected from Darcy’s law, the results in Figure 5.11 show that the models experiencing large increase in permeability in the crack zone have higher fluid velocity inside the crack zone. This increased fluid velocity inside the crack zone justifies the increase in damage associated with permeability increase. When permeability increases and fluid flows inside the crack, the fluid flow inside the crack acts as a secondary crack opening effect, i.e. the mechanics of the model become more similar to hydraulic fracture.

5.4 Summary and conclusion

In this Chapter, we present the computational implementation of the 4-field ($u - P - \bar{\sigma}^eq - \bar{\kappa}$) implementation of the non-local transport damage model. The 4-field formulation allows better physical description of the processes occurring in geomaterials because it eliminates the essential assumptions of the 3-field formulation which are: 1) similar length scale for transport and damage, 2) similar driving stress/strain measures of the evolution of damage and permeability. The mixed finite element formulation of the 4-field implementation is discussed in detail. A polynomial pressure projection stabilization scheme is adopted in order to allow the use of equal order approximations for all projected variables. An implicit scheme with adaptive time stepping is used to evolve the system in time, where at every time step Newton’s method with analytical derivation
of the Jacobian matrix, is employed to solve the nonlinear system.

Numerical examples of hydraulic fracture and consolidation are shown to demonstrate the significance of the 4-field formulation. In the hydraulic fracture example, the transport length scale is increases while the damage length scale is constant. The increased transport length scale reflects the presence of inter-granular fluid transport network. The results show that the presence of larger permeability length scale leads to wider and shorter hydraulic fractures. We also noticed the formation of a wide fluid-pool that dissipates fluid through crack walls rather than advancing the crack tip in the presence of a larger permeability length scale. In the consolidation example, we investigated the different possible configurations of modeling permeability evolution while the damage evolution was controlled by shear cracking. We concluded that if the permeability evolution is driven by shear crack or mode-I crack opening, the crack will be eventually water-filled and the model will experience a secondary hydraulic-fracture-like effect which accelerates the damage propagation.
5.4. SUMMARY AND CONCLUSION

(a) Damage ($D$): Case I

(b) Non-local permeability ($\tilde{\kappa}$): Case I

(c) Damage ($D$): Case II

(d) Non-local permeability ($\tilde{\kappa}$): Case II

(e) Damage ($D$): Case III

(f) Non-local permeability ($\tilde{\kappa}$): Case III

Figure 5.7: Damage ($D$) and non-local permeability $\tilde{\kappa}$ at $3.3 \times 10^3$ s for Cases I, II and III for $b_2 = 1 \times 10^{-15}$. 
Figure 5.8: Damage ($D$) and non-local permeability $\tilde{\kappa}$ at $3.3 \times 10^3$ s for Cases I, II and III for $b_2 = 1 \times 10^{-17}$. 
Figure 5.9: Damage evolution at points A and B shown on Figure 5.6 for $b_2 = \{1 \times 10^{-15}, 1 \times 10^{-17}\}$.
Figure 5.10: Non-local permeability evolution at points A and B shown on Figure 5.6 for $b_2 = \{1 \times 10^{-15}, 1 \times 10^{-17}\}$. 
Figure 5.11: Fluid velocity magnitude evolution at points A and B shown on Figure 5.6 for $b_2 = \{1 \times 10^{-15}, 1 \times 10^{-17}\}$. 

(a) Point A, $b_2 = 1 \times 10^{-15}$ 
(b) Point B, $b_2 = 1 \times 10^{-15}$ 
(c) Point A, $b_2 = 1 \times 10^{-17}$ 
(d) Point B, $b_2 = 1 \times 10^{-17}$
Chapter 6

Modeling hydraulic fracture of glaciers using continuum damage mechanics

6.1 Introduction

In this study, we propose a formulation to incorporate the effects of water pressure in crevasses, based on the principles of continuum damage mechanics and poromechanics. This new approach considers the effect of water pressure inside damaged ice in the crevassed zones as an additional damage effect, which we call “hydrostatic damage”. For the sake of proof of concept, we use the viscoelastic constitutive damage evolution model for polycrystalline ice previously proposed [120, 122], but we note that the formulation we propose can be used in conjunction with other constitutive damage models. The constitutive model is based on the small strain assumption and the additive decomposition of strain into its elastic and viscous components, which is valid in this context because the total simulation time is relatively small (hours to days) and the accumulated elastic and viscous strain components are reasonably small. The proposed formulation is used to model the propagation of surface crevasses and the simultaneous propagation of surface and basal crevasses in grounded glaciers. The rest of the paper is organized as follows: first, the
6.2. MODEL FORMULATION

viscoelastic damage model incorporating the hydrostatic damage effect is presented; second, the physical geometry and boundary conditions are described along with the results from benchmark studies and several representative numerical examples. In the appendices, we present a simpler, uniaxial derivation of the model unencumbered by the tensor notation that clouds the more general derivation and show how the model can be applied to the simpler, purely viscous, rheologies more commonly used in glaciology.

6.2 Model formulation

In this section, we review the viscoelastic constitutive damage model for polycrystalline ice under dry conditions, previously presented by [120], and then extend it for wet (saturated) conditions within the framework of Biot’s poroelastic theory [16, 244]. We refer the readers to the appendices for a simpler derivation based on idealized stress states.

6.2.1 Viscoelastic rheology of undamaged ice

Assuming small elastic deformations, we additively decompose the total strain tensor $\varepsilon$ into elastic and viscous components

$$\varepsilon_{kl} = \varepsilon^e_{kl} + \varepsilon^v_{kl}, \quad (6.1)$$

where $\varepsilon^e_{kl}$ is the elastic strain (time-independent and recoverable) component and $\varepsilon^v_{kl}$ is the viscous (time-dependent and irrecoverable) component. Making the usual assumption that bulk glacier ice is isotropic, owing to its random polycrystalline microstructure, the elastic stress-strain relationship (multi-axial Hooke’s law) is given by

$$\varepsilon^e_{kl} = \frac{1}{E} \left[ \sigma_{kl} - \nu \left( \sigma_{ii} \delta_{kl} - \sigma_{kl} \right) \right], \quad (6.2)$$

where $\sigma_{kl}$ denote components of the Cauchy stress tensor, $E$ is Young’s modulus, $\nu$ is Poisson’s ratio, $\delta_{kl}$ is the Kronecker’s delta, and repeated indices imply summation. The above equation can
be rewritten in the form

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}^e,$$  \hspace{1cm} (6.3)

where the fourth-order elasticity tensor $C_{ijkl}$ is defined as

$$C_{ijkl} = \frac{E}{2(1+\nu)} (\delta_{ij}\delta_{jk} + \delta_{ik}\delta_{jl}) + \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij}\delta_{kl}. \hspace{1cm} (6.4)$$

Denoting the deviatoric stress, $\sigma_{kl}^{\text{dev}} = \sigma_{kl} - \sigma_{ii}\delta_{kl}/3$, the viscous rheology can be expressed using the power-law creep relationship

$$\dot{\varepsilon}_{kl}^{\nu} = A \tau_e^{n-1} \sigma_{kl}^{\text{dev}}, \hspace{1cm} (6.5)$$

where $A$ is the temperature dependent viscosity coefficient, $n$ is the flow law exponent and the dot decoration represents the material derivative and $\tau_e^2 = \frac{3}{2} \sigma_{ij}^{\text{dev}} \sigma_{ij}^{\text{dev}}$ is the (von Mises) stress invariant. Because in a Maxwell viscoelastic model the stress felt by the viscous and elastic elements is the same, provided that the elasticity tensor does not vanish, the stress can be computed from Equation (6.3) and then used directly in Equation (6.5) to compute the viscous strain rate.

**6.2.2 Viscoelastic rheology of damaged ice**

We assume isotropic damage of ice under tension to simplify the formulation. We introduce a scalar internal state variable $D$, such that its evolution from $D = 0$ to 1 represents the deterioration of ice from the fully intact undamaged state to the completely damaged state. For $0 < D < 1$, we can define an effective stress $\bar{\sigma}_{ij}$ in the material that is larger than the average stress $\sigma_{ij}$ defined by

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{1-D}. \hspace{1cm} (6.6)$$
6.2. MODEL FORMULATION

Following the hypothesis of equivalent strain [245], the stress-strain relationship for damaged ice, in terms of the effective stress, can be expressed as

$$\bar{\sigma}_{ij} = C_{ijkl}\varepsilon_{kl}. \quad (6.7)$$

The damage modified viscous strain rate tensor is now defined in terms of the effective stress as

$$\dot{\varepsilon}_{kl}^v = A\bar{\tau}^{(n-1)}_e\bar{\sigma}_{kl}^{\text{dev}}, \quad (6.8)$$

with the effective deviatoric stress $\bar{\sigma}_{kl}^{\text{dev}} = \bar{\sigma}_{kl} - \bar{\sigma}_{ii}\delta_{kl}/3$ and $\bar{\tau}^2_e = \frac{3}{2}\bar{\sigma}_{ij}^{\text{dev}}\bar{\sigma}_{ij}^{\text{dev}}$. Substituting these definitions into Equation (6.8), the damage magnified viscous strain rate is

$$\dot{\varepsilon}_{kl}^v = \frac{A}{(1-D)^n}\bar{\tau}^{(n-1)}_e\bar{\sigma}_{kl}^{\text{dev}}. \quad (6.9)$$

Thus, the presence of damage leads to a non-linear strain rate enhancement factor of $(1 - D)^{-n}$.

6.2.3 Effect of pore pressure on the rheology of damaged ice

The previous sections described a constitutive creep damage models for polycrystalline ice, analogous to those developed by [119], [120] and, with the exception of the damage production law, [123]. In this section, we extend the continuum damage model to incorporate hydraulic fracture under wet (saturated) conditions where water can penetrate into microfractures (see Figure 6.1). Recalling, Biot’s theory of poroelasticity [16, 244], the relationship between the homogenized Cauchy stress $\sigma_{ij}$ and macroscopic solid effective stress $\bar{\sigma}_{ij}$ under saturated conditions can be defined as

$$\sigma_{ij} = (1 - \phi)\bar{\sigma}_{ij} - \phi P^h \delta_{ij}, \quad (6.10)$$

where $P^h$ represents the pressure of water filling the pores of a permeable medium, and $\phi$ is the average porosity of the medium. Assuming that damage and porosity are equivalent in isotropi-
cally damaged ice as a first approximation, we can extend the definition of the effective stress $\bar{\sigma}_{ij}$ (defined in Equations (6.6) and (6.7)) to express the homogenized Cauchy stress as

$$\sigma_{ij} = (1 - D) \bar{\sigma}_{ij} - DP^h \delta_{ij}. \quad (6.11)$$

Figure 6.1: Schematic showing the main features of the model. Surface and basal crevasses are present in the grounded ice.

Note that in the dry case $P^h = 0$ and Equation (6.11) reduces to the original definition of effective stress defined in Equation (6.6). For simplicity, we further assume that water flow into pores is sufficiently rapid so that the water pressure $P^h$ in the microvoids and microcracks in damaged ice is hydrostatic. This implies

$$P^h = \rho_f g \langle z \rangle, \quad (6.12)$$

where $\rho_f$ is the fluid density, $g$ is the gravitational acceleration, $\langle \rangle$ denote Macaulay brackets defined such that $\langle \chi \rangle = \chi$ when $\chi > 0$ and $\langle \chi \rangle = 0$ when $\chi < 0$, and the hydraulic head $z$ is the vertical distance between the water surface level $z_0$ and the level of the material point $z_1$ (i.e. $z = z_0 - z_1$). Combining equations (6.12) and (6.11), we can now write the macroscopic stress-strain relationship as

$$\sigma_{ij} = (1 - D)C_{ijkl} \varepsilon_k^l - \rho_f g \langle z \rangle D \delta_{ij}. \quad (6.13)$$

When $D = 0$ the above equation for Cauchy stress reduces to that of the undamaged material and when $\langle z \rangle = 0$, that is, when the hydraulic head is below the material point, the equation reduces to
that of the damaged material under dry conditions. Note that the effective solid stress $\bar{\sigma}_{ij}$ increases under saturated conditions, as given by

$$\bar{\sigma}_{ij} = C_{ijkl} \varepsilon_{kl} = \frac{\sigma_{ij}}{1 - D} + \frac{D}{1 - D} P^h \delta_{ij}. \quad (6.14)$$

The damage modified viscous strain rate tensor under wet conditions is given by

$$\dot{\varepsilon}_{kl}^v = A \bar{\tau}_v (n - 1) \bar{\sigma}_{kl}^{\text{dev}}, \quad (6.15)$$

with the effective deviatoric stress $\bar{\sigma}_{kl}^{\text{dev}} = \bar{\sigma}_{kl} - \bar{\sigma}_{ii} \delta_{kl}/3$ and $\bar{\tau}_v = \frac{3}{2} \bar{\sigma}_{ij}^{\text{dev}} \bar{\sigma}_{ij}^{\text{dev}}$. Recalling that neither the von Mises stress, $\bar{\tau}_e$, nor the deviatoric stress, $\sigma_{kl}^{\text{dev}}$, depend on pore pressure, Equation (6.15) shows that unlike the elastic rheology, the viscous component of the rheology is invariant to the inclusion of pore pressure. This is illustrated more explicitly for the uniaxial example in Appendices A and B.

