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COMPLEXITY OF INDEFINITE ELLIPTIC PROBLEMS

by

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Abstract

This paper deals with the optimal solution of a linear regularly-elliptic 2m-th order boundary-value problem $Lu = f$, with $f \in H^r(\Omega)$, $r \geq -m$. Suppose that the problem is indefinite, i.e., the variational form of the problem involves a weakly-coercive bilinear form. Of particular interest is the strength of finite element information (FEI) of degree k and the quality of the finite element method (FEM) using that information. The error is measured in the Sobolev ℓ norm $(0 \leq \ell \leq m)$; we assume that $k \geq 2m - 1 - \ell$. Both the normed and seminormed cases are considered, in which an a priori bound is given on the Sobolev r-norm and seminorm of f, respectively. In the normed case, the FEM is quasi-optimal iff $k \geq 2m - 1 + r$, but FEI is always quasi-optimal information (i.e., the spline algorithm using FEI is a quasi-optimal algorithm). In the seminormed case, we give a very restrictive necessary and sufficient condition for the FEM to have finite error. When the FEM has finite error for the seminormed case, it is quasi-optimal iff $k \geq 2m - 1 + r$; however, FEI is always quasi-optimal information for the seminormed case.

1. Introduction

This paper is a theoretical study of the optimal solution of the variational form of 2m-th order linear regularly-elliptic boundary-value problems $Lu = f$ with $f \in H^r(\Omega)$, $\Omega \subset \mathbb{R}^N$, $r \geq -m$ having homogeneous boundary conditions (see Section 2). Such problems are to be solved using information of cardinality at most n. (In this Introduction, we use words such as information, cardinality, quasi-optimal, etc., without definition; they are defined rigorously in Section 3.)

In [15], this problem was considered under the following conditions:

(i) The problem is definite; i.e., its variational form involves a coercive bilinear form (which is thus an inner product over the space of functions satisfying the essential boundary conditions).

(ii) Error is measured in the "energy norm" generated by this inner product (which is equivalent to the Sobolev m-norm).

(iii) An a priori bound is given on the Sobolev r-norm of f.

Of particular interest was the optimality of the finite element method (FEM) of degree k, as well as the optimality of finite element information (FEI) (see Section 4). The main result was that the FEM is quasi-optimal among all algorithms iff $k \geq 2m - 1 + r$. However, FEI is always quasi-optimal information; that is, the spline algorithm using FEI is always quasi-optimal.

How crucial are the conditions in [15]?

Condition (i) disallows problems such as the Helmholtz problem: given $f \in H^r(\Omega)$, find $u : \bar{\Omega} \to \mathbb{R}$ such that

$$\text{(1.1)} \qquad \begin{aligned} \Delta u + \lambda u &= f \quad &&\text{in } \Omega \\ u &= 0 \quad &&\text{on } \partial\Omega, \end{aligned}$$

where λ is not an eigenvalue of $-\Delta$. If λ is bigger than the smallest eigenvalue of $-\Delta$, this problem does not yield a coercive bilinear form.

Condition (ii) is that the energy norm was used. Although this norm is equivalent to the Sobolev m-norm, the constant which measures this equivalence may be so large that a very good energy-norm solution may not be sufficiently accurate in the m-norm. (This would appear to be the situation in boundary layer problems.) Moreover, it is sometimes of more interest to use other norms, such as the L_2-norm for measuring displacement error.

Assumption (iii) is a standard assumption on partial differential equations and the FEM [1,3,8]. However, for many other problems, one often only assumes an a priori bound on the Sobolev r-seminorm (see the examples and the annotated bibliography of [11]). If we wish to place the elliptic boundary-value problem into a complexity hierarchy consisting of such problems, it too must be solved under the assumption that an a priori bound on the Sobolev r-seminorm is given.

In this paper, the results of [15] are extended by weakening conditions (i), (ii), and (iii) above. We assume:

(i)' The problem is <u>indefinite</u>: its variational form involves a weakly coercive bilinear form.

(ii)' Error is measured in the Sobolev ℓ norm, where $0 \leq \ell \leq m$.

(iii)' Both the <u>normed</u> and <u>seminormed</u> <u>cases</u> are considered, i.e., an a priori bound is given on the Sobolev r-norm and the r-seminorm of $f \in H^r(\Omega)$ (respectively), the seminormed case making sense iff r is a non-negative integer.

Again, we will be interested in the optimality of the FEM of degree k, as well as that of FEI. We assume in this paper that

$$k \geq 2m - 1 - \ell; \tag{1.2}$$

see Section 4 for further information.

It turns out that replacing (i) by (i)' causes almost no difficulty, while replacing (ii) by (ii)' may be done via a variant of the Aubin-Nitsche duality argument [5, pp. 136-139]. Hence in the normed case, the main results of [15] still hold when (i) and (ii) are replaced by (i)' and (ii)'; see Section 5 for details.

The situation is different when replacing (iii) by (iii)', i.e., going from the normed to the seminormed case. Consider the problems

$$-u'' + u = f \text{ in } (0,1) \text{ with } u(0) = u(1) = 0 \tag{1.3}$$

and

(1.4) $-u'' + u = f$ in $(0,1)$ with $u'(0) = u'(1) = 0$,

where

(1.5) $f \in H^1(0,1)$ and $\int_0^1 [f'(x)]^2 dx \le 1$.

No matter what value is given for k, the FEM has infinite error for (1.3), but there exists an algorithm using FEI which has finite error. In fact, FEI is always quasi-optimal information for (1.3); that is, the spline algorithm using FEI is a quasi-optimal algorithm. On the other hand, the FEM always has finite error for (1.4) and is (in fact) quasi-optimal when $k \ge 2$; however, FEI is always quasi-optimal information for (1.4).

We discuss the seminormed case in detail in Section 6. We show that the FEM has finite error iff $P_{r-1}(\Omega) \subseteq L\mathcal{S}_n$, where $P_{r-1}(\Omega)$ and $\mathcal{S}_n$ respectively denote the space of polynomials of degree at most $r - 1$ over Ω and the finite element space of dimension n. (Note that this condition is very restrictive in practice; see e.g. Remark 6.1.) When this is the case, we find that the FEM is quasi-optimal among all algorithms using FEI iff $k \ge 2m - 1 + r$. However, FEI is always quasi-optimal information for the seminormed case.

In Section 7, we discuss the complexity of obtaining ε-approximations. We show that (in both the normed and seminormed cases) the penalty for using the FEM if $k < 2m - 1 + r$ is unbounded as $\varepsilon \to 0$. Since this is an asymptotic measure, we also wish to know whether there is a penalty for using the FEM for fixed ε of moderate size when $k < 2m - 1 + r$. We consider a simple model problem, and show that the complexity of the "spline algorithm" using FEI is less than that of the FEM whenever $\varepsilon < \varepsilon_0$, where $\varepsilon_0 \doteq 0.227$.

Finally, in Section 8, we summarize our work, and point out some possible extensions and open questions.

2. The Variational Boundary-Value Problem

In what follows, we use the standard notation for Sobolev spaces, inner products, and norms, multi-indices, etc., found in Ciarlet [5]. Fractional- and negative-order Sobolev spaces are defined by Hilbert-space interpolation and duality, respectively; see Chapter 2 of [3] and Chapter 4 of [8] for details.

Let $\Omega \subset \mathbb{R}^N$ be a bounded, simply connected, C^∞ region. Define the uniformly strongly elliptic operator

$$Lv := \sum_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta} D^\beta v)$$

with real coefficients $a_{\alpha\beta} \in C^\infty(\bar{\Omega})$ such that $a_{\alpha\beta} = a_{\beta\alpha}$. In order to have appropriate boundary conditions, define a normal family of operators

$$B_j v := \sum_{|\alpha|\le q_j} b_{j\alpha} D^\alpha v \qquad (0 \le j \le m-1)$$

(with real coefficients $b_{j\alpha} \in C^\infty(\partial\Omega)$), where

$$0 \le q_0 \le \ldots \le q_{m-1} \le 2m-1,$$

which covers L on $\partial\Omega$. To make the boundary-value problem to be solved be self-adjoint, we let

$$m^* := \min\{j : q_j \ge m\}$$

and require that

$$\{q_j\}_{j=0}^{m^*-1} \cup \{2m-1-q_j\}_{j=m^*}^{m-1} = \{0,\ldots,m-1\}.$$

(See Chapter 3 of [3], Chapter 5 of [8] for further definitions and illustrative examples.) We are interested in solving the elliptic boundary-value problem

given $f \in H^r(\Omega)$, where $r \geq -m$, find $u : \bar{\Omega} \to \mathbb{R}$ such that

(2.1) $$Lu = f \quad \text{in } \Omega$$
$$B_j u = 0 \quad (0 \leq j \leq m-1) \text{ on } \partial\Omega.$$

Let

$$H_E^m(\Omega) := \{v \in H^m(\Omega) : B_j v = 0 \quad (0 \leq j \leq m^* - 1)\}$$

denote the space of $H^m(\Omega)$-functions satisfying the essential boundary conditions. We define a symmetric, continuous bilinear from B on $H_E^m(\Omega)$ by

$$B(v,w) := \sum_{|\alpha|,|\beta| \leq m} \int_\Omega a_{\alpha\beta} D^\alpha v D^\beta w.$$

In [15], we assumed that B was $H_E^m(\Omega)$-coercive, i.e., that there exists $\gamma > 0$ such that

$$B(v,v) \geq \gamma \|v\|_m^2 \quad \text{for } v \in H_E^m(\Omega).$$

When $m = 0$, the conditions on L yield that B is $L_2(\Omega)$-coercive. However, for $m \geq 1$, there exist elliptic boundary-value problems which do not yield a bilinear form that is $H_E^m(\Omega)$-coercive (such as the Dirichlet problem for the Helmholtz equation).

