Demazure-Lusztig Operators and Metaplectic Whittaker Functions on Covers of the General Linear Group

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ABSTRACT

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There are two different approaches to constructing Whittaker functions of metaplectic groups over non-archimedean local fields. One approach, due to Chinta and Offen for the general linear group and to McNamara in general, represents the spherical Whittaker function in terms of a sum over a Weyl group. The second approach, by Brubaker, Bump and Friedberg and separately by McNamara, expresses it as a sum over a highest weight crystal.

This work builds a direct, combinatorial connection between the two approaches. This is done by exploring both in terms of Demazure and Demazure-Lusztig operators associated to the Weyl group of an irreducible root system. The relevance of Demazure and Demazure-Lusztig operators is indicated by results in the non-metaplectic setting: the Demazure character formula, Tokuyama's theorem and the work of Brubaker, Bump and Licata in describing Iwahori-Whittaker functions.

The first set of results is joint work with Gautam Chinta and Paul E. Gunnells. We define metaplectic Demazure and Demazure-Lusztig operators for a root system of any type. We prove that they satisfy the same Braid relations and quadratic relations as their nonmetaplectic analogues. Then we prove two formulas for the long word in the Weyl group. One is a metaplectic generalization of Demazure's character formula, and the other connects the same expression to Demazure-Lusztig operators. Comparing the two results to McNamara's construction of metaplectic Whittaker functions results in a formula for the Whittaker functions in the spirit of the Demazure character formula.

The second set of results relates to Tokuyama's theorem about the crystal description of type A characters. We prove a metaplectic generalization of this theorem. This establishes a combinatorial link between the two approaches to constructing Whittaker functions for metaplectic covers of any degree. The metaplectic version of Tokuyama's theorem is proved as a special case of a stronger result: a crystal description of polynomials produced by sums of Demazure-Lusztig operators acting on a monomial. These results make use of the Demazure and Demazure-Lusztig formulas above, and the branching structure of highest weight crystals of type A. The polynomials produced by sums of Demazure-Lusztig operators acting on a monomial are related to Iwahori fixed Whittaker functions in the nonmetaplectic setting.

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Chapter 1

Introduction

This thesis explores the relationship between different approaches to constructing Whittaker functions on a metaplectic group over a non-archimedean local field. The main result is a metaplectic analogue of Tokuyama's Theorem and a crystal description of certain polynomials related to Iwahori-Whittaker functions. This relies on formulas of metaplectic Demazure and Demazure-Lusztig operators, which we prove in joint work with Gautam Chinta and Paul E. Gunnells.

In this chapter, we start by giving an overview of the motivation in Section 1.1. In Section 1.2 we introduce some notation and background necessary to state the main results in Section 1.3. We will outline the structure of the thesis in Section 1.4. Some further motivation and proposed applications are discussed in section 1.5.

1.1 Motivation

The study of metaplectic groups was initiated by Matsumoto [28]. Analytic number theory, in particular questions about the mean values of L-functions led to research on multiple Dirichlet series, which in turn motivated interest in Whittaker coefficients of metaplectic Eisenstein series. Kubota [24] was the first to closely examine Eisenstein series on higher covers of GL_2, and the theory of associated Whittaker functions was further developed by Kazhdan and Patterson [22]. In recent years, this development gained further impetus from unexpected connections to other areas, such as combinatorial representation theory, the geometry of Schubert varieties, and solvable lattice models. While the theory of metaplectic Whittaker functions is familiar in the case of double covers of reductive groups, it is less well understood in the case of higher covers.

1.1.1 The Casselman-Shalika formula

Whittaker functions are higher dimension generalizations of Bessel functions and are associated to principal series representations of a reductive group over a local field. The Casselman-Shalika formula is an explicit formula for values of a spherical Whittaker function over a local p-adic field in terms of a character of a reductive group. It is a central result in understanding the local and global theory of automorphic forms and their L-functions. A metaplectic analogue, describing Whittaker functions on n-fold metaplectic covers of a reductive group, has similar significance in the study of Dirichlet series of several variables. For example, Whittaker coefficients of GL_3 have a surprising connection, through the theory of double Dirichlet series, to a classical arithmetic problem: the representation of integers as quadratic forms. (See section 1.5.2.) Different approaches to generalize the Casselman-Shalika formula to the metaplectic setting have recently emerged.

1.1.2 Metaplectic analogues

Chinta-Offen [15] and McNamara [29] generalize the Casselman-Shalika formula by replacing the character with a metaplectic analogue: a sum over the Weyl group involving a modified action of the Weyl group that depends on the metaplectic cover. Brubaker-Bump-Friedberg [6] and McNamara [30] express a type A Whittaker function as a sum over a crystal base. Both constructions produce the Whittaker function as a polynomial determined by combinatorial data: the root datum of the group, a dominant weight, and the degree n of the metaplectic cover. The first one handles all types of root datum, while the second one makes it possible to compute the coefficients of the polynomial individually.

1.1.3 Combinatorial link

Various authors have worked on generalizing the crystal approach to root systems of other types: Chinta and Gunnells for type D [13], Beineke [1], Brubaker, Bump, Chinta, Gunnells [4], Frechette, Friedberg and Zhang [16] for type B and C, McNamara [30] working, less explicitly, with crystal bases in general. The resulting formulas are all significantly more intricate than the type A construction in [7]. The fact that the descriptions are purely combinatorial in nature, and rely heavily on Weyl group combinatorics and on the structure of the crystal graph, indicates that deeper properties of these constructions can be understood using methods of combinatorial representation theory. In this thesis we develop a combinatorial understanding of the relationship of the two approaches described in section 1.1.2. One of the possible applications of this is to understand how the crystal approach extends to other types.

1.2 Background

In this section, we introduce notation and discuss some background. We restrict ourselves to what is necessary to state the main results in the next section. Much of this background will be covered in more detail later.

Let Φ be a root system and Λ the corresponding weight lattice. We identify $\mathbb{C}(\Lambda)$ with a ring of rational functions $\mathbb{C}(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_{r+1})$ and $\mathbf{x}^{\alpha_1} = x_1/x_2$. The Weyl group W is generated by σ_i simple reflections. Let $w_0 \in W$ be the long element. The highest weight crystal $\mathcal{C}_{\lambda+\rho}$ will be introduced in Chapter 3. For now, we say it is a graph whose vertices are in bijection with a basis of the irreducible representation of highest weight $\lambda + \rho$, where $\lambda \in \Lambda$ is dominant and ρ is the Weyl vector. The number n denotes the degree of the metaplectic cover of a split reductive algebraic group corresponding to Φ.

1.2.1 Tokuyama's Theorem

Both approaches to constructing Whittaker functions described in section 1.1.2, i.e. summing over the Weyl group and summing over a crystal graph, also make sense in the nonmetaplectic setting. In this special case, a theorem of Tokuyama provides a combinatorial link between them [32]. This theorem will be discussed in detail in Section 3.5. It is a deformation of the Weyl character formula in type A :

$$\mathbf{x}^{\rho} \cdot \prod_{\alpha \in \Phi^{+}} (1 - v \cdot \mathbf{x}^{\alpha}) \cdot s_{\lambda}(\mathbf{x}) = \sum_{b \in \mathcal{C}_{\lambda+\rho}} G(b) \cdot \mathbf{x}^{\mathrm{wt}(b)}, \tag{1.2.1}$$

where s_λ is the Schur function. The left hand side essentially agrees with the Casselman-Shalika formula for Whittaker functions. (There the deforming parameter v is specialized to q^{-1}, where q is the order of the residue field of the non-archimedean local field underlying the group.) On the right hand side, the Gelfand-Tsetlin coefficients $G(b)$ are determined from the position of b in the crystal, using Berenstein-Zelevinsky-Littelmann paths (see Section 3.3 for details). This reproduces the construction of the same Whittaker function as a sum over a crystal base (Brubaker-Bump-Friedberg [6]). Thus explicitly relating the two constructions in section 1.1.2 is in essence proving a metaplectic analogue of Tokuyama's Theorem.

This motivation also provides guidance as to what the more general statement should be. On the left hand side of (1.2.1), the Schur function can be expressed by the Weyl character formula as

$$s_{\lambda}(\mathbf{x}) = \frac{1}{\mathbf{x}^{\rho} \cdot \prod_{\alpha \in \Phi^{+}} (1 - \mathbf{x}^{\alpha})} \cdot \sum_{w \in S_r} (-1)^{\ell(w)} \cdot \mathbf{x}^{w(\lambda+\rho)}. \tag{1.2.2}$$

Chinta-Offen [15] show what a correct analogue of the right hand side in (1.2.2) is, by using an action of the Weyl group W on $\mathbb{C}(\Lambda)$ introduced by Chinta-Gunnells in [12].

This action depends on n. The definition and properties of the Chinta-Gunnells action form the contents of Chapter 2.

On the right hand side of (1.2.1), in the metaplectic crystal description (Brubaker-Bump-Friedberg [6]) the coefficients $G(b)$ depend on n via Gauss sums. The method of determining these Gelfand-Tsetlin coefficients from the position of the element b in the crystal is given in Section 3.4.

1.2.2 Demazure operators

The proof of Tokuyama's theorem (Tokuyama [32]), which uses Pieri rules, does not generalize directly to the metaplectic setting. Below, we reformulate it as a stronger statement (1.3.3) about Demazure-Lusztig operators. Demazure operators $\mathcal{D}_w$ and Demazure-Lusztig operators $\mathcal{T}_w$ are a set of operators on $\mathbb{C}(\Lambda)$ corresponding to elements of the Weyl group. Their relevance is indicated by work of Littelmann and Kashiwara [21] giving character formulas on a crystal, and of Brubaker, Bump and Licata [8] relating them to Iwahori-fixed Whittaker functions.

Our reformulated statement (1.3.3) below is a crystal description of certain polynomials that, in the nonmetaplectic setting, are related to Iwahori-fixed Whittaker functions by [8]. In addition, the proof of this stronger statement generalizes to metaplectic covers.

The definitions of the operators $\mathcal{D}_w$ and $\mathcal{T}_w$ involve the action of the Weyl group W on $\mathbb{C}(\Lambda)$. This is the natural action inherited from the action of W on the weight lattice. As in the case of the metaplectic Casselman-Shalika formula, the normal permutation action is to be replaced by the Chinta-Gunnells action, and thus depends on n. The first step towards reformulating Tokuyama as indicated above is to create metaplectic analogues of these operators.

1.3 Results

1.3.1 Metaplectic Formulas

In joint work with Gautam Chinta and Paul E. Gunnells, we are able to define metaplectic Demazure and Demazure-Lusztig operators $\mathcal{D}_w$ and $\mathcal{T}_w$ that satisfy the same braid relations and quadratic relations as their nonmetaplectic analogues in [8]. The results of this joint work form the contents of Chapter 4. We prove the following two formulas:

the **Demazure formula** (Theorem 4.2.1):

$$\mathcal{D}_{w_0} = \frac{1}{\Delta} \cdot \sum_{w \in W} (-1)^{\ell(w)} \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{m(\alpha)\alpha} \cdot w, \tag{1.3.1}$$

and the **Demazure-Lusztig formula** (Theorem 4.2.2):

$$\sum_{u \leq w_0} \mathcal{T}_u = \Delta_v \cdot \mathcal{D}_{w_0}. \tag{1.3.2}$$

Here

$$\Delta = \prod_{\alpha \in \Phi^+} \left(1 - \mathbf{x}^{m(\alpha)\alpha}\right) \quad \text{and} \quad \Delta_v = \prod_{\alpha \in \Phi^+} \left(1 - v \cdot \mathbf{x}^{m(\alpha)\alpha}\right)$$

are the metaplectic versions of the Weyl denominator and its deformation from the left hand side of (1.2.1), $m(\alpha) = \dfrac{n}{\gcd(n, ||\alpha||)}$ and $\leq$ denotes the Bruhat order. The summation in (1.3.2) is over the entire Weyl group W, this form eases comparison with (1.3.3).

The formula (1.3.1) is a generalization of a result in Fulton's book on Young tableaux [17]. That result is the special case of (1.3.1) when Φ is of type A and $n = 1$. Both (1.3.1) and (1.3.2) are identities of operators on $\mathbb{C}(\Lambda)$ for Φ of any type. Together they interpret the left hand side of Tokuyama's theorem (1.2.1) in terms of Demazure-Lusztig operators. This will be explained in Section 4.2.

Though (1.3.1) and (1.3.2) look like their nonmetaplectic analogues, the dependence on the degree n is hidden in the group action. The definitions of $\mathcal{D}_w$ and $\mathcal{T}_w$ make use of the Chinta-Gunnells action of the Weyl group on $\mathbb{C}(\Lambda)$.

Example. Let $n = 2$ and Φ be of type A_1. Then $\mathbb{C}(\Lambda) = \mathbb{C}(x_1, x_2)$, and $W = \{1, \sigma_1\} \cong \mathbb{Z}/2\mathbb{Z}$. The action of the nontrivial element $\sigma_1 \in W$ on any $f(x_1, x_2) \in \mathbb{C}(\Lambda)$ is given by

$$\sigma_1(f(x_1,x_2)) = \frac{x_2}{x_1} \cdot \frac{x_1 - \sqrt{v} \cdot x_2}{x_2 - \sqrt{v} \cdot x_1} \cdot \frac{f(x_2,x_1) + f(-x_2,-x_1)}{2} + \frac{x_2}{x_1} \cdot \frac{f(x_2,x_1) - f(-x_2,-x_1)}{2},$$

and the Demazure and Demazure-Lusztig operators are defined by

$$\mathcal{D}_{\sigma_1}(f) = \frac{f - \frac{x_1^2}{x_2^2} \cdot \sigma_1(f)}{1 - \frac{x_1^2}{x_2^2}}; \qquad \mathcal{T}_{\sigma_1}(f) = \left(1 - v \cdot \frac{x_1^2}{x_2^2}\right) \cdot \mathcal{D}_{\sigma_1}(f) - f.$$

Work of Peter McNamara [29] relates Whittaker functions to the local factors of Weyl group multiple Dirichlet series constructed in [12]. Our theorems above recover the formulas from [12], thus obtaining a description of Whittaker functions similar to Demazure's character formula. We shall make the connection explicit in Chapter 5.

1.3.2 Metaplectic Tokuyama and Iwahori-Whittaker polynomials

In this thesis, the operators, relations, and (1.3.2) above are used to prove a crystal description of sums of Demazure-Lusztig operators in type A. More precisely, we prove (Theorem 6.2.1) that

$$\left(\sum_{u \le w} \mathcal{T}_u\right) \mathbf{x}^{w_0(\lambda)} = \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{1.3.3}$$

Here $\mathcal{C}_{\lambda+\rho}^{(w)}$ is the Demazure crystal corresponding to w within the highest weight crystal $\mathcal{C}_{\lambda+\rho}$. The coefficient $G(v)$ is the usual Gelfand-Tsetlin coefficient, described for non-metaplectic and metaplectic cases in [7].

The statement (1.3.3) provides the combinatorial link between the approaches to constructing metaplectic Whittaker functions described in section 1.1.2. It follows from the two formulas (1.3.1) and (1.3.2) above that the special case of this statement for

$w = w_0$ and $n = 1$ is exactly Tokuyama's theorem. The statement is formally stronger than Tokuyama even in the nonmetaplectic setting, and provides a metaplectic analogue for higher n. The $w = w_0$ special case of this identity is present when works of Brubaker-Bump-Friedberg-Hoffstein [6], Chinta-Gunnells-Offen [12, 15] and McNamara [30, 29] are combined, but our result (1.3.3) provides a much more direct connection. In addition, the operators

$$\sum_{u \leq w} \mathcal{T}_u$$

are related to the construction of Iwahori-Whittaker functions by [8], so the appearance of the same operator in (1.3.3) indicates that constructions in [8] may have metaplectic analogues as well.

In Chapters 6 and 7, we prove formula (1.3.3) for any degree n of metaplectic cover and for any w that is a "beginning section" of a particular long word:

$$w_0 = \underbrace{\sigma_1\sigma_2\sigma_1 \cdots \sigma_{r-1} \cdots \sigma_1\sigma_r \cdots \sigma_{r-k}}_{w}\sigma_{r-k-1} \cdots \sigma_1.$$

The proof consists of two parts. In Chapter 6, the statement of (1.3.3) is reduced to a simpler statement, about the crystal description of

$$(\mathcal{T}_{\sigma_r}\mathcal{T}_{\sigma_{r-1}} \cdots \mathcal{T}_{\sigma_1})\mathbf{x}^{w_0(\lambda)}. \tag{1.3.4}$$

That statement is then proved by induction in Chapter 7. It is interesting to note that while this proof of (1.3.3) avoids using the Pieri rules, it does rely on branching rules by exploiting the structure of a type A highest weight crystal.

1.4 Structure

The contents of the chapters are as follows.

Chapters 2 and 3 are dedicated to presenting background. In 2 we introduce the Chinta-Gunnells action, discuss some of its properties, and relate the definition given here to those in the literature. In Chapter 3, we turn our attention to crystals. We describe the bijection between elements of type A highest weight crystals and Gelfand-Tsetlin patterns and Berenstein-Zelevinsky-Littelmann paths. We give the definitions of the Gelfand-Tsetlin coefficients that appear in (1.2.1) and in (1.3.3). Finally, we present Tokuyama's theorem, and interpret it in the language of crystals.

Chapters 4 and 5 contains joint work with Gautam Chinta and Paul E. Gunnells. Chapter 4 is where the results of Section 1.3.1 are presented. We define metaplectic versions of Demazure and Demazure-Lusztig operators, and prove two formulas for the long word. In addition, we interpret Tokuyama's theorem in this language. In Chapter 5 we define Whittaker functions on a metaplectic group over a non-archimedean local field, and relate the formulas of Chapter 4 to constructions of Whittaker functions in the literature. (See Section 1.1.2.)

Chapters 6 and 7 are dedicated to the statement and proof of the main theorem described in Section 1.3.2 ((1.3.3), or Theorem 6.2.1). As sketched above, this is done in two stages. In Chapter 6, we use the branching structure of type A highest weight crystals to reduce the statement of Theorem 6.2.1, (1.3.3) to a statement about (1.3.4). Chapter 7 contains the (rather technical) proof of this simpler statement.

1.5 Further questions of interest

We end this chapter by discussing avenues of further research, where the methods and results of this thesis could be applied.

1.5.1 The Alcove Path Model

The construction of Whittaker functions as a sum over the Weyl group [15, 29] has the key feature that the Weyl group functional equations satisfied by the Whittaker function become very apparent. These functional equations play a key role in the analytic construction of global multiple Dirichlet series constructed from the Whittaker functions.

Moreover, they have proven successful in studying certain affine analogues. (See Section 1.5.4 below.)

The functional equations are less explicit in the description by crystal graphs. However, the crystal construction gives explicit formulas for individual coefficients of the Whittaker function. Reasons for trying to understand these coefficients are elaborated on in sections 1.5.2 and 1.5.4. The crystal formulas are all significantly more complicated outside of type A. Some preliminary work by Beazley and Brubaker suggests that perhaps the alcove path model is better suited for creating a construction that generalizes the type A crystal approach. We would like to use Demazure and Demazure-Lusztig operators to give a metaplectic Casselman-Shalika formula in terms of the alcove path model. If this can be done, the resulting construction might better reflect the Weyl group symmetry of Whittaker functions.

1.5.2 The Littlewood-Richardson Rule

The Littlewood-Richardson rule gives an explicit combinatorial description of the coefficients that appear when decomposing a tensor product of representations of GL_r into a direct sum of irreducibles. (Equivalently, in the decomposition of the product of two Schur functions into a linear combination of Schur functions.) Having a combinatorial understanding of the metaplectic analogue of the Schur function induces one to look for a metaplectic analogue of the Littlewood-Richardson rule. Such an analogue would lead to a deeper understanding of more classical arithmetic questions for the following reason. As referred to in section 1.1.1, Whittaker functions of GL_3 appear as coefficients of Dirichlet series of two variables, and as such, are related to arithmetic questions of a classical flavor. The coefficients are solutions of problems of counting integral points on flag varieties associated to quadratic forms [14]. For example, the main step in the proof of the result in [14], the computation of the orthogonal period of a $\mathrm{GL}_3(\mathbb{Z})$ Eisenstein series boils down to Gauss' three squares theorem.

1.5.3 Iwahori-Whittaker functions

In [8], the authors use Demazure and Demazure-Lusztig operators to compute values of Iwahori-Whittaker functions in terms of Hecke algebras, the geometry of Bott-Samelson varieties and the combinatorics of Macdonald polynomials. The analogies between these topics are intertwined with the combinatorics of the Bruhat order on the Weyl group, and identities satisfied by the Demazure and Demazure-Lusztig operators. The metaplectic operators introduced in out joint work with Gautam Chinta and Paul E. Gunnells satisfy analogous identities to the original nonmetaplectic ones. Thus replacing the operators in that paper with their metaplectic analogues, some of the results will automatically generalize to the metaplectic setting. Furthermore, since the non-metaplectic version of the operator in (1.3.3) is related in [8] to Iwahori-Whittaker functions, it is natural to ask if the explicit crystal description of this operator given by (1.3.3) leads to a more explicit understanding of metaplectic Iwahori-Whittaker functions.

1.5.4 Affine Weyl group multiple Dirichlet Series

Recent work of Bucur-Diaconu [9], Lee-Zhang [25] and Whitehead [33] attempts to extend the theory of multiple Dirichlet series to the affine setting. There the theory of Eisenstein series is not (yet) available. These authors construct multiple Dirichlet series that satisfy functional equations corresponding to an affine Weyl group. The coefficients of these power series can be explicitly related to character sums and coefficients of L-functions [33]. Some of our methods may lead to a combinatorial understanding of these coefficients.

Chapter 2

The Chinta-Gunnells action

This chapter is dedicated to the Chinta-Gunnells action. This "metaplectic" action of a Weyl group on a ring of rational functions plays a crucial role in the following chapters. It is the main ingredient in the definition of metaplectic analogues of Demazure and Demazure-Lusztig operators in Chapter 4.

The same group action was used by Chinta and Gunnells [11, 12] to construct Weyl group multiple Dirichlet series. Later, Chinta-Offen [15] constructed Whittaker functions on a metaplectic cover of GL_r over a p-adic field as a Weyl group sum using the same action for type A root systems. McNamara generalized this to other unramified reductive groups over local fields [29].

Here we cover the definition and elementary properties of the Chinta-Gunnells action. Relevant notation about root systems and corresponding Weyl groups is introduced in Section 2.1. Then the action is defined explicitly in Section 2.2 for an irreducible reduced root system of any type. The full generality will be of use in Chapter 4. In Chapter 6 and Chapter 7 we will restrict our attention to type A root systems. To facilitate explicit computations there, we introduce alternate notation in Section 2.4.

The fact that the action defined in Section 2.2 indeed defines an action of the Weyl group was proved in Chinta-Gunnells [12] and the arguments in McNamara [29] provide another proof. We do not prove this here. Instead, in Section 2.5 we explicitly relate the notation and Definition 2.2.1 to the notation and definitions in Chinta-Gunnells [12] and, for convenience, to Chinta-Offen [15].

2.1 Preliminaries

Here we define notation and discuss ingredients of Definition 2.2.1. We use [19] as a reference for facts about root systems and the Weyl group.

Let Φ be an irreducible reduced root system of rank r with Weyl group W. Choose an ordering of the roots and let $\Phi = \Phi^+ \cup \Phi^-$ be the decomposition into positive and negative roots. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be the set of simple roots, and let σ_i be the Weyl group element corresponding to the reflection through the hyperplane perpendicular to α_i. Define

$$\Phi(w) = \{\alpha \in \Phi^+ : w(\alpha) \in \Phi^-\}. \tag{2.1.1}$$

Let Λ be a lattice containing Φ as a subset. Let $\mathcal{A} = \mathbb{C}[\Lambda]$ be the ring of Laurent polynomials on Λ. Let $\mathcal{K}$ be the field of fractions of $\mathcal{A}$. The action of W on the lattice Λ induces an action of W on $\mathcal{K}$: we put

$$(w, x^\lambda) \longmapsto x^{w\lambda} =: w.x^\lambda, \tag{2.1.2}$$

and then extend linearly and multiplicatively to all of $\mathcal{K}$. We will always denote this action using the lower dot

$$(w, f) \longmapsto w.f$$

to distinguish it from the metaplectic W-action on $\mathcal{K}$ constructed below in (2.2.3). We sometimes refer to this as the "nonmetaplectic" group action.

Take a W-invariant $\mathbb{Z}$-valued quadratic form Q defined on Λ, and define a bilinear form $B(\alpha, \beta) = Q(\alpha + \beta) - Q(\alpha) - Q(\beta)$. Fix a positive integer n. The integer n determines a collection of integers $\{m(\alpha) : \alpha \in \Phi\}$ by

$$m(\alpha) = n/\gcd(n, Q(\alpha)), \tag{2.1.3}$$

and a sublattice $\Lambda_0 \subset \Lambda$ by

$$\Lambda_0 = \{\lambda \in \Lambda : B(\alpha, \lambda) \equiv 0 \bmod n \text{ for all simple roots } \alpha\}. \tag{2.1.4}$$

Lemma 2.1.1. *For any simple root α, we have $m(\alpha)\alpha \in \Lambda_0$.*

Proof. The proof is straightforward, proceeding case-by-case for irreducible root systems of each type. It depends on the fact that Q and B are determined up to a scalar multiple by the invariance under the Weyl group. For details, see Appendix A. $\square$

Let $\lambda \mapsto \bar{\lambda}$ be the projection $\Lambda \to \Lambda/\Lambda_0$ and $(\Lambda/\Lambda_0)^*$ be the group of characters of the quotient lattice. Any $\xi \in (\Lambda/\Lambda_0)^*$ induces a field automorphism of $\mathcal{K}/\mathbb{C}$ by setting $\xi(x^\lambda) = \xi(\bar{\lambda}) \cdot x^\lambda$ for $\lambda \in \Lambda$. This leads to the direct sum decomposition

$$\mathcal{K} = \bigoplus_{\bar{\lambda} \in \Lambda/\Lambda_0} \mathcal{K}_{\bar{\lambda}} \tag{2.1.5}$$

where $\mathcal{K}_{\bar{\lambda}} = \{f \in \mathcal{K} : \xi(f) = \xi(\bar{\lambda}) \cdot f \text{ for all } \xi \in (\Lambda/\Lambda_0)^*\}$

The next ingredient of Definition 2.2.1 is a set of complex parameters $v, g_0, \ldots, g_{n-1}$ satisfying

$$g_0 = -1 \text{ and } g_i g_{n-i} = v^{-1} \text{ for } i = 1, \ldots, n-1. \tag{2.1.6}$$

For all other j we define $g_j := g_{r_n(j)}$, where $0 \leq r_n(j) < n-1$ denotes the remainder upon dividing j by n.

