EXPANSION OF A FILTRATION WITH A STOCHASTIC PROCESS
A HIGH-FREQUENCY TRADING PERSPECTIVE

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ABSTRACT

Expansion of a filtration with a stochastic process

A high-frequency trading perspective

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A theory of expansion of filtrations has been developed since the 1970s to model dynamic probabilistic problems with asymmetric information. It has found a special echo in mathematical finance around the concept of *insider trading*, which has appeared in return very convenient for expressing the abstract properties of augmentations of filtrations. Research has historically focused on two particular classes of expansions, *initial* and *progressive* expansions, corresponding to additional information generated respectively by a random variable and a random time. Although they can reproduce some stylized facts in the insider trading paradigm, those two types of expansions are too restrictive to model quantitatively dynamic phenomenons of contemporary interest such as the topical high-frequency trading. In order to model such a continuous flow of information Kchia and Protter (2015) introduce augmentations of filtrations where the additional information is generated by a stochastic process.

This thesis complements the pioneering work of Kchia and Protter (2015) with an analysis of the *information drift* appearing in the transformation of semimartingales, which leads to a quantitative valuation of the additional information. In the preliminary chapters we introduce the general framework of expansions of filtrations and present the information drift as a key proxy to the value of information by characterizing its existence as a no-arbitrage condition and expressing problems the value increase of optimization problems associated with additional information as one of its integrals. The theoretical core of this thesis is formed by two series of convergence theorems for semimartingales and their information drifts under a new topology on filtrations, from which we derive the transformation of semimartingales when the filtration is augmented with a stochastic process as well as a computational method to estimate the information drift. We finally study several dynamical examples of anticipative expansions of a Brownian filtration with stochastic processes, where the information drift does or does not exist, and set the foundations for an ongoing application to estimating the advantage of high-frequency traders on the general market.

RÉSUMÉ

Expansion d'une filtration avec un processus stochastique

Une perspective sur le trading à haute-fréquence

Léo Neufcourt

Une théorie des expansions de filtrations s'est développée depuis les années 1970 autour de l'école de probabilité française pour étudier les problèmes à univers aléatoires où l'information est asymétrique. Elle a trouvé un écho particulier en mathématiques financières autour du concept de *délit d'initié*, qui en retour est apparu très utile pour illustrer les propriétés abstraites des grossissements de filtrations. La recherche s'est historiquement concentrée sur deux classes d'expansions particulières, le *grossissement initial* et le *grossissement progressif*, qui correspondent à l'ajout d'information générée respectivement par une variable aléatoire et un temps aléatoire. Bien qu'ils puissent reproduire certains faits stylisés du paradigme du délit d'initié, ces deux types de grossissements sont trop restrictifs pour modéliser quantitativement certains phénomènes contemporains tels que le trading à haute fréquence souvent décrié. Pour modéliser un tel flux d'information, il apparaît naturel de considérer des expansions de filtrations avec un processus stochastique, qui sont introduites pour la première fois dans le travail pionnier de Kchia and Protter (2015).

C'est le point de vue que nous adoptons dans cette thèse, où nous complémentons

les résultats de Kchia and Protter (2015) avec une analyse du *drift*[1] *d'information*, qui nous conduit à une méthode quantitative pour estimer la valeur d'une information privée. Dans les chapitres préliminaires nous introduisons le cadre général des expansions de filtrations, et nous démontrons l'importance du drift d'information en caractérisant son existence comme une condition de non-arbitrage ainsi qu'en exprimant l'accroissement de valeur associé à un incrément d'information comme une de ses intégrales pour des problèmes d'optimisation simples. Le cœur de cette thèse est constitué de deux séries de théorèmes de convergence des semimartingales et de leur drifts d'information pour une nouvelle topologie sur l'espace des filtrations, dont nous déduisons la transformation des semimartingales lorsque la filtration est augmentée avec un processus stochastique, ainsi qu'une méthode quantitative d'estimation du drift d'information. Nous étudions enfin plusieurs exemples dynamiques d'expansions anticipatives de filtrations Browniennes avec un processus stochastique, admettant ou non un drift d'information, et nous posons les bases d'une application en cours pour estimer l'avantage des traders à haute fréquence sur le reste du marché.

[1]glissement

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List of abbreviations and symbols

Abbreviations

BM : Brownian Motion

ELMM : Equivalent Local Martingale Measure (see Theorem 2.2)

FINRA : Financial Industry Regulatory Authority (in the US)

HF : High-Frequency

HFT : High-Frequency Trading (see Section 2.3)

LOB : Limit Order Book (see Paragraph 2.3.1)

NA1 : No Arbitrage of the First Kind (see Definition 2.1)

NFLVR : No Free Lunch with Vanishing Risk (see Definition 2.1)

resp. : respectively

SDE(s) : Stochastic Differential Equation(s)

Symbols

X^t : stochastic process X stopped at a stopping time τ (see Definition 1.4)

$[X, Y]$: quadratic covariation of semimartingales X and Y (see Paragraph 1.1.4)

$(X \cdot Y)$: formally $t \mapsto \int_0^t X_u dY_u$, in the sense of a stochastic integral (see Definition 1.14) or a Riemann-Stieltjes integral (see Paragraph 1.1.4)

$\mathcal{E}(X)$: stochastic exponential of a local martingale X (see Paragraph 1.1.5)

$\mathbb{L}_p^0$: space of finite valued random variables topologized with the convergence in probability (see Definition 1.11)

$\mathcal{S}(\mathcal{Y})$ **or** $\mathcal{S}$: $\mathcal{Y}$-predictable semimartingale integrands (see Definition 1.12)

$\mathcal{S}_{ucp}$: space $\mathcal{S}$ topologized with the uniform convergence in probability (see Definition 1.12)

$\mathcal{S}^1(\mathcal{Y}, M)$ **or** $\mathcal{S}^1(M)$: $\mathcal{Y}$-predictable processes which are in $L^1(d\mathbb{P} \times d[M, M])$, for a given $\mathcal{Y}$-local martingale M (see Definition 1.15)

$\mathcal{S}^2(\mathcal{Y}, M)$ **or** $\mathcal{S}^2(M)$: $\mathcal{Y}$-predictable processes which are in $L^2(d\mathbb{P} \times d[M, M])$, for a given $\mathcal{Y}$-local martingale M (see Definition 1.15)

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À mes parents formidables …

Introduction

Expansions of filtrations and information drift The problem of expansions of filtrations can be formulated easily: how do the dynamical properties of stochastic processes change when additional information is added to the filtration? The seminal article by Itô (1978) first skimmed this question when studying the extension of the stochastic integral to anticipative integrands and established that a drift appears in the Brownian motion when the filtration anticipates its value at a future time. A classical theory of expansions of filtrations was subsequently developed around the French probability school to study the transformation of semimartingales, easily reduced to the particular cases of integrable martingales, when a *general filtration* is augmented with *additional information*. Brémaud and Yor (1978) prove a first major result: the martingale property is preserved by an expansion of the filtration if and only if general filtration and additional information are conditionally independent. It follows that most expansions do not preserve the martingale property, and the next important question has been whether a martingale remain a semimartingale for an expansion of the filtration, also known as Jacod's Hypothesis $(\mathscr{H}')$. Indeed, semimartingales are the most general processes on which dynamics can be considered: not only is the semimartingale decomposition a natural way to decompose a stochastic dynamics, but semimartingales also compose the most general class of processes on which integration and Stochastic Differential Equations (SDEs) can be consistently defined. The problem of conservation of semimartingales has appeared challenging and a characterization of its general form is yet to be found. One can get a first glimpse of the challenge involved by considering the classical Bitcheler-Dellacherie characterization of semimartingales (Theorem), where expanding the filtration requires integrability on a larger class of semimartingale integrands. It is notable that the same argument implies immediately that the converse phenomenon, namely *filtration shrinkage*, preserves semimartingales (see Section 1.4 or Stricker (1977), Föllmer and Protter (2011)).

In the interesting case where a semimartingale remains a semimartingale for an augmented filtration, its decomposition is modified with an additional finite variation term, which is characteristic of the change of its dynamics between the two filtrations. Existence of a local martingale measure, or a local martingale deflator, requires this finite variation process to admit a density with respect to the quadratic variation of the semimartingale, the *information drift* (see Definition 1.21 and Theorem 1.20), which is also the volatility of the process defining the equivalent change of local martingale measure. Hence the information drift appears as a key proxy to the quantitative value of additional information. A general representation of the information drift in term of conditional probabilities has been recently established in Ankirchner, Dereich, and Imkeller (2006) and Ankirchner (2005), with techniques of embedding of probability spaces inspired by the search of coupling measures solving weakly conditioned SDEs (see Baudoin (2002)).

Initial and progressive expansions. Two types of expansions have been principally studied in the past, *initial expansions* and *progressive expansions*. Initial expansions are the simplest type of expansions as they correspond to augmenting the filtration with a random variable at the initial time. Most of the abundant literature on initial expansions relies on the results of Jacod (1985), who proves that semimartingales are preserved by an initial expansion as long as the conditional distributions of the random variable are dominated by a common non-random measure, and admit in the continuous case an information drift which can be represented explicitly. Progressive expansions, which correspond to the smallest expansions for which a given random time becomes a stopping time, have been first studied by Barlow (1979) and Jeulin (1980), who proved to preserve semimartingales up to the random time, and over the whole time interval if the stopping time is honest. Initial and progressive expansions are unified by Kchia, Larsson, and Protter (2013) who show that Jacod's condition is also sufficient for progressive expansions. The reader can also refer to Protter (2004, Chapter VI) or Jeanblanc (2010) for overviews of main results on initial and progressive expansions.

Dynamical expansions. Initial expansions have been successfully used to develop toy models reproducing qualitative expectations, and progressive expansions can catch some situations where the extra information available to the insider trader comes from a continuous flow of knowledge. but they have not been sophisticated enough to build quantitative models for real world problems, many of which can however be well described by expansions of filtrations with stochastic processes. A first family of dynamical expansions is studied independently in Corcuera et al. (2004) and Kchia and

Protter (2015) where a Brownian filtration is expanded with some future value perturbed with an independent Gaussian noise. The transformation of semimartingales can there be deduced from the theory of initial expansions, and one can use Gaussian computations to obtain explicit formulas for the information drift. In a more systematic approach, Kchia, Larsson, and Protter (2013) proves the validity of Jacod's condition for successive expansions with random variables at random times, i.e. a jump process, and extends the formula for the information drift. Meanwhile a theory of convergence of filtrations has been developped through Barlow and Protter (1990), Antonelli and Kohatsu-Higa (2000), Coquet, Mémin, and Mackevičius (2000), Coquet, Mémin, and Slominski (2001) and Mémin (2003), where the stability of the decomposition of semimartingales is studied for several notions of convergence of filtrations.

The pioneer article by Kchia and Protter (2015) combines expansions of filtrations and convergence of filtrations to study expansions generated by a stochastic process as the limit of expansions with jump processes, for which it extends the validity of Jacod's condition under integrability conditions on the successive information drifts induced by the discrete approximations. However their non-constructive argument based on weak convergence of filtrations does not allow to identify the form of the limit decomposition or check the existence of an information drift with respect to the limit filtration. This problem has been one of the motivations of the present thesis, and it is solved in Chapter 3, where we can construct the limit information drift by considering a convergence of filtrations based on L^p norms, better suited to linear projections due to their convexity.

* * * * *

The insider trading paradigm. Expansions of filtration are the canonical framework for insider trading, and mathematical finance has provided in return a suggestive language to describe the properties of expansions of filtrations. Insider trading is a tenuous notion. The SEC (U.S. Securities and Exchange Commission), emphasizing that insider trading recovers both legal and illegal behaviors, proposes the following definition on its website:

> "The legal version is when corporate insiders - officers, directors, and employees - buy and sell stock in their own companies. When corporate insiders trade in their own securities, they must report their trades to the SEC. [...] Illegal insider trading refers generally to buying or selling a security, in breach of a fiduciary duty or other relationship of trust and confidence, while in possession of material, nonpublic information about the security. Insider trading violations may also include 'tipping' such information, securities trading by the person 'tipped', and securities trading by those who misappropriate such information."(Securities and Commission (2017))

For the purposes of this thesis we leave the legal discussion aside, and by *insider trading* we simply refer to the market phenomenon in which some agents have access to more information than the general public. The first modern mathematical model for insider trading was proposed by Kyle (1985) in discrete time with three types of agents, market makers, insider traders and noise traders, with the purpose of understanding the assimilation of the insider information by the whole market. Kyle's model was extended to continuous time in Back (1992) and Back (1993). Applications of expansions of filtrations to insider trading were subsequently studied extensively with two market agents, a regular trader and an insider. We refer the reader to the PhD theses of Wu (1999), Aksamit (2014), Falafala (2014) for comprehensive presentations of the abundant insider trading literature.

Arbitrage and statistical arbitrage. The widespread use of risk-neutral pricing in finance presupposes absence of arbitrage opportunities. Supposing this is satisfied for the general public, a natural question is whether there exists arbitrage opportunities for the insider. No Arbitrage of the First kind (NA1) and No Free Lunch with Vanishing Risk (NFLVR), two of the most common mathematical translations of arbitrage, both require conservation of semimartingales and existence of an information drift (see for instance Acciaio, Fontana, and Kardaras (2016), Fontana (2015), Fontana (2014), and Falafala (2014)), in which case the change of risk neutral probability is the stochastic exponential of the information drift up to an orthogonal process. Imkeller (2002) also exhibits concrete examples of free lunches in initial and progressive expansions for which the conservation of semimartingales fails.

In a well posed model without arbitrage *stricto sensu*, the advantage associated to additional information lies in *statistical arbitrage*, in the general sense of opportunities for higher gain at a lower risk with high probability. The potential gain of an insider from such statistical arbitrage can be evaluated in hard cash by comparing the value of portfolio optimization problems obtained for the two filtrations. Suggestive results are obtained for simple portfolio optimization problems under asymetric information in Karatzas and Pikovsky (1996) and Biagini and Øksendal (2005) via anticipative stochastic calculus and stochastic forward integral. More recent developments can be found in Nunno et al. (2009), Aase, Bjuland, and Øksendal (2010), Aase, Bjuland, and

Øksendal (2011) or Øksendal and Zhang (2012), as well as Amendinger, Imkeller, and Schweizer (1998), Amendinger (2000), Ankirchner (2005) and Ankirchner, Dereich, and Imkeller (2006). Our Chapter 2 proposes also a new practical example where the value associated to additional information can be expressed explicitly with the information drift. Expansions of filtrations have also found recent applications beyond mathematical finance such as causal optimal transport (see Acciaio, Veraguas, and Zalashko (2016)).

* * * * *

Outline. In this thesis we would like to propose a quantitative method to estimate the value associated with a dynamical anticipation of the filtration, *via* an understanding of the transformation of the semimartingale and of the information drift. Our successive chapters study the different aspects of this problem and are mostly independent, with the exception of Chapter 1 where we build our general setup. Chapters 3 and 4 compose the theoretical core of this work whereas Chapters 2 and 5 propose examples and applications.

Chapter 1 presents classical properties of martingales, semimartingales (Section 1.1) and expansions of filtrations (Section 1.2). We first motivate the problem of the transformation of semimartingales and introduce the information drift and review the relationships found between existence of a local martingale measure, conservation of semimartingales and information drift by their necessity for the (Section 1.3), complementing the results found in the literature with a proof of the simultaneous existences a local martingale deflator and an information drift (Theorem 1.20). We also prove several reductions of the problem often stated *en passant* and establish general preliminary results.

Chapter 2 translates expansions of filtrations in a financial setting *via* the paradigm of insider trading and expresses the value of the additional information with the information drift. We first study arbitrage opportunities (Section 2.1) and prove the equivalence between the existence of the information drift and absence of arbitrage of the first kind (Theorem 2.6), which is the financial counterpart of Theorem 1.20. We then consider statistical arbitrage and express the additional value of simple optimization problems for the insider involving integrals of the information drift, in two simple examples: the classical logarithmic utility maximization, and a new practical constrained optimization (Section 2.2). We also introduce the application to insider trading, which we motivate with evidence of the advantage of high-frequency traders over the general public in limit order book data (Section 2.3).

Chapter 3 obtains the decomposition of semimartingales and their information drift when the filtration is expanded with a stochastic process using discrete approximations. For this purpose we study the stability of semimartingales through convergence of filtrations and complement Kchia and Protter (2015) with a convergence theorem for the information drifts, obtained by relaxing the topology on filtration to a convergence based on L^p norms. We first introduce (Section 3.1) our convergence of filtrations as a dual pointwise convergence of stochastic processes and derive sufficient conditions under which (i) semimartingales are conserved (Theorem 3.10) (ii) their decomposition is stable (Theorems 3.11) (iii) there exists an information drift for the limit filtration (Theorem

3.12). In the case of expansions generated by a stochastic process (Section 3.2) we approximate the augmented filtration with successive expansions generated by discrete samples of the process and apply the previous convergence results to obtain Theorems 3.10, 3.12 and 3.22, which are probably the most significant results of this chapter.

Chapter 4 extends the approach proposed in Ankirchner, Dereich, and Imkeller (2006) to represent the information drift as the diffusion coefficient in the predictable representation of the conditional probabilities. After technical preiminaries (Section 4.1) and a study of extensions of measures (Section 4.2) we revisit the argument of Ankirchner, Dereich, and Imkeller (2006) (Theorem 4.17) for a refining sequence of filtrations and show that our representation of the information drift is stable if there is a uniform L^p bound on the information drifts (Theorem 4.19), from which we deduce a new convergence theorem. We also prove a duality between convergence of filtrations and convergence of measures (Theorem 4.26) as well as a new representation of the information drift as the volatility of the Jacod conditional likelihood (Theorem 4.31 in Section 4.4). We conclude with a computational method based on functional Itô calculus to estimate the information drift for dynamical expansions (Theorem 4.41 in Section 4.5).

Finally Chapter 5. considers several examples of dynamical anticipations of a Brownian filtration, inspired by our search for a model for high-frequency trading (see Section 2.3). We first prove a formula for the information drift of a Brownian motion based on its conditional expectation with respect to the augmented filtration, using the determinism of its quadratic variation (Section 5.1). We then review the classical example of a fixed anticipation of the Brownian filtration perturbed with an independent dynamical noise, which can be solved either by the theory of initial expansions (Corollary 1.38) or by explicit Gaussian computations together, and we give new explicit conditions for existence of a local martingale measure, which is the most classical standard for absence of arbitrage (Section 5.2). The last sections investigate the existence and the form of the information drift in more authentic examples of dynamical anticipative expansions, considering successively fixed (Section 5.3) and random anticipations (Section 5.4).

The appendices gather somehow isolated but nevertheless interesting results. In Appendix A we obtain a formula for the information drift in a simple case of initial expansion by introducing local times. Appendix B is a first attempt to apply expansions of filtrations to solving degenerate SDEs, and relates the information drift to the potential difference between the solutions adapted to two nested filtrations.

Our main results are indicated with $(*)$.

Chapter 1

Expansions of filtrations

In this whole thesis we consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a general time interval I of the form $[0, \infty)$, $[0, T)$ or $[0, T]$ for some $T > 0$, for which we always denote $\sup I := T \in (0, +\infty]$. We classically assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is complete (i.e. $\mathcal{A}$ contains every $\mathbb{P}$-null sets), and that all random variables and stochastic processes take values in a Polish space $(E, \mathcal{E})$. We also consider a (reference) filtration $\mathcal{F}$ which we assume satisfies the usual hypotheses (i.e. $\mathcal{F}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Other notations are defined inside results or at the beginning of chapters and sections, however S and M consistently denote respectively a $\mathcal{F}$-semimartingale and a local martingale whereas $\mathcal{F}$ denotes a generic filtration and $\mathcal{G}$ is an expansion of $\mathcal{F}$.

* * * * *

This chapter formally introduces and motivates the problem of expansions of filtrations to build the general setup of this thesis. Most of our argument here follows the literature, still we review some proofs of particular interest, and prove also two easy results that we have not found as such, namely the equivalence between the existences of a local martingale deflator and of the information drift, and the absence of impact of the usual hypotheses on an expansion of filtration.

We start by recalling classical properties of semimartingales upon which the theory of expansions of filtrations is built. Proofs are omitted and can be found in Protter (2004, Chapters II & III).

1.1 Martingales and semimartingales

In this paragraph we denote a generic filtration $\mathcal{Y}$.

1.1.1 Semimartingales as decomposable processes

Definition 1.1 (Martingale). *A $\mathcal{Y}$-adapted stochastic process M such that $\forall t \in I, \mathbb{E}|M_t| < \infty$ is called a $\mathcal{Y}$-**martingale** if*

$$\forall s \leq t, \mathbb{E}[M_t | \mathcal{Y}_s] = M_s.$$

Martingales are the mathematical representation of a "fair" dynamical game in a stochastic environment. Many interesting properties are conserved if we relax the integrability requirement which leads to the class of local martingales.

Definition 1.2 (Localizing sequence). *A $\mathcal{Y}$-**localizing sequence**, or localizing sequence if there is no ambiguity, is a non-decreasing sequence $(\tau_n)_{n\geq 1}$ of $\mathcal{Y}$-stopping times diverging to $+\infty$.*

Definition 1.3 (Local martingale). *A $\mathcal{Y}$-adapted stochastic process M is called a $\mathcal{Y}$-**local martingale** if there exists a $\mathcal{Y}$-localizing sequence $(\tau_n)_{n\geq 1}$ such that for every $n \geq 1$ the* *stopped process M^{τ_n} is a $\mathcal{Y}$-martingale.*

We recall the definition of a stopped process for completeness:

Definition 1.4 (Stopped process). *Given a stochastic process X and a random time τ, the stopped process X^τ is defined by $X^\tau_\cdot = X_{\cdot\wedge\tau}$.*

Definition 1.5 (Finite variation process). *A $\mathcal{Y}$-adapted (respectively $\mathcal{Y}$-predictable) stochastic process A is called a $\mathcal{Y}$-**finite variation process** (respectively $\mathcal{Y}$-**predictable finite variation process**) if A has bounded variations on every compact time interval, i.e. if $\int_0^t |dA_u| < \infty$ a.s. for every $t \in I$.*

Definition 1.6 (Semimartingale). *A stochastic process S is called a $\mathcal{Y}$-**semimartingale** if it can be decomposed as $S =: S_0 + M + A$, where M is a $\mathcal{Y}$-local martingale and A a finite variation process, with $M_0 = A_0 = 0$.*

*We call such a decomposition $S =: S_0 + M + A$ **a $\mathcal{Y}$-semimartingale decomposition** of S.*

When there is no ambiguity we omit the dependency on the filtration $\mathcal{Y}$.

1.1.2 Special semimartingales

Proposition-Definition 1.7 (Special semimartingale). *A $\mathcal{Y}$-semimartingale S is called a $\mathcal{Y}$-**special** semimartingale if it can be decomposed as $S = S_0 + M + A$, where M is a $\mathcal{Y}$-local martingale and A a $\mathcal{Y}$-**predictable** finite variation process, with $M_0 = A_0 = 0$.*

Such M and A are unique (up to a modification), and we call the decomposition $S = S_0 + M + A$ ***the $\mathcal{Y}$-canonical decomposition*** *of M.*

Here again we omit the dependency on the filtration $\mathcal{Y}$ if there is no ambiguity.

Proposition 1.8. *A semimartingale S is special if and only if one of the following statements hold:*

(i) $(\sup_{s\leq t} |\Delta S_t|)_{t\in I}$ *is locally integrable*

(ii) $(\sup_{s\leq t} |S_t|)_{t\in I}$ *is locally integrable*

Corollary 1.9. *A continuous semimartingale is a special semimartingale.*

Proposition 1.10. *Let $\mathcal{Y}$ be a filtration satisfying the usual hypotheses and let S be a continuous $\mathcal{Y}$-semimartingale with canonical decomposition $S = S_0 + M + A$. Then M and A are continuous.*

1.1.3 Semimartingales as stochastic integrators

Equivalently, semimartingales arise as the class of natural integrators for simple predictable processes, as long as we request such integration to be continuous with respect to the uniform convergence in probability - this is necessary for the expected dominated convergence of the integrals of a converging sequence of integrands uniformly bounded in probability.

Definition 1.11 (Space $\mathbb{L}^0_p$). *We denote $\mathbb{L}^0_p$ the space of finite valued random variables topologized with the convergence in probability.*

Definition 1.12 (Space $\mathcal{S}$). *Given a filtration $\mathcal{Y}$, we denote $\mathcal{S}(\mathcal{Y})$, or $\mathcal{S}$ when there is no ambiguity, the space of $\mathcal{Y}$-predictable semimartingale integrands, i.e. the closure of the class of simple predictable processes with respect to the norm of uniform convergence in probability.*

We also denote S_{ucp} the space S topologized with the uniform convergence in probability.

By a simple predictable process we mean here a process H of the form $H_t := \sum_{i=1}^k h_i 1_{t \leq t_i}$ for every $t \in I$, where $k \in \mathbb{N}$ and h_i is a finite $\mathcal{Y}_{t_i}$-measurable random variable for every $i \leq k$.

Theorem 1.13 (Bitcheler-Dellacherie characterization of semimartingales). *A stochastic process S is a semimartingale if and only if the stochastic integration*

$$\begin{array}{rccc} I_S : & \mathcal{S}_{ucp} & \longrightarrow & \mathbb{L}_p^0 \\ & H & \longmapsto & \sum_{i=1}^k h_i (S^\tau_{t_{i+1}} - S^\tau_{t_i}) \end{array}$$

is continuous for every fixed time $\tau \in I$.

Definition 1.14 (Stochastic integral). *For a semimartingale S and a process $H \in \mathcal{S}$, we define the stochastic integral $H \cdot S$ as the continuous process given by*

$$(H \cdot S)_t := I_{S^t}(H) =: \int_0^t H_u dS_u, t \in I.$$

Note that this justifies our assertion that $\mathcal{S}(\mathcal{Y})$ is the "natural" class of semimartingale integrands (see also Protter (2004), Chapter II).

1.1.4 Quadratic (co-)variation

Given two semimartingales Y and Z we classically denote $[Y, Z]$ be the quadratic covariation process of Y with Z, which is a finite variation process and has continuous paths as long as Y or Z does.

We also denote $H \cdot [Y, Z]$ the Riemann-Stieltjes integral defined as

$$(H \cdot [Y, Z])_t := \int_0^t H_s d[Y, Z]_s, t \in I.$$

Definition 1.15. *We denote $\mathcal{S}^1(\mathcal{Y}, M)$ and $\mathcal{S}^2(\mathcal{Y}, M)$, or $\mathcal{S}^1(M)$ and $\mathcal{S}^2(M)$, or $\mathcal{S}^1$ and $\mathcal{S}^2$ according to the context, the subspaces of $\mathcal{S}(\mathcal{Y})$ composed by the processes $H \in \mathcal{S}$ satisfying respectively*

$$\mathbb{E} \int_0^T |H_s| d[M, M]_s < \infty,$$
$$\mathbb{E} \int_0^T H_s^2 d[M, M]_s < \infty.$$

1.1.5 Stochastic exponential

Given a local martingale Y we define the stochastic exponential $\mathcal{E}(Y)$ by

$$\mathcal{E}(Y)_t := e^{Y_t - \frac{1}{2}[Y,Y]_t}, t \in I.$$

It follows from Itô's formula that $\mathcal{E}(Y)$ is the solution of the stochastic differential equation

$$Z_t = e^{Y_0} + \int_0^{\cdot} Z_u dY_u, t \in I,$$

which also shows that $\mathcal{E}(Y)$ is a local martingale.