In this paper, we assume instead that B is weakly $H_E^m(\Omega)$-coercive [8, pg. 310]. Since B is symmetric, this means that there exists $\gamma > 0$ such that for any nonzero $v \in H_E^m(\Omega)$, there exists nonzero

(2.2) $w \in H_E^m(\Omega)$ such that

$$B(v,w) \geq \gamma \|v\|_m \|w\|_m .$$

The following lemma gives a condition which is sufficient to establish weak coercivity. (The result appears to be well-known; its proof for arbitrary m is a straightforward modification of the proof for the case $m = 1$ which is found in [3, Chapter 5]:)

Lemma 2.1. Let $m \geq 1$. Suppose that

(i) the only solution of (2.1) with $f = 0$ is $u = 0$

and

(ii) B is $[H_E^m(\Omega), L_2(\Omega)]$-coercive [8, pg. 301]; that is, there exist $\gamma_0 > 0$ and $\gamma_1 \geq 0$ such that

$$|B(v,v) \geq \gamma_0 \|v\|_m^2 - \gamma_1 \|v\|_0^2 \qquad \forall\ v \in H_E^m(\Omega).$$

Then B is weakly $H_E^m(\Omega)$-coercive. $\square$

Remark 2.1. Suppose that B_j is the jth normal derivative operator $(0 \leq j \leq m - 1)$, so that (2.1) is a Dirichlet problem and $H_E^m(\Omega) = H_0^m(\Omega)$. Then (2.3) is Gårding's inequality (see e.g. [1]) which follows immediately from the conditions on L. Hence, B is weakly $H_0^m(\Omega)$-coercive provided that (i) holds in Theorem 2.1. For example, the Helmholtz problem (1.1) is weakly $H_0^1(\Omega)$-coercive if λ is not an eigenvalue of $-\Delta$. $\square$

We now define the variational boundary-value problem as follows. Let $r \geq -m$. We wish to solve the problem

$$(2.4)\qquad \begin{gathered}\text{given } f \in H^{r}(\Omega),\ \text{find } u = Sf \in H^{m}_{E}(\Omega) \text{ such that} \\ B(u,v) = (f,v)_0 = \int_\Omega fv \quad \forall\, v \in H^{m}_{E}(\Omega).\end{gathered}$$

From the Generalized Lax-Milgram Theorem [3, Theorem 5.2.1], S is a Hilbert space isomorphism of $H^{-m}(\Omega)$ onto $H^{m}_{E}(\Omega)$, and so $S : H^{r}(\Omega) \to H^{m}_{E}(\Omega)$ is a bounded linear injection. Since B is only assumed to be weakly coercive (i.e., we do not know that $B(v,v) \geq \gamma\|v\|_m^2$ holds), the problem (2.4) is said to be <u>indefinite</u>.

It is useful to recall the "shift theorem" ([3, Chapter 3], [8, Chapter 5]), which states that since $f \in H^{r}(\Omega)$, we have $Sf \in H^{m}_{E}(\Omega) \cap H^{2m+r}(\Omega)$, and there exists a constant $\sigma > 0$, independent of f, such that

$$(2.5)\qquad \sigma^{-1}\|Sf\|_{2m+r} \leq \|f\|_r \leq \sigma\|Sf\|_{2m+r}.$$

If $r > N/2$, then the shift theorem, Sobolev's embedding theorem, and an m-fold integration by parts yield that $u = Sf$ is the solution to (2.1).

3. Information and Algorithms

In this section, we briefly define some of the concepts mentioned in the Introduction. A more leisurely description may be found in [15]; most of the terminology and results are taken from [11].

Recall that we are trying to approximate $S : H^r(\Omega) \to H_E^m(\Omega)$, where $r \geq -m$. We are only allowed to sample a finite amount of "information" about problem elements $f \in H^r(\Omega)$. Here, information $\mathfrak{N}$ of cardinality n is a linear surjection $\mathfrak{N} : H^r(\Omega) \to \mathbb{R}^n$. By an algorithm φ using $\mathfrak{N}$, we then mean a (possibly-nonlinear) mapping $\varphi : D_\varphi \subset \mathfrak{N}(H^r(\Omega)) \to H_E^m(\Omega)$; the class of algorithms using $\mathfrak{N}$ is denoted $\Phi(\mathfrak{N})$.

Given information $\mathfrak{N}$ and an algorithm $\varphi \in \Phi(\mathfrak{N})$, the quality of the approximations produced by φ is then measured by its (worst-case) H^{ℓ}-error with respect to a given set $\mathfrak{F}$ of problem elements

$$e_\ell(\varphi, \mathfrak{F}) := \sup_{f \in \mathfrak{F}} \|Sf - \varphi(\mathfrak{N}f)\|_\ell, \tag{3.1}$$

where $0 \leq \ell \leq m$. In this paper, we consider the normed case, where $\mathfrak{F}$ is the unit ball $BH^r(\Omega)$ of $H^r(\Omega)$ defined by

$$BH^r(\Omega) := \{f \in H^r(\Omega) : \|f\|_r \leq 1\},$$

and the seminormed case, where $\mathfrak{F}$ is the unit semiball $\mathcal{B}H^r(\Omega)$ of $H^r(\Omega)$ given by

$$\mathcal{B}H^r(\Omega) := \{f \in H^r(\Omega) : |f|_r \leq 1\},$$

(the latter for r a non-negative integer). In either case, there exists a Hilbert space H and a bounded linear surjection $T : H^r(\Omega) \to H$ such that

$$\mathfrak{F} = \{f \in H^r(\Omega) : \|Tf\| \leq 1\}. \tag{3.2}$$

(Indeed, choose $H = H^r(\Omega)$ and $T = I$, the identity operator in the normed case. The seminormed case is covered in [14, Section 5].) Note that $\ker T = 0$ in the normed case and $\ker T = P_{r-1}(\Omega)$ in the seminormed case.

We then wish to determine the <u>optimal</u> $H^{\ell}(\Omega)$-<u>error</u>, $e_{\ell}(n,\mathfrak{F})$, <u>of algorithms</u> using the given information n

$$e_{\ell}(n,\mathfrak{F}) := \inf_{\varphi \in \Phi(n)} e_{\ell}(\varphi,\mathfrak{F}).$$

From Chapter 2 of [11],

$$e_{\ell}(n,\mathfrak{F}) = \sup_{h \in \mathfrak{F} \cap \ker n} \|Sh\|_{\ell}, \tag{3.3}$$

which makes it easier to determine $e_{\ell}(n,\mathfrak{F})$. An algorithm $\varphi^{oe} \in \Phi(n)$ such that

$$e_{\ell}(\varphi^{oe},\mathfrak{F}) = e_{\ell}(n,\mathfrak{F})$$

is then said to be an <u>optimal error algorithm</u> using n.

<u>Remark 3.1</u>. We briefly discuss the nature of optimal error algorithms. If $\ker n \cap \ker T \neq 0$, then $e_{\ell}(n,\mathfrak{F}) = +\infty$ [11, Theorem 2.3.1] (recall that $\mathfrak{F}$ and T are related by (3.2)). So we assume that $\ker n \cap \ker T = 0$. Then $T \ker n$ is a closed

subspace of H [2, Proposition 6.1]. For each integer i, $1 \le i \le n$, let $z_i \in H^r(\Omega)$ satisfy

$$N(z_i) = e_i = \text{ith unit vector in } \mathbb{R}^n$$

$$Tz_i \text{ is orthogonal to } T \ker N.$$

Then the <u>spline algorithm</u>

$$\varphi^s(Nf) := v \cdot Nf \qquad v = [Sz_1 \ \ldots \ Sz_n]^T$$

is a (linear) optimal error algorithm using N [11, Chapter 4]. □

<u>Remark 3.2</u>. Although the procedure above tells us how to construct optimal error algorithms, it may be very difficult to follow in practical situations. Usually, we are willing to settle for algorithms that are only optimal to within a constant factor (independent of the cardinality of the information used) rather than optimal error algorithms. More precisely, let $\{N_n\}_{n=1}^{\infty}$ be a sequence of information operators with card $N_n \le n$, and let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of algorithms with $\varphi_n \in \Phi(N_n)$. We say that $\{\varphi_n\}_{n=1}^{\infty}$ is a <u>quasi-optimal</u> sequence of algorithms <u>using</u> $\{N_n\}_{n=1}^{\infty}$ if

$$e_i(\varphi_n, \mathfrak{F}) = \Theta(e_i(N_n, \mathfrak{F})) \quad \text{as} \quad n \to \infty.^*$$

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*We use the Ω- and Θ-notations, as well as the standard O-notation. For functions f and g, we write

$$f = \Omega(g) \quad \text{iff} \quad g = O(f)$$

and

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \text{ and } g = O(f).$$

(The terminology is taken from the finite-element literature, see e.g. [13].) □

Just as we may ask for optimal algorithms using given information, one may ask which information of a given cardinality is best. Let

$$e_\ell(n,\mathfrak{F}) := \inf\{e_\ell(\Pi,\mathfrak{F}) \;:\; \text{card } \Pi \leq n\}$$

denote the nth minimal error. We say that Π_n^* of cardinality at most n is an nth optimal information if

$$e_\ell(\Pi_n^*,\mathfrak{F}) = e_\ell(n,\mathfrak{F}).$$

From Chapters 2 and 3 of [11], we have

$$e_\ell(n,\mathfrak{F}) = d_n(S(\mathfrak{F}),H_E^\ell(\Omega)), \tag{3.4}$$

where d_n is the Kolmogorov n-width and $H_E^\ell(\Omega)$ is the completion of $H_E^m(\Omega)$ in the $H^\ell(\Omega)$ norm.

