Limited Arbitrage, Gains from Trade and Arrow's Theorem

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LIMITED ARBITRAGE, GAINS FROM TRADE AND ARROW'S THEOREM

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The expression *limited arbitrage* is used to describe economies where only bounded, or limited, opportunities for gains are available to the traders from their initial endowments. This concept was rigorously defined in [4], [5] and shown to be central to the problem of resource allocation; it is also linked to the social diversity of the economy [7]. It turns out that a simple geometric interpretation can be given to limited arbitrage: here I show that it is equivalent to bounding the gains from trade, namely the sum of utilities increases which the traders can achieve from their initial endowments (Proposition 2, Section 1). From this geometry a somewhat unexpected new link emerges: a close connection with Arrow's impossibility theorem [1]. I establish that markets have limited arbitrage if and only in they have no Condorcet triples beyond certain utility levels (Proposition 3, Section 3). This means that on choices of great importance, irrational or intransitive behavior does not arise. Since Condorcet triples are the building blocks of Arrow's theorem, limited arbitrage appears to be at the core of social choice theory. The connection between limited arbitrage and the concept of no-arbitrage used in financial markets is discussed in Section 2.

The geometry of limited arbitrage provides therefore a well-defined connection between two classic forms of resource allocation which have been considered separate and almost antagonistic until now: markets and public choices. The concept is fundamental for a market's operation: limited arbitrage is both necessary and sufficient for the existence of a competitive equilibrium in Arrow-Debreu markets with or without short sales [4], with finite or infinitely many commodities [10]. It is also fundamental for social choice: limited arbitrage has been shown to be equivalent to the contractibility of spaces of preferences [6], a condition which is necessary and sufficient for the existence of social choice rules which are continuous, anonymous and respect unanimity, [2], [5], [8], [3]. It is somewhat surprising that while this latter set of axioms of social choice, introduced in [2], is different from Arrow's, the property of limited arbitrage is closely connected with both.

1. Limited Arbitrage and Gains from Trade. To offer a formal perspective one needs a few definitions. An economy E has $H \geq 2$ traders who trade $N \geq 2$ commodities or assets, so that the trading space is R^N; when short sales are not allowed the trading space is instead R^{N+}. In the following I shall focus on markets with short sales: markets without short sales are covered in [4]. A trader i is described by an initial endowment $\Omega_i \in R^N$, and by a preference represented by a utility function $u_i : R^N \to R, .u_i(0) = 0$, which is concave, increasing and satisfies mild regularity conditions, see the Appendix. Everything in this paper is ordinal, namely independent of the utility representation; therefore without loss of generality we may consider utilities where $\sup_{\{x:x \in R^N\}} u_i(x) = \infty$.

One wishes to identify those trading opportunities which could yield unbounded utility increases for the *ith* trader. These are described by net trades in $A_i = \{y \in R^N :$

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$\forall \lambda > 0$, $u_i(\Omega_i + \lambda y) > u_i(\Omega_i)$ and $\lim_{\lambda \to \infty} u_i(\Omega_i + \lambda y) = \infty\}$, a concept new to the literature, which contains global information about the trader and is therefore called a *global cone.* The trader's *market cone* is the set of all those prices at which all trading opportunities in A_i are unaffordable, $D_i = \{p \in R^{N+} : < p, (y - \Omega_i) > > 0\}$.[1] The existence of both competitive equilibrium[2] and social choice rules is shown to depend on the relation between the traders' market cones [4], [5], [10]; this relation also provides a framework for measuring social diversity [7].

DEFINITION 1. *The market economy E has limited arbitrage when all its market cones intersect:* $\bigcap_{i=1}^{H} D_i \neq \phi$.