1.2 Hypotheses $(\mathscr{H})$ and $(\mathscr{H}')$

Definition 1.16. *A filtration $\mathcal{G}$ is called an expansion of the filtration $\mathcal{F}$ if for every $t \in I$ we have $\mathcal{F}_t \subset \mathcal{G}_t$.*

Remark 1.17. *We can always decompose an expansion $\mathcal{G}$ of $\mathcal{F}$ as $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t, t \in I$ where $\mathcal{H}$ is another generic filtration - one can take for instance $\mathcal{H} := \mathcal{G}$! Conversely $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t, t \in I$ always define an expansion of $\mathcal{F}$.*

It has been of particular interest whether a given expansion $\mathcal{G}$ of $\mathcal{F}$ satisfies the following properties:

Hypothesis $(\mathscr{H})$: Every $\mathcal{F}$-martingale is a $\mathcal{G}$-martingale.

Hypothesis $(\mathscr{H}')$: Every $\mathcal{F}$-semimartingale is a $\mathcal{G}$-semimartingale.

1.2.1 Martingales and Hypothesis $(\mathscr{H})$

A natural characterization of Hypothesis $(\mathscr{H})$ can be found in Brémaud and Yor (1978).

Theorem 1.18 (Brémaud and Yor). *Let $\mathcal{G}$ be an expansion of $\mathcal{F}$. $(\mathscr{H})$ holds if and only if, for every $t \in I$, $\mathcal{G}_t$ and $\mathcal{F}_\infty$ are independent conditionally to $\mathcal{F}_t$ (where $\mathcal{F}_\infty := \bigcap_{u \in I} \mathcal{F}_u$).*

1.2.2 Semimartingales and Hypothesis $(\mathscr{H}')$

Theorem 1.18 shows that Hypothesis $(\mathscr{H})$ is a strong requirement, which has been relaxed into Hypothesis $(\mathscr{H}')$. The reader can picture from the Bitcheler-Dellacherie characterization of semimartingales that the conservation of semimartigales through an expansion of the filtration is far from automatic, as the class of predictable processes on which semimartingale integration is required to be continuous is *a priori* smaller for $\mathcal{F}$ than for an expansion $\mathcal{G}$.

Understanding Hypothesis $(\mathscr{H}')$ has appeared to be a challenging problem and a characterization of its general form is yet to be found. A natural way to approach this question is to study the transformation of subclasses of semimartingales.

1.3 Transformation of semimartingales and information drift

Suppose given an expansion $\mathcal{G}$ of the filtration $\mathcal{F}$ and a semimartingale S. Following our previous argument we are interested in the properties of S with respect to $\mathcal{G}$, in particular:

(i) whether S is a $\mathcal{G}$-semimartingale, and

(ii) if so, what is its $\mathcal{G}$-semimartingale decomposition.

Contrary to filtration shrinkage (see Paragraph 1.4) the transformation of semimartingales through an expansion of the filtration reduces to the case of local martingales.

Lemma 1.19. *Let $\mathcal{G}$ be an expansion of $\mathcal{F}$. If S is an $\mathcal{F}$-(special) semimartingale with (canonical) decomposition $S = S_0 + M + A$, then S is a $\mathcal{G}$-(special) semimartingale if and only if M is a $\mathcal{G}$-(special) semimartingale. If moreover M is an $\mathcal{G}$-(special) semimartingale with decomposition $L + C$, then S is an $\mathcal{G}$-(special) semimartingale with (canonical) decomposition $S_0 + L + (A + C)$.*

Proof. The $\mathcal{F}$-(predictable) finite variation process A is also a $\mathcal{G}$-(predictable) finite variation process and the first claim follows. If M is a $\mathcal{G}$-(special) semimartingale with (canonical) decomposition $L + C$ then $S = L + (A + C)$ where L is a $\mathcal{G}$-local martingale and $(A + C)$ is a $\mathcal{G}$-(predictable) finite variation process. □

One motivation for studying the transformation of a $\mathcal{F}$-local martingale M comes from the question of the existence of an Equivalent Local Martingale Measure (ELMM) for M with respect to the filtration $\mathcal{G}$, namely whether M can be turned into a $\mathcal{G}$-local martingale by an equivalent change of probability - which is a key property of stochastic models for financial markets (see our financial perspective in Chapter 2).

Existence of an ELMM means precisely the existence of a positive and uniformly integrable martingale Z such that ZM is a $\mathcal{G}$-local martingale. By dropping the integrability requirement on Z, we obtain a *local martingale deflator* which characterizes the transformation of M.

1.3.1 Local martingale deflator and information drift

Theorem 1.20 ($*$). *Let M be a $\mathcal{F}$-local martingale and $\mathcal{G}$ an expansion of $\mathcal{F}$. The following assertions are equivalent:*

(i) There exists a positive $\mathcal{G}$-local martingale Z such that ZM is a $\mathcal{G}$-local martingale.

(ii) There exists a $\mathcal{G}$-predictable process $\alpha \in \mathcal{S}(\mathcal{G})$ such that M is a $\mathcal{G}$-semimartingale with decomposition

$$M = \widetilde{M} + \int_0^\cdot \alpha_s d[M, M]_s.$$

In that case α is unique $d\mathbb{P} \times d[M, M]$-a.s., and we have the following correspondences:

(a) $d[Z, M]_t = \alpha_t d[M, M]_t, d\mathbb{P} \times d[M, M]$-a.s.

(b) Z is of the form $Z = \mathcal{E}(\alpha \cdot M + L) = \mathcal{E}(\alpha \cdot M)\mathcal{E}(L)$, where L is a $\mathcal{G}$-local martingale with $[L, M] = 0, d\mathbb{P} \times d[M, M]$-a.s.

Definition 1.21 (Information drift[1]). *The $\mathcal{G}$-predictable process $\alpha \in \mathcal{S}(\mathcal{G})$ defined by Theorem 1.20, i.e. such that M is a $\mathcal{G}$-semimartingale with decomposition*

$$M = \widetilde{M} + \int_0^\cdot \alpha_s d[M, M]_s,$$

*is called the **information drift** of the filtration $\mathcal{G}$ for M (with respect to filtration $\mathcal{F}$).*

Note that this is equivalent to $M - \int_0^\cdot \alpha_s d[M, M]_s$ being a local martingale.

[1] This most appropriate expression can already be found in Biagini and Øksendal (2005) and Ankirchner, Dereich, and Imkeller (2006), which the author regrets deeply

We will see throughout this thesis that the information drift is a key object to understand both the impact of an expansion of the filtration on the dynamics of stochastic processes and the potential value associated with it.

Definition 1.22 (Local martingale deflator). *The positive $\mathcal{G}$-local martingale Z defined by Theorem 1.20, i.e. such that ZM is a $\mathcal{G}$-local martingale, is called a **local martingale deflator** for the semimartingale M (with respect to the filtration $\mathcal{G}$).*

The proof of Theorem 1.20 relies on the next lemma.

Lemma 1.23. *Let Z be a positive local martingale and let M be a general stochastic process. Then $M^Z := ZM$ is a local martingale if and only if $M - \int_0^\cdot \frac{1}{Z_s} d[Z, M]_s$ is a local martingale.*

Proof. First note that, as Z is a positive semimartingale, M^Z is a semimartingale if and only M is (this follows for instance from Itô's formula).

By Itô's formula $d(\frac{1}{Z})_t = \frac{-1}{Z_t^2} dZ_t + \frac{1}{Z_t^3} d[Z, Z]_t$. Hence

$$dM_t = M_t^Z d(\frac{1}{Z_t}) + \frac{1}{Z_t} dM_t^Z + d[M^Z, \frac{1}{Z}]_t = \frac{-M_t^Z}{Z_t^2} dZ_t + \frac{1}{Z_t} dM_t^Z + dA_t,$$

where $dA_t := \frac{M_t^Z}{Z_t^3} d[Z, Z]_t - \frac{1}{Z_t^2} d[M^Z, Z]_t$.

Since $\int \frac{-M_s^Z}{Z_s^2} dZ_s$ defines a local martingale, it follows that M^Z is a local martingale if and only if $M - A$ is. Finally the identity $d[Z, ZM] = Md[Z, Z] + Zd[Z, M]$ leads to $A = -\int_0^\cdot \frac{1}{Z_s} d[Z, M]_s$. $\square$

Proof of Theorem 1.20. Suppose first that Z is a positive local martingale such that ZM is a $\mathcal{G}$-local martingale. It follows from the lemma that $M - \int_0^{\cdot} \frac{1}{Z_t} d[Z, M]_s$ is a $\mathcal{G}$-local martingale. The Kunita-Watanabe inequality implies that there exists a predictable càdlàg process β such that $d[Z, M] = \beta d[M, M]$, and $\alpha := -\frac{\beta}{Z}$ is such that $d[Z, M] = \alpha Z d[M, M]$ and $M - \int_0^{\cdot} \alpha_s d[M, M]_s$ is a local martingale. Moreover by the predictable representation theorem (see Revuz and Yor (2005, Chapter V)) there exist a predictable process $J \in \mathcal{S}$ and a local martingale L such that $[L, M] = 0$ and $Z_t = Z_0 + \int_0^t J_s Z_s dM_t + \int_0^t Z_s dL_t$. It follows from $[Z, M] = [(JZ) \cdot M + Z \cdot L, M] = (JZ) \cdot [M, M] + Z \cdot [L, M] = (JZ) \cdot [M, M]$ that $J = \alpha, d\mathbb{P} \times dt$-a.s. and that $Z = \mathcal{E}(\alpha \cdot M + L) = \mathcal{E}(\alpha \cdot M)\mathcal{E}(L)$.

Now suppose conversely that $M - \int_0^{\cdot} \alpha_s d[M, M]_s$ is a local martingale. The positive local martingale $Z^\alpha := \mathcal{E}(\alpha \cdot M)$ satisfies $dZ_t^\alpha = \alpha Z_t^\alpha dM_t$ and $[Z^\alpha, M] = [(\alpha Z^\alpha) \cdot M, M] = (\alpha Z^\alpha) \cdot [M, M]$. Hence $M - \int_0^{\cdot} \frac{1}{Z_{s-}^\alpha} d[Z^\alpha, M]_s$ is a local martingale, and $Z^\alpha M$ is a local martingale by Lemma 1.23. □

1.3.2 Equivalent local martingale measure

We now link the properties of the information drift to the classical concept of Equivalent Local Martingale Measure (ELMM), which is close for our purposes to a local martingale deflator. Indeed, the Radon-Nykodym derivative corresponding to an equivalent local martingale measure is a local martingale deflator, and conversely a local martingale deflator defines an ELMM if and only if it is uniformly integrable, which can be checked for example with Novikov or Kamazaki's conditions. The results in this paragraph hold for any filtration when it is not specified.

Definition 1.24 (ELMM). *An Equivalent Local Martingale Measure (ELMM) for a stochastic process M is a probability $\mathbb{Q} \sim \mathbb{P}$ such that M is a local martingale under $\mathbb{Q}$.*

Proposition 1.25. *Let M be a stochastic process.*

(i) If $\mathbb{Q}$ is an ELLM for M, then $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is a local martingale deflator for M.

(ii) If a local martingale deflator (see Theorem 1.20) Z is uniformly integrable, then we can define an ELLM $\mathbb{Q}$ by $\mathbb{Q} := Z \cdot \mathbb{P}$.

Proof. This is an immediate consequence of the definitions of an ELLM (Definition 1.24 and local martingale deflator (Definition 1.22). □

Theorem 1.26. *Let $\mathcal{G}$ be an expansion of $\mathcal{F}$ and M be a $\mathcal{F}$-local martingale. If M has an information drift α and $\mathcal{E}(\alpha \cdot M)$ is uniformly integrable, then we can define an ELLM $\mathbb{Q}$ by $\mathbb{Q} := \mathcal{E}(\alpha \cdot M) \cdot \mathbb{P}$.*

Proof. This is an immediate combination of Theorem 1.20 and Proposition 1.25. □

Lemma 1.27. *Let M be a local martingale and $\alpha \in \mathcal{S}$. The following statements are equivalent:*

(i) $\mathcal{E}(\alpha \cdot M)$ is a martingale

(ii) $\mathcal{E}(\alpha \cdot M)$ is a uniformly integrable martingale

(iii) $\mathbb{E}\mathcal{E}(\alpha \cdot M) = 1$

Proof. A stochastic exponential $\mathcal{E}(\alpha \cdot M)$ is a local martingale, hence a supermartingale, and in particular

$$\mathbb{E}\mathcal{E}(\alpha \cdot M) \leq 1.$$

□

When M is continuous, the uniform integrability of a stochastic exponential process can finally be checked classically with Novikov's and Kamazaki's criterion.

Theorem 1.28 (Kamazaki). *Let M be a continuous local martingale and $\alpha \in \mathcal{S}$. The exponential martingale $\mathcal{E}(\alpha \cdot M)$ is (uniformly) integrable if*

$$\sup_{\tau} \mathbb{E} e^{\frac{1}{2}\int_0^{\tau} \alpha_s^2 dM_s} < \infty,$$

where the sup *is taken over all stopping times in I.*

Proof. We omit the proof of this classical result for which we refer the reader to Protter (2004) (Part III, Chapter 8, Theorem 44). □

It is often more practical to check the slightly stronger condition provided by Novikov.

Theorem 1.29 (Novikov). *Let M be a continuous local martingale and $\alpha \in \mathcal{S}$. The exponential martingale $\mathcal{E}(\alpha \cdot M)$ is (uniformly) integrable if*

$$\mathbb{E} e^{\frac{1}{2}\int_I \alpha_s^2 d[M,M]_s} < \infty.$$

Proof. We only need to show that the hypothesis implies Kamazaki's condition. Letting $Z := \alpha \cdot M$, if follows from the definition of the stochastic exponential that

$$e^{Z_t} = \mathcal{E}(Z)_t e^{\frac{1}{2}[Z,Z]_t}$$

so that Cauchy-Schwarz inequality together with the bound $\mathbb{E}\mathcal{E}(Z) \leq 1$ imply that for every $t \in I$

$$\mathbb{E}e^{Z_t} \leq \mathbb{E}e^{\frac{1}{2}[Z,Z]_t} \leq \mathbb{E}e^{\frac{1}{2}[Z,Z]_\infty},$$

where we denote $[Z,Z]_\infty$ the monotone limit of $[Z,Z]_t$ as t increases. □

1.4 Filtration shrinkage

The converse phenomenon, *filtration shrinkage*, studied in particular in Stricker (1977) and Föllmer and Protter (2011), brings an interesting light on expansions of filtrations.

1.4.1 Optional projections

Proposition-Definition 1.30 (Optional and predictable projections). *Let $\mathcal{Y}$ be a filtration satisfying the usual hypotheses and Y a càdlàg process. If there exists a $\mathcal{Y}$-localizing sequence that makes Y integrable, then there exists a unique càdlàg process ${}^{(o)}Y$ (up to a modification) such that for every $t \in I$, ${}^{(o)}Y_t = \mathbb{E}[Y|\mathcal{Y}_t]$ $a.s.$, which we call the **optional projection** of Y onto $\mathcal{Y}$.*

If furthermore Y is a $\mathcal{Y}$-semimartingale then there exists a unique $\mathcal{Y}$-predictable pro-

cess ${}^{(p)}Y$ *(up to a modification) such that* ${}^{(o)}Y - {}^{(p)}Y$ *is a* $\mathcal{Y}$*-martingale - the compensator of* ${}^{(o)}Y$ *- and we call* ${}^{(p)}Y$ *the* ***predictable projection*** *of* Y *onto* $\mathcal{Y}$.

Proof. See for instance Dellacherie and Meyer (1980, p. 112 & following). □

When there is no ambiguity we may abusively denote $\mathbb{E}[Y_.|\mathcal{Y}_.]$ the optional projection of Y onto $\mathcal{Y}$.

1.4.2 Filtration shrinkage and transformation of semimartingales

In this paragraph we let $\mathcal{Y} \subset \mathcal{Z}$ be two filtrations containing $\mathcal{F}$ and satisfying the usual hypotheses. The transformation of martingales through a filtration shrinkage is relatively straightforward.

Proposition 1.31. *The optional projection of a* $\mathcal{Z}$*-martingale on* $\mathcal{Y}$ *is a* $\mathcal{Y}$*-progressively measurable process and a* $\mathcal{Y}$*-martingale.*

Proof. It is immediate to verify that for every martingale Z a process Y satisfying $\forall t \in I, Y_t = \mathbb{E}[Z_t|\mathcal{Z}_t]$ a.s. is a $\mathcal{Y}$-martingale. □

Theorem 1.32 (Stricker's theorem). *A* $\mathcal{Z}$*-semimartingale is a* $\mathcal{Y}$*-semimartingale as long as it is adapted.*

Note that $\mathcal{Y}$ *and* $\mathcal{Z}$ *do not need to satisfy the usual hypotheses here.*

Proof. For us, this is a direct consequence of the Bitcheler-Dellacherie characterization of semimartingales. □

Föllmer and Protter (2011) refine Stricker's theorem with a study of the semimartingale decomposition in the smaller filtration. The hypotheses required on the semimartingale are summarized by Protter (see Protter (2004)) in the notion of $\mathcal{Y}$-**special**, $\mathcal{Z}$-**semimartingale**.

Definition 1.33. *A $\mathcal{Z}$-special semimartingale is called a $\mathcal{Y}$-**special**, $\mathcal{Z}$-**semimartingale** if the process M in its canonical decomposition $S = S_0 + M + A$ is a $\mathcal{Z}$-martingale up to a $\mathcal{Y}$-localizing sequence.*

Theorem 1.34 (Föllmer and Protter (2011))**.** *If S is a $\mathcal{Y}$-special, $\mathcal{Z}$-semimartingale with canonical decomposition $S =: S_0 + M + A$, then ${}^{(o)}S$ is a $\mathcal{Y}$-semimartingale with decomposition $S = {}^{(o)}M + {}^{(o)}A$, where the optional projections are taken with respect to $\mathcal{Y}$.*

In the case where $\mathcal{Y}$ and $\mathcal{Z}$ contain $\mathcal{F}$ the previous theorem has an interesting corollary for continuous $\mathcal{F}$-semimartingales.

Theorem 1.35. *Let S be a continuous $\mathcal{F}$-semimartingale and suppose that $\mathcal{F} \subset \mathcal{Y} \subset \mathcal{Z}$.*

If S is a $\mathcal{Z}$-semimartingale with (canonical) decomposition $S =: M + A$, then S is a $\mathcal{Y}$-semimartingale with (canonical) decomposition $M = {}^{(o)}M + {}^{(o)}A$, where the optional projections are taken with respect to $\mathcal{Y}$.

Proof. Since S is a $\mathcal{F}$-semimartingale, there exists a $\mathcal{F}$-localizing sequence that makes S, M and A integrable so that the optional projections ${}^{(o)}M$ and ${}^{(o)}A$ are well defined. On the other hand S is locally a $\mathcal{Z}$-quasimartingale so that by Rao's theorem it is also a $\mathcal{Y}$-quasimartingale (see for instance Protter (2004)), and admits the canonical decomposition

${}^{(o)}S = ({}^{(o)} + {}^{(o)}A - {}^{(p)}A) + {}^{(p)}A$ (as an application of Proposition 1.31). The conclusion follows from the continuity of S. □

We can also derive from the filtration shrinkage theorem a useful tool to study the information drift of nested enlargements.

Lemma 1.36 (Extension lemma). *Let M be a continuous $\mathcal{F}$-local martingale and suppose that $\mathcal{F} \subset \mathcal{Y} \subset \mathcal{Z}$. For every $\alpha \in \mathcal{S}(\mathcal{Z})$, if $M - \int_0^{\cdot} \alpha_s d[M, M]_s$ is a $\mathcal{Z}$-local martingale, then $M - \int_0^{\cdot} \mathbb{E}[\alpha_s|\mathcal{Y}_s]d[M, M]_s$ is a $\mathcal{Y}$-local martingale.*

If furthermore W is a $\mathcal{F}$-Brownian motion and $W - \int_0^{\cdot} \alpha_s ds$ is a $\mathcal{Z}$-Brownian motion, then $W - \int_0^{\cdot} \mathbb{E}[\alpha_s|\mathcal{Y}_s]ds$ is a $\mathcal{Y}$-Brownian motion.

Here $\mathbb{E}[\alpha_{\cdot}|\mathcal{Y}_{\cdot}]$ implicitly stands for the $\mathcal{Y}$-progressively measurable version of the optional projection of α, which is in $\mathcal{S}(\mathcal{Y})$.

Proof. The proof for local martingales is identical to the one for Brownian motion. Define $\overline{W} := W - \int_0^{\cdot} \mathbb{E}[\alpha_s|\mathcal{Y}_s]ds$, where $\mathbb{E}[\alpha_{\cdot}|\mathcal{Y}_{\cdot}]$ is a $\mathcal{Y}$-progressively measurable version of the process (see Proposition 1.30). For $s < t$,

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[\overline{W}_t|\mathcal{Y}_t]|\mathcal{Y}_s] &= \mathbb{E}[\overline{W}_t|\mathcal{Y}_s] \\
&= \mathbb{E}[\mathbb{E}[\overline{W}_t|\mathcal{Z}_s]|\mathcal{Y}_s] \\
&= \mathbb{E}[\overline{W}_s|\mathcal{Y}_s] \\
&= \overline{W}_s
\end{aligned}$$

so that $\mathbb{E}[\overline{W}_{\cdot}|\mathcal{Y}_{\cdot}]$ is a $\mathcal{Y}$-martingale, and we can conclude according to Levy's characterization. □

Note that a slighty different version of this result can be found in Corcuera et al. (2004, Proposition 1).

1.5 Expansion with independent information

The case of enlargement with independent information is simple.

Proposition 1.37 (Augmentation with independent information). *Let $\mathcal{J}$ be a filtration independent from $\mathcal{F}_\infty$. Then a $\mathcal{F}$-local martingale (resp. martingale, semimartingale) M is a $\mathcal{F} \vee \mathcal{J}$-local martingale (resp. martingale, semimartingale).*

Proof. By Proposition 1.39 the case of semimartingales follows from the case of local martingales, which reduces to the case of martingales up to a $\mathcal{F}$-localizing sequence we can suppose that makes M integrable.

For every $t \geq 0$, since $\mathcal{F}_t$ and $\mathcal{J}_t$ are independent, we have

$$\mathbb{E}[M_t|\mathcal{F}_t \vee \mathcal{J}_t] = \mathbb{E}[M_t|\mathcal{F}_t] = M_t \; a.s.,$$

so that M is a $\mathcal{F} \vee \mathcal{J}$-martingale (up to the same $\mathcal{F}$-localizing sequence). □

Combining Proposition 1.37 and Lemma 1.36 we can understand the impact of an independent perturbation of an expansion.

Corollary 1.38. *Suppose given expansions $\mathcal{Y}, \mathcal{G}$ of $\mathcal{F}$ and another filtration $\mathcal{J} \perp\!\!\!\perp \mathcal{G}$ such that $\mathcal{F} \subset Y \subset \mathcal{G} \vee \mathcal{J}$. If S is a continuous $\mathcal{F}$-semimartingale (or local-martingale) and a $\mathcal{G}$-semimartingale with (canonical) decomposition $S =: M + \int_0^{\cdot} \alpha_s d[M,M]_s$, where $\alpha \in \mathcal{S}(\mathcal{G})$, then it is a $\mathcal{Y}$-semimartingale with (canonical) decomposition*

$$S = {}^o M + \int_0^{\cdot} \mathbb{E}[\alpha_s|\mathcal{Y}_s] d[M,M]_s.$$

For instance, if W is a $\mathcal{F}$-Brownian motion and $W - \int_0^{\cdot} \alpha_s ds$ is a $\mathcal{G}$-Brownian motion for some $\alpha \in \mathcal{S}(\mathcal{G})$, then $W - \int_0^{\cdot} \mathbb{E}[\alpha_s|\mathcal{Y}_s] ds$ is a $\mathcal{Y}$-Brownian motion.

Proof. Since $\mathcal{J}$ is independent from $\mathcal{G}$, then $S - \int_0^{\cdot} \alpha_s d[M,M]_s$ defines a local martingale for $\mathcal{G} \vee \mathcal{J}$. As $\mathcal{Y} \subset \mathcal{G} \vee \mathcal{J}$, the proof follows from the Lemma 1.36.

The case of Brownian motion follows from Levy's characterization. □

The reader will find an application of this result in Section 5.2.

1.6 About the usual hypotheses

In many applications it is expected that filtrations satisfy the usual hypotheses. This is in general a minor assumption, and we show in this paragraph that it makes no difference in regards to our problem.

Since semimartingales and local martingales are defined up to the choice of a version, it is straightforward that adding null sets to the filtration does not alter their properties (this is also a consequence of Proposition 1.37). The next proposition tackles the question of right-continuity.

Proposition 1.39. *Let $\check{\mathcal{Y}}$ be a filtration and $\mathcal{Y}$ the right-continuous filtration defined by $\mathcal{Y}_t := \bigcap_{u>t} \check{\mathcal{Y}}_u$. For every $\check{\mathcal{Y}}$-adapted process S, the following assertions are equivalent:*

(i) S is a $\check{\mathcal{Y}}$-semimartingale with decomposition $M + A$

(ii) S is a $\mathcal{Y}$-semimartingale with decomposition $M + A$

Proof. We first let S be a $\check{\mathcal{Y}}$-semimartingale with decomposition $M + A$ and prove with the direct implication. It is sufficient to prove that M is a $\mathcal{Y}$-local martingale, and since $\check{\mathcal{Y}}$-stopping times remain $\mathcal{Y}$-stopping times, and we can assume that M is a $\check{\mathcal{Y}}$-martingale. Fix $0 \leq s \leq t$. $\left(\mathbb{E}[M_t|\check{\mathcal{Y}}_{s+\frac{1}{n}}]\right)_{n\geq 1}$ defines a discrete backwards martingale which therefore converges almost surely to $\mathbb{E}[M_t|\bigcap_{n\geq 1}\check{\mathcal{Y}}_{s+\frac{1}{n}}] = \mathbb{E}[M_t|\mathcal{Y}_s]$. On the other hand as M is a $\check{\mathcal{Y}}$-martingale $\mathbb{E}[M_t|\check{\mathcal{Y}}_{s+\frac{1}{n}}] = M_{s+\frac{1}{n}} \xrightarrow[n\to\infty]{a.s.} M_s$. Identifying the two limits proves that M is a $\mathcal{Y}$-martingale.

Conversely, let S be a $\mathcal{Y}$-semimartingale with decomposition $M + A$. It is sufficient to prove that M is $\check{\mathcal{Y}}$-adapted, and as M is continuous, $M_t = \lim_{n\to\infty} M_{t-1/n}$ so that M_t is measurable with respect to $\bigvee_{n\geq 1}\mathcal{Y}_{s-1/n} \subset \check{\mathcal{Y}}_s$. □

In practice considering expansions satisfying the usual hypotheses allows to apply widespread fundamental results for which they are required, such as martingale representation and predictable representation theorems, or the properties of special semimartingales.