Remark 3.3. From Chapter 2 of [11], we see how to construct an nth optimal information operator Π_n^*. However, Π_n^* involves knowing eigenvectors of K^*K, where $K = ST^\dagger$ (the dagger representing the pseudoinverse). Since these are difficult to determine in practice, we once again are willing to settle for optimality to within a constant factor. More precisely, let $\{\Pi_n^*\}_{n=1}^\infty$ be a a sequence of information operators with card $\Pi_n^* \leq n$. Then $\{\Pi_n^*\}_{n=1}^\infty$ is quasi-optimal if

$$e_\ell(\Pi_n^*,\mathfrak{F}) = \Theta(e_\ell(n,\mathfrak{F})) \quad \text{as} \quad n \to \infty. \quad \square$$

4. Finite Element Methods and Information

In this Section, we show how the finite element method (FEM) fits into the setting of the previous section. We describe the finite element information (FEI) which the FEM uses, and recall some quasi-optimality results from [15] for the case where assumptions (i), (ii), and (iii) from Section 1 hold.

We let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be a regular family of finite element subspaces of degree k. That is, $\mathcal{S}_n$ is an n-dimensional subspace of $H_E^m(\Omega)$ consisting of piecewise polynomials of degree k over a triangulation $\mathcal{T}_n$ of Ω, where $\{\mathcal{T}_n\}_{n=1}^{\infty}$ is regular [8, pg. 132]. Of course, since Ω is C^{∞}, we must make an additional assumption about boundary elements to guarantee that $\mathcal{S}_n \subseteq H_E^m(\Omega)$ in the situation where (2.1) is not a Neumann problem. (For instance, we may use curved elements as in [6].)

Remark 4.1. As indicated in (1.2) of the Introduction, we assume that $k \geq 2m - 1 - \ell$ in this paper. This technical assumption is needed for the proofs of some upper bounds which appear below. However, there are a number of situations where this holds automatically:

(i) If $\max(0, m - 1) \leq \ell \leq m$, (1.2) holds by [15, Lemma 4.1]. In particular, (1.2) holds when $m = 0$ (where $\ell = 0$) or $m = 1$ (where $0 \leq \ell \leq 1$).

(ii) If $N \geq 2$ and triangular elements are used, the results of [16] and the fact that $\ell \geq 0$ show that (1.2) holds.

Hence, the only possiblity that (1.2) will fail is when $N = 1$ or when rectangular elements are used. However, it is possible for $k = m$ in either of these settings; hence (1.2) can indeed fail to hold when $N = 1$ or when rectangular elements are used. $\square$

The finite element method (FEM) using $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is then defined as follows:

given $f \in H^r(\Omega)$ and a non-negative integer n, let $u_n \in \mathcal{S}_n$ satisfy

$$(4.1) \qquad B(u_n, s) = (f,s)_0 \qquad \forall\, s \in \mathcal{S}_n.$$

It is well known (see e.g. [3, Chapter 6] for the case $m = 1$, the general case being similar) that if $\{\mathcal{T}_n\}_{n=1}^{\infty}$ is quasi-uniform [8, pg. 272], then B is weakly coercive on $\{\mathcal{S}_n\}_{n=1}^{\infty}$ in the sense of Theorem 8.1 of [8], and hence there exists a unique solution $u_n \in \mathcal{S}_n$ to (4.1). Moreover, in this quasi-uniform case, we may use (1.2) and [8, Theorems 8.2 and 8.6] to see that there exists a positive constant C (independent of f, u, n and u_n) such that

$$(4.2) \qquad \|u - u_n\|_m \le C \inf_{s \in \mathcal{S}_n} \|u - s\|_m$$

and

$$(4.3) \qquad \|u - u_n\|_\ell \le C n^{-(\mu+m-\ell)/N} \|f\|_r \qquad \forall\, f \in H^r(\Omega),$$

where

$$\mu := \min(k + 1 - m,\; m + r).$$

We now show how the FEM fits into the setting of Section 3. Let $\{s_1, \ldots, s_n\}$ be a basis for $\mathcal{S}_n$. Then (4.1) means that $u_n \in \mathcal{S}_n$ satisfies

(4.4) $$B(u_n, s_i) = (f, s_i)_0 \qquad 1 \le i \le n.$$

Hence u_n depends only on the <u>finite</u> <u>element</u> <u>information</u> (FEI) Π_n determined by $\mathcal{S}_n$, where

$$\Pi_n f := [(f, s_1)_0 \ \dots \ (f, s_n)_0]^T \qquad \forall\ f \in H^r(\Omega).$$

We define an algorithm $\varphi_n \in \Phi(\Pi_n)$ by

$$\varphi_n(\Pi_n f) := u_n,$$

where $u_n \in \mathcal{S}_n$ satisfies (4.4); we will refer to φ_n as being the FEM using the FEI Π_n wherever this will cause no confusion. Hence, (4.3) now becomes

(4.5) $$\|Sf - \varphi_n(\Pi_n f)\|_\ell \le Cn^{-(\mu+m-\ell)/N}\|f\|_r \qquad \forall\ f \in H^r(\Omega)$$

in the quasi-uniform case.

We now relate the results of [15], where we assumed that (i), (ii), and (iii) from the Introduction held, so that $\|\cdot\|_m$ and the energy norm are equivalent on $H^m_E(\Omega)$:

<u>Theorem</u> <u>4.1</u>. Let B be $H^m_E(\Omega)$-coercive.

(i) $e_m(\varphi_n, BH^r(\Omega)) = \Omega(n^{-\mu/N})$, with "$\Omega$" replaced by "$\Theta$" in the quasi-uniform case.

(ii) $e_m(n, BH^r(\Omega)) = \Theta(n^{-(m+r)/N})$.

(iii) $e_m(\Pi_n, BH^r(\Omega)) = \Omega(n^{-(m+r)/N})$, with "$\Omega$" replaced by "$\Theta$" in the quasi-uniform case. $\square$

Hence, for the case where B is $H^m_E(\Omega)$-coercive, the error is measured in the energy norm, and the set of problem elements consists of the unit ball of $H^r(\Omega)$, the FEM is quasi-optimal using FEI iff $k \geq 2m - 1 + r$, while the FEI is always quasi-optimal information (in the quasi-uniform case).

5. Analysis of the Normed Case

In the next two sections, we extend the results in Theorem 4.1 to the case where B is weakly coercive and where error is measured in the norm $\|\cdot\|_{\ell}$ (where $0 \leq \ell \leq m$ and (1.2) holds). In this section, we consider the case where $\mathfrak{F} = BH^{r}(\Omega)$, the unit ball in $H^{r}(\Omega)$.

We first determine the nth minimal error.

Theorem 5.1. $e_{\ell}(n, BH^{r}(\Omega)) = \Theta(n^{-(r+2m-\ell)/N})$ as $n \to \infty$.

Proof: From (3.4), we have

(5.1) $$e_{\ell}(n, BH^{r}(\Omega)) = d_{n}(SBH^{r}(\Omega), H_{E}^{\ell}(\Omega)).$$

Let

$$H_{E}^{r+2m}(\Omega) = \{u \in H^{r+2m}(\Omega) : B_{j}u = 0 \quad 1 \leq j \leq m^{*} - 1)\}$$

denote the class of all $H^{r+2m}(\Omega)$-functions satisfying the essential boundary conditions. For any $\theta > 0$, let

(5.2) $$X(\theta) := \{u \in H_{E}^{r+2m}(\Omega) : \|u\|_{r+2m} \leq \theta\} = \theta BH_{E}^{r+2m}(\Omega).$$

Then the shift theorem (2.5) yields

(5.3) $$X(\sigma^{-1}) \subseteq SBH^{r}(\Omega) \subseteq X(\sigma),$$

and so (5.1) and (5.3) yield

(5.4) $$d_{n}(X(\sigma^{-1}), H_{E}^{\ell}(\Omega)) \leq e_{\ell}(n, BH^{r}(\Omega)) \leq d_{n}(X(\sigma), H_{E}^{\ell}(\Omega)).$$

But (5.2) implies that

$$(5.5)\quad d_n(X(\theta), H_E^{\ell}(\Omega)) = \theta d_n(BH_E^{r+2m}(\Omega), H_E^{\ell}(\Omega)) \qquad \forall\ \theta > 0,$$

so that (5.4) and (5.5) yield

$$(5.6)\quad \sigma^{-1} d_n(BH_E^{r+2m}(\Omega), H_E^{\ell}(\Omega)) \le e_{\ell}(n, BH^r(\Omega)) \le \sigma d_n(BH_E^{r+2m}(\Omega), H_E^{\ell}(\Omega)).$$

By Theorem 2.5.1 of [3],

$$(5.7)\quad d_n(BH_E^{r+2m}(\Omega), H_E^{\ell}(\Omega)) = [\Theta(d_n(BH_E^{1/2}(\Omega), L_2(\Omega)))]^{2(r+2m-\ell)}.$$

But $H_E^{1/2}(\Omega) = H^{1/2}(\Omega) = H_0^{1/2}(\Omega)$, and so

$$(5.8)\quad d_n(BH_E^{1/2}(\Omega), L_2(\Omega)) = d_n(BH_0^{1/2}(\Omega), L_2(\Omega))$$
$$= [\Theta(d_n(BH_0^{1}(\Omega), L_2(\Omega)))]^{1/2},$$

the last by another application of Theorem 2.5.1 of [3]. Hence (5.6), (5.7), and (5.8) yield

$$(5.9)\quad e_{\ell}(n, BH^r(\Omega)) = [\Theta(d_n(BH_0^{1}(\Omega), L_2(\Omega)))]^{r+2m-\ell}.$$

But the results of Jerome [7] yield

$$(5.10)\quad d_n(BH_0^{1}(\Omega), L_2(\Omega)) = \Theta(n^{-1/N}).$$

The theorem follows immediately from (5.9) and (5.10). □

We next show that the usual estimate (4.5) of the error of the FEM is sharp. Recall that $\mu = \min(k + 1-m,\ m + r)$.