6.2.4 Damage evolution law

To complete the constitutive damage model description, we need to specify the damage evolution law. There is large uncertainty in the appropriate specification of a damage evolution law, but our formulation in theory is more general and does not depend on the particular choice of evolution law. Nonetheless, for definiteness we adopt a power-law damage rate function analogous to that first introduced by [119] and used by [120, 121]:

$$\dot{D} = \frac{B \langle \chi \rangle^r}{(1 - D)^{k_\sigma}}, \quad (6.16)$$

where the damage rate coefficient $B$ is a (possibly temperature dependent) model parameter, the exponent $r$ is a chosen constant, the exponent $k_\sigma$ is a stress dependent parameter and $\chi$ is the
Hayhurst equivalent stress given by [119]

\[
\chi = \begin{cases} 
\alpha \bar{\sigma}^{(1)} + \beta \bar{\tau}_e - (1 - \alpha - \beta) \bar{\sigma}_{kk}, & \text{if } \bar{\sigma}^{(1)} > 0, \\
0, & \text{if } \bar{\sigma}^{(1)} \leq 0.
\end{cases}
\]  

(6.17)

In the above equation, \( \alpha \) and \( \beta \) are material parameters that control the damage growth mechanism (a detailed discussion on their selection is provided in [119], \( \bar{\sigma}^{(1)} \) is the largest effective principal stress and \( \bar{\tau}_e \) is the effective von Mises stress corresponding to the solid matrix of porous ice. In this paper, we consider failure occurs due to tensile stresses only so that ice remains intact in compression. Hence, damage only accumulates when \( \bar{\sigma}^{(1)} \) is greater than zero. Creep experiments on polycrystalline materials (including metals and ice at high temperatures) illustrate that the rate of creep damage growth increases drastically as we approach full collapse. To be consistent with previous studies [119, 120] we introduce a stress dependent exponent of the form

\[
k_{\sigma} = \begin{cases} 
 k_1 + k_2 \bar{\sigma}_{ii}, & \text{for } 0 \leq \bar{\sigma}_{ii} \leq 1 \text{ MPa}, \\
 k_1 + k_2, & \text{for } \bar{\sigma}_{ii} > 1 \text{ MPa}, \\
 0, & \text{for } \bar{\sigma}_{ii} < 0 \text{ MPa}.
\end{cases}
\]  

(6.18)
6.2.5 Mechanical Equilibrium

Assuming small deformations, the mechanical equilibrium can be described by the standard viscoelastic boundary value problem in the computational domain $\Omega$ as

$$
\sigma_{i,j} + b_i = 0, \quad \text{in } \Omega, \quad (6.19)
$$

$$
\varepsilon^e_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m}) - \varepsilon^v_{mn} \quad \text{in } \Omega, \quad (6.20)
$$

$$
\sigma_{ij} = C^{hd}_{ijmn} \varepsilon^e_{mn}, \quad \text{in } \Omega, \quad (6.21)
$$

$$
\sigma_{ij}n_j = \bar{t}_i \quad \text{on } \Gamma_u, \quad (6.22)
$$

$$
u_i = \bar{u}_i \quad \text{on } \Gamma_t, \quad (6.23)
$$

where $b_i$ is the body forces vector; $\bar{u}_i$ denotes any prescribed displacements conditions corresponding to free slip or zero slip on the domain boundary $\Gamma_u$, $\bar{t}_i$ denotes any prescribed traction conditions corresponding to seawater pressure on the domain boundary $\Gamma_t$, respectively; and $n_j$ denotes the outward normal to the boundary $\Gamma_t$. Equation 7.1b is the static equilibrium equation in solid mechanics which resembles the stationary Stokes approximation from the fluid mechanics point of view. The viscous strain $\varepsilon^v_{mn}$ in Equation (6.20) is calculated from the evolution law in Equation (6.15) which defines $\varepsilon^v_{mn}$. In equation (6.21), $C^{hd}_{ijmn}$ denotes the hydro-damage modified fourth-order elasticity tensor defined as

$$
C^{hd}_{ijmn} = \left( (1 - D)\delta_{km}\delta_{ln} - D^{byd}\delta_{kl}\delta_{mn} \right) C_{ijkl}. \quad (6.24)
$$
CHAPTER 6. MODELING HYDRAULIC FRACTURE OF GLACIERS USING CONTINUUM DAMAGE MECHANICS

Using the relations $C_{ijkl}\delta_{kl} = \frac{E}{(1-2\nu)}\delta_{ij} = 3\kappa\delta_{ij}$ and $\bar{\sigma}_{qq} = 3\kappa\varepsilon_{pp}^e$, the above equation can be derived from equation (6.13) as follows,

$$\sigma_{ij} = (1-D)C_{ijkl}\varepsilon_{kl}^e - \rho_f g(z)D\delta_{ij},$$

$$= (1-D)C_{ijkl}\varepsilon_{kl}^e - \frac{\rho_f g(z)D}{3\kappa}C_{ijkl}\delta_{kl},$$

$$= (1-D)C_{ijkl}\varepsilon_{kl}^e - \frac{\rho_f g(z)D}{3\kappa\varepsilon_{pp}^e}\varepsilon_{pp}^e C_{ijkl}\delta_{kl},$$

$$= (1-D)C_{ijkl}\delta_{km}\delta_{ln}\varepsilon_{mn}^e - \frac{\rho_f g(z)D}{\bar{\sigma}_{qq}}C_{ijkl}\delta_{kl}\delta_{mn}\varepsilon_{mn}^e,$$

$$= \left(1-D\right)\delta_{km}\delta_{ln} - \frac{\rho_f g(z)D}{\bar{\sigma}_{qq}}\delta_{kl}\delta_{mn}\right)C_{ijkl}\varepsilon_{mn}^e,$$

where $\kappa$ denotes the bulk modulus of elasticity and the hydrostatic or hydraulic damage component $D^{hyd}$ is defined as

$$D^{hyd} = \frac{\rho_f g(z)D}{\bar{\sigma}_{qq}}.$$

Evidently, the hydraulic damage $D^{hyd}$ is non-zero only when there is some existing damage and is equal to the ratio of the effective fluid pore pressure $P^hD = \rho_f g(z)D$ and solid matrix pressure $\bar{\sigma}_{qq} = 3\kappa\varepsilon_{pp}^e$. Subjected to plane strain assumptions, the solution to the nonlinear boundary value problem defined by equations (7.1b) – (6.23) is obtained using the Galerkin finite element method detailed in [120–122]. In the present finite element implementation, four-node bilinear quadrilateral elements were used to discretize the unknown displacement field and four-point Gauss quadrature rule is used for integration. The internal state and history variables (e.g., damage, viscous strains) are stored at the quadrature points and an explicit forward Euler scheme is used to update these variables in time. We note that choosing higher order elements or finer resolutions is generally recommended in case higher accuracy is required.
6.3 Numerical Examples

In this section, we demonstrate the numerical results obtained from the finite element simulation of surface and basal crevasse propagation. First, we present a benchmark example to demonstrate the capability of the model to accurately calculate the hydraulic forces on the crevasse walls and the resulting stress field in the ice. Second, we investigate the propagation of surface crevasses, as well as the simultaneous propagation of both surface and basal crevasses for different boundary conditions.

6.3.1 Model geometry and parameters

We idealize the geometry of grounded marine-terminating glaciers as rectangular slabs of ice in contact with water, as shown in Figure 6.2. We apply a free slip boundary condition in the horizontal direction at the bottom edge of the slab (assuming minimal friction from the bed) and in the vertical direction on the left edge of the slab (where the ice slab is connected to the larger glacier). We apply a fixed (or zero displacement) boundary condition in the horizontal direction on the left edge of the slab. Assuming the influx of ice is independent of depth, this set of boundary conditions is translationally invariant in the horizontal direction and hence independent of the inflow velocity. Furthermore, the choice of zero displacement or velocity boundary condition is justified because we are interested in calculating the stress and deformation rates defined by the displacement or velocity gradients, thus, they are independent of the inflow velocity or displacement condition at the left edge. We denote the depth of the surface and basal crevasses by $d_s$ and $d_b$, respectively, and the initial ice thickness by $H$. The initial notches play the role of pre-existing weaknesses or starter cracks in linear elastic fracture mechanics and provide the seeds for localized damage propagation. This assumption prevents the growth of non-physical damage areas on the top of the slab in areas where the FEM discretization may be coarse to reduce the computational burden. Alternative crack or damage initiation schemes can be employed (e.g., seeding the glacier with random
Figure 6.2: Schematic drawing of the idealized grounded ice slab with dimensions and boundary conditions.

defects), but our scheme (i.e., seeding crevasses using notches) allows us to easily perform model sensitivity studies. Additionally, in all the following numerical examples, once damage localizes near the initial notches, we allow hydrostatic (or hydraulic) damage to occur only in the vicinity of the crack path. The slab length \( L = 2500 \) m is set to five times the ice thickness \( H = 500 \) m and the initial surface or basal crevasses (notches) are prescribed at mid-length, to avoid edge effects at the inflow and outflow boundaries. The depths of the initial crevasses (notch) are set to 8\% of the initial slab thickness \( H \). The piezometric head or hydraulic head in surface crevasses \( h_s \) is defined as the height of the water column measured from the top edge of the slab, consequently, the water pressure at the bottom of the surface crevasse is proportional to \( (h_s + d_s) \). We specifically allow for \( h_s > 0 \) to examine the effect of a supra-glacial lake filling a crevasse, although it simulates an unphysical example because we do not model the lake nor the topography necessary to sustain the lake. The piezometric head in the basal crevasse is assumed to be equal to the height of water level on the right edge of the slab denoted by \( h_w \). All the relevant material and model parameters listed in Table 6.1 are assumed from [120] for ice at -10\(^\circ\). The homogeneous ice density \( \rho_i = 910 \) Kg/m\(^3\) and water density \( \rho_f = 1000 \) Kg/m\(^3\). The value of the damage coefficient parameter \( B \) in equation 6.16, controlling the damage rate, is varied in the numerical studies so as to assess model
6.3. NUMERICAL EXAMPLES

Table 6.1: Values of mechanical and damage parameters for ice at $-10^\circ$. The parameter $A$ is the viscosity coefficient retrieved from [122] by taking $A = \frac{3}{2}K_N$; where $K_N$ is the viscosity coefficient defined in [122].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>9500</td>
<td>MPa</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.35</td>
<td>-</td>
</tr>
<tr>
<td>$r$</td>
<td>0.43</td>
<td>-</td>
</tr>
<tr>
<td>$n$</td>
<td>3.1</td>
<td>-</td>
</tr>
<tr>
<td>$A$</td>
<td>$3.84 \times 10^{-7}$</td>
<td>MPa$^{-n}$ s$^{-1}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.21</td>
<td>-</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.63</td>
<td>-</td>
</tr>
<tr>
<td>$D_{cr}$</td>
<td>0.45</td>
<td>-</td>
</tr>
</tbody>
</table>

6.3.2 Benchmark simulation

To demonstrate the capability of the proposed damage model to consistently calculate the hydraulic forces on the crevasse walls we consider a rectangular ice slab ($2500 \times 500$ m) initialized with a surface and a basal crevasse in two different approaches. In the first approach, the crevasses are defined by notches and the hydraulic forces at the finite element nodes lying on the crevasse walls are calculated from the hydrostatic pressure distribution using the line integral, $\int_{\Gamma} P_h d\Gamma$, as indicated in Figure 6.3. Thus, in the first approach the geometrical features of the crevasses are explicitly meshed and the corresponding results provide a reference solution or benchmark. In the second approach, the crevasses are defined by fully damaged elements by specifying $D = 1$ inside the crevasse zone and the hydraulic forces at the finite element nodes lying on the crevasse walls are calculated from the stress distribution using the area integral, $\int_{\Omega} \rho_f g(z) D \delta_{ij} d\Omega$ (as given by the second term in Eq. 6.13). Thus, in the second approach the geometrical features of the crevasses are implicitly defined by the damage variable and the results are compared with those obtained from the first approach. The following parameters were used in this study: crevasse depths $d_s = d_b = 25$ m; the piezometric head $h_s = 0$ and $h_w = 0.5H$. The total hydraulic forces
calculated from both approaches on either side of the crevasse are the same: 3126.9 kN for the surface crevasse and 59411.8 kN for the basal crevasse. The horizontal stress distributions computed using the two approaches, shown in Figure 6.4, are identical to within numerical error. Thus, this benchmark investigation indicates that our damage model is capable of accurately describing the stress state of an ice slab with fully damaged crevasses (i.e., when $D = 1$). This example also demonstrates the main advantage of the continuum damage mechanics description that completely eliminates the need for remeshing as the fracture propagates.

![Diagram of hydraulic pressure distribution](image)

Figure 6.3: Hydraulic pressure distribution on the surface and basal crevasses, assuming complete ice material failure ($D = 1$) of the elements in the crevasse zone, which is represented by a notch in this figure. The values of the hydraulic pressure $P_h$ are given at the bottom and top of the surface and basal crevasses.

### 6.3.3 Effect of hydro-damage on surface crevasse propagation

We first conducted several simulations to investigate the effect of hydro-damage on the depth $d_s$ to which surface crevasses penetrate in relation to the seawater depth $h_w$. We consider three different values of $h_w/H = \{0, 0.5, 0.8\}$ and take $h_s = 0$ to activate hydraulic damage under wet conditions. Figure 6.5 shows snapshots of damage contours (red zones indicate completely damaged ice) for $h_w/H = 0.8$ and damage coefficient $B = 10^{-5}$ MPa$^{-r_s}$. These simulation results emphasize the localized nature of crevasse propagation in glaciers driven by stress concentrations near pre-existing defects. Next, we conducted model sensitivity studies by varying the value of...
6.3. NUMERICAL EXAMPLES

Figure 6.4: Snapshot of horizontal stress, $\sigma_{xx}$, contours in the linear elastic configuration using two methods; the first (Panel (a)) models crevasses as notches and hydraulic forces are applied as nodal forces; and the second (Panel (b)) represents the equivalent proposed damage mechanics technique. Panel (c) represents the stress $\sigma_{xx}$ profile along the slab centerline where the height of the material point is measured from the bottom of the slab.