1.7 Initial expansion with a random variable

Initial expansions compose the simplest class of exansions of filtrations.

Definition 1.40 (Initial expansion). *The initial expansion of $\mathcal{F}$ with a random variable L at some stopping time τ is the smallest right-continuous filtration containing $\mathcal{F}$ that for which L is measurable at time τ, i.e.*

$$\mathcal{G}_t := \bigcap_{\epsilon>0} \left(\mathcal{F}_{t+\epsilon} \vee \sigma(L) 1_{t+\epsilon \geq \tau}\right).$$

The time τ, at which the additional information becomes available in $\mathcal{G}$, appears of secondary importance, and most authors study the case $\tau = 0$. Note that the case of initial expansions with a "large" σ-algebra $\mathcal{H}_0$ has also been considered: results are qualitatively similar, and the main difference is that explicit formulas can't be found if the nature of $\mathcal{H}_0$ is not known.

Proposition-Definition 1.41 (Jacod's condition). *A random variable L satisfies **Jacod's condition** if for almost every $t \in I$ there exists a (non-random) σ-finite measure η_t on the Polish space $(E, \mathcal{E})$ in which it takes values such that*

$$P_t(\omega, L \in .) \ll \eta_t(.) \; a.s.$$

η_t can be assumed to be a constant η, or $\mathbb{P}(L \in .)$.

If L satisfies Jacod's condition we define the conditional densities

$$q_t^L(\omega, x) := \frac{P_t(\omega, L \in dx)}{\eta(dx)},$$

which we also call the ***Jacod's conditional densities****.*

Proof. See Jacod (1985). □

Theorem 1.42 (Jacod (1985)). *Suppose that L satisfies Jacod's condition. Then:*

(i) every $\mathcal{F}$-semimartingale is a $\mathcal{G}$-semimartingale

(ii) if M is a continuous $\mathcal{F}$-local martingale, then it is a $\mathcal{G}$-semimartingale with decomposition

$$M = \widetilde{M} + \int_0^{\cdot} \gamma_s(\omega, L(\omega)) d[M, M]_s$$

where $\widetilde{M}$ is a $\mathcal{G}$-local martingale and γ is the family of $\mathcal{G}$-predictable process in $\mathcal{S}(\mathcal{G})$ given by

$$\gamma_t(., x) = \frac{1}{q_t(., x)} \frac{d[q(., x), M]_t}{d[M, M]_t}.$$

1.8 Progressive expansion with a random time

Definition 1.43 (Progressive expansion). *A progressive expansion of $\mathcal{F}$ with a random time $\tau \in [0, \infty]$ is the smallest right-continuous filtration $\mathcal{G}$ containing $\mathcal{F}$ for which τ is a stopping time, given by*

$$\mathcal{G}_t := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \vee (t + \epsilon)).$$

Although they are of intrinsic and applied interest, progressive expansions with random times are not directly related to our argument, and we have nothing to add to the literature. For conciseness, we refer the reader to Barlow (1979) and Jeulin (1980) for an historical perspective, as well as Kchia, Larsson, and Protter (2013) and Kchia and Protter (2015) for an overview of modern results as well as a unified approach to initial and progressive expansions.

Chapter 2

A financial perspective

In this chapter we let $\mathcal{G}$ be an expansion of the filtration $\mathcal{F}$, for which we conveniently denote $\mathcal{G}_t =: \mathcal{F}_t \vee \mathcal{H}_t$, and we let S be a $\mathcal{F}$-semimartingale with decomposition $S =: S_0 + M + A$.

* * * * *

In the paradigm of *insider trading*, the filtration $\mathcal{F}$ represents the *public information* (which is supposed homogeneous in a mean-field approximation), the expansion $\mathcal{G} := \mathcal{F} \vee \mathcal{H}$ represent the *insider information* of a particular agent with better information, and the semimartingale S represents the price process of a security (or a vector of price processes for various securities).

Note here that **all** usual models for the security prices are semimartingales, including parametric or stochastic volatility models as well as Markov or non-Markov diffusions, with or without jumps. The semimartingale property is actually necessary to evaluate the payoff of continuous time strategies as it requires integration with respect to the price process (see Paragraph 1.1.3).

We emphasize that although expansions of filtrations have been extensively related to insider trading, from which we consistently borrow some terminology, it can actually recover a broad spectrum of imbalances between game players, inside or outside financial markets, such as the sophistication level, or the topical trading frequency, where the additional information of the informed player can be modeled either with a direct information on the future or access to *hidden variables*.

2.1 Absence of arbitrage(s) and risk-neutral measure

Absence of arbitrage - "no free money without risk" - is a key feature of financial markets, and a fundamental axiom at the macroscopic scale for many practitioners (see for instance Delbaen and Schachermayer (1994) for a discussion). It has been translated into mathematics with slight variations.

Definition 2.1 (Arbitrage).

(i) A (strict) arbitrage is a strategy $H \in \mathcal{S}$ such that

$$V := \int_I H_u dS_u \geq 0 \text{ a.s. and } \mathbb{P}(V > 0) > 0.$$

(ii) An arbitrage of the first kind (or unbounded profit with bounded risk) is a family of strategies $H^\epsilon \in \mathcal{S}, \epsilon > 0$, such that there exists a non-negative random variable V with $\mathbb{P}(V > 0) > 0$ for which

$$\forall \epsilon > 0, \forall t \in I, \epsilon + \int_0^t H_u^\epsilon dS_u \geq 0 \text{ a.s and } \epsilon + \int_I H_u^\epsilon dS_u \geq V \text{ a.s.}$$

(iii) A free lunch with vanishing risk is a family of strategies $H^\epsilon \in \mathcal{S}, \epsilon > 0$, such that there exists a positive number V for which

$$\forall \epsilon > 0, \epsilon + \int_I H_u^\epsilon dS_u \geq 0 \text{ a.s. and } \mathbb{P}(\int_I H_u^\epsilon dS_u \geq V) > 0$$

We refer to absence of such arbitrage respectively as NA (No Arbitrage), NA1 (No Arbitrage of the First kind), and NFLVR (No Free Lunch with Vanishing Risk).

Theorem 2.2 below, also known as the (First) Fundamental Theorem of Asset Pricing (FTAP), turned NFLVR into a gold standard in mathematical finance by justifying the risk neutral pricing, widespread among market practitioners. However NA1 has been appeared particularly meaningful in the context of stochastic portfolio theory (see Kardaras (2012)). Those three notions of arbitrage are nevertheles closely related: NFLVR is equivalent to the combination of NA and NA1 (see Fontana (2015)), and Kabanov, Kardaras, and Song (2016) also shows that under NA1 a local martingale deflator is always a local martingale numéraire for a probability measure arbitrarily close in total variation.

Theorem 2.2 (FTAP(Delbaen and Schachermayer (1994))). *NFLVR is equivalent to the existence of an equivalent local martingale measure (for S), i.e. a probability $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{A}$ such that S is a $\mathbb{Q}$-local martingale.*

The FTAP shows that assuming NFLVR for the public market is equivalent to supposing that the price process S is the $\mathcal{F}$-local martingale M, up to an equivalent change of probability. The next question is naturally whether the filtration $\mathcal{G}$ allows in that case for arbitrage opportunities.

Corollary 2.3. *Suppose that the filtration $\mathcal{F}$ satisfies NFLVR for S. Then the filtration $\mathcal{G}$ satisfies NFLVR for S if and only if there exists an equivalent probability under which M is a $\mathcal{G}$-semimartingale.*

The uniform integrability required in NFLVR to define an equivalent change of probability is not always meaningful and often difficult to check. It is dropped in the weaker condition NA1, which admits an almost-as-elegant characterization.

Theorem 2.4 (Weak FTAP (Kabanov, Kardaras, and Song (2016))). *NA1 for a S is equivalent to the existence of a positive local martingale Z such that ZS is a local martingale.*

Corollary 2.5. *Suppose that the filtration $\mathcal{F}$ satisfies NA1 for S. Then the filtration $\mathcal{G}$ satisfies NA1 for S if and only if there exists a $\mathcal{G}$-positive local martingale Z such that ZM is a local martingale, i.e. $\mathcal{G}$ satisfies NA1 for M.*

Our Theorem 1.20 can now be translated in terms of NA1.

Theorem 2.6 (∗). *$\mathcal{G}$ satisfies NA1 for the $\mathcal{F}$-local martingale M if and only if $\mathcal{G}$ admits an information drift for M, i.e. a process $\alpha \in \mathcal{S}(\mathcal{G})$ such that $M - \int_0^\cdot \alpha_s d[M,M]_s$ is a $\mathcal{G}$-local martingale.*

We can also translate Theorem 1.20, which states that NFLVR is satisfied for the filtration $\mathcal{G}$ if $\mathcal{E}(\alpha \cdot M)$ is a uniformly integrable martingale. Under the assumption that the market is complete, this condition is necessary and sufficient.

Theorem 2.7. *Suppose that the markets are complete, i.e. $\mathcal{F}_t := \bigcap_{u>t} \sigma(M_s, s \leq u)$, and that NA1 holds for the local martingale M. Then NFLVR is satisfied for the filtration $\mathcal{G}$ if and only if $\mathcal{E}(\alpha \cdot M)$ is a uniformly integrable martingale.*

Additionally, even in incomplete markets, one has a way to check for a violation of NFLVR without discussing uniform integrability of the local martingale deflator.

Theorem 2.8 (Immediate arbitrage[1]). *$(NFLVR)$ is violated if* $\mathbb{P}(\int_0^T \alpha_s^2 d[M,M]_s = \infty) > 0$. *We say that such situation creates an* ***immediate arbitrage*** *(opportunity).*

Proof. See for instance Imkeller (2002, Proposition 1.1). □

2.2 Portfolio optimization with additional information

Theorem 2.6 shows that absence of (strict) arbitrage opportunities with the insider information requires the existence of the information drift, which defines in that case the predictable component of the dynamics of the price process M for the insider.

Suppose accordingly that M is a $\mathcal{G}$-semimartingale with decomposition $M = \widetilde{M} + \alpha \cdot [M,M]$ for some process $\alpha \in \mathcal{S}(\mathcal{G})$. We attempt here to understand the statistical value of the additional information beyond strict arbitrage by expressing quantitatively the statistical advantage of the insider in some simple portfolio optimization problems with asymmetric information.

Maximization of logarithmic utility Extending to general semimartingales the classical problem of logarithmic utility maximization introduced in the seminal article of Karatzas and Pikovsky (1996), where the optimization program of an agent with initial wealth x

[1]The term can already be found in Imkeller (2002)

and filtration $\mathcal{Y}$ is given by

$$sup_{H \in \mathcal{S}(\mathcal{Y})} \mathbb{E} \log \left(x + (H \cdot M)_T \right)$$
$$s.t.\ x + H \cdot M \geq 0 \ on \ I$$

Ankirchner, Dereich, and Imkeller (2006) proves that, if $\mathfrak{u}(x, \mathcal{Y})$ is the value of the above problem,

$$\mathfrak{u}(x, \mathcal{G}) - \mathfrak{u}(x, \mathcal{F}) = \mathbb{E} \int_0^T \alpha_s^2 d[M, M]_s.$$

Constrained maximization of returns It seems to us closer to practice to consider instead a classical risk-return framework, where the optimization program of an agent with filtration $\mathcal{Y}$, initial wealth x and risk aversion λ is given by

$$sup_{H \in \mathcal{S}(\mathcal{Y})} V^{\lambda}(x, \mathcal{Y}) := x + \mathbb{E}(H \cdot M)_T - \lambda \mathbb{V}ar(H \cdot M)_T$$
$$s.t.\ x + H \cdot M \geq 0 \ on \ I.$$

Let $\mathfrak{v}^{\lambda}(x, \mathcal{Y})$ be the value of the above problem. For $H \in \mathcal{S}(\mathcal{Y})$, let $V(x, H) := \mathbb{E}[H \cdot M]$ be the corresponding expected return process, and if H^* is an optimal strategy also let $v^{\lambda}(x, \mathcal{Y}) = V(x, H^*)$.

For an agent with filtration $\mathcal{F}$, $\mathbb{P}$ is a risk-neutral probability and $H^* = 0$ is an optimal strategy, with corresponding expected return $V(x, H^*) = x$ and risk-adjusted return $V^{\lambda}(x, H^*) = x$.

However, for an agent with filtration $\mathcal{G}$,

$$\mathbb{E}[H \cdot M] = \mathbb{E}\big[H \cdot \widetilde{M} + \int_0^{\cdot} \alpha_s H_s d[M,M]_s - \lambda \int_0^{\cdot} H_s^2 d[M,M]_s\big]$$

and is maximal for the $\mathcal{G}$-predictable strategy $H_s^* = \frac{\alpha_s}{2\lambda}, s \in I$, with corresponding expected return

$$V(x, H^*) = x + \mathbb{E}\int_0^{\cdot} \alpha_s H_s^* d[M,M]_s = x + \mathbb{E}\int_0^{\cdot} \frac{\alpha_s^2}{2\lambda} d[M,M]_s$$

and risk-adjusted return

$$V^{\lambda}(x, H^*) = x + \frac{1}{2}\mathbb{E}\int_0^{\cdot} \alpha_s H_s^* d[M,M]_s = x + \mathbb{E}\int_0^{\cdot} \frac{\alpha^2}{4\lambda} d[M,M]_s.$$

We can conclude that

$$v^{\lambda}(x, \mathcal{G}) - v^{\lambda}(x, \mathcal{F}) = \mathbb{E}\int_0^{T} \frac{\alpha_s^2}{2\lambda} d[M,M]_s. \tag{2.1}$$

We emphasize that $\mathbb{E}\int_0^T \frac{\alpha_s^2}{2\lambda} d[M,M]_s$ is therefore the $ value of the additional information, as it corresponds to the average additional value of the portfolio of a trader with information $\mathcal{G}$ (behaving optimally and with the same risk aversion).

The difference between the optimal risk-adjusted returns is also proportional to the L^2 norm of α:

$$\mathfrak{v}^\lambda(x, \mathcal{G}) - \mathfrak{v}^\lambda(x, \mathcal{F}) = \mathbb{E} \int_0^T \frac{\alpha_s^2}{4\lambda} d[M, M]_s. \tag{2.2}$$

Other examples One can search formulas for more general utility functions by taking the expectation of the decomposition of $u(H \cdot M)$ via Itô's formula.

2.3 High-frequency trading

The framework of expansion of filtrations with a stochastic process, which we study in the next chapters, can apply in particular to model the informational advantage of a high-frequency trader.

Our starting point is the postulate that high-frequency traders can observe the limit order book by posting and canceling limit orders and observing the market response (see for instance Lewis (2014)). According to Lewis (and many followers), the information contained in the supply curve provides crucial advantages to a HF trader, one of them being *front-running* opportunities for which the following cynical definition can be found on the mainstream investment website www.investopedia.com:

> "'Electronic front-running' [...] involves a HFT firm racing ahead of a large client order on an exchange, scooping up all the shares on offer at various other exchanges (if it is a buy order) or hitting all the bids (if it is a sell order), and then turning around and selling them to (or buying them from) the client and pocketing the difference."(Elvis Picardo (2016))

More recently complains arose against another type of HF behavior, *spoofing*, which according to Richard Ketchum, FINRA's chairman and chief executive, corresponds to "a type of manipulation that involves faking orders for a security to deceive the market by creating the illusion of demand" (Barlyn and Ajmera (2016)).

In this section, we first rely on a high-frequency (HF) analysis of limit order book data, which will be further developed in an upcoming article Neufcourt, Protter, and Wang (2017), to show that the knowledge of some features of the supply curve, often alleged for HF traders, allows indeed to predict the direction of the next price moves with an accuracy far above usual standards, and to build elementary low-exposure front-running strategies with returns that many quantitative traders could only dream of. We propose in a second part a model to describe this phenomenon based of an expansion of filtration with an anticipative stochastic process.

2.3.1 The limit order book

In most stock exchanges, market trades for a given security are executed at the atomic level through a Limit Order Book (LOB). The LOB is composed of two priority lists, one for the 'bid' side (buyers of the security) and one for the 'ask' side (sellers of the security), with which market agents can interact *via* two types of orders, limit and market orders. Experts will notice that we omit the various - and technical - parameters of the limit and market orders: they are not unrelated to our argument, but negligible in our elementary approach.

Limit orders come with a fixed *limit price* on either the 'bid' or the 'ask' side and are inserted in the limit order book according to the lexicographical order for price and time: for a given price, the First-In-First-Out (FIFO) rule is applied, but a limit order with a better price will be placed before older limit orders with a less competitive price. Once placed in the LOB limit orders wait to be executed at a price at least as advantageous as their limit price. We emphasize that on the 'bid' side (respectively 'ask' side) orders with a higher price (respectively lower price) have a higher priority, and orders with identical prices are executed with First-In-First-Out priority. On the contrary, market orders come with a side ('buy' or 'sell') but no fixed price, and are executed directly against the best matching limit order in the corresponding side of the LOB.

Admissible prices take values in a discrete set $\mathcal{P}$ with a fixed **price tick** δ: $\mathcal{P} := \{n\delta, n = m, \ldots, M\}$ for some integers $m < M$. Naturally if at any time the best limit bid order and the best limit ask orders are compatible they are executed against each other. It follows that the **ask price** $(a_t)_{t\geq 0}$ (lowest price among all ask limit orders) and the **bid price** $(b_t)_{t\geq 0}$ (highest price among all bid limit orders) have a strictly positive difference, the **spread** $s_t := a_t - b_t > 0, t \geq 0$. We also define the **midprice** $m_t := \frac{a_t+b_t}{2}$, which satisfies $b_t < m_t < a_t$.

The aggregation of all limit orders forms the supply curve $\mathfrak{S} : [0,\infty) \times \mathcal{P} \to [0,\infty)$; precisely, $\mathfrak{S}_t(p)$ is the number of shares available at time t and price tick p in the LOB, which we can model at the market microstructure level with a **finite** family of càdlàg jump processes (obviously highly correlated).

2.3.2 Front-running the supply curve in a market replay

We study here evidence in LOB market data of the statistical advantage associated to information on the supply curve. Our empirical analysis relies on high-frequency data of the limit order book of ten stocks of the NASDAQ-INET stock exchange between August 1, 2013 and August 31, 2013.

In this first approach we only suppose that we can observe when the number of orders at the best bid and best ask pass a given threshold, and evaluate a simple prediction method for the direction of the next price jump as well as an elementary front-running strategy.

2.3.2.1 Prediction of price direction from LOB imbalances

With the notations introduced previously, we consider the prediction $\hat{\Delta}$ defined at every time $t \geq 0$ by:

$$\hat{\Delta}_t := 1_{\{\mathfrak{S}_t(b_t) - \mathfrak{S}_t(a_t) \geq h^+\}} - 1_{\{\mathfrak{S}_t(a_t) - \mathfrak{S}_t(b_t) \geq h^-\}}.$$

In other words, we predict:

- if $\mathfrak{S}_t(a_t) - \mathfrak{S}_t(b_t) \geq h^+$, the price will go down
- if $\mathfrak{S}_t(a_t) - \mathfrak{S}_t(b_t) \leq h^-$, the price will go up

Note that this is a partial prediction, in the sense that we make a prediction only at selected times when one of the two conditions is satisfied, which means that we are confident enough in our prediction on the direction of the next price jump. Here h^+ and h^- are positive thresholds that could be optimized over many parameters - but we fix them at $h^+ = -h^- = 2000$ without any deep analysis as it seemed a reasonable sensitivity given the usual quantities of orders at the main ticks.

Samples of the daily success rates achieved with this method are given in Table 2.1 (several stocks on a the same day) and Figure 2.1a (one stock on different days).

Stock	CSCO	DELL	F	GE	MSFT	T
Prediction success	0.87	0.84	0.84	0.86	0.86	0.83

Table 2.1: Prediction success for several NASDAQ stocks on August 1, 2013

We emphasize that such figures are much higher than usual success rate - for a top-tier quantitative hedge fund trader, 60% of prediction success over a significant period would already be exceptional.

2.3.2.2 Front-running strategies from LOB imbalances

We go further by using the information on the supply curve to design systematic high-frequency trading strategies, which we implement on the limit order book data at a low volume that - we assume - would not impact the agents' behavior. In this first approach we consider only one elementary directional strategy combining one limit order and one market order, which we name a *butterfly order.*

Definition 2.9 (Butterfly order). *Given a security and a time $t \geq 0$ we define a positive (resp. negative) butterfly order as the combination of*

- *a market order to buy (resp. sell) one share of the security at the current ask price a_t (resp. at the current bid price b_t)*

- *a limit order to sell (resp. buy) one share, placed one tick above the current ask price (resp. one tick below the current bid price)*

Our simple front-running strategy is given in Algorithm 1. $\hat{\Delta}$ (which is actually very similar to the $\hat{\Delta}$ predicting the price direction in the previous paragraph) is used here as a meta-rule to define a safeguard preventing concurrent strategies to run simultaneously, which would lead to increasing the risk, by simply paying the spread as soon as the market goes against our prediction. This is probably sub-optimal - which only reinforces our argument that it is easy for a trader to benefit from the knowledge of the supply curve.

Here again the positive thresholds h_a, h_b, H_a and H_b could be optimized, as well as the number of butterfly orders N sent each time (which could be for instance naturally studied as $N(\mathfrak{S}_t(a_t), \mathfrak{S}_t(b_t)))$. For simplicity we choose for h_a, h_b, H_a and H_b constants values depending on the average volume in the LOB for each security.

The choice of N is of relative importance: the returns do not depend on N (both the initial investment and the final value are proportional to it), but N scales the potential value of the portfolio, and needs in the same time to be small enough to interfere little with the behavior of other agents interacting with the LOB. , We also choose a constant order volume $N := 100$, based on the following remarks:

- there are no “reasonable” orders of size inferior than 100
- in view of LOB data it is reasonable to consider that an order of size 100 would not modify significantly the behavior of the market agents

Algorithm 1: A simple front-running strategy

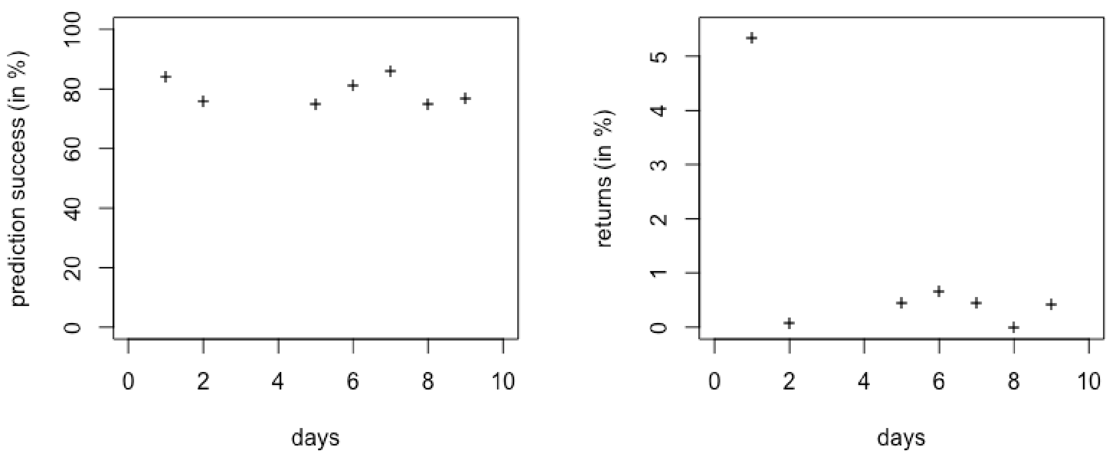
```
Δ̂ ← 0 for every t do
    if we have no active limit order then
        Δ̂ ← 0
    if 𝔖_t(a_t) ≤ h_a and 𝔖_t(b_t) ≥ H_b then
        if Δ̂ := 0 then
            buy N positive butterfly orders
            Δ̂ ← 1
        else if Δ̂ := −1 then
            cancel all limit orders
            clear position at market price
            Δ̂ ← 0
    else if 𝔖_t(b_t) ≤ h_b and 𝔖_t(a_t) ≥ H_a > 0 then
        if Δ̂ := 0 then
            buy N negative butterfly orders
            Δ̂ ← −1
        else if Δ̂ := +1 then
            cancel all limit orders
            clear position at market price
            Δ̂ ← 0
```

Samples of the daily returns achieved with this simple strategy are given in Table 2.2 (several stocks on a the same day) and Figure 2.1b (one stock on different days).

Stock	CSCO	DELL	F	GE	MSFT	T
Opening price (in \$)	25.68	12.91	16.93	24.64	32.01	35.38
Closing price (in \$)	25.90	13.13	17.03	24.85	31.61	35.65
Final value (in \$)	-	0.69	-	-	0.11	0.58
Daily return (in %)	-	5.34	-	-	0.34	1.64
Maximum risk (in \$)	0.02					

Table 2.2: Front-running strategy performance for several NASDAQ stocks on August 1, 2013

Returns. We define our returns very conservatively as $R_t := \frac{V_t}{S_0}$, where V_t is the value of our portfolio, and S_0 the value of the stock (at time 0). Here again we emphasize



(a) (b)

Figure 2.1: Prediction success (2.1a) and front-running strategy performance (2.1b) for DELL stock between August 1 and August 10, 2013

that such returns (which moreover do not take into account any potential leverage from margin trading) are very high, with 1% daily returns correponding to $1.01^{252} \approx 123\%$ annualized returns. As a comparison, Renaissance achieved the best performance among all US hedge funds in 2016 with annualized returns of 19.3% (see Porzecanski, Kumar, and Bloomberg (2016)).

Risk. In practice, our risk exposure is practically null: indeed, our conservative strategy doesn't allow to take positions with a risk higher than two price ticks, i.e. $\$0.02$, neither to take any position when the spread is more than one tick ($\$0.01$) - which explains that our strategy doesn't take any position at all for some stocks.

Limitations. The following limitations of our approach are to be noted:

1. Our strategy relies on the fact that the midprice m_t can only jump to $m_t \pm \frac{\delta}{2}$. In

practice this is true when the spread is equal to one, hence at most times for the most liquid stocks, but would need to be reviewed to allow non-zero positions for less liquid stocks.

2. We also implicitly suppose that posting limit orders is free (this is often true) and that canceling limit orders is also free (this is more approximate; the practice seems to be a fee of order 1% for canceling limit orders).

2.3.3 A model with expansions of filtrations

The performances of the prediction method and front-running strategy in the previous paragraph rely the key assumption that we can observe some features of the supply curve $\mathfrak{S}_t$ at every instant $t \geq 0$ (additionally to the public information). In other words the strategy that we designed is not measurable with respect the public filtration $\mathcal{F}$ - classically $\mathcal{F}_t := \sigma(S_u, u \leq t)$ where S is still our asset price - but only with respect to the expansion

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\mathfrak{S}_u(p), u \leq t, p \in \mathcal{P}).$$

We actually needed not to add the information generated by the whole supply curve $\mathfrak{S}$, but only by the processes $X_t(H, K) := 1_{\{\mathfrak{S}_t(a_t) \leq H\}} 1_{\{\mathfrak{S}_t(b_t) \leq K\}}$, for a few constants H, K (precisely 8 couples), namely

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(X^i_u, u \leq t, i = 1, \ldots, 8),$$

which is precisely the expansion of $\mathcal{F}$ with the processes $X^i, i = 1, \ldots, 8$ - and significantly smaller.