<u>Theorem 5.2</u>.

(i) $e_{\ell}(\varphi_n, BH^r(\Omega)) = \Omega(n^{-(\mu+m-\ell)/N})$ as $n \to \infty$.

(ii) If $\{\Delta_n\}_{n=1}^{\infty}$ is quasi-uniform, then

$$e_\ell(\varphi_n, BH^r(\Omega)) = \Theta(n^{-(\mu+m-\ell)/N}) \quad \text{as} \quad n \to \infty.$$

Proof: First note that Theorem 5.1 yields

$$(5.11) \qquad e_\ell(\varphi_n, BH^r(\Omega)) \geq e_\ell(n, BH^r(\Omega)) = \Theta(n^{-(r+2m-\ell)/N}).$$

It remains to show

$$(5.12) \qquad e_\ell(\varphi_n, BH^r(\Omega)) = \Omega(n^{-(k+1-\ell)/N}) \quad \text{as} \quad n \to \infty,$$

since (5.11) and (5.12) yield (i), while (i) and the usual estimate (4.5) yield (ii).

The proof of (5.12) is very similar to that of (4.17) of [15]. Let Ω^0 be the interior of a hypercube such that $\bar{\Omega}^0 \subset \Omega$, and let

$$\mathcal{T}_n^0 := \{K \in \mathcal{T}_n : K \subset \bar{\Omega}^0\}.$$

Choose $u \in H_E^{r+2}$[illegible]$)$ such that

$$u(x) = \frac{1}{(k+1)!} x_1^{k+1} \qquad \forall\ x \in \bar{\Omega}^0.$$

In what follows, we define (for any region $K \subset \mathbb{R}^N$) $|\cdot|_{\ell,K}$ to be the usual seminorm [5, (3.1.2)] for non-negative integers ℓ, while for non-integral values of $\ell \geq 0$, we define $|\cdot|_{\ell,K}$ by the Sloboditskiĭ technique, i.e.,

$$|v|_{\ell,K}^2 = \sum_{|\alpha|=[\ell]} \int_K \int_K \frac{[D^\alpha v(x) - D^\alpha v(\xi)]^2}{|x-\xi|^{N+2(\ell-[\ell])}} \, d\xi \, dx$$

(see [8, pg. 96]). In any case, we have $\|\cdot\|_{\ell,K} \geq |\cdot|_{\ell,K}$. We write $P_k(K)$ for the polynomials of degree at most k over K.

Let $K \in \mathcal{J}_n^0$. We claim that there is a constant $C_1 > 0$, independent of K and n, such that

$$(5.13) \qquad \inf_{s \in P_k(K)} |u - s|^2_{\ell,K} \geq C_1^2 \, \mathrm{vol}(K)^{2(k+1-\ell)/N + 1}.$$

To show (5.13), note that K is the affine image of a "reference element" $\hat{K}$ which is independent of K. It is straightforward to check that the functionals

$$\hat{v} \mapsto |\hat{v}|_{k+1,\hat{K}} \qquad \text{and} \qquad \hat{v} \mapsto \inf_{\hat{s} \in P_k(\hat{K})} |\hat{v} - \hat{s}|_{\ell,\hat{K}}$$

are seminorms on $P_{k+1}(\hat{K})$. Since $\ell \leq m \leq k$ (the last by Lemma 4.1 of [15]), they have the same kernel $P_k(\hat{K})$. Since $P_{k+1}(\hat{K})$ is finite-dimensional, there is a constant $\hat{C}_1 = \hat{C}_1(k,m,\hat{K}) > 0$ such that

$$(5.14) \qquad \inf_{\hat{s} \in P_k(\hat{K})} |\hat{v} - \hat{s}|_{\ell,\hat{K}} \geq \hat{C}_1 |\hat{v}|_{k+1,\hat{K}} \qquad \forall\, \hat{v} \in P_{k+1}(\hat{K}).$$

As in [15], we may then use Theorem 3.1.2 of [5] and (5.14) to conclude that (5.13) holds.

Let

$$\Omega_n := \mathrm{int} \cup \{K : K \in \mathcal{J}_n^0\}.$$

We then use (5.13) to see that

$$\inf_{s\in S_n} |u - s|_\ell^2 \geq \sum_{K\in\mathcal{J}_n^0} \inf_{s\in P_k(K)} |u - s|_{\ell,K}^2$$

(5.15) $$\geq C_1^2 \sum_{K\in\mathcal{J}_n^0} \operatorname{vol}(K)^{2(k+1-\ell)/N + 1}$$

$$\geq C_1^2 \left[\frac{\operatorname{vol}(\bar{\Omega}_n)}{\#\mathcal{J}_n^0}\right]^{2(k+1-\ell)/N},$$

the last because

$$\sum_{K\in\mathcal{J}_n^0} \operatorname{vol}(K) = \operatorname{vol}(\bar{\Omega}_n).$$

Since $\Omega_n \subset \Omega^0$ and $\lim_{n\to\infty} \operatorname{vol}(\bar{\Omega}_n) = \operatorname{vol}(\bar{\Omega}^0)$, there is an $n_0 > 0$ such that

(5.16) $$\operatorname{vol}(\bar{\Omega}_n) \geq \frac{1}{2} \operatorname{vol}(\bar{\Omega}^0) \qquad \forall\ n \geq n_0.$$

Hence (5.15) and (5.16) yield that there is a $C_2 > 0$, independent of n, such that

(5.17) $$\inf_{s\in S_n} |u - s|_\ell \geq C_2(\#\mathcal{J}_n^0)^{-(k+1-\ell)/N} \qquad \forall\ n \geq n_0.$$

But $\#\mathcal{J}_n^0 = O(n)$ (see (4.14) of [15]), and so

(5.18) $$\inf_{s\in S_n} |u - s|_\ell \geq C_3 n^{-(k+1-\ell)/N} \qquad \forall\ n \geq n_0;$$

where $C_3 > 0$ is independent of n.

Now let $f = Lu$. Then f is a nonzero element of $H^r(\Omega)$, since u is a nonzero element of $H_E^{r+2m}(\Omega)$. Let

$$f^* := f/\|f\|_r,$$

so that

$$Sf^* = u/\|f\|_r.$$

Then $\varphi(\eta_n f^*) \in \mathcal{S}_n$ and $\|\cdot\|_\ell \geq |\cdot|_\ell$ yield

$$\|Sf^* - \varphi_n(\eta_n f^*)\|_\ell \geq |Sf^* - \varphi_n(\eta_n f^*)|_\ell$$

(5.19)
$$\geq \inf_{s \in \mathcal{S}_n} |Sf^* - s|_\ell$$

$$= \frac{1}{\|f\|_r} \inf_{s \in \mathcal{S}_n} |u - s|_\ell,$$

since $\mathcal{S}_n$ is a subspace of $H_E^m(\Omega)$. Hence (5.18) and (5.19) yield

(5.20) $$\|Sf^* - \varphi_n(\eta_n f^*)\|_\ell \geq \frac{C_3}{\|f\|_r} n^{-(K+1-\ell)/N} \qquad \forall\ n \geq n_0.$$

Since $f^* \in BH^r(\Omega)$ and $\|f\|_r > 0$ is independent of n, (3.1) implies (5.12). ∎

We now ask when the FEM is quasi-optimal using FEI. We see that this is the case iff $k \geq 2m - 1 + r$, while FEI is always quasi-optimal information, in

<u>Theorem</u> <u>5.3</u>.

(i) $e_\ell(\eta_n, BH^r(\Omega)) = \Omega(n^{-(r+2m-\ell)/N})$ as $n \to \infty$.