Remark 2.1.2. In applications, the parameters $g_0, \ldots, g_{n-1}$ are Gauss sums. See Section 2.3 below.

2.2 Definition and basic properties

We now define an action of the Weyl group W on $\mathcal{K}$:

Definition 2.2.1. For $f \in \mathcal{K}_{\bar{\lambda}}$ and the generator $\sigma_\alpha \in W$ corresponding to a simple root α, define

$$\sigma_i(f) = \frac{\sigma_i.f}{1 - vx^{m(\alpha_i)\alpha_i}} \cdot \Big[x^{-r_{m(\alpha_i)}\left(-\frac{B(\lambda,\alpha_i)}{Q(\alpha_i)}\right)\cdot\alpha_i} \cdot (1-v) \\ - v \cdot g_{Q(\alpha_i)-B(\lambda,\alpha_i)} \cdot x^{(1-m(\alpha_i))\alpha_i} \cdot (1 - x^{m(\alpha_i)\alpha_i}) \Big] \tag{2.2.1}$$

where λ is any lift of $\bar{\lambda}$ to Λ.

It is easy to see that the quantity in brackets depends only on $\bar{\lambda}$. We extend the definition of σ_α to $\mathcal{K}$ by additivity. One can check that with this definition, $\sigma_\alpha^2(f) = f$ for all $f \in \mathcal{K}$. Furthermore it is proven in Chinta-Gunnells [12] (see also McNamara [29]) that this action satisfies the defining relations of W: if $(m_{i,j})$ is the Coxeter matrix for W, then

$$(\sigma_i\sigma_j)^{m_{i,j}}(f) = f \quad \text{for all } i, j \text{ and } f \in \mathcal{K}. \tag{2.2.2}$$

Therefore (2.2.1) extends to an action of the full Weyl group W on $\mathcal{K}$, which we denote

$$(w, f) \longmapsto w(f). \tag{2.2.3}$$

(For the explicit translation between our notation and that of Chinta-Gunnells [12], see Section 2.5.)

We remark that if $n = 1$, the action (2.2.1) collapses to the usual action (2.1.2) of W on $\mathcal{K}$.

The fact that the quantity in brackets in (2.2.1) depends only on $\bar{\lambda}$ and not λ translates to the following lemma:

Lemma 2.2.2. *Let $f \in \mathcal{K}$ and $h \in \mathcal{K}_0$. Then for any $w \in W$,*

$$w(hf) = (w.h) \cdot w(f).$$

Here $w.h$ means the action of (2.1.2), whereas $\cdot$ denotes multiplication in $\mathcal{K}$. □

Lemma 2.2.2 is used repeatedly in the proofs in Chapter 4 and Chapter 6. It is important to note that the action of W on $\mathcal{K}$ defined by (2.2.1) is $\mathbb{C}$-linear, but is *not* by endomorphisms of that ring, i.e. it is not in general multiplicative. The point of Lemma 2.2.2 is that if we have a product of two terms hf, the first of which satisfies $h \in \mathcal{K}_0$, then in (2.2.1) we can apply w to the product hf by performing the usual permutation action on h and then acting on f by the twisted W-action.

2.3 Gauss sums

As mentioned in Remark 2.1.2, applications of the group action of Definition 2.2.1 (e.g. in the construction of Weyl group multiple Dirichlet series in Chinta-Gunnells [12], or of metaplectic Whittaker functions in Chinta-Offen [15]) define the complex parameters $v, g_0, \ldots, g_{n-1}$ as Gauss sums. In Brubaker-Bump-Friedberg [7], similar Gauss sums ($g^\flat$ and $h^\flat$) are used to define Gelfand-Tsetlin coefficients on a crystal graph. (This will be discussed in detail in Chapter 3.) To facilitate comparison of results, we recall the various choices here.

We start by recalling the definition of the functions $g^\flat$ and $h^\flat$ in Brubaker-Bump-Friedberg [7]. Chapter 1 of that book is the source of the following notation and definitions. For facts about the power residue symbol we use Brubaker-Bump-Friedberg [5] as a reference. We will then examine the conditions (2.1.6) and choose the parameters $v, g_0, \ldots, g_{n-1}$ to satisfy these.

Let F be an algebraic number field containing the group μ_{2n} of $2n$-th roots of unity. Let S be a finite set of places of F, large enough that it contains all the places that are Archimedean or ramified over $\mathbb{Q}$, and the ring of S-integers $\mathfrak{o}_S = \{x \in F \mid |x|_v \leq 1 \text{ for } v \notin S\}$ is a principal ideal domain. Let ψ be a character on F_S of conductor $\mathfrak{o}_S$.

For any $m, c \in \mathfrak{o}_S$, $c \neq 0$, consider the n-th power residue symbol $\left(\frac{m}{c}\right)_n$. Recall that $\left(\frac{m}{c}\right)_n$ is zero unless m is prime to c. It is multiplicative, i. e.

$$\left(\frac{m}{c}\right)_n \cdot \left(\frac{m}{b}\right)_n = \left(\frac{m}{bc}\right)_n .$$

If p is a prime and m is coprime to p, then $\left(\frac{m}{p}\right)_n$ is the element of μ_n satisfying

$$\left(\frac{m}{p}\right)_n \equiv m^{\frac{\mathbb{N}p-1}{n}} \mod p.$$

With the notation above, we may define the Gauss sum

$$g(m,c) = \sum_{a \mod c} \left(\frac{a}{c}\right)_n \psi\left(\frac{am}{c}\right). \tag{2.3.1}$$

Fix a p prime in $\mathfrak{o}_S$, and let q be the cardinality of the residue field $\mathfrak{o}_S/p\mathfrak{o}_S$. We assume $q \equiv 1$ modulo $2n$. Define $g(a) = g(p^{a-1}, p^a)$ and $h(a) = g(p^a, p^a)$ for any $a > 0$. In this case we have

$$g(a) = \sum_{b \mod p^a} \left(\frac{b}{p^a}\right)_n \psi\left(\frac{b}{p}\right) = q^{a-1} \cdot \sum_{b \mod p} \left(\frac{b}{p}\right)_n^a \psi\left(\frac{b}{p}\right)$$

and

$$h(a) = \sum_{b \mod p^a} \left(\frac{b}{p^a}\right)_n \psi(b) = q^{a-1} \cdot \sum_{b \mod p} \left(\frac{b}{p}\right)_n^a \cdot 1 = \begin{cases} 0 & n \nmid a; \\ (q-1) \cdot q^{a-1} & n | a. \end{cases}$$

We are now ready to define the functions $g^\flat$ and $h^\flat$. These are used in Brubaker-Bump-Friedberg [7] to assign Gelfand-Tsetlin coefficients to elements of a crystal graph. We will recall those definitions in Chapter 3, and make use of the properties of $g^\flat$ and $h^\flat$ in Chapter 6 and Chapter 7.

Let

$$g^\flat(a) = q^{-a} \cdot g(a) \quad \text{and} \quad h^\flat(a) = q^{-a} \cdot h(a). \tag{2.3.2}$$

Then we have

$$h^\flat(a) = \begin{cases} 0 & n \nmid a; \\ 1 - \frac{1}{q} & n | a. \end{cases} \tag{2.3.3}$$

and

$$g^\flat(a) = q^{-1} \cdot \sum_{b \mod p} \left(\frac{b}{p}\right)_n^a \psi\left(\frac{b}{p}\right). \tag{2.3.4}$$

Notice that (2.3.4) implies that $g^\flat(a)$ and $h^\flat(a)$ both only depend on the residue of a modulo n.

If a is divisible by n then

$$g^\flat(a) = -q^{-1}, \tag{2.3.5}$$

and if $0 < a < n$ then

$$g^\flat(a) \cdot g^\flat(n-a) = q^{-1}. \tag{2.3.6}$$

Now let us turn to the conditions imposed on the parameters $v, g_0, \ldots, g_{n-1}$ in Section 2.1. According to (2.1.6), these must satisfy $g_0 = -1$ and $g_i g_{n-i} = v^{-1}$ for $1 \leq i \leq n-1$. Now we can choose these parameters by modifying the functions $g^\flat$ and $h^\flat$.

Take $v = q^{-1}$ and

$$g_i = v^{-1} \cdot g^\flat(i) = q \cdot g^\flat(i) = \sum_{b \mod p} \left(\frac{b}{p}\right)_n^i \psi\left(\frac{b}{p}\right) \text{ for } i = 1, \ldots, n-1. \tag{2.3.7}$$

Then (2.3.5) implies $g_0 = q \cdot (-q^{-1}) = -1$ and(2.3.6) implies

$$g_i g_{n-i} = v^{-2} \cdot g^\flat(i) \cdot g^\flat(n-i) = v^{-2} \cdot v = v^{-1}.$$

We pause to compare this choice with the literature.

The definition of $\mathfrak{g}^\psi(i)$ on page 425 of Chinta-Offen [15] is almost the same as that of g_i above; the only difference is that in our notation, $\mathfrak{g}^\psi(i)$ corresponds to the character ψ^{-1}. This implies $\mathfrak{g}^\psi(i) = g_{-i}$.

In Chinta-Gunnells [12], the parameters $\gamma(i)$ (indexed by integers modulo n) have a slightly different condition imposed on them:

$$\gamma(0) = -1 \quad \text{and} \quad \gamma(i) \cdot \gamma(n-i) = q^{-1}. \tag{2.3.8}$$

Notice that by (2.3.5) and (2.3.6) these conditions are satisfied if $\gamma(i) = g^\flat(i)$.

We summarize the different choices of parameters in the following claim. The notation $t^n = v = q^{-1}$ is introduced for later convenience.

Claim 2.3.1. *If $n \nmid a$, then $h^\flat(a) = 0$, and*

$$v \cdot g_a = q^{-1} \cdot \mathfrak{g}^\psi(-a) = \gamma(a) = g^\flat(a) = q^{-1} \cdot \sum_{b \mod p} \left(\frac{b}{p}\right)_n^a \psi\left(\frac{b}{p}\right).$$

However, if $n|a$, then $h^\flat(a) = 1 - v$, $\gamma(a) = g_a = g_0 = -1$, and $g^\flat(a) = -q^{-1} = -v = -t^n$.

To see the comparison of Definition 2.2.1 with Chinta-Offen [15] and Chinta-Gunnells [12] in more detail, see Section 2.5 below.

2.4 Notation in type A

If the root system Φ is of type A_r, all roots in Φ are of the same length. Consequently, the notation involved in Definition 2.2.1 becomes simpler. Chapter 6 and Chapter 7 contain explicit computations with the metaplectic group action. We introduce notation that is specific to type A_r for convenience.

Since all roots are of the same length, $m(\alpha) = n/\gcd(n, Q(\alpha))$ is the same for every root. In particular, in Chapter 6 and Chapter 7 we shall have $Q(\alpha) = 1$ and hence $m(\alpha) = n$. As usual,

$$\Phi = \{e_i - e_j \in \mathbb{R}^{r+1} \mid 1 \leq i \neq j \leq r+1\},$$

$$\Delta = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq r\},$$

where $e_1, \ldots e_{r+1}$ is the standard basis of $\mathbb{R}^{r+1}$, and take

$$\Lambda = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}) \in \mathbb{Z}^{r+1}\}.$$

We give an example of a $W \cong S_{r+1}$-invariant, integer-valued quadratic form. For $\lambda \in \Lambda$, (and c arbitrary), let

$$\begin{aligned} Q(\lambda) &= -\sum_{h<j} \lambda_h \lambda_j - c \cdot \left(\sum_{h=1}^{r+1} \lambda_h\right)^2 \\ &= -(1+2c) \cdot \sum_{h<j} \lambda_h \lambda_j - c \cdot \sum_{h=1}^{r+1} \lambda_h^2. \end{aligned} \tag{2.4.1}$$

Then certainly Q (and thus B) are integer valued on Λ and $n\Lambda \subseteq \Lambda_0$. Furthermore, it is easy to check that $Q(\alpha_i) = 1$ and $B(\alpha_i, \lambda) = \lambda_i - \lambda_{i+1}$.

Let $\mathbf{x} = (x_1, \ldots, x_r, x_{r+1})$. We may identify $\mathcal{K}$ with $\mathbb{C}(x_1, \ldots, x_{r+1}) = \mathbb{C}(\mathbf{x})$ by writing

$$x_i = \mathbf{x}^{e_i}.$$

In general, for $\lambda = \sum_i \lambda_i e_i \in \Lambda$ as above, we write $\mathbf{x}^\lambda = x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdots x_{r+1}^{\lambda_{r+1}}$.

With this notation, Definition 2.2.1 above can be rewritten for the action of σ_i on $f = \mathbf{x}^\lambda$ as

$$\sigma_i(f) = \frac{\sigma_i . f}{1 - v \cdot \left(\frac{x_i}{x_{i+1}}\right)^n} \cdot \left[\left(\frac{x_i}{x_{i+1}}\right)^{-r_n(\lambda_{i+1}-\lambda_i)} \cdot (1-v) \right. \\ \left. - v \cdot g_{1+\lambda_{i+1}-\lambda_i} \cdot \left(\frac{x_i}{x_{i+1}}\right)^{1-n} \cdot \left(1 - \left(\frac{x_i}{x_{i+1}}\right)^n\right)\right] \tag{2.4.2}$$

We may now compare (2.4.2) to equation (9.1) in Chinta-Offen [15]. In light of Section 2.3 we see that the two agree exactly.

2.5 Equivalent definitions

In this section, we relate Definition (2.2.1) to definitions of the Chinta-Gunnells action in the literature. We have already seen that it agrees with the definition in (9.1) of Chinta-Offen [15]. We show that both are equivalent to the definition in Chinta-Gunnells [12]. We rely on the latter fact to prove that Definition (2.2.1) in fact produces an action of the Weyl group on $\mathcal{K}$ (Theorem 3.2 in [12]). To see this, we examine three aspects of each definition: the notation for the function field where the Weyl-group action is defined, the complex parameters $v, g_0, \ldots, g_{n-1}$, and the quadratic form on $\Lambda \otimes \mathbb{R}$.

2.5.1 Notation for the function field

Statements and notation below follow Chinta-Gunnells [12]. Let Λ denote the root lattice. In Chinta-Gunnells [12], the inner product on $\Lambda \otimes \mathbb{R}$ is normalized such that $\langle \alpha, \alpha \rangle = \|\alpha\|^2 = 1$ for any short root α. Let us fix a positive integer $n^{(0)}$, and define

$$\hat{m}(\alpha) = n^{(0)} / \gcd(n^{(0)}, \|\alpha\|^2). \tag{2.5.1}$$

Take a collection of complex numbers $\gamma(i) \in \mathbb{C}$ (indexed modulo $n^{(0)}$) such that $\gamma_0 = -1$ and $\gamma(i) \cdot \gamma(-i) = 1/q$ if $i \not\equiv 0$ modulo $n^{(0)}$. Further, take $\theta = \sum_{i=1}^r (l_i + 1)\omega_i$ to be a (strictly) dominant weight, a linear combination of the fundamental weights ω_i. For any $\lambda \in \Lambda$ we may write it as an integral linear combination

$$\lambda = k_1\alpha_1 + \cdots + k_r\alpha_r.$$

Let $d(\lambda) = k_1 + \cdots + k_r$, and for any $w \in W$, define

$$w \bullet \lambda = w(\lambda - \theta) + \theta$$

and

$$\mu_{\theta,i}(\lambda) = d(\sigma_i \bullet \lambda - \lambda) \in \mathbb{Z}.$$

Then

$$\sigma_i \bullet \lambda = \sigma_i\lambda + (l_i + 1)\alpha_i,$$

and thus, with λ as above, we have

$$\mu_{\theta,i}(\lambda) = d(\sigma_i\lambda - \lambda + (l_i + 1)\alpha_i) = l_i + 1 - \frac{2\langle\lambda, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle},$$

$$\mu_{\theta,i}(\sigma_i \bullet \lambda) = -\mu_{\theta,i}(\lambda).$$

In Chinta-Gunnells [12], $\mathbb{C}[\Lambda]$ is identified with $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ and a change of variable action is defined on the latter ((3.8) in [12]). This makes the map

$$\mathbf{x}^\lambda \to q^{d(\lambda)} \cdot x_1^{k_1} \cdots x_r^{k_r}$$

an intertwining operator with the action (2.1.2) on $\mathbb{C}[\Lambda]$ and the change of variable action on $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$. (These are both left-actions of the Weyl group.) Thus x_i corresponds to $q^{-1} \cdot \mathbf{x}^{\alpha_i}$.

With $v = q^{-1}$ and choosing $\theta = \sum_{i=1}^r \omega_i$ equations (3.12) and (3.13) in [12] are replaced by

$$\mathcal{P}_{\lambda,i}(x^{\alpha_i}) = x^{\left(1 - r_{\hat{m}(\alpha_i)}(1 - 2\langle\lambda,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle)\right)\cdot\alpha_i} \cdot \frac{1 - v}{1 - vx^{\hat{m}(\alpha_i)\alpha_i}}. \tag{2.5.2}$$

and

$$\mathcal{Q}_{\sigma_i \bullet \lambda,i}(x^{\alpha_i}) = \gamma(||\alpha_i||^2 - 2\langle\lambda, \alpha_i\rangle) \cdot x^{\alpha_i} \cdot \frac{1 - x^{-\hat{m}(\alpha_i)\alpha_i}}{1 - v \cdot x^{\hat{m}(\alpha_i)\alpha_i}}. \tag{2.5.3}$$

With these ingredients, and writing $f = x^\lambda$, we may give the definition (2.5.4) below.

$$\sigma_i(f) = \frac{\sigma_i.f}{1 - vx^{\hat{m}(\alpha_i)\alpha_i}} \cdot \Big(x^{\left(1 - r_{\hat{m}(\alpha_i)}(1 - 2\langle\lambda,\alpha_i\rangle\langle\alpha_i,\alpha_i\rangle)\right)\cdot\alpha_i} \cdot (1-v) + \\ + \gamma(||\alpha_i||^2 - 2\langle\lambda,\alpha_i\rangle) \cdot x^{(1-\hat{m}(\alpha_i))\alpha_i} \cdot (x^{\hat{m}(\alpha_i)\alpha_i} - 1) \Big) . \tag{2.5.4}$$

2.5.2 Complex parameters

We rewrite this with the numbers g_i. Notice that the g_i are also indexed modulo $n^{(0)}$, and we have $g_0 = -1$ and $g_i \cdot g_{-i} = v^{-1} = q$ if $i \not\equiv 0$ modulo $n^{(0)}$. Thus we may write $\gamma(i) = \mu(i) \cdot g_i$, where

$$\mu(i) = \begin{cases} 1 & \text{if } \; i \equiv 0 \mod n^{(0)}; \\ v = (q^{-1}) & \text{if } \; i \not\equiv 0 \mod n^{(0)}. \end{cases}$$

Rewriting (2.5.4) in terms of the g_i gives the following.

$$\sigma_i(f) = \frac{\sigma_i.f}{1 - vx^{\hat{m}(\alpha_i)\alpha_i}} \cdot \Big(x^{\left(1 - r_{\hat{m}(\alpha_i)}(1 - 2\langle\lambda,\alpha_i\rangle\langle\alpha_i,\alpha_i\rangle)\right)\cdot\alpha_i} \cdot (1-v) + \\ + \mu(||\alpha_i||^2 - 2\langle\lambda,\alpha_i\rangle) \cdot g_{||\alpha_i||^2 - 2\langle\lambda,\alpha_i\rangle} \cdot x^{(1-\hat{m}(\alpha_i))\alpha_i} \cdot (x^{\hat{m}(\alpha_i)\alpha_i} - 1) \Big) . \tag{2.5.5}$$

Now, definition (2.5.5) is a little awkward in two ways. One is the appearance of the factor $\mu(\|\alpha_i\|^2 - 2\langle\lambda,\alpha_i\rangle)$ in the second term, which one would like to replace by a factor of v. The other is the coefficient

$$1 - r_{\hat{m}(\alpha_i)} \left(1 - \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \right)$$

of α_i in the exponent of the first term, which one would like to replace with

$$-r_{\hat{m}(\alpha_i)} \left(- \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \right) = 1 - r_{\hat{m}(\alpha_i)} \left(1 - \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \right) + \\ + \begin{cases} (-\hat{m}(\alpha_i)) & \text{if } \; 1 - \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \equiv 0 \mod \hat{m}(\alpha_i); \\ 0 & \text{if } \; 1 - \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \equiv 0 \mod \hat{m}(\alpha_i); \end{cases} \tag{2.5.6}$$

Thus we would like the part in parentheses to read

$$x^{-r_{\hat{m}(\alpha_i)}\left(-\frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\right)\cdot\alpha_i}\cdot(1-v)+v\cdot g_{||\alpha_i||^2-2\langle\lambda,\alpha_i\rangle}\cdot x^{(1-\hat{m}(\alpha_i))\alpha_i}\cdot(x^{\hat{m}(\alpha_i)\alpha_i}-1). \tag{2.5.7}$$

Observe that (2.5.7) is equal term-by term to the part enclosed in parentheses in (2.5.5) in every case, except when

$$\|\alpha_i\|^2-2\langle\lambda,\alpha_i\rangle\equiv 0 \mod n^{(0)} \iff 1-\frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\equiv 0 \mod \hat{m}(\alpha_i)=\frac{n^{(0)}}{\gcd(n^{(0)},\|\alpha_i\|^2)}.$$

But in this case, the two "mistakes" cancel out: the part enclosed in parentheses in (2.5.5) reads

$$x^{\alpha_i}\cdot(1-v)+(-1)\cdot x^{(1-\hat{m}(\alpha_i))\alpha_i}\cdot(x^{\hat{m}(\alpha_i)\alpha_i}-1)=x^{(1-\hat{m}(\alpha_i))\cdot\alpha_i}\cdot\left(1-v\cdot x^{\hat{m}(\alpha_i)\alpha_i}\right)$$

while (2.5.7) gives

$$x^{(1-\hat{m}(\alpha_i))\cdot\alpha_i}\cdot(1-v)+v\cdot(-1)\cdot x^{(1-\hat{m}(\alpha_i))\alpha_i}\cdot(x^{\hat{m}(\alpha_i)\alpha_i}-1)=x^{(1-\hat{m}(\alpha_i))\cdot\alpha_i}\cdot\left(1-v\cdot(x^{\hat{m}(\alpha_i)\alpha_i}\right).$$

These are exactly the same.

Thus we may indeed rewrite the definition of the group action as

$$\begin{aligned}\sigma_i(f)=\frac{\sigma_i.f}{1-vx^{\hat{m}(\alpha_i)\alpha_i}}\cdot\Bigg(&x^{-r_{\hat{m}(\alpha_i)}\left(-\frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\right)\cdot\alpha_i}\cdot(1-v)+\\&+v\cdot g_{\|\alpha_i\|^2-2\langle\lambda,\alpha_i\rangle}\cdot x^{(1-\hat{m}(\alpha_i))\alpha_i}\cdot(x^{\hat{m}(\alpha_i)\alpha_i}-1)\Bigg).\end{aligned} \tag{2.5.8}$$

2.5.3 Different quadratic forms

This is the last step of the translation from Chinta-Gunnells [12]. Instead of using the inner product $\langle .,.\rangle$ normalized such that short roots have norm one, we wish to write the group action in terms of the Weyl group invariant quadratic form Q and the bilinear form $B(x,y)=Q(x+y)-Q(x)-Q(y)$. Since any two W-invariant quadratic forms on $\Lambda\otimes\mathbb{R}$ differ by a scalar multiple, if α_0 is a short root, we have

$$Q(\lambda)=Q(\alpha_0)\cdot\|\lambda\|^2,\quad B(\lambda,\mu)=2Q(\alpha_0)\cdot\langle\lambda,\mu\rangle. \tag{2.5.9}$$

Now (2.1.3) defines $m(\alpha)$ for any root α as

$$m(\alpha) = n/\gcd(n, Q(\alpha)) = n/\gcd(n, Q(\alpha_0) \cdot \|\alpha\|^2).$$

To relate this to $\hat{m}(\alpha)$ defined using $\langle .,. \rangle$ above in (2.5.1), notice that

$$\gcd(n, Q(\alpha_i)) = \gcd(n, Q(\alpha_0) \cdot \|\alpha_i\|^2) = \gcd(n, Q(\alpha_0)) \cdot \gcd\left(\frac{n}{\gcd(n, Q(\alpha_0))}, \|\alpha\|^2\right)$$

hence

$$m(\alpha) = \frac{n}{\gcd(n, Q(\alpha_0))} \cdot \frac{1}{\gcd\left(\frac{n}{\gcd(n,Q(\alpha_0))}, \|\alpha\|^2\right)}.$$

This means that if we choose

$$n^{(0)} = \frac{n}{\gcd(n, Q(\alpha_0))}, \tag{2.5.10}$$

then we have $m(\alpha) = \hat{m}(\alpha)$ for any $\alpha \in \Phi$. By (2.5.9) we have

$$\frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \frac{B(\lambda, \alpha_i)}{Q(\alpha_i)}.$$

Furthermore,

$$\|\alpha_i\|^2 - 2\langle \lambda, \alpha_i \rangle = \frac{1}{Q(\alpha_0)} \cdot (Q(\alpha_i) - B(\lambda, \alpha_i)).$$

Thus if $g'_0, g'_1 \ldots, g'_{n^{(0)}-1}$ are the fixed Gauss-sums corresponding to $n^{(0)}$ with the choice in (2.5.10), and $g_0, g_1, \ldots g_{n-1}$ are as defined previously in Section 2.3, then

$$g'_{\|\alpha_i\|^2 - 2\langle \lambda, \alpha_i \rangle} = g_{Q(\alpha_i) - B(\lambda, \alpha_i)}.$$

Thus we may re-write (2.5.8) yet again as

$$\sigma_i(f) = \frac{\sigma_i.f}{1 - vx^{m(\alpha_i)\alpha_i}} \cdot \left(x^{-r_{m(\alpha_i)}\left(-\frac{B(\lambda,\alpha_i)}{Q(\alpha_i)}\right) \cdot \alpha_i} \cdot (1-v) + \right. \\ \left. + v \cdot g_{Q(\alpha_i) - B(\lambda, \alpha_i)} \cdot x^{(1-m(\alpha_i))\alpha_i} \cdot \left(x^{m(\alpha_i)\alpha_i} - 1\right) \right). \tag{2.5.11}$$

This agrees with Definition 2.2.1 exactly.