The damage coefficient $B = \{10^{-3}, 10^{-4}, 10^{-5}\}$ MPa$^{-r}s^{-1}$ and the corresponding time steps used in the analysis are $dt = \{0.1, 1, 10\}$ s, respectively. The time step is chosen sufficiently small (on the order of seconds) to ensure accuracy and stability of the explicit time update scheme used for computing damage and viscous strain evolution. Crevasse depths’ were computed under dry (no hydrodamage - NHD) and wet (hydrodamage - HD) conditions. In Figure 6.6, the normalized sur-
face crevasse depths \(d_s/H\) are plotted against simulation time (in hours) for different normalized seawater depths \(h_w/H\). Note that the depth of the surface crevasse \(d_s\) at a given time is measured as the vertical distance from the top of the slab to the farthest finite element node where the damage exceeds the critical damage value, that is, \(D > D_{cr}\) (see Table 6.1). The following conclusions can be drawn from Figure 6.6:

1. Water-filled surface crevasses experience more damage at any particular time and propagate to greater depths (for a given color compare the solid and dashed lines in Fig. 6.6), including full-depth fractures indicating calving events (i.e., \(d_s/H = 1\)); these results conform with Nye zero-stress results in [128] and LEFM results in [112];

2. The value of the damage coefficient \(B\) does not significantly change the final crevasse depth (compare the different colored dashed lines in Fig. 6.6), but it affects the rate at which crevasses propagates, consequently, it affects the time interval between calving events. However, model predicted damage (or crevasse) propagation rates are poorly calibrated by field or laboratory experiments. This result also demonstrates that the small strain assumption does not influence the conclusions drawn from our modeling studies; because similar crevasse penetration depths are retrieved for the case of \(B = 10^{-3}\) (where the accumulated viscous strains are small) and the case of \(B = 10^{-5}\) (where the accumulated viscous strains are larger).

We next performed a sequence of simulations to investigate the stability of marine-terminating glaciers in relation to the piezometric head \(h_s\) in surface crevasses. We consider three different values of \(h_s = \{0, 25, 50\}\) m (where \(h_s > 0\) corresponds to the presence of a supra-glacial lake) and recorded the temporal evolution of surface crevasses for different seawater depths \(h_w\). In Figure 6.7, we plot the normalized maximum (or final) crevasse depth \(d_{s,max}/H\) and the total time (in hours) elapsed till maximum crevasse depth is attained as a function of the normalized seawater depth \(h_w/H\), under dry conditions and under wet conditions for three different values of piezometric
6.3. NUMERICAL EXAMPLES

Figure 6.5: Snapshot of damage contours at different time steps for the isolated surface crevasse propagation model with $h_w/H = 0.8$. These results were simulated using $B = 10^{-5}$ MPa$^{-r_s}$. These simulations illustrate that:

1. Under dry conditions, through thickness surface crevasse propagation is not observed, regardless of the seawater level (blue line in Fig. 7a); whereas, under wet conditions through thickness surface crevasse propagation always occurs except when the seawater level is sufficiently high ($h_w > 0.8$, as indicated by green, red and brown solid lines in Figure 6.7). Thus, meltwater in surface crevasses destabilizes the glacier by driving through thickness crevasse propagation. These conclusions agree with the Nye zero-stress model in [128] that water filled surface crevasses are highly likely to reach the bottom of the slab.

2. An increase of seawater height generally decreases the rate of crevasse propagation (more pronounced when $h_w > 0.5$ in Figures 6.6 and 6.7), consequently, it increases the total time elapsed till maximum crevasse depth is attained. Thus, the seawater level has a stabilizing effect on crevasse propagation as it applies a compressive crack-closing pressure. These results provide a qualitative measure of the conditions which lead to faster vs. slower crevasse propagation, although the quantitative damage propagation rate remains poorly calibrated.

The numerical Nye-zero depth is calculated as the depth of the material point (from the top surface) at which the horizontal tensile stress vanishes ($\sigma_{xx} = 0$) and the values were recorded prior to damage propagation for all the simulations. The final crevasse depths retrieved from our damage
Figure 6.6: The evolution of surface crevasse with time under different values of near terminus water depth \(h_w\). The figures show simulation results for different values of \(B\), the parameter in Equation (6.16). The tag “HD” in the legend means including Hydraulic Damage and “NHD” means No Hydraulic Damage.

simulations were always within 10% or closer to those predicted by the Nye zero stress model.
6.3. NUMERICAL EXAMPLES

6.3.4 Effect of hydro-damage on surface and basal crevasse propagation

We next investigated the effect of hydro-damage on the simultaneous propagation of surface and basal crevasses. Unless the ocean water level is sufficiently high \( h_w/H > 0.8 \), surface crevasses always form in our simulation and it is not possible to have isolated basal crevasses without surface crevasses. This could be changed by setting a finite threshold for the largest principal stress as opposed to using a zero threshold as we did here. We find that, in accordance with the findings of [116] and [27], basal crevasses do not propagate unless they are water-filled.

Through finite element simulation, we estimated surface and basal crevasse depths by varying the seawater height \( h_w/H = \{0, 0.25, 0.6, 0.7, 0.8, 0.9\} \) for two scenarios: (1) only basal crevasse is water-filled and surface crevasse is dry; and (2) both surface and basal crevasses are water-filled. The plots of normalized maximum (or final) crevasse depths \( d_{s,max} \) and \( d_{b,max} \) versus the normalized seawater height are shown in Figures 6.8 and 6.9 for the two scenarios, respectively. In the case of a dry surface crevasse and water-filled basal crevasse (see Fig. 6.8), our results indicate that the maximum total crevasse depth \( d_{s,max} + d_{b,max} \) decreases as the seawater level increases until
$h_w/H = 0.8$, but then increases as the seawater level further increases $h_w/H > 0.8$. This is because the maximum surface crevasse depth decreases as seawater level increases, whereas, the maximum basal crevasse depth becomes significant only at high seawater levels, when the water pressure is sufficient to induce a tensile stress at the basal crack tips. Thus, our simulation results in Figure 6.8 are in good agreement with those published in [27]. In Figures 6.8 and Figure 6.9, we plotted the time taken for the crevasses to reach the maximum (or final) depth as function of $h_w/H$. These results show that the crevasses propagation times are smallest for $h_w/H = 0$ and $h_w/H = 0.9$, indicating that the corresponding crevasse propagation rates are the largest for $h_w/H = 0$ and $h_w/H = 0.9$, which is attributed to the existence of higher tensile stresses at the surface and basal crack tips, respectively. An important point to note from Figure 6.9 is that surface and basal crevasses propagate to a greater depth (or height) when both are water-filled, thus, indicating a mutually positive effect. To further investigate the effect of water-filled surface crevasses on basal crevasse propagation and vice-versa, we plot the temporal evolution of surface and basal crevasses in Figure 6.10 for water level $h_w/H = 0.25$. The results in Figure 6.10 illustrate that the basal

![Graph of final crevasse depth vs. $h_w/H$ and simulation time vs. $h_w/H$.](image)

Figure 6.8: Final crevasses’ depths ($d_{s_{\text{max}}}^\text{max}$ for surface and $d_{b_{\text{max}}}^\text{max}$ for basal) and corresponding simulation times under different values of near terminus water depth $h_w$; in these simulations, only the basal crevasses are water filled while the surface crevasses are dry. These results were simulated using $B = 10^{-4}$ MPa$^{-1}$s$^{-1}$. 

crevasses propagate to a greater depth (or height) when both are water-filled, thus, indicating a mutually positive effect. To further investigate the effect of water-filled surface crevasses on basal crevasse propagation and vice-versa, we plot the temporal evolution of surface and basal crevasses in Figure 6.10 for water level $h_w/H = 0.25$. The results in Figure 6.10 illustrate that the basal

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6.3. NUMERICAL EXAMPLES

Figure 6.9: Final crevasses’ depths ($d_s^{\text{max}}$ for surface and $d_b^{\text{max}}$ for basal) and corresponding simulation times under different values of side water pressure $h_w$; in these simulations, both surface and basal crevasses are water filled. These results were simulated using $B = 10^{-4}$ MPa$^{-r_s-1}$.

crevasse propagation is triggered when the surface crevasse approaches the bottom of the slab and alters the stress at the basal crevasse tip. The main conclusion of our study is consistent with previous studies: glaciers with dry surface crevasses are more stable than those with water-filled surface crevasses, and the situation worsens when the basal crevasses are water-filled. In this study, we did not exclusively investigate conditions that enable the propagation of only a basal crevasse without a surface crevasse. Because we assume that glacier ice has zero tensile strength, surface crevasses will always form while simulating an extending glacier and so it is not possible to have isolated basal crevasses without surface crevasses. However, the model can account for the tensile strength for ice by specifying a stress threshold for damage initiation [119], which disables the formation of surface crevasses at very low deformation rates. Similarly, including a sliding law instead of a free-slip boundary condition would quantitatively alter our simulation results. Nonetheless, the hydraulic damage methodology we have developed is directly applicable to more complex geometries, boundary conditions and rheologies.
Figure 6.10: Damage propagation for the case of $h_{w}/H = 0.25$ from Figure 6.9 (surface and basal crevasses are water filled). The top plot shows crevasses propagation with time and the following are snapshots of damage contours at different time steps. These results were simulated using $B = 10^{-4}$ MPa$^{-1}$s$^{-1}$.

### 6.4 Conclusion

In this paper, we demonstrated the viability of a continuum damage approach to model the propagation of crevasses filled with water. Using the poromechanics concept of effective solid stress, a viscoelastic damage model is developed to simulate hydraulic fracture of glacier ice, within a Lagrangian finite element framework. The advantages of using the proposed damage mechanics
6.4. CONCLUSION

approach over the LEFM approach are: (1) the incorporation of a viscoelastic constitutive law to model the time dependent mechanical behavior of ice, and (2) the elimination of adaptive remeshing or mesh moving procedures to model crevasse propagation. Although the model is developed for the small strain case assuming additive decomposition of strain, it can be extended to the finite strain case assuming multiplicative decomposition of the deformation gradient tensor [246].

Several numerical examples and sensitivity studies are considered to analyze the effects of water-filled surface and basal crevasses on the process of iceberg calving from idealized grounded glaciers. The finite element simulations considered different cases of ocean water levels, presence of surface lakes and variable piezometric water levels at basal crevasses. Several conclusions about the crevasse propagation could be drawn from the numerical results. The water filled crevasses tend to propagate further and faster than dry ones. Basal crevasses require a sufficiently high water pressure to start propagating, which is in accordance with the findings of previous studies. However, in contrast to studies that ignore the feedback between surface and basal crevasses, water filled surface and basal crevasses alter the stress state and interactively stimulate crevasse propagation deeper into the glacier, thus, speeding up the fracture process.
Chapter 7

Adaptive modeling of damage growth using a coupled FEM/BEM approach

7.1 Introduction

In this paper, linear elastic BEM is coupled with FEM to model localized damage growth in structures efficiently. Such approach is in particular relevant to large structural domains and small localized damaged zones. Non-local damage is essential to avoid mesh dependent results and to keep the well-posedness of the solution [247, 248]. An integral-type nonlocal continuum damage [121] with adapting FEM mesh is used to model multiple damage zones and follow their propagation in the structure. Strong form coupling, BEM hosted, is achieved using Lagrange multipliers and the solution is obtained by a monolithic Newton method that is used to solve the coupled system. In order to avoid the calculation of inaccurate values of non-local damage at the interface between FEM and BEM subdomains, we preserve a safe non-damage zone of finite elements along the interface.

The paper is organized as follows. Section 7.2 reviews the BEM formulation for linear elastic problems and finite element modeling of the brittle damage model by Mazar [249]. Then, the cou-
pling procedure is detailed in section 7.3. Numerical implementations that validate the proposed model are presented in section 7.5. The method is applied to multiple fractures growth benchmark problems and shows good agreement with the literature.

7.2 Problem formulation

7.2.1 Problem Statement

Consider an elastic domain \( \Omega \) as shown in Figure 7.1 with the boundary \( \partial \Omega = \Gamma \); and \( \Gamma = \Gamma_u \cup \Gamma_t \) where \( \Gamma_u \) is the part of boundary with prescribed displacements and \( \Gamma_t \) is the part of boundary with prescribed tractions. For \( u \) displacements and \( t \) tractions, a standard elastostatic boundary value problem is defined as:

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad \text{in} \quad \Omega \quad (7.1a)
\]

\[
\sigma_{ij,j} + b_i = 0 \quad \text{in} \quad \Omega \quad (7.1b)
\]

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \text{in} \quad \Omega \quad (7.1c)
\]

\[
\sigma_{ij} n_j = \bar{t}_i \quad \text{on} \quad \Gamma_t \quad (7.1d)
\]

\[
u_{i} = \bar{u}_{i} \quad \text{on} \quad \Gamma_u \quad (7.1e)
\]

where \( \sigma_{ij} \) denotes Cauchy stress; \( \varepsilon_{ij} \) denotes strain tensor; \( C_{ijkl} \) is the elastic stiffness tensor; \( b_i \) is the body forces vector; \( \bar{u}_i, \bar{t}_i \) denote prescribed boundary displacements and traction respectively and \( n_j \) denotes the boundary normal direction. Einstein’s indicial notation is followed in this paper; with repetition of indices indicating summation.