(ii) If $\{\Delta_n\}_{n=1}^{\infty}$ is quasi-uniform, then

$$e_\ell(\eta_n, BH^r(\Omega)) = \Theta(n^{-(r+2m-\ell)/N}) \text{ as } n \to \infty.$$

Proof: Using Theorem 5.1, we have

$$e_\ell(\Lambda_n, BH^r(\Omega)) \geq e_\ell(n, BH^r(\Omega)) = \Theta(n^{-(r+2m-\ell)/N}) \quad \text{as} \quad n \to \infty,$$

establishing (i). To establish (ii), we let $z \in BH^r(\Omega) \cap \ker \Lambda_n$, i.e.,

$$(z,s)_0 = 0 \qquad \forall\, s \in \mathcal{S}_n \text{ and } \|z\|_r \leq 1. \tag{5.21}$$

Let $g \in H^{-\ell}(\Omega) \subset H^{-m}(\Omega)$ (since $\ell \leq m$). Symmetry of B and (5.21) yield

$$\begin{aligned}(Sz,g)_0 &= B(Sz,Sg) \\ &= (z,Sg)_0 \\ &= (z,Sg - s)_0 \qquad \forall\, s \in \mathcal{S}_n,\end{aligned}$$

and so

$$|(Sz,g)_0| \leq \|z\|_r \|Sg - s\|_{-r} \qquad \forall\, s \in \mathcal{S}_n. \tag{5.22}$$

By Theorem 4.1.1 of [3], there exists $s \in \mathcal{S}_n$ such that

$$\|Sg - s\|_{-r} \leq C_1 n^{-\lambda/N} \|Sg\|_{2m-\ell} \leq C_1 \sigma n^{-\lambda/N} \|g\|_{-\ell}, \tag{5.23}$$

where $C_1 > 0$ is independent of n, g, and z,

$$\lambda = \min(r + 2m - \ell,\ k + 1 + r) = r + 2m - \ell \tag{5.24}$$

(by (1.2)), and (2.5) was used to establish the right-hand inequality. Hence, (5.21)-(5.24) yield

$$\frac{|(Sz,g)_0|}{\|g\|_{-\ell}} \leq C_1 \sigma n^{-(r+2m-\ell)/N}. \tag{5.25}$$

Since $g \in H^{-\ell}(\Omega)$ is arbitrary, (5.25) implies

$$(5.26) \qquad \|Sz\|_{\ell} = \sup_{g \in H^{-\ell}(\Omega)} \frac{|(Sz,g)_0|}{\|g\|_{-\ell}} \le C_1 \sigma n^{-(r+2m-\ell)/N}.$$

Since $z \in BH^r(\Omega) \cap \ker N_n$ is arbitrary, (3.3) and (5.26) yield

$$e_{\ell}(N_n, BH^r(\Omega)) = \sup_{z \in BH^r(\Omega) \cap \ker N_n} \|Sz\|_{\ell} \le C_1 \sigma n^{-(r+2m-\ell)/N},$$

which, along with (i), yields (ii). ☐

Hence in the quasi-uniform case, the FEI is always quasi-optimal, information, while the FEM is quasi-optimal using the FEI iff $k \ge 2m - 1 + r$. Thus the spline algorithm (see Section 3) using FEI is always quasi-optimal among all algorithms, while the FEM is quasi-optimal among all algorithms iff $k \ge 2m - 1 + r$.

6. Analysis of the Seminormed Case

In this Section, we see how the results of [15] extend to the case where B is weakly coercive, the error being the worst-case $H^{\ell}(\Omega)$-error over $BH^r(\Omega)$ (the set of $H^r(\Omega)$-functions f such that $|f|_r \leq 1$) where $0 \leq \ell \leq m$. In this Section, we assume that r is a non-negative integer.

Let

$$n^* := \dim P_{r-1} = \binom{N + r - 1}{r - 1}$$

denote the problem index [11, pg. 31]. We then have an estimate of the n-th minimal error in

Theorem 6.1.

(i) If $n < n^*$, then $e_{\ell}(n, BH^r(\Omega)) = +\infty$.

(ii) $e_{\ell}(n, BH^r(\Omega)) = \Theta(n^{-(r+2m-\ell)/N})$ as $n \to \infty$.

Proof: (i) follows immediately from Theorem 2.3.2 of [11]. To establish (ii), let

$$\hat{H}^r(\Omega) = H^r(\Omega)/P_{r-1}(\Omega)$$

under the quotient norm. Then [14, Lemma 5.3]

$$\begin{aligned} e_{\ell}(n, BH^r(\Omega)) &= O(e_{\ell}(n - n^*, B\hat{H}^r(\Omega))) \\ &= O(e_{\ell}(n - n^*, BH^r(\Omega))) \end{aligned} \tag{6.1}$$

where "B" denotes unit ball and the second step is because $B\hat{H}^r(\Omega) \subseteq BH^r(\Omega)$. But Theorem 3.1 yields

$$(6.2)\qquad \begin{aligned} e_{\ell}(n - n^*, \mathcal{B}H^r(\Omega)) &= \Theta((n - n^*)^{-(r+2m-\ell)/N}) \\ &= \Theta(n^{-(r+2m-\ell)/N}) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence (6.1) and (6.2) yield

$$(6.3)\qquad e_{\ell}(n, \mathcal{B}H^r(\Omega)) = O(n^{-(r+2m-\ell)/N}) \quad \text{as} \quad n \to \infty.$$

On the other hand, $BH^r(\Omega) \subset \mathcal{B}H^r(\Omega)$ yields

$$(6.4)\qquad e_{\ell}(n, \mathcal{B}H^r(\Omega)) \geq e_{\ell}(n, BH^r(\Omega)) = \Theta(n^{-(r+2m-\ell)/N}) \quad \text{as} \quad n \to \infty.$$

The estimate (ii) then follows from (6.3) and (6.4). $\square$

We next investigate the error of the FEM. We show that either the FEM has infinite error, or the estimates of Theorem 5.2 hold.

Theorem 6.2.

(i) If $SP_{r-1}(\Omega) \not\subseteq \mathcal{S}_n$, then $e_{\ell}(\varphi_n, \mathcal{B}H^r(\Omega)) = +\infty$.

(ii) If $SP_{r-1}(\Omega) \subseteq \mathcal{S}_n$ for all sufficiently large n, then

$$e_{\ell}(\varphi_n, \mathcal{B}H^r(\Omega)) = \Omega(n^{-(\mu+m-\ell)/N}) \quad \text{as} \quad n \to \infty,$$

where the "Ω" may be changed to "Θ" when $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is quasi-uniform.

Proof:

(i) Let $SP_{r-1}(\Omega) \not\subseteq \mathcal{S}_n$. Since the range of φ_n is $\mathcal{S}_n$, there exists $f \in P_{r-1}(\Omega)$ such that $\varphi_n(\Pi_n f) \neq Sf$. Since φ_n is linear and $P_{r-1}(\Omega) = \ker|\cdot|_r = \ker T$ (see (3.2)), $e_{\ell}(\varphi_n, \mathcal{B}H^r(\Omega)) = +\infty$ by Lemma 3.2.2 of [11].

(ii) Let $SP_{r-1}(\Omega) \subseteq \mathcal{S}_n$. Then Theorem 5.2(i) yields

$$e_\ell(\varphi_n,\mathcal{B}H^r(\Omega)) \geq e_\ell(\varphi_n,BH^r(\Omega)) = \Theta(n^{-(\mu+m-\ell)/N}).$$

Suppose now, in addition, that $\{\Delta_n\}_{n=1}^{\infty}$ is quasi-uniform. We need only show that

(6.5) $$e_\ell(\varphi_n,\mathcal{B}H^r(\Omega)) = O(n^{-(\mu+m-\ell)/N}) \text{ as } n \to \infty.$$

Let $f \in \mathcal{B}H^r(\Omega)$. Then

(6.6) $$f = f_1 + f_2$$

where

$$f_1 \in P_{r-1}(\Omega) \text{ and } f_2 \in \hat{H}^r(\Omega) = H^r(\Omega)/P_{r-1}(\Omega).$$

Recall [5, Theorem 3.1.1] that there is a constant $C_1 > 0$ such that

(6.7) $$\|\cdot\|_r \leq C_1|\cdot|_r \text{ on } \hat{H}^r(\Omega).$$

Since $f_1 \in P_{r-1}(\Omega)$ and $f \in \mathcal{B}H^r(\Omega)$, we have

$$\|f_2\|_r \leq C_1|f_2|_r = C_1|f|_r \leq C_1,$$

so that (4.5) yields

(6.8) $$\|Sf_2 - \varphi_n(\Lambda_n f_2)\|_\ell \leq Cn^{-(\mu+m-\ell)/N}\|f_2\|_r \leq C_2 n^{-(\mu+m-\ell)/N}.$$

Now $f_1 \in P_{r-1}(\Omega)$ and $SP_{r-1}(\Omega) \subseteq \mathcal{S}_n$, so that $Sf_1 \in \mathcal{S}_n$. Using (4.2), we have

$$\|Sf_1 - \varphi_n(\Pi_n f_1)\|_\ell \le \|Sf_1 - \varphi_n(\Pi_n f_1)\|_m$$

$$(6.9) \qquad \le C \inf_{s \in S_n} \|Sf_1 - s\|_m$$

$$= 0.$$

Since S, φ_n, and Π_n are linear, (6.6), (6.8), and (6.9) yield

$$\|Sf - \varphi_n(\Pi_n f)\|_\ell \le C_2 n^{-(\mu+m-\ell)/N}.$$

Since $f \in \mathcal{B}H^r(\Omega)$ is arbitrary, this yields (6.5). □

Remark 6.1. We illustrate the different possibilities in Theorem 6.2 by considering the model problems (1.3) and (1.4), where we have $r = 1$ and $m = 1$. Hence we define two solution operators, by letting $S_1 : H^1(0,1) \to H_0^1(0,1)$ and $S_2 : H^1(0,1) \to H^1(0,1)$ be the solution operators corresponding to (1.3) and (1.4), respectively. That is, for any $f \in H^1(0,1)$,

$$u = S_1 f \quad \text{satisfies} \quad -u'' + u = f \quad \text{in } (0,1) \qquad u(0) = u(1) = 0,$$

and

$$u = S_2 f \quad \text{satisfies} \quad -u'' + u = f \quad \text{in } (0,1) \qquad u'(0) = u'(1) = 0.$$

(Note that S_1 and S_2 differ only in their boundary conditions.) We claim that the FEM has infinite error for S_1, but has finite error for S_2. Again, keep in mind that these problems are being solved for all $f \in H^1(0,1)$ such that $|f|_1 \le 1$.