This means that there exists one price, the same for all traders, at which the trades they can afford only increase their utilities by limited, or bounded, amounts. The concept of limited arbitrage can also be interpreted in terms of gains from trade, defined as the maximum increment in the sum of utilities which the traders can achieve by reallocating the economy's resources:

$$\text{Gains from trade } = G(E) = Sup\left(\sum_{i=1}^{H} u_i(x_i) - u_i(\Omega_i)\right),$$

where for all i $u_i(x_i) \geq u_i(\Omega_i)$ and $\sum_{i=1}^{H}(x_i - \Omega_i) = 0$.

PROPOSITION 2. *An economy E satisfies limited arbitrage if and only if it has bounded gains from trade, namely* $G(E) < \infty$.

For a proof see the Appendix. The geometry of limited arbitrage is simple: it means that the traders' global cones cannot contain net trades which add up to zero: $\sim \exists\ x_i, x_j$ such that $x_i + x_j = 0$, $x_i \in A_i$ and $x_j \in A_j$. In other words: all global cones A_i must lie on one side of a given price hyperplane.

Figure 1 illustrates an economy E_1 with two traders and two assets which has limited arbitrage. Its global cones are A_1 and A_2 and the price line p leaves both cones on one side. Therefore net trades in directions which lead to unbounded utility gains are unaffordable by all traders from their initial endowments at price p. The gains from trade in this economy $G(E_1)$ are bounded.

The economy of Figure 2 does not satisfy limited arbitrage: there are two directions of net trades $w_1' \in A_1$ and $w_1 \in A_2$, yielding unbounded increases in utility and which sum up to zero. Therefore, there is no price p at which all net trades in A_1 and in A_2 are unaffordable from initial endowments. The gains from trade in this economy are unbounded.

The boundedness of possible gains from trade, which we now know to be equivalent to limited arbitrage, is fundamental to the existence of a competitive equilibrium: it is necessary and sufficient [4], [7], [10]. Intuitively this is reasonable: an economy such as that in Figure 2, where traders wish to take unboundedly large and opposed trading positions, cannot reach an equilibrium. Desired trades are just too diverse to be accommodated within the same economy.

[1] Global cones and market cones were introduced in Chichilnisky [4], as was the concept of limited arbitage.

[2] Limited arbitrage is the first necessary and sufficient condition for the existence of a competitive equilibrium, and it applies to markets with or without short sales and with finite [4] or infinitely many commodities [10]. Sufficient conditions for existence of a competitive equilibrium with short sales and infinitely many commodities were provided in Chichilnisky and Heal [9].

2. Limited Arbitrage and No-Arbitrage. In financial markets an *arbitrage opportunity* exists when individuals can make unbounded gains at no cost, or, equivalently, by taking no risks. For example, buying an asset in a market where its price is low while simultaneously selling it at another where its price is high can lead to unbounded gains at no risk to the trader. *No-arbitrage* means that such opportunities do not exist, and it provides a standard way of pricing a financial asset: precisely so that no arbitrage opportunities should arise between this and other related assets. Since trading does not cease until all arbitrage opportunities are extinguished, at a market clearing equilibrium there is no-arbitrage.

The simplest illustration of the link between limited arbitrage and no-arbitrage is an economy E where the traders' initial endowments are zero, $\Omega_i = 0$ for $i = 1, 2$. Here *no-arbitrage* at the initial endowments means that there are no trades which could increase the traders' utility at zero cost: gains from trade in E must be zero. By contrast, E has *limited arbitrage* when no trader can increase utility beyond a given bound at zero cost; as seen in Proposition 2 of Section 1, gains from trade are bounded. In summary: no-arbitrage requires that there should be *no* gains from trade at zero cost, while limited arbitrage requires that there should be only bounded or *limited* gains from trade.

The two concepts are related but nonetheless quite different. No-arbitrage is a market clearing condition: it is used to describe an allocation at which there is no further reason to trade. It can be applied at the initial allocations, but then it means that there is no reason for trade: the economy is autarchic and therefore not very interesting. By contrast, limited arbitrage is applied only to the economy's initial data, the traders' endowments and preferences, and it does not imply that the economy is autarchic. Quite to the contrary, it is valuable in predicting whether the economy can ever reach a competitive equilibrium, and allows us to do this simply by examining the economy's initial conditions. This is the subject of the next section.