Hence, in a general extend, modeling the information of a HF trader as an expansion of the filtration with a stochastic process, of the form $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(X_u, u \leq t)$, is very appropriate. If we are able to access the information drift of the asset price S with respect to the expansion $\mathcal{G}$, we are also able to evaluate the statistical advantage of the HF trader, for instance with the techniques of Section 2.2. The problem is now to propose a model for the process X itself.

We obviously want X to be anticipative, which in our simple model corresponds to making X depend of the future of S. One approach, focused on the supply curve, would be to consider the dynamics of S at the microstructure level, express it relatively to $\mathfrak{S}$, or X, and deduce an inverse expression for X.

We can also take some distance from the microstructure phenomenon and the supply curve to consider more directly that the HF trader "sees" the price with some advance δ, which corresponds typically to take X_t to be some anticipated asset price $S_{t+\delta}$ modulate with some noise, e.g. with an expansion

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\mathcal{S}_{u+\delta_u} + N_u, u \leq t),$$

where δ (the HF advance) is a non-negative stochastic process independent from $\mathcal{F}$ and N a noise process also independent from $\mathcal{F}$. We refer the reader to Chapter 5 for a further study of those models.

One could also believe that at an intermediary scale X should behave very much like a diffusion or the solution of a SDE. A naive idea would be SDE of the form

$$dX_t = a(t, X_t)d[S, S]_t + b(t, X_t)dS_t.$$

However, unless the SDE is degenerate, we have $\sigma(X_u, u \leq t) = \sigma(S_u, u \leq t)$, and similarly to Paragraph 5.3.1 such a model induces strict arbitrage and there is in general no information drift. The case of degenerate SDEs is skimmed in Appendix B. More generally SDEs of the form

$$dX_t = a(t, X_t)d[Y, Y]_t + b(t, X_t)dY_t,$$

where Y is a stochastic process depending on S, could also be appropriate.

Chapter 3

Convergence of filtrations and expansion with a stochastic process

In this chapter $\mathcal{G}$ denotes an expansion of $\mathcal{F}$ and M is a $\mathcal{F}$ local martingale.

* * * * *

We study here the stability of semimartingales, their decomposition and their information drifts when the filtrations converge, for the main purpose of applying it to study the properties of an expansion of a filtration with a stochastic process. This chapter composes the core of the upcoming article Neufcourt and Protter (2017).

3.1 Convergence of filtrations and semimartingales

Kchia and Protter (2015) show that semimartingales are preserved by the pointwise weak convergence of σ-algebras introduced in Coquet, Mémin, and Slominski (2001) as long as the successive finite variation terms are uniformly bounded in L^1. However weak convergence of σ-algebras is not enough to access the semimartingale decomposition in the limit filtration.

Considering here a stronger notion of convergence derived from L^p norms, which is more adapted to conditional expectations due to its convexity, we obtain sensible conditions under which the stability of both the semimartingale property and the semimartingale decomposition.

We show in Section 3.2 that this convergence is also satisfied by the filtrations successively generated by refining samples of a large class of càdlàg process.

3.1.1 Convergence of σ-algebras and fitrations

Recall the notation

$$Y^n \xrightarrow[n\to\infty]{L^p} Y \iff \mathbb{E}[|Y^n - Y|^p] \xrightarrow[n\to\infty]{} 0.$$

Definition 3.1. *Let $p \geq 1$. We say that a sequence of σ-algebras $(\mathcal{Y}^n)_{n\geq 1}$ converges in L^p to a σ-algebra $\mathcal{Y}$ if one the following equivalent conditions are satisfied:*

(i) $\forall B \in \mathcal{Y}, \mathbb{E}[1_B|\mathcal{Y}_n] \xrightarrow[n\to\infty]{L^p} 1_B$

(ii) $\forall Y \in L^p(\mathcal{Y}, \mathbb{P}), \mathbb{E}[Y|\mathcal{Y}_n] \xrightarrow[n\to\infty]{L^p} Y$

We say that a sequence of filtrations $(\mathcal{Y}^n)_{n\geq 1}$ converges in L^p to a filtration $\mathcal{Y}$ if $\forall t \in I, \mathcal{Y}^n_t \xrightarrow[n\to\infty]{L^p} \mathcal{Y}_t$.

Remark 3.2. *The convergence above defines a topology on respectively the class of countably generated σ-algebras and the class of countably generated filtrations. In particular it also defines a topology on the class of filtrations generated by a (countable number of) càdlàg* *stochastic processes. Although it is not clear for us whether the whole class of σ-algebras (or filtrations) can be turned into a topological space for the above notion convergence, we still call it a topology for convenience.*

In this chapter we mostly use convergences in L^1 and L^2 although we can probably obtain weaker hypothesis for our convergence theorems by considering L^p spaces for $1 < p < 2$. Note that there is no uniqueness of the limits in the above definition: for instance, any sequence of σ-algebras converges in L^p to the trivial filtration, and any σ-algebra contained in a limiting σ-algebra is also a limiting σ-algebra.

The remainder of this subdivision establishes simple properties of convergence of filtrations in L^p, following corresponding properties of convergence of random variables, and using similar techniques as in Kchia and Protter (2015).

Proposition 3.3. *Let $1 \leq p \leq q$. Convergence of σ-algebras in L^q implies convergence in L^p.*

Remark 3.4 (Weak convergence). *Weak convergence in the sense of Coquet, Mémin, and Slominski (2001) and Kchia and Protter (2015) corresponds to requesting convergence in probability instead of L^p in the previous definition. It is also easy to see that weak convergence is weaker than convergence of σ-algebras in L^p for any $p \geq 1$.*

Proposition 3.5. *For every* ***non-decreasing*** *sequence of σ-algebras $(\mathcal{Y}^n)_{n\geq 1}$ and every $p \geq 1$ we have*

$$\mathcal{Y}_n \xrightarrow[n\to\infty]{L^p} \bigvee\nolimits_{n\in\mathbb{N}} \mathcal{Y}_n.$$

Proof. Let $\mathcal{Y} := \bigvee_{n\in\mathbb{N}} \mathcal{Y}_n$ and consider $Y \in L^p(\mathcal{Y}, \mathbb{P}), p \geq 1$. $(\mathbb{E}[Y|\mathcal{Y}^n])_{n\geq 1}$ is a closed (and uniformly integrable) martingale which converges to $\mathbb{E}[Y|\mathcal{Y}]$ a.s. and in L^p. □

This is also the consequence of the following more general property.

Proposition 3.6. *Let $\mathcal{Y}$ be a σ-algebra and $(\mathcal{Y}^n)_{n\geq 1}, (\mathcal{Z}^n)_{n\geq 1}$ two sequences of σ-algebras.*

$$\text{If } \mathcal{Y}^n \xrightarrow[n\to\infty]{L^2} \mathcal{Y} \text{ and } \mathcal{Y}^n \subset \mathcal{Z}^n \text{ then } \mathcal{Z}^n \xrightarrow[n\to\infty]{L^2} \mathcal{Y}.$$

Proof. Let $Y \in L^2(\mathcal{Y}, \mathbb{P})$. Since $\mathbb{E}[Y|\mathcal{Z}^n]$ minimizes $\|U - Y\|_{L^2}$ over all $\mathcal{Z}^n$-measurable random variables U, we have

$$\|\mathbb{E}[Y|\mathcal{Y}^n] - Y\|_{L^2} \geq \|\mathbb{E}[Y|\mathcal{Z}^n] - Y\|_{L^2}.$$

□

Corollary 3.7. *Let $\check{\mathcal{Y}}$ be a filtration, $(\check{\mathcal{Y}}^n)_{n\geq 1}$ a sequence of filtrations and consider the sequence of right-continuous filtrations defined by $\mathcal{Y}^n_t := \bigcap_{u>t} \check{\mathcal{Y}}^n_u, t \in I$.*

$$\text{If } \check{\mathcal{Y}}^n \xrightarrow[n\to\infty]{L^2} \check{\mathcal{Y}} \text{ then } \mathcal{Y}^n \xrightarrow[n\to\infty]{L^2} \check{\mathcal{Y}}.$$

Proposition 3.8. *Let $\mathcal{Y}$ be a σ-algebra, $(\mathcal{Y}^n)_{n\geq 1}$ a sequence of σ-algebras and $(Z^n)_{n\geq 1}$ a sequence of random variables.*

$$\text{If } \mathcal{Y}^n \xrightarrow[n\to\infty]{L^1} \mathcal{Y} \text{ and } Z^n \xrightarrow[n\to\infty]{L^1} Z\text{, then } \mathbb{E}[Z^n|\mathcal{Y}^n] \xrightarrow[n\to\infty]{L^1} \mathbb{E}[Z|\mathcal{Y}].$$

Proof.

$$\Big|\mathbb{E}[Z|\mathcal{Y}] - \mathbb{E}[Z^n|\mathcal{Y}^n]\Big| \leq \Big|\mathbb{E}[Z|\mathcal{Y}] - \mathbb{E}[Z|\mathcal{Y}^n]\Big| + \mathbb{E}[\Big|Z - Z_n\Big||\mathcal{Y}]$$

The first term converges to zero because of convergence of filtrations, and the second term converges to 0 in L^1 by the tower property of conditional expectations. $\square$

Proposition 3.9. *Let $\mathcal{Y}, \mathcal{Z}, (\mathcal{Y}^n)_{n\geq 1}, (\mathcal{Z}^n)_{n\geq 1}$ be σ-algebras and $p \geq 1$. If $\mathcal{Y}^n \xrightarrow[L^p]{n\to\infty} \mathcal{Y}$ and $\mathcal{Z}^n \xrightarrow[L^p]{n\to\infty} \mathcal{Z}$, then*

$$\mathcal{Y}^n \vee \mathcal{Z}^n \xrightarrow[L^p]{n\to\infty} \mathcal{Y} \vee \mathcal{Z}.$$

Proof. This is actually the core of the proof of Proposition 2.4 of Coquet, Mémin, and Mackevičius (2000) where the conclusion is reached for weak convergence. $\square$

3.1.2 Stability of semimartingale property

Theorem 3.10 ((∗) Stability of semimartingale property).

Let $(\mathcal{G}^n)_{n\geq 1}$ be a sequence of filtrations and suppose that M is a $\mathcal{G}^n$ semimartingale with decomposition $M =: M^n + A^n$ for every $n \geq 1$.

$$\textit{If } \mathcal{G}^n \xrightarrow[n\to\infty]{L^2} \mathcal{G} \textit{ and } \sup_n \mathbb{E}\int_0^T d|A^n_t| < \infty, \textit{ then } M \textit{ is a } \mathcal{G}\textit{-semimartingale.}$$

Proof. We show that M is a $\mathcal{G}$-quasimartingale (see Protter (2004), Chapter III).

Let H be a simple $\mathcal{G}$-predictable process, $H_t := \sum_{i=1}^p h_i 1_{]u_i,u_{i+1}]}(t)$, where h_i is $\mathcal{G}_{u_i}$-measurable, $0 \leq u_1 < ... < u_{p+1} = T$, and $\forall i = 1...p, |h_i| \leq 1$.

Define $H^n_t := \mathbb{E}[H_t|\mathcal{G}^n_t] = \sum_{i=1}^p h^n_i 1_{]u_i,u_{i+1}]}(t)$, with $h^n_i := \mathbb{E}[h_i|\mathcal{G}^n_{t_i}]$. H^n is a simple $\mathcal{G}^n$-predictable process bounded by 1, hence

$$|\mathbb{E}[(H^n\cdot M)_T]| = |\mathbb{E}\int_0^T H^n_t dA^n_t| \leq \mathbb{E}\int_0^T |dA^n_t| \leq \sup_n \mathbb{E}\int_0^T |dA^n_t|.$$

Now

$$\mathbb{E}[((H-H^n)\cdot M)_T] = \sum_{i=1}^{p} \mathbb{E}[(h_i - h_i^n)(M_{u_{i+1}} - M_{u_i})]$$

and using the Cauchy-Schwarz inequality

$$|\mathbb{E}[((H-H^n)\cdot M)_T]| \leq \sum_{i=1}^{p} \mathbb{E}[|h_i - h_i^n||M_{u_{i+1}} - M_{u_i}|]$$

$$\leq \sum_{i=1}^{p} \sqrt{\mathbb{E}[(h_i - h_i^n)^2]\mathbb{E}[(M_{u_{i+1}} - M_{u_i})^2]}$$

$$\leq \sqrt{\sup_{i=1\ldots p} \mathbb{E}[(h_i - h_i^n)^2]} \sum_{i=1}^{p} \sqrt{\mathbb{E}[(M_{u_{i+1}} - M_{u_i})^2]}.$$

Then using another time the Cauchy-Schwarz inequality

$$\sum_{i=1}^{p} \sqrt{\mathbb{E}[(M_{u_{i+1}} - M_{u_i})^2]} \leq \sqrt{p \sum_{i=1}^{p} \mathbb{E}[(M_{u_{i+1}} - M_{u_i})^2]} < \infty.$$

Moreover

$$\sup_{i=1\ldots p} \mathbb{E}[(h_i - h_i^n)^2] \xrightarrow[n\to\infty]{} 0$$

as a consequence of the convergence of σ-algebras $\mathcal{G}_t^n \xrightarrow[n\to\infty]{} \mathcal{G}_t, 0 \leq t \leq T$ and the bound $\mathbb{E}[h_i^2] \leq 1 < \infty$.

Hence we can conclude that

$$\mathbb{E}[((H-H^n)\cdot M)_T] \xrightarrow[n\to\infty]{} 0$$

so that

$$|\mathbb{E}[(H\cdot M)_T]| \leq \sup_n \mathbb{E}\int_0^T |dA_t^n|$$

and M is a $\mathcal{G}$-quasimartingale by Bichteler-Dellacherie characterization (see Protter (2004)). □

The last theorem shows that the semimartingale property is conserved through the convergence of filtrations in L^2, as long as the sequence of finite variation terms is absolutely uniformly bounded. The next result proves that if the sequence of finite variation terms converges, convergence of filtrations in L^1 is sufficient.

Theorem 3.11 ($*$). *Let $(\mathcal{G}^n)_{n\geq 1}$ be a sequence of filtrations and suppose that M is a $\mathcal{G}^n$ semimartingale with decomposition $M =: M^n + A^n$ for every $n \geq 1$.*
If $\mathcal{G}^n \xrightarrow[n\to\infty]{L^1} \mathcal{G}$ and $\mathbb{E}[\int_0^T d|A_t^n - A_t|] \xrightarrow[n\to\infty]{} 0$ then M is a $\mathcal{G}$-semimartingale with decomposition $M =: \widetilde{M} + A$ (where $\widetilde{M} := M - A$).

Proof. Since A^n is $\mathcal{G}$-predictable $\forall n \geq 1$ its limit A is also $\mathcal{G}$-predictable. Moreover $\int_0^T |dA_t| \leq \int_0^T |dA_t^n - dA_t| + \int_0^T |dA_t^n| < \infty$ so A has finite variations. We also have $\forall t \in [0,T]$, $A_t^n \xrightarrow[n\to\infty]{L^1} A_t$ and by a localization argument we can assume without loss of generality that M and A are bounded.Hence according to Proposition 3.8 we also have

$$\forall s \leq t, 0 = \mathbb{E}[(M_t - A_t^n) - (M_s - A_s^n)|\mathcal{G}_s^n] \xrightarrow[n\to\infty]{L^1} \mathbb{E}[(M_t - A_t) - (M_s - A_s)|\mathcal{G}_s],$$

which shows that $M - A$ is a $\mathcal{G}$-martingale. □

3.1.3 Convergence of information drift in $\mathcal{S}^2$

In this paragraph we focus on convergence of square-integrable information drift. We rely on the Hilbert structure of $\mathcal{S}^2(M) \subset L^2(\Omega, \mathcal{A}, d[M,M] \times d\mathbb{P})$ (see Definition 1.12) to show that a uniform bound on the norms of the information drifts is sufficient to prove the existence of a limit, which is the information drift of the limit filtration.

Theorem 3.12 ((∗) Convergence of information drifts). *Let $(\mathcal{G}^n)_{n\geq 1}$ be a* ***non-decreasing*** *sequence of filtrations and suppose that M is a $\mathcal{G}^n$-semimartingale with decomposition $M =: M^n + \int_0^\cdot \alpha_s^n d[M,M]_s$ for every $n \geq 1$ for some process $\alpha^n \in \mathcal{S}(\mathcal{G}^n)$.*
If $\mathcal{G}^n \xrightarrow[n\to\infty]{L^2} \mathcal{G}$ and $\sup_{n\geq 1} \int_0^T (\alpha_u^n)^2 d[M,M]_u < \infty$ then M is a $\mathcal{G}$-semimartingale with decomposition

$$M =: \widetilde{M} + \int_0^\cdot \alpha_s d[M,M]_s,$$

where $\alpha \in \mathcal{S}^2(\mathcal{G}, M)$.

Proof. Suppose first that there exists such α. Up to a localizing sequence, we can assume that $[M,M]$ is bounded. With $A^n := \int_0^\cdot \alpha_s^n d[M,M]_s$ and $A := \int_0^\cdot \alpha_s d[M,M]_s$, we have

$$\int_0^T d|A_t^n - A_t| = \int_0^T |\alpha_s^n - \alpha_s| d[M,M]_s \leq [M,M]_T \int_0^T (\alpha_s^n - \alpha_s)^2 d[M,M]_s \xrightarrow[n\to\infty]{} 0$$

such that M has decomposition $\widetilde{M} + \int_0^\cdot \alpha_s d[M,M]_s$ by Theorem 3.11.

Now if $m \leq n$ we have in the Hilbert space $L^2(M) := L^2(\Omega, \mathcal{A}, d[M,M] \times d\mathbb{P})$

the following orthogonality:

$$\begin{aligned}\mathbb{E}\int_0^T(\alpha^m-\alpha^n)\alpha^n d[M,M]_s &= \mathbb{E}\int_0^T \alpha^n[(dM_t-dM_t^m)+(dM_t-dM_t^n)]\\ &= \mathbb{E}\int_0^T \alpha^n(dM_t^m-dM_t^n)\\ &= 0\end{aligned}$$

because α^n is $\mathcal{G}^n$ and $\mathcal{G}^m$ predictable which implies that both stochastic integrals are L^2 martingales with expectation 0.

Hence

$$\|\alpha^m\|^2_{L^2(M)} = \|\alpha^n\|^2_{L^2(M)} + \|\alpha^m-\alpha^n\|^2_{L^2(M)}$$

so the sequence $(\|\alpha^n\|_{L^2(M)})_{n\geq 1}$ is increasing and bounded thus has finite limit $\sup_{n\in\mathbb{N}}\|\alpha^n\|^2_{L^2(M)} < \infty$.

It follows from

$$\|\alpha^m-\alpha^n\|^2_{L^2(M)} = \|\alpha^m\|^2_{L^2(M)} - \|\alpha^n\|^2_{L^2(M)}$$

that $(\alpha^n)_{n\geq 1}$ is Cauchy in $L^2(M)$ and converges to some $\alpha \in L^2(M)$ which is predictable and hence in $\mathcal{S}^2(\mathcal{G}, M)$. □

3.1.4 Stability of the semimartingale decomposition

We give a partial result in the converse direction, namely what is the decomposition in the limit filtration when the semimartingale property is obtained non-constructively as in Theorem 3.10. Coquet, Mémin, and Slominski (2001) and Coquet, Mémin, and Mackevičius (2000) give a result based on weak convergence and Skorohod's J^1 topology (see Coquet, Mémin, and Slominski (2001)). We prove here a similar result involving L^1 convergence. We would ideally like to find convergence of the information drifts, but only obtain in general the convergence of the integrated finite variation process.

Theorem 3.13 (∗). *Let $(\mathcal{G}^n)_{n\geq 1}$ be a* ***non-decreasing*** *sequence of right-continuous filtrations such that $\forall t \in I, \mathcal{G}_t \subset \bigvee_{n\geq 1} \mathcal{G}_t^n$. If M is a $\mathcal{G}^n$ semimartingale with decomposition $M =: M^n + A^n$ for every $n \geq 1$, and M is a $\mathcal{G}$-semimartingale with decomposition $M =: \widetilde{M} + A$, then*

$$\forall t \in I, A_t^n \xrightarrow[n\to\infty]{L^1} A_t.$$

Proof. Since M is a continuous $\mathcal{F}$-adapted $\mathcal{G}$-semimartingale, the filtration shrinkage theorem in Föllmer and Protter (2011) assures that M is a $\mathcal{G}^n$-special semimartingale with decomposition $M = {}^{(n)}\widetilde{M} + {}^{(n)}A$, where ${}^{(n)}$ stands for the optional projection with respect to $\mathcal{G}^n$.

It follows from the unicity of the semimartingale decomposition of M in the right-continuous filtration $\mathcal{G}^n$ that $A^n = {}^{(n)}A$, and Proposition 3.8 proves the pointwise convergence $\forall t \in I, \mathbb{E}[A_t|\mathcal{G}_t^n] \xrightarrow[n\to\infty]{L^1} A_t$. □

3.2 Expansion with a stochastic process

In this section we consider an augmentation of the filtration $\mathcal{F}$ with a càdlàg process X on a bounded time interval $I := [0,T], T > 0$. Hence we let $\mathcal{H}$ be the natural filtration of X, namely

$$\mathcal{H}_t = \sigma(X_s, s \leq t),$$

and we define $\check{\mathcal{G}}$ as the "raw" augmentation of $\mathcal{F}$ with X, namely

$$\check{\mathcal{G}}_t := \mathcal{F}_t \vee \mathcal{H}_t.$$

We are initially interested in the semimartingale properties of M in $\check{\mathcal{G}}$. However it is often expected that filtrations satisfy the usual hypotheses, and accordingly we prefer consider $\mathcal{G}$, the smallest right-continuous filtration containing $\mathcal{F}$ to which X is adapted, defined as

$$\mathcal{G}_t := \bigcap_{u>t} \check{\mathcal{G}}_u.$$

In any case, Proposition 1.39 in 1 shows that $\mathcal{G}$ and $\check{\mathcal{G}}$ are essentially equivalent for semimartingales.

In the remainder of this section we first approximate $\check{\mathcal{G}}$ by the discrete augmentations generated by finite samples of X and we apply the results of Kchia, Larsson, and Protter (2013), where Jacod's condition for initial expansion is extended to discrete expansions at stopping times. We then apply the results of our previous section to derive the stability of the semimartingale property and decomposition of M in $\mathcal{G}$ from the semimartingale decompositions in the intermediate filtrations.

3.2.1 Approximations with discrete expansions

3.2.1.1 Discretization of the filtrations

Let $(\pi^n)_{n\geq 1}$ be a refining sequence of subdivisions of $[0,T]$ with mesh size converging to 0. We also denote $\pi^n =: (t_i^n)_{i=0}^{\ell(n)}$, with $0 = t_0^n < \ldots < t_{\ell(n)}^n < t_{\ell(n)+1}^n = T$. For every $n \geq 1$ we define a discrete càdlàg process X^n by

$$X_t^n := \sum_{i=0}^{\ell(n)} X_{t_{t_i^n}} 1_{t_i^n \leq t < t_{i+1}^n}.$$

We also define the non-decreasing sequence of filtration $\mathcal{H}^n$ generated by X^n,

$$\mathcal{H}_t^n := \sigma(X_s^n, s \leq t) = \sigma(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \ldots, X_{t_{\ell(n)+1}^n} - X_{t_{\ell(n)}^n}),$$

as well as the expansions $\check{\mathcal{G}}^n$ and $\mathcal{G}^n$ given by

$$\check{\mathcal{G}}_t^n := \mathcal{F}_t \vee \mathcal{H}_t^n \quad \text{and} \quad \mathcal{G}_t^n = \bigcap_{u>t} \check{\mathcal{G}}_u^n.$$

The interest of the approximation of $\mathcal{H}$ with "discrete" filtrations $\mathcal{H}^n$ is that discrete augmentations can be well described under Jacod's condition. Jacod's condition has been accepted since the 1980s as a satisfactory answer to the initial expansion problem, and its validity has been extended in Kchia, Larsson, and Protter (2013) to successive initial expansions at random times.

3.2.1.2 Jacod's condition

Definition 3.14 (Jacod's condition). *A random variable L satisfies **Jacod's condition** if for almost every $t \in I$ there exists a (non-random) σ-finite measure η_t on the Polish space $(E, \mathcal{E})$ in which it takes values such that*

$$P_t(\omega, L \in .) \ll \eta_t(.)\ a.s.$$

Hence in the sequel we will assume that the η given by the generalized Jacod criterion is invariant with time.

Lemma 3.15. *Let $\mathcal{Y}$ be a general filtration and $(\eta_t)_{t\in I}$ a family of σ-finite measures such that*

$$P_t(\omega, .) \ll \eta_t(.) \text{ on } \mathcal{Y}_t, d\mathbb{P} \times d[M, M] - a.s.$$

Then the same statement holds for a some σ-finite measure η invariant with time, which can be taken to be $\mathbb{P}$.

Proof. This statement is slightly more general than Jacod's original version which corresponds to taking $\mathcal{Y} := \sigma(L)$ but the proof is mostly identical.

Let $p_t(\omega, \omega') := \frac{dP_t(\omega,.)}{d\eta_t}(\omega')$ and define $a_t(\omega') := \int_\Omega p_t(\omega, \omega')d\mathbb{P}(\omega)$ and $\widetilde{p}_t(\omega, \omega') := \frac{p_t(\omega,\omega')}{a_t(\omega')} 1_{a_t(\omega')>0}$ (with the convention $\infty \times 0 = 0$).

Then for any $A \in \mathcal{Y}_t$ we have

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{E}P_t(., A) \\
&= \int_\Omega P_t(\omega, A) d\mathbb{P}(\omega) \\
&= \int_\Omega \int_A P_t(\omega, d\omega') d\mathbb{P}(\omega) \\
&= \int_\Omega \int_A p_t(\omega, \omega') \eta_t(d\omega') d\mathbb{P}(\omega) \\
&= \int_A \int_\Omega p_t(\omega, \omega') d\mathbb{P}(\omega) \eta_t(d\omega') \\
&= \int_A a_t(\omega') \eta_t(d\omega')
\end{aligned}$$

so that $\mathbb{P}(d\omega') = a_t(\omega')\eta_t(\omega')$ and $P_t(\omega, d\omega') = p_t(\omega, \omega')\eta_t(d\omega') = \widetilde{p}_t(\omega, \omega')\mathbb{P}(d\omega')$. □

Lemma 3.15 applied to the σ-algebra generated by L shows that the family of measures η_t in Jacod's condition can be assumed to be a constant measure η. If L satisfies Jacod's condition we can define the *Jacod's conditional densities* by

$$q_t^L(\omega, x) := \left.\frac{dP_t(\omega, .)}{d\eta(.)}\right|_{\sigma(L)} (L)^{-1}(x) \tag{3.1}$$

Lemma 3.16. *The conditional density $q_t^L(., L(.))$ is positive a.s. on I.*

Proof. The result is proved in Paragraphs 1.6 to 1.11 of Jacod (1985). As Jacod's argument has often been ommitted in the literature we succinctly summarize it with our notations.