Remark 2.5.1. It follows from the relationship between the parameters $g'_0, g'_1 \ldots, g'_{n^{(0)}-1}$ and $g_0, g_1, \ldots g_{n-1}$ that the action given in Definition 2.2.1 is the metaplectic action corresponding to the cover of degree $n^{(0)} = \frac{n}{\gcd(n, Q(\alpha_0))}$ in the sense of Chinta-Gunnells [12].

Remark 2.5.2. In Chinta-Gunnells [12], Λ denotes the root lattice. The lattice Λ may be bigger, for example the weight lattice in Chapter 6. For the definition to make sense, all that is necessary is for $\dfrac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$ to be an integer. And this holds exactly when λ is an integer linear combination of the fundamental weights ω_i, since by definition

$$\frac{2\langle \omega_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}.$$

Furthermore, since $\langle \alpha_i, \alpha_i \rangle$ is an integer, this implies

$$||\alpha_i||^2 - 2\langle \lambda, \alpha_i \rangle$$

is also an integer.

Chapter 3

Crystals and Gelfand-Tsetlin patterns

Tokuyama's theorem (Theorem 3.5.2) is the immediate motivation for the results in Chapter 6, in particular Theorem 6.2.1. As preparation for Tokuyama's theorem, and for Chapter 6, the rest of this chapter is dedicated to discussing facts about type A highest weight crystals and Gelfand-Tsetlin patterns.

Gelfand-Tsetlin patterns can be parametrized in more than one way. A pattern is an array of integers, hence the entries themselves give a parametrization. Alternately, one may take a corresponding "Γ-array" (Definition 3.2.2). Gelfand-Tsetlin patterns with a fixed top row are in bijection with vertices of a highest weight crystal. The bijection is via Berenstein-Zelevinsky-Littelmann paths. We will see in this chapter that moving back and forth between the various parametrizations (patterns, Γ-arrays, BZL-paths and vertices of a crystal) is not particularly difficult. Hence one may choose the parametrization that is most convenient in a given context.

Crystals are introduced in Section 3.1. Gelfand-Tsetlin patterns, their Γ-arrays and some relevant notions are defined in Section 3.2. The bijection between Gelfand-Tsetlin patterns, vertices of a crystal, and Berenstein-Zelevinsky-Littelmann paths is explained in Section 3.3. A method of assigning coefficients to Gelfand-Tsetlin patterns is explained in Section 3.4. This has both a "nonmetaplectic" and a "metaplectic" version. The nonmetaplectic coefficients appear in the statement of Tokuyama's Theorem 3.5.2; we will make use of the metaplectic versions in Theorem 6.2.1, which provides a metaplectic analogue of Tokuyama's Theorem.

The source for most of this material is Chapter 2 of Brubaker-Bump-Friedberg [7]. Discussion of the background on crystal graphs, on Gelfand-Tsetlin patterns, and on Berenstein-Zelevinsky-Littelmann paths follow the presentation there closely, but in less detail. In some cases we give Brubaker-Bump-Friedberg [7] as a reference, but in doing so, implicitly rely on other sources. In particular, for the combinatorial definition of a crystal graph, we use Hong-Kang [18] and Kashiwara [20]. For the correspondence between Gelfand-Tsetlin patterns and highest weight crystals, Berenstein-Zelevinsky [2, 3], Littelmann [26], or Lusztig [27] are further references. Section 3.5 relies on Tokuyama [32].

3.1 Highest weight crystals

We present the general, combinatorial definition of a crystal according to Kashiwara [20], then immediately restrict our attention to type A highest weight crystals. We omit all discussion of the relationship between crystals and representations of $U_q(\mathfrak{g})$ (the quantized universal enveloping algebra of a Lie algebra) and simplify notation accordingly. (For example, we follow Brubaker-Bump-Friedberg [7] in denoting the Kashiwara operators by e_i and f_i, instead of the usual $\widetilde{e_i}$ and $\widetilde{f_i}$.)

Let P be a free $\mathbb{Z}$-module and I an index set. Let $\alpha_i \in P$ and $h_i \in P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ for $i \in I$. Let $(\cdot,\cdot) : P \times P \to \mathbb{Q}$ be a bilinear symmetric form, and let $\langle \cdot,\cdot \rangle : P^* \times P \to \mathbb{Z}$ denote the canonical pairing. Assume $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$, $\langle h_i, \lambda \rangle = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $i \in I$ and $\lambda \in P$, and $(\alpha_i, \alpha_j) \leq 0$ for $i, j \in I$, $i \neq j$.

A crystal $\mathcal{C}$ is a set B endowed with a weight function $\mathrm{wt} : B \to P$, functions $\varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}$, $\varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}$ and Kashiwara operators $e_i : B \to B \sqcup \{0\}$,

$f_i : B \to B \sqcup \{0\}$ for every $i \in I$. Elements of B are called elements or vertices of the crystal. A crystal satisfies the following axioms.

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for every $i \in I$

(ii) If $e_i(b) \neq 0$, then

$$\varepsilon_i(e_i(b)) = \varepsilon_i(b) - 1,$$

$$\varphi_i(e_i(b)) = \varphi_i(b) + 1,$$

$$\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i.$$

(iii) If $f_i(b) \neq 0$, then

$$\varepsilon_i(e_i(b)) = \varepsilon_i(b) + 1,$$

$$\varphi_i(e_i(b)) = \varphi_i(b) - 1,$$

$$\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) - \alpha_i.$$

(iv) For $b_1, b_2 \in B$, we have $b_2 = f_i(b_1)$ if and only if $b_1 = e_i(b_2)$.

(v) If $\varphi_i(b) = -\infty$, then $e_i(b) = f_i(b) = 0$.

(Let $-\infty + n = -\infty$ for every $n \in \mathbb{Z}$.)

We restrict our attention to type A_r crystals. Recall notation from Section 2.4; choose P to be the weight lattice Λ identified with $\mathbb{Z}^{r+1}$, and take $I = \{1, 2, \ldots, r\}$ as the index set. Let α_i be the simple roots, and define $(\cdot, \cdot)$ as B from 2.4. Let $h_i = \mathbf{e}_i^* - \mathbf{e}_{i+1}^*$ for $i \in I$ where, as before, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{r+1}$ denotes the standard basis of $\mathbb{R}^{r+1}$ and $\mathbf{e}_1^*, \mathbf{e}_2^*, \ldots, \mathbf{e}_{r+1}^*$ the standard dual basis. (We use $\mathbf{e}_i$ here to distinguish the basis vectors from the Kashiwara operators e_i.) Then all assumptions on the defining data are satisfied.

We next describe type A_r highest weight crystals. Recall that the weight

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1})$$

is dominant if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1}$; strongly dominant if $\lambda_1 > \lambda_2 > \cdots > \lambda_{r+1}$; λ is effective if $\lambda_{r+1} \geq 0$. There is a partial ordering on $\mathbb{Z}^{r+1}$ where $\mu \preccurlyeq \lambda$ if and only if $\lambda - \mu$ lies in the cone generated by simple roots.

For every dominant weight λ there is a corresponding crystal graph $\mathcal{C}_\lambda$ with highest weight λ. The function wt maps the vertices of $\mathcal{C}_\lambda$ to weights of the representation of $\mathfrak{gl}_{r+1}(\mathbb{C})$ with highest weight λ. The number of vertices with weight μ is equal to the multiplicity of the weight μ in the representation. Furthermore, the Kashiwara operators determine a directed graph structure on $\mathcal{C}_\lambda$. There is an edge $v \xrightarrow{i} w$ if and only if $f_i(v) = w \neq 0$. We say this edge is labeled with i.

The description in particular implies that $\mathcal{C}_\lambda$ has exactly one element $v_{highest}$ with weight λ (this is the "highest" element). If w_0 is the longest element of the type A_r Weyl group S_{r+1}, then $w_0\lambda = (\lambda_{r+1}, \lambda_r, \ldots, \lambda_2, \lambda_1)$, and $\mathcal{C}_\lambda$ has exactly one element v_{lowest} with weight $w_0\lambda$.

The edges labeled with the same $i \in I$ determine disjoint "i-strings" in the crystal. These are themselves isomorphic to type A_1 highest weight crystals. The functions ε_i and φ_i determine where a vertex is within an i-string. That is,

$$\varepsilon_i(b) = \max\{n \geq 0 |\ e_i^n b \neq 0\},$$

$$\varphi_i(b) = \max\{n \geq 0 |\ f_i^n b \neq 0\}.$$

The following fact about highest weight crystals will be of much use later, in particular in Section 6.3. When all edges labeled with r are removed from a crystal $\mathcal{C}_\lambda$, the resulting graph is a disjoint union of connected components. These connected components are themselves isomorphic to type A_{r-1} crystals. To get the weight function corresponding to these subcrystals, just restrict $\mathrm{wt} : \mathcal{C}_\lambda \to \mathbb{Z}^{r+1}$ to the appropriate subcrystal, and omit the last component of the result.

In section 3.3 we will explain how the vertices of a crystal $\mathcal{B}_\lambda$ can be parametrized by the set of Gelfand-Tsetlin patterns with top row λ, and how some of the edge structure can be recovered from the so-called Γ-array of these patterns, via Berenstein-Zelevinsky-Littelmann paths.

We conclude this section by an example.

Example 3.1.1. Figure 1 shows a crystal of type A_2 corresponding to highest weight $(3,1,0)$. Here the index set is $I=\{1,2\}$. The red edges correspond to the label 1, while the green edges correspond to the label 2. Figure 2 shows the image of the same crystal under the weight map.

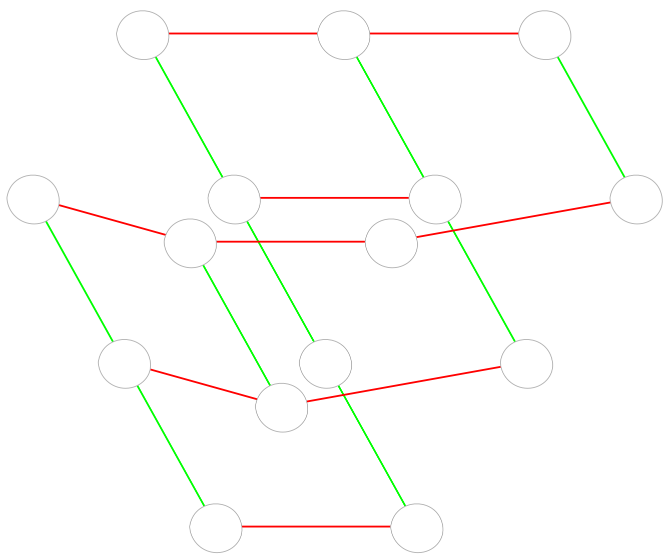

Figure 1: The crystal $\mathcal{C}_{(3,1,0)}$.

3.2 Gelfand-Tsetlin patterns

In this section, we define Gelfand-Tsetlin patterns, and introduce the Γ-array and the weight associated to a pattern.

Definition 3.2.1. A Gelfand-Tsetlin pattern of rank r and top row λ is an array of

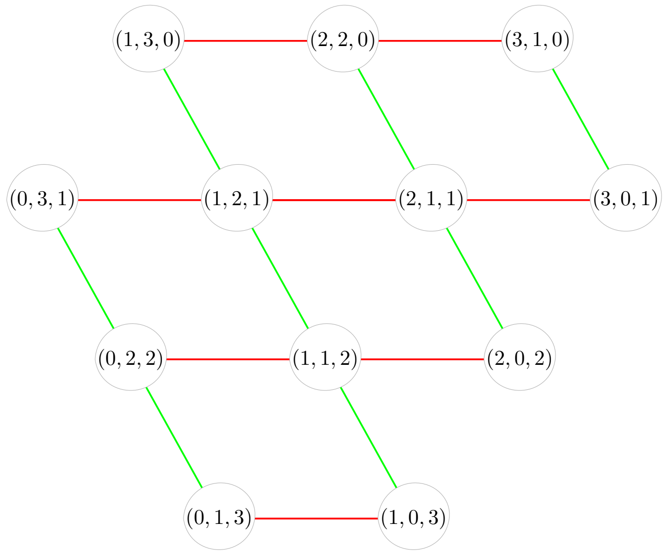

Figure 2: The weights of the $\mathfrak{gl}_3(\mathbb{C})$ representation of highest weight $(3,1,0)$.

nonnegative integers

$$\mathfrak{T} = \begin{pmatrix} a_{00} & & a_{01} & & a_{02} & \cdots & a_{0,r-1} & & a_{0r} \\ & a_{11} & & a_{12} & & \cdots & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{pmatrix}$$

where the top row is

$$\lambda = (a_{00}, a_{01}, \ldots, a_{0,r-1}, a_{0,r}) = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}),$$

and rows are non-increasing and interleave:

$$a_{i-1,j-1} \geq a_{ij} \geq a_{i-1,j}.$$

Definition 3.2.2. Let $\mathfrak{T}$ be a Gelfand-Tsetlin pattern as in Definition 3.2.1. For every $1 \leq i \leq r$, let

$$\Gamma_{ij} = \Gamma_{ij}(\mathfrak{T}) = \sum_{k=j}^{r} (a_{i,k} - a_{i-1,k}).$$

This gives the Γ-array of $\mathfrak{T}$

$$\Gamma(\mathfrak{T}) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1r} \\ & \Gamma_{22} & \dots & \Gamma_{2r} \\ & & \ddots & \vdots \\ & & & \Gamma_{1r} \end{bmatrix}.$$

The following remark will be of use in establishing the bijection between elements of a highest-weight crystal and Gelfand-Tsetlin patterns with a fixed top row.

Lemma 3.2.3. *Note that given the top row, the entries of the Gelfand-Tsetlin pattern* $\mathfrak{T}$ *can be recovered from the entries of* $\Gamma(\mathfrak{T})$. *That is, given* $a_{0,i}$ *and* $\Gamma_{i,j}$ *for* $1 \le i \le j \le r$, *one can compute each* $a_{i,j}$.

Proof. This is easy to see by induction on i and (for fixed i) by $j - i$, since

$$a_{i,r} = \Gamma_{i,r} + a_{i-1,r} \quad \text{and} \quad a_{i,j} = \Gamma_{i,j} - \Gamma_{i,j+1} + a_{i-1,j}.$$

□

Since the entries of the Gelfand-Tsetlin pattern $\mathfrak{T}$ satisfy $0 \le a_{i,k} - a_{i-1,k} \le a_{i-1,k-1} - a_{i-1,k}$, we have

$$0 \le \Gamma_{ir} \le a_{i-1,r-1} - a_{i-1,r}; \quad \forall i \le l \le r-1 \ \Gamma_{i,l+1} \le \Gamma_{i,l} \le \Gamma_{i,l+1} + a_{i-1,l-1} - a_{i-1,l}; \tag{3.2.1}$$

so the rows in $\Gamma(\mathfrak{T})$ are nonnegative, non-increasing and there is an upper bound on the difference of consecutive entries in a row. Whether these inequalities are strict or not influences the definition of Gelfand-Tsetlin coefficient assigned to a pattern $\mathfrak{T}$. We introduce some relevant terminology here.

Definition 3.2.4. Depending on whether the inequalities in equation 3.2.1 are strict or not, we may *decorate* the entries of $\Gamma(\mathfrak{T})$. Each entry may be *undecorated*, *circled*, *boxed*, or *both*. The table below shows the ("right-leaning") rules for decorating $\Gamma(\mathfrak{T})$. (If $j = r$, take $\Gamma_{i,r+1} = 0$.)

$\Gamma_{i,j+1} = \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}$	Γ_{ij} is circled	
$\Gamma_{i,j+1} < \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}$	Γ_{ij} is undecorated	(3.2.2)
$\Gamma_{i,j+1} < \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}$	Γ_{ij} is boxed	
$\Gamma_{i,j+1} = \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}$	Γ_{ij} is circled and boxed	

We can also phrase this as decorating the entries of the Gelfand-Tsetlin pattern $\mathfrak{T}$ itself. Only the entries below the top row are decorated.

$a_{i-1,j} = a_{ij} < a_{i-1,j-1}$	a_{ij} is circled	
$a_{i-1,j} < a_{ij} < a_{i-1,j-1}$	a_{ij} is undecorated	(3.2.3)
$a_{i-1,j} < a_{ij} = a_{i-1,j-1}$	a_{ij} is boxed	
$a_{i-1,j} = a_{ij} = a_{i-1,j-1}$	a_{ij} is circled and boxed	

Let d_i denote the sum of the entries in the i-th row of $\mathfrak{T}$, that is,

$$d_i = d_i(\mathfrak{T}) = \sum_{j=i}^{r} a_{ij}. \tag{3.2.4}$$

Then we may define the *weight* of a Gelfand-Tsetlin pattern $\mathfrak{T}$.

$$\mathrm{wt}(\mathfrak{T}) := (d_r, d_{r-1} - d_r, \ldots, d_0 - d_1) \tag{3.2.5}$$

In Section 3.3, we shall see that the bijection between vertices of $\mathcal{C}_\lambda$ and Gelfand-Tsetlin patterns with top row λ respects the weight function on both sets (Proposition 3.3.2).

We again conclude by an example.

Example 3.2.5. Consider Gelfand-Tsetlin patterns of top row $(3, 1, 0)$. One example of these is

$$\mathfrak{T} = \begin{pmatrix} 3 & & 1 & & 0 \\ & 3 & & 1 & \\ & & 2 & & \end{pmatrix}.$$

The corresponding Γ-array is

$$\Gamma(\mathfrak{T}) = \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}. \tag{3.2.6}$$

The sums of elements in the rows of the pattern $\mathfrak{T}$ are $d_0 = 4$, $d_1 = 4$ and $d_2 = 2$, hence

$$\mathrm{wt}(\mathfrak{T}) = (d_2, d_1 - d_2, d_0 - d_1) = (2, 2, 0).$$

3.3 Berenstein-Zelevinsky-Littelmann paths

A Berenstein-Zelevinsky-Littelmann path corresponding to an element v in the highest-weight crystal $\mathcal{C}_\lambda$ is a path in the graph theoretic sense. It starts from v and steps along the directed edges of the crystal. This corresponds to applying successive Kashiwara operators f_i to v. The choice of the indices of these Kashiwara operators will be dictated by the choice of a long word in the type A_r Weyl group $W \cong S_{r+1}$. This is explained in detail below. The notation continues to follow Brubaker-Bump-Friedberg [7]. We include an explicit type A_2 example after the general explanation.

Let

$$w_0 = \sigma_1\sigma_2\sigma_1 \cdots \sigma_{r-1} \cdots \sigma_1\sigma_r \cdots \sigma_1. \tag{3.3.1}$$

This is our reduced expression of choice for the longest element in S_{r+1} (our "favourite long word"). Let $1 \le \Omega_i \le r$ $(1 \le i \le N = \ell(w_0))$ be the indices so that

$$w_0 = \sigma_{\Omega_1}\sigma_{\Omega_2} \cdots \sigma_{\Omega_N}, \tag{3.3.2}$$

is the same reduced expression as in (3.3.1), i.e. $\Omega_1 = 1,\ \Omega_2 = 2,\ \Omega_3 = 1,\ \ldots,\ \Omega_N = 1$.

Let v be any element of the highest weight crystal $\mathcal{C}_\lambda$. The BZL path corresponding to v will be built as follows. Recall that for any element $w \in \mathcal{C}_\lambda$ and any $1 \le i \le r$, we may have either $f_i(w) \in \mathcal{C}_\lambda$, in which case $\mathrm{wt}(f_i(w)) = \mathrm{wt}(w) - \alpha_i$, or $f_i(w) = 0$. Let b_1 be a largest integer such that $f_{\Omega_1}^{b_1} v \neq 0$. Let $v_1 = f_{\Omega_1}^{b_1} v$, and similarly for $i = 2, \ldots, N$ let b_i be the largest integer such that $(v_i :=) f_{\Omega_i}^{b_i} v_{i-1} \neq 0$. We may write these integers into an array.

$$BZL(v) = BZL_\Omega(v) = \begin{bmatrix} b_{\binom{r}{2}+1} & b_{\binom{r}{2}+2} & \cdots & b_{\binom{r+1}{2}} \\ & b_{\binom{r-1}{2}+1} & \cdots & b_{\binom{r}{2}} \\ & \ddots & & \\ & & b_2 & b_3 \\ & & & b_1 \end{bmatrix} \tag{3.3.3}$$

Example 3.3.1. Let $r = 2$, and $\lambda = (3, 1, 0)$. We have $w_0 = \sigma_1\sigma_2\sigma_1$. Let $v = v_{(2,2,0)}$ be the single vertex of $\mathcal{C}_{(3,1,0)}$ with $\mathrm{wt}(v) = (2, 2, 0)$ (see Figure 1 and Figure 2). Then $b_1 = 1$, $b_2 = 3$ and $b_3 = 1$, and the BZL array of v is

$$BZL(v) = \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}. \tag{3.3.4}$$

Notice that this is the same as the Γ-array in (3.2.6). The BZL path corresponding to v is as shown on Figure 3.

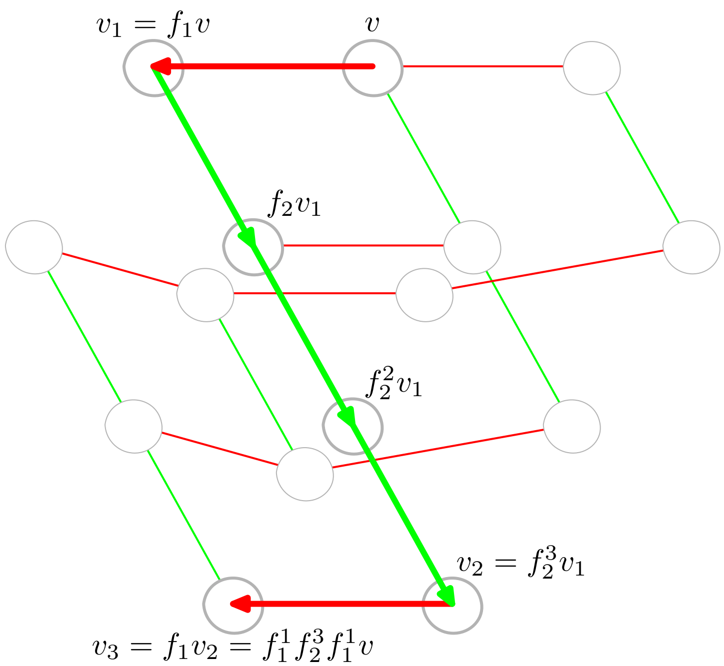

Figure 3: The Berenstein-Zelevinsky-Littelmann path of $v_{(2,2,0)} \in \mathcal{C}_{(3,1,0)}$.

We are now ready to state the correspondence between elements of a crystal and Gelfand-Tsetlin patterns of a fixed top row.

Proposition 3.3.2. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1})$ be a dominant weight, $\mathcal{C}_\lambda$ the crystal with highest weight λ.*

(i) For any $v \in \mathcal{C}_\lambda$ the BZL-path of v "ends" in the lowest element $v_{lowest} \in \mathcal{C}_\lambda$, i.e. $v_{\binom{r+1}{2}} = v_{lowest}$.

(ii) A vertex v can be recovered from $BZL(v)$.

(iii) For any $v \in \mathcal{C}_\lambda$ and $BZL(v) = (b_i)_{1 \le i \le \binom{r+1}{2}}$ as above, we have

$$\mathrm{wt}(v) - \mathrm{wt}(v_{lowest}) = \sum_{i=1}^{\binom{r+1}{2}} b_i \cdot \alpha_{\Omega_i}. \tag{3.3.5}$$

(iv) Elements of the crystal $\mathcal{C}_\lambda$ are in bijection with Gelfand-Tsetlin patterns with top row λ. The correspondence is given by assigning $\mathfrak{T}(v)$ to v if and only if

$$BZL(v) = \Gamma(\mathfrak{T}). \tag{3.3.6}$$

(v) With the correspondence as in (3.3.6), we have

$$\mathrm{wt}(v) = \mathrm{wt}(\mathfrak{T}(v)). \tag{3.3.7}$$

Proof. Parts of this proposition are proved throughout Chapter 2 of Brubaker-Bump-Friedberg [7]. In particular, Lemma 2.1 of [7] proves (i) and (ii); Proposition 2.3 of [7] proves (iii) and (v). The correspondence in (iv) is proved through correspondence with Young-tableaux. Some of the relevant proofs in [7] use Berenstein and Zelevinsky [2, 3], Kirillov and Berenstein [23], Littelmann [26] and Lusztig [27] as a reference. □

3.4 Gelfand-Tsetlin coefficients

In this section, we describe a way to assign various coefficients to any element of a highest-weight crystal $\mathcal{C}_\lambda$. We fix a positive integer n. The "nonmetaplectic" coefficients, corresponding to $n = 1$ will appear in Tokuyama's Theorem 3.5.2 below. The general version will appear in the statement of Theorem 6.2.1, which gives a crystal description of certain polynomials coming from Iwahori-Whittaker functions. The definitions of the "metaplectic" coefficients make use of the Gauss sums $g^\flat(a)$ and $h^\flat(a)$ defined in Section 2.3.

Recall that in Section 3.2 we defined decorations of a Gelfand-Tsetlin pattern $\mathfrak{T}$ and the corresponding array $\Gamma(\mathfrak{T})$ in (3.2.2) and (3.2.3). We now make use of these decorations.