The point C in Figure 7.1 indicates a stress concentration point where damage is likely to occur. In damage mechanics, the elastic constitutive law is modified to accommodate for the effects of damage. Theory of damage mechanics is typically based on the effective stress principle [250], in which the damage zone is transformed into an “equivalent”, damage free linear elastic zone with
CHAPTER 7. ADAPTIVE MODELING OF DAMAGE GROWTH USING A COUPLED FEM/BEM APPROACH

Figure 7.1: General structure with domain $\Omega$; loading and boundary conditions are denoted by $\Gamma_l$ and $\Gamma_u$ respectively. The point C indicates a stress concentration point where damage is likely to initiate.

reduced area. In this work we assume an isotropic damage approach, in which the damage variable $D$, is defined as:

$$D = \frac{A_D}{A}$$ (7.2)

where $A_D$ is the reduced area accounting for micro cracks and voids. The damage variable presents the ratio of the material that has failed and can be no longer accounted for stress resistance. Upon stress redistribution, the effective stress $\bar{\sigma}$ is defined as:

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{1 - D}$$ (7.3)

In the case of isotropic damage, as in this study, the constitutive law in equation (7.1a) can be rewritten as:

$$\bar{\sigma}_{ij} = \tilde{C}_{ijkl}\varepsilon_{kl} = (1 - D)C_{ijkl}\varepsilon_{kl}$$ (7.4)
According to the type of material and the loading conditions, different damage models can be used to analyze failure. In order to implement the concept of damage, a damage constitutive law is required to define the type of material failure behavior. In the work presented herein, Mazar's damage law [249] is utilized. Being originally described for concrete, it is useful for describing damage in quasi-brittle materials [251]. This law describes damage as a function of the equivalent strain \( \varepsilon^* \) as follows:

\[
D(\varepsilon^*) = \begin{cases} 
1 - \left[ \frac{\varepsilon_D(1-a)}{\varepsilon^*} + \frac{a}{\exp(b(\varepsilon^* - \varepsilon_D))} \right] & \text{if } \varepsilon^* \geq \varepsilon_D \\
0 & \text{if } \varepsilon^* < \varepsilon_D 
\end{cases}
\] (7.5)

where \( \varepsilon_D \) is the damage threshold strain beyond which no damage occurs. In addition to the damage threshold, \( a \) and \( b \) are material dependent parameters that are retrieved from experimental data. The factor which controls damage, the equivalent strain, is a scalar measure that describes the overall deformation in the material. In this model, we deploy the measure introduced by Lemaitre [252] which describes the equivalent strain as a function of the principal strains \( \{ \varepsilon_I, \varepsilon_{II}, \varepsilon_{III} \} \) as follows:

\[
\varepsilon^* = \sqrt{\langle \varepsilon_I \rangle^2 + \langle \varepsilon_{II} \rangle^2 + \langle \varepsilon_{III} \rangle^2}
\] (7.6)

where the macaulay brackets \( \langle \varepsilon \rangle \) indicate that only the positive (tensile) components of strain is considered to cause damage. This is valid for brittle materials which have much higher strength in compression than in tension, e.g. concrete.

For larger values of damage, modeling damage accumulation as a local material point variable introduces loss of well-posedness of the model [247, 248, 253] which leads to numerical solutions that may be non-physical and/or mesh dependent results. As a result, the concept of the non-local damage [74] was introduced to average the damage using a weighting function for calculating the effective non-local damage. For a given material point \( X_p \), the non-local damage, \( D_{nl}(X_p) \), is
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defined as [121]:

\[ D_{nl}(X_p) = \frac{\int_{\Omega} \phi(X_p - X) D(X) d\Omega}{\int_{\Omega} \phi(X_p - X) d\Omega} \]  (7.7)

where \( \phi \) is a weighting function, commonly chosen as an exponential bell function [254, 255]:

\[ \phi(X_p - X) = \exp \left( -\frac{||X_p - X||^2}{l_c^2} \right) \]  (7.8)

where \( l_c \) is the characteristic length parameter, which reflects results of experimental data [256].

Upon loading, the area around point C (in Figure 7.1) will be the first to experience damage. Therefore, the key idea, illustrated in Figure 2, is to decompose the domain along the dashed line into a damaged domain (in the vicinity of point C where stress concentrations are expected) and a healthy domain (which includes the rest of the structure). The healthy domain, is a linear elastic damage free subdomain and hence can efficiently be modeled by a BEM method while the damaged zone by a detailed FEM subdomain.

7.2.2 BEM formulation for linear elastostatics

Substituting equation (7.1b) into equation (7.1a), one arrives at Navier’s equation, which is the linear elasticity differential equation in terms of displacements:

\[ \frac{1}{1 - 2\nu} u_{j,ji} + u_{i,(jj} + \frac{1}{\mu} b_i = 0 \]  (7.9)

where \( \mu \) is the shear modulus and \( \nu \) is Poisson’s ratio.

The fundamental solutions of Navier’s equation are Green’s function for displacement and traction respectively, which describe the response due to a point load in an infinite domain. For
two dimensional plain strain problems, Kelvin’s fundamental solutions are given by [135]:

\[
H_{ij}(\xi, x) = \frac{1}{8\pi\mu(1 - \nu)} \left[ (1 - 4\nu)\delta_{ij} \ln \frac{1}{r} + r, r, \right]
\tag{7.10a}
\]

\[
G_{ij}(\xi, x) = \frac{-1}{4\pi(1 - \nu)} r, n, \left\{ (1 - 2\nu)\delta_{ij} + 2n_i r_j - n_j r_i \right\} + (1 - 2\nu)(n_i r_j - n_j r_i)
\tag{7.10b}
\]

where \(\xi\) and \(x\) are the collocation and field boundaries respectively, and \(r\) is the distance between collocation and field points. The kernels \(H_{ij}\) and \(G_{ij}\) are the fundamental solutions for tractions and displacements respectively.

The boundary element method has been used for the analysis of many engineering applications. Early work on the solution of Boundary Integral Equation (BIE) was reported by Fox and Goodwin in 1953 [257]; however the name Boundary Element Method (BEM) was first coined by Brebbia in [258]. The first practical implementations were presented by Brebbia and Dominguez [259], Banerjee and Butterfield [260], Cruse [261] and others.

Consider the domain to be modeled using BEM, \(\Omega_{BE}\), as shown in Figure 7.2a, with the boundary \(\partial\Omega_{BE} = \Gamma_{BE}\); and \(\Gamma_{BE} = \Gamma_{Bu} \cup \Gamma_{Bt}\) where \(\Gamma_{Bu}\) is the part of boundary with prescribed displacements and \(\Gamma_{Bt}\) is the part of boundary with prescribed traction.

In the absence of body forces, the displacement boundary integral equation can be written as [259]:

\[
c_{ij}(\xi)u_j(\xi) + \int_{\Gamma_{BE}} H_{ij}(\xi, x)u_j(x) d\Gamma_{BE} = \int_{\Gamma_{BE}} G_{ij}(\xi, x)t_j(x) d\Gamma_{BE}
\tag{7.11}
\]

The value of \(c_{ij}(\xi)\) depends on the boundary geometry [259], for \(\xi\) on a smooth boundary it has been proven to be \(c_{ij}(\xi) = (1/2)\delta_{ij} [262]\); \(u_j(x)\) and \(t_j(x)\) represent the boundary displacements and traction respectively. Discretizing \(\Gamma_{BE}\) using quadratic boundary elements, we get the boundary mesh in Figure 7.2a. For the discretized domain, equation (7.11) may be written in the matrix form as:

\[
[H]\{u\} = [G]\{t\}
\tag{7.12}
\]
where \( \mathbf{u} \) and \( \mathbf{t} \) are nodal displacement and traction vectors, respectively. For \( N_{BE} \) boundary elements, \( \mathbf{H} \) and \( \mathbf{G} \) present the integrals of the fundamental solutions in equation (7.10), using quadratic shape functions, \( h_j(s) \), as follows:

\[
[H] = \sum_{e=1}^{N_{BE}} \sum_{i=1}^{3} \left[ \int_{\Gamma_e} H_{ij}(\xi, x) h_j(s) d\Gamma_e \right] + c_{ij}(\xi) \tag{7.13a}
\]

\[
[G] = \sum_{e=1}^{N_{BE}} \sum_{i=1}^{3} \left[ \int_{\Gamma_e} G_{ij}(\xi, x) h_j(s) d\Gamma_e \right] \tag{7.13b}
\]

A standard BEM procedure following equation (7.12) is to rearrange the equation such that the unknown displacements and tractions are on the left hand side, with the final form:

\[
[A] \{x\} = \{b\} \tag{7.14}
\]
where $A$ has columns from $H$ and $G$, $x$ is the mixed unknowns vector and $b$ is the vector containing prescribed values of tractions and displacements.

### 7.2.3 FEM formulation for non-local damage

Consider the domain to be modeled using FEM in Figure 7.2b. The domain is denoted as $\Omega_{FE}$ with the boundary $\partial \Omega = \Gamma_{FE}$; and $\Gamma_{FE} = \Gamma_{Fu} \cup \Gamma_{Ff}$ where $\Gamma_{Fu}$ is the part of boundary with prescribed displacements and $\Gamma_{Ff}$ is the part of boundary with prescribed forces. Discretizing $\Omega_{FE}$, we get the finite element mesh in Figure 7.2b. The discretization can be done using any type of finite elements, however, to insure compatibility with BEM, we employ quadratic elements to achieve continuity with the quadratic boundary elements.

For the discretized FEM domain in Figure 7.3a, the integrals in equation 7.7 reduce to summations over integration points lying within the radius $l_c$, as shown in Figure 7.3b. The non-local damage variable can be calculated as follows [121]:

$$D_{nl}(X_p) = \frac{\sum_{i=1}^{N_{lc}} \Phi(X_p - X_{lc}) D(X_{lc})}{\sum_{i=1}^{N_{lc}} \Phi(X_p - X_{lc})}$$ (7.15)

where $X_p$ denote the current integration (gauss) point and $N_{lc}$ denote the number of all other integration points $X_{lc}$ lying within the radius $l_c$. Using the effective constitutive stiffness tensor from equation (7.4) and substituting the damage variable with the non-local damage $D_{nl}(X_p)$ calculated from equation (7.15), the modified constitutive law from equation (7.4) can be rewritten as:

$$\bar{C} = (1 - D_{nl})C.$$  

The Galerkin approximation of the weak form of equation (7.1), using $N$ and $B$ to denote shape functions and their derivatives, reduces to:

$$\sum_{e=1}^{N_{FE}} \int_{\Omega_e} B^T \bar{C} B \Omega_d u + \sum_{e=1}^{N_{FE}} \int_{\Omega_e} N^T t \Omega_d + \sum_{e=1}^{N_{FE}} F$$ (7.16)

in the absence of body forces, with $t$ and $F$ denoting tractions and point loads on $\Gamma_{Ff}$ for $N_{FE}$
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number of finite elements, equation (7.16) can be rewritten in the following global matrix form:

\[
[\bar{K}] \{u\} = \{P\}
\]

(7.17)

where \( \bar{K} \) is the damaged stiffness matrix, \( u \) is the nodal displacement vector and \( P \) is the nodal force vector summing the effects of all the boundary tractions and forces.

Figure 7.3: Non-local damage calculation in the FEM subdomain. Figure (a) shows the integration points in the FEM mesh and Figure (b) shows the calculation of damage at an integration point by averaging over a circle with radius of \( l_c \).

7.3 Coupling procedure

7.3.1 Strong coupling using Lagrange multipliers

Consider the coupling of FEM and BEM domains as in Figure 2; the dotted line denotes the interface boundary \( \Gamma_I \). The domain boundaries are defined as \( \partial \Omega = \Gamma_{FE} \cup \Gamma_I \) and \( \partial \Omega_{BE} = \Gamma_{BE} \cup \Gamma_I \). The variables on the interface zone are: \( u_{IF} \) and \( F_{IF} \) which represent displacement and forces on the FEM side of the interface, respectively; and \( u_{IB}, t_{IB} \) which represent displacement and traction.
on the BEM side of the interface, respectively. In the following, for any state variable the subscripts
$B$ and $F$ denote BEM and FEM domain variables respectively.