Let $\tau^L(\omega, x) := \inf\{t \geq 0 : q_t^L(\omega, x) = 0\}$. For a given (ω, x), $q_t^L(\omega, x) > 0$ and $q_{t-}^L(\omega, x) > 0$ for $t < \tau^L(\omega, x)$, and $q_t^L(\omega, x) = q_{t-}^L(\omega, x) = 0$ for $t \geq \tau^L(\omega, x)$. Since

$\tau^L(.,L(.))$ is a stopping time the indicator $1_{\tau^L(.,L(.))<t}$ is predictable and we have the representation

$$\mathbb{P}(\tau^L(.,L(.))<t)=\mathbb{E}1_{\tau^L(.,L(.))<t}=\mathbb{E}\int q_{t^-}(\omega,x)1_{\tau^L(.,x)<t}\,\eta(dx)=0$$

so that $q_t(.,L(.))>0$ a.s. for $t\in I$. □

3.2.1.3 Discrete expansions and semimartingales

Recall that $(E,\mathcal{E})$ is a generic Polish space and denote $(E,\mathcal{E})^n:=(E^n,\mathcal{E}^{\otimes n})$ the product Polish space ($n\geq 1$). The following theorem is an application of Kchia, Larsson, and Protter (2013) in the particular case where the expansion times are deterministic.

Theorem 3.17 (Kchia, Larsson, and Protter (2013)). *Let $n\geq 1$ and suppose given a random variable $(L_1,...,L_n)$ on $(E,\mathcal{E})^n$ that satisfy Jacod's criterion. Let q^k be the conditional density q^{L_k} of L^k defined by Equation (3.1). Let $0\leq\tau_1\leq...\leq\tau_n\leq\tau_{n+1}=T$ be a family of fixed times, and define the augmentation*

$$\mathcal{Y}_t:=\bigcap_{u>t}(\mathcal{F}_u\vee\sigma(L_k1_{t\geq\tau_k},k=1...n)),t\in[0,T].$$

Then M is a $\mathcal{Y}$-semimartingale with decomposition $M=:\widetilde{M}+A$, where

$$A_t(\omega):=\sum_{k=1}^{n}\int_{\tau_k}^{\tau_{k+1}}1_{s\leq t}\left(\frac{d[q^k(\omega,.),M]_s}{q^k_{s_-}(\omega,.)}\right)(L^k(\omega))$$

(with notation $\tau_{n+1}=T$).

In the setting of the augmentation with the process X, applying Theorem 3.17 to the discrete augmentations $\mathcal{G}^n$ leads to the next corollary (similarly to the argument in Kchia and Protter (2015)).

Corollary 3.18. *Let $n \geq 1$. Suppose that $\big(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, ..., X_{t_{\ell(n)+1}^n} - X_{t_{\ell(n)}^n}\big)$ satisfies Jacod's condition and for $k \leq \ell(n)$ let $q^{k,n}$ be the conditional density of $L^k := \big(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, ..., X_{t_{k+1}^n} - X_{t_k^n}\big)$ defined by Equation (3.1). Then M is a $\mathcal{G}^n$-semimartingale on $[0,T]$ with decomposition*

$$M =: M^n + \int_0^{\cdot} \alpha_s^n d[M,M]_s,$$

where α^n is the $\mathcal{G}^n$-predictable process defined by

$$\alpha_t^n(\omega) := \sum_{k=0}^{\ell(n)} 1_{t_k^n \leq t < t_{k+1}^n} \frac{d[q^{k,n}(.,x_k), M]_s}{q_{s-}^{k,n}(.,x_k)}\Bigg|_{x_k=(X_{t_0}, X_{t_1}-X_{t_0}, \cdots, X_{t_k}-X_{t_{k-1}})}. \tag{3.2}$$

We emphasize that, taking η to be the law of L, the conditional densities $q^{k,n}$ can be computed explicitly from the conditional finite dimensional distributions of X as

$$q_t^{k,n}(\omega, x) := \frac{P_t\big(\omega, (X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, ..., X_{t_{k+1}^n} - X_{t_k^n}) \in dx\big)}{\mathbb{P}\big((X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, ..., X_{t_{k+1}^n} - X_{t_k^n}) \in dx\big)}$$

3.2.2 Expansion with a process and semimartingales

This section contains probably the most significant theorem of the thesis. We apply here the results of Section 3.1 on convergence of filtrations together with Corollary 3.18 to obtain the semimartingale property and decomposition of M in the filtration $\mathcal{G}$ augmented with the process X.

We briefly postpone to Paragraph 3.2.3 the necessary discussion on the minor technical assumptions required on the process X to fit in our framework by introducing the class $\mathbb{X}^\pi$.

Definition 3.19 (Class $\mathbb{X}^\pi$). *We say that the càdlàg process X is of Class $\mathbb{X}^\pi$, or of Class $\mathbb{X}$ if there is no ambiguity, if*

$$\forall t \in I, \mathcal{H}^n_{t^-} \xrightarrow[n\to 1]{L^2} \mathcal{H}_{t^-}.$$

The notation $\mathbb{X}^\pi$ emphasizes the potential dependency of this property in the discretization π defined earlier. Note that the usual hypothesis need not apply here on the filtrations $\mathcal{H}^n$ and $\mathcal{H}$. The next proposition summarizes the results of Paragraph 3.2.3 to convince the reader that most usual processes are of Class $\mathbb{X}$.

Proposition 3.20. *The càdlàg process X is of Class $\mathbb{X}$ if one of the following holds:*

(i) $\mathbb{P}(\Delta X_t \neq 0) = 0$

(i') X *is continuous*

(ii) $\mathcal{H}$ *is (quasi-) left continuous.*

(ii') X *is a Hunt process (e.g a Levy process)*

(iii) X *jumps only at totally inaccessible stopping times*

(iv) π *contains all the fixed times of discontinuity of* X *after a given rank*

The reader will notice that all conditions except the last depends only on the properties of X, and not on the choice of the discretization π. The next lemma extends the convergence of the filtrations $\mathcal{H}^n$ to the filtrations $\mathcal{G}^n$.

Lemma 3.21. *If X is of Class $\mathbb{X}$, then $\mathcal{G}^n \xrightarrow[n\to\infty]{L^2} \check{\mathcal{G}}$.*

Proof. It follows from the definition of Class $\mathbb{X}$ that $\mathcal{H}^n \xrightarrow[n\to\infty]{L^2} \mathcal{H}$. By Proposition 3.9 we also have $\check{\mathcal{G}}^n \xrightarrow{n\to\infty} \check{\mathcal{G}}$, and by Corollary 3.7 the convergence extends to the right-continuous filtrations $\mathcal{G}^n$ and $\mathcal{G}^n \xrightarrow[L^2]{n\to\infty} \check{\mathcal{G}}$. □

We can now prove the main result of this section.

Theorem 3.22 ((∗) Expansion with a stochatic process). *Suppose that*

(i) X is of Class $\mathbb{X}$,

(ii) $\forall n \geq 1$, $\left(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, ..., X_{t^n_{\ell(n)+1}} - X_{t^n_{\ell(n)}}\right)$ satisfies Jacod's condition,

and let α^n be as defined by Equation (3.2) in Corollary 3.18. Then:

1. *If $sup_{n\geq 1}\mathbb{E}\int_0^T |\alpha_t^n| d[M,M]_t < \infty$, M is a continuous $\mathcal{G}$-semimartingale.*

2. *If $\sup_n \mathbb{E}\int_0^T (\alpha_s^n)^2 d[M,M]_s < \infty$, M is a continuous $\mathcal{G}$-semimartingale with decomposition*

$$M =: \widetilde{M} + \int_0^{\cdot} \alpha_s d[M,M]_s$$

where $\alpha \in \mathcal{S}^2(\mathcal{G}, M)$ and $\mathbb{E}\int_0^T (\alpha_t^n - \alpha_t)^2 d[M,M]_t \xrightarrow[n\to\infty]{} 0$.

We recall that $\alpha \in \mathcal{S}^2(\mathcal{G}, M)$ means that α is $\mathcal{G}$-predictable and $\mathbb{E}\int_0^T \alpha_t^2 d[M,M]_s < \infty$.

Proof of the theorem. Corollary 3.18 shows that, for every $n \geq 1$, M is a $\mathcal{G}^n$ semimartingale with decomposition $M =: M^n + \int_0^{\cdot} \alpha_s^n d[M,M]_s$ for a $\mathcal{G}^n$-predictable process α^n.

Lemma 3.21 establishes the convergence $\mathcal{G}^n \xrightarrow[n\to\infty]{L^2} \mathcal{G}$, and the two points of the theorem now follow respectively from Theorem 3.10 and Theorem 3.12. □

Proposition 3.13 leads to a partial result in the converse direction.

Theorem 3.23. *If M is a $\mathcal{G}$-semimartingale with decomposition $M =: \widetilde{M} + A$, then we have the pointwise convergence:*

$$\forall t \geq 0, A_t^n \xrightarrow[n\to\infty]{L^1} A_t,$$

where $A_t^n := \int_0^t \alpha_s^n d[M,M]_s, t \in I$.

3.2.3 Description of class $\mathbb{X}$

In this paragraph we derive several sufficient conditions for X to be of class $\mathbb{X}$. A particular subset of processes of class $\mathbb{X}$ is easy to characterize.

3.2.3.1 Class $\mathbb{X}_0$

Definition 3.24. *We say that X is of Class $\mathbb{X}_0$ if $\mathcal{H}_{t-} = \bigvee_{n\geq 1} \mathcal{H}_{t-}^n$.*

Proposition 3.25. *If X is of Class $\mathbb{X}_0$, it is also of Class $\mathbb{X}$*

Proof. This is a direct corollary of Proposition 3.5, which shows that if $\mathcal{H}_t = \bigvee_{n\geq 1} \mathcal{H}_t^n$, then for every $n \geq 1$ $\mathcal{H}_t^n \xrightarrow[n\geq 1]{L^p} \mathcal{H}_t$. Note that we need not the hypothesis that π^n contains all fixed time of discontinuity of X as is required in Lemma 5 and Example 1 of Kchia and Protter (2015). □

Proposition 3.26. *The process X is of Class $\mathbb{X}_0$ if its natural filtration $\mathcal{H}$ is quasi left-continuous.*

Proof. In general we have the inclusion

$$\mathcal{H}_{t^-} \subset \bigvee_{n \geq 1} \mathcal{H}_t^n \subset \mathcal{H}_t.$$

It follows that X is of Class $\mathbb{X}_0$ if $\mathcal{H}$ is left continuous. More generally the second inclusion is tight if $\mathcal{H}$ is quasi-left continuous. This is for instance the case if X is a Lévy, Feller or more generally a càdlàg Hunt Markov process, or any integral of such process. □

Corollary 3.27. *Hunt processes (which include Lévy processes) are of class $\mathbb{X}_0$.*

Proof. See for instance Protter (2004, p. 36). Note that if X is a Hunt process, the $\mathbb{P}$-completion of $\mathcal{H}$ is also right-continuous, but we cannot in general conclude that $\check{\mathcal{G}}$ is also right-continuous. □

3.2.3.2 Class $\mathbb{X}$

This paragraph shows that class $\mathbb{X}$ contains càdlàg processes without fixed points of discontinuity (in particular continuous processes) and Hunt processes. We start with a characterization of convergence of filtrations in L^p with finite dimensional distributions for filtrations generated by càdlàg process.

Proposition 3.28. *Let $\mathcal{Y} := \sigma(Y_t, 0 \leq t \leq T)$ for some càdlàg measurable process Y and $p \geq 1$.*

For a given sequence $\mathcal{Y}^n, n \geq 1$ of σ-algebras, $\mathcal{Y}^n \xrightarrow[n\to\infty]{L^p} \mathcal{Y}$ if and only if for all $k \in \mathbb{N}$,$u_1, ..., u_k \in [0,T]$, f bounded continuous function

$$\mathbb{E}[f(Y_{u_1}, ..., Y_{u_k})|\mathcal{Y}^n] \xrightarrow[n\to\infty]{L^p} f(Y_{u_1}, ..., Y_{u_k})$$

Proof. The proof is similar to the characterization of weak convergence in Lemma 3 of Kchia and Protter (2015). □

Proposition 3.29. *Let $Y, Y^n, n \geq 1$ be càdlàg stochastic processes and $\mathcal{Y}, \mathcal{Y}^n$ their respective natural filtrations. Suppose that*

$$\forall k \in \mathbb{N}, 0 \leq u_1 \leq ... \leq u_k, (Y^n_{u_1}, Y^n_{u_2}, ..., Y^n_{u_k}) \xrightarrow[n\to\infty]{d} (Y_{u_1}, Y_{u_2}, ..., Y_{u_k})$$

Then, for every $p \geq 1$,

$$\mathcal{Y}^n \xrightarrow[n\to\infty]{L^p} \mathcal{Y}.$$

Proof. Fix $p \geq 1, 0 \leq u_1 \leq ... \leq u_k \leq t$ and let f a bounded continuous function. By the Minkowski inequality (and because $f(Y^n_{u_1}, Y^n_{u_2}, ..., Y^n_{u_k})$ is $\mathcal{Y}^n_t$-measurable) we have

$$\begin{aligned}
&\left|\mathbb{E}[f(Y_{u_1}, Y_{u_2}, ..., X_{u_k})|\mathcal{Y}^n_t] - f(Y_{u_1}, Y_{u_2}, ..., Y_{u_k})\right|^p \\
&\leq \left|\mathbb{E}[f(Y_{u_1}, Y_{u_2}, ..., Y_{u_k}) - f(Y^n_{u_1}, Y^n_{u_2}, ..., Y^n_{u_k})|\mathcal{Y}^n_t]\right|^p \\
&+ \left|f(Y^n_{u_1}, Y^n_{u_2}, ..., Y^n_{u_k}) - f(Y_{u_1}, Y_{u_2}, ..., Y_{u_k})\right|^p
\end{aligned}$$

and consequently

$$\mathbb{E}\Big|\mathbb{E}[f(Y_{u_1}, Y_{u_2}, ..., Y_{u_k})|\mathcal{Y}_t^n] - \mathbb{E}[f(Y_{u_1}, Y_{u_2}, ..., Y_{u_k})|\mathcal{Y}_t]\Big|$$
$$\leq 2\mathbb{E}[\Big|f(Y_{u_1}, Y_{u_2}, ..., Y_{u_k}) - f(Y^n_{u_1}, Y^n_{u_2}, ..., Y^n_{u_k})\Big|].$$

Since f is bounded, for every $p \geq 1$, $f(Y_{u_1}, Y_{u_2}, ..., Y_{u_k}) \xrightarrow[n\to\infty]{L^p} f(Y^n_{u_1}, Y^n_{u_2}, ..., Y^n_{u_k})$ as long as $(Y^n_{u_1}, Y^n_{u_2}, ..., Y^n_{u_k}) \xrightarrow[n\to\infty]{d} (Y_{u_1}, Y_{u_2}, ..., Y_{u_k})$ so we can conclude from Proposition 3.28. □

Using this characterization, we show that convergence in probability of the underlying processes is sufficient for convergence of filtrations, under a technical assumption on the jumps.

Proposition 3.30. *If X is a càdlàg process with $\Delta X_t = 0$ a.s. for every $t \in I$, then X is of class $\mathbb{X}$.*

Proof. The hypothesis of Proposition 3.29 is satisfied as long as we have $\forall t \in I$, $Y_t^n \xrightarrow[n\to\infty]{\mathbb{P}} Y_t$ and $\mathbb{P}(\Delta Y_t \neq 0)$. With our discretization setting this is a consequence of the convergence $X^n \xrightarrow[n\to\infty]{a.s.} X$ (which actually holds if π^n contain the fixed points of discontinuity of X after some fixed rank). □

Remark 3.31 (Kchia and Protter (2015)). *The proof of the last proposition can be put in perspective with Lemmas 3 and 4 of Kchia and Protter (2015) which derive sufficient conditions for weak convergence of the filtrations generated by càdlàg processes, where the underlying process are required to converge in probability for the Skorohod J^1 topology. In comparison we only ask for a pointwise convergence.*

Chapter 4

Information drift and conditional probabilities

In this chapter $(\Omega, \mathcal{A}, \mathbb{P})$ is a standard (complete) Borel probability space, $\mathcal{H}$ a general filtration, and we define the expansions $\check{\mathcal{G}}$ and $\mathcal{G}$ of $\mathcal{F}$ as

$$\check{\mathcal{G}}_t := \mathcal{F}_t \vee \mathcal{H}_t \text{ and } \mathcal{G}_t := \textstyle\bigcap_{u>t} \check{\mathcal{G}}_u.$$

M is a **continuous** $\mathcal{F}$-local martingale.

* * * * *

We transpose here our problem of transformation of semimartingales in terms of a diffusion coefficient k of the conditional probabilities between $\mathcal{F}$ and $\mathcal{G}$ with respect to M. We first follow Ankirchner, Dereich, and Imkeller (2006) in a slightly more general framework (Theorem 4.17) to derive from the Hilbert property of L^2 spaces a sufficient condition on k ensuring that M admits an information drift with respect to $\mathcal{G}$ (see Definition 1.21). Based on a preliminary study of extensions of measures (Theorem 4.10) we also consider the case where the filtration $\mathcal{H}$ can be written as the increasing limit of a sequence of (simpler) filtrations, i.e. as

$$\forall t \in I, \mathcal{H}_t := \bigvee_{n\geq 1} \mathcal{H}_t^n,$$

where we can rely on the properties of the filtrations $\mathcal{H}^n$ to express the information drift as a converging limit of expressions involving k and deduce a computational method to estimate the information drift (Theorem 4.19).

4.1 Representation of the conditional probabilities

The approach of this chapter relies on the predictable representation of the conditional probabilities.

4.1.1 Existence of a regular family of conditional probabilities

Definition 4.1. *We define a regular family of $\mathcal{F}$-conditional probabilities as a function*

$$I \times \Omega \times \mathcal{A} \longrightarrow [0,1]$$

$$(t, \omega, A) \longmapsto P_t(\omega, A)$$

which is measurable with respect to $\mathcal{B}_{\mathcal{T}} \otimes \mathcal{F}_{\mathcal{T}} \otimes \mathcal{A}$ such that

(i) $\forall t \in I, A \in \mathcal{A}, \omega \mapsto P_t(\omega, A)$ is a version of $\mathbb{P}(A|\mathcal{F}_t)$

(ii) $\forall A \in \mathcal{A}$, $(t, \omega) \mapsto P_t(\omega, A)$ has càdlàg paths.

(iii) $A \mapsto P_t(\omega, A)$ is a probability measure on $(\Omega, \mathcal{A})$, $d\mathbb{P} \times d[M, M]$-a.s.

(see Dellacherie and Meyer (1980))

Remark 4.2. *The above definition implies that for every $A \in \mathcal{A}$, $(t, \omega) \mapsto P_t(\omega, A)$ is a càdlàg $\mathcal{F}$-martingale.*

Proposition 4.3. *There exist a regular family of $\mathcal{F}$-conditional probabilities.*

Proof. Since the probability space is a Borel space there exist a regular version of the conditional probabilities with respect to $\mathcal{F}$, i.e. a function $(t, \omega, A) \longmapsto P_t(\omega, A)$ such that

(i) for every $t \in I$ and $A \in \mathcal{A}$, $\omega \mapsto P_t(\omega, A)$ is a version of $\mathbb{P}(A|\mathcal{F}_t)$

(ii) for every $t \in I$ and $\omega \in \Omega$ $P_t(\omega, .)$ is a (random) probability measure on $(\Omega, \mathcal{A})$

Theorem 4, Chapter VI of Dellacherie-Meyer (Dellacherie and Meyer (1980)) shows that we can construct a modification of the conditional probabilities which has càdlàg paths. The proof of the theorem moreover shows that the modification alters the process only on a negligible set, so that the regularity is conserved $d\mathbb{P} \times d[M, M]$-a.s. (this is asserted without proof in Ankirchner, Dereich, and Imkeller (2006)). □

Until the end of this chapter P denotes a regular version of the $\mathcal{F}$-conditional probabilities.

4.1.2 Predictable representation of the conditional probabilities

Lemma 4.4. $\forall A \in \mathcal{A}$, *there exist a unique (up to a modification) decomposition*

$$P_t(\omega, A) = \mathbb{P}(A) + \int_0^t k_s(\omega, A) dM_s + L_t(\omega, A), \tag{4.1}$$

where $k(., A)$ is a $\mathcal{F}$-predictable process with $\mathbb{E} \int_I k_s(., A)^2 d[M, M]_s < \infty$, and $L(., A)$ is a $\mathcal{F}$-local martingale with $[L(., A), M] \equiv 0$.

Proof. We let $A \in \mathcal{A}$ and drop the dependency in A for simplicity.

Uniqueness: If $k \cdot M + L = 0$ then considering the quadratic variation we have $k^2 \cdot [M, M] + [L, L] = 0$, so that $k \equiv 0$ and $L \equiv 0$ a.s.

Existence: As a consequence of the Kunita-Watanabe inequality there exist a process k such that $[M, P] =: k \cdot [M, M]$. Hence if we take $L := P - k \cdot M$ we have $P = k \cdot M + L$, and $[M, L] = [M, P] - k \cdot [M, M] = 0$.

k is in $\mathcal{S}^2(\mathcal{F}, M)$ since $(k^2) \cdot [M, M] + [L, L] = [P, P] < \infty$.

References: Revuz and Yor (2005, Chapter V, Paragraph 4) or Lowther (2010). □

Proposition 4.5. *There exist functions*

$$I \times \Omega \times \mathcal{A} \longrightarrow \mathbb{R}$$

$$(t, \omega, A) \longmapsto k_t(\omega, A)$$

and

$$I \times \Omega \times \mathcal{A} \longrightarrow \mathbb{R}$$

$$(t, \omega, A) \longmapsto L_t(\omega, A)$$

satisfying

(i) $\forall t \in I, \omega \in \Omega, A \in \mathcal{A}$, $P_t(\omega, A) = \mathbb{P}(A) + \int_0^t k_s(\omega, A) dM_s + L_t(\omega, A)$

(ii) $\forall A \in \mathcal{A}$, $(t, \omega) \mapsto k_t(\omega, A)$ *is a* $\mathcal{F}$*-predictable process with càdlàg paths*

(iii) $\forall A \in \mathcal{A}$, $(t,\omega) \mapsto L_t(\omega, A)$ *is a* $\mathcal{F}$*-local martingale with càdlàg paths and* $[L(.,A), M] = 0$

(iv) $k_t(\omega, .)$ *and* $L_t(\omega, .)$ *are finite signed measures on* $(\Omega, \mathcal{A})$*,* $d\mathbb{P} \times d[M,M]$*-a.s.*

Proof. This proposition is a consequence of the preceding lemma. We only need to consider modifications of the processes k and L to ensure càdlàg paths as it is done in Dellacherie and Meyer (1980, Chapter VI, Theorem 4). As equality $d\mathbb{P} \times d[M,M]$-a.s. is weaker than indistinguishability, the uniqueness in Lemma 4.4 ensures the σ-additivity of k and L in their last argument, so that k and L are finite signed measures. □

In the sequel we fix k and L such as given in Proposition 4.5.

4.1.3 Embedding formula

Lemma 4.6 (Embedding formula). *If* $(t,\omega,\omega') \longmapsto \theta_t(\omega,\omega')$ *is a* $\mathcal{F} \otimes \mathcal{G}$*-predictable process, then*

$$\int\int\int \theta_t(\omega,\omega')P_t(\omega, d\omega')d[M,M]_t d\mathbb{P}(\omega) = \int\int \theta_t(\omega,\omega)d[M,M]_t(\omega)d\mathbb{P}(\omega)$$

as long as θ *is non-negative or* $\int\int \theta_t(\omega,\omega)d[M,M]_t(\omega)d\mathbb{P}(\omega) < \infty$.

Proof. Let $0 \leq u < v$, $A \in \mathcal{F}_u$, $B \in \mathcal{G}_u$ and $\theta_t(\omega,\omega') := 1_{]u,v]}(t)1_A(\omega)1_B(\omega')$.

Then

$$
\begin{aligned}
&\int\int\int \theta_t(\omega,\omega')P_t(\omega,d\omega')d[M,M]_t d\mathbb{P}(\omega)\\
&= \int\int_u^v 1_A(\omega)P_t(\omega,B)d[M,M]_t d\mathbb{P}(\omega)\\
&\Big(= \mathbb{E}\int_u^v 1_A \mathbb{E}(1_B|\mathcal{F}_t)d[M,M]_t\Big)\\
&\Big(= \mathbb{E}\mathbb{E}\Big(\int_u^v 1_A 1_B d[M,M]_t|\mathcal{F}_t\Big)\Big)\\
&= \int\int_u^v 1_A(\omega)1_B(\omega)d[M,M]_t d\mathbb{P}(\omega)\\
&= \int\int \theta_t(\omega,\omega)d[M,M]_t d\mathbb{P}(\omega)
\end{aligned}
$$

The two formal equalities inside parenthesis are only illustrative ; the passage from Line 2 to Line 5 is for instance justified in term of predictable projections by Theorem 57 in Dellacherie and Meyer (1980, Chapter VI).

We can finally extend this equality to general $\mathcal{F}\otimes\mathcal{G}$-predictable processes via a monotone class argument. □

4.2 Absolutely continuous extension of measures

4.2.1 The case of finite measures

In this paragraph we consider a probability measure π and a finite measure μ. We also let $(\mathcal{Y}^n)_{n\geq 1}$ be a non-decreasing sequence of σ-algebras (i.e. a discrete time filtration), and define $\mathcal{Y} := \bigvee_{n\geq 1} \mathcal{Y}^n$.

We suppose that, for every $n \geq 1$, $\mu \ll \pi$ on $\mathcal{Y}^n$, and we want to show that $\mu \ll \pi$ holds on the whole σ-algebra $\mathcal{Y}$.

Lemma 4.7. *The process $(Z^n)_{n\geq 1}$ defined by $Z^n := \frac{d\mu}{d\pi}\Big|_{\mathcal{Y}^n}$ is a discrete-time martingale.*

Proof. $\forall A \in \mathcal{Y}^n$ we have the identity $\int_A Z^{n+1} d\pi = \mu(A) = \int_A Z^n d\pi$, which establishes the martingale property. □

Lemma 4.8. *Z converges a.s. to some integrable random variable Z^∞.*

Proof. Doob's martingale convergence theorem the non-negative martingale $(Z^n)_{n\geq 1}$ converges. □

Proposition 4.9. *$\mu \ll \pi$ on $\mathcal{Y}$ if and only if Z is a closed martingale, if and only if Z is uniformly integrable.*

In that case $Z^\infty = \frac{d\mu}{d\pi}\Big|_{\mathcal{Y}}$.

Proof. This proof was inspired by comments on www.math.stackexchange.com by Byron Schmuland[1] as well anonymous mathematicians identified as Mathoman and Saz. It also appears available in Jacod and Protter (2004).

It is a classical result that the discrete martingale Z is closed if and only if it is uniformly integrable. If Z is closed by Z^∞, then by the tower property of conditional expectation μ coincides with the measure $\widetilde{\mu}$ defined by $\widetilde{\mu}(A) := \int_A Z^\infty d\pi$ on $\bigcup_{n\geq 1} \mathcal{Y}^n$,

[1]Renown probabilist at University of Alberta

hence they are identical on $\mathcal{Y} = \sigma(\bigcup_{n\geq 1} \mathcal{Y}^n)$.