3. Limited Arbitrage and Arrow's Theorem. I shall show next that the traders' preferences in the economy E satisfy limited arbitrage if and only if they contain no *Condorcet triples* of large utility values. Condorcet triples are building blocks of Arrow's impossibility theorem, and are at the root of the social choice problem. Thus limited arbitrage eliminates the source of Arrow's impossibility theorem for choices of large utility values.

DEFINITION 1. *A Condorcet triple is a collection of three preferences over a choice set X, represented by utilities $u_i : X \to R$, $i = 1, 2, 3$, and three choices α, β, γ within a feasible set $Y \subset X$ such that $u_1(\alpha) > u_1(\beta) > u_1(\gamma)$, $u_2(\gamma) > u_2(\alpha) > u_2(\beta)$, and $u_3(\beta) > u_3(\gamma) > u_3(\alpha)$.*

Within an economy E, the social choice problem is about the choice of allocations: choices are in $X = R^{N \times H}$. An allocation $(x_1 ... x_H)$ is *feasible* if $\sum_i (x_i - \Omega_i) = 0$. Preferences over allocations are induced naturally by the traders' preferences over private consumption: $u_i(x_1 ... x_H) \geq u_i(y_i ... y_H) \Leftrightarrow u_i(x_i) \geq u_i(y_i)$.

DEFINITION 2. *In an economy E a family of preferences $\{u_1 ... u_H\}$ has a Condorcet triple of size k if there exists three feasible allocations $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k) \in X \subset R^{N \times 3}$; $\beta^k = (\beta_1^k, \beta_2^k, \beta_3^k)$ and $\gamma^k = (\gamma_1^k, \gamma_2^k, \gamma_3^k)$, and three preferences $u_1^k, u_2^k, u_3^k \in \{u_1 ... u_H\}$ which define a Condorcet triple, and such that each trader achieves at least a utility level k at each choice:* $\min_{i=1,2,3}\{[u_i^k(\alpha_i^k), u_i^k(\beta_i^k), u_i(\gamma_i^k)]\} > k$.

The following shows that limited arbitrage eliminates Condorcet triples on matters of great importance, namely on those with utility level approaching the supremum of utilities:

PROPOSITION 3. *Let E be a market economy E with no bounds on short sales. Then E has social diversity if and only if its traders' preferences have Condorcet triples of every size.*[3] *Equivalently, E has limited arbitrage if and only for some $k > 0$, the traders' preferences have no Condorcet triples of size larger than k.*

A proof is in the Appendix: it relies on the fact that limited arbitrage is equivalent to bounded gains from trade, Proposition 2 of Section 1.