The potential energy of the coupled system can be written as:

$$
\Pi_{tot} = \Pi_F + \Pi_B + \Pi_I
$$

where $\Pi_{tot}$ is the total energy of the whole system; $\Pi_F$ and $\Pi_B$ represents the potential energies of
the FEM and BEM domains (excluding the interface), respectively, and $\Pi_I$ represents the potential
energy components from the interface zone. For any elastic domain $\Omega$, the potential energy $\Pi_\Omega$
can be written as:

$$
\Pi_\Omega = \frac{1}{2} \int_\Omega \sigma_{ij} \varepsilon_{ij} \, d\Omega - \int_{\Gamma} u_i t_i \, d\Gamma
$$

Using the matrix form derived in equation (7.17), equation (7.19) can be written for the FEM
domain as:

$$
\Pi_F = \frac{1}{2} u_f^T K_F u_f - u_f^T P
$$

In order to write $\Pi_B$ similarly, we first need to find the BEM stiffness matrix $K_B$ which can be
obtained by rearranging equation (7.12) to be in the following form:

$$
[M][G]^{-1}[H]\{u_B\} = [M]\{t\}
$$

where $M$ is a matrix with similar structure as the standard consistent mass matrix, defined in terms
of $h_i$, the BEM 1D shape functions, as follows [142]:

$$
M_{ij} = \sum_{e=1}^{N_{BE}} \int_{\Gamma^e} h_i h_j d\Gamma^e
$$
implying that:

$$[M]\{t\} = \{F_B\} \quad (7.23)$$

where \(F_B\) is the boundary force vector. Furthermore, the stiffness matrix \(K_B\), can be defined as:

$$K_B = [M][G]^{-1}[H] \quad (7.24)$$

Hence, \(\Pi_B\) can be written as:

$$\Pi_B = \frac{1}{2}u_B^KK_Bu_B - u_B^TF_B \quad (7.25)$$

For strong coupling, using Lagrange multipliers, \(\Lambda\), \(\Pi_I\) is written as [263]:

$$\Pi_I = \Lambda^T(u_{IB} - u_{IF}) + u_{IB}(\Lambda^T - F_{IB}) + u_{IF}(-\Lambda^T + F_{IF}) \quad (7.26)$$

where \(F_{IB}\) are interface BEM forces. Using the definitions from equations (7.18), (7.20), (7.25) and (7.26), the variation of \(\Pi_{tot}\) is defined as:

$$\delta\Pi_{tot} = \delta u_F^T(Ku_F - P) + \delta u_B^T(K_Bu_B - F_B) +$$

$$\delta\Lambda^T(u_{IB} - u_{IF}) + \delta u_{IB}(\Lambda^T - F_{IB}) + \delta u_{IF}(-\Lambda^T + F_{IF}) \quad (7.27)$$

Applying the minimum potential energy theory to equation (7.27), we arrive at the residual vector

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\[ r = \{ r(u_F), r(u_B), r(u_{IB}, u_{IF}), r(F_{IB}), r(F_{IF}) \}^T, \]
and the following system of equations:

\[ r(u_F) = \bar{K} u_F - P = 0 \] (7.28a)
\[ r(u_B) = K_B u_B - F_B = 0 \] (7.28b)
\[ r(u_{IB}, u_{IF}) = u_{IB} - u_{IF} = 0 \] (7.28c)
\[ r(F_{IB}) = \Lambda^T - F_{IB} = 0 \] (7.28d)
\[ r(F_{IF}) = -\Lambda^T + F_{IF} = 0 \] (7.28e)

Equation (7.28a) is the same as equation (7.17). Equation (7.28b) is the BEM system of equations. In order to avoid the inversion of the \( G \) matrix required to compute \( K_B \) as in equation (7.24), we can reverse the operations done in equation (7.21) to get back to the form in equation (7.12) which is represented in the final form in equation (7.14). Equation (7.28c) implies compatibility, or strong displacement coupling, between the FEM and BEM domains. The Lagrange multipliers from equations (7.28d) and (7.28e) can be eliminated; and using the relation from equation (7.23) we arrive at the following equilibrium equation:

\[ [M] \{ t_{IB} \} = F_{IB} = -F_{IF} \] (7.29)

Compiling equations (7.28), (7.29) and (7.14) we arrive the final system of equations \( r = \bar{A}\bar{x} - \bar{b}, \) defined as:

\[ \begin{bmatrix} \bar{r}_0 \\ \bar{r}_1 \end{bmatrix} = \begin{bmatrix} \bar{K} & \bar{\Lambda}_{01} \\ \bar{\Lambda}_{10} & \bar{\Lambda}_{11} \end{bmatrix} \begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \end{bmatrix} - \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \end{bmatrix} \] (7.30)
where the components of the matrices in equation (7.30) are defined as follows:

\[
\bar{A}_{01} = \begin{bmatrix} 0 & L_1 \end{bmatrix} ; \quad \bar{A}_{10}^T = \begin{bmatrix} 0 & L_2 \end{bmatrix}
\]

(7.31a)

\[
\bar{A}_{11} = \begin{bmatrix} A & 0 \\ L_3 & L_4 \end{bmatrix}
\]

(7.31b)

\[
\bar{x}_0 = \begin{bmatrix} u_F \\ u_{IF} \end{bmatrix} ; \quad \bar{b}_0 = \begin{bmatrix} F \\ 0 \end{bmatrix}
\]

(7.31c)

\[
\bar{x}_1 = \begin{bmatrix} x \\ u_{IB} \\ t_{IB} \\ F_{IF} \end{bmatrix} ; \quad \bar{b}_1 = \begin{bmatrix} b \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

(7.31d)

where the submatrix 0 is used to denote a vector or a matrix of zeros with the appropriate size and the submtrices \(L_1, L_2, L_3\) and \(L_4\) are defined as:

\[
L_1 = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} ; L_2 = \begin{bmatrix} 0 & -I \\ 0 & 0 \end{bmatrix}
\]

(7.32a)

\[
L_3 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & M \end{bmatrix} ; L_4 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}
\]

(7.32b)

### 7.3.2 Solution procedure

In the absence of damage, the system of equations in equation (7.30) is linear and the solution is straight forward and converges in one solution step. However, in the presence of damage, the stiffness matrix in equation (7.17) becomes a function of the displacement and the system is non-linear. In order to solve the non-linear system of equations using a Newton-Raphson method we
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need to solve the system \( J \delta \tilde{x} = -r \) which is detailed as follows:

\[
\begin{bmatrix}
T & \tilde{A}_{01} \\
\tilde{A}_{10} & \tilde{A}_{11}
\end{bmatrix}
\begin{bmatrix}
\delta \tilde{x}_{0}^{i+1} \\
\delta \tilde{x}_{1}^{i+1}
\end{bmatrix}
= -
\begin{bmatrix}
r_{0}^{i} \\
r_{1}^{i}
\end{bmatrix}
\]

(7.33)

where the \( J \) is the Jacobian of the system \( J = \frac{\partial r}{\partial \tilde{x}} \), the system in equation (7.33) is solved iteratively and the solution vector \( \tilde{x}^{i+1} \) at iteration \( i \) is computed as a function of the increment vector \( \delta \tilde{x}^{i+1} \) as follows:

\[
\tilde{x}^{i+1} = \tilde{x}^{i} + \delta \tilde{x}^{i+1}
\]

(7.34)

for iteration \( i \) until convergence. Due to the localized non-linearity in the FEM domain only, the Jacobian matrix \( J \) has similar components as \( \tilde{A} \) except that the damage stiffness \( \tilde{K} \) in \( \tilde{A} \) is replaced by the tangent matrix of the FEM component \( T \), which is defined as follows:

\[
T = \frac{\partial r(u_{F})}{\partial u_{F}} = \tilde{K}(u_{F}) + \frac{\partial \tilde{K}(u_{F})}{\partial u_{F}} u_{F}
\]

(7.35)

in order to compute the second term in equation (7.35) we need to compute the derivative of the stiffness matrix, in equation (7.16), with respect to nodal displacements. In the case of small deformations, the shape functions \( N \) and their derivatives \( B \) are not functions of nodal displacements. In our case, where we have a material non-linearity, the the stiffness tensor \( \tilde{C} \) is function of damage and hence displacements. Therefore, the tangent matrix \( T \) can be computed as:

\[
T = \sum_{e=1}^{N_{FE}} \int_{\Omega_{e}} B^{T} \tilde{C} B \ d\Omega_{e} + \sum_{e=1}^{N_{FE}} \int_{\Omega_{e}} B^{T} \frac{\partial \tilde{C}}{\partial u_{F}} B \ d\Omega_{e} u_{F}
\]

(7.36)

where the effective stiffness tensor derivative is obtained by using the chain rule:

\[
\frac{\partial \tilde{C}}{\partial u_{F}} = \frac{\partial \tilde{C}}{\partial D} \frac{\partial D}{\partial u_{F}} = - \frac{\partial D}{\partial u_{F}} C
\]

(7.37)
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The system in equation (7.33) can be implemented in a non-linear solution algorithm to solve the problem. However, a direct implementation would be computationally expensive because it requires the solution of the whole system of equations in each iteration of the non-linear solution.

Different solutions may be available to improve the solution of equations process. Usually, iterative solvers are more efficient to deal with finite element sparse and symmetric systems [132, 133, 264]. However, iterative solvers may be less applicable in our case because of the BEM components which are fully populated and unsymmetric [265].

In this study, we use the Schur complement [266] of $J$ to solve the problem more efficiently by taking into account that only the FEM matrices are affected by the nonlinear damage component. Hence, before every nonlinear loading step, one can store the LU factors corresponding to the BEM part and solve the BEM part of the matrix quickly. In details, we reduce the size of the system of equations by condensing $\bar{x}_1$.

First, we rewrite the second row in equation (7.33), to make $\delta \bar{x}_1$ the subject, as follows:

$$
\delta \bar{x}_1^{i+1} = \tilde{A}^{-1}_{11} \left[ -r_1^i - \tilde{A}_{01} \delta \bar{x}_0^{i+1} \right] \tag{7.38}
$$

Substituting $\delta \bar{x}_1$ from equation (7.38) into the first row in equation (7.33) to get the final residual $r^*$ as follows:

$$
S \delta \bar{x}_0^{i+1} = -r^* \tag{7.39}
$$

where $S$ is the Schur complement of $J$ with respect to $\tilde{A}_{11}$ defined as:

$$
S = T - \tilde{A}_{01} \tilde{A}_{11}^{-1} \tilde{A}_{10} \tag{7.40}
$$

and $r^*$ is the new residual vector defined as:

$$
r^* = r_0^i - \tilde{A}_{01} \tilde{A}_{11}^{-1} r_1^i \tag{7.41}
$$

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In the practical implementation of this approach, the inverse of the $\tilde{A}_{11}$ is required to be computed at each step. In order to perform this in a computationally efficient way, we can compute and store the LU decomposition of $\tilde{A}_{11}$ and get the solutions for $\tilde{A}_{01}\tilde{A}_{11}^{-1}\tilde{A}_{10}$ and $\tilde{A}_{01}\tilde{A}_{11}^{-1}r_1$ by forward and backward substitution whenever required. Algorithm (4) details the monolithic solution approach including the Schur complement calculations.

Algorithm 4 Non-linear analysis algorithm - Monolithic solution

1: Initialize all variables
2: Calculate BEM influence matrices: $H, G$ Eq. (7.13)
3: Calculate $M$ Eq. (7.22)
4: Calculate and store LU decomposition of $\tilde{A}_{11}$
5: for $j=1:N_{load}$ do Loop 1: Loop on load steps
6: Update $\tilde{b}$ and $\tilde{A}$, then compute the residual $r$ Eq. (7.30)
7: $i = 0$
8: while $\|\{r\}\|_2/\|\{x_0,x_1\}\|_2 < $tolerance do Loop 2: Non-linear solution loop
9: Calculate $r^*$, using LU factors Eq. (7.41)
10: Solve $S\delta x_0^{i+1} = -r^*$ using LU factors Eq. (7.39)
11: Solve $\tilde{A}_{11}\delta x_1^{i+1} = \left[-r_1^{i} - \tilde{A}_{01}\delta x_0^{i+1}\right]$, using LU factors, Eq. (7.38)
12: Assemble global $\delta \tilde{x}^{i+1} = \{\delta x_0^{i+1}; \delta x_1^{i+1}\}$
13: Update $\tilde{x}^{i+1} = \tilde{x}^i + \delta \tilde{x}^{i+1}$, Eq. (7.34)
14: Calculate the residual $\{r^{i+1}\}$ Eq. (7.30)
15: $i = i + 1$
16: end while
17: end for

7.3.3 Adaptive damage growth

The aforementioned BEM/FEM approach is useful in describing damage growth in large scale structures, where the damage is localized in a small portion(s) of the structure. The damage susceptible region is the finite elements domain. The smaller the portion covered by FEM, the more computationally efficient the presented model will be. However, when the damage approaches the boundaries of the FEM domain, the solution is no longer valid because areas from the BEM domain would be expected to experience damage as well.

To this end, an adaptive nonlocal damage growth is proposed, such that whenever the damage
approaches the interface boundary $\Gamma_I$, new FEM elements are added and the interface zone between FEM and BEM is rediscretized. In this study, this process is referred to as adaptive remeshing. Note that since $\Gamma_I$ is growing the discretized system of equations in equation (7.33) varies as well. Hence the system of equations becomes larger with the damage growth.

Direct implementation of this approach may violate the averaging properties of the nonlocal integral operator [267]. One possible remedy to this problem is to keep a damage free zone along the interface, in which a layer of undamaged elements is used as a buffer zone. This procedure is demonstrated graphically in Figure 7.4; as the red integration point exhibits a value of damage larger than zero, nodes on the interface boundary $\Gamma_I$ will lie within the radius of the characteristic length of non-local damage $l_c$ and remeshing is required. The vector $l_s$ denotes the list of distances between each integration point and nodes on the interface boundary.

After remeshing, the boundary element influence matrices $[H]$ and $[G]$ need to be recalculated. The flow chart in Figure 7.5 demonstrates the solution of the non-linear problem incorporating the adaptive remeshing process.

7.4 Convergence of the coupled solution

In order to verify the proposed model and understand the behaviour of the interaction between the BEM and FEM domains, we perform several convergence studies. These studies include the linear and non-linear behaviour of the model. For each convergence study we investigate the effect of refining the FEM mesh only while keeping BEM mesh coarse, refining the BEM mesh only while keeping FEM coarse, and refining both meshes simultaneously. The benchmark problem used in these studies is that of a large plate with a small centered circular hole under uniform tension $\sigma_\infty$, known as the Kirsch problem. The plate, of $(10m \times 10m)$ size and a $1m$ diameter hole, is shown in Figure 7.6a. This problem has been investigated thoroughly in the elasticity and fracture mechanics literature [268, 269]. The concrete material parameters are chosen as: Young’s modulus $E = 30,000N/m^2$ and Poisson’s ratio $\nu = 0.2$. 