Conversely suppose that $\mu \ll \pi$ on $\mathcal{Y}$. By the Radon-Nykodym theorem there exists a $\mathcal{Y}$-measurable and integrable random variable Y such that $\forall A \in \mathcal{Y}, \mu(A) = \int_A Y d\pi$. In particular

$$\forall A \in \mathcal{Y}^n, \int_A Z^n d\pi = \mu(A) = \int_A Y d\pi$$

which implies by the characterization of conditional expectation that $\mathbb{E}[Y|\mathcal{Y}^n] = Z^n$, i.e. that Z is closed by Y. It follows that $Z^n \to Y$ a.s. and in L^1, and we conclude that by identifying the limits $Y = Z^\infty$ a.s. □

The reader may check that in the above the measures π and μ need only be defined on $\mathcal{Y}$.

4.2.2 The case of finite signed measures

We now extend Proposition 4.9 of the previous paragraph to the case where if μ is only a finite signed measure into the next theorem.

Theorem 4.10. *Let π be a probability measure, μ a finite signed measure on some σ-algebra $\mathcal{Y}$, where $\mathcal{Y} = \bigvee_{n\geq 1} \mathcal{Y}^n$ for some non-decreasing sequence of σ-algebras $(\mathcal{Y}^n)_{n\geq 1}$. Suppose that for every $n \geq 1$, $\mu \ll \pi$ on $\mathcal{Y}^n$ and define $Z^n := \frac{d\mu}{d\pi}\Big|_{\mathcal{Y}^n}$, which as a discrete time martingale converges a.s. to some random variable Z^∞. Then the following assertions are equivalent:*

(i) $\mu \ll \pi$ on $\mathcal{Y}$

(ii) Z is a closed martingale

(iii) Z is uniformly integrable

Each of the three assertions also implies that $Z^n \xrightarrow[n\to\infty]{} Z^\infty$ a.s. and uniformly in L^1, and that $\frac{d\mu}{d\pi}\Big|_{\mathcal{Y}} = Z^\infty$.

Proof. If μ is a finite signed measure there exist two finite measures μ^+ and μ^- with disjoint supports such that $\mu = \mu^+ - \mu^-$. Furthermore $\mu \ll \pi$ if and only if both $\mu^+ \ll \pi$ and $\mu^- \ll \pi$, in which case $\frac{d\mu}{d\pi} = \frac{d\mu^+}{d\pi} - \frac{d\mu^-}{d\pi}$, and we can conclude from the case of positive measures (Proposition 4.9). □

4.3 Information drift and Hypothesis $(\mathscr{K})$ on conditional probabilities

This section studies the relationship between the information drift and the representation of conditional probabilities found in Section 2 of Ankirchner, Dereich, and Imkeller (2006), which is based on the following assumption:

Hypothesis 4.11 (Hypothesis $(\mathscr{K})$).

$$k_t(\omega, .) \ll P_t(\omega, .) \text{ on } \mathcal{H}_{t^-}, d\mathbb{P} \times d[M, M] - a.s.$$

We recall that $P_t(\omega, .)$ denotes a regular family of conditional probabilities such as given by Proposition 4.3. Note that for simplicity the notation omits the dependency of Hypothesis $(\mathscr{K})$ in the equivalence class of the real measure derived from $d[M, M]$.

Remark 4.12. *Since k is defined $d\mathbb{P} \times d[M, M]$-a.s., the previous can be stated without loss of generality as $k_t(\omega, .) \ll P_t(\omega, .)$ for every (t, ω).*

Remark 4.13. *It is equivalent in Hypothesis* $(\mathscr{K})$ *to assume the domination* $k_t(\omega,.) \ll P_t(\omega,.)$ *on* $\mathcal{H}_{t^-}$ *or* $\check{\mathcal{G}}_{t^-}$.

Indeed, suppose that it holds on $\mathcal{H}_{t^-}$. *For every* $\omega \in \Omega$, $t \in I$, $F \in \mathcal{F}_t$ *we have the representation*

$$P_t(\omega, F) = 1_F(\omega) = \mathbb{P}(F) + L^{\mathcal{F}}(\omega, F),$$

with $L^{\mathcal{F}}(\omega, F) := (1 - \mathbb{P}(F))1_F(\omega) - \mathbb{P}(F)(1 - 1_F(\omega))$ *and* $k_t(\omega, F) = 0$.
Moreover for every $F \in \mathcal{F}_{t^-}$ *and* $H \in \mathcal{H}_{t^-}$ *we have* $P_t(\omega, F \cap H) = 1_F(\omega)P_t(\omega, H)$. *Hence the uniqueness in the predictable representation in Proposition 4.5 (and Lemma 4.4) implies that* $k_t(\omega, F \cap H) = 1_F(\omega)k_t(\omega, H)$, $d\mathbb{P} \times d[M, M]$*-a.s.*
We conclude that F *and* H *are orthogonal for* $k_t(\omega, .)$, *which therefore coincides with a tensor product on* $\mathcal{F}_t \vee \mathcal{H}_t$. *Furthermore if* $P_t(\omega, F \cap H) = 0$, *then* $1_F(\omega) = 0$ *or* $P_t(\omega, H) = 0$, *so that* $k_t(\omega, F \cap H) = 1_F(\omega)k_t(\omega, H) = 0$, *which proves the absolute continuity of* k_t.

4.3.1 Construction of the information drift under Hypothesis $(\mathscr{K})$

Definition 4.14. *We say that a* σ*-algebra* $\mathcal{Y}$ *is countably generated if for every* $t \in I$, *there exists events* $B_n, n \in \mathbb{N}$ *such that* $\mathcal{H}_t = \sigma(B_n, n \in \mathbb{N})$.

We say that a filtration $\mathcal{Y}$ *is countably generated if for every* $t \in I$ *the* σ*-algebra* $\mathcal{Y}_t$ *is countably generated.*

Note that the separability property of Borel spaces assures that $\mathcal{A}$ is countably generated, and consequently every filtration on $(\Omega, \mathcal{A})$ is countably generated.

We start with a preliminary construction of a sequence of discrete approximations of the filtrations $\mathcal{G}$

Lemma 4.15. *Let $\mathcal{H}$ be a countably generated filtration and $\mathbb{T} := (t_i^n)_{i,n\geq 0}$ a family of times. There exists a family $(G_i^n)_{i,n\geq 0}$ of finite sets of disjoint measurable events, with positive measure for $d\mathbb{P} \times d[M,M]$-a.e. , such that*

(i) $\mathcal{H}_{t^-} = \sigma(G_i^n, i, n \geq 0, t_i^n = t)$ *for every* $t \in \mathbb{T}$

(ii) $G_i^n \subset G_{i+1}^n$ *for every* $i, n \geq 0$

(iii) if $t_i^m = t_j^n$ *with* $m \leq n, i \leq j$ *then* $G_i^m \subset G_j^n$

(iv) $\bigcup_{i,n\geq 0} G_i^n$ *is a partition of* Ω

Proof. Since for every $t \in \mathbb{T}, \mathcal{H}_{t^-}$ is countably generated and $\{i, n \geq 0 : t_i^n = t\}$ is countable, there exists a set of disjoint measurable events $\{A_{i,n} : i, n \geq 0\}$ such that $\mathcal{H}_{t^-} = \sigma(A_i^n, i, n \geq 0, t_i^n = t)$. $\mathcal{H}$ is non-decreasing with time and $\mathcal{H}_{t^-} = \sigma(A_j^n, i, j, n \geq 0, j \leq i, t_i^n \leq t)$.

Hence, defining the sets $G_i^n := \{A_j^m, j \leq i, m \leq n\}$, we have

$$\mathcal{H}_{t^-} = \sigma(\bigcup_{n\geq 0} \bigcup_{i\geq 0: t_i^n = t} G_i^n) = \sigma(G_i^n, i, n \geq 0, t_i^n \leq t).$$

By construction the family G is non-decreasing in both indices. The A_i^n are disjoints, and as $\Omega \in \mathcal{H}_{t^-}$ it follows $\bigcup_{i,n\geq 0} A_i^n = \Omega$ so that $\bigcup_{i,n\geq 0} G_i^n$ is a partition of Ω.

Finally we can always suppose without losing generality that the sets G_i^n contain no null set of $P_t(\omega, .)$, $d\mathbb{P} \times d[M,M]$-a.s. $\square$

The next lemma (which reformulates Lemma 2.3 in Ankirchner, Dereich, and Imkeller (2006)) uses the partitions of the probability space constructed in Lemma 4.15 to define a Radon-Nykodym derivative as a limit.

Lemma 4.16. *Suppose* $k_t(\omega, .) \ll P_t(\omega, .)$ *on* $\mathcal{H}_{t^-}$, $d\mathbb{P} \times d[M,M] - a.s.$. *Then there exists an* $\mathcal{F} \otimes \mathcal{G}$*-predictable process* γ *such that,* $d\mathbb{P} \times d[M,M]$*-a.s.,*

$$\gamma_t(\omega, \omega') = \frac{dk_t(\omega, .)}{dP_t(\omega, .)}\bigg|_{\mathcal{H}_{t^-}} (\omega').$$

Proof. Let $(T^n)_{n\geq 0}$ be a refining family of increasing sequence of times $T^n =: (t_i^n)_{i\geq 0}$, e.g. $t_i^n := \frac{i}{2^n}$, Let $\mathbb{T} := \{t_i^n, i \geq 0, n \geq 0\}$ and let $(G_i^n)_{i,n\geq 0}$ be a family of finite sets of events such as provided by Lemma 4.15.

The processes defined by

$$\gamma_t^n(\omega, \omega') := \sum_{i\geq 0} \sum_{A \in G_i^n} 1_{]t_i^n, t_{i+1}^n]}(t) 1_A(\omega') \frac{k_t(\omega, A)}{P_t(\omega, A)}$$

are $\mathcal{F} \otimes \mathcal{G}$-predictable, and on the other hand define a discrete time martingale in (ω', n). Therefore the sequence converges as $n \to \infty$ to a limit $\gamma_t(\omega, \omega')$, which is defined (as is k) for every $\omega' \in \Omega$, $d\mathbb{P}(\omega) \times d[M,M]$-a.s.

For every $A \in \mathcal{H}_{t^-}$,

$$\begin{aligned}
\int_A \gamma_t(\omega, \omega') dP_t(\omega') &= \lim_{n\to\infty} \inf_{A \subset B \in G_i^n, i \geq 0} \int_B \gamma_t^n(\omega, \omega') dP_t(\omega') \\
&= \inf_{A \subset B \in G_i^n, i,n \geq 0} k_t(\omega, B) \\
&= k_t(\omega, A),
\end{aligned}$$

and $\gamma_t(\omega,\omega') = \frac{dk_t(\omega,.)}{dP_t(\omega,.)}\Big|_{\mathcal{H}_{t-}}(\omega')$, $d\mathbb{P}\times d[M,M]\times d\mathbb{P}$-a.s. □

Theorem 4.17. *Suppose that $k_t(\omega,.) \ll P_t(\omega,.)$ on $\mathcal{H}_t$, $d\mathbb{P}\times d[M,M]$-a.s. Then M has an information drift $\alpha \in \mathcal{S}(\mathcal{G})$, which is a $\mathcal{G}$-predictable version of the Radon-Nykodym derivative $(\omega,t)\mapsto \frac{dk_t(\omega,.)}{dP_t(\omega,.)}\Big|_{\mathcal{H}_{t-}}(\omega)$.*

Note that this also means according to our definition that α is the information drift for the filtration $\mathcal{G}$ (see Section 1.3).

Remark 4.18. *It follows directly from the embedding formula in Lemma 4.6 that*

$$\int\int\int \gamma_t^2(\omega,\omega')P_t(\omega,d\omega')d[M,M]_t d\mathbb{P}(\omega) = \int\int \alpha_t^2(\omega)d[M,M]_t d\mathbb{P}(\omega).$$

4.3.2 Convergence of filtrations and Hypothesis $(\mathscr{K})$

We recall that k is given by Equation (4.1).

Theorem 4.19. *Let $\mathcal{H}^n$ be a sequence of filtrations such that $\forall t\in I, \mathcal{H}_t := \bigvee_{n\geq 1}\mathcal{H}_t^n$ and suppose that*

$$\forall n\geq 1, t\in I,\ k_t(\omega,.) \ll P_t(\omega,.) \text{ on } \mathcal{H}_{t-}^n.$$

Then, with $\mathcal{G}^n := \bigcap_{u>t}(\mathcal{F}_u\vee\mathcal{H}_u^n)$:

(i) For every $n\geq 0$, M has an information drift $\alpha\in\mathcal{S}(\mathcal{G}^n)$ with respect to $\mathcal{G}_n$, given by $\alpha^n(\omega) := \frac{dk_t(\omega,.)}{dP_t(\omega,.)}\Big|_{\mathcal{H}_{t-}^n}(\omega)$

(ii) If $\sup_{n\geq 1}\mathbb{E}\int_0^T|\alpha_s^n|^p d[M,M]_t < \infty$ for some $p>1$, M has an information drift $\alpha\in\mathcal{S}(\mathcal{G})$ with respect to $\mathcal{G}$, satisfying $\alpha^n \xrightarrow[n\to\infty]{} \alpha$ a.s. and in $L^p(\Omega\times I, d\mathbb{P}\times d[M,M])$.

Proof. Let $n \geq 1$ and suppose that $\forall t \in I, k_t(\omega, .) \ll P_t(\omega, .)$ on $\mathcal{H}^n_{t^-}$. By the Radon-Nykodym theorem there exist a $\mathcal{H}^n$-predictable process $\gamma^n(\omega, .)$ such that, $\forall A \in \mathcal{H}^n_t$, $k_t(\omega, A) := \int_A \gamma^n_t(\omega, \omega')P_t(\omega, d\omega')$. Additionally Theorem 4.17 establishes that M has an information drift α^n with respect to $\mathcal{G}^n$, given by $\alpha^n_s(\omega) := \gamma^n_s(\omega, \omega)$.

On the other hand, $(\omega', n) \mapsto \gamma^n_s(\omega, \omega')$ defines a discrete time martingale. As a consequence of Theorem 4.10, for fixed $\omega \in I$ and $t \in I$, $k_t(\omega, .) \ll P_t(\omega, .)$ on $\mathcal{H}_{t^-}$ holds if and only if the discrete time martingale $(\omega', n) \mapsto \gamma^n_s(\omega, \omega')$ is uniformly integrable. This is for instance the case if we have, for some $p > 1$,

$$\sup_{n\geq 1} \int_\Omega |\gamma^n_s(\omega, \omega')|^p P_t(\omega, d\omega') < \infty.$$

Finally by the embedding formula in Lemma 4.6,

$$\begin{aligned} & \int_\Omega \int_0^T \int_\Omega |\gamma^n_s(\omega, \omega)|^p P_t(\omega, d\omega') d[M, M]_s d\mathbb{P}(\omega) \\ = & \int_0^T \int_\Omega |\gamma^n_s(\omega, \omega)|^p d[M, M]_s d\mathbb{P}(\omega) \\ = & \int_0^T \int_\Omega |\alpha^n_s(\omega)|^p d[M, M]_s d\mathbb{P}(\omega) \end{aligned}$$

Therefore, if $\sup_{n\geq 1} \int_0^T \int_\Omega |\alpha^n_s(\omega)|^p d[M, M]_t d\mathbb{P}(\omega) < \infty$, then it follows from Theorem 4.10 that $k_t(\omega, .) \ll P_t(\omega, .)$ on $\mathcal{H}_t$, $d\mathbb{P} \times d[M, M] - a.s.$, with a Radon-Nykodym density given by $\gamma_t(\omega, .)$.

Theorem 4.10 also shows that $\gamma^n_t(\omega, .) \xrightarrow[n\to\infty]{a.s., L^p} \gamma_t(\omega, .)$, which implies by the embedding formula in Lemma 4.6 that $\alpha^n \xrightarrow[n\to\infty]{} \alpha$ a.s. and in $L^p(\Omega \times I, d\mathbb{P} \times d[M, M])$. □

Theorem 4.10 actually allows us to give a refined statement.

Theorem 4.20. *Suppose $\mathcal{H} = \bigvee \mathcal{H}_t^n$. Then the following assertions are equivalent*

(i) $k_t(\omega, .) \ll P_t(\omega, .)$ *on* $\mathcal{H}_{t^-}$

(ii) for every $n \geq 1$, $k_t(\omega, .) \ll P_t(\omega, .)$ *on* $\mathcal{H}_{t^-}^n$ *and* $\left(\frac{dk_t(\omega,.)}{dP_t(\omega,.)}\Big|_{\mathcal{H}_{t^-}^n}(\omega) \right)_{n\geq 1}$ *is uniformly integrable*

4.3.3 Necessity of Hypothesis $(\mathscr{K})$ under a uniform integrability condition

We now show, in the converse direction, that Hypothesis 4.11 is actually necessary when the information drift is uniformly integrable, by constructing a Radon-Nykodym density for $k_t(\omega, .)$ with respect to $P_t(\omega, .)$. We follow again Ankirchner, Dereich, and Imkeller (2006) with relaxed hypotheses.

In this paragraph let $t_i^n := \frac{i}{2^n}$, $\mathbb{T} := (t_i^n, i \geq 0, n \geq 0)$ and G_i^n be as provided by Lemma 4.15. Define also

$$Z_t^n(\omega, \omega') := \sum_{i\geq 0} \sum_{A \in G_i^n} 1_{]t_i^n, t_{i+1}^n]}(t) 1_A(\omega') \frac{k_t(\omega, A)}{P_t(\omega, A)}.$$

Lemma 4.21. $d\mathbb{P} \times d[M, M]$*-a.s.,* $(Z^n(\omega, .))_{n\geq 0}$ *is a martingale bounded in* $L^1(d\mathbb{P})$*, which converges a.s. to some random variable* Z^∞*.*

Proof. Fix $\omega \in \Omega$ and $t \in I$. Since the discretizations G_i^n are refining with $n \geq 1$, it follows from the additivity of $P_t(\omega, .)$ that $(Z^n(\omega, .))_{n\geq 0}$ is a discrete time martingale, and Doob's martingale convergence theorems justifies the remainder of the statement. For an alternative proof see Section 2 of Ankirchner, Dereich, and Imkeller (2006). □

4.3.3.1 Uniform integrability assumption

Proposition 4.22. *Fix $\omega \in \Omega$ and $t \in I$. If $(Z_t^n(\omega,.))_{n\geq 0}$ defines a uniformly integrable martingale, then $k_t(\omega,.) \ll P_t(\omega,.)$ on $\mathcal{H}_{t^-}$.*

Proof. Similarly as in the proof of Proposition 4.9, the uniform integrability of $(Z_t^n(\omega,.))_{n\geq 0}$ implies the uniform convergence of the discrete-time martingale, and the reader can check easily that the limit is a Radon-Nykodym density for $k_t(\omega,.)$ with respect to $P_t(\omega,.)$ on $\mathcal{H}_{t^-}$. □

Theorem 4.23. *Let $t_i^n := \frac{i}{2^n}, \mathbb{T} := (t_i^n, i \geq 0, n \geq 0)$ and let G_i^n be as provided by Lemma 4.15. Define*

$$Z_t^n(\omega,\omega') := \sum_{i\geq 0}\sum_{A\in G_i^n} 1_{]t_i^n,t_{i+1}^n]}(t)1_A(\omega')\frac{k_t(\omega,A)}{P_t(\omega,A)}.$$

Then Hypothesis 4.11 holds if and only if the discrete time martingale $(\omega',n) \mapsto Z_t^n(\omega,\omega')$ is uniformly integrable, $d\mathbb{P}\times d[M,M]$-a.s.

Proof. In the light of Proposition 4.22 is is sufficient to suppose that $k_t(\omega,.) \ll P_t(\omega,.)$ on $\mathcal{H}_{t^-}$ and observe that $Z_t(\omega,.)^n$ converges to the Radon-Nykodym derivative $\left.\frac{dk_t(\omega,.)}{dP_t(\omega,.)}\right|_{\mathcal{H}_{\sqcup^-}}$ as $n\to\infty$. Alternatively the reader can also apply directly Theorem 4.10. □

4.3.3.2 Square integrability assumption

We showed that Hypothesis $(\mathscr{K})$ is equivalent to the uniform integrability of the discrete time martingale $(\omega',n) \mapsto Z_t^n(\omega,\omega')$. In the particular case where the information drift is square integrable, it provides a L^2 bound on Z^n and we can recover the necessary condition in Ankirchner, Dereich, and Imkeller (2006).

Lemma 4.24. *Let* $0 \leq s < t$ *and suppose that there exist a* $\mathcal{G}$*-predictable process* α *with*

$$\mathbb{E}\int_s^t \alpha_u^2 d[M,M]_u < \infty$$

such that $M - \alpha \cdot [M,M]$ *is a* $\mathcal{G}$*-local martingale. If* $\mathcal{P} := \{A_1, ..., A_n\}$ *is a partition of* Ω *into* $\check{\mathcal{G}}_s$*-measurables sets, then*

$$\mathbb{E}\int_s^t \sum_{k=1}^n (\frac{k_u}{P_u})^2(., A_k)1_{A_k} d[M,M]_u \leq 4\mathbb{E}\int_s^t \alpha_u^2 d[M,M]_u.$$

Proof. See Lemma 2.8 in Ankirchner, Dereich, and Imkeller (2006). □

Lemma 4.24 shows that Z^n is a square integrable martingale, in particular uniformly integrable, so that Proposition 4.22 applies and we recover Theorem 2.10 in Ankirchner, Dereich, and Imkeller (2006).

Theorem 4.25. *Suppose that* M *has an information drift* α *with respect to* $\mathcal{G}$*, satisfying*

$$\int_\Omega \int_0^T \alpha_s(\omega)^2 d[M,M]_s d\mathbb{P}(\omega) < \infty.$$

Then Hypothesis 4.11 is satisfied, i.e.

$$k_t(\omega, .) \ll P_t(\omega, .) \text{ on } \check{\mathcal{G}}_{t^-}, d\mathbb{P} \times d[M,M] - a.s.$$

4.3.4 Trace isometry for square-integrable information drift and densities

We conclude this section by exhibiting an isometry between the information drift and the Radon-Nykodym derivative.

Theorem 4.26 (Trace isometry). *The two following assertions are equivalent.*

(i) M *has an information drift* $\alpha \in \mathcal{S}(\mathcal{G})$ *with respect to* $\mathcal{G}$*, satisfying* $\mathbb{E} \int_I \alpha_s^2 d[M,M]_s < \infty$.

(ii) There exists a $\mathcal{F} \otimes \mathcal{G}$*-predictable processes* γ *such that* $k_t(\omega, .) = \int_. \gamma(\omega, \omega') P_t(\omega, d\omega')$ *on* $\mathcal{H}_{t^-}$ *(or equivalently on* $\check{\mathcal{G}}_{t^-}$*) and* $\mathbb{E} \int_I \gamma_s^2(\omega, \omega) d[M,M]_s < \infty$.

If one of the two statements above hold, the processes γ *and* α *are unique, satisfy* $\alpha(\omega) = \gamma(\omega, \omega)$ *a.s., as well as the trace isometry formula*

$$\int \int \int \gamma_t^2(\omega, \omega') P_t(\omega, d\omega') d[M,M]_t d\mathbb{P}(\omega) = \int \int \alpha_t^2(\omega) d[M,M]_t d\mathbb{P}(\omega).$$

Proof. $(ii) \implies (i)$ is proved by Theorem 4.17 together with Remark 4.18. $(i) \implies (ii)$ is proved by Theorem 4.25, together with Remark 4.18. The last remark is finally a simple statement of Remark 4.18. □

4.4 Representation of Jacod's conditional densities with the information drift

In this section we express the Radon-Nykodym density associated to Hypothesis 4.11 as a stochastic logarithm of the flow of the conditional likelihood, or equivalently as an infinitesimal generator of the conditional density in Jacod's Hypothesis.

We first introduce a generalization of Jacod's condition for initial expansions (see Jacod (1985), Paragraph 1.7 or 3.2.1.2).

Hypothesis 4.27 (Hypothesis ($\mathscr{J}$)). *$d\mathbb{P} \times d[M,M] - a.s.$, there exist a σ-finite measure η_t on $\mathcal{H}_{t^-}$ such that*

$$P_t(\omega,.) \ll \eta_t \text{ on } \mathcal{H}_{t^-}.$$

Remark 4.28. *It is immediate that Hypothesis ($\mathscr{J}$) cannot hold on the filtration $\mathcal{F}$, as for every $F \in \mathcal{F}_t$ we have $P_t(\omega,.) = 1_F(\omega)$ which defines a singular measure a.s. with respect to any different measure.*

Lemma 4.29. *If the enlargement satisfies the generalized Jacod criterion, we can take the measure η to be the initial probability distribution P_0, which in particular is invariant with time.*

Proof. We adapt the proof of Jacod's original result in Jacod (1985) in the case of an initial enlargement, and prove that for each $t \in [0,T]$, $\eta_t \ll P_0$ on $\mathcal{H}_{t^-}$.

Define $a_t(\omega') := \int_\Omega p_t(\omega,\omega')d\mathbb{P}(\omega)$ and $q_t(\omega,\omega') = \frac{p_t(\omega,\omega')}{a_t(\omega')}1_{a_t(\omega')>0}$. Then for any $A \in \mathcal{H}_{t^-}$ we have

$$\begin{aligned} P_0(A) &= \mathbb{E}P_t(A) \\ &= \int_\Omega P_t(\omega, A)d\mathbb{P}(\omega) \\ &= \int_\Omega \int_A P_t(\omega, d\omega')d\mathbb{P}(\omega) \\ &= \int_\Omega \int_A p_t(\omega,\omega')\eta_t(d\omega')d\mathbb{P}(\omega) \\ &= \int_A \int_\Omega p_t(\omega,\omega')d\mathbb{P}(\omega)\eta_t(d\omega') \\ &= \int_A a_t(\omega')\eta_t(d\omega') \end{aligned}$$

so that $P_0(d\omega') = a_t(\omega')\eta_t(\omega')$ and $P_t(\omega, d\omega') = p_t(\omega,\omega')\eta_t(d\omega') = q_t(\omega')P_0(d\omega')$. $\square$

According to Lemma 4.29 we may implicitly assume in the sequel that the η_t given by the generalized Jacod criterion are invariant with time, in which case we will simply write η.

Definition 4.30. *If Hypothesis ($\mathscr{J}$) is satisfied, we define Jacod conditional likelihood of a filtration $\mathcal{Y}$, or the Jacod conditional likelihood when there is no ambiguity, as a càdlàg version of the Radon-Nykodym derivative $p_t(\omega,\omega') := \frac{dk_t(\omega,.)}{dP_t(\omega,.)}(\omega')$, i.e. satisfying $P_t(\omega, A) =: \int_A p_t(\omega,\omega')\eta_t(d\omega')$.*

The next theorem, which provides a new characterization of Hypothesis 4.11, is the main result of this section.