To see that the FEM has infinite error for S_1, note that

$S_1(P_0(0,1))$ is spanned by the solution of

$$-z'' + z = 1 \quad \text{in } (0,1) \qquad z(0) = z(1) = 0,$$

the solution of which is

$$z(x) := 1 - \left(\frac{e - 1}{e^2 - 1}\right)e^x - \left(\frac{e^2 - e}{e^2 - 1}\right)e^{-x}.$$

Since z is not a polynomial, we have $S_1(P_0(0,1)) \not\subseteq \mathcal{S}_n$, no matter how big k (the degree of the space $\mathcal{S}_n$) or n (the dimension of $\mathcal{S}_n$) are. Hence, the FEM has infinite error for the problem S_1.

We now consider the problem S_2. We find that $S_2(P_0(0,1)) = P_0(0,1)$, since the only solution to

$$-z'' + z \quad 1 \quad \text{in } (0,1) \qquad z'(0) = z'(1) = 0$$

is $z(x) := 1$. Since $k \geq 1$ [15, Lemma 4.1] and there are no essential boundary conditions for this problem, we have $S_2(P_0(0,1)) \subseteq \mathcal{S}_n$ for all $n \geq 1$ and any choice of k. $\square$

Remark 6.2. Since $\mathcal{S}_n \subset H^m_E(\Omega)$, the condition $SP_{r-1}(\Omega) \subseteq \mathcal{S}_n$ is equivalent to the condition $P_{r-1}(\Omega) \subseteq L\mathcal{S}_n$. In situations where the explicit form of the solution operator is unknown (i.e., most cases which arise in practice), it will generally be easier to verify whether $P_{r-1}(\Omega) \subseteq L\mathcal{S}_n$ than whether $SP_{r-1}(\Omega) \subseteq \mathcal{S}_n$. $\square$

Remark 6.3. The Condition $SP_{r-1}(\Omega) \subset \mathcal{S}_n$ is very restrictive, since it is not generally the case that the solution u of the problem $Lu = f$ (with f polynomial) is a piecewise polynomial satisfying the boundary conditions. (For example, we saw that the solution $u = S_1 f$ of (1.3) with $f = 1$ involves exponential functions in Remark 6.1.) It would be extremely unlikely to have $SP_{r-1}(\Omega) \subset S_n$ in most situations, especially those where Ω has a complicated boundary or the coefficients $a_{\alpha\beta}$ of L are nonpolynomial. Hence, we see that the FEM has finite error only under exceptional circumstances. □

Remark 6.4. Instead of fixing n and varying f (in our worst-case setting), we can fix f and increase n. This yields the asymptotic setting studied in Trojan [12]. Using results from [12], it is easy to show that for the seminormed case, the $H^{\ell}(\Omega)$-error of the FEM is always $\Theta(n^{-(r+2m-\ell)/N})$ as $n \to \infty$ in the asymptotic setting. □

Hence, there are situations in which the FEM has infinite error, no matter how big k and n are. Is this a feature of the FEM itself, or is it a feature of the information which the FEM uses? In the remainder of this section, we show that the fault lies with the FEM rather than with the FEI. In fact, we will show that FEI is quasi-optimal information.

In order to do this, we first establish

Lemma 6.1. There exists an integer $n_0 \geq 0$ and a constant $C > 0$ such that for any $n \geq n_0$,

$$\|z\|_r \leq C|z|_r \qquad \forall\, z \in \ker \Pi_n .$$

Proof: If $r = 0$, this is immediate.

Suppose now that $r \geq 1$. If the conclusion is false, then there is a subsequence $\{z_{n_i} \in \ker \Pi_{n_i}\}_{i=1}^{\infty}$ such that

$$(6.10) \qquad \|z_{n_i}\|_r = 1 \qquad \text{and} \qquad \lim_{i\to\infty} |z_{n_i}|_r = 0.$$

Following the proof of [8, Theorem 3.1.1], the Rellich-Kondrasov compactness theorem yields that there exists $z \in P_{r-1}(\Omega)$ and a subsequence, which we again denote $\{z_{n_i} \in \ker \Pi_{n_i}\}_{i=1}^{\infty}$, such that

$$(6.11) \quad \lim_{i\to\infty} z_{n_i} = z \text{ in } H^r(\Omega) \qquad (\text{and thus in } L_2(\Omega)).$$

Hence we see that

$$(6.12) \qquad \|z\|_r = 1.$$

We claim that $z = 0$, contradicting (6.12). Indeed, let $\varepsilon > 0$. Using denseness of $C_0^{\infty}(\Omega)$ in $L_2(\Omega)$, there exists $w \in C_0^{\infty}(\Omega)$ such that

$$(6.13) \qquad \|z - w\|_0 < \frac{1}{3}\varepsilon.$$

Since $C_0^{\infty}(\Omega) \subseteq H_E^m(\Omega) \cap H^1(\Omega)$, the standard results (found in, e.g., [8, Chapter 6]) yield that there is a $C_1 > 0$ (independent of z and w) such that for any $j > 0$, there exists $s_j \in S_j$ for which

$$\|w - s_j\|_0 \leq C_1 j^{-1/N} |w|_1.$$

Hence, there is an index $i_0(\varepsilon)$ such that for any $i \geq i_0(\varepsilon)$, there exists $s_{n_i} \in \mathcal{S}_{n_i}$ satisfying

(6.14) $$\|w - s_{n_i}\|_0 < \frac{1}{3}\varepsilon.$$

From (6.11) and $\|\cdot\|_0 \le \|\cdot\|_r$, there is an index $i_1(\varepsilon)$ such that for any $i \ge i_1(\varepsilon)$, there exists $z_{n_i} \in \ker \Pi_{n_i}$ for which

(6.15) $$\|z - z_{n_i}\|_0 < \frac{1}{3}\varepsilon.$$

Let $i_2(\varepsilon) = \max\{i_0(\varepsilon), i_1(\varepsilon)\}$. Then (6.13)-(6.15) and the triangle inequality yield

(6.16) $$\|z_{n_i} - s_{n_i}\|_0 < \varepsilon \qquad \forall\ i \ge i_2(\varepsilon).$$

But $z_{n_i} \in \ker \Pi_{n_i} = H^r(\Omega) \cap (L_2(\Omega)/\mathcal{S}_{n_i})$ and $s_{n_i} \in \mathcal{S}_{n_i}$. Hence

$$(z_{n_i}, s_{n_i})_0 = 0$$

which yields

$$\|z_{n_i}\|_0^2 \le \|z_{n_i}\|_0^2 - 2(z_{n_i}, s_{n_i})_0 + \|s_{n_i}\|_0^2$$
$$= \|z_{n_i} - s_{n_i}\|_0^2 < \varepsilon^2$$

(from (6.16)). Thus for any $\varepsilon > 0$, there is an index $i_2(\varepsilon)$ for which

$$\|z_{n_i}\|_0 < \varepsilon \qquad \forall\ i \ge i_2(\varepsilon).$$

Hence

(6.17) $$\lim_{i\to\infty} z_{n_i} = 0 \quad \text{in} \quad L_2(\Omega).$$

From (6.11) and (6.17), we have $z = 0$, the desired contradiction. $\square$

We are now ready to show that the FEI is always quasi-optimal information for the seminormed case.

Theorem 6.3.

(i) $e_{\ell}(\mathfrak{n}_n, \mathcal{B}H^r(\Omega)) = \Omega(n^{-(r+2m-\ell)/N})$ as $n \to \infty$.

(ii) If $\{\mathfrak{I}_n\}_{n=1}^{\infty}$ is quasi-uniform, then

$$e_{\ell}(\mathfrak{n}_n, \mathcal{B}H^r(\Omega)) = \Theta(n^{-(r+2m-\ell)/N}) \text{ as } n \to \infty.$$

Proof: Since $\operatorname{card} \mathfrak{n}_n = n$, Theorem 6.1 yields

$$e_{\ell}(\mathfrak{n}_n, \mathcal{B}H^r(\Omega)) \geq e_{\ell}(n, \mathcal{B}H^r(\Omega)) = \Theta(n^{-(r+2m-\ell)/N}) \text{ as } n \to \infty,$$

establishing (i). To prove (ii), let $z \in \mathcal{B}H^r(\Omega) \cap \ker \mathfrak{n}_n$. For any $g \in H^{-\ell}(\Omega)$, we use (5.22) to see that

$$(6.18) \qquad |(Sz,g)_0| \leq \|z\|_r \inf_{s \in \mathcal{S}_n} \|Sg - s\|_{-r}.$$

By (5.23) and (5.24), there is a $C_1 > 0$ (independent of n, g, and z) such that

$$(6.19) \qquad \inf_{s \in \mathcal{S}_n} \|Sg - s\|_{-r} \leq C_1 n^{-(r+2m-\ell)/N} \|g\|_{-\ell}.$$

By Lemma 6.1 and $z \in \mathcal{B}H^r(\Omega)$, we have

$$(6.20) \qquad \|z\|_r \leq C_2 |z|_r \leq C_2,$$

where C_2 is independent of n and z. Hence (6.18)-(6.20) yield

$$(6.21) \qquad \frac{|(Sz,g)_0|}{\|g\|_{-\ell}} \le Cn^{-(r+2m-\ell)/N},$$

where C is independent of n, z, and g. Taking the supremum over all nonzero $g \in H^{-\ell}(\Omega)$, (6.21) implies

$$(6.22) \qquad \|Sz\|_\ell \le Cn^{-(r+2m-\ell)/N}.$$

Since $z \in \mathcal{B}H^r(\Omega) \cap \ker \Lambda_n$ is arbitrary, (3.3) and (6.22) yield

$$e_\ell(\Lambda_n, \mathcal{B}H^r(\Omega)) \le Cn^{-(r+2m-\ell)/N},$$

which, along with (i), establishes (ii). ☐

Hence (in the quasi-uniform case), the spline algorithm using FEI is quasi-optimal among all algorithms in the seminormed case.