4. Appendix. Definitions: A market *economy E* is defined by its trading space and its traders $E = \{X, \Omega_i \in R^{N+}, u_i : X \to R, i = 1...H\}$, where $X = R^N$, or $X = R^{N+}$ when no short sales are allowed. The traders' *preferences* $u_i : X \to R$ are continuous, concave and increasing: $x \geq y \Rightarrow u_i(x) \geq u_i(y)$ and $u_i(0) = 0$. When the trading space $X = R^{N+}$, if an indifference surface of positive utility intersects the boundary of R^{N+} all indifference surfaces of higher utility do too. When the trading space $X = R^N$ preferences are smooth (C^2), $\exists \varepsilon, K > 0 : \forall x \in R^N, ||Du(x)|| > \varepsilon$, and $||D^2u(x)|| < K$, and the directions of gradients of an indifference surface which is not bounded below form a closed set. This includes Cobb-Douglas, CES, strictly concave and linear preferences, and preferences with indifferences which intersect the axis, and which contain halflines. *Global cones A_i* and *market cones D_i* were defined in Section 1. The *market cone ∂D_i* of an economy E with trading space R^{N+} is $\partial D_i = D_i \cap S(E)$ if $S(E) \subset N$, and $\partial D_i = D_i$ otherwise, where $N = \{v \in R^N : \exists i \text{ with } < v, \Omega_i >= 0\}$, and where $S(E)$ is the set of supports to individually rational allocations: $S(E) = \{v \in R^N : \exists (x_1...x_H) \in R^{H \times N+} \text{ with } \sum(x_i - \Omega_i) = 0,\ u_i(x_i) \geq u_i(\Omega_i) \text{ for all } i, \text{ and } \forall z_i \in R^{N+}, u_i(z_i) \geq u_i(x_i) \Rightarrow < v, z_i - x_i > \geq 0\}$. The *utility set* is $U(E) = \{U_1, ...U_H \in R^{H+}+ : \forall i = 1...H, \exists x_1...x_H \text{ with } U_i = u_i(x_i) \geq u_i(\Omega_i) \text{ where } \sum_{i=1}^{H}(x_i - \Omega_i) \leq 0\}$. The *Pareto frontier* $P(E) = \{V = V_1...V_H \in U(E) :\sim \exists W_1...W_H \in U(E) \text{ with } W_j \geq U_j\ \forall j \text{ and for some } h, W_h > U_h\}$.

PROPOSITION 1. *The global cones A_i of the economy E are open convex sets.*

Proof. A sequence $(v^n)_{n=1,2...} \subset C(A_i) =$ the complement of A_i, defines halflines $(\Gamma^n)_{n=1,2...}$, with $Sup_{\{x:x\in\Gamma^n\}}(u_i(x)) < \infty\ \forall n$. By the assumptions on u_i, $\forall\ n\ \exists y \in \Gamma^n :< Du_i(y), w >= 0$ if $w \in \Gamma^n$. Concavity of u_i implies that $\forall w \in \Gamma^n < Du_i(\lambda y), w > \leq 0\ \forall\ \lambda > 1$. Assume that on two halflines $\Gamma^n \neq \Gamma^m$ the utility u_i is eventually constant: $\exists y^n \in \Gamma^n$ and $y^m \in \Gamma^m$ such that $\forall \lambda > 1 < Du_i(\lambda y^n), w >= 0\ \forall w \in \Gamma^n$, and $< Du_i(\lambda y^m), w >= 0\ \forall w \in \Gamma^m$, and $u_i(y^n) < u_i(y^m)$. Let Π be a supporting hyperplane for the preferred set of u_i at λy^m; this determines a halfspace Λ of $R^N : \forall q \in \Lambda$, $u_i(q) < u_i(\lambda y^m)$; note that Π contains an unbounded segment of Γ^m, and Λ an unbounded segment of Γ^n. Therefore $\forall\ K > 0\ \exists\ z^K \in \Gamma^n$ and $w^K \in \Pi : ||z^K - w^K|| > K$ and $\forall K$, $u_i(z^K) = u_i(y^n)$ and $u_i(w^K) = u_i(y^m)$. Since by assumption $\exists \varepsilon > 0 : \forall x$, $||Du_i(x)|| > \varepsilon$, $\forall K$ the distance between z^K and $\{w \in R^N : u_i(w) = u_i(y^m)\}$ is bounded: $\exists T > 0 : \forall K$, $||z^K - w^K|| < T$, a contradiction. The contradiction arises from assuming that u_i is eventually constant on Γ^n and Γ^m with $n \neq m$; therefore $\exists n_o : \forall j \geq n_o\ \exists y \in \Gamma^j :< Du_i(\lambda y), w > < 0\ \forall w \in \Gamma^j$ and $\forall \lambda > 1$. By concavity of u_i, this implies that along the halfline Γ defined by $v = \lim^n v^n$, u_i is bounded, so that $v \in C(A_i)$. Thus $C(A_i)$ is closed and A_i open. Convexity is immediate.□