Theorem 4.31. *Assume that Hypothesis ($\mathscr{J}$) holds. Then the two statements in Theorem 4.26 are also equivalent to:*

(iii) There exists a $\mathcal{F} \times \mathcal{G}$-predictable processes γ such that the conditional densities admit the representation $p_t(.,\omega') = p_0(\omega')\mathcal{E}\big((\gamma(.,\omega') \cdot M)_t\big)$ on $\mathcal{H}_{t-}$, and which satisfies $\mathbb{E}\int_I \gamma_s^2(\omega,\omega)d[M,M]_s < \infty$.

Remark 4.32. *The representation in Theorem 4.31 implies that $p_t(\omega,\omega)$ defines an equivalent local martingale measure for the semimartingale M in the filtration $\mathcal{G}$.*

Theorem 4.31 is a consequence of the next two propositions.

Proposition 4.33. *Suppose that Hypothesis ($\mathscr{J}$) is satisfied, and that there exists an $\mathcal{F} \otimes \mathcal{G}$-predictable process γ such that $p_t(\omega,\omega') = p_0(\omega')\mathcal{E}\big(\gamma(\omega,\omega') \cdot M(\omega)\big)_t$ and $\int_\Omega \int_0^T \gamma_t(\omega,\omega)^2 d[M,M]_t(\omega)d\mathbb{P}(\omega) < \infty$.*

Then $k_t(\omega,A) := \int_A \gamma_t(\omega,\omega')P_t(\omega,d\omega')$ is such that the conditional probabilities $P_t(\omega,d\omega')$ admit the representation given in Equation (4.1), and satisfy $k_t(\omega,.) \ll P_t(\omega,.)$ on $\mathcal{H}_{t-}$, $d\mathbb{P} \times d[M,M] - a.s.$ and $\frac{k_t(\omega,d\omega')}{P_t(\omega,d\omega')} = \gamma_t(\omega,\omega')$.

Proof. $(t,\omega) \longmapsto p_t(\omega,.)$ is a local martingale satisfying the SDE

$$p_t(\omega,.) = p_0(.) + \int_0^t \gamma_s(\omega,.)p_s(\omega,.)dM_s$$

so that

$$P_t(\omega, A) = P_0(A) + \int_A \int_0^t \gamma_s(\omega,\omega')p_s(\omega,\omega')dM_s(\omega)\eta(d\omega')$$
$$= P_0(A) + \int_0^t \int_A \gamma_s(\omega,\omega')p_s(\omega,\omega')\eta(d\omega')dM_s(\omega)$$
$$= P_0(A) + \int_0^t \int_A \gamma_s(\omega,\omega')P_s(\omega,d\omega')dM_s(\omega).$$

The change of the order of integration is justified by the *embedding formula* in Lemma 4.6 which provides the integrability

$$\int_\Omega \int_0^T \int_\Omega \gamma_t(\omega,\omega')^2 P_s(\omega,d\omega')d[M,M]_t(\omega)d\mathbb{P}(\omega) = \int_\Omega \int_0^T \gamma_t(\omega,\omega)^2 d[M,M]_t(\omega)d\mathbb{P}(\omega) < \infty.$$

□

Proposition 4.34. *Suppose that Hypothesis* $(\mathscr{J})$ *is satisfied and that* $k_t(\omega,.) \ll P_t(\omega,.)$ *on* $\mathcal{H}_{t^-}$, $d\mathbb{P} \times d[M,M]$*-a.s., with* $\gamma_t(\omega,\omega') := \frac{k_t(\omega,d\omega')}{P_t(\omega,d\omega')}$ *such that* $\int_\Omega \int_0^T \gamma_t(\omega,\omega)^2 d[M,M]_t(\omega)d\mathbb{P}(\omega) < \infty$. *Then,* $d\mathbb{P} \times d[M,M]$*-a.s.,* $P_t(\omega,.)$ *has its density on* $\mathcal{H}_{t^-}$ *given by*

$$p_t(\omega,\omega') = p_0(\omega')\mathcal{E}\big(\gamma(\omega,\omega').M(\omega)\big)_t,$$

i.e.

$$\forall A \in \mathcal{H}_{t^-}, P_t(\omega,A) = \int_A p_0(\omega')\mathcal{E}\big(\gamma(\omega,\omega').M(\omega)\big)_t d\mathbb{P}(\omega').$$

Proof. The process $(t,\omega) \longmapsto \gamma_t(\omega,.)$ is $\mathcal{F}$-adapted and, $d\mathbb{P} \times d[M,M]$-a.s., for each $A \in$

$\mathcal{H}_{t^-}$, we have

$$k_t(\omega, A) = \int_A \gamma_s(\omega, \omega') P_s(\omega, d\omega'),$$

so that $P_t(\omega, A)$ has the representation

$$P_t(\omega, A) = P_0(A) + \int_0^t \int_A \gamma_s(\omega, \omega') P_s(\omega, d\omega') dM_s = P_0(A) + \int_0^t \int_A \gamma_s(\omega, \omega') p_s(\omega, \omega') \eta(d\omega') dM_s.$$

According to the *embedding formula*,

$$\int_\Omega \int_0^T \int_\Omega \gamma_t(\omega, \omega')^2 P_s(\omega, d\omega') d[M, M]_t(\omega) d\mathbb{P}(\omega) = \int_\Omega \int_0^T \gamma_t(\omega, \omega)^2 d[M, M]_t(\omega) d\mathbb{P}(\omega) < \infty,$$

thus with probability 1 we can switch the orders of integration and

$$P_t(\omega, A) = P_0(A) + \int_A \int_0^t \gamma_s(\omega, \omega') p_s(\omega, \omega') dM_s \eta(d\omega').$$

It follows that the density $p_s(\omega, \omega')$ is a solution of the SDE

$$p_t(\omega, \omega') = p_0(\omega') + \int_0^t \gamma_s(\omega, \omega') p_s(\omega, \omega') dM_s.$$

Hence we obtain $P_t(\omega, d\omega') = P_0(d\omega') e^{\int_0^t \gamma_s(\omega, \omega') dM_s - \frac{1}{2} \int_0^t \gamma_s(\omega, \omega')^2 ds}$, which in particular shows that $P_t(\omega, .) \ll P_0(.)$ on $\mathcal{H}_{t^-}$. □

4.5 Estimation of the information drift via functional Itô calculus

In this section we combine the characterizations of the information drift in Theorems 4.17 and 4.19 in the previous sections to the functional Itô calculus to derive a computational method to estimate the information drift.

The functional Itô calculus was introduced in Dupire (2009), and further developed in Cont and Fournié (2010a), Cont and Fournié (2010b) and Cont and Fournié (2013), to extend the stochastic calculus to function of the whole path of Brownian motion and more generally semimartingales. An earlier and independent approach can be found in Ahn (1997).

The idea is the following: if we suppose that the filtration $\mathcal{F}$ is generated by a (family of) càdlàg semimartingale(s) ζ, we can write the conditional probabilities P_t as $P_t(\omega, .) = \mathfrak{f}(t, \zeta^t, .)$ for some measurable application $\mathfrak{f}$, which forms a family of non-anticipative functionals of the semimartingale ζ. We can therefore apply the functional Itô calculus to compute the information drift as a diffusion coefficient.

In this section we suppose that $E = \mathbb{R}^d$, and that $I = [0, T)$ for some $T > 0$ (so that it is an open interval). For $t \in I$ we let $\mathcal{D}([0, t])$ (resp. $\mathcal{D}'([0, t])$ be the space of processes taking values in E (respectively in the space of positive definite matrices on E) with càdlàg paths.

Definition 4.35 (Non-anticipative functionals). *A non-anticipative functional* $\mathfrak{f}$ *on* $\mathcal{D}([0,T[) \times \mathcal{D}'([0,T[)$ *is a family of functions*

$$\mathfrak{f}_t : \mathcal{D}([0,t]) \times \mathcal{D}'([0,t]) \longrightarrow E,$$

such that $\mathfrak{f}_.(\theta_., a_.)$ *is* $\mathcal{F}$*-adapted.*

We call $\mathbb{D}^0$ *the space of non-anticipative functionals.*

Definition 4.36 (Space $\mathbb{D}$). *We define* $\mathbb{D}$ *as the space of functionals* $\mathfrak{f} := (\mathfrak{f}_t, t \in I) \in \mathbb{D}^0$ *which are locally Lipschitz, uniformly on* $[0,T[$*, for the metric* d_∞ *defined by:*

$$\forall t \leq t', x \in \mathfrak{S}, d_\infty((x,t),(x',t')) = \sup_{u \in [t,t']} |x_{u \wedge t} - x'_u| + |t' - t|$$

Definition 4.37 (Horizontal derivative). *We say that a non-anticipative functional* $\mathfrak{f} \in \mathbb{D}$ *admits an horizontal derivative if there exists* $\mathcal{D}_t\mathfrak{f}$ *such that for every* $t \in I$*,* $(x,v) \in \mathcal{D}([0,t]) \times \mathcal{D}([0,t])$*,*

$$\mathcal{D}_t\mathfrak{f}(x,v) = \lim_{h \to 0} \frac{\mathfrak{f}_{t+h}(x,v) - \mathfrak{f}_t(x,v)}{h},$$

in which case we call $\mathcal{D}_t\mathfrak{f}$ *the* ***horizontal derivative*** *of* $\mathfrak{f}$*.*

Definition 4.38 (Vertical derivative). *We say that a non-anticipative functional* $\mathfrak{f} \in \mathbb{D}$ *admits a vertical derivative if there exists* $\nabla_\omega\mathfrak{f}$ *such that for every* $(x,v) \in \mathcal{D}([0,t]) \times \mathcal{D}([0,t])$*,*

$$\nabla_\omega\mathfrak{f}(x,v) = \lim_{h \to 0} \frac{\mathfrak{f}_t(x^t + h\delta_t, v^t) - \mathfrak{f}_t(x^t, v^t)}{h},$$

in which case we call $\mathcal{D}_t\mathfrak{f}$ *the* ***vertical derivative*** *of* $\mathfrak{f}$*.*

Theorem 4.39 (Functional Itô formula). *Let f be a non-anticipative functional in $\mathbb{D}$ and suppose that it admits a horizontal derivative $\mathcal{D}_t\mathfrak{f}$ and vertical derivative $\nabla_\omega\mathfrak{f}$ which are also in $\mathbb{D}$.*

Then for every càdlàg semimartingale Y we have:

$$\mathfrak{f}(T, Y^T) = \mathfrak{f}(0, Y^0) + \int_0^T \mathcal{D}_t\mathfrak{f}(t, Y^t)dt + \frac{1}{2}\int_0^T \nabla_\omega^2\mathfrak{f}(t, Y^{t-})d[Y,Y]^c(t)$$

$$+ \sum_{0<t\leq T} [\mathfrak{f}(t, Y_t) - f(t, Y^{t-}) - \nabla_\omega\mathfrak{f}(t, Y^{t-})]\Delta Y_t + \int_0^T \nabla_\omega\mathfrak{f}(t, Y^{t-})dY_t.$$

Proof. see Cont and Fournié (2010a) or Cont and Fournié (2013) (or Cont and Fournié (2010b)) □

We can apply this representation to the family of conditional probabilities P_t. Suppose that ζ is a continuous semimartingale and that $\mathcal{F}$ is the smallest filtration satisfying the usual hypothesis to which ζ is adapted. In that case we can write $P_t(\omega, .) = \mathfrak{f}(t, \zeta^t, .), t \in I$.

Theorem 4.40 ($*$). *Suppose that the conditional probabilities P_t are of the form $P_t(\omega, .) := \mathfrak{f}_t(\zeta^t(\omega), .), t \in I$ for some (non-anticipative) functional $\mathfrak{f} \in \mathbb{D}$ which admits a horizontal derivative $\mathcal{D}_t\mathfrak{f}$ and vertical derivative $\nabla_\omega\mathfrak{f}$ that are also in $\mathbb{D}$.*

If $\nabla\mathfrak{f}(\zeta^t, .) \ll \mathfrak{f}(\zeta^t, .)$ on $\check{\mathcal{H}}_{t\_}$, then M has an information drift $\alpha \in \mathcal{S}(\mathcal{G})$, with $\mathbb{E}\int_0^T \alpha_u^2 d[M,M]_s < \infty$.

Proof. This is an application of the sufficiency of Hypothesis ($\mathscr{K}$) for the existence of the information drift (Theorem 4.17) together with the functional Itô representation (Theorem 4.39). □

Additionally we can deduce from Theorem 4.19 a computational method to estimate the information drift.

Theorem 4.41 ((∗) Estimation of the information drift). *Suppose that* $\mathcal{H}_t := \bigvee_{n\geq 1} \mathcal{H}_t^n$ *for every* $t \in I$ *and that for every* $n \geq 1, t \in I$,

$$P_t(\omega, .) = \mathfrak{f}_t^n(\zeta^t(\omega), .) \text{ on } \mathcal{H}^n,$$

where $\mathfrak{f}^n$ *is a non-anticipative functional. Suppose also that* $\mathfrak{f}^n$ *admits a horizontal derivative* $\mathcal{D}_t f$ *and vertical derivative* $\nabla_\omega f_t$ *that are also in* $\mathbb{D}$.

Let $\mathcal{G}_t := \bigcap_{u>t}(\mathcal{F}_t \vee \mathcal{H}_t)$. *If for every* $n \geq 1$ *we have* $\nabla f(\zeta^t, .) \ll f(\zeta^t, .)$ *on* $\check{\mathcal{G}}_{t_-}$, *then*

(i) For every $n \geq 1$, M *has an information drift* $\alpha^n \in \mathcal{S}(\mathcal{G}^n)$ *with respect to* $\mathcal{G}^n$, *given by* $\alpha_t^n := \left.\frac{dk_t(\omega,.)}{dP_t(\omega,.)}\right|_{\mathcal{H}_{t-}^n}$.

(ii) If $\sup_{n\geq 1} \mathbb{E}\int_0^T |\alpha_s^n|^p d[M,M]_t < \infty$ *for some* $p > 1$, *then* M *has an information drift* $\alpha^n \in \mathcal{S}(\mathcal{G}^n)$ *with respect to* $\mathcal{G}^n$, *satisfying* $\alpha^n \xrightarrow[n\to\infty]{} \alpha$ *a.s. and in* $L^p(\Omega \times I, d\mathbb{P} \times d[M,M])$.

Chapter 5

Examples

In this chapter we consider a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a Brownian motion on a finite time interval $I := [0,T], T > 0$. We let $\mathcal{F}$ be the right-continuous filtration defined by

$$\mathcal{F}_t := \bigcap_{u>t} \sigma(W_s, s \leq u), t \in I,$$

for which W is a Brownian motion. Consistently with the general notations of this thesis we consider the two expansions $\check{\mathcal{G}}$ and $\mathcal{G}$ of $\mathcal{F}$ defined by

$$\check{\mathcal{G}}_t := \mathcal{F}_t \vee \sigma(X_u, u \leq t) \text{ and } \mathcal{G}_t := \textstyle\bigcap_{u>t} \check{\mathcal{G}}$$

for several given càdlàg process X.

* * * * *

We study here several toy examples of dynamic anticipations of a Brownian filtration. We have in mind our high-frequency trading setting (see Section 2.3) where we want to model the mechanisms by which a high-frequency trader can use his speed to guess the next price moves in the limit order book .

We first prove a formula for the information drift of a Brownian motion in terms of its conditional expectation with respect to the augmented filtration (Section 5.1), and motivate the examples of this section by the perspective of high-frequency trading (Section 2.3). As a first example (Section 5.2) we review the case of an anticipation of the Brownian motion a fixed time perturbed with an independent noise, where we complement the literature with explicit conditions for the existence of a local martingale measure, i.e. absence of arbitrage in the classical sense of NFLVR (see Chapter 2). The remaining sections follow our search towards an example of arbitrage-free and continuously anticipative expansion, where we successively consider expansions with fixed anticipation (Section 5.3) and expansions with a random anticipation (Section 5.4).

5.1 Information drift of Brownian motion

Theorem 5.1 ($*$). *Let W be a $\mathcal{F}$-Brownian motion and suppose that there exists a process $\alpha \in \mathfrak{H}^1(\mathcal{G})$ such that $M - \int_0^\cdot \alpha_s d[M,M]_s$ is a $\mathcal{G}$-Brownian motion. Then we have:*

(i) $\alpha_s = \lim_{\substack{t \to s \\ t > s}} \mathbb{E}[\frac{W_t - W_s}{t-s} | \mathcal{G}_s]$

(ii) $\alpha_s = \frac{\partial}{\partial t} \mathbb{E}[W_t | \mathcal{G}_s]\Big|_{t=s}$

Proof. Since W is $\mathcal{G}$-adapted it is clear that (i) and (ii) are equivalent. Suppose that $M - \int_0^\cdot \alpha_u du$ is a $\mathcal{G}$-martingale. It follows that

$$\forall s \leq t, \mathbb{E}\Big[W_t - W_s - \int_s^t \alpha_u | \check{\mathcal{G}}_s\Big] du = 0,$$

hence

$$\forall s \leq t,\ \mathbb{E}\Big[W_t - W_s|\check{\mathcal{G}}_s]$$
$$= \mathbb{E}[\int_s^t \alpha_u du|\check{\mathcal{G}}_s\Big]$$
$$= \int_s^t \mathbb{E}[\alpha_u|\check{\mathcal{G}}_s]du.$$

By differentiating with respect to t we now obtain

$$\mathbb{E}[\alpha_t|\mathcal{G}_s] = \frac{\partial}{\partial t}\mathbb{E}[W_t|\mathcal{G}_s]$$

which proves (ii). □

Remark 5.2.

1. *Theorem 5.1 holds for general augmentations $\mathcal{G}$ of the Brownian filtration $\mathcal{F}$.*
2. *It can be naturally extended to compute the information drift of martingales with deterministic quadratic variation.*
3. *For more general semimartingales, see for instance Protter (2004, Exercises 27 and 30, p. 150), as well as Janson, M'Baye, and Protter (2010).*

Corollary 5.3 (∗). *Suppose that for every $s \leq t$ we have a representation of the form*

$$\mathbb{E}[W_t|\mathcal{G}_s] = \int_0^s X_u \mu_{s,t}(du),$$

where X is a $\mathcal{G}$-measurable stochastic process and $(\mu_{s,t})_{s\leq t}$ a family of signed measures

s-adapted to $\mathcal{G}$. Then

$$\alpha_s = \int_0^s X_u \partial_t \mu_{s,t}\Big|_{t=s}(du) = \lim_{\substack{t \to s \\ t > s}} \int_0^s X_u \frac{1}{t-s} \mu_{s,t}(du)\Big|_{t=s}.$$

Proof. This is an application of the lemma preceding. □

Remark 5.4. *Note that in Corollary 5.3 the measure $\mu_{s,s}$ must satisfy $\int_0^s (X_u - W_s)\mu_{s,s}(du) = 0$ a.s.*

5.2 A suggestive example

We obtain a toy example of an expansion with a process by perturbating an initial expansion with a fixed future value with an independent Gaussian *noise*, which according to Corollary 1.38 does not alterate the properties of semimartingales. Several variations of this example have also been considered in Kchia and Protter (2015, Section 6.2.3), Ankirchner (2005, Example 5.2.2) and Corcuera et al. (2004). In this section we prove the following theorem.

Theorem 5.5 (∗). *Suppose that $X_t := W_1 + \epsilon N_t$, where $\epsilon > 0$ and N is a continuous Gaussian process with mean $\mathbb{E}[N_t] = 0$ and covariance $\mathbb{E}[N_t N_s] = \phi(1 - (s \wedge t))$.*

1. *$W_t - \int_0^t \frac{X_s - W_s}{(1-s)+\epsilon^2\phi(1-s)}$ defines a $\mathcal{G}$-Brownian motion on $[0, 1]$.*

   In other words, W has an information drift with respect to $\mathcal{G}$ given by

$$\alpha_t := \frac{X_s - W_s}{(1-s) + \epsilon^2 \phi(1-s)}.$$

2. *Additionally,*

   (i) *if* $\frac{\phi(1-s)}{1-s} \geq m > 0$ *as* $s \to 1$*, then there exists an ELMM on* $[0, \tau^*]$*, where* τ^* *is given by the coupling equation*

$$\frac{\tau}{(1-\tau)} < (1 + m\epsilon^2)^2 \tag{5.1}$$

   (ii) *if* $\frac{\phi(1-s)}{1-s} \to \infty$ *as* $s \to 1$*, then there exists an ELMM on* $[0, 1[$.

   (iii) *if* $\frac{\phi(1-s)}{1-s} \leq M$ *for some* $M > 0$ *as* $s \to 1$*, then there cannot exist an ELLM on* $[0, 1]$.

Proof. Given the expression of the conditional expectation $\mathbb{E}[W_t|\mathcal{G}_s]$ in Lemma 5.7, the formula for the information drift given in Corollary 5.3 leads precisely to 1. One can alternatively obtain it from the Gaussian computations of Paragraph 5.2.1.

To prove 2 we start from the particular case of Brownian noise where $\phi(1-s) := (1-s)$. In that case, Proposition 5.11 shows that the coupling condition (5.1) is sufficient to ensure that there exists an ELMM on $[0, \tau]$. 2(i) and 2(iii) are deduced respectively in Corollaries 5.12 and 5.14, and 2(ii) is simply a consequence of 2(i). □

Remark 5.6 (Brownian noise). *In Theorem 5.5 the case of Brownian noise corresponds to taking* $N_t := V_{1-t}$*, where* $(V_t)_{t\in[0,1]}$ *is another* $\mathcal{F}$*-Brownian Motion independent from* W*. Note that we can construct a Brownian motion* V *orthogonal to* W *with* $V_t := W_t - \int_0^t \frac{W_s}{s} ds$ *(see Wu (1999, Chapter 3)).*

5.2.1 Computing the information drift (Proof of 1.)

Lemma 5.7.

$$\mathbb{E}[W_t|\mathcal{G}_s] = W_s + \frac{t-s}{1-s+\epsilon^2\phi(1-s)}(X_s - W_s)$$

Proof. Let $s < t$.

$$\begin{aligned}
\mathbb{E}[W_t|\mathcal{G}_s] &= \mathbb{E}[W_t|W_s, X_s] \\
&= \mathbb{E}[Z_t|W_s, W_1 - W_s + \epsilon N_s] \\
&= \lambda W_s + \mu(W_1 - W_s + \epsilon N_s)
\end{aligned}$$

for some $\lambda, \mu \in \mathbb{R}$, due to the linearity of conditional expectation for Gaussian random variables.

$$\begin{aligned}
\mathbb{E}[W_t W_s] &= \lambda \mathbb{E}[W_s^2] \\
&= \mathbb{E}[W_s^2]
\end{aligned}$$

implies $\lambda = 1$.

$$\begin{aligned}
\mathbb{E}[W_t(W_1 - W_s + \epsilon N_s] &= \mu(\mathbb{E}[(W_1 - W_s)^2] + \epsilon^2 \mathbb{E}[N_s^2]) \\
&= \mathbb{E}[(W_t - W_s)^2]
\end{aligned}$$

implies $\mu = \frac{t-s}{1-s+\epsilon^2\mathbb{E}[N_s^2]} = \frac{t-s}{1-s+\epsilon^2\phi(1-s)}$.

Finally:

$$\mathbb{E}[W_t|\mathcal{G}_s] \quad = \quad W_s + \frac{t-s}{1-s+\epsilon^2\phi(1-s)}(X_s - W_s)$$

□

5.2.2 Equivalent risk-neutral probability (Proof of 2.)

Lemma 5.8. *Consider the $\mathcal{G}$-local martingale $Z := \mathcal{E}(\alpha \cdot \widetilde{W})$, where $\alpha_t := \frac{X_t - W_t}{(1-t)+\epsilon^2\phi(1-t)}$ and $\widetilde{W}$ is a $\mathcal{G}$-Brownian Motion. If $\mathbb{E}\left[e^{\frac{1}{2}\int_0^\tau \alpha_s^2 ds}\right] < \infty$, then Z defines an equivalent local martingale measure for W with respect to the filtration $\mathcal{G}$ on $[0, \tau]$.*

Proof. It follows from Theorems 1.20 and 5.1 that Z is a positive local martingale deflator, and it is an ELMM as application of Novikov criterion (see Theorem 1.29). □

This leads us to estimate $\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \alpha_s^2 ds}\right]$.

Remark 5.9.

$$\alpha_t := \frac{X_t - W_t}{(1-t)+\epsilon^2\phi(1-t)} \sim \mathcal{N}\Big(\mu = 0, \sigma^2 = \frac{1}{(1-t)+\epsilon^2\phi(1-t)}\Big).$$

5.2.2.1 Brownian noise

We first consider the case of Brownian noise, with $N_t := V_{1-t}$ for some Brownian motion V independent from W, which corresponds to taking $\phi(1-s) = (1-s)$.

According to Lemma 5.8 we want to evaluate

$$\mathbb{E}^{\mathbb{P}}\big[e^{\frac{\kappa}{2}\int_0^\tau (\frac{W_s}{s})^2 ds}\big],$$

where

$$\kappa := \frac{1}{(1+\epsilon^2)^2} \in (0,1) \tag{5.2}$$

In the Brownian case,

$$\alpha_t := \frac{X_t - W_t}{(1+\epsilon^2)(1-t)} \sim \frac{W'_t}{(1+\epsilon^2)(1-t)} \sim \mathcal{N}\Big(0, \frac{1}{1+\epsilon^2 \frac{\phi(1-t)}{1-t}}\Big),$$

where W' is another Brownian motion. Hence the expectation becomes

$$\mathbb{E}^{\mathbb{P}}\big[e^{\frac{\kappa}{2}\int_\eta^1 (\frac{W_s}{s})^2 ds}\big]$$

with $\eta := 1-\tau \in (0,1)$.

The next lemma is a classical result about the moment of the χ^2 distribution.