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First, we analyze the behaviour of the linear elastic model, for which the convergence of two parameters is studied. The first is Stress Concentration Factor (SCF) at the top of the hole, denoted by point A in Figure 7.6a, for which the reference is the analytical value $\sigma_{xa} = 3\sigma_\infty$. The second parameter is the error in displacements computed over the whole domain. For this purpose, displacement measurement points are taken at predefined points everywhere within the problem domain (in both the BEM and FEM subdomains), as shown in Figure 7.6b; the reference solution

Figure 7.4: Illustration of adaptive damage growth. The interface $\Gamma_I$ grows as damage approach the boundaries. Note that a damage-free finite element zone is kept between damaged and non damaged regions.
is obtained by a very fine finite element mesh (element size of 0.075m). The results of the convergence studies are presented in Figure 7.7. Note that we employ quadratic BEM elements and quadratic FEM elements.

It can be seen from Figure 7.7a that refining the local FEM mesh leads to convergence of the local stress even if the BEM mesh is kept coarse; however, as expected, refining the BEM mesh only does not affect the local SCF results. Furthermore, analyzing Figure 7.7b, it can be seen that refining the BEM mesh only leads to convergence of displacement over the global domain even if FEM mesh is still coarse. Therefore, we conclude that a local FEM mesh is efficient in capturing local stress concentrations. However, if high accuracy of fields is required all over the
7.4. CONVERGENCE OF THE COUPLED SOLUTION

Figure 7.6: Modeling the Kirsch problem. Figure (a) shows the geometry and boundary conditions, the plate is supported by rollers on top and bottom and subject to far-field stress $\sigma_{\infty} = 0.157\text{Pa}$. Stress intensity factor $\sigma_{\infty} A$ will be calculated at point A. Figure (b) shows the coupled FEM-BEM mesh used in the analysis; the blue points are the points at which displacement is calculated for the convergence study and the black points are the boundary element nodes.

domain, which is the case with the proposed adaptive remeshing approach, then BEM and FEM finer meshes are required altogether.

Figure 7.7: Convergence of the coupled BEM/FEM model to Kirsch problem

The second set of studies involves the formation of damage at the stress concentration point A. The fine FEM model is used as a reference solution. The damage model used is the one described in equation (7.5) where its parameters are chosen as follows: damage threshold strain $\varepsilon_D = 0.01$, \[\varepsilon_D = 0.01,\]
CHAPTER 7. ADAPTIVE MODELING OF DAMAGE GROWTH USING A COUPLED FEM/BEM APPROACH

\[ a = 0.4, \ b = 2 \times 10^4, \ l_c = 0.35m. \]

The damage propagation results of the reference FEM mesh and the coupled FEM-BEM are shown in Figure 7.8. The convergence of two parameters is investigated as well. The first is the maximum value of damage, which is similar to the localized SCF in the absence of damage studied previously; the second is the displacement error all-over the domain. The reference in both cases is the results from the fine FEM mesh from Figure 7.8a. The results of convergence are presented in Figure 7.9.

Similar to the linear elastic model results, we can see that refining the FEM mesh only leads to accurate local results only, while refinement of BEM mesh is required to calculate accurate results over the whole domain.

![Figure 7.8: Damage growth in the Kirsch problem](image)

The results of the convergence studies verify the proposed coupling approach. Moreover, we conclude from the convergence plots that one has to refine both the FEM and BEM mesh in order to get accurate damage propagation behaviour.
7.5. ADAPTIVE DAMAGE GROWTH EXAMPLES

The numerical examples in this section demonstrate the advantage of the proposed technique to model efficiently adaptive damage growth. The BEM advantage is realized in the linear elastic domain while the local behaviour of damage is accurately captured by a superimposed local FEM mesh. The BEM and FEM domain are coupled using Lagrange multipliers as detailed in Section 3.

The first example is the L-bracket problem investigated for damage failure. The second example demonstrates the propagation of two symmetric cracks in the four-point bending test, as reported in the literature [270].

7.5.1 L-bracket problem

This problem was investigated for damage failure by James and Waisman [271]. The geometry and loading of the bracket are demonstrated in Figure 7.10. The concrete material parameters are taken as: Young’s modulus $E = 30,000N/m^2$, Poisson’s ratio $\nu = 0.2$; and the damage model parameters are: damage threshold strain $\varepsilon_D = 10^{-4}$, $a = 0.4$, $b = 2 \times 10^4$, $l_c = 0.25m$. The adaptive

Figure 7.9: Convergence of the coupled BEM/FEM damage model to Kirsch problem

(a) Convergence of Damage at point A

(b) Convergence of displacements at the points defined in Figure 7.6b

Figure 7.9: Convergence of the coupled BEM/FEM damage model to Kirsch problem
CHAPTER 7. ADAPTIVE MODELING OF DAMAGE GROWTH USING A COUPLED FEM/BEM APPROACH

propagation of damage is presented in Figure 7.11. The results match the damage propagation in [271]. As expected the damage is initiated at the re-entrant corner and propagates into a mode I configuration.

![Figure 7.10: L-Bracket geometry, boundary conditions and loads. The Bracket is fixed at the top edge and a displacement $u_z$ is applied at the middle of the right edge.](image)

A force-displacement curve, describing the global response of the structure is shown in Figure 7.12. We illustrate the response for local and nonlocal damage, considering three different FEM meshes, in Figures 7.12a and 7.12b, respectively. The force and displacement are recorded at the application point of the displacement. It can see from the results in Figure 7.12 that the proposed non-local damage model reliably describes the behaviour of the structure while the local model is sensitive to the mesh size. In this example, the tolerance required for the convergence of the non-linear solution in algorithm 4 is set to $1 \times 10^{-11}$.

7.5.2 Beam with two notches subjected to four point bending

The beam in Figure 7.14 was first analyzed experimentally and numerically by Bocca [270], and later by many authors in the field of fracture mechanics [272]. The forces $P$ are applied to a rigid loading frame which transfers the load to points A,B,C and D. Horizontal translation is
7.5. ADAPTIVE DAMAGE GROWTH EXAMPLES

Figure 7.11: Snapshots of adaptive damage propagation for the L-Bracket problem.

constrained at points A,B,C and D. Two initial notches are prescribed along the beam’s centerline, as shown in Fig. 7.14. All the material and damage parameters are chosen to be the same as in the previous example.

The initial finite element meshing of the problem close to the notches is presented in Figure 7.15. The problem is modeled using the proposed adaptive damage growth approach. The damage growth is presented in Figure 7.16. The damage propagation results are demonstrated in Figure 7.16, where two cracks emanate from the notches and propagate into the beam. The results conform with the crack propagation path presented in [270, 272]. Force-displacement plots demonstrating softening of the structure for local and non-local damage are presented in Figures 7.13a and 7.13b. Similar to observations from the previous example, we can see the consistency the of the non-local
Figure 7.12: Force-displacement curves of the L-Bracket problem using local and nonlocal damage models. Force and displacement are recorded at the application point of the force. Note that in both local and non-local models, damage initiates at displacement of (0.06 m); however, the later behaviour is different.

damage model behavior versus the sensitivity of the local damage model results to finite element size. In order to get the complete damage propagation in Figure 7.16, the tolerance required for the convergence of the non-linear solution in algorithm 4 is set to $1 \times 10^{-15}$. The tolerance is
taken lower than the previous example to decrease the accumulated error and hence allow for more accurate computations of elastic fields at later loading stages.

Figure 7.13: Force-displacement curves of the beam problem using local and nonlocal damage models. Force and displacement are recorded at the application point of the force. Note that in both local and non-local models, damage initiates at displacement of (0.04 m); however, the later behaviour is different.
CHAPTER 7. ADAPTIVE MODELING OF DAMAGE GROWTH USING A COUPLED FEM/BEM APPROACH

Figure 7.14: Four point beam bending test geometry and boundary conditions. All dimensions are given in meters.

Figure 7.15: Initial meshing of the four point bending problem

Figure 7.16: Snapshots of adaptive damage propagation for the four point bending problem
7.5.3 Discussion of the results

Continuum damage mechanics provides a nice framework for modeling fracture processes in structures. Modeling the damage of the structures in the examples using FEM is computationally expensive; especially at later stages of crack propagation which require fine discretization of the FEM domain to be able to model the stress fields around the crack accurately. Using the proposed coupling of BEM with FEM to model localized damage in structures reduces the computational cost of the problem. Moreover, using very fine finite element meshes, which may be required as in the example in section 7.5.2, is more feasible with the coupled system.

The presented coupling FEM and BEM model is proved to be successful in modeling linear and non-linear continuum damage behaviour. The model is able to track multiple damage/fracture initiation and propagation in structures. Furthermore, the model captures softening behaviour due to damage which is consistent with previous results from the literature for both local and non-local damage.

7.6 Summary and conclusion

An adaptive approach to model the damage growth is presented in this paper. Linear elastic BEM is coupled with FEM model which incorporates non-local damage to describe material failure problems. The presented coupling is a BEM hosted domain decomposition approach, obtained by Lagrange multiplies that enforce compatibility between BEM and FEM. BEM offers the flexibility of modeling large domains efficiently while the nonlinear damage growth is accurately accounted by a local FEM mesh.

As damage grows, the size of the finite element domain grows adaptively to capture more parts of the BEM domain that are getting past their linear limits. The formulation of the non-linear problem is explained in this paper. In this implementation, Newton’s approach is used to monolithically solve the coupled non-linear system of equations. In order to use the advantage of the localized
CHAPTER 7. ADAPTIVE MODELING OF DAMAGE GROWTH USING A COUPLED FEM/BEM APPROACH

non-linearity in the FEM part of the domain, the system of equations size is reduced by getting the Schur complement with respect to FEM variables only. The solution of equations is performed on the reduced system of equations and the algorithm is set to iterate until the convergence of the global residual vector.

The model is validated for linear and non-linear problems by a set of convergence studies based on the analysis of the Kirsch plate problem. The reference solution is taken to be a fine finite element mesh. Then, the proposed model is used to analyze two benchmark fracture mechanics problems. The proposed coupling model successfully detected the initiation of the fracture/damage and the damage propagation is in agreement with previous results from the literature. The softening behaviour is examined for local and non-local damage and the results conform with the expected softening behaviour in both cases.

The advantage of the proposed model is that it provides a solution to the analysis of the fracture mechanics problems with a reduced computational cost. The coupling of FEM and BEM allows the use of very fine FEM around in the non-linear behaviour zone while using the advantage of modeling the rest of the domain, which behaves linearly, using BEM. This approach is expected to be useful in analyzing problems in which large structures are subject to localized damage or cracking. As shown in the numerical results, the proposed model is able to capture both the local damage growth as well as its global effect on the structure.
Chapter 8

Summary and conclusions

In this thesis, we attempted to improve several aspects of the continuum damage mechanics (CDM) formulations for modeling fracture in geomaterials. The thesis can be divided into three main parts.

8.1 Non-local damage transport

The first part is a novel non-local damage transport formulation for modeling fracture in porous media. The mathematical basis of the model were derived analytically from thermodynamic principles and volumetric homogenization in Chapter 2. The thermodynamically consistent formulation is used to derive the new state laws and regularization equation are derived and the non-local damage poroelasticity constitutive relationships are provided. The solid-damage and fluid-viscous flow energy dissipation functions are derived analytically. Chapters 3 and 4 provide a three-field implementation of the non-local transport damage model and Chapter 5 provides a four-field implementation of the model. The three-field formulation couples the non-local transport and non-local deformation equation into a single scalar equation by assuming that: 1) non-local damage and transport have a similar length scale, and 2) the driving local stress/strain measures controlling damage and permeability evolution are the same.

Chapter 3 provides a strain-based definition of the non-local damage transport model. The for-
CHAPTER 8. SUMMARY AND CONCLUSIONS

The formulation of the non-linear mixed finite element algorithm is presented and the analytical derivation of the Jacobian matrix is shown. The model is used to analyze 1d, 2d hydraulic fracture and 2d consolidation examples. The results show that the developed non-local damage transport is capable of capturing the essential features of modeling consolidation and hydraulic fracture. The results also mesh-independence.

Chapter 4 provides a stress-based definition of the non-local damage transport model. The formulation of the non-linear mixed finite element algorithm is presented and the analytical derivation of the Jacobian matrix is shown. The developed algorithm is used to analyze the different energy dissipation mechanisms associated hydraulic fracture under different configurations of material and loading parameters. The numerical simulations examine in details the effect of incorporating non-local transport in hydraulic fracture modeling.

Chapter 5 provides a decoupled damage-permeability evolution definition of the non-local damage transport model. The formulation of the non-linear mixed finite element algorithm is presented and the analytical derivation of the Jacobian matrix is shown. The numerical results show the significance of the four-field formulation. First, modeling hydraulic fracture with a transport length scale higher than damage length scale lead to the formation of a fluid pool that dissipates fluid through the fracture wall and hinders hydraulic fracture propagation. The elevated transport length scale indicates the presence of a wide inter-granular fluid-transport network. Second, the four-field formulation is used to model different configurations of driving stresses for the evolution of damage and permeability in consolidation problems. The results show that if the permeability increases inside a shear crack, the crack will be water-filled and a hydraulic-fracture-like phenomenon that accelerates damage growth.

8.2 CDM formulation for hydraulic fracture of glaciers

The second part is a novel poro-visco-elastic damage approach developed to analyze hydraulic fracture of glaciers. The presence of water-filled crevasses is known to increase the penetration
8.3. ADAPTIVE FEM/BEM DOMAIN DECOMPOSITION APPROACH FOR DAMAGE GROWTH

depth of crevasses and this has been hypothesized to play an important role controlling iceberg calving rate. Here, we develop a continuum-damage-based poro-mechanics formulation that enables the simulation of water-filled basal and/or surface crevasse propagation. The formulation incorporates a scalar isotropic damage variable into a Maxwell-type viscoelastic constitutive model for glacial ice and the effect of the water pressure on fracture propagation using the concept of effective solid stress. We illustrate the model by simulating quasi-static hydro-fracture in idealized rectangular slabs of ice in contact with the ocean. Our results indicate that water-filled basal crevasses only propagate when the water pressure is sufficiently large and that the interaction between simultaneously propagating water-filled surface and basal crevasses can have a mutually positive influence leading to deeper crevasse propagation which can critically affect glacial stability. Therefore, this study supports the hypothesis that hydraulic fracture is a plausible mechanism for the accelerated breakdown of glaciers.