[3] Without loss of generality assume that for all i, $Sup_{\{x;x\in R^N\}} u_i(x) = \infty$

THEOREM 2. *Let E be an economy without bounds on short sales. The Pareto frontier of the economy E is bounded if and only if the economy satisfies limited arbitrage. In particular, the economy E has bounded gains from trade, $G(E) < \infty$, if and only if it has limited arbitrage.*

Proof. By contradiction. Assume E has limited arbitrage. If $P(E)$ were *not* bounded there would exist a sequence of net trades $(z_1^j...z_H^j)_{j=1,2...}$ such that $\forall j$, $\sum_{h=1}^{H} z_h^j = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for some h.[4] It suffices to consider the case where $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for all h.[5] Consider two exhaustive and exclusive cases: Case 1 and Case 2. *Case 1: For infinitely many j's, $z_h^j \in A_h$ for all h.* Limited arbitrage requires that there exists a hyperplane that leaves all the cones A_h on one side for all h, and this contradicts the fact that $z_h^j \in A_h$ for all h and $\sum_{h=1}^{H} z_h^j = 0$. Since the contradiction arises from the assumption that $P(E)$ is unbounded, $P(E)$ must be bounded in this case. *Case 2: From some j onwards, $z_h^j \notin A_h$ for some h.* Consider the sequence $\{z_h^j/\|z_h^j\|\}_{j=1,2,...} \subset S^{N-1}$, the $N-1$ sphere in R^N. Since S^{N-1} is compact, it follows that there exists a subsequence, denoted also $\{z_h^j/\|z_h^j\|\}_{j=1,2,...}$ such that $\lim_{j\to\infty} z_h^j/\|z_h^j\| = \alpha_h \in S^{N-1}$ for all $h = 1...H$. Assume first that $\alpha_h \notin A_h$. Note that it suffices to consider utilities with indifference surfaces not bounded below, since when they are bounded below, $P(E)$ is always a bounded set. Then, by assumption, the directions of gradients of each indifference surface define a closed set. Since we assumed that $\alpha_h \notin A_h$, it follows that $Sup_{\lambda \in R^+}(u_h(\Omega_h + \lambda\alpha_h) < \infty$. This, together with the assumption on the utilities, implies that if Γ is the halfline defined by the vector α_h , either $\exists w \in \Gamma$ where the gradient $Du_h(z)$ is orthogonal to Γ, or else the utility u_h achieves a maximum at $y \in \Gamma$, and is a constant beyond y. These two alternatives are exhaustive and I will show that in both it is impossible that $\alpha_h = \lim_j z_h^j/\|z_h^j\|$ with $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j) = \infty$. If the gradient $Du_h\,(w)$ is orthogonal to Γ at some point w, and for $\lambda > 1$ $Du_h(\lambda w)$ projected on Γ is negative, then it is also negative in a neighborhood. Therefore for directions β sufficiently close to α_h ,$\exists K > 0$ such that $\sup_{x\in\Upsilon}(u_h(x)) < K$, a contradiction. The second alternative is that u_i utility achieves a maximum at w and remains constant thereafter on Γ. Similar reasoning, using convexity, shows that $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) = \infty$ cannot hold either. Since these two alternatives are exhaustive, $\alpha_h \notin A_h$ is impossible, so that $\alpha_h \in A_h$ for all h where $z_h^j \notin A_h$ from some j onwards. Therefore, in Case 2, $\forall h = 1...H$ the vectors $\alpha_h \in A_h^c$, the closure of A_h, and, for some h, $\alpha_h \in A_h$. Since the cones A_h are open by Proposition 1 in the Appendix, there exist nearby vectors $\beta_1...\beta_H$ s.t. $\sum_h \beta_h = 0$ and $\beta_h \in A_h$ for all h, contradicting limited arbitrage. Limited arbitrage thus implies that $P(E)$ is bounded. The reciprocal is immediate.□

PROPOSITION 3. *Let E be an economy without bounds on short sales and $H \geq 3$ traders. E has Condorcet triple of all sizes if and only if it does not satisfy limited arbitrage.*

Proof. Let E have limited arbitrage. For each $k > 0$, let $(\alpha^k, \beta^k, \gamma^k) \in R^{3\times N\times H}$ and $u_1^k, u_2^k, u_3^k \subset \{u_1...u_H\}$ be a Condorcet triple of size k. Without loss assume that $\forall i$, $\Omega_i = 0$, and choose a utility representation : $\forall i$, $Sup_{\{x:x\in R^N\}}(u_i(x)) = \infty$.

[4] Without loss of generality, we normalized utilities so that $\sup_{x\in R^N}(u_h(x)) = \infty$.

[5] By the assumptions on preferences if $\exists(u_1^j...u_H^j)_{j=1,2...} : \forall j, \sum_{h=1}^{H} u_h^j = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + u_h^j)) \to \infty$ for some h, $u_h(\Omega_h + u_h^j) \geq u_h(\Omega_h)\ \forall h$, then $\exists(z_1^j...z_H^j)_{j=1,2...}$: $\forall j, \sum_{h=1}^{H} z_h^j = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for all h.

The three allocations are feasible $\forall k$, e.g. $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k) \in R^{N\times 3}$, $\sum_{i=1}^{3}(\alpha_i^k) = 0$, and $\lim_{k\to\infty}(\min_{i=1,2,3}(u_i(\alpha_1^k), u_i(\beta_i^k), u_i(\gamma_i^k)) = \infty$. There exist therefore three traders called 1, 2, and 3 and a corresponding sequence of allocations $(\theta^k)_{k=1,2...} = (\theta_1^k, \theta_2^k, \theta_3^k)_{k=1,2...}$: $\forall k$, $\sum_i \theta_i^k = 0$ and $\forall i = 1,2,3$, $\sup_{k\to\infty} u_i(\theta_i^k) = \infty$. This implies that E has unbounded gains from trade, which contradicts Theorem 6. Therefore E cannot have Condorcet triples of every size.

Conversely, if E has no limited arbitrage, there exist three traders, called 1, 2, 3, with preferences u_1, u_2, u_3 and three vectors in R^N, $a \in A_1, b \in A_2$, $c \in A_3$, which add up to zero. For any integer $k > 0$, and small $\varepsilon > 0$ consider the vector $\Delta = (\varepsilon, ..., \varepsilon) \in R^{N+}$ and the following three allocations: $\alpha^k = (ka, kb - 2\Delta, kc + 2\Delta)$, $\beta^k = (ka - \Delta, kb, kc + \Delta)$ and $\gamma^k = (ka - 2\Delta, kb - \Delta, kc + 3\Delta)$; each allocation is feasible, e.g. $ka + kb - 2\Delta + kc + 2\Delta = k(a + b + c) = 0$. For each $k > 0$ the three allocations $\alpha^k, \beta^k, \gamma^k$ and the three utilities u_1, u_2, u_3 define a Condorcet triple of size $m(k)$, with $\lim_{k\to\infty} m(k) = \infty$.□

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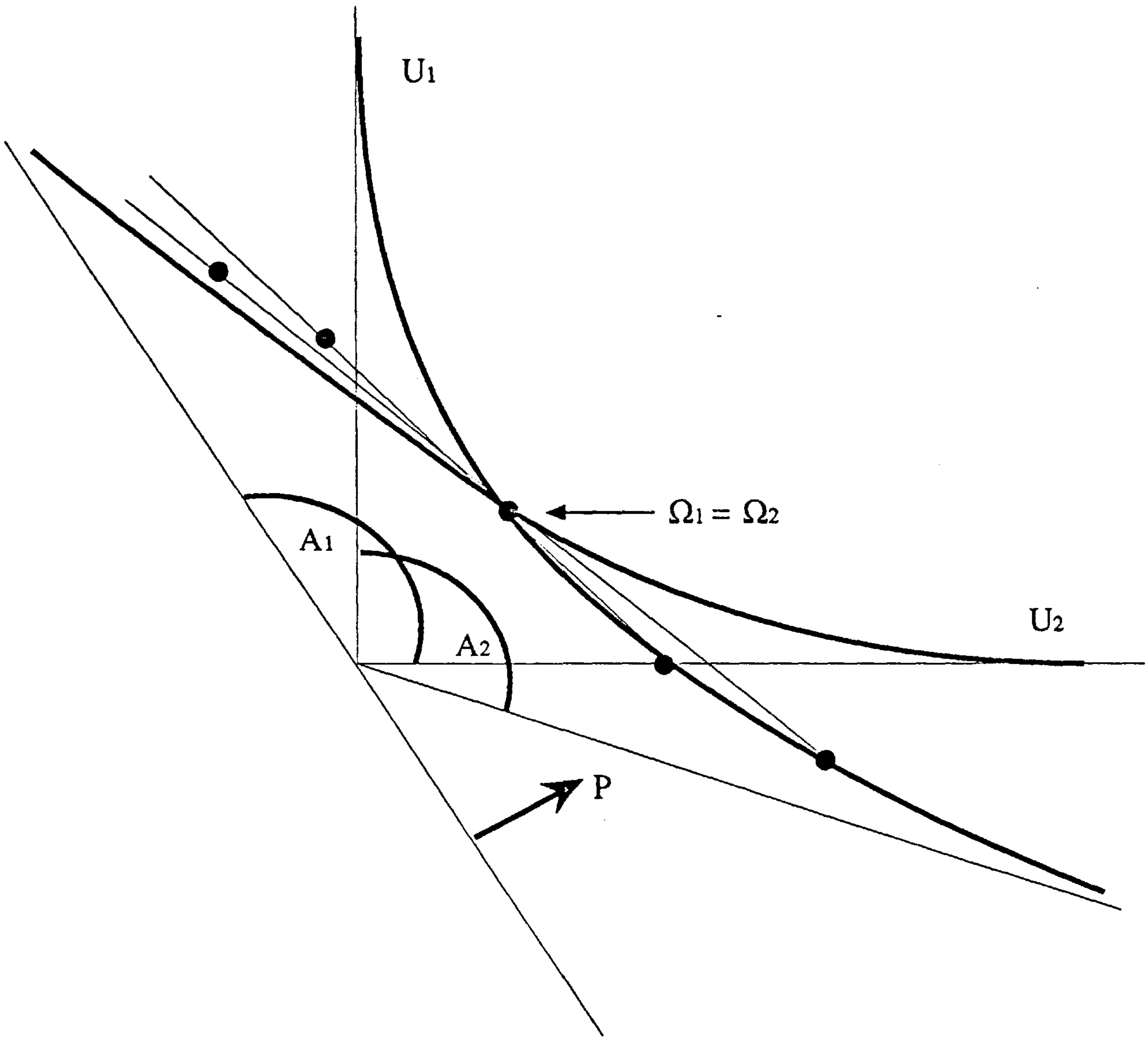

Figure1: limited arbitrage is satisfied. The two global cones lie in the halfspace defined by P. There are no feasible trades that increase utilities without limit: these would consist of pairs of points symmetrically placed about the common initial endowment, and as shown such pairs of points lead to utility values below those of the endowments at a bounded distance from the initial endowments.

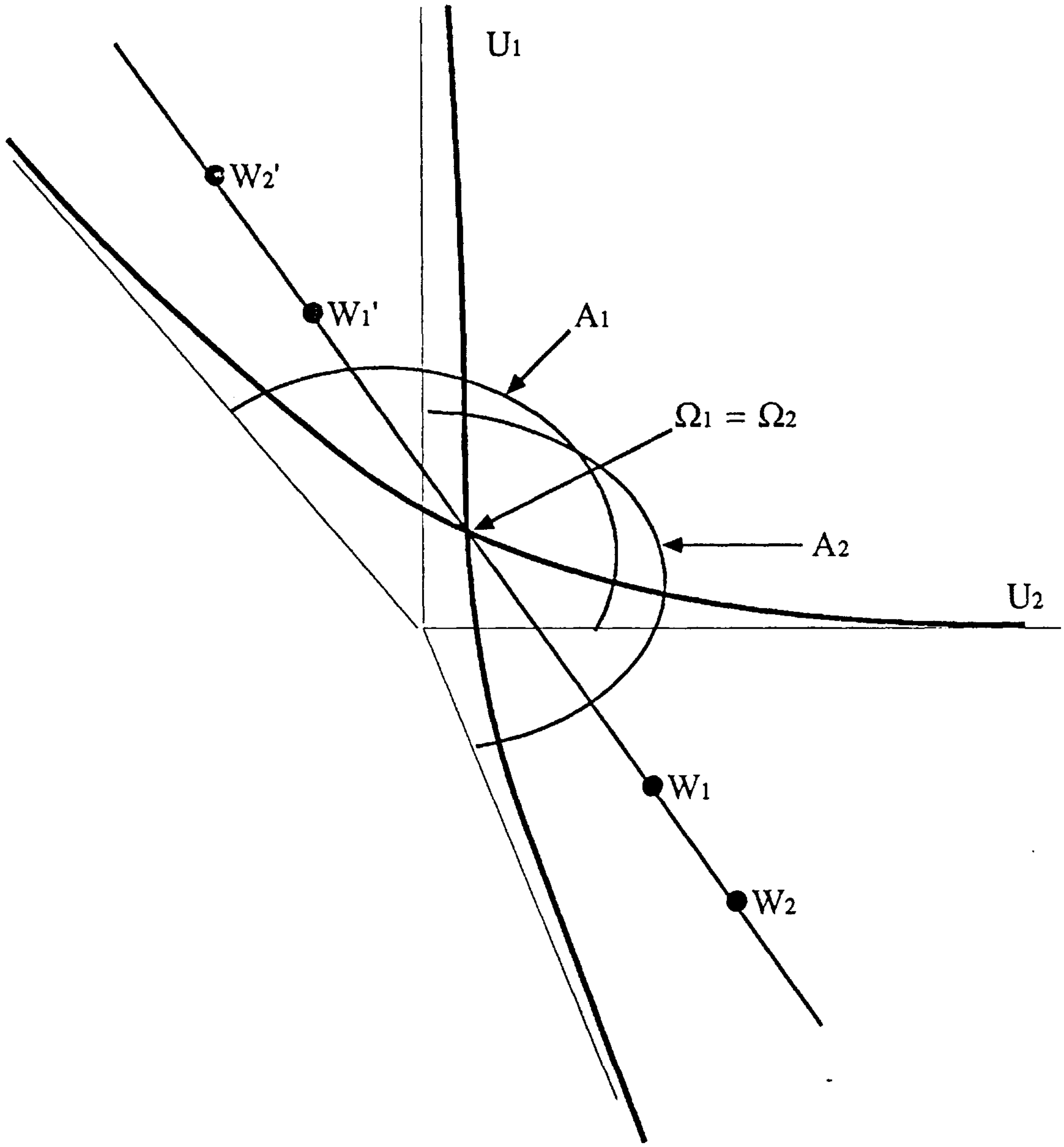

Figure 2: Limited arbitrage does not hold. The global cones are not contained in a half space, and there are sequences of feasible allocations such as W_1 and W_1', W_2 and W_2', which produce unbounded utilities.