As noted in Lemma 3.2.3, once the top row λ is fixed, a pattern $\mathfrak{T}$ can be recovered from $\Gamma(\mathfrak{T})$. Since many of the computations in what follows will involve a fixed top row and a fixed positive integer n, we will sometimes suppress λ and n from the notation. We shall have

$$G^{(n,\lambda)}(\mathfrak{T}) = G^{(n)}(\mathfrak{T}) = G(\mathfrak{T}) = G^{(n,\lambda)}(\Gamma(\mathfrak{T})) = G^{(\lambda)}(\Gamma(\mathfrak{T})) = G^{(n,\lambda)}(\Gamma) = G^{(\lambda)}(\Gamma) = G(\Gamma)$$

when we understand $\mathfrak{T}$ to be a Gelfand-Tsetlin pattern with top row λ. We sometimes may also write

$$G^{(n,\lambda)}(v) = G^{(\lambda)}(v) = G(v)$$

when $v \in \mathcal{C}_\lambda$ is the crystal element corresponding to $\mathfrak{T}$ according to Proposition 3.3.2.

Definition 3.4.1. Let $\mathfrak{T}$ be a Gelfand-Tsetlin pattern with top row λ, $\Gamma(\mathfrak{T}) = (\Gamma_{ij})_{1\leq i\leq j\leq r}$ the corresponding Γ-array as in Definition 3.2.2. Then we define the degree n Gelfand-Tsetlin coefficient corresponding to $\mathfrak{T}$ as

$$G^{(n)}(\mathfrak{T}) = \prod_{1\leq i\leq j\leq r} g_{ij}^n(\mathfrak{T}), \tag{3.4.1}$$

where $g_{ij}(\mathfrak{T}) = g_{ij}^{(n)}(\mathfrak{T})$ is given below.

$$g_{ij}(\mathfrak{T}) = \begin{cases} 1 & \Gamma_{i,j+1} = \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ h^\flat(\Gamma_{ij}) & \Gamma_{i,j+1} < \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ g^\flat(\Gamma_{ij}) & \Gamma_{i,j+1} < \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ 0 & \Gamma_{i,j+1} = \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \end{cases} \tag{3.4.2}$$

The coefficient depends strongly on n. To elucidate this, we give the examples of $n = 1$ and $n = 2$ explicitly below. Recall that $t^n = v = q^{-1}$, where q is the cardinality of a residue field $\mathfrak{o}_S/p\mathfrak{o}_S$ as explained in Section 2.3.

Example 3.4.2. In the nonmetaplectic case (i.e. when $n = 1$), the factors $g_{ij}^{(n)}(\mathfrak{T})$ of the Gelfand-Tsetlin coefficient $G^{(n)}(\mathfrak{T})$ are defined as follows.

$$g_{ij}^{(1)}(\mathfrak{T}) = \begin{cases} 1 & a_{i-1,j} = a_{ij} & a_{ij} \text{ is circled} \\ 1-t & a_{i-1,j} < a_{ij} < a_{i-1,j-1} & a_{ij} \text{ is undecorated} \\ -t & a_{i-1,j} < a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is boxed.} \\ 0 & a_{i-1,j} = a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is circled and boxed.} \end{cases} \tag{3.4.3}$$

We can also phrase this in terms of $\Gamma(\mathfrak{T})$.

$$g_{ij}^{(1)}(\mathfrak{T}) = \begin{cases} 1 & \Gamma_{i,j+1} = \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ 1-t & \Gamma_{i,j+1} < \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ -t & \Gamma_{i,j+1} < \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ 0 & \Gamma_{i,j+1} = \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \end{cases} \tag{3.4.4}$$

Using these explicit definitions, it is easy to compute the Gelfand-Tsetlin coefficient of the pattern in Example 3.2.5. Recall that the pattern corresponds to the single element of $\mathcal{C}_{(3,1,0)}$ of weight $(2,2,0)$.

$$\mathfrak{T}(v_{(2,2,0)}) = \begin{pmatrix} 3 & & 1 & & 0 \\ & 3 & & 1 & \\ & & 2 & & \end{pmatrix}, \quad \text{and} \quad \Gamma(\mathfrak{T}(v_{(2,2,0)})) = \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}.$$

Here a_{11} and a_{12} (or Γ_{11} and Γ_{12}) are boxed, while a_{22} (or Γ_{22}) is undecorated. Thus we have

$$G^{(1)}(\mathfrak{T}(v_{(2,2,0)})) = (-t)^2 \cdot (1-t).$$

It is also worth writing down the definition in the simplest metaplectic case, i.e. when $n = 2$.

Example 3.4.3. Let $n = 2$. Then the factors $g_{ij}^{(n)}(\mathfrak{T})$ of the Gelfand-Tsetlin coefficient $G^{(n)}(\mathfrak{T})$ are defined as follows.

$$g_{ij}^{(2)}(\Gamma) = \begin{cases} 1 & a_{i-1,j} = a_{ij} & a_{ij} \text{ is circled} \\ 1-t^2 & a_{i-1,j} < a_{ij} < a_{i-1,j-1}; 2 \mid \Gamma_{i,j} & a_{ij} \text{ is undecorated} \\ 0 & a_{i-1,j} < a_{ij} < a_{i-1,j-1}; 2 \nmid \Gamma_{i,j} & a_{ij} \text{ is undecorated} \\ -t^2 & a_{i-1,j} < a_{ij} = a_{i-1,j-1}; 2 \mid \Gamma_{i,j} & a_{ij} \text{ is boxed.} \\ t & a_{i-1,j} < a_{ij} = a_{i-1,j-1}; 2 \nmid \Gamma_{i,j} & a_{ij} \text{ is boxed.} \\ 0 & a_{i-1,j} = a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is circled and boxed.} \end{cases} \tag{3.4.5}$$

Again, we can express the coefficients in terms of how the inequalities of the $\Gamma_{i,j}$ are satisfied:

$$g_{ij}^{(2)}(\mathfrak{T}) = \begin{cases} 1 & \Gamma_{i,j+1} = \Gamma_{i,j} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ 1-t^2 & \Gamma_{i,j+1} < \Gamma_{i,j} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j};\ 2 \mid \Gamma_{i,j} \\ 0 & \Gamma_{i,j+1} < \Gamma_{i,j} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j};\ 2 \nmid \Gamma_{i,j} \\ -t^2 & \Gamma_{i,j+1} < \Gamma_{i,j} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j};\ 2 \mid \Gamma_{i,j} \\ t & \Gamma_{i,j+1} < \Gamma_{i,j} = \Gamma_{i,j+1} + \lambda_j - \lambda_{j+1} + 1;\ 2 \nmid \Gamma_{i,j} \end{cases} \tag{3.4.6}$$

Notice that the factors depend on the residue of Γ_{ij} modulo $n = 2$.

Returning to the example of $v_{(2,2,0)} \in \mathcal{C}_{(3,1,0)}$, we see that since $\Gamma_2 2 = 1$ is undecorated and odd,

$$G^{(2)}(\mathfrak{T}(v_{(2,2,0)})) = t^2 \cdot 0 = 0.$$

3.5 Tokuyama's Theorem

Tokuyama's theorem (Theorem 3.5.2 below) is a deformation of the Weyl Character formula (in type A). It also includes, as special cases, the Gelfand-Tsetlin parametrization of a basis of highest-weight representations of GL_{r+1}, and Stanley's formula about singular values of Hall-Littlewood polynomials and generating functions of strict Gelfand-Tsetlin patterns (Tokuyama [32]).

The statement, in its original form, relates a Schur function to a generating function of strict Gelfand patterns. This is easily rephrased (Proposition 3.5.4) to relate a sum over a Weyl group to a sum over a highest weight crystal. This second form is more convenient for the purposes of generalizing the theorem to the metaplectic setting.

In the sections above, we followed notation from Brubaker-Bump-Friedberg [7], because that is most convenient to use for metaplectic definitions of Gelfand-Tsetlin coefficients. The notation and approach in Tokuyama's paper [32] is slightly different. Here we review the notation and definitions from Tokuyama's paper and compare it to the notation above. We shall phrase Tokuyama's theorem using both and explain why the two versions are equivalent.

Let $\mathbf{x} = (x_1, \ldots, x_r, x_{r+1})$, $\mathbf{z} = (z_1, \ldots, z_r, z_{r+1})$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1})$ and let

$$\rho = (r, r-1, \ldots, 1, 0) \tag{3.5.1}$$

be the Weyl vector. Let $s_\lambda(\mathbf{x})$ (or $s_\lambda(\mathbf{z})$) denote the Schur function associated to the highest-weight representation of GL_{r+1} with highest weight λ. Recall that a Gelfand-Tsetlin pattern is an array of the form

$$\mathfrak{T} = \begin{pmatrix} a_{00} & & a_{01} & & a_{02} & \cdots & a_{0,r-1} & & a_{0r} \\ & a_{11} & & a_{12} & & \cdots & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{pmatrix}$$

where rows are non-increasing and interleave:

$$a_{i-1,j-1} \geq a_{ij} \geq a_{i-1,j}.$$

As in Tokuyama [32], we say a pattern $\mathfrak{T}$ is strict if $a_{i-1,j-1} > a_{i-1,j}$ holds for every $1 \leq i \leq j \leq r$. Following notation there, let $\mathrm{G}(\lambda)$ denote the set of Gelfand-Tsetlin patterns with top row λ, and let $\mathrm{SG}(\lambda)$ be the set of strict Gelfand-Tsetlin patterns with top row λ.

Remark 3.5.1. Note that according to the decoration defined in Definition 3.2.4 (in particular, in (3.2.3)), a Gelfand-Tsetlin pattern $\mathfrak{T}$ is strict if and only it has no entries that are both circled and boxed. Notice also that in every version of Gelfand-Tsetlin coefficients, an entry that is both circled and boxed corresponds to a factor of zero. This implies that summing over $\mathrm{G}(\lambda)$ is no different from summing over $\mathrm{SG}(\lambda)$ as long as the terms of the sum involve the Gelfand-Tsetlin coefficients defined in Section 3.4 above.

Recall that (3.2.5) defines the weight $\mathrm{wt}(\mathfrak{T})$ of a Gelfand-Tsetlin pattern $\mathfrak{T}$ as an element of $\mathbb{Z}^{r+1}$. If d_i is the sum of elements in the i-th row of $\mathfrak{T}$ as in (3.2.4), then

$$\mathrm{wt}(\mathfrak{T}) = (d_r, d_{r-1} - d_r, \ldots, d_0 - d_1). \tag{3.5.2}$$

In Tokuyama [32], we have

$$M(\mathfrak{T}) = (d_0 - d_1, d_1 - d_2, \ldots, d_{r-1} - d_r, d_r). \tag{3.5.3}$$

For a weight $\mu = (\mu_1, \mu_2, \ldots, \mu_{r+1})$ we may write

$$\mathbf{x}^\mu = x_1^{\mu_1} \cdot x_2^{\mu_2} \cdots x_{r+1}^{\mu_{r+1}}.$$

Recall that in 3.4.1 the (nonmetaplectic) Gelfand-Tsetlin coefficient $G(\mathfrak{T}) = G^{(1)}(\mathfrak{T})$ was defined as a product of factors $g_{ij}(\mathfrak{T})$ described in 3.4.3. Let us treat t as an indeterminate for the time being. Then the factor $g_{ij}(\mathfrak{T})$ corresponding to an entry is as follows.

$$g_{ij}(\mathfrak{T}) = \begin{cases} 1 & a_{i-1,j} = a_{ij} & a_{ij} \text{ is circled} \\ 1-t & a_{i-1,j} < a_{ij} < a_{i-1,j-1} & a_{ij} \text{ is undecorated} \\ -t & a_{i-1,j} < a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is boxed.} \\ 0 & a_{i-1,j} = a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is circled and boxed.} \end{cases}$$

In Tokuyama [32], the entry a_{ij} is called "special" if $a_{i-1,j} < a_{ij} < a_{i-1,j-1}$, that is, exactly if it is undecorated according to 3.2.3. The entry is called "lefty" if $a_{i-1,j} = a_{ij}$. This corresponds to boxed entries in our notation. (In particular for strict patterns, boxed entries that are not also circled, see Remark 3.5.1.)

With this notation we are ready to state Tokuyama's theorem in both the notation in Tokuyama [32] and in the notation introduced in previous sections.

Theorem 3.5.2. *(Tokuyama's theorem) Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}) \in \mathbb{Z}^{r+1}$ *where* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1} \geq 0$, *and* $\rho = (r, r-1, \ldots, 1, 0)$, $\mathrm{SG}(\lambda+\rho)$, *and* $M(\mathfrak{T})$ *as defined above. Let* $s(\mathfrak{T})$ *be the number of special entries of* $\mathfrak{T}$ *and* $l(\mathfrak{T})$ *the number of lefty entries.*

$$s_\lambda(\mathbf{z}) \cdot \prod_{1 \leq i < j < r+1} (z_i - t \cdot z_j) = \sum_{\mathfrak{T} \in \mathrm{SG}(\lambda+\rho)} (1-t)^{s(\mathfrak{T})} \cdot (-t)^{l(\mathfrak{T})} \cdot \mathbf{z}^{M(\mathfrak{T})}. \tag{3.5.4}$$

In the notation introduced in previous sections of this chapter, this can be re-written as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \sum_{\mathfrak{T} \in \mathrm{G}(\lambda+\rho)} G(\mathfrak{T}) \cdot \mathbf{x}^{\mathrm{wt}(\mathfrak{T})}. \tag{3.5.5}$$

The first form of the equation, (3.5.4) is Theorem 2.1 of [32], substituting $-t$ for t. We explain why (3.5.5) is equivalent. Notice that (thinking of t as an indeterminate)

$$G(\mathfrak{T}) = (1-t)^{s(\mathfrak{T})} \cdot (-t)^{l(\mathfrak{T})}$$

holds for any strict Gelfand-Tsetlin pattern. (This follows from (3.4.3) and that "special" entries are undecorated, and "lefty" entries are boxed as explained above.) Furthermore, by Remark 3.5.1 we have

$$G(\mathfrak{T}) = 0 \quad \text{if } \mathfrak{T} \in \mathrm{G}(\lambda+\rho) \setminus \mathrm{SG}(\lambda+\rho).$$

From (3.5.2) and (3.5.3) we see that the components of $\mathrm{wt}(\mathfrak{T})$ are exactly the components of $M(\mathfrak{T})$ in reverse order. So if we write

$$x_1 = z_{r+1}, \; x_2 = z_r, \; \ldots, \; x_r = z_2, \; x_{r+1} = z_1 \tag{3.5.6}$$

we have

$$\mathbf{x}^{\mathrm{wt}(\mathfrak{T})} = \mathbf{z}^{M(\mathfrak{T})}.$$

From this it is clear that with the choice as in (3.5.6) the right hand sides of (3.5.4) and (3.5.5) agree. It remains to check that the left hand sides agree as well. Notice that with the choice as in (3.5.6), we have $s_\lambda(\mathbf{x}) = s_\lambda(\mathbf{z})$. This is clear, for example, from the Weyl Character Formula for Schur functions:

$$s_\lambda(\mathbf{x}) = \frac{\begin{vmatrix} x_1^{\lambda_1+r} & x_2^{\lambda_1+r} & \cdots & x_{r+1}^{\lambda_1+r} \\ x_1^{\lambda_2+r-1} & x_2^{\lambda_2+r-1} & \cdots & x_{r+1}^{\lambda_2+r-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_{r+1}} & x_2^{\lambda_{r+1}} & \cdots & x_{r+1}^{\lambda_{r+1}} \end{vmatrix}}{\begin{vmatrix} x_1^{r} & x_2^{r} & \cdots & x_{r+1}^{r} \\ x_1^{r-1} & x_2^{r-1} & \cdots & x_{r+1}^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{0} & x_2^{0} & \cdots & x_{r+1}^{0} \end{vmatrix}}. \tag{3.5.7}$$

Finally, (3.5.6) implies

$$\prod_{1\le i<j<r+1} (z_i - t\cdot z_j) = \prod_{1\le i<j<r+1} (x_j - t\cdot x_i).$$

In the remainder of this section, we focus our attention in reformulating Theorem 3.5.2 in terms of a sum over the Weyl group and a sum over a highest-weight crystal. This is done separately for the two sides.

3.5.1 The left hand side of Tokuyama's theorem as a Weyl group sum

Recall notation relevant to type A root systems in Section 2.4. This notation and the Weyl Character Formula for Schur functions (3.5.7) will allow us to rewrite the left hand side of (3.5.5) as a sum over the Weyl group. Recall that in type A_r, the Weyl group

$W \cong S_{r+1}$ acts on $\Lambda = \mathbb{Z}^{r+1}$ by permuting the coordinates. Thus, if w_0 denotes the long element of the Weyl group, we have

$$\prod_{1 \le i < j < r+1} (x_j - t \cdot x_i) = \prod_{1 \le i < j < r+1} x_j \cdot (1 - t \cdot \frac{x_i}{x_j}) \tag{3.5.8}$$

$$= x_2 \cdot x_3^2 \cdots x_r^{r-1} \cdot x_{r+1}^r \cdot \prod_{1 \le i < j < r+1} (1 - t \cdot \frac{x_i}{x_j}) \tag{3.5.9}$$

$$= \mathbf{x}^{w_0 \cdot \rho} \cdot \prod_{\alpha \in \Phi^+} (1 - t \cdot \mathbf{x}^\alpha). \tag{3.5.10}$$

Now by (3.5.7) we have

$$s_\lambda(\mathbf{x}) = \frac{\sum_{w \in W} \operatorname{sgn}(w) \cdot w.(\mathbf{x}^{\lambda+\rho})}{\prod_{1 \le i < j < r+1} (x_i - x_j)}$$

where $\operatorname{sgn}(w) = (-1)^{\ell(w)}$, and $\ell(w)$ is the number of simple reflections in a reduced expression of w. This can be rewritten as

$$s_\lambda(\mathbf{x}) = \frac{\sum_{w \in W} \operatorname{sgn}(w) \cdot w.(\mathbf{x}^{\lambda+\rho})}{\mathbf{x}^{w_0 \rho} \cdot \operatorname{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} (1 - \mathbf{x}^\alpha)}.$$

Thus the left hand side of (3.5.5) can be rewritten as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \le i < j < r+1} (x_j - t \cdot x_i) = \frac{\prod_{\alpha \in \Phi^+} (1 - t \cdot \mathbf{x}^\alpha)}{\prod_{\alpha \in \Phi^+} (1 - \mathbf{x}^\alpha)} \cdot \operatorname{sgn}(w_0) \cdot \sum_{w \in W} \operatorname{sgn}(w) \cdot w.(\mathbf{x}^{\lambda+\rho}).$$

To simplify notation, we introduce the following shorthand for the Weyl denominator and its deformation:

$$\Delta_t = {\Delta_t}^{(1)} = \prod_{\alpha \in \Phi^+} \left(1 - t \cdot x^\alpha\right). \tag{3.5.11}$$

We write Δ for Δ_t when $t = 1$. With this notation we have

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \le i < j < r+1} (x_j - t \cdot x_i) = \frac{\Delta_t}{\Delta} \cdot \operatorname{sgn}(w_0) \cdot \sum_{w \in W} \operatorname{sgn}(w) \cdot w.(\mathbf{x}^{\lambda+\rho}). \tag{3.5.12}$$

This form in (3.5.12) produces the left-hand side of Tokuyama's theorem as the result of an operator,

$$\frac{\Delta_t}{\Delta} \cdot \operatorname{sgn}(w_0) \cdot \left(\sum_{w \in W} \operatorname{sgn}(w) \cdot w \right)$$

acting on a monomial $\mathbf{x}^{\lambda+\rho}$. In Chapter 4 we will concern ourselves with identities of operators. These operators act on a ring of rational functions, and the operators themselves are linear combinations of elements of the Weyl group, or rational functions acting by multiplication. (For example, in 3.5.12 the operator "multiplication by $\mathbf{x}^{\rho}$" is composed with a linear combination of Weyl group elements. All are acting on $\mathbb{C}(\mathbf{x})$ in the usual nonmetaplectic way.)

As a consequence of Theorem 4.2.1 and Theorem 4.2.2, we will be able to produce the polynomial in 3.5.12 (and its metaplectic analogues) by acting with Demazure or Demazure-Lusztig operators on a monomial. The monomial will be $\mathbf{x}^{w_0\lambda}$. To facilitate that, we modify (3.5.12) a little further. Substituting ww_0 for w we may write (3.5.12) as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \le i < j < r+1} (x_j - t \cdot x_i) = \frac{\Delta_t}{\Delta} \cdot \sum_{w \in W} \mathrm{sgn}(w) \cdot w.(\mathbf{x}^{w_0(\lambda+\rho)})$$

Now, to write this as a linear combination of Weyl group elements acting on $\mathbf{x}^{w_0\lambda}$, notice that as operators,

$$w \cdot \mathbf{x}^{w_0\rho} = (w.\mathbf{x}^{w_0\rho}) \cdot w = \mathbf{x}^{ww_0\rho} \cdot w.$$

Recall the notation $\Phi(w) = \{\alpha \in \Phi^+ : w(\alpha) \in \Phi^-\}$ from (2.1.1). Since

$$\Phi(w^{-1}) = w(\Phi^-) \cap \Phi^+ = ww_0(\Phi^+) \cap \Phi^+,$$

we have

$$ww_0\rho - w_0\rho = \frac{1}{2} \cdot \sum_{\alpha \in ww_0(\Phi^+)} \alpha - \frac{1}{2} \cdot \sum_{\alpha \in \Phi^-} \alpha = \sum_{\alpha \in ww_0(\Phi^+) \cap \Phi^+} \alpha = \sum_{\alpha \in \Phi(w^{-1})} \alpha,$$

whence

$$w \cdot \mathbf{x}^{w_0\rho} = \mathbf{x}^{w_0\rho} \cdot \mathbf{x}^{ww_0\rho - w_0\rho} \cdot w = \mathbf{x}^{w_0\rho} \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{\alpha} \cdot w.$$

Thus we may rewrite (3.5.12) as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \le i < j < r+1} (x_j - t \cdot x_i) = \mathbf{x}^{w_0\rho} \cdot \frac{\Delta_t}{\Delta} \cdot \sum_{w \in W} \left(\mathrm{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{\alpha} \right) \cdot w.(\mathbf{x}^{w_0(\lambda)}). \quad (3.5.13)$$

Remark 3.5.3. In this chapter, we defined "the Weyl vector" as $\rho = (r, r-1, \ldots, 1, 0)$. As a result, we have

$$\mathbf{x}^{\rho} = (x_1 \cdot x_2 \cdots x_{r+1})^r \cdot \prod_{\alpha \in \Phi^+} \mathbf{x}^{\frac{1}{2}\alpha}.$$

In the computations above, the factor $(x_1 \cdot x_2 \cdots x_{r+1})^r$ was never written out explicitly, but all of the equations hold as written. The factor $x_1 \cdot x_2 \cdots x_{r+1}$ is symmetric under W, so as an operator, it commutes with any element of the Weyl group.

3.5.2 The right hand side of Tokuyama's theorem as a sum over a crystal

Now we turn to the right-hand side of Tokuyama's theorem. The correspondence between elements of a crystal of highest weight $\lambda + \rho$ and Gelfand-Tsetlin patterns of top row $\lambda + \rho$ was established in Section 3.3, in particular, in Proposition 3.3.2. Following this parametrization, we may write $G(v) = G^{(1)}(v) := G^{(1)}(\mathfrak{T}(v))$ where $\mathfrak{T}(v)$ is the Gelfand-Tsetlin pattern corresponding to $v \in \mathcal{C}_{\lambda+\rho}$. We also have $\mathrm{wt}(\mathfrak{T}(v)) = \mathrm{wt}(v)$ by Proposition 3.3.2. Thus we may rewrite the right hand side of (3.5.5) as

$$\sum_{\mathfrak{T} \in \mathrm{G}(\lambda+\rho)} G(\mathfrak{T}) \cdot \mathbf{x}^{\mathrm{wt}(\mathfrak{T})} = \sum_{v \in \mathcal{C}_{\lambda+\rho}} G(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{3.5.14}$$

3.5.3 The crystal version of Tokuyama's theorem

The following "crystal version" of Tokuyama's theorem is a direct consequence of (3.5.5), (3.5.13) and (3.5.14).

Proposition 3.5.4. *Let λ be a dominant, effective weight. Then we have*

$$\mathbf{x}^{w_0\rho} \cdot \frac{\Delta_t}{\Delta} \cdot \sum_{w \in W} \left(\mathrm{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{\alpha} \right) \cdot w.(\mathbf{x}^{w_0(\lambda)}) = \sum_{v \in \mathcal{C}_{\lambda+\rho}} G(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{3.5.15}$$

Proposition 3.5.4 is convenient to generalize, and we shall return to it in Section 4.2.1 and again in Chapter 6. We illustrate it by returning to the example of $\mathcal{C}_{(3,1,0)}$.

Example 3.5.5. Let $r = 2$, and take $\lambda = (1, 0, 0)$. Then $\rho = (2, 1, 0)$ and $\lambda + \rho = (3, 1, 0)$. We have $\mathbf{x}^{w_0(\rho)} = x_2 x_3^2$, $\mathbf{x}^{w_0(\lambda)} = x_3$, $W = S_3 = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$ and

$$\frac{\Delta_t}{\Delta} = \frac{\left(1 - t \cdot \frac{x_1}{x_2}\right)\left(1 - t \cdot \frac{x_2}{x_3}\right)\left(1 - t \cdot \frac{x_1}{x_3}\right)}{\left(1 - \frac{x_1}{x_2}\right)\left(1 - \frac{x_2}{x_3}\right)\left(1 - \frac{x_1}{x_3}\right)}$$

Recall that

$$\mathbf{x}^{w_0\rho} \cdot \left(\operatorname{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{\alpha}\right) \cdot w.(\mathbf{x}^{w_0(\lambda)}) = w.(\mathbf{x}^{w_0(\lambda+\rho)}).$$

Thus the left hand side of (3.5.15) is

$$\frac{(x_2 - t \cdot x_1)(x_3 - t \cdot x_2)(x_3 - t \cdot x_1)}{(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)} \cdot \left(x_2 x_3^3 - x_1 x_3^3 - x_2^3 x_3 + x_1^3 x_3 + x_1 x_2^3 - x_1^3 x_2\right).$$

The right hand side can be read off from Figure 4 below.