8.3 Adaptive FEM/BEM domain decomposition approach for damage growth

The third part is a coupled Boundary Element Method (BEM) and Finite Element Method (FEM) for modeling localized damage growth in structures. BEM offers the flexibility of modeling large domains efficiently while the nonlinear damage growth is accurately accounted by a local FEM mesh. An integral-type nonlocal continuum damage mechanics with adapting FEM mesh is used to model multiple damage zones and follow their propagation in the structure. Strong form coupling, BEM hosted, is achieved using Lagrange multipliers. Since the non-linearity is isolated in the FEM part of the system of equations, the system size is reduced using Schur complement approach, then, the solution is obtained by a monolithic Newton method that is used to solve both domains simultaneously. The coupled BEM/FEM approach is verified by a set of convergence studies, where the reference solution is obtained by a fine FEM. In addition, the method is applied
to multiple fractures growth benchmark problems and shows good agreement with the literature.

8.4 Future work

The work presented in this thesis can be extended in several different directions. Possible future research projects are:

- Calibrating the additional constitutive parameters in the thermodynamic model via a more physics enriched model e.g. DEM or multiscale model
- Modeling Hydraulic Fracture in Floating Ice Shelves
- Adaptive Domain Decomposition for Damage Propagation in Poroelastic Media
- Improving finite element formulation to avoid spurious pressure solution
- Linking the developed model to tomographic data to lead to Data Driven Geomechanics modeling
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Appendix A

Derivatives used in the Jacobian matrix in the 3-field formulation

A.1 General derivatives

\[ \frac{\partial \alpha(D)}{\partial D} = -\frac{K}{K_s} \]  
(A.1)

\[ \frac{\partial 1/(M(D))}{\partial D} = \frac{2\alpha(D)\frac{\partial K}{\partial D}}{K^u - K(D)} - \frac{K\alpha(D)^2}{[K^u - K(D)]^2} \]  
(A.2)

\[ \frac{\partial D}{\partial \tilde{\kappa}} = \frac{\partial \tilde{\xi}_{eq}}{\partial \tilde{\kappa}} \]  
(A.3)

\[ \frac{\partial \alpha(D)}{\partial \tilde{\kappa}} = \frac{\partial \alpha(D)}{\partial D} \frac{\partial D}{\partial \tilde{\kappa}}; \quad \frac{\partial 1/(M(D))}{\partial \tilde{\kappa}} = \frac{\partial 1/(M(D))}{\partial D} \frac{\partial D}{\partial \tilde{\kappa}} \]  
(A.4)

\[ \frac{\partial \alpha(D)}{\partial t} = \frac{\partial \alpha(D)}{\partial D} \frac{\partial D}{\partial \tilde{\kappa}} \frac{\partial \tilde{\kappa}}{\partial t}; \quad \frac{\partial 1/(M(D))}{\partial t} = \frac{\partial 1/(M(D))}{\partial D} \frac{\partial D}{\partial \tilde{\kappa}} \frac{\partial \tilde{\kappa}}{\partial t} \]  
(A.5)
A.2 STRAIN BASED MODEL

\[
\frac{\partial^2 \alpha(D)}{\partial \tilde{\kappa} \partial t} = \frac{\partial \alpha(D)}{\partial D} \frac{\partial D}{\partial \tilde{\kappa}} \frac{\partial \tilde{\varepsilon}_{eq}}{\partial \tilde{\kappa}} \frac{\partial^2 \tilde{\kappa}}{\partial t} + \frac{\partial \alpha(D)}{\partial D} \frac{\partial D}{\partial \tilde{\varepsilon}_{eq}} \frac{\partial \tilde{\varepsilon}_{eq}}{\partial \tilde{\kappa}} \frac{\partial^2 \tilde{\kappa}}{\partial t} + \frac{\partial^2 \alpha(D)}{\partial \tilde{\kappa}^2} \frac{\partial \tilde{\kappa}}{\partial \tilde{\varepsilon}_{eq}} \frac{\partial \tilde{\varepsilon}_{eq}}{\partial \tilde{\kappa}} \frac{\partial^2 \tilde{\kappa}}{\partial t} = 0 \quad (A.6)
\]

where the first, second and third term are equal to zero because the second order derivatives are equal to zero, while the second order derivative from the third term is:

\[
\frac{\partial^2 D}{\partial \tilde{\kappa} \partial \tilde{\varepsilon}_{eq}} = \frac{\partial^2 D}{\partial (\tilde{\varepsilon}_{eq})^2} \frac{\partial \tilde{\varepsilon}_{eq}}{\partial \tilde{\kappa}} \quad (A.7)
\]

Similarly, the derivative \(\frac{\partial^2 1/M(D)}{\partial \tilde{\kappa} \partial t}\) can be written as:

\[
\frac{\partial^2 1/M(D)}{\partial \tilde{\kappa} \partial t} = \frac{\partial 1/M(D)}{\partial D} \frac{\partial D}{\partial \tilde{\kappa}} \frac{\partial \tilde{\varepsilon}_{eq}}{\partial \tilde{\kappa}} \frac{\partial \tilde{\kappa}}{\partial t} \quad (A.8)
\]

### A.2 Strain based model

\[
\frac{\partial \tilde{\varepsilon}_{eq}}{\partial \varepsilon} = \begin{cases} 
\frac{\partial \chi}{\partial \varepsilon} & \text{if } \chi \geq 0 \\
0 & \text{otherwise} 
\end{cases} \quad (A.9)
\]

\[
\frac{\partial \kappa(\tilde{\varepsilon}_{eq})}{\partial \tilde{\varepsilon}_{eq}} = \left[ b_2 b_3 (\tilde{\varepsilon}_{eq})^{b_3-1} \right] \kappa_0 \quad (A.10)
\]

\[
\frac{\partial g(\tilde{\varepsilon}_{eq})}{\partial \tilde{\varepsilon}_{eq}} = c \left[ k_1 k_2 \exp(-k_2 \tilde{\varepsilon}_{eq}) \right] \quad (A.11)
\]
APPENDIX A. DERIVATIVES USED IN THE JACOBIAN MATRIX IN THE 3-FIELD FORMULATION

\[
\frac{\partial D}{\partial \tilde{\varepsilon}^{eq}} = \begin{cases} 
0 & \text{if } \tilde{\varepsilon}^{eq} \leq \varepsilon_{i}^{eq} \\
\frac{\varepsilon_{f}^{eq} \varepsilon_{i}^{eq}}{(\varepsilon_{f}^{eq})^2 (\varepsilon_{f}^{eq} - \varepsilon_{i}^{eq})} & \text{if } \varepsilon_{i}^{eq} \leq \tilde{\varepsilon}^{eq} \leq \dot{\varepsilon}^{eq} \\
0 & \text{if } \dot{\varepsilon}^{eq} \leq \varepsilon^{eq}
\end{cases}
(A.12)
\]

\[
\frac{\partial^2 D}{\partial (\tilde{\varepsilon}^{eq})^2} = \begin{cases} 
0 & \text{if } \tilde{\varepsilon}^{eq} \leq \varepsilon_{i}^{eq} \\
-2\varepsilon_{f}^{eq} \varepsilon_{i}^{eq} & \text{if } \varepsilon_{i}^{eq} \leq \tilde{\varepsilon}^{eq} \leq \dot{\varepsilon}^{eq} \\
0 & \text{if } \dot{\varepsilon}^{eq} \leq \varepsilon^{eq}
\end{cases}
(A.13)
\]

\[
\frac{\partial \varepsilon^{eq}}{\partial \kappa} = \frac{1}{b_3 b_2 \kappa_0} \left[ \frac{[\tilde{\kappa}/\kappa_0] - b_1}{b_2} \right] \left( \frac{1}{b_3} - 1 \right)
(A.14)
\]

\[
\frac{\partial^2 \varepsilon^{eq}}{\partial \kappa^2} = \frac{\left( \frac{1}{b_3} - 1 \right)}{b_3 (b_2 \kappa_0)^2} \left[ \frac{[\tilde{\kappa}/\kappa_0] - b_1}{b_2} \right] \left( \frac{1}{b_3} - 2 \right)
(A.15)
\]

A.3 Stress based model

\[
\frac{\partial \sigma^{eq}}{\partial \varepsilon} = \begin{cases} 
\frac{\partial \chi^{i}}{\partial \varepsilon} & \text{if } \chi \geq 0 \\
0 & \text{otherwise}
\end{cases}
(A.16)
\]

\[
\frac{\partial \kappa(\sigma^{eq})}{\partial \sigma^{eq}} = \left[ b_2 b_3 (\sigma^{eq})^{b_3 - 1} \right] \kappa_0
(A.17)
\]

\[
\frac{\partial g(\sigma^{eq})}{\partial \sigma^{eq}} = \begin{cases} 
0 & \text{if } \sigma^{eq} \leq g^{i} \\
(l^k)^2 (1 - k_1) \frac{1}{g^f - g^i} & \text{if } g^i \leq \sigma^{eq} \leq g^f \\
0 & \text{if } \sigma^{eq} \geq g^f
\end{cases}
(A.18)
\]
A.3. STRESS BASED MODEL

\[ \frac{\partial D}{\partial \tilde{\sigma}^{eq}} = \begin{cases} 
0 & \text{if } \tilde{\sigma}^{eq} \leq S_i \\
 \frac{S_i^i}{(\tilde{\sigma}^{eq})^2(S_i^i - S_f)} & \text{if } S_i^i \leq \tilde{\sigma}^{eq} \leq S_f^h \\\n0 & \text{if } S_f^h \leq \tilde{\sigma}^{eq} 
\end{cases} \quad (A.19) \]

\[ \frac{\partial^2 D}{\partial (\tilde{\sigma}^{eq})^2} = \begin{cases} 
0 & \text{if } \tilde{\sigma}^{eq} \leq S_i \\
\frac{-2S_i^i}{(\tilde{\sigma}^{eq})^3(S_i^i - S_f)} & \text{if } S_i \leq \tilde{\sigma}^{eq} \leq S_f^h \\\n0 & \text{if } S_f^h \leq \tilde{\sigma}^{eq} 
\end{cases} \quad (A.20) \]

\[ \frac{\partial \tilde{\sigma}^{eq}}{\partial \kappa} = \frac{1}{b_3 b_2 \kappa_0} \left[ \frac{[\tilde{\kappa}/\kappa_0] - b_1}{b_2} \right] \left( \frac{1}{b_3} - 1 \right) \quad (A.21) \]

\[ \frac{\partial^2 \tilde{\sigma}^{eq}}{\partial \kappa^2} = \frac{\left( \frac{1}{b_3} - 1 \right)}{b_3 (b_2 \kappa_0)^2} \left[ \frac{[\tilde{\kappa}/\kappa_0] - b_1}{b_2} \right] \left( \frac{1}{b_3} - 2 \right) \quad (A.22) \]
Appendix B

Analytical derivations of steady state solutions for hydraulic fracture in 1d poroelastic column

By assuming local damage, the governing PDE system in Equations (3.9) to (3.11) for one-dimensional poroelasticity reduces to:

\[ \frac{\partial}{\partial z} \left( \sigma^s z - \alpha(D) P \right) = 0 \]  \hspace{1cm} (B.1a)

\[ \frac{1}{M(D)} \frac{\partial P}{\partial t} + P \frac{\partial}{\partial t} \left( \frac{1}{M(D)} \right) + \alpha(D) \frac{\partial \varepsilon_z}{\partial t} + \varepsilon_z \frac{\partial \alpha(D)}{\partial t} - \frac{\partial}{\partial z} \left[ \kappa(\varepsilon_z) \frac{\partial P}{\partial z} \right] = 0 \]  \hspace{1cm} (B.1b)

where \( \sigma^s z \) is the solid component of the total stress. By substituting the definition of the damage evolution law from Equation (3.5) into the definition of the stress tensor for a 1d poroelastic column
in Equation (3.26) and simplifying, $\sigma_z$ can be written as:

$$
\sigma_z = \begin{cases} 
E \varepsilon_z & \text{if } \varepsilon < \varepsilon_i^{eq} \\
(a\varepsilon + b)\theta(\varepsilon^{eq} - \varepsilon_z) + ((1 - D_{max})E\varepsilon_z)\theta(\varepsilon_z - \varepsilon^{eq}) & \text{if } \varepsilon_z \geq \varepsilon_i^{eq}
\end{cases}
$$

(B.2)

where $\theta$ is the Heaviside step function, $D_{max}$ is the maximum value of damage permitted, $a$ and $b$ are constants defined as:

$$a = - \frac{D_{max}E\varepsilon_i^{eq}}{\varepsilon^{eq} - \varepsilon_i^{eq}}$$

(B.3)

$$b = -E \frac{\varepsilon_i^{eq} + (-1 + D_{max})\varepsilon_i^{eq}^2}{-\varepsilon^{eq} + \varepsilon_i^{eq}}$$

(B.4)

By substituting the stress-strain relationship in Equation (B.2) into the PDE system in Equation (B.1), assuming incompressible constituents ($\alpha = 1, M = \infty$) and steady state behaviour by dropping time derivatives, the PDE system can be written as follows.