7. Complexity Analysis

In this section, we discuss the complexity of finding ε-approximations to the solution of the variational boundary-value problem, as well as the penalty for using the FEM when $k < 2m - 1 + r$.

Let $\varepsilon > 0$. An algorithm $\varphi \in \Phi(\mathfrak{N})$ produces an ε-approximation to the problem $(S,\mathfrak{F})$ in the $H^{\ell}(\Omega)$-norm if

$$e_{\ell}(\varphi,\mathfrak{F}) \leq \varepsilon.$$

The complexity $\text{comp}(\varphi)$ of an algorithm $\varphi \in \Phi(\mathfrak{N})$ is defined via the model of computation discussed in Chapter 5 of [11]. (Informally, we assume that any linear functional can be evaluated with finite cost c_1, and that the cost of an arithmetic operation is unity.) It then turns out that if $\mathfrak{N}$ has cardinality n, then

(7.1) $$\text{comp}(\varphi) \geq nc_1 + n - 1 \qquad \forall\, \varphi \in \Phi(\mathfrak{N}),$$

while if φ is linear, then

(7.2) $$\text{comp}(\varphi) \leq nc_1 + 2n - 1;$$

see [11, Chapter 5, Section 2]. We then define, for $\varepsilon > 0$, the ε-complexity $\text{COMP}_{\ell}(\varepsilon,\mathfrak{F})$ of the problem $(S,\mathfrak{F})$ in the $H^{\ell}(\Omega)$-norm to be

$$\text{COMP}_{\ell}(\varepsilon,\mathfrak{F}) = \inf\{\text{comp}(\varphi) : e_{\ell}(\varphi,\mathfrak{F}) \leq \varepsilon\}.$$

If φ^{*} is an algorithm for which

(7.3) $$e_{\ell}(\varphi^{*},\mathfrak{F}) \leq \varepsilon \quad \text{and} \quad \text{comp}(\varphi^{*}) = \text{COMP}_{\ell}(\varepsilon,\mathfrak{F}),$$

then φ^* is said to be an <u>optimal</u> <u>complexity</u> <u>algorithm</u> for ε-approximation of the problem $(S,\mathcal{F})$ in the $H^{\ell}(\Omega)$-norm.

<u>Remark</u> <u>7.1</u>. Not surprisingly, it is difficult to determine optimal complexity algorithms. We will generally willing to settle for optimality to within a constant factor, independent of ε. We say that a family $\{\varphi_\varepsilon^*\}_{\varepsilon>0}$ of algorithms has <u>quasi-optimal-complexity</u> for the problem $(S,\mathcal{F})$ iff

$$e_\ell(\varphi_\varepsilon^*,\mathcal{F}) \le \varepsilon \quad \text{for all sufficiently small} \quad \varepsilon > 0$$

and

$$\mathrm{comp}(\varphi_\varepsilon^*) = \Theta(\mathrm{COMP}_\ell(\varepsilon,\mathcal{F})) \quad \text{as} \quad \varepsilon \to 0. \quad \square$$

Recall that φ_n is the FEM of degree k using the FEI $\mathcal{N}_n$ based on the finite element subspace $\mathcal{S}_n$. We assume that $\{\mathcal{T}_n\}_{n=1}^{\infty}$ is quasi-uniform, where $\mathcal{T}_n$ is the triangulation of Ω upon which $\mathcal{S}_n$ is based. Let φ_n^s denote the spline algorithm using $\mathcal{N}_n$ (see Remark 3.1). We let

$$\mathrm{FEM}_\ell(\varepsilon,\mathcal{F}) = \inf\{\mathrm{comp}(\varphi_n) : e_\ell(\varphi_n,\mathcal{F}) \le \varepsilon\}$$

denote the algorithmic complexity of the FEM for the problem $(S,\mathcal{F})$ in the $H^{\ell}(\Omega)$-norm, and

$$\mathrm{SPLINE}_\ell(\varepsilon,\mathcal{F}) = \inf\{\mathrm{comp}(\varphi_n^s) : e_\ell(\varphi_n^s,\mathcal{F}) \le \varepsilon\}$$

denote the algorithmic complexity of the spline algorithm using the FEI for this problem. Using the results of Sections 5 and 6, (7.1), and (7.2), we find

Theorem 7.1.

(i) $COMP_{\ell}(\varepsilon,\mathfrak{F}) = \Theta(\varepsilon^{-N/(r+2m-\ell)})$ as $\varepsilon \to 0$ for $\mathfrak{F} = BH^r(\Omega)$ and for $\mathfrak{F} = \mathcal{B}H^r(\Omega)$.

(ii) $SPLINE_{\ell}(\varepsilon,\mathfrak{F}) = \Theta(\varepsilon^{-N/(r+2m-\ell)})$ as $\varepsilon \to 0$ for $\mathfrak{F} = BH^r(\Omega)$ and for $\mathfrak{F} = \mathcal{B}H^r(\Omega)$.

(iii) $FEM_{\ell}(\varepsilon,BH^r(\Omega)) = \Theta(\varepsilon^{-N/(\mu+m+\ell)})$ as $\varepsilon \to 0$.

(iv) (a) If there exists no integer $n \geq 0$ for which $SP_{r-1} \subseteq \mathcal{S}_n$, then

$$FEM_{\ell}(\varepsilon,\mathcal{B}H^r(\Omega)) = +\infty \qquad \forall\ \varepsilon > 0.$$

(b) If there exists no integer $n_0 \geq 0$ such that $SP_{r-1} \subseteq \mathcal{S}_n \quad \forall\ n \geq n_0$, then there exists $\varepsilon_0 > 0$ such that

$$FEM_{\ell}(\varepsilon,\mathcal{B}H^r(\Omega)) = +\infty \qquad 0 \leq \varepsilon \leq \varepsilon_0.$$

(c) If there exists an integer $n_0 \geq 0$ such that $SP_{r-1} \subseteq \mathcal{S}_n \quad \forall\ n \geq n_0$, then

$$FEM_{\ell}(\varepsilon,\mathcal{B}H^r(\Omega)) = \Theta(\varepsilon^{-N/(\mu+m-\ell)}) \quad \text{as} \quad \varepsilon \to 0. \quad \square$$

Hence, we may draw the following conclusions:

Corollary 7.1.

(i) The spline algorithm using the FEI is always quasi-optimal, for both the normed and seminormed cases.

(ii) The FEM is quasi-optimal for the normed case iff $k \geq 2m - 1 + r$. If $k < 2m - 1 + r$, then

$$\frac{FEM_{\ell}(\varepsilon, BH^{r}(\Omega))}{COMP_{\ell}(\varepsilon, BH^{r}(\Omega))} = \Theta((\frac{1}{\varepsilon})^{\lambda N}) \quad \text{as} \quad \varepsilon \to 0,$$

where

(7.4) $$\lambda = \frac{1}{k + 1 - \ell} - \frac{1}{r + 2m - \ell} > 0,$$

and so

$$\lim_{\varepsilon \to 0} \frac{FEM_{\ell}(\varepsilon, BH^{r}(\Omega))}{COMP_{\ell}(\varepsilon, BH^{r}(\Omega))} = +\infty .$$

(iii) The FEM is quasi-optimal for the seminormed case iff $k \geq 2m - 1 + r$ and $SP_{r-1} \subseteq S_n$ for all sufficiently large n. If $k < 2m - 1 + r$ and $SP_{r-1} \subseteq S_n$ for all sufficiently large n, then

$$\frac{FEM_{\ell}(\varepsilon, BH^{r}(\Omega))}{COMP_{\ell}(\varepsilon, BH^{r}(\Omega))} = \Theta((\frac{1}{\varepsilon})^{\lambda N}) \quad \text{as} \quad \varepsilon \to 0,$$

where λ is given by (7.4); if $SP_{r-1} \not\subseteq S_n$ for all n sufficiently large, then

$$FEM_{\ell}(\varepsilon, BH^{r}(\Omega)) = +\infty \quad \text{for all} \quad \varepsilon > 0 \quad \text{sufficiently small.} \quad \square$$

Hence when k is too small for a given value of r, there is an infinite asymptotic penalty for using the FEM instead of the spline algorithm. Corollary 7.1 implies that there is an

$\varepsilon_0 > 0$ such that

(7.5) $\quad \text{SPLINE}_{\ell}(\varepsilon,\mathfrak{F}) < \text{FEM}_{\ell}(\varepsilon,\mathfrak{F})$ for $0 < \varepsilon < \varepsilon_0$.

What is the value of ε_0? If ε_0 is unreasonably small, it may turn out that it is more reasonable to use the FEM for "practical" values of ε. We determine the value of ε_0 for a model problem in

Example 7.1. Let $N = 1$, $\Omega = (0,\pi)$, $m = 1$, $r = 1$, $H^1_E(\Omega) = H^1_0(0,\pi)$, and consider the bilinear form $B : H^1_0(0,\pi) \times H^1_0(0,\pi) \to \mathbb{R}$ defined by

$$B(v,w) := \int_0^\pi v'w' \qquad \forall\ v,w \in H^1_0(0,\pi).$$

Hence for $f \in H^1(0,\pi)$, $u = Sf$ is the variational solution to the problem

$$-u'' = f \quad \text{in } (0,\pi)$$
$$u(0) = u(\pi) = 0.$$

We choose $\mathcal{S}_n$ to be the n-dimensional subspace of $H^1_0(0,\pi)$ consisting of piecewise linear polynomials with nodes at $x_j = \frac{j\pi}{n+1}$ $(0 \le j \le n + 1)$, so that $k = 1$.