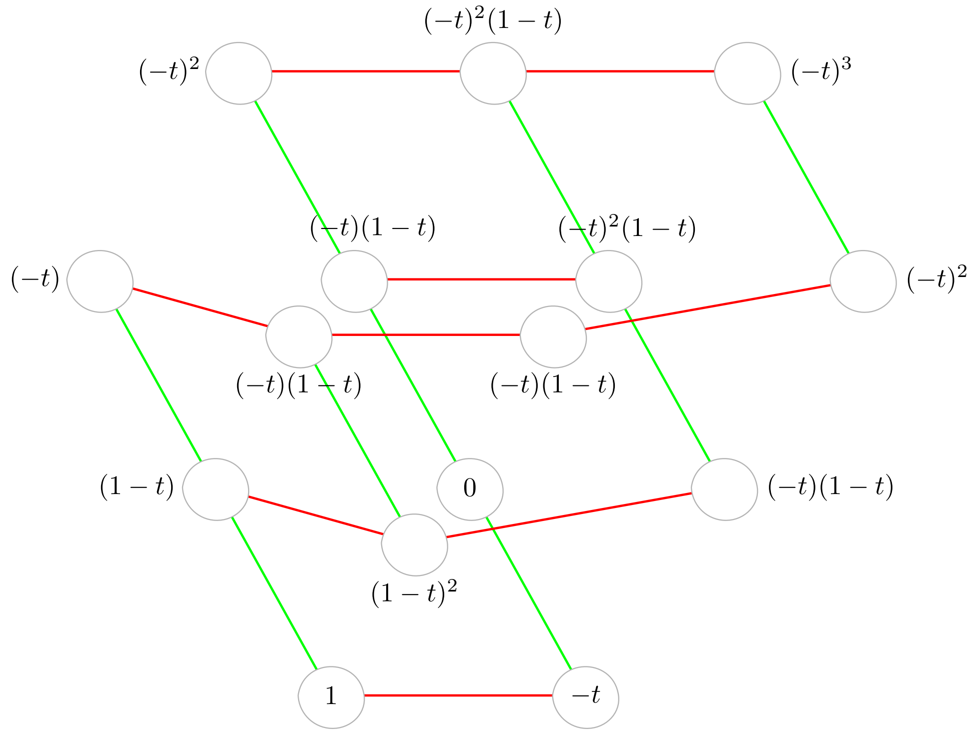

Figure 4: The right hand side of Tokuyama's theorem, on $\mathcal{C}_{(3,1,0)}$.

Chapter 4

Demazure and Demazure-Lusztig Operators

Metaplectic Demazure and Demazure-Lusztig operators are our primary tool in phrasing and proving a metaplectic analogue of Tokuyama's theorem, and in describing Iwahori-Whittaker type polynomials in terms of highest weight crystals in Chapter 6. This chapter introduces the metaplectic operators in Section 4.1 and discusses a few elementary properties. The main results of this chapter are the Demazure Formula (Theorem 4.2.1) and the Demazure-Lusztig Formula (Theorem 4.2.2). These are stated in Section 4.2. Section 4.2 also relates the definitions, elementary properties and statements of Theorems 4.2.1 and 4.2.2 to classical operators by reviewing relevant results in the literature; in particular the Demazure character formula (Fulton [17], Brubaker-Bump-Licata [8]). Theorem 4.2.1 and Theorem 4.2.2 are proven in Section 4.3 and Section 4.4 respectively.

4.1 Definitions of Metaplectic operators

The definitions below make use of the Chinta-Gunnells action introduced in Chapter 2. Both the Demazure operators and the Demazure-Lusztig operators are divided difference operators on $\mathcal{K}$.

For $1 \leq i \leq r$ and $f \in \mathcal{K}$ define the *Demazure operators* by

$$\mathcal{D}_i(f) = \mathcal{D}_{\sigma_i}(f) = \frac{f - x^{m(\alpha_i)\alpha_i} \cdot \sigma_i(f)}{1 - x^{m(\alpha_i)\alpha_i}}, \tag{4.1.1}$$

and the *Demazure-Lusztig operators* by

$$\begin{aligned} \mathcal{T}_i(f) = \mathcal{T}_{\sigma_i}(f) &= \left(1 - v \cdot x^{m(\alpha_i)\alpha_i}\right) \cdot \mathcal{D}_i(f) - f \\ &= \left(1 - v \cdot x^{m(\alpha_i)\alpha_i}\right) \cdot \frac{f - x^{m(\alpha_i)\alpha_i} \cdot \sigma_i(f)}{1 - x^{m(\alpha_i)\alpha_i}} - f. \end{aligned} \tag{4.1.2}$$

When there is no danger of confusion, we write more simply

$$\mathcal{D}_i = \frac{1 - x^{m(\alpha_i)\alpha_i}\sigma_i}{1 - x^{m(\alpha_i)\alpha_i}} \qquad \text{and} \qquad \mathcal{T}_i = \left(1 - v \cdot x^{m(\alpha_i)\alpha_i}\right) \cdot \mathcal{D}_i - 1,$$

that is, a rational function h in the above equations is interpreted to mean the "multiplication by h" operator. The rational functions here are in $\mathcal{K}_0$.

We prove in 4.1.2 below that the operators $\mathcal{D}_i$ and $\mathcal{T}_i$ satisfy the same braid relations as the σ_i. Consequently, we can define $\mathcal{D}_w$ and $\mathcal{T}_w$ for any $w \in W$ as follows. Let $w = \sigma_{i_1} \cdots \sigma_{i_l}$ be a reduced expression for w in terms of simple reflections. Then we define

$$\mathcal{D}_w = \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_l} \qquad \text{and} \qquad \mathcal{T}_w = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_l}.$$

In this section we prove the *quadratic relations* (Proposition 4.1.1) and *braid relations* (Proposition 4.1.2) satisfied by the Demazure and Demazure-Lusztig operators.

Proposition 4.1.1. *The operators $\mathcal{D}_i$ and $\mathcal{T}_i$ ($1 \leq i \leq r$) satisfy the following quadratic relations:*

(i) $\mathcal{D}_i^2 = \mathcal{D}_i$;

(ii) $\mathcal{T}_i^2 = (v-1)\mathcal{T}_i + v$.

In addition, we have

$$\mathcal{D}_i x^{m(\alpha_i)\alpha_i} \mathcal{D}_i = -\mathcal{D}_i. \tag{4.1.3}$$

Proof. To simplify the notation, we drop the subscripts and write $\mathcal{D}, \alpha$, and σ, and abbreviate $m(\alpha)$ to m. We start by proving (i). Using the definition of $\mathcal{D}$ and Lemma 2.2.2, we have

$$\begin{aligned}\mathcal{D}^2 &= \Big(\frac{1-x^{m\alpha}\sigma}{1-x^{m\alpha}}\Big)^2 \\ &= \Big(\frac{1}{(1-x^{m\alpha})^2} + \frac{x^{m\alpha}}{1-x^{m\alpha}} \cdot \frac{x^{-m\alpha}}{1-x^{-m\alpha}}\Big) \cdot 1 \\ &\qquad + \Big(\frac{-x^{m\alpha}}{(1-x^{m\alpha})^2} + \frac{-x^{m\alpha}}{1-x^{m\alpha}} \cdot \frac{1}{1-x^{-m\alpha}}\Big) \cdot \sigma \\ &= \frac{1}{1-x^{m\alpha}} \cdot \Big(\frac{1}{1-x^{m\alpha}} + \frac{1}{1-x^{-m\alpha}}\Big) \cdot 1 \\ &\qquad + \frac{-x^{m\alpha}}{1-x^{m\alpha}} \cdot \Big(\frac{1}{1-x^{m\alpha}} + \frac{1}{1-x^{-m\alpha}}\Big) \cdot \sigma.\end{aligned}$$

Since

$$\frac{1}{1-x^{m\alpha}} + \frac{1}{1-x^{-m\alpha}} = 1$$

we obtain $\mathcal{D}^2 = \mathcal{D}$. This completes the proof of (i). We continue with the proof of (4.1.3), by direct computation.

$$\begin{aligned}\mathcal{D}x^{m\alpha}\mathcal{D} &= \mathcal{D}\left(\frac{x^{m\alpha} - x^{2m\alpha}\sigma}{1-x^{m\alpha}}\right) \\ &= \frac{x^{m\alpha} - x^{2m\alpha}\sigma}{(1-x^{m\alpha})^2} - \frac{\sigma - x^{-m\alpha}}{(1-x^{m\alpha})(1-x^{-m\alpha})} \\ &= \frac{x^{m\alpha}\sigma - 1}{1-x^{m\alpha}} \\ &= -\mathcal{D}.\end{aligned}$$

Now we are ready to prove (ii). First notice that proving $\mathcal{T}^2 = (v-1)\mathcal{T} + v$ is equivalent to proving

$$(\mathcal{T}+1)^2 = \mathcal{T}^2 + 2\mathcal{T} + 1 = (1+v)\cdot(\mathcal{T}+1).$$

And we may compute $(\mathcal{T}+1)^2$ using $\mathcal{D}^2 = \mathcal{D}$ and (4.1.3) above.

$$\begin{aligned}(\mathcal{T}+1)^2 &= (1 - v\cdot x^{m\alpha})\cdot\mathcal{D}\cdot(1 - v\cdot x^{m\alpha})\cdot\mathcal{D}\\ &= (1 - v\cdot x^{m\alpha})\cdot\left(\mathcal{D}^2 - v\cdot\mathcal{D}\cdot x^{m\alpha}\cdot\mathcal{D}\right)\\ &= (1 - v\cdot x^{m\alpha})\cdot(\mathcal{D} - v\cdot(-\mathcal{D}))\\ &= (1+v)\cdot(1 - v\cdot x^{m\alpha})\cdot\mathcal{D} = (1+v)\cdot(\mathcal{T}+1).\end{aligned}$$

□

We pause to point out the key role played by Lemma 2.2.2 in the proof of Proposition 4.1.1: the action of σ_i on an arbitrary rational function is given by the complicated formula (2.2.1), but thanks to Lemma 2.2.2 we can pass the operator σ past the monomial $x^{m(\alpha)\alpha}$, after acting on this monomial by the usual permutation action. This fact will be used repeatedly.

Proposition 4.1.2. *Suppose $(\sigma_i\sigma_j)^{m_{i,j}} = 1$ is a defining relation for W. Then*

$$\mathcal{D}_i\mathcal{D}_j\mathcal{D}_i\cdots = \mathcal{D}_j\mathcal{D}_i\mathcal{D}_j\cdots, \tag{4.1.4}$$

$$\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i\cdots = \mathcal{T}_j\mathcal{T}_i\mathcal{T}_j\cdots, \tag{4.1.5}$$

where there are $m_{i,j}$ factors on both sides of (4.1.4)–(4.1.5).

Proof. Both statements boil down to explicit computations with rank 2 root systems, and in fact are special cases of Theorems 4.2.1 and 4.2.2. We explain what happens in detail with (4.1.4) in A_2, which is typical of all the computations. Since all roots have the same length, we lighten notation by putting $m = m(\alpha)$.

By definition,

$$\mathcal{D}_1 = \frac{1 - x^{m\alpha_1}\sigma_1}{1 - x^{m\alpha_1}}.$$

Next we apply $\mathcal{D}_2$ and use $\sigma_2(\alpha_1) = \alpha_1 + \alpha_2$:

$$\mathcal{D}_2\mathcal{D}_1 = \frac{1 - x^{m\alpha_1}\sigma_1}{(1 - x^{m\alpha_2})(1 - x^{m\alpha_1})} - \frac{x^{m\alpha_2}\sigma_2 - x^{m(\alpha_1+2\alpha_2)}\sigma_2\sigma_1}{(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1+\alpha_2)})}$$

Finally we apply $\mathcal{D}_1$ to obtain

$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_1 = \frac{1 - x^{m\alpha_1}\sigma_1}{(1 - x^{m\alpha_2})(1 - x^{m\alpha_1})^2} - \frac{x^{m\alpha_2}\sigma_2 - x^{m(\alpha_1+2\alpha_2)}\sigma_2\sigma_1}{(1 - x^{m\alpha_1})(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1+\alpha_2)})}$$
$$- \left(\frac{x^{m\alpha_1}\sigma_1 - 1}{(1 - x^{m\alpha_1})(1 - x^{m(\alpha_1+\alpha_2)})(1 - x^{-m\alpha_1})} - \frac{x^{m(2\alpha_1+\alpha_2)}\sigma_1\sigma_2 - x^{m(2\alpha_1+2\alpha_2)}\sigma_1\sigma_2\sigma_1}{(1 - x^{m\alpha_1})(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1+\alpha_2)})} \right),$$

which simplifies to

$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_1 = \frac{1 - x^{m\alpha_1}\sigma_1 - x^{m\alpha_2}\sigma_2 + x^{m(2\alpha_1+\alpha_2)}\sigma_1\sigma_2 + x^{m(\alpha_1+2\alpha_2)}\sigma_2\sigma_1 - x^{m(2\alpha_1+2\alpha_2)}\sigma_1\sigma_2\sigma_1}{\Delta}, \tag{4.1.6}$$

where $\Delta = (1 - x^{m\alpha_1})(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1+\alpha_2)})$. The final formula (4.1.6) clearly depends only on the longest word in the Weyl group for A_2 and not on the reduced expression used to define it, which proves (4.1.4). □

The computation above checks the Demazure formula (Theorem 4.2.1) in the special case $\Phi = A_2$. The product Δ is a metaplectic analogue of the Weyl denominator Δ, as introduced in (3.5.11). We extend this notation to the metaplectic analogue of Δ_v. Let

$$\Delta_v = \Delta_v{}^{(n)} = \prod_{\alpha\in\Phi^+} \left(1 - v \cdot x^{m(\alpha)\alpha}\right). \tag{4.1.7}$$

If $v = 1$ we again write simply $\Delta_v = \Delta$.

4.2 Demazure and Demazure-Lusztig formulas

Now we are ready to state the two main theorems of this chapter. In these theorems, both sides of the equalities are to be understood as identities of operators on $\mathcal{K}$.

Theorem 4.2.1. *For the long element w_0 of the Weyl group W we have*

$$\mathcal{D}_{w_0} = \frac{1}{\Delta} \cdot \sum_{w\in W} \operatorname{sgn}(w) \cdot \prod_{\alpha\in\Phi(w^{-1})} x^{m(\alpha)\alpha} \cdot w.$$

Theorem 4.2.2. *We have*

$$\Delta_v \cdot \mathcal{D}_{w_0} = \sum_{w\in W} \mathcal{T}_w.$$

We prove Theorem 4.2.1 in Section 4.3 and Theorem 4.2.2 in Section 4.4.

To provide motivation for the two theorems above, we explain how they relate to previous results about Demazure and Demazure-Lusztig operators in the nonmetaplectic setting. This has two components.

First, Theorem 4.2.1 is a generalization of the Demazure Character Formula (see Fulton [17]) to arbitrary type of root systems and arbitrary metaplectic degree n. We make this connection explicit in Section 4.2.1. The same section also clarifies how the two theorems above relate Tokuyama's theorem (Proposition 3.5.4) to Theorem 6.2.1 in the next chapter.

Second, it is worth noting that the properties of metaplectic Demazure and Demazure-Lusztig operators are analogous to properties of their nonmetaplectic counterparts. As a reference for this we cite Brubaker-Bump-Licata [8]. We also recall the definitions of Demazure operators on crystals from Kashiwara [21]. These remarks on the classical definitions are presented in Section 4.2.2.

4.2.1 Tokuyama's theorem and the Demazure Character Formula

In this section, we restrict our attention to type A_r and $n = 1$, and thus make use of notation introduced in Section 2.4. The restriction $n = 1$ means in particular that the Weyl group $W \cong S_{r+1}$ acts on $\mathbb{C}(\mathbf{x}) = \mathbb{C}(\Lambda)$ by permuting the variables.

It follows from Theorem 4.2.1 and Theorem 4.2.2 that the left hand side of Tokuyama's theorem can be rewritten in terms of Demazure and Demazure-Lusztig operators. We make this explicit here.

Recall that the left hand side of Tokuyama's theorem may be rewritten to involve a sum over the Weyl group as in (3.5.13):

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \le i < j < r+1} (x_j - v \cdot x_i) = \mathbf{x}^{w_0 \rho} \cdot \frac{\Delta_v}{\Delta} \cdot \sum_{w \in W} \left(\operatorname{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{\alpha} \right) \cdot w.(\mathbf{x}^{w_0(\lambda)}).$$

Since $n = 1$, we have $t = t^1 = v = q^{-1}$. (Here we are only concerned with the left hand side of Tokuyama's theorem, hence v continues to denote q^{-1} and not an element of a highest-weight crystal.) In light of Theorem 4.2.1 we may write (3.5.13) as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \le i < j < r+1} (x_j - v \cdot x_i) = \mathbf{x}^{w_0\rho} \cdot \Delta_v \cdot \mathcal{D}_{w_0}(\mathbf{x}^{w_0(\lambda)}). \tag{4.2.1}$$

Further, using Theorem 4.2.2 this can be written as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \le i < j < r+1} (x_j - v \cdot x_i) = \mathbf{x}^{w_0\rho} \cdot \left(\sum_{w \in W} \mathcal{T}_w \right) (\mathbf{x}^{w_0(\lambda)}). \tag{4.2.2}$$

Comparing this with Proposition 3.5.4 we arrive to yet another form of Tokuyama's theorem.

Proposition 4.2.3. *Let λ be a dominant, effective weight and let $n = 1$. Then we have*

$$\mathbf{x}^{w_0\rho} \cdot \left(\sum_{w \in W} \mathcal{T}_w \right) (\mathbf{x}^{w_0(\lambda)}) = \sum_{v \in \mathcal{C}_{\lambda+\rho}} G(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{4.2.3}$$

The fact that the left hand side of Tokuyama's theorem is reproduced by $\mathcal{D}_{w_0}$ is essentially the Demazure Character Formula. In this sense, Theorem 4.2.1 is a generalization of the Demazure Character Formula to the metaplectic situation. We explain this in type A. The Demazure Character Formula (Fulton [17]) in type A is the following identity of operators on $\mathbb{C}(\mathbf{x})$

$$\partial_{w_0} = \frac{1}{\prod_{1 \le i < j \le r+1} (x_j - x_i)} \cdot \sum_{w \in W} \mathrm{sgn}(w) \cdot w. \tag{4.2.4}$$

Here ∂_w are a version of nonmetaplectic Demazure operators defined similarly to $\mathcal{D}_w$. That is,

$$\partial_{\sigma_i} = \frac{1 - \sigma_i}{1 - \mathbf{x}^{\alpha_i}}; \tag{4.2.5}$$

these satisfy the same braid relations as the $\mathcal{D}_{\sigma_i}$ and one may define ∂_{w_0} analogously to $\mathcal{D}_{w_0}$. It is clear from (4.2.4) and the Weyl Character Formula for Schur functions that

$$s_\lambda(\mathbf{x}) = \partial_{w_0} \mathbf{x}^{\lambda+\rho}.$$

Comparing (4.2.4) to the $n = 1$ special case of Theorem 4.2.1, the only difference is the presence of the factor

$$\prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{\alpha}$$

in the coefficient of $w \in W$. This discrepancy is caused by a "shift" by the Weyl vector ρ. Multiplication by $\mathbf{x}^{\rho}$ does not commute with acting on a monomial by a Weyl group element. To see how changing the order of these operators results in different expressions for the Schur function and the expression of the left hand side of Tokuyama's theorem, see Section 3.5.1. The shift also appears in the contrasting Definition 4.1.1 with (4.2.5): as operators,

$$\mathcal{D}_{\sigma_i} = \partial_{\sigma_i} \cdot \mathbf{x}^{-\alpha_i}.$$

4.2.2 Classical operators

In the nonmetaplectic case, i.e. when $n = 1$, we have $m(\alpha_i) = 1$ for every $\alpha_i \in \Phi$. Hence Definitions 4.1.1 and 4.1.2 reduce to

$$\mathcal{D}_i(f) = \frac{f - x^{\alpha_i} \cdot \sigma_i(f)}{1 - x^{\alpha_i}} \text{ and } \mathcal{T}_i(f) = (1 - v \cdot x^{\alpha_i}) \cdot \frac{f - x^{\alpha_i} \cdot \sigma_i(f)}{1 - x^{\alpha_i}} - f. \tag{4.2.6}$$

These "nonmetaplectic" operators are well known. One may also take the very similar definitions

$$\mathcal{D}_i(f) = \frac{f - x^{-\alpha_i} \cdot \sigma_i(f)}{1 - x^{-\alpha_i}} \text{ and } \mathcal{T}_i(f) = \left(1 - v \cdot x^{-\alpha_i}\right) \cdot \frac{f - x^{-\alpha_i} \cdot \sigma_i(f)}{1 - x^{-\alpha_i}} - f. \tag{4.2.7}$$

(both in the nonmetaplectic and in the metaplectic setting). The quadratic and braid relations satisfied by the metaplectic operators are analogous to the relations satisfied by these classical versions. One may consult Brubaker-Bump-Licata [8] as a source. The quadratic relations and braid relations given in Propositions 4.1.1 and 4.1.2 are analogous to the ones proved in Propositions 6 and 5 of Brubaker-Bump-Licata [8], respectively.

Furthermore, one may define Demazure operators on highest weight crystals. We present the definition of Kashiwara [21] as an example. Recall the definition of the highest weight crystal $\mathcal{C}_\lambda$ and the relevant notation from Section 3.1. The Demazure operator $\mathscr{D}_i$ acts on $\mathbb{Z}^{\mathcal{C}_\lambda}$ (the free $\mathbb{Z}$-module generated by elements of the crystal $\mathcal{C}_\lambda$) linearly. For an element $b \in \mathcal{C}_\lambda$ it is defined by

$$\mathscr{D}_i b = \begin{cases} \sum_{0 \le k \le \langle h_i, \mathrm{wt} b\rangle} f_i^k b & \text{if } \langle h_i, \mathrm{wt} b\rangle \ge 0; \\ -\sum_{1 \le k \le -\langle h_i, \mathrm{wt} b\rangle} e_i^k b & \text{if } \langle h_i, \mathrm{wt} b\rangle < 0. \end{cases} \tag{4.2.8}$$

4.3 Proof of Theorem 4.2.1

We now turn to the proof of Theorem 4.2.1. Before we can begin, we require more notation. The following is [10, Proposition 21.10], applied to $\Phi(w^{-1})$ instead of $\Phi(w)$:

Proposition 4.3.1. *Let $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N}$ be a reduced expression for $w \in W$. Then the set*

$$\Phi(w^{-1}) = \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^-\}$$

consists of the elements

$$\alpha_{i_1},\ \sigma_{i_1}(\alpha_{i_2}),\ \sigma_{i_1}\sigma_{i_2}(\alpha_{i_3}), \ldots,\ \sigma_{i_1}\cdots\sigma_{i_{N-1}}(\alpha_{i_N}),$$

where the α_i are the simple roots.

Let $p\colon \Phi \to \mathcal{K}_0$ be a map. We say p is *W-intertwining* if for any $\beta \in \Phi$ and $w \in W$, we have

$$p(w\beta) = w.p(\beta).$$

Proposition 4.3.1 has the following corollary, useful for the proof of Theorems 4.2.1 and 4.2.2:

Corollary 4.3.2. *Assume $p\colon \Phi \to \mathcal{K}_0$ is W-intertwining, and suppose $w \in W$ has a reduced expression $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N}$. Then we have the following equality of operators on $\mathcal{K}$:*

$$p(\alpha_{i_1})\sigma_{i_1} \cdot p(\alpha_{i_2})\sigma_{i_2} \cdots p(\alpha_{i_N})\sigma_{i_N} = \left(\prod_{\alpha \in \Phi(w^{-1})} p(\alpha)\right) \cdot w. \tag{4.3.1}$$

Proof. Making repeated use of Lemma 2.2.2, we can re-order the operators on the left of (4.3.1) by passing all the σ_{i_j}s to the right and all elements of $\mathcal{K}_0$ to the left. After this, the left of (4.3.1) becomes

$$p(\alpha_{i_1}) \cdot (\sigma_{i_1}.p(\alpha_{i_2})) \cdot (\sigma_{i_1}\sigma_{i_2}.p(\alpha_{i_3})) \cdots \big(\sigma_{i_1} \cdots \sigma_{i_{n-1}}.p(\alpha_{i_N})\big) \cdot \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_N}.$$

Here $\sigma_{i_1} \cdots \sigma_{i_N}$ is a reduced expression for w. Moreover, by Proposition 4.3.1,

$$\alpha_{i_1},\ \sigma_{i_1}(\alpha_{i_2}),\ \sigma_{i_1}\sigma_{i_2}(\alpha_{i_3}), \ldots,\ \sigma_{i_1} \cdots \sigma_{i_{N-1}}(\alpha_{i_N})$$

enumerates $\Phi(w^{-1})$. As a consequence the corresponding elements of $\mathcal{K}_0$, namely

$$p(\alpha_{i_1}),\ p(\sigma_{i_1}(\alpha_{i_2})),\ p(\sigma_{i_1}\sigma_{i_2}(\alpha_{i_3})), \ldots,\ p(\sigma_{i_1} \cdots \sigma_{i_{N-1}}(\alpha_{i_N})),$$

have product $\prod\limits_{\alpha \in \Phi(w^{-1})} p(\alpha)$. Since the map p is W-intertwining, these are exactly the factors appearing on the left of (4.3.1). □

We now begin the proof of Theorem 4.2.1. First notice that by Lemma 2.2.2, any composition of the operators $\mathcal{D}_i$ can be written as a $\mathcal{K}_0$-linear combination of the operators $w \in W$. Hence we can write

$$\mathcal{D}_{w_0} = \sum_{w \in W} R_w \cdot w, \tag{4.3.2}$$

for some choice of rational functions $R_w \in \mathcal{K}_0$.

Let $l\colon W \to \mathbb{Z}$ denote the length function on W. It is a standard fact about finite Coxeter groups that for any $1 \leq j \leq r$, we can find a reduced expression $\sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_{l(w_0)}}$ for the longest word w_0 with $i_1 = j$. By Proposition 4.1.2, we have $\mathcal{D}_{w_0} = \mathcal{D}_j\mathcal{D}_{w_j}$ for $w_j = \sigma_{i_2} \ldots \sigma_{i_{l(w_0)}}$. Since $\mathcal{D}_j^2 = \mathcal{D}_j$ (Proposition 4.1.1), we have $\mathcal{D}_j\mathcal{D}_{w_0} = \mathcal{D}_{w_0}$. In other words,

$$\frac{1 - x^{m(\alpha_j)\alpha_j}\sigma_j}{1 - x^{m(\alpha_j)\alpha_j}}\mathcal{D}_{w_0} = \mathcal{D}_{w_0}.$$

It follows that $\sigma_j\mathcal{D}_{w_0} = \mathcal{D}_{w_0}$. Now apply σ_j to both sides of (4.3.2). Since each $R_w \in \mathcal{K}_0$, Lemma 2.2.2 implies

$$\mathcal{D}_{w_0} = \sum_{w \in W} (\sigma_j.R_w) \cdot \sigma_j w. \tag{4.3.3}$$

Comparing coefficients in (4.3.2) and (4.3.3) and using the fact that the elements of W are linearly independent as operators on $\mathcal{K}$, we obtain $\sigma_j.R_w = R_{\sigma_j w}$. Thus

$$u.R_w = R_{uw} \quad \forall u, w \in W \tag{4.3.4}$$

To finish the proof of Theorem 4.2.1, it suffices to compute R_{w_0}; the remaining coefficients can then be computed using (4.3.4). In fact we shall prove the following:

Lemma 4.3.3. *For $w \in W$, we have*

$$R_w = \frac{\operatorname{sgn}(w)}{\Delta} \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha}.$$

Proof. To start, assume the statement is true for $w = w_0$:

$$R_{w_0} = \frac{\operatorname{sgn}(w_0)}{\Delta} \prod_{\alpha \in \Phi^+} x^{m(\alpha)\alpha}. \tag{4.3.5}$$

By (4.3.4), we have

$$\begin{aligned}
R_{uw_0} = u.R_{w_0} &= u.\left(\operatorname{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} \frac{x^{m(\alpha)\alpha}}{1 - x^{m(\alpha)\alpha}}\right) \\
&= \operatorname{sgn}(w_0) \cdot \prod_{\alpha \in u(\Phi^+)} \frac{x^{m(\alpha)\alpha}}{1 - x^{m(\alpha)\alpha}} \\
&= R_{w_0} \operatorname{sgn}(u) \prod_{\alpha \in \Phi(u^{-1})} x^{-m(\alpha)\alpha}
\end{aligned}$$

Let $u = ww_0$. Then $\Phi(u^{-1}) = \Phi^+ \cap w(\Phi^+)$. Hence

$$\begin{aligned}
R_w &= R_{w_0} \operatorname{sgn}(ww_0) \prod_{\alpha \in \Phi^+ \cap w(\Phi^+)} x^{-m(\alpha)\alpha} \\
&= \frac{\operatorname{sgn}(w)}{\Delta} \prod_{\alpha \in \Phi^+} x^{m(\alpha)\alpha} \prod_{\alpha \in \Phi^+ \cap w(\Phi^+)} x^{-m(\alpha)\alpha} \\
&= \frac{\operatorname{sgn}(w)}{\Delta} \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha}.
\end{aligned}$$

Thus the proof will be complete if we show (4.3.5).

Begin by writing

$$\mathcal{D}_i = p_1(\alpha_i) - p_2(\alpha_i)\sigma_i,$$

where $p_1, p_2 \colon \Phi \to \mathcal{K}_0$ are defined by

$$p_1(\beta) = \frac{1}{1 - x^{m(\beta)\beta}}, \quad p_2(\beta) = \frac{x^{m(\beta)\beta}}{1 - x^{m(\beta)\beta}}.$$

Given a reduced expression $w_0 = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N}$, it is easy to see that

$$R_{w_0}w_0 = \operatorname{sgn}(w_0)p_2(\alpha_{i_1}) \cdot \sigma_{i_1} \cdot p_2(\alpha_2) \cdot \sigma_{i_2} \cdots p_2(\alpha_{i_N}) \cdot \sigma_{i_N}.$$

Since the map p_2 is readily shown to be W-intertwining, it follows from the above equality and Corollary 4.3.2 that

$$R_{w_0} = \operatorname{sgn}(w) \prod_{\alpha \in \Phi^+} p_2(\alpha) = \frac{\operatorname{sgn}(w_0)}{\Delta} \prod_{\alpha \in \Phi^+} x^{m(\alpha)\alpha}.$$

This completes the proof of the lemma, and thus of Theorem 4.2.1. □

4.4 Proof of Theorem 4.2.2

In this section we prove Theorem 4.2.2:

$$\Delta_v \cdot \mathcal{D}_{w_0} = \sum_{w \in W} \mathcal{T}_w.$$

Let $\widetilde{\mathcal{T}} = \sum_{w \in W} \mathcal{T}_w$. We begin with some lemmas.

Lemma 4.4.1. *For any $1 \le i \le r$ we have*

$$\mathcal{T}_i \cdot (\Delta_v \mathcal{D}_{w_0}) = v \cdot (\Delta_v \mathcal{D}_{w_0})$$

Proof. Since the simple reflection σ_i permutes the elements of $\Phi^+ \setminus \{\alpha_i\}$, the operator $\mathcal{D}_i$ commutes with

$$\prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} (1 - vx^{m(\beta)\beta}) = \frac{\Delta_v}{1 - vx^{m(\alpha_i)\alpha_i}}.$$

Consequently,

$$(1 - vx^{m(\alpha_i)\alpha_i})\mathcal{D}_i\Delta_v = \Delta_v\mathcal{D}_i(1 - vx^{m(\alpha_i)\alpha_i}). \tag{4.4.1}$$

Take a reduced expression for the long element, $w_0 = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N}$ satisfying $i_1 = i$. Thus $\mathcal{D}_{w_0} = \mathcal{D}_i\mathcal{D}_{w_i}$ for $w_i = \sigma_{i_2}\cdots\sigma_{i_N}$. Using this and (4.4.1),

$$(\mathcal{T}_i + 1)\cdot(\Delta_v\mathcal{D}_{w_0}) = \Delta_v\mathcal{D}_i(1 - vx^{m(\alpha_i)\alpha_i})\mathcal{D}_i\mathcal{D}_{w_i}.$$

The idempotency of $\mathcal{D}_i$ (Proposition 4.1.1) and (4.1.3) imply

$$\mathcal{D}_i(1 - vx^{m(\alpha_i)\alpha_i})\mathcal{D}_i = (1 + v)\mathcal{D}_i.$$

Putting everything together, we conclude that $(\mathcal{T}_i + 1)\cdot(\Delta_v\mathcal{D}_{w_0}) = (1 + v)\cdot(\Delta_v\mathcal{D}_{w_0})$. □

Lemma 4.4.2. *For any $1 \le i \le r$ we have*

$$\mathcal{T}_i \cdot \widetilde{\mathcal{T}} = v \cdot \widetilde{\mathcal{T}}.$$

Proof. Recall that $l\colon W \to \mathbb{Z}$ is the length function on W. Then for any element $w \in W$ and any simple reflection σ_i, we have $l(\sigma_i w) = l(w) \pm 1$. Partition W into $C_1 \cup C_2$, where

$$C_1 = \{w \in W : l(\sigma_i w) = l(w) - 1\},$$
$$C_2 = \{w \in W : l(\sigma_i w) = l(w) + 1\}.$$

Then the map $w \mapsto \sigma_i w$ defines a bijection between C_1 and C_2. We compute

$$\begin{aligned}\mathcal{T}_i\widetilde{\mathcal{T}} &= \mathcal{T}_i \cdot \left(\sum_{w\in C_1}\mathcal{T}_w + \sum_{w\in C_2}\mathcal{T}_w\right)\\ &= \mathcal{T}_i \cdot \left(\sum_{w\in C_2}\mathcal{T}_i\mathcal{T}_w + \sum_{w\in C_2}\mathcal{T}_w\right)\\ &= \sum_{w\in C_2}\mathcal{T}_i^2\mathcal{T}_w + \sum_{w\in C_2}\mathcal{T}_i\mathcal{T}_w.\end{aligned}$$

The second sum above is simply $\sum_{w\in C_1}\mathcal{T}_w$. In the first, we use the quadratic relation $\mathcal{T}_i^2 = (v-1)\mathcal{T}_i + v$ of Proposition 4.1.1 to write

$$\sum_{w\in C_2}\mathcal{T}_i^2\mathcal{T}_w = (v-1)\sum_{w\in C_2}\mathcal{T}_i\mathcal{T}_w + v\sum_{w\in C_2}\mathcal{T}_w = (v-1)\sum_{w\in C_1}\mathcal{T}_w + v\sum_{w\in C_2}\mathcal{T}_w.$$

Thus $\mathcal{T}_i \cdot \widetilde{\mathcal{T}} = v \cdot \widetilde{\mathcal{T}}$. □

Lemma 4.4.3. *Let*

$$\widetilde{\mathcal{R}} = \sum_{w \in W} R_w \cdot w$$

be an operator on $\mathcal{K}$ that is a linear combination of the Weyl group elements with coefficients $R_w \in \mathcal{K}_0$. Assume $\widetilde{\mathcal{R}}$ is an eigenclass for $\mathcal{T}_i$ with eigenvalue v:

$$\mathcal{T}_i \cdot \widetilde{\mathcal{R}} = v \cdot \widetilde{\mathcal{R}}.$$

Then for every $w \in W$, we have

$$R_w = \frac{1 - vx^{m(\alpha_i)\alpha_i}}{1 - vx^{-m(\alpha_i)\alpha_i}} \cdot \sigma_i.R_{\sigma_i w}.$$

Proof. The proof is a straightforward computation. Begin with

$$\mathcal{T}_i R_w w = [q_1(\alpha_i) - q_2(\alpha_i)\sigma_i] \, R_w w = q_1(\alpha_i) R_w w - q_2(\alpha_i)(\sigma_i.R_w)\sigma_i w$$

with q_1, q_2 defined by

$$q_1(\beta) = \frac{1 - vx^{m(\beta)\beta}}{1 - x^{m(\beta)\beta}} - 1, \qquad \text{and} \qquad q_2(\beta) = \frac{(1 - vx^{m(\beta)\beta})x^{m(\beta)\beta}}{1 - x^{m(\beta)\beta}}. \tag{4.4.2}$$

Summing over $w \in W$, we get

$$\mathcal{T}_i \cdot \widetilde{\mathcal{R}} = \sum_{w \in W} [q_1(\alpha_i) R_w - q_2(\alpha_i)\sigma_i.R_{\sigma_i w}] \, w$$

But we also have $\mathcal{T}_i \cdot \widetilde{\mathcal{R}} = v \cdot \widetilde{\mathcal{R}}$, so comparing coefficients yields

$$q_1(\alpha_i) R_w - q_2(\alpha_i)\sigma_i.R_{\sigma_i w} = vR_w.$$

Solving for R_w completes the proof. □

Lemma 4.4.3 has the following easy and useful corollary.

Corollary 4.4.4. *Let*

$$\widetilde{\mathcal{R}} = \sum_{w \in W} R_w \cdot w, \quad \widetilde{\mathcal{S}} = \sum_{w \in W} S_w \cdot w$$

be two operators on $\mathcal{K}$ that are linear combinations of the Weyl-group elements with coefficients $R_w, S_w \in \mathcal{K}_0$. Assume that

$$\mathcal{T}_i \cdot \widetilde{\mathcal{R}} = v \cdot \widetilde{\mathcal{R}}, \quad \mathcal{T}_i \cdot \widetilde{\mathcal{S}} = v \cdot \widetilde{\mathcal{S}}$$

for every $i, 1 \leq i \leq r$. Assume further that we have $R_{w_0} = S_{w_0}$ for the long element $w_0 \in W$. Then $\widetilde{\mathcal{R}} = \widetilde{\mathcal{S}}$ as operators on $\widetilde{\mathcal{A}}$.

Proof. We show that $R_w = S_w$ for every $w \in W$. This can be seen by descending induction on the length of w. For $l(w)$ maximal we have $R_{w_0} = S_{w_0}$ by assumption. Now assume $l(\sigma_i w) = l(w) + 1$, and $R_{\sigma_i w} = S_{\sigma_i w}$. It follows from Lemma 4.4.3 that

$$R_w = \frac{1 - vx^{m(\alpha_i)\alpha_i}}{1 - vx^{-m(\alpha_i)\alpha_i}} \cdot \sigma_i . R_{\sigma_i w} = \frac{1 - vx^{m(\alpha_i)\alpha_i}}{1 - vx^{-m(\alpha_i)\alpha_i}} \cdot \sigma_i . S_{\sigma_i w} = S_w,$$

thus $R_w = S_w$. This completes the proof. □

We now turn to the proof of Theorem 4.2.2. Applying Lemmas 4.4.1 and 4.4.2 to the operators $\Delta_v \mathcal{D}_{w_0}$ and $\widetilde{\mathcal{T}}$, we have

$$\mathcal{T}_i \cdot (\Delta_v \mathcal{D}_{w_0}) = v \cdot \Delta_v \mathcal{D}_{w_0}, \quad \mathcal{T}_i \cdot \widetilde{\mathcal{T}} = v \cdot \widetilde{\mathcal{T}}$$

for every $1 \leq i \leq r$. It follows from the definitions that as operators on $\mathcal{K}$, both $\Delta_v \mathcal{D}_{w_0}$ and $\widetilde{\mathcal{T}}$ can be written as a linear combination of elements of W with coefficients in $\mathcal{K}_0$. Let us write

$$\Delta_v \mathcal{D}_{w_0} = \sum_{w \in W} R_w \cdot w \quad \text{and} \quad \widetilde{\mathcal{T}} = \sum_{w \in W} S_w \cdot w$$

for some $R_w, S_w \in \mathcal{K}$. We shall show that if $w_0 \in W$ is the long element of the Weyl group, then $R_{w_0} = S_{w_0}$. By Corollary 4.4.4, this suffices to prove the theorem.

The long coefficient R_{w_0} of $\Delta_v \mathcal{D}_{w_0}$ is easily read off from Theorem 4.2.1:

$$R_{w_0} = \operatorname{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} \frac{\left(1 - v \cdot x^{m(\alpha)\alpha}\right) \cdot x^{m(\alpha)\alpha}}{\left(1 - x^{m(\alpha)\alpha}\right)}. \tag{4.4.3}$$

To determine the coefficient S_{w_0} we again use the property of W-intertwining maps from Corollary 4.3.2 and argue as in the proof of Lemma 4.3.3. First, note that the only term in $\widetilde{\mathcal{T}} = \sum_{w \in W} T_w = \sum_{w \in W} S_w \cdot w$ that contributes to the coefficient S_{w_0} is T_{w_0}. (All the other T_w have fewer than $l(w_0)$ simple reflections appearing in them.) To examine T_{w_0}, fix a reduced expression for the long element: $w_0 = \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_N}$. Let us again write

$$\mathcal{T}_i = q_1(\alpha_i) - q_2(\alpha_i)\sigma_i$$

where $q_1, q_2 \colon \Phi \to \mathcal{K}_0$ are defined in (4.4.2). It is clear that the map q_2 is W-intertwining. The only contribution to S_{w_0} from $T_{w_0} = \mathcal{T}_{i_1}\mathcal{T}_{i_2} \cdots \mathcal{T}_{i_N}$ is from

$$q_2(\alpha_{i_1})\sigma_{i_1} \cdot q_2(\alpha_{i_2})\sigma_{i_2} \cdots q_2(\alpha_{i_N})\sigma_{i_N}.$$

Using Corollary 4.3.2, we conclude

$$S_{w_0} = \operatorname{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} q_2(\alpha) = \operatorname{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} \frac{\left(1 - v \cdot x^{m(\alpha)\alpha}\right) \cdot x^{m(\alpha)\alpha}}{1 - x^{m(\alpha)\alpha}}. \tag{4.4.4}$$

Comparing (4.4.3) and (4.4.4) we see that indeed $R_{w_0} = S_{w_0}$, as desired. This completes the proof of Theorem 4.2.2.

Chapter 5

Whittaker functions

In this chapter, we relate the formulas of Chapters 4 and 6 to Whittaker functions on certain metaplectic groups. The purpose is to make the motivation behind those results explicit.

In Section 5.1, we recall some background on metaplectic groups and their unramified principal series representations. In Sections 5.2 and 5.3, the main source is Section 9 of Chinta-Offen [15] and Section 15 of McNamara [30]. The results there relate Whittaker functions to the local factors of Weyl group multiple Dirichlet series constructed in Chinta-Gunnells [12]. Comparing those results to Theorems 4.2.1 and 4.2.2 will result in a description of Whittaker functions in the spirit of Demazure's character formula. We make this description explicit in type A using formulas of Chinta-Offen [15]; work in progress by McNamara [29] suggests that the same description is valid in a more general setting.

In Section 5.4 we recall a result of McNamara [30] that relates a metaplectic Whittaker function to the local factors of type A Weyl group multiple Dirichlet series constructed in Brubaker-Bump-Friedberg [6]. Sections 5.3 and 5.4 together explain how the results of Chapter 6, in particular the metaplectic version of Tokuyama's theorem (Theorem 6.2.2) accomplish the goal outlined in Section 1.1.3. The proof of Theorem 6.2.2 presented in Chapters 6 and 7 is combinatorial, thus Theorem 6.2.2 establishes a combinatorial link between the separate approaches to constructing metaplectic Whittaker functions.

Section 5.5 aims to further elucidate the motivation behind Theorem 6.2.1. In that section, we summarize the relationship between Demazure-Lusztig operators and (non-metaplectic) Iwahori-fixed Whittaker functions, explored in Brubaker-Bump-Licata [8].

5.1 Metaplectic groups and Principal Series

We start by introducing notation and recalling the construction of unramified principal series representations on metaplectic groups. This presentation follows McNamara [30, 29, 31]. (Those sources may be consulted for more details.)

5.1.1 Notation

Some of this notation has already been introduced in Chapter 2. Let F be a local field containing the n^{th} roots of unity. Denote the group of n^{th} roots of unity in F by μ_n, and fix, once and for all, an identification of μ_n with the complex n^{th} roots of unity. Let $\mathcal{O}$ denote the ring of integers and $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$ with uniformizer ϖ. Let q denote the order of the residue field $\mathcal{O}/\mathfrak{p}$. We assume that $q \equiv 1$ modulo $2n$. (This implies that F contains the $2n^{\text{th}}$ roots of unity, and -1 is an n^{th} power.)

Let $(,) = (,)_{F,n} : \ F^\times \times F^\times \to \mu_n(F)$ be the n^{th} order Hilbert symbol. It is a bilinear form on $F^\times$ that defines a nondegenerate bilinear form on $F^\times/F^{\times n}$ and satisfies

$$(x, -x) = (x, y)(y, x) = 1, \ x, \ y \in F^\times.$$

The fact that $-1 \in F^{\times n}$ implies that $(\varpi, -1) = 1$. Let ψ_F be an additive character on F with conductor $\mathcal{O}$. We may define the Gauss sum

$$g_i = \sum_{u \in \mathcal{O}^\times/(1+\mathfrak{p})} (u, \varpi^i) \cdot \psi_F(\varpi^{-1}u). \tag{5.1.1}$$

This implies that g_i depends only on the residue class of i modulo n, $g_0 = -1$ and $g_i g_{n-i} = q$ if $n \nmid i$.

Let G be a connected reductive group over F. Assume that G is split and arises as the special fiber of a group scheme $\mathbf{G}$ defined over $\mathbb{Z}$. Let $K = \mathbf{G}(\mathcal{O})$ be a maximal compact subgroup, T a maximal split torus and Λ the group of cocharacters of T. Let B be a Borel subgroup containing T, let U be the unipotent radical of B, and $U^- \subseteq B^-$ the opposite subgroup to U.

Let Φ denote the roots of T in G, with $\Delta \subset \Phi$ the set of simple roots. The Weyl group W of Φ acts on Λ. As in Section 2.1, we fix a W-invariant, integer valued quadratic form Q on Λ, and define the sublattice $\Lambda_0 \subset \Lambda$ as in (2.1.4):

$$\Lambda_0 = \{\lambda \in \Lambda : B(\alpha, \lambda) \equiv 0 \bmod n \text{ for all simple roots } \alpha\}. \tag{5.1.2}$$

5.1.2 The metaplectic cover

Let $\widetilde{G}$ be the n-fold metaplectic cover of G, as defined in Section 2 of McNamara [29]. This means that there is an exact sequence

$$1 \to \mu_n \to \widetilde{G} \to G \to 1. \tag{5.1.3}$$

Denote the inverse image of any subgroup $J \subset G$ with a tilde: $\widetilde{J}$. It is known that (5.1.3) splits canonically over U and U^- (Proposition 4.1 of McNamara [31]). The sequence does not, in general, split over K, but the assumption above on q implies that it does (Theorem 4.2 of [31], or 2.5 of McNamara [29]). Fix a splitting $\widetilde{K} = \mu_n \times K$, and identify K with its image in $\widetilde{G}$.

Let H be the centralizer of $T \cap K$ in $\widetilde{T}$. The lattice Λ (respectively, Λ_0) can be identified with $\widetilde{T}/(\mu_n \times (T \cap K))$ (respectively, $H/(\mu_n \times (T \cap K))$). The assumptions on G imply that H is Abelian (Lemma 2.9 of McNamara [29]), and in fact, $H/(T \cap K) \cong \mu_n \times \Lambda_0$ (although not canonically). Choose a lift of Λ into $\widetilde{G}$; denote this lift by $\lambda \mapsto \varpi^\lambda$.

5.1.3 Unramified Principal Series Representations

The unramified principal series representations are parametrized by complex-valued characters χ of Λ_0. Given such a character, we may obtain a character of H from the surjection

$H \to \mu_n \times \Lambda_0$, letting the roots of unity act faithfully. By inducing this character from H to $\widetilde{T}$, we get the representation $(\pi_\chi, i(\chi))$. The *unramified principal series* representation $I(\chi)$ of $\widetilde{G}$ is formed using normalized induction of this representation $(\pi_\chi, i(\chi))$ to $\widetilde{G}$. That is, we have

$$I(\chi) = \{f = \widetilde{G} \to i(\chi) : f(bg) = \delta^{1/2}(b)\pi_\chi(b)f(g),\ b \in \widetilde{B},\ g \in \widetilde{G},\ f \text{ locally constant}\},$$

where δ is the modular quasicharacter of $\widetilde{B}$. Then $\widetilde{G}$ acts on $I(\chi)$ by right translation. One may prove that $I(\chi)^K$ is one-dimensional; a nonzero element $\phi_K \in I(\chi)^K$ is called a *spherical vector*.

5.2 Whittaker functions

Let $\psi : U^- \to \mathbb{C}$ be an unramified character. By definition this means that the restriction of ψ to each of the root subgroups $U_{-\alpha}$, $\alpha \in \Delta$ is a character of $U_{-\alpha} \cong F$ with conductor $\mathcal{O}$. Then the function $\widetilde{G} \to i(\chi)$ defined by

$$g \mapsto \int_{U^-} \phi_K(ug)\psi(u)du \tag{5.2.1}$$

is the $i(\chi)$*-valued Whittaker function with character* ψ. One may obtain a complex-valued Whittaker function by composing the map (5.2.1) with a linear functional $\xi \in i(\chi)^*$. We now construct certain functionals (in $i(\chi)^*$) to arrive at very explicit formulas for the complex-valued Whittaker function. Recall that $\phi_K \in I(\chi)^K$ is our fixed spherical vector; let $v_0 = \phi_K(1)$. It turns out that there is an isomorphism $I(\chi)^K \cong i(\chi)^{\widetilde{T}\cap K}$. Let A be a set of coset representatives for $\widetilde{T}/H$; our assumptions imply that we may assume they each have the form ϖ^λ for some $\lambda \in \Lambda$. The vectors $\{\pi_\chi(a)v_0 :\ a \in A\}$ give a basis of $i(\chi)$.

Let $\widetilde{\chi} = \widetilde{T} \to \mathbb{C}$ be an extension of χ to $\widetilde{T}$ satisfying $\widetilde{\chi}(th) = \widetilde{\chi}(t)\chi(h)$ for any $t \in \widetilde{T}$, $h \in H$. The extension $\widetilde{\chi}$ determines a functional $\xi_{\widetilde{\chi}} \in i(\chi)^*$ by

$$\xi_{\widetilde{\chi}}(\pi_\chi(a)v_0) = \widetilde{\chi}(a).$$

Since we assumed each a has the form ϖ^λ for some $\lambda \in \Lambda$, we may write $\widetilde{\chi}(\lambda)$ instead of $\widetilde{\chi}(a)$. Composing the linear functional $\xi_{\widetilde{\chi}} \in i(\chi)^*$ with the map in (5.2.1) , we arrive at the complex-valued Whittaker function

$$\mathcal{W} = \mathcal{W}_{\widetilde{\chi}} : \; g \mapsto \xi_{\widetilde{\chi}} \left(\int_{U^-} \phi_K(ug)\psi(u)du \right). \tag{5.2.2}$$

It is a consequence of the construction that $\mathcal{W}$ satisfies

$$\mathcal{W}(\zeta ugk) = \zeta\phi(u)\mathcal{W}(g), \zeta \in \mu_n, \; u \in U, \; g \in \widetilde{G}, \; k \in K.$$

This fact, together with the Iwasawa decomposition $G = UTK$ implies that it suffices to compute $\mathcal{W}$ on $\widetilde{T}$.

5.3 Evaluation in terms of Demazure operators

We are almost ready to evaluate the Whittaker function in terms of Demazure and Demazure-Lusztig operators. In this section, we maintain the general notation. The main result the section relies on, Theorem 5.3.1 (Theorem 4 of Chinta-Offen [15]) expresses Whittaker-functions in type A. However, work in progress by McNamara [29] indicates that a similar result holds in a more general setting. Thus it is expected that Theorem 5.3.3 holds for any group satisfying the assumptions in Section 5.1.1 above.

As mentioned before, set $v = q^{-1}$ in Definition 2.2.1 of the metaplectic group action. The Weyl group action on Λ may be interpreted on $\widetilde{\chi}$ via the identification $\widetilde{\chi}(\varpi^\lambda) = x^\lambda$. Define

$$c_{w_0}(x) = \frac{\prod_{\alpha \in \Phi^+}(1 - q^{-1} \cdot x^{m(\alpha)\alpha})}{\prod_{\alpha \in \Phi^+}(1 - x^{m(\alpha)\alpha})}. \tag{5.3.1}$$

(Here $m(\alpha) = n/\gcd(m, Q(\alpha))$ is as defined in (2.1.3).) Furthermore (following notation in Chinta-Offen [15]), let

$$j(w, x) = \frac{\prod_{\alpha \in \Phi^+}(1 - x^{m(\alpha)\alpha})}{\prod_{\alpha \in w(\Phi^+)}(1 - x^{m(\alpha)\alpha})} \tag{5.3.2}$$

We have the following formula of Chinta-Offen:

Theorem 5.3.1. *(Chinta-Offen [15], Theorem 4) Let λ be a dominant coweight. Then*

$$(\delta^{-1/2}\mathcal{W}_{\widetilde{\chi}})(\varpi^{\lambda}) = c_{w_0}(x) \cdot \sum_{w \in W} j(w,x) \cdot w(x^{w_0\lambda})$$

where w acts on x^{λ} as in Definition 2.2.1.

To reformulate the right hand side in terms of Demazure operators, we first rewrite $j(w,x)$ in a more familiar form.

Lemma 5.3.2. *For any element w of the Weyl group, we have*

$$j(w,x) = \operatorname{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha}, \tag{5.3.3}$$

Proof. First note that $w(\Phi^+) \cap \Phi^- = -\Phi(w^{-1})$. This and $\operatorname{sgn}(w) = \operatorname{sgn}(w^{-1})$ implies that

$$\begin{aligned} j(w,x) = & \prod_{\alpha \in \Phi(w^{-1})} \frac{(1 - x^{m(\alpha)\alpha})}{(1 - x^{-m(\alpha)\alpha})} \\ = & \prod_{\alpha \in \Phi(w^{-1})} (-x^{m(\alpha)\alpha}) \\ = & \operatorname{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha}. \end{aligned}$$

□

Combining Theorem 5.3.1 with Theorems 4.2.1 and 4.2.2 we arrive at an expression of the Whittaker function in terms of Demazure or Demazure-Lusztig operators.

Theorem 5.3.3. *For λ a dominant coweight, we have*

$$\begin{aligned} (\delta^{-1/2}\mathcal{W}_{\widetilde{\chi}})(\varpi^{\lambda}) = & \prod_{\alpha \in \phi+} (1 - q^{-1} \cdot x^{m(\alpha)\alpha}) \mathcal{D}_{w_0}(x^{w_0\lambda}) \\ = & \sum_{w \in W} \mathcal{T}_w(x^{w_0\lambda}). \end{aligned} \tag{5.3.4}$$

5.4 Whittaker functions and crystals

In this section, we restrict our attention to type A, and review how metaplectic Whittaker functions can be expressed as a sum over a highest weight crystal. The source of this material is McNamara [30], especially Section 8. The results there relate Whittaker functions to the local factors of Weyl group multiple Dirichlet series constructed in Brubaker-Bump-Friedberg [6].

Let $\lambda = (\lambda_1, \ldots, \lambda_{r+1})$ denote a dominant weight. McNamara [30] computes the integral I_λ, which, up to a relatively trivial constant factor, is the same as $(\delta^{-1/2}\mathcal{W}_{\widetilde{\chi}})(\varpi^\lambda)$, as a sum over the highest weight crystal $\mathcal{C}_{\lambda+\rho}$. The crystal $\mathcal{C}_{\lambda+\rho}$ is parametrized in terms of Lusztig data. For our purposes, the Gelfand-Tsetlin parametrization of $\mathcal{C}_{\lambda+\rho}$ presented in Chapter 3 is more convenient; translating between the two parametrizations is not difficult. We start by presenting the result in the notation of McNamara [30], then we convert to the notation of Chapter 3.

Recall that the root system of type A_r has $\binom{r+1}{2}$ positive roots, indexed by pairs (i, j) where $1 \le i < j \le r+1$. Let $\mathbf{m}$ be an $\binom{r+1}{2}$ tuple of integers, indexed by positive roots. By Proposition 8.3 of McNamara [30], the elements of $\mathcal{C}_{\lambda+\rho}$ are in bijection with tuples of integers $\mathbf{m} = (m_{i,j})_{1\le i<j\le r+1}$ that satisfy $m_{i,j} \ge 0$ and

$$\sum_{k=j}^{r+1} m_{i,k} \le \lambda_i - \lambda_{i+1} + 1 + \sum_{k=j}^{r} m_{i+1,k+1}. \tag{5.4.1}$$

We write $\mathbf{m} \in \mathcal{C}_{\lambda+\rho}$ for tuples that satisfy the conditions above. These tuples may be decorated as follows. For every $\alpha = \alpha_{i,j} \in \Phi^+$ we say $m_\alpha = m_{i,j}$ is *circled* if $m_{i,j} = 0$, and *boxed* if equality holds in (5.4.1), i.e. if

$$\sum_{k=j}^{r+1} m_{i,k} = \lambda_i - \lambda_{i+1} + 1 + \sum_{k=j}^{r} m_{i+1,k+1}.$$

Let

$$r_{i,j} = r_\alpha = \sum_{k\le i} m_{k,j}. \tag{5.4.2}$$

Then for any $\mathbf{m} \in \mathcal{C}_{\lambda+\rho}$ and $\alpha \in \Phi^+$ we may use the functions $h^\flat$ and $g^\flat$ defined in Section 2.3 to define a coefficient corresponding to the decorations. This weight function is

$$w(\mathbf{m}, \alpha) = \begin{cases} 1 & \text{if } m_\alpha \text{ is circled, but not boxed,} \\ h^\flat(r_\alpha) & \text{if } m_\alpha \text{ is not circled and not boxed,} \\ g^\flat(r_\alpha) & \text{if } m_\alpha \text{ is boxed, but not circled,} \\ 0 & \text{if } m_\alpha \text{ is both circled and boxed} \end{cases} \tag{5.4.3}$$

Choose parameters $\mathbf{x} = (x_1, \dots, x_{r+1})$ such that $\chi(\varpi^\lambda) = \mathbf{x}^\lambda$ for the unramified χ used to define the principal series representation. We are ready to state McNamara's [30] result calculating the metaplectic Whittaker function.

Theorem 5.4.1. *(McNamara [29], Theorem 8.6) The value of the integral I_λ which calculates the metaplectic Whittaker function is zero unless λ is dominant; and for dominant λ it is given by*

$$I_\lambda = \sum_{\mathbf{m} \in \mathcal{C}_{\lambda+\rho}} \prod_{\alpha \in \Phi^+} w(\mathbf{m}, \alpha) \cdot x^{m_\alpha \alpha}. \tag{5.4.4}$$

We reformulate this result in the language of Gelfand-Tsetlin patterns and Γ-arrays. Recall that the elements of $\mathcal{C}_{\lambda+\rho}$ are in bijection with Gelfand-Tsetlin patterns $\mathfrak{T}$ of top row $\lambda+\rho$, or, equivalently, Γ-arrays $\Gamma(\mathfrak{T}) = (\Gamma_{h,k})_{1 \le h \le k \le r}$ that satisfy certain inequalities. (The precise statements were presented in Section 3.2.) To translate Theorem 5.4.1, all we have to do is give a bijection between elements $\mathbf{m} = v \in \mathcal{C}_{\lambda+\rho}$ and arrays Γ. It is much simpler to write down a bijection between the Lusztig parametrization of $\mathcal{C}_{-w_0\lambda+\rho}$ and the Gelfand-Tsetlin parametrization of $\mathcal{C}_{\lambda+\rho}$. (Here w_0 continues to denote the long element of the Weyl group. Note that $-w_0\lambda$ is dominant if and only if λ is.)

Note that the only place where λ appears in the parametrization by Lusztig-data is the upper bound for the $m_{i,j}$. When parametrizing $\mathcal{C}_{-w_0\lambda+\rho}$ instead of $\mathcal{C}_{\lambda+\rho}$, the condition (5.4.1) is replaced by

$$\sum_{k=j}^{r+1} m_{i,k} \le \lambda_{r+1-i} - \lambda_{r+2-i} + 1 + \sum_{k=j}^{r} m_{i+1,k+1}; \tag{5.4.5}$$

and $m_{i,j}$ is boxed if this is satisfied with an equality.

We give the bijection between $\mathbf{m} \in \mathcal{C}_{-w_0\lambda+\rho}$ and $\Gamma(\mathfrak{T}_v)$ for $v \in \mathcal{C}_{\lambda+\rho}$ through the $r_{i,j}$ defined in (5.4.2). Let

$$\mathbf{m} \mapsto \Gamma(\mathfrak{T}) \quad \text{if} \quad \Gamma_{h,k} = r_{r+1-k,r+2-h}. \tag{5.4.6}$$

First note that (h,k) satisfy $1 \le h \le k \le r$ if and only if $i = r+1-k$ and $j = r+2-h$ satisfy $1 \le i < j \le r+1$. Further,

$$m_{i,j} = r_{i,j} - r_{i-1,j}. \tag{5.4.7}$$

The bijection may be expressed in terms of the corresponding Γ-array as

$$m_{i,j} = \Gamma_{h,k} - \Gamma_{h,k+1}, \tag{5.4.8}$$

or from entries of the Gelfand-Tsetlin pattern $\mathfrak{T} = (a_{h,k})_{0\le h\le k\le r}$ as

$$m_{i,j} = a_{h,k} - a_{h-1,k}. \tag{5.4.9}$$

Thus $m_{i,j}$ is circled if and only if $\Gamma_{h,k} = \Gamma_{h,k+1}$, i.e. $\Gamma_{h,k}$ is circled by Definition 3.2.4. Similarly, $m_{i,j}$ is boxed if and only if

$$\begin{aligned}
\sum_{t=j}^{r+1} m_{i,t} =& \lambda_{r+1-i} - \lambda_{r+2-i} + 1 + \sum_{t=j}^{r} m_{i+1,t+1} \\
\sum_{t=j}^{r+1} (a_{r+2-t,k} - a_{r+1-t,k}) =& \lambda_k - \lambda_{k+1} + 1 + \sum_{t=j}^{r} (a_{r+1-t,k-1} - a_{r-t,k-1}) \\
\sum_{s=0}^{r+1-j} (a_{s+1,k} - a_{s,k}) =& \lambda_k - \lambda_{k+1} + 1 + \sum_{s=1}^{r+1-j} (a_{s,k-1} - a_{s-1,k-1}) \\
a_{r+2-j,k} - a_{0,k} =& \lambda_k - \lambda_{k+1} + 1 + a_{r+1-j,k-1} - a_{0,k-1} \\
a_{h,k} - a_{0,k} =& \lambda_k - \lambda_{k+1} + 1 + a_{h-1,k-1} - a_{0,k-1} \\
(a_{0,k-1} - a_{0,k}) - (\lambda_k - \lambda_{k+1} + 1) =& a_{h-1,k-1} - a_{h,k} \\
0 =& a_{h-1,k-1} - a_{h,k},
\end{aligned} \tag{5.4.10}$$

i.e. if and only if $\Gamma_{h,k}$ is boxed.

Comparing the definition of the coefficient $w(\mathbf{m}, \alpha)$ in (5.4.3) with the definition of the Gelfand-Tsetlin coefficients in Section 3.4, we find that for any $\mathbf{m} \in \mathcal{C}_{-w_0\lambda+\rho}$ and corresponding $v \in \mathcal{C}_{\lambda+\rho}$, we have

$$\prod_{\alpha \in \Phi^+} w(\mathbf{m}, \alpha) = G^{(n,\lambda+\rho)}(v). \tag{5.4.11}$$

We next wish to compare the monomial

$$\prod_{\alpha \in \Phi^+} \mathbf{x}^{m_\alpha \alpha} \tag{5.4.12}$$

corresponding to $\mathbf{m} \in \mathcal{C}_{-w_0\lambda+\rho}$ in (5.4.4) to $\mathbf{x}^{\mathrm{wt}(v)}$ as defined in Section 3.3. Recall that by part (iii) of Proposition 3.3.2, if $w_0(\lambda+\rho)$ is the lowest weight of $\mathcal{C}_{\lambda+\rho}$, then we have

$$\mathrm{wt}(v) - w_0(\lambda+\rho) = \sum_{1 \le h \le k \le r} \Gamma_{h,k} \cdot \alpha_{r+1-k,r+2-k}. \tag{5.4.13}$$

We may rewrite this in terms of the corresponding $r_{i,j} = \Gamma_{r+1-k,r+2-h}$ and $m_{i,j}$ as follows.

$$\begin{aligned} \mathrm{wt}(v) - w_0(\lambda+\rho) &= \sum_{1 \le h \le k \le r} \Gamma_{h,k} \cdot \alpha_{r+1-k,r+2-k} \\ &= \sum_{1 \le i < j \le r+1} r_{i,j} \cdot \alpha_{i,i+1} \\ &= \sum_{1 \le t \le i < j \le r+1} m_{t,j} \cdot \alpha_{i,i+1} \\ &= \sum_{1 \le t < j \le r+1} m_{t,j} \cdot \sum_{i=t}^{j-1} \alpha_{i,i+1} \\ &= \sum_{1 \le t < j \le r+1} m_{t,j} \cdot \alpha_{i,j} \\ &= \sum_{\alpha \in \Phi^+} m_\alpha \alpha. \end{aligned} \tag{5.4.14}$$

The following proposition is a direct consequence of Theorem 5.4.1, (5.4.11) and (5.4.14).

Proposition 5.4.2. *The value of the integral $I_{-w_0\lambda}$ which calculates the metaplectic Whittaker function is zero unless λ is dominant; and for dominant λ it is given by*

$$I_{-w_0\lambda} = \mathbf{x}^{-w_0(\lambda+\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{5.4.15}$$

The comparison of Theorem 5.3.3 and Proposition 5.4.2 motivates the metaplectic analogue of Tokuyama's Theorem (Theorem 6.2.2) in the next chapter.

5.5 Iwahori-fixed Whittaker functions

In this section, we summarize some results of Brubaker-Bump-Licata [8] concerning the relationship between Iwahori fixed Whittaker functions and Demazure Lusztig operators. This provides further motivation for Theorem 6.2.1. In particular, it sheds some light on why operators of the form

$$\sum_{u \leq w} \mathcal{T}_u \tag{5.5.1}$$

might be of interest. The analogous nature of the metaplectic Demazure and Demazure-Lusztig operators to their classical counterparts (see Section 4.2.2) leads one to hope that some of the results of Brubaker-Bump-Licata [8] will extend to the metaplectic setting. However, that is a topic for future research (see Section 1.5.3), and we make no explicit claims in this direction here.

We start by recalling notation. Throughout this section, we follow the presentation of Brubaker-Bump-Licata [8], but omit many of the details. Let G be a split, reductive Chevalley group over a non-archimedean local field F. Let q, K, B, U, T, B^- and U^- be as before. Let J be the Iwahori subgroup: the preimage of $B^-(\mathbb{F}_q)$ in K under the mod ϖ reduction map $K \to G(\mathbb{F}_q)$.

We have

$$T(F) \twoheadrightarrow T(F)/T(\mathcal{O}) \cong X_*(T) \cong X^*(\hat{T}) = \Lambda. \tag{5.5.2}$$

For any λ in the cocharacter lattice Λ let $a_\lambda \in T(F)$ be a representative of the corresponding coset in $T(F)/T(\mathcal{O})$. For $\mathbf{z} \in \hat{T}(\mathbb{C})$ and $\lambda \in \Lambda$, denote by $\mathbf{z}^\lambda$ the application of

λ to $\mathbf{z}$. For any $t \in T(F)$, the element of $X^*(\hat{T})$ corresponding to t by the homomorphism (5.5.2) may be applied to $\mathbf{z}$. Denote this by $\tau_{\mathbf{z}}(t)$. This interprets $\tau_{\mathbf{z}}$ as an unramified character of T, and $\mathbf{z} \mapsto \tau_{\mathbf{z}}$ establishes a (W-invariant) isomorphism between unramified $\mathbb{C}$-valued characters of T and $\hat{(T)}(\mathbb{C})$.

For an unramified character $\tau = \tau_{\mathbf{z}}$, denote by $I(\tau) = I(\mathbf{z})$ the space of the induced principal series representation, i.e. the space of locally constant functions $f : G(F) \to \mathbb{C}$ that satisfy $f(bg) = (\delta^{1/2}\tau)(b)f(g)$ for any $b \in B(F)$. Then $G(F)$ acts on $I(\mathbf{z})$ by right translation, let $\pi = \pi_{\mathbf{z}}$ denote this representation. Denote by $M(\tau) = M(\mathbf{z})$ the space $I(\mathbf{z})^J$ of Iwahori fixed vectors in $I(\mathbf{z})$. Under certain conditions on τ (referring to the generic position of τ) we may assume that $I(\mathbf{z})$ is irreducible and $M(\mathbf{z})$ has dimension equal to the order $|W|$ of the Weyl group.

We may define a basis $\{\Phi_w^{\mathbf{z}}\}_{w\in W}$ as follows. For any element $g \in G(F)$, we may write $g = bw'k$ with $b \in B$, $w \in N(T) \cap K$ (thought of as $w' \in W$ by abuse of notation) and $k \in J$. Let

$$\Phi_w^{\mathbf{z}}(bw'k) := \begin{cases} \delta^{1/2}\tau_{\mathbf{z}}(b) & \text{if } w' = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\Phi_w^{\mathbf{z}}\}_{w\in W}$ is a basis of $M(\mathbf{z})$. An other basis, $\{\widetilde{\Phi}_w^{\mathbf{z}}\}_{w\in W}$ may be defined from as

$$\widetilde{\Phi}_w^{\mathbf{z}}(bw'k) := \begin{cases} \delta^{1/2}\tau_{\mathbf{z}}(b) & \text{if } w' \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

(Here $\leq$ denotes the Bruhat order.)

We will be interested in the values of a Whittaker functional $\Omega_{\mathbf{z}}$ on elements of these two bases of $M(\mathbf{z})$. Let ψ be a character of $U^-(F)$ as before. Consider the Whittaker functional $\Omega_{\mathbf{z}}$ on the module $M(\mathbf{z})$ with respect to ψ defined by

$$\Omega_{\mathbf{z}}(f) = \int_{U^-(F)} f(u)\psi(u)^{-1}du. \tag{5.5.3}$$

We are almost ready to state the results from Brubaker-Bump-Licata [8] (Theorem 5.5.1 and Corollary 5.5.2 below) that provide the link between Iwahori-fixed Whittaker functions and the operators in (5.5.1) above. Define

$$\mathcal{W}_{\lambda,w}(\mathbf{z}) = \delta^{-1/2}(a_\lambda)\Omega_{\mathbf{z}^{-1}}(\pi(a_{-\lambda})\Phi_w^{z^{-1}}). \tag{5.5.4}$$

(In [8] this function is defined in terms of the contragredient $I(\mathbf{z}^{-1})$ of $I(\mathbf{z})$ and with the inclusion of the constant $\delta^{-1/2}(a_\lambda)$ for convenience.) Further, define

$$\begin{aligned}\widetilde{\mathcal{W}}_{\lambda,w}(\mathbf{z}) =& \sum_{y\leq w}\mathcal{W}_{\lambda,y}(\mathbf{z}) \\ =&\delta^{-1/2}(a_\lambda)\Omega_{\mathbf{z}^{-1}}(\pi(a_{-\lambda})\widetilde{\Phi}_w^{z^{-1}}).\end{aligned} \tag{5.5.5}$$

The operator $\mathfrak{T}_w$ for $w \in W$ is the classical analogue of the Demazure-Lusztig operator. It is defined for a simple reflection σ_i by

$$\mathfrak{T}_i f(\mathbf{z}) := (1 - q^{-1}\mathbf{z}^{-\alpha_i})(1-\mathbf{z}^{-\alpha_i})^{-1}(f(\mathbf{z}) - \mathbf{z}^{-\alpha_i}f(\sigma_i\mathbf{z})) - f(\mathbf{z}),$$

and extended to all Weyl group elements in the usual way. (These may be interpreted as operators on the ring $\mathcal{O}(\hat{T})$ of regular functions on $\hat{T}$.)

We are now ready to state a result of Brubaker-Bump-Licata [8], and the corollary that is relevant to us.

Theorem 5.5.1. *(Theorem 1 from Brubaker-Bump-Licata [8]) For any dominant weight λ, we have*

$$\mathcal{W}_{\lambda,1}(\mathbf{z}) = \mathbf{z}^\lambda.$$

Furthermore, if $w \in W$ and σ_i is a simple reflection such that $\sigma_i w > w$ by the Bruhat order, then

$$\mathcal{W}_{\lambda,\sigma_i w} = \mathfrak{T}_i\mathcal{W}_{\lambda,w}(\mathbf{z}).$$

The following straightforward corollary illustrates the relevance of operators as in (5.5.1).

Corollary 5.5.2. *For any dominant weight λ and $w \in W$, we have*

$$\widetilde{\mathcal{W}}_{\lambda,w}(\mathbf{z}) = \left(\sum_{y\leq w}\mathfrak{T}_y\right)\mathbf{z}^\lambda.$$

□

Chapter 6

Metaplectic Tokuyama and Demazure-Lusztig polynomials

In this chapter we restrict our attention to type A root systems. The main result is Theorem 6.2.1, a description of polynomials, coming from Iwahori-Whittaker functions (Chapter 5), as a sum over a highest weight crystal. These polynomials result from applying the operator

$$\sum_{u \leq w} \mathcal{T}_u \tag{6.0.1}$$

on a monomial, as explained in Chapter 5. The operators $\mathcal{T}_u$ are the metaplectic Demazure-Lusztig operators as defined in Chapter 4. This is a generalization of the theorem by Tokuyama in two ways. We have seen in Section 4.2.1 how, in the nonmetaplectic situation, the left hand side of Tokuyama's theorem can be reproduced with an operator like (6.0.1) when $w = w_0$ is the long word of the Weyl group. Theorem 6.2.1 holds for any degree n of the metaplectic cover, hence the special case $w = w_0$ is a metaplectic analogue of Tokuyama's theorem. Moreover, in Theorem 6.2.1 we allow w to be any "beginning section" of our favourite long word (3.3.1).

This chapter and Chapter 7 are dedicated to the statement and proof of Theorem 6.2.1.

After recalling some notation in Section 6.1 we state the main result and the connection with Tokuyama's theorem explicitly in Section 6.2.

The strategy for proving this statement is to use the "branching" structure of type A highest-weight crystals. The main theorem expresses a polynomial as a sum over a highest weight crystal of type A_r. When edges of this crystal that correspond to the r-th Kashiwara operator are removed, the crystal falls apart into connected components. The components are themselves highest-weight crystals of type A_{r-1}. Using this "branching" and Theorem 4.2.1, one may replace Theorem 6.2.1 with statements relating smaller expressions of Demazure-Lusztig operators to subcrystals.

To phrase the simpler statements and prove that they imply Theorem 6.2.1, we will make use of branching properties of Demazure crystals. These are summarized in Proposition 6.3.3. Section 6.3 is dedicated to the proof of that Proposition, and (in 6.3.5) to setting up convenient notation. Once the computation of Gelfand-Tsetlin coefficients on Demazure-subcrystals is well understood, the fact that the simpler statements imply Theorem 6.2.1 will be explained in Section 6.4. Some simple lemmas, necessary along the way, are proved in Section 6.5.

The process described above proves that Theorem 6.2.1 is equivalent to a statement about a single string $\mathcal{T}_r\mathcal{T}_{r-1}\cdots\mathcal{T}_1$ acting on a monomial. This statement can be proved independently, via induction. This induction proof forms the contents of Chapter 7.

6.1 Notation

Most of the conventions and notation particular to type A have already been introduced. See Section 2.4 in particular for general notation for the root system, weight lattice and group action; Chapter 3 for notation on highest weight crystals, Gelfand-Tsetlin patterns, Γ-arrays, Berenstein-Zelevinsky-Littelmann paths and Tokuyama's theorem. For convenience we recall some of the notation here.

The Weyl group W in type A_r is isomorphic to S_{r+1}. It is generated by simple reflections $\sigma_1, \sigma_2, \ldots, \sigma_r$ corresponding to the simple roots α_i. For any $w \in W$ the number of simple reflections in a reduced expression of W is $\ell(w) = |\Phi(w)|$. There is a unique longest element in W that we continue to denote by w_0.

As in (3.3.1) we pick "our favourite long word" in the following reduced expression for w_0.

$$w_0 = w_0^{(r)} = \sigma_1\sigma_2\sigma_1 \cdots \sigma_{r-1} \cdots \sigma_1\sigma_r \cdots \sigma_1 = \sigma_{\Omega_1}\sigma_{\Omega_2} \cdots \sigma_{\Omega_N}. \tag{6.1.1}$$

The statement of Theorem 6.2.1 is phrased for particular elements w of the Weyl group. As discussed, the special case of $w = w_0$ is of particular interest. The other relevant elements are "beginning sections" of the long word in (6.1.1) above. The element w is a beginning section if $w = \sigma_{\Omega_1}\sigma_{\Omega_2} \cdots \sigma_{\Omega_l}$ for some $l (= \ell(w)) \leq N$. Since the proof of Theorem 6.2.1 is by induction on r, we will sometimes assume that w is a beginning section of $w_0^{(r)}$, but not of $w_0^{(r-1)}$. In this case w is of the form

$$w = w_0^{(r-1)}\sigma_r \cdots \sigma_{r-k}. \tag{6.1.2}$$

We will continue to use the notation $k = \ell(w) - \ell(w_0^{(r-1)}) - 1$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1})$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1}$ (λ dominant). We assume for now that $\lambda_{r+1} \geq 0$ (λ effective). (In some of the statements below this last condition will be omitted.) We have $w_0(\lambda) = (\lambda_{r+1}, \lambda_r, \ldots, \lambda_2, \lambda_1)$. Let $\rho = \rho_r = (r, r-1, \ldots, 1, 0)$. We will use $d(\lambda) = \lambda_1 + \cdots + \lambda_{r+1}$, for example, $d(\rho_r) = \binom{r+1}{2}$. Denote $\mathbf{x} = (x_1, \ldots, x_r, x_{r+1})$ and $\mathbf{y} = (x_1, \ldots, x_r)$. Let $v = q^{-1} = t^n$ as introduced in 2.3. We will primarily use t^n, both to emphasize the dependence on metaplectic degree, and to avoid confusion with v possibly denoting a vertex of a crystal graph. Recall (4.1.7) for the metaplectic definition of the Weyl denominator. We continue to use this in the form

$$\Delta_t = \Delta_t^{(n,r)} = \prod_{\alpha \in \Phi+} (1 - t^n \cdot \mathbf{x}^{n \cdot \alpha}) = \prod_{1 \leq i < j \leq r+1} \left(1 - t^n \cdot \frac{x_i^n}{x_j^n}\right).$$

Definitions of $\mathcal{D}_u$ and $\mathcal{T}_u$ for $u \in W$ are as defined in Chapter 4. Since in type A_r we have $m(\alpha) = n$ for every $\alpha \in \Phi$, these have a simpler form. For every σ_i simple reflection corresponding to a simple root α_i the Demazure-Lusztig operator is

$$\mathcal{T}_i = \mathcal{T}_{\sigma_i} = (1 - t^n \cdot \mathbf{x}^{n\cdot\alpha_i}) \cdot \mathcal{D}_i - 1,$$

where $\mathcal{D}_i$ is the Demazure operator

$$\mathcal{D}_i = \mathcal{D}_{\sigma_i} = \frac{1 - \mathbf{x}^{n\cdot\alpha_i} \cdot \sigma_i}{1 - \mathbf{x}^{n\cdot\alpha_i}}.$$

Here $\sigma_i = \sigma_i^{(n)}$ acts on polynomials by the Chinta-Gunnells metaplectic action, as in Section 2.4, in particular, (2.4.2). This depends on the degree n. For $n = 1$, the element σ_i acts simply by exchanging the variables x_i and x_{i+1}. For $n = 2$, the element $\sigma_i = \sigma_i^{(2)}$ acts on a monomial $f(\ldots, x_i, x_{i+1}, \ldots) = x_i^a x_{i+1}^b$ as follows:

$$(\sigma_i(f))(\ldots, x_i, x_{i+1}, \ldots) = \begin{cases} f(\ldots, x_{i+1}, x_i, \ldots) \cdot \frac{x_{i+1}}{x_i} \cdot \frac{x_i - t\cdot x_{i+1}}{x_{i+1} - t\cdot x_i}, & \text{if } 2 \mid a - b \\ f(\ldots, x_{i+1}, x_i, \ldots) \cdot \frac{x_{i+1}}{x_i}, & \text{if } 2 \nmid a - b. \end{cases}$$

An important ingredient in the statement of Theorem 6.2.1 is the Demazure crystal.

Definition 6.1.1. Let $\mathcal{C}_{\lambda+\rho}$ be a crystal of highest weight $\lambda+\rho$, and for any $v \in \mathcal{C}_{\lambda+\rho}$, let $b_i(v)$ denote the i-th entry of the Berenstein-Zelevinsky-Littelmann array of v, $BZL(v)$, as defined in Section 3.3. (Recall that this is the length of the i-th segment of the BZL-path of v.) Let w be a beginning section of the long word (6.1.1). Then

$$\mathcal{C}_{\lambda+\rho}^{(w)} = \{v \in \mathcal{C}_{\lambda+\rho} \mid b_i(v) = 0 \text{ for all } i > l(w)\}. \tag{6.1.3}$$

To define a crystal structure $\mathcal{C}_{\lambda+\rho}^{(w)}$, we contend that as a directed graph it is a full subgraph of $\mathcal{C}_{\lambda+\rho}$. That is, the edges of $\mathcal{C}_{\lambda+\rho}^{(w)}$ are exactly the edges of $\mathcal{C}_{\lambda+\rho}$ with both endpoints in $\mathcal{C}_{\lambda+\rho}^{(w)}$.

Definition 6.1.1 means that an element $v \in \mathcal{C}_{\lambda+\rho}$ belongs to $\mathcal{C}_{\lambda+\rho}^{(w)}$ if and only if the BZL-path of v reaches v_{lowest} after the first $l(w)$ steps. With w a beginning segment of

$w_0^{(r)}$ but not of $w_0^{(r-1)}$ as in (6.1.2), this means that $v \in \mathcal{C}_{\lambda+\rho}$ if and only if

$$BZL(v) = BZL_\Omega(v) = \begin{bmatrix} b_{\binom{r}{2}+1} & b_{\binom{r}{2}+2} & \cdots & b_{\binom{r}{2}+k} & 0 & \cdots & 0 \\ & b_{\binom{r-1}{2}+1} & & \cdots & & & b_{\binom{r}{2}} \\ & & \ddots & & & & \\ & & & & & b_2 & b_3 \\ & & & & & & b_1 \end{bmatrix}. \quad (6.1.4)$$

The structure of $\mathcal{C}_{\lambda+\rho}^{(w)}$ will be discussed in detail in Section 6.3. For now, we content ourselves with an example in type A_2.

Example 6.1.2. Recall the crystal $\mathcal{C}_{3,1,0}$ of highest weight $(3,1,0)$ from Example 3.1.1. The Demazure subcrystal corresponding to $w = \sigma_1\sigma_2$ is the highlighted part of the crystal in Figure 5.

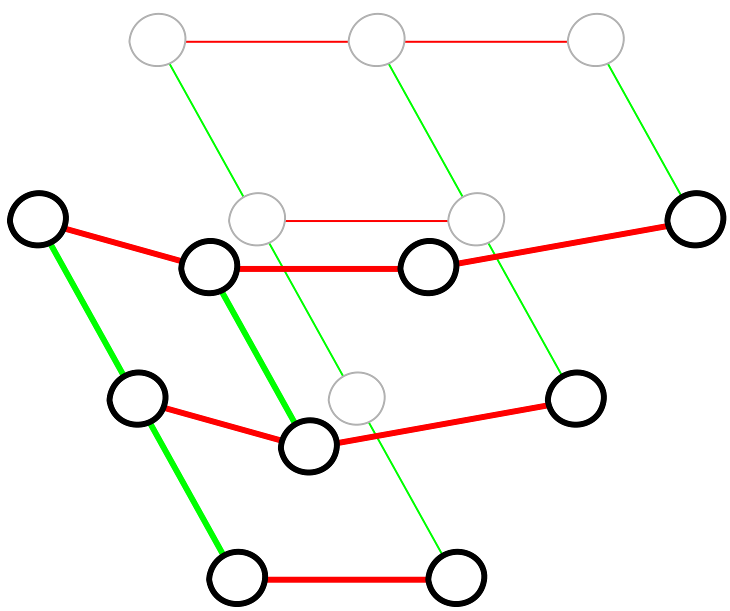

Figure 5: The Demazure crystal $\mathcal{C}_{(3,1,0)}^{(\sigma_1\sigma_2)}$ within $\mathcal{C}_{(3,1,0)}$.

Remark 6.1.3. Much, but not all of the notation introduced above, in previous chapters, and in what follows, depends on the value of n. In particular the σ_i, $\mathcal{D}_i$, $\mathcal{T}_i$ and $G^{(\lambda,w)}(v)$

have definitions that depend on n, but w_0, W, $\mathcal{C}^{(w)}_{\lambda+\rho}$ and $\mathrm{wt}(v)$ do not. Having given the definitions, and sometimes examples for $n = 1$ and $n = 2$, we will suppress n from the notation. When reading the statements and proofs, one should keep in mind that the meaning varies with n. The entire argument (about a sequence of statements being pairwise equivalent) is about a(n arbitrarily) fixed n.

6.2 Main Theorem

We are ready to state the main theorem and compare it to Tokuyama's theorem.

Theorem 6.2.1. *Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_{r+1})$ be any dominant, effective weight, $\rho = (r, r-1, \dots, 1, 0)$, w a beginning section of the long word. Then*

$$\left(\sum_{u \leq w} \mathcal{T}_u\right) \mathbf{x}^{w_0(\lambda)} = \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}^{(w)}_{\lambda+\rho}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{6.2.1}$$

Here $\leq$ is the Bruhat order, $G^{(n,\lambda+\rho)}(v) = G(v)$ is the Gelfand-Tsetlin coefficient corresponding to v (as defined in Section 3.4), and $\mathcal{C}^{(w)}_{\lambda+\rho}$ is the Demazure-crystal corresponding to w as is Definition 6.1.1.

The proof of this statement is by induction on r. It will be convenient to assume w has length at least $\binom{r}{2}$, i. e. is as in (6.1.2). We may make this assumption without loss of generality by Remark 6.3.4. Call the statement of Theorem 6.2.1 for such a w (but for any λ dominant, effective weight) $\mathbf{IW^{(n)}_{r,k}}$ (we will usually suppress n from the notation). Note that the pair of positive integers (r, k) where $0 \leq k < r$ encodes the choice of w. Proving $IW_{r,k}$ for any pair $0 \leq k < r$ is sufficient to prove Theorem 6.2.1.

As mentioned above, a special case of this statement is a metaplectic analogue of Tokuyama's theorem. We now make this statement explicit.

Let $k = r - 1$, then $w = w_0^{(r)}$, hence $\{u \in W \mid u \leq w\} = W$ and $\mathcal{C}^{(w)}_{\lambda+\rho} = \mathcal{C}_{\lambda+\rho}$. By Proposition 4.2.3, Tokuyama's theorem states that if $n = 1$,

$$\left(\sum_{u \in W} \mathcal{T}_u\right) (\mathbf{x}^{w_0(\lambda)}) = \mathbf{x}^{-w_0\rho} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}} G^{(1)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{6.2.2}$$

Call the statement that (6.2.2) holds for every λ dominant, effective weight $\mathbf{Tok_r^{(1)}}$. It is clear from comparing (6.2.1) to (6.2.2) that $IW^{(1)}_{r,r-1} = Tok^{(1)}_r$. This motivates the name we give the following result.

Theorem 6.2.2. *(Tokuyama's Theorem, Metaplectic Version.) Let $\lambda = (\lambda_1, \ldots, \lambda_{r+1})$ be any dominant, effective weight and $\rho = (r, \ldots, 1, 0)$. Then*

$$\left(\sum_{u\in W} \mathcal{T}_u\right)(\mathbf{x}^{w_0(\lambda)}) = \mathbf{x}^{-w_0\rho} \cdot \sum_{v\in\mathcal{C}_{\lambda+\rho}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{6.2.3}$$

Call the statement of Theorem 6.2.2 (for fixed r, but any λ dominant, effective weight) $\mathbf{Tok_r}^{(n)}$ (again, we usually suppress n from the notation).

Now the statement that the Metaplectic Version of Tokuyama's theorem is the $w = w_0$ special case of Theorem 6.2.1 can be formulated as

$$IW^{(n)}_{r,r-1} = Tok^{(n)}_r. \tag{6.2.4}$$

It is clear that Theorem 6.2.1 implies Theorem 6.2.2.

We now describe the strategy for proving Theorem 6.2.1 ($IW_{r,k}$) and Theorem 6.2.2 (Tok_r). We phrase two more, similar statements, $M_{r,k}$ (Proposition 6.4.3) and $N_{r,k}$ (Proposition 6.4.4). All four statements make sense for r, k integers, $0 \le k < r$. The statements $M_{r,k}$ and $N_{r,k}$ involve smaller expressions of Demazure-Lusztig operators on the left hand side. $N_{r,r-1}$ describes the polynomial $(\mathcal{T}_r\mathcal{T}_{r-1}\cdots\mathcal{T}_1)(\mathbf{x}^{w_0(\lambda)})$. By Lemma 6.4.7, $N_{r,r-1}$ for any r implies $N_{r,k}$ for any pair of integers $0 \le k < r$. Having that, in turn, implies $M_{r,k}$ for any pair (Lemma 6.4.6). The proof of $IW_{r,k}$ is then by induction on r. For the induction step, one shows that $M_{r,k}$ and $IW_{r-1,r-2}$ together imply $IW_{r,k}$. This will be explained in Section 6.4 in detail. The statement $N_{r,r-1}$ (Proposition 7.0.4) is proved in Chapter 7, thus completing the proof of Theorem 6.2.1 and Theorem 6.2.2.

In order to phrase the statements $M_{r,k}$ and $N_{r,k}$, we must examine the branching structure of the Demazure subcrystal $\mathcal{C}^{(w)}_{\lambda+\rho}$ in more detail. The guiding principle is that if we remove the edges corresponding to the r-th Kashiwara operator from $\mathcal{C}^{(w)}_{\lambda+\rho}$, we get the disjoint union of highest weight crystals of type A_{r-1}. This will allow us to replace the right hand side of $IW_{r,k}$ with the sum of Tokuyama-type expressions, all involving crystals of type A_{r-1}.

The necessary ingredients are discussed and illustrated in Section 6.3. Proposition 6.3.3, in particular part (iv), summarizes the results of that section. In Section 6.3.5 we introduce notation that is convenient for phrasing $M_{r,k}$ and $N_{r,k}$. The statement of Proposition 6.3.3, the example in Section 6.3.1, Definition 6.3.9 and the statement of Lemma 6.3.10 are of particular interest. The rest of Section 6.3 is included for completeness.

6.3 Demazure crystals

In this section we examine the structure and Gelfand-Tsetlin coefficients of the Demazure crystal $\mathcal{C}^{(w)}_{\lambda+\rho}$.

The following branching rule of type A_r highest-weight crystals is well known. (See, for example, (2.4) in Brubaker-Bump-Friedberg [7].) It plays a key role in the rest of this chapter.

Proposition 6.3.1. *When all the edges of $\mathcal{C}_{\lambda+\rho}$ labeled by r are removed, the connected components of the result are all isomorphic to highest weight crystals $\mathcal{C}_\mu$ of type A_{r-1}. Omitting the last component of* wt $:\mathcal{C}_{\lambda+\rho}\to\mathbb{Z}^{r+1}$*, and restricting it to a connected component gives the weight function on that component:*

$$\mathrm{wt}_\mu : \mathcal{C}_\mu \to \mathbb{Z}^r. \tag{6.3.1}$$

The highest weights μ that appear in this decomposition are dominant and interleave with $\lambda+\rho$. We identify the highest weight crystal $\mathcal{C}_\mu$ with the appropriate subcrystal of $\mathcal{C}_{\lambda+\rho}$. That is, we have

$$\mathcal{C}_{\lambda+\rho} = \bigcup_{\mu} \mathcal{C}_\mu. \tag{6.3.2}$$

and the (disjoint) union is over all $\mu=(\mu_1,\mu_2\ldots,\mu_r)$ such that

$$\lambda_1+r\geq\mu_1\geq\lambda_2+r-1\geq\cdots\geq\lambda_r+1\geq\mu_r\geq\lambda_{r+1}. \tag{6.3.3}$$

An element $v \in \mathcal{C}_{\lambda+\rho}$ belongs to $\mathcal{C}_\mu$ in the disjoint union (6.3.2) if the second row of the pattern $\mathfrak{T}(v)$ is $(a_{11}, a_{12}, \ldots, a_{1r}) = \mu$. (Here $\mathfrak{T}(v)$ is the Gelfand-Tsetlin pattern with top row $\lambda + \rho$ corresponding to v as in Proposition 3.3.2.)

Example 6.3.2. If $\lambda+\rho = (3,1,0)$, then the weights μ are $(3,1)$, $(3,0)$, $(2,1)$, $(2,0)$, $(1,1)$ and $(1,0)$. Figure 6 shows the corresponding components of $\mathcal{C}_{(3,1,0)}$. These are of Cartan type A_1. The highest element of each string is labeled with the corresponding weight μ.

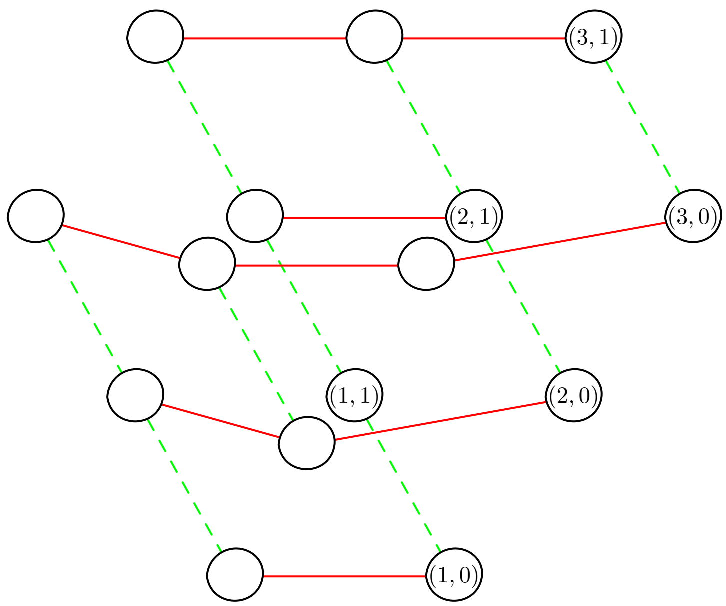

Figure 6: The A_1 components (1-strings) of $\mathcal{C}_{(3,1,0)}$.

Now let w be as in (6.1.2) and the Demazure crystal $\mathcal{C}^{(w)}_{\lambda+\rho}$ as in Definition 6.1.1. Then the analogue of Proposition 6.3.1 is true for $\mathcal{C}^{(w)}_{\lambda+\rho}$. In addition, if $\mathcal{C}_\mu$ is a component in the decomposition (6.3.2), then both the weights and the Gelfand-Tsetlin coefficients on $\mathcal{C}_\mu$ only depend on the crystal $\mathcal{C}_\mu$. The precise statement of this fact forms the contents of the proposition below.

Proposition 6.3.3. *Let $w = w_0^{(r-1)} \cdots \sigma_r \cdots \sigma_{r-k}$ and $\mathcal{C}^{(w)}_{\lambda+\rho}$ the corresponding Demazure crystal. Then the following statements hold.*

(i) When the edges labeled by r are removed from $\mathcal{C}^{(w)}_{\lambda+\rho}$ are removed, it is a disjoint union of highest-weight crystals of type A_{r-1} :

$$\mathcal{C}_{\lambda+\rho} = \bigcup_{\mu} \mathcal{C}_{\mu}. \tag{6.3.4}$$

Here the union is over all $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ that interleave with $\lambda + \rho$ and $\mu_j = \lambda_{j+1} + r - j$ for $j > k + 1$.

(ii) For any element $v \in \mathcal{C}_{\mu}$ in a component in (6.3.4) we have

$$\mathbf{x}^{\mathrm{wt}(v)} = \mathbf{y}^{\mathrm{wt}_{\mu}(v)} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)}. \tag{6.3.5}$$

(iii) If v_* is the lowest element of a component $\mathcal{C}_{\mu}$ in (6.3.4) and $v \in \mathcal{C}_{\mu}$ is any element, then we have

$$G^{(n,\lambda+\rho)}(v) = G^{(n,\mu)}(v) \cdot G^{(n,\lambda+\rho)}(v_*). \tag{6.3.6}$$

(iv) With notation as in the previous parts, we may write

$$\mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}^{(w)}_{\lambda+\rho}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)} = \sum_{\mu} G^{(n,\lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \left(\mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})} \cdot \sum_{v \in \mathcal{C}_{\mu}} G^{(n,\mu)}(v) \cdot \mathbf{y}^{\mathrm{wt}_{\mu}(v)} \right). \tag{6.3.7}$$

Here the sum is over all $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ that interleave with $\lambda + \rho$ and $\mu_j = \lambda_{j+1} + r - j$ for $j > k + 1$.

Proof. The remainder of this section will be dedicated to the proof of this Proposition. Part (i) is proved in 6.3.2, part (ii) in 6.3.3, part (iii) in 6.3.4. Part (iv) is a straightforward consequence of the first three parts. □

Note that part (iv) of Proposition 6.3.3 produces the right hand side of $IW_{r,k}$ as a sum, where the summands contain copies of the right hand side of T_{r-1}, multiplied by constants and powers of x_{r+1}.

Remark 6.3.4. In proving Theorem 6.2.1, we restrict our attention to the case where the Weyl group element w that is a beginning section of $w_0^{(r)}$ has length at least $\binom{r}{2}$. This leads to no loss of generality, because for w shorter than that, the statement is essentially the same as an instance of the theorem in type A_{r-1}. If w is a beginning section of $w_0^{(r)}$ and $\ell(w) \leq \binom{r}{2}$, then in fact w is a beginning section of the long word $w_0^{(r-1)}$ in type A_{r-1}. Let $\lambda' = (\lambda_2, \lambda_3, \ldots, \lambda_{r+1})$. We show that if $\ell(w) \leq \binom{r}{2}$, then the statement (6.2.1) for type A_r, λ and w is the same statement for type A_{r-1}, λ' and w, except both sides are multiplied by $x_{r+1}^{\lambda_1}$. On the left hand side, this is true because $\mathcal{T}_1, \ldots, \mathcal{T}_{r-1}$ all commute with multiplication by x_{r+1}. As for the right hand side of the equation, in the decomposition (6.3.2), $\mathcal{C}_{\lambda+\rho}^{(w)}$ is contained in the component $\mathcal{C}_{\lambda'}$ of the lowest element, $v_{lowest} \in \mathcal{C}_{\lambda+\rho}$. The lowest element of that component is $v_* = v_{lowest}$. We have $G^{\lambda+\rho}(v_{lowest}) = 1$. Now the same argument that proves Proposition 6.3.3 shows that the right hand side of (6.2.1) for λ and type A_r is just $x_{r+1}^{\lambda_1}$ times the analogous statement for λ' and type A_{r-1}.

Proving Proposition 6.3.3 is straightforward once the branching rule in Proposition 6.3.1 and characteristics of elements of $\mathcal{C}_\mu$ are well understood. The objects involved, however, have many different parametrizations (for example vertices of a crystal can be represented by Gelfand-Tsetlin patterns, BZL-paths or Γ-arrays). This makes notation slightly cumbersome. Hence before discussing these facts in general, we look at an explicit example in 6.3.1. We will introduce simpler notation in Section 6.3.5.

6.3.1 An example

Let, once again, $r = 2$, $\lambda + \rho = (3, 1, 0)$ and $w = \sigma_1\sigma_2$ (so $k = 0$). Then $\mathcal{C}_{\lambda+\rho}^{(w)} = \mathcal{C}_{(3,1,1)}^{(\sigma_1\sigma_2)}$ is as in Figure 5. Under the branching 6.3.2 it is the union of three crystals of type A_1, of highest weight $(3, 0)$, $(2, 0)$ and $(1, 0)$, respectively.

Determining the highest weights $(3, 0)$, $(2, 0)$ and $1, 0$ is easy from Proposition 3.3.2 and (6.1.4). Recall that a vertex belongs to the Demazure crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$ if $BZL(v) =$

$\Gamma(\mathfrak{T}(v))$ has zeros in the last $l(w_0) - l_w = k+1$ places in the first row. We have $\lambda + \rho = (a_{00}, a_{01}, a_{02}) = (3,1,0)$. This means that the component $\mathcal{C}_\mu$ of (6.3.2) belongs to $\mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,1)}$ if $\mu = (\mu_1, \mu_2)$ satisfies $\Gamma_2 = \mu_2 - a_{0,2} = 0$, hence $\mu_2 = 0$. Figure 7 shows the three components of $\mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,0)}$.

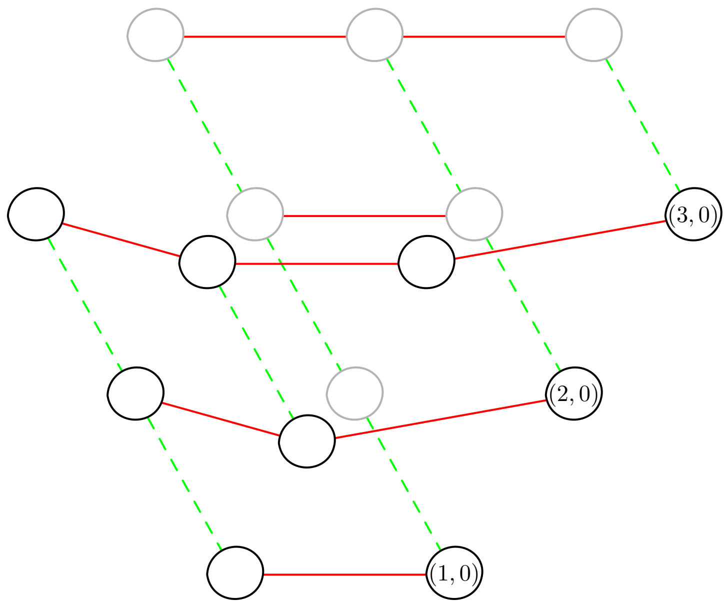

Figure 7: Branching within $\mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,0)}$.

Let $n=1$. The Gelfand-Tsetlin coefficients assigned to the vertices of $\mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,1)}$ can be read off of Figure 8.

Let us restrict our attention to the top A_1 string, $\mathcal{C}_{(3,0)} \subseteq \mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,0)}$. For the vertices $v \in \mathcal{C}_{(3,0)} \subseteq \mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,0)}$ we have

$$\mathfrak{T}(v) = \begin{pmatrix} 3 & & 1 & & 0 \\ & 3 & & 0 & \\ & & a_{22} & & \end{pmatrix}, \quad \text{and} \quad \Gamma(\mathfrak{T}(v)) = BZL(v) = \begin{bmatrix} 3 & 0 \\ & \Gamma_{22} \end{bmatrix},$$

where $\Gamma_{22} = a_{22} - a_{12} = a_{22}$.

The table below show the vertices of this string, and the corresponding Gelfand-