Lemma 5.10. *If $U \sim \chi^2(n)$ then $\mathbb{E}[e^{\lambda U}] = \frac{1}{(1-2\lambda)^{n/2}}$ for $\lambda \in (0, \frac{1}{2})$*

Proposition 5.11 (Coupling condition for Brownian noise). *Suppose that $\phi(1-s) := (1-s)$. Then $\mathbb{E}\big[e^{\frac{1}{2}\int_0^\tau \alpha_s^2 ds}\big] < \infty$ if one of the following equivalent conditions hold:*

(i) $\kappa < \frac{\eta}{1-\eta}$

(ii) $\frac{\tau}{1-\tau} < (1+\epsilon^2)^2$

(iii) $\tau < \tau^* := \frac{1}{1+\kappa}$

Proof. Applying Jensen inequality and the lemma preceding:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\frac{\kappa}{2}\int_\tau^1 (\frac{W_s}{s})^2 ds}] &= \mathbb{E}^{\mathbb{P}}[e^{\frac{(1-\tau)\kappa}{2}\int_\tau^1 (\frac{W_s}{s})^2 \frac{ds}{1-\tau}}] \\
&\leq \mathbb{E}^{\mathbb{P}}[\frac{1}{1-\tau}\int_\tau^1 e^{\frac{(1-\tau)\kappa}{2}(\frac{W_s}{s})^2} ds] \\
&= \frac{1}{1-\tau}\int_\tau^1 \mathbb{E}^{\mathbb{P}}[e^{\frac{(1-\tau)\kappa}{2}(\frac{W_s}{s})^2}] ds \\
&= \frac{1}{1-\tau}\int_\tau^1 \mathbb{E}^{\mathbb{P}}[e^{\frac{(1-\tau)\kappa}{2s}U^2}] ds, U \sim \chi^2(1) \\
&= \frac{1}{1-\tau}\int_\tau^1 \frac{1}{\sqrt{1-\frac{(1-\tau)\kappa}{s}}} ds
\end{aligned}$$

which is finite if and only if $\frac{(1-\tau)\kappa}{2s} < \frac{1}{2}, \forall s \in (\tau, 1)$, i.e. $\kappa < \frac{\tau}{(1-\tau)}$ □

Corollary 5.12 (Coupling condition for mid memory noise). *Suppose that* $\frac{\phi(1-s)}{1-s} \geq m$ *for some* $m > 0$ *at* $s \to 1$*. If*

$$\frac{\tau}{1-\tau} < (1 + (1 + m\epsilon^2)^2),$$

which also writes

$$\tau \leq \tau^* := \frac{1}{1 + (1 + m\epsilon^2)^2},$$

then $\mathbb{E}\left[e^{\frac{1}{2}\int_0^\tau \alpha_s^2 ds}\right] < \infty$*.*

Remark 5.13. *Note that the above coupling condition is satisfied automatically if* $\tau < \frac{1}{2}$*.*

Corollary 5.14 (Immediate arbitrage). *Suppose that* $\frac{\phi(1-s)}{1-s} \leq M$ *for some* $M > 0$ *at* $s \to 1$*. Then* $\mathbb{E}\int_0^1 \alpha_u^2 du = \infty$*.*

This means in particular that there is a free lunch with vanishing risk at time 1*, and that there cannot be an ELMM on* $[0, 1]$ *with respect to* $\mathcal{G}$*.*

Proof. By the monotone convergence theorem the integral is the same as

$$\int_0^1 \mathbb{E}\alpha_s^2 ds \geq \int_0^1 \frac{1}{(1 + M\epsilon^2)^2(1 - s)} ds = \infty.$$

It follows that this is a situation of immediate arbitrage (see Theorem 2.8), which in particular violates NFLVR, i.e. the existence of an ELMM. □

Corollary 5.14 applies in particular to the the case of Brownian noise W.

Remark 5.15. *Corollary 5.14 means that there will be arbitrage opportunity at time 1 for an insider who has access to the information contained in* $\mathcal{G}$ *unless he stops receiving information strictly before the time of interest (which can be, according to the model, the maturity time of a European call, the date of a public announcement, ...). The reader can refer to Theorem 2.8 or Imkeller (2002) for more on immediate arbitrage.*

For generalizations of this example admitting closed formulas one can consult Corcuera et al. (2004) who considers anticipative signals of the more general form $F(W_T)$ as well as Kchia and Protter (2015) who consider expansions with a backward diffusion.

5.3 Anticipative expansions

We move now our focus to authentic expansions with stochastic processes.

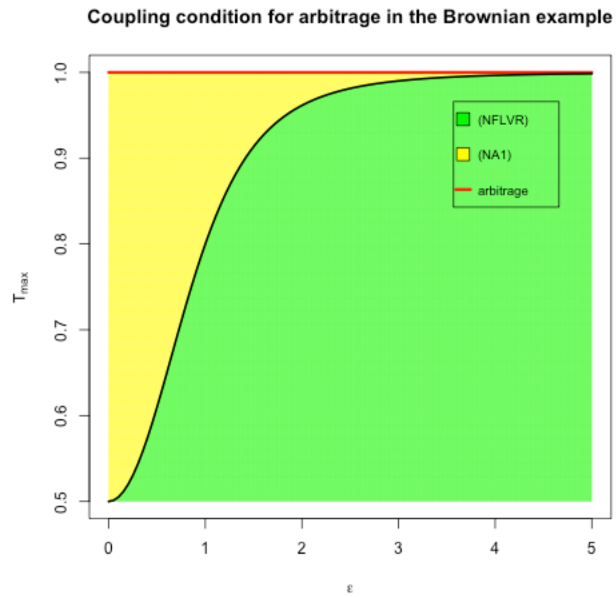


Figure 5.1: Arbitrage opportunities according to final expansion time τ and noise coefficient ϵ in the suggestive example

5.3.1 Fixed anticipation

We start by recalling what goes wrong in the very naive situation where the filtration is augmented with the Brownian motion anticipated deterministically.

Proposition 5.16. *Suppose that X is given by $X_t := W_{t+\delta}$ for some fixed real number $\delta > 0$. Then W is **not** a $\mathcal{G}$-semimartingale.*

Proof. $\widetilde{W} := (W_{t+\delta} - W_\delta)_{t \in I}$ is a $\mathcal{G}$-Brownian motion, and in particular a $\mathcal{G}$-semimartingale. Moreover the process $\widetilde{\widetilde{W}} := \widetilde{W} - W$ has quadratic variation $[\widetilde{\widetilde{W}}, \widetilde{\widetilde{W}}]_t = 2(\delta \wedge t)$.

Hence if W were to be a $\mathcal{G}$-semimartingale, then $(\widetilde{\widetilde{W}}_t)_{t \geq \delta}$ would also be a $\mathcal{G}$-semimartingale with constant quadratic variation, hence a finite variation process, which is impossible since the Brownian motion has paths of infinite variation on every non-empty interval. □

5.3.2 Deterministic anticipation

The previous argument easily extends to deterministic time-dependent non-decreasing processes $(\delta(t))_{t\in[0,T]}$.

Proposition 5.17. *Suppose that X is given by $X_t := W_{t+\delta(t)}$ for some deterministic, non-negative and a.e. non-null process (function) $\delta := (\delta(t))_{t\in[0,T]}$. Then W is **not** a $\mathcal{G}$-semimartingale (after* $\inf\{u \geq 0 : \delta(u) > 0\}$*)*

Proof. Let $s \leq t$. We have $\mathbb{E}[W_t|\mathcal{G}_s] = W_{t\wedge(s+\delta(s))}$. Hence if W were a $\mathcal{G}$-semimartingale, its decomposition $W = M + A$ should satisfy

$$W_{t\wedge(s+\delta(s))} = M_s + \mathbb{E}[A_t|\mathcal{G}_s].$$

In particular $\mathbb{E}[A_t|\mathcal{G}_s]$ would need to be constant and equal to A_s for $t \in [s, s+\delta(s)]$. It would follow that A is a $\mathcal{G}$-martingale with finite variation and would be a trivial process so that W would be a $\mathcal{G}$-martingale which is impossible. □

5.3.3 Fixed anticipation with white noise

A variation of the fixed anticipation consists in adding noise to the anticipative signal, and the case of white noise is enlightening to understand what happens.

Theorem 5.18. *Let N be a stochastic process, independent from $\mathcal{F}$, with $\mathbb{E}[N_t] = 0$ for every $t \geq 0$, and suppose that N satisfies strong law of large numbers, i.e. for every $s \geq 0$ there exists a decreasing sequence $\epsilon_n \xrightarrow[n\to\infty]{} 0$ such that $\frac{1}{n}\sum_{k=1}^{n} N_{s+\epsilon_k} \xrightarrow[n\to\infty]{a.s.} 0$.*
If X is given by $X_t := W_{t+\delta} + N_t$ for some fixed real number $\delta > 0$, then:

(i) $\sigma(W_{u+\delta}, s \leq t) \subset \sigma(X_u, u \leq t)$

(ii) W *is* ***not*** *a $\mathcal{G}$-semimartingale (after time δ)*

Proof. This argument was found with the gracious help of Jean Jacod. Due the continuity of W and Cesaro's theorem (whose original proof can be found in Cauchy (1821)), we have

$$\forall s \geq 0, p \geq 1, \frac{1}{n}\sum_{k=p}^{n+p} W_{s+\delta+\frac{1}{k}} \xrightarrow[n\to\infty]{a.s.} W_{s+\delta}.$$

It follows from the strong law of large numbers that

$$\forall s \geq 0, p \geq 1, \frac{1}{n}\sum_{k=p}^{n+p} X_{s+\epsilon_k} \xrightarrow[n\to\infty]{a.s.} W_{s+\delta}.$$

Hence $W_{s+\delta}$ is measurable by the filtration $\sigma(X_u, u < s+\epsilon)$ for every $\epsilon > 0$, which proves (i).

Now let $\widetilde{\mathcal{G}} := \bigcap_{u>t} \sigma(W_s, s \leq u+\delta)$ (which is an expansion considered in Paragraph 5.3.1. W is $\widetilde{\mathcal{G}}$-measurable and is not a $\widetilde{\mathcal{G}}$ semimartingale by Proposition 5.18. As $\widetilde{\mathcal{G}} \subset \mathcal{G}$ it follows from (the contrapositive of) Stricker's theorem (see Protter (2004)) that W cannot be a $\mathcal{G}$-semimartingale. □

An example of such process N is a general white noise, in the sense of a stochastic process with expectation 0 such that $(N_t)_{t\in[0,T]}$ is an i.i.d. family (the standard white noise corresponds to normal marginals). More generally we can choose any process with sufficient mixing property, such as any fractional Brownian motion.

5.3.4 Randomized constant anticipation

Proposition 5.19. *Suppose that X is given by $X_t := W_{t+\delta}$ for some positive random variable δ. Then*

1. *δ is $\mathcal{G}_\delta$ measurable*
2. *W is **not** a $\mathcal{G}$-semimartingale (after time δ)*

Proof. For every time $s > \delta$ and sequence $\epsilon_n \to_{n\to\infty} 0$ $(0 < \epsilon_n \geq s - \delta$ we have

$$\lim_{n\to\infty} inf\{x \geq 0 : \forall k \in 1, \dots, n, W_{s-x-\epsilon_k+\delta} = W_{s-\epsilon_k}\} = \delta.$$

Therefore δ is $\mathcal{G}_s$-measurable, and since $\mathcal{G}$ is right-continuous it is also $\mathcal{G}_\delta$-measurable. This proves (i), and (ii) follows from Proposition 5.16 which proves that the $\mathcal{G}$-semimartingale property is violated after time δ. □

5.4 Random anticipation

We now consider a non-negative stochastic process δ with continuous paths and independent from W. We let $X_t := W_{t+\delta_t}$ and defines the filtrations $\check{\mathcal{G}}$ and $\mathcal{G}$ as

$$\check{\mathcal{G}} := \mathcal{F}_t \vee \sigma(X_s, s \leq t), \quad \mathcal{G}_t := \bigcap_{u>t} (\mathcal{F}_u \vee \sigma(X_s, s \leq u)), t \in I.$$

We will suppose that $(t+\delta_t)_{t\in I}$ is non-decreasing, which does not affect the generality.

5.4.1 Computation of the information drift

Let $\tau(s,t)$ be the smallest solution of $u+\delta_u = t$ for $u \geq 0$.

Lemma 5.20.

$$\mathbb{E}[W_t|\mathcal{G}_s] = \mathbb{E}[W_{\tau(s,t)+\delta_{\tau(s,t)}}|\mathcal{G}_s].$$

Proof. If follows from the definition of $\tau(s,t)$ that $W_{\tau(s,t)+\delta_{\tau(s,t)}}$ coincides with W_t when $t \leq s+\delta_s$. On the event $t > s+\delta_s$:

$$\begin{aligned}
\mathbb{E}[W_t|\mathcal{G}_s]1_{t>s+\delta_s} &= W_{s+\delta_s} + \mathbb{E}[W_t - W_{s+\delta_s}|\mathcal{G}_s]1_{t>s+\delta_s} \\
&= W_{s+\delta_s} + \mathbb{E}[\mathbb{E}(W_t - W_{s+\delta_s}|\mathcal{G}_s,\delta_s)|\mathcal{G}_s]1_{t>s+\delta_s} \\
&= W_{s+\delta_s} \\
&= W_{\tau(s,t)+\delta_{\tau(s,t)}}1_{t>s+\delta_s}
\end{aligned}$$

where we used the independence of the increments of the Brownian motion to obtain that on every event $\delta_s = u, u \leq t-s$ we have

$$\mathbb{E}(W_t - W_{s+\delta_s}|\mathcal{G}_s,\delta_s) = \mathbb{E}(W_t - W_{s+u}|\mathcal{G}_s) = \mathbb{E}(W_t - W_{s+u}) = 0.$$

□

Corollary 5.21.

$$\mathbb{E}[W_t|\mathcal{G}_s] = \int_0^s W_{u+\delta_u}\mathbb{P}(\tau(s,t)\in du|\mathcal{G}_s)$$

Proof. This is a reformulation of Lemma 5.20 □

Remark 5.22. $\mathbb{P}(\tau(s,t) \in du|\mathcal{G}_s)$ *in 5.21 can be interpreted as the likelihood of* $\tau(s,t)$ *given the path of* $(W_u, W_{u+\delta_u})_{u\leq s}$.

Theorem 5.23. *Let* δ *be a non-negative stochastic process* δ *independent from* W *with continuous paths, and suppose that* $X_t := W_{t+\delta_t}$. *For every* $s \geq t$ *define the random time* $\tau(s,t) := \inf\{u \geq s : u + \delta_u = t\}$, *suppose that it admits a* $\mathcal{G}_s$*-conditional density (with respect to Lebesgue measure) and let* $\mathbb{P}(\tau(s,t) \in du|\mathcal{G}_s) =: f(u;s,t)du$. *Then the information drift* α *is given by*

$$\alpha_s = \int_0^s W_{u+\delta_u} \partial_t f(u;s,t)\Big|_{t=s} du.$$

Proof. This theorem is a consequence of Corollary 5.3 and Corollary 5.21. □

Conclusion

The classical theory of expansion of filtration, mainly developed around initial and progressive expansion, has been lacking of general and quantitative results. With this thesis we extend the innovative results of Kchia and Protter (2015) and Ankirchner, Dereich, and Imkeller (2006) to make a step towards both directions: expansions with a stochastic process are in our opinion general enough - although it is of course easy to construct expansions that do not fit in that framework - and can be used to model contemporary problems such as high-frequency trading.

One of the leading purposes of the insider trading theory is probably to develop a method to detect (illegal) insider trading. However to our knowledge the only attempt is made by Grorud and Pontier (1998) and Grorud and Pontier (1997), who propose a statistical test for insider trading based on the change of optimal strategy induced by an initial expansion in a final-logarithmic-utility-maximization problem. We are enthusiastic that the methods in this thesis could be applied to develop a more quantitative test for asymmetric information, for instance in relation to the trading frequency.

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Appendix A

A formula for the information drift via local times

Let W be a Brownian motion, $\mathcal{F}$ its natural filtration, or equivalently $\mathcal{F}_t := \bigcap_{u>t} \sigma(W_s, s \leq u)$. Let δ be a non-null random time, independent from W, and fix a time $\tau > 0$. We consider the process $X_t := W_{\tau+\delta} 1_{t \geq \tau}$ and the filtrations defined by $\check{\mathcal{G}} := \mathcal{F}_t \vee \sigma(X_s, s \leq t)$ and $\mathcal{G}_t := \bigcap_{u>t} (\mathcal{F}_u \vee \sigma\big(X_s, s \leq u)\big)$.

We also denote $\pi(x, y, t)$ the transition density of the Brownian motion, L_t^x the local time of W at point x on $(0, t)$, $F_t(x) := P(\delta_t \leq x)$, and we suppose that F is differentiable and that its derivative f is differentiable.

The goal here is to compute the information drift α of $\mathcal{G}$ as $\mathcal{F}$-conditional density of X according to the method in Chapter 4.

Let $\tau \leq t$.

$$
\begin{aligned}
&P(X_\tau \leq x|\mathcal{F}_t)\\
=&P(W_{\tau+\delta} \leq x|\mathcal{F}_t)\\
=&\int_0^\infty P(W_{\tau+u} \leq x|\mathcal{F}_t)dF_\tau(u)\\
=&\int_0^{t-\tau} 1_{\{W_{\tau+u}\leq x\}} f(u)du + \int_{t-\tau}^\infty \int_{-\infty}^x \pi(W_t, y, \tau+u-t)dyf(u)du
\end{aligned}
$$

The conditioning 'disappears', for the first term because is $\mathcal{F}_t$-measurable, and for the second term because of the Markov property of the Brownian motion (BM); π is defined the transition density of the BM.

The following lemma is a consequence of integration by parts for Lebesgue integrals.

Lemma A.1. *Let $0 \leq a \leq b$, $0 \leq s$, A a Borel set and g a differentiable function. Then:*

$$\int_a^b 1_{\{W_u - W_s \in A\}} du = \int_{W_s+A} (L_b^x - L_a^x)dx$$

$$\int_a^b 1_{\{W_u - W_s \in A\}} g(u)du = g(b)\int_{W_s+A} (L_b^x - L_a^x)dx - \int_{W_s+A}\int_a^b (L_u^x - L_a^x)g'(u)dudx.$$

Proof. The first claim is a consequence of:

$$\begin{aligned}\int_a^b 1_{\{W_u - W_s \in A\}} du &= \int_a^b 1_{\{W_u \in (W_s + A)\}} du \\ &= \int_0^b 1_{\{W_u \in (W_s + A)\}} du - \int_0^a 1_{\{W_u \in (W_s + A)\}} du \\ &= \int_{W_s + A} L_b^x dx - \int_{W_s + A} L_a^x dx \\ &= \int_{W_s + A} (L_b^x - L_a^x) dx\end{aligned}$$

For the second claim, let us denote $\Phi(t) := \int_a^t 1_{\{W_u - W_s \in A\}} du$. We have, by the integration by parts formula for Lebesgue integrals:

$$\begin{aligned}\int_a^b 1_{\{W_u - W_s \in A\}} g(u) du &= \int_a^b \Phi'(u) g(u) du \\ &= [\Phi(u) g(u)]_a^b - \int_a^b \Phi(u) g'(u) du \\ &= g(b)\Phi(b) - \int_a^b \Phi(u) g'(u) du\end{aligned}$$

and we can conclude because $\Phi(b)$ is exactly the expression computed in the first claim; the change of integration orders is justified as long as g' is bounded from above or below.

□

Hence by the lemma we have

$$\int_0^{t-\tau} 1_{\{W_{\tau+u} \leq x\}} f(u) du = f(t-\tau) \int_{-\infty}^x (L_t^y - L_\tau^y) dy - \int_{-\infty}^x \int_\tau^t (L_y^y - L_\tau^y) f'(u-\tau) du dy$$

and

$$P(X_\tau \leq x|\mathcal{F}_t) = f(t-\tau)\int_{-\infty}^{x}(L_t^y - L_\tau^y)dy - \int_{-\infty}^{x}\int_{\tau}^{t}(L_u^y - L_\tau^y)f'(u-\tau)dudy$$
$$+\int_{-\infty}^{x}\int_{t-\tau}^{\infty}\pi(W_t, y, \tau+u-t)f(u)dudy$$

Therefore there exists a conditional density given by

$$q_t^0(\omega, y) := f(t-\tau)(L_t^y - L_\tau^y) - \int_{\tau}^{t}(L_u^y - L_\tau^y)f'(u-\tau)du + \int_{t-\tau}^{\infty}\pi(W_t, y, \tau+u-t)f(u)du$$

and

$$\begin{aligned}\alpha_t :=& \frac{1}{q_t(\omega,y)}\frac{d[q(\omega,y),W]_t}{dt}\Bigg|_{y=X_\tau}\\ =& \frac{\int_{t-\tau}^{\infty}\frac{\partial\pi}{\partial x}(W_t,y,\tau+u-t)f(u)du}{\int_{t-\tau}^{\infty}\pi(W_t,y,\tau+u-t)f(u)du}\Bigg|_{y=X_\tau}\\ =& \int_{t-\tau}^{\infty}(\frac{1}{\pi}\frac{\partial\pi}{\partial x})(W_t,y,\tau+u-t)\ell_\tau(y,u|t,W_t)du\Bigg|_{y_0=X_\tau}\end{aligned}$$

where $\ell_\tau(y,u|t,W_t)$ is the likelihood of $(W_{\tau+\delta},\delta)$ given $\mathcal{F}$, i.e. the density of $(W_{\tau+\delta},\delta)$ given W_t.

Proposition A.2. *With the notations introduced in this paragraph, we have*

$$\alpha_t = \int_{t-\tau}^{\infty}(\frac{1}{\pi}\frac{\partial\pi}{\partial x})(W_t,y,\tau+u-t)\ell_\tau(y,u|t,W_t)du\Bigg|_{y_0=X_\tau}$$

Appendix B

An application to optimal stochastic control on degenerate SDEs

In this appendix we give simple result linking the integrability of the information drift to the difference between two solutions of a simple optimization control problem on a degenerate SDE which are adapted to respectively to the two usual filtrations of interest: the Brownian filtration and the filtration generated by the degenerate diffusion. We are confident that similar results can be formulated in higher dimensions.

B.1 Setup

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space, W a Brownian motion on $[0, T]$, $T > 0$ and $\mathcal{G}$ its natural filtration, namely

$$\mathcal{G}_t := \sigma(W_s, s \leq t),$$

potentially completed with the null sets so that it satisfies the usual hypotheses. We also consider the unique solution Z of an SDE of the form

$$dZ_t = b(t, Z_t)dt + \sigma(t, Z_t)dW_t$$

where the coefficients b and σ are locally Lipschitz in both variables.

We consider the case where $\sigma(t, Z_t)$ is degenerate, in which case the filtration $\mathcal{F}$ generated by Z,

$$\mathcal{G}_t := \sigma(Z_s, s \leq t),$$

may be strictly smaller than the Brownian filtration $\mathcal{G}$.

We consider here a one dimensional Brownian motion but the application of the theory of finite utility filtrations may potentially be extended to Brownian motion of higher dimensions.

B.2 From $\mathcal{G}$ to $\mathcal{F}$

It follows from the definition of Z that it is a $\mathcal{G}$-semimartingale with decomposition given by the SDE. By Stricker's theorem since it is adapted to $\mathcal{F}$ it is also a $\mathcal{F}$-semimartingale, with some decomposition $Z = M + A$.

The case of W. For W, it is by definition a $\mathcal{G}$-martingale hence its optional projection on $\mathcal{F}$ is a $\mathcal{F}$-martingale (by Föllmer and Protter (2011)) that we denote ${}^oW$.

Hence W has an orthogonal decomposition as $W =^o W + (W -^o W)$ where the first term is a $\mathcal{F}$-martingale.

It further follows that

$$dZ_t = b(t, Z_t)dt + \sigma(t, Z_t)d^o W_t + \sigma(t, Z_t)d(W -^o W)_t$$

B.3 From $\mathcal{F}$ to $\mathcal{G}$

We consider the optimization problem

$$\sup_H \mathbb{E}[U(x + \int_0^T H_s dZ_s)],$$

where U is a utility function.

Utility function. We consider a utility function $U : \mathbb{R} \to [-\infty, \infty)$ which is non-decreasing and concave. We suppose that $u^* := \inf\{y \in \mathbb{G} : U(y) > -\infty\} < \infty$, that U is strictly increasing and strictly concave on (u^*, ∞), and that $\lim_{+\infty} U = +\infty$ (see also Ankirchner (2005) below Definition 7.1.2).

Admissible strategies and optimization problem. Classically the sup is taken over $\mathcal{G}$-predictable strategies H such that the stochastic integral $\int_0^\cdot H_s dZ_s$ has paths bounded from below by a deterministic constant with probability one, which we denote $\mathfrak{H}(\mathcal{G})$. This leads to the problem

$$u(\mathcal{G}) := \sup_{H \in \mathfrak{H}(\mathcal{G})} \mathbb{E}[U(x + \int_0^T H_s dZ_s)]. \tag{B.1}$$

The problem where the admissible strategies are restricted to the space $\mathfrak{H}_s(\mathcal{G})$ of simple processes in $\mathfrak{H}(\mathcal{G})$ (corresponding to *buy-and-hold* strategies) is also of interest.

$$u_s(\mathcal{G}) := \sup_{H \in \mathfrak{H}_s(\mathcal{G})} \mathbb{E}[U(x + \int_0^T H_s dZ_s)] \tag{B.2}$$

See Definition 7.1.1 in Ankirchner (2005) for a definition of admissible strategies.

Change of filtration Given that $\mathcal{F}$ is *a priori* strictly smaller than $\mathcal{G}$ we can consider the same optimization problems restricted to $\mathcal{F}$-predictable integrands.

$$u(\mathcal{F}) := \sup_{H \in \mathfrak{H}(\mathcal{F})} \mathbb{E}[U(x + \int_0^T H_s dZ_s)] \tag{B.3}$$

$$u_s(\mathcal{F}) := \sup_{H \in \mathfrak{H}_s(\mathcal{F})} \mathbb{E}[U(x + \int_0^T H_s dZ_s)] \tag{B.4}$$

(See Definition 7.1.3 and Definition 8.1.1 in Ankirchner (2005))

Finite utility filtrations and semimartingales. We have $u(\mathcal{F}) \leq u(\mathcal{G})$ and $u_s(\mathcal{F}) \leq u_s(\mathcal{G})$ but a priori no equality.

Supposing that $u(\mathcal{G}) < \infty$ we can apply the theory of finite utility filtrations in Ankirchner (2005) (see also Ankirchner, Dereich, and Imkeller (2006)). By Theorem 9.1.4 in Ankirchner (2005) there exist a $\mathcal{G}$-predictable process α such that Z is a

$\mathcal{G}$-semimartingale with decomposition

$$Z = (M - \int_0^{\cdot} \alpha_s d[M,M]_s) + (A + \int_0^{\cdot} \alpha_s d[M,M]_s),$$

and

$$\int_0^T \alpha_s^2 d[M,M]_s < \infty.$$

Since it is continuous Z is a $\mathcal{G}$-special semimartingale, and the unicity of its semimartingale decomposition leads to

$$M = \int_0^{\cdot} \sigma(s, Z_s) dW_s + \int_0^{\cdot} \alpha_s d[Z,Z]_s$$

and

$$A = \int_0^{\cdot} b(s, Z_s) ds - \int_0^{\cdot} \alpha_s d[Z,Z]_s.$$

In particular we can express the semimartingale decomposition of Z in $\mathcal{F}$ with b, σ and α.

Theorem 9.1.4 in Ankirchner (2005) also applies to $\mathcal{F}$ and proves as a side result that A is also of the form $A_t = \int_0^{\cdot} \beta_s d[Z,Z]_s$ for some $\mathcal{F}$-predictable process β that also satisfies

$$\int_0^T \beta_s d[M,M]_s < \infty.$$

Note also that $[M,M] = [Z,Z] = \int_0^t \sigma(s, Z_s)^2 ds$. In particular, $d[Z,Z]_t = 0$ (or is singular) at each point at which σ is singular.

B.4 To go further...

An generally interesting problem would be to express or bound α or its integrals. It would be meaningful as well to relate it to $u(\mathcal{G}) - u(\mathcal{F})$. A nice formula is found in the case of **logarithmic utility** in Karatzas and Pikovsky (1996) (Wiener integrals) and Ankirchner, Dereich, and Imkeller (2006) (general semimartingales):

$$u(\mathcal{G}) - u(\mathcal{F}) = \int_0^T \alpha_s^2 d[Z, Z]_s.$$

In view of the general results obtained in Ankirchner, Dereich, and Imkeller (2006) and Ankirchner (2005) on finite utility filtrations, we are confident that there is more to be said.