We wish to determine ε_0 such that (7.5) holds with $\ell = 1$ and $\mathfrak{F} = BH^1(0,\pi)$. This is similar to Example 6.1 of [15], the only difference being that in [15], we measured error by the energy norm (which is the H^1-seminorm $|\cdot|_1$), while here we use the H^1-norm $\|\cdot\|_1$. Using the Poincare inequality [9]

$$\|\cdot\|_0 \le |\cdot|_1 \quad \text{on} \quad H_0^1(0,\pi),$$

we have

(7.6) $$|\cdot|_1 \le \|\cdot\|_1 \le \sqrt{2}\ |\cdot|_1 \quad \text{on} \quad H_0^1(0,\pi).$$

Hence (6.25) and (6.32) of [15], along with (7.6) imply

(7.7) $$e_1(\varphi_n, BH^1(0,\pi)) \ge \frac{\pi}{\sqrt{12}\ (n+1)} \doteq \frac{.90689968}{n+1},$$

and

(7.8) $$e_1(\varphi_n^s, BH^1(0,\pi)) \le \sqrt{2}\ \left(\frac{1}{n+1}\right)^2 \doteq \frac{1.4142136}{(n+1)^2}.$$

Using (7.1) and (7.7), we have

(7.9) $$\text{FEM}_1(\varepsilon, BH^1(0,\pi)) \ge (c_1+1)\left(\frac{\pi}{\sqrt{12}}\ \varepsilon^{-1} - 1\right) - 1,$$

while (7.2) and (7.8) yield

(7.10) $$\text{SPLINE}_1(\varepsilon, BH^1(0,\pi)) \le (c_1+2)(\sqrt[4]{2}\ \varepsilon^{-1/2} - 1) - 1.$$

Thus (7.9) and (7.10) yield that

(7.11) $$\text{SPLINE}_1(\varepsilon, BH^1(0,\pi)) < \text{FEM}_1(\varepsilon, BH^1(0,\pi)) \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0,$$

where ε_0 is the smallest positive solution ε of

$$(c_1+1)\left(\frac{\pi}{\sqrt{12}}\ \varepsilon^{-1} - 1\right) = (c_1+2)(\sqrt[4]{2}\ \varepsilon^{-1/2} - 1).$$

Some algebra yields

$$\varepsilon_0 = \varepsilon_0(c_1) = \left[\sqrt[4]{2}\ (\tfrac{1}{2} c_1 + 1) - \sqrt{\sqrt{2}\ (\tfrac{1}{2} c_1 + 1)^2 - \frac{\pi}{\sqrt{12}} (c_1 + 1)}\right]^2 .$$

We now examine the value of $\varepsilon_0(c_1)$ under various assumptions on the cost c_1 of evaluating a linear functional (noting that for $c_1 \geq 0$, $\varepsilon_0(c_1)$ increases with c_1). Clearly $c_1 \geq 0$, so that

$$\varepsilon_0(c_1) \geq \varepsilon_0(0) = \left[\sqrt[4]{2} - \left(\sqrt{2} - \frac{\pi}{\sqrt{12}}\right)^{1/2}\right]^2 \doteq 0.22747884.$$

This tells us that (7.11) holds for all ε less than (roughly) 0.227. Next, we assume that $c_1 \geq 1$, i.e., that evaluating a linear functional is at least as hard as an arithmetic operation (it would be hard to imagine otherwise). In this case,

$$\varepsilon_0(c_1) \geq \varepsilon_0(1) = \left[\frac{3}{2}\sqrt[4]{2} - \left(\frac{9}{4}\sqrt{2} - \frac{\pi}{\sqrt{3}}\right)^{1/2}\right]^2 \doteq 0.37714081.$$

Finally, it is reasonable to assume that $c_1 \gg 0$, i.e., that evaluating a linear functional is <u>much</u> harder than an arithmetic operation [11, pg. 85]. One may check that

$$\lim_{c_1 \to \infty} \varepsilon_0(c_1) = \frac{1}{24}\pi^2\sqrt{2} \doteq 0.58157202,$$

giving an estimate of $\varepsilon_0(c_1)$ for large values of c_1.

Based on this example, it seems reasonable to conjecture that (7.5) generally holds for "reasonable" values of ε_0. (However, see the discussion at the end of [15, Section 6] for some comments about this conjecture.)

8. Summary, Extensions, and Open Questions

We have shown that FEI of degree k is always quasi-optimal information for indefinite linear elliptic problems $Lu = f$ under the following conditions:

(i) Error is measured in the Sobolev ℓ-norm, where $0 \le \ell \le m$.

(ii) Either $\|f\|_r \le 1$ $(r \ge -m)$ or $|f|_r \le 1$ (r a nonnegative integer).

(iii) $k \ge 2m - 1 - \ell$.

However, the FEM is not always quasi-optimal among all algorithms using FEI. In the normed case $\|f\|_r \le 1$, the FEM is quasi-optimal iff $k \ge 2m - 1 + r$. In the seminormed case $|f|_r \le 1$, the FEM has finite error iff the finite element subspace contains SP_{r-1}; if this holds, then the FEM is quasi-optimal iff $k \ge 2m - 1 + r$. In the case where $k < 2m - 1 + r$, the asymptotic penalty for using the FEM is infinite.

What happens when we try to weaken the assumptions above?

The natural weakening of (i) is to allow ℓ to satisfy the inequality $\ell \le m$. The proofs of Theorems 5.2(i) and 6.2(ii) (the lower bound for the FEM) do not hold, since there seems to be no natural definition of the Sobolev ℓ-seminorm for negative values of ℓ. In the case where $\{S_n\}_{n=1}^{\infty}$ is uniform and $H^m(\Omega) = H_E^m(\Omega)$ (i.e., a Neumann problem), the results in [10] show that these results do hold for negative ℓ in this special situation. We conjecture that this is true in general, i.e., the lower bounds for the FEM given in these theorems hold for any $i \leq m$. However, the other results in this paper do hold for any i such that $i \leq m$, provided (iii) still holds.

Condition (ii) may be weakened in a number of ways. Rather than use the norm (or seminorm) over $H^r(\Omega)$, we may use $\|\cdot\|_{r,p}$ or $|\cdot|_{r,p}$, the $W^{r,p}(\Omega)$ norm and seminorm. Alternatively, we may decide to use the norm in the Besov space $B_p^{r,s}(\Omega)$ (see [4]). If we let $\mathfrak{F}$ be the unit ball (or semiball) in one of these spaces, is FEI still quasi-optimal information? When is the FEM quasi-optimal?

Finally, what happens when (iii) no longer holds? In this case, the bounds that may be established using the techniques of this paper are no longer tight. Although (iii) holds for $i \geq 0$ in most cases of practical interest, it is important to find out what happens when (iii) is false, which can occur when rectangular elements are used or when error is measured in negative norms.

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References

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems. Van Nostrand, Princeton, NJ, 1965.

[2] P. M. Anselone and P. J. Laurent, "A general method for the construction of interpolating or smoothing spline functions," Numer. Math. v. 12, 1968, pp. 66-82.

[3] I. Babuška and A. K. Aziz, "Survey lectures on the mathematical foundations of the finite element method," in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A. K. Aziz, ed.). Academic Press, New York, 1972, pp. 3-359.

[4] P. L. Butzer and H. Berens, Semi-groups of Operators and Approximation. Springer-Verlag, Berlin, 1957.

[5] P. G. Ciarlet, The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam, 1978.

[6] P. G. Ciarlet and P. A. Raviart, "Interpolation theory over curved elements," Comput. Methods Appl. Mech. Engrg. v. 1, 1972, pp. 217-249.

[7] J. W. Jerome, "Asymptotic estimates of the L_2 n-width," J. Math. Anal. Appl., v. 22, 1968, pp. 449-464.

[8] J. T. Oden and J. N. Reddy, An Introduction to the Mathematical Theory of Finite Elements, Wiley-Interscience, New York, 1976.

[9] M. Schultz, Spline Analysis. Prentice-Hall, Englewood Cliffs, NJ, 1973.

[10] G. Strang and G. Fix, "A Fourier analysis of the finite element variational method," in Constructive Aspects of Functional Analysis, Part II, C.I.M.E., Rome, 1973.

[11] J. F. Traub and H. Woźniakowski, A General Theory of Optimal Algorithms. Academic Press, New York, 1980.

[12] J. M. Trojan, "Asymptotic model for linear problems" (in preparation).

[13] L. B. Wahlbin, "Quasi-optimality of the $\overset{\circ}{H}{}^1$ projection into finite element spaces," in Lectures on the Numerical Solution of Partial Differential Equations: Proceedings of the Special Year in Numerical Analysis (I. Babuska, T.-P. Liu, J. Osborn, eds.). Department of Mathematics, University of Maryland, College Park, MD, Lecture Notes #20, 1981.

[14] A. G. Werschulz, "Does increased regularity lower complexity?", Math. Comp., January, 1984 (to appear).

[15] A. G. Werschulz, "Finite element methods are not always optimal" (submitted for publication).

[16] A. Ženíšek, "Hermite interpolation on simplexes in the finite element method," in Proceedings EquaDiff 3, J. E. Purkyně University, Brno, pp. 271-277, 1972.