- The equilibrium equation:

$$
\begin{align*}
\frac{E}{\varepsilon_z} & = \alpha \frac{\partial P}{\partial z} & \text{if } \varepsilon_z < \varepsilon_i^{eq} \\
\frac{\partial \varepsilon_z}{\partial z} & [\theta(\varepsilon_z - \varepsilon^{eq})E (1 - D_{max}) (1 + \varepsilon_z) + a\theta(\varepsilon^{eq} - \varepsilon_z)] & \text{if } \varepsilon_z \geq \varepsilon_i^{eq}
\end{align*}
$$

(B.5)

- fluid flow continuity equation:

$$
\frac{\partial}{\partial z} \left[ \kappa(\varepsilon_z) \frac{\partial P}{\partial z} \right] = 0
$$

(B.6)

where $\delta$ is Dirac Delta function. The boundary conditions for the poroelastic column subjected
to fluid fracturing shown in Figure are:

\[ E \varepsilon_z|_{z=H} = \bar{\sigma} \]  
\[ u_z|_{z=0} = 0 \]  
\[ P|_{z=H} = 0 \]  
\[ \kappa \frac{\partial P}{\partial z}|_{z=0} = q \]  

The derivation of the analytical solution is continued below, considering the poroelastic model (LM) and local damage with constant permeability (LDCP).

**B.1 Linear poroelastic model permeability (LM)**

In the absence of damage \((D = 0)\) and assuming constant permeability by setting \(b_2 = 0\), hence \(\kappa = \kappa_0\). By integrating Equation (B.6) twice and applying the boundary conditions indicated mentioned in Equations (B.7)-(B.10), the fluid pressure in steady state is:

\[ P(z) = \bar{q}(z-H) \]  

where \(\bar{q} = q/\kappa_0\). The pressure gradient \(\frac{\partial P}{\partial z} = \bar{q}\) can be substituted in the case of \(\varepsilon_z < k_i\) in Equation (B.5), and the linear strain \(\varepsilon_z^L\) in steady state can be derived as:

\[ \varepsilon_z^L = \frac{\alpha \bar{q}}{E}(z-H) + \frac{\bar{\sigma}}{E} \]  

**B.2 Local damage constant permeability (LDCP)**

In this model we allow the material to experience local damage while the permeability is kept constant by setting \(b_2 = 0\), hence \(\kappa = \kappa_0\). In the presence of local damage, the material points experiencing high vertical strain \(\varepsilon_z > \varepsilon_i^{eq}\) experience damage. Using this definition and the strain...
in the LM model at steady state given in Equation (B.12), we can find the value of $L$ for which $\varepsilon_z = \varepsilon_i^{eq}$ at $z = L$, as:

$$L = \frac{1}{\alpha \bar{q}} (E\varepsilon_i^{eq} - \bar{\sigma}) + H$$  \hspace{1cm} (B.13)

In order to solve the second case of Equation (B.5), ($\varepsilon_z \geq \varepsilon_i^{eq}$), we multiply both sides by $dz$ and separation of variables may be used as follows:

$$[\theta(\varepsilon_z - \hat{\varepsilon}^{eq})E(1 - D^{max})(1 + \varepsilon_z) + a\theta(\hat{\varepsilon}^{eq} - \varepsilon_z) - (a\varepsilon_z + b)\delta(\hat{\varepsilon}^{eq} - \varepsilon_z)] d\varepsilon_z = \alpha \frac{\partial P}{\partial z} dz$$  \hspace{1cm} (B.14)

where the pressure derivative on the right hand side $\frac{\partial P}{\partial z} = \bar{q}$ from Equation (B.11). Equation (B.14) can be integrated on both sides to lead to:

$$a\varepsilon_z - (a\varepsilon_z + b - (1 - D^{max})E\varepsilon_z)\theta(\varepsilon_z - \hat{\varepsilon}^{eq}) = \alpha \bar{q}z + C$$  \hspace{1cm} (B.15)

where $C$ is the integration constant which can be found using the continuity of the strain condition at $z = L$. By using $\varepsilon_z|_{z=L} = \varepsilon_L^z = \varepsilon_L^i = \varepsilon_i^{eq}$ and substituting the definition of $\varepsilon_L^z$ from Equation (B.12), and also noting that $\theta(\varepsilon_i^{eq} - \hat{\varepsilon}^{eq}) = 0$ because $\varepsilon_i^{eq} < \hat{\varepsilon}^{eq}$, the constant $C$ is found as $C = \alpha \bar{q}L - a\varepsilon_i^{eq}$. Hence, Equation (B.15) can be rewritten as:

$$a\varepsilon_z - (a\varepsilon_z + b - (1 - D^{max})E\varepsilon_z)\theta(\varepsilon_z - \hat{\varepsilon}^{eq}) = \alpha \bar{q}(z - L) + a\varepsilon_i^{eq}$$  \hspace{1cm} (B.16)

after substituting the expression of $C$. Equation (B.16) may have two solutions, the first is when $\varepsilon_i^{eq} < \varepsilon_z^N < \hat{\varepsilon}^{eq}$ which leads to $\theta(\varepsilon_z^N - \hat{\varepsilon}^{eq}) = 0$, and the second is $\varepsilon_z > \hat{\varepsilon}^{eq}$ which leads to $\theta(\varepsilon_z^N - \hat{\varepsilon}^{eq}) = 1$. By substituting the value of $\theta(\varepsilon_z - \hat{\varepsilon}^{eq})$ for each case and simplifying, the two possible
solutions can be written as:

\[
\begin{align*}
\varepsilon_{\text{Ni}}^N(z) &= \frac{1}{a}(\alpha q(z - L) + a\varepsilon_i^{eq}) \quad \text{if } \varepsilon_i^{eq} < \varepsilon_c < \hat{\varepsilon}^{eq} \\
\varepsilon_{\text{Nj}}^N(z) &= \frac{\alpha q(z-L) + a\varepsilon_i^{eq} + b}{E(1-D_{\text{max}})} \quad \text{if } \varepsilon_i > \hat{\varepsilon}^{eq}
\end{align*}
\] (B.17)

The solution in Equation (B.17) is examined for existence of solutions, the real constraints of the solution are: \(\varepsilon_{i}^{L} < \varepsilon_{i}^{eq}, \varepsilon_{i}^{eq} < \varepsilon_{i}^{Ni} < \hat{\varepsilon}^{eq}\) and \(\varepsilon_{i}^{Nj} > \hat{\varepsilon}^{eq}\). Applying these constraints leads to the following conditions for the existence of solution:

\[
a > 0 \text{ and } \frac{E\varepsilon_i^{eq} + \alpha \bar{q}(H - z)}{\bar{\sigma}} < \frac{aD_{\text{max}}\varepsilon_i^{eq}(\varepsilon_i^{eq} - \varepsilon_f^{eq}) + ((D_{\text{max}} - 1)\varepsilon_f^{eq} - D_{\text{max}}\varepsilon_i^{eq})(E\varepsilon_i^{eq} + \alpha \bar{q}(H - z))}{(D_{\text{max}} - 1)\varepsilon_f^{eq} - D_{\text{max}}\varepsilon_i^{eq}} \quad \text{and} \quad b > \frac{a\alpha \varepsilon_i^{eq}\bar{q}(\varepsilon_i^{eq} - \varepsilon_f^{eq}) - E\varepsilon_i^{eq}(\alpha \bar{q}D_{\text{max}}\varepsilon_i^{eq} - (D_{\text{max}} - 1)\varepsilon_f^{eq}(\alpha \bar{q} + \alpha \bar{q}))}{\alpha \bar{q}((D_{\text{max}} - 1)\varepsilon_f^{eq} - D_{\text{max}}\varepsilon_i^{eq})} \quad \text{and} \quad b > \frac{-a\bar{q}(D_{\text{max}} - 1)\varepsilon_f^{eq} - D_{\text{max}}\varepsilon_i^{eq})(\alpha \bar{q}(z - H) + \bar{\sigma})}{\alpha \bar{q}((D_{\text{max}} - 1)\varepsilon_f^{eq} - D_{\text{max}}\varepsilon_i^{eq})}
\] (B.18)

By examining these conditions, they can not be satisfied because \(a\) has to be \(a < 0\) according to the definition in Equation (B.3). There is no solution of the PDE in the strong form for strain \(\varepsilon_{i}\) in the region \(z < L\). This means that once the material starts failing the process is unstable and there is no mechanism by which damage may stop.
Appendix C

Simplified uniaxial derivation of viscoelastic damage model with pore pressure

In this section, we illustrate that the viscoelastic damage model used in this paper describes a damaged Maxwell viscoelastic material under a uniaxial stress state, thus, proving that it is suitable for representing the non-Newtonian fluid like behavior of glacial ice with damage evolution. Assuming the uniaxial macroscopic loading in the real stress state, the full stress tensor $\sigma_{kl}$ and the deviatoric stress tensor $\sigma_{kl}^{\text{dev}}$ become

$$
[\sigma_{kl}] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
[\sigma_{kl}^{\text{dev}}] = \begin{bmatrix} \frac{2}{3}\sigma_{11} & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_{11} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{11} \end{bmatrix},
$$

(C.1)

hence, the elastic strain component $\varepsilon_{11}^e$ for undamaged ice is given by

$$
\varepsilon_{11}^e = \frac{1}{E} \sigma_{11},
$$

(C.2)
where $E$ is the Young’s modulus of undamaged ice and the viscous strain rate in Equation (6.5) can be written in the form

$$\dot{\varepsilon}_{11}^v = \frac{1}{\eta} \sigma_{11}, \quad \text{where} \quad \eta = \left(\frac{2}{3} A (\sigma_{11})^{(n-1)} \right)^{-1}. \quad (C.3)$$

Differentiating the elastic component with respect to time and assuming small elastic deformations, we can add the elastic and viscous components to write the total strain rate as

$$\dot{\varepsilon}_{11} = \dot{\varepsilon}_{11}^e + \dot{\varepsilon}_{11}^v = \frac{\dot{\sigma}_{11}}{E} + \frac{\sigma_{11}}{\eta}. \quad (C.4)$$

The above equation represents a one-dimensional Maxwell viscoelastic element that can be represented by a spring with stiffness $E$ and a dashpot with viscosity $\eta$ connected in series.

Including the effect of damage in the spring and dashpot under dry conditions, we can write the elastic and viscous strain rates as

$$\dot{\varepsilon}_{11}^e = \frac{\dot{\sigma}_{11}}{(1-D)E}, \quad \dot{\varepsilon}_{11}^v = \frac{\sigma_{11}}{(1-D)^n \eta}. \quad (C.5)$$

The total strain rate can now be expressed as

$$\dot{\varepsilon}_{11} = \frac{\dot{\sigma}_{11}}{E^d} + \frac{\sigma_{11}}{\eta^d}, \quad (C.7)$$

where the “damaged” modulus $E^d = (1-D)E$ and a dashpot with “damaged” viscosity $\eta^d = (1-D)^n \eta$. Next, recalling Equation (6.13) upon including the effect of pore pressure $P^h$ under wet conditions we find the only non-zero component of the stress tensor is $\sigma_{11} = (1-D)E \varepsilon_{11}^e - D P^h$. 

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Rewriting the above equation and neglecting the contribution of damage rate $\dot{D}$ (small damage rate assumption), the elastic strain rate then becomes

$$\dot{\varepsilon}_{11}^e = \frac{\dot{\sigma}_{11} + D\dot{p}_h}{E^d}, \quad (C.8)$$

and upon using Equation (6.14), the effective solid stress becomes

$$[\bar{\sigma}_{kl}] = \begin{bmatrix}
\frac{\sigma_{11}}{1-D} + \frac{D}{1-D}p_h & 0 & 0 \\
0 & \frac{D}{1-D}p_h & 0 \\
0 & 0 & \frac{D}{1-D}p_h
\end{bmatrix}. \quad (C.9)$$

The inclusion of pore-pressure leads to a three-dimensional effective stress state. Because the deviatoric stress is unaffected by an additional hydrostatic stress, the viscous strain rate $\dot{\varepsilon}_{11}^v$ is unaffected by pore-pressure $p_h$ and so the total strain rate becomes

$$\dot{\varepsilon}_{11} = \frac{\dot{\sigma}_{11} + D\dot{p}_h}{E^d} + \frac{\sigma_{11}}{\eta^d}. \quad (C.10)$$

The damage evolution rate for ice under wet conditions remains

$$\dot{D} = \frac{B\langle\chi\rangle^r}{(1-D)^{k\sigma}}, \quad (C.11)$$

but the Hayhurst equivalent stress $\chi$ can now be written in the form

$$\chi = \begin{cases}
\alpha \frac{\sigma_{11} + Dp_h}{1-D} + \beta \frac{\sigma_{11}}{1-D} + (1 - \alpha - \beta) \frac{\sigma_{11} + 3Dp_h}{1-D}, & \text{if } \frac{\sigma_{11} + Dp_h}{1-D} > 0, \\
0, & \text{if } \frac{\sigma_{11} + Dp_h}{1-D} \leq 0,
\end{cases} \quad (C.12)$$

from which it is apparent that pore pressure increases $\chi$ and allows damage to initiate under con-
ditions experiencing a lower tensile stress state.
Appendix D

Purely viscous uniaxial model

In the absence of elastic strains, the proposed model can be proven to behave as a viscous material. The purely viscous uniaxial model corresponds to taking the limit $E \to \infty$ in Equation (C.10). This leads to the relation

$$\dot{\varepsilon}_{11}^v = \frac{\sigma_{11}}{(1-D)^n \eta}, \quad (D.1)$$

or, more intuitively

$$\sigma_{11} = (1-D)^n \eta \dot{\varepsilon}_{11}^v. \quad (D.2)$$

The only place pore pressure enters the model now is through the damage metric defined in Equation (C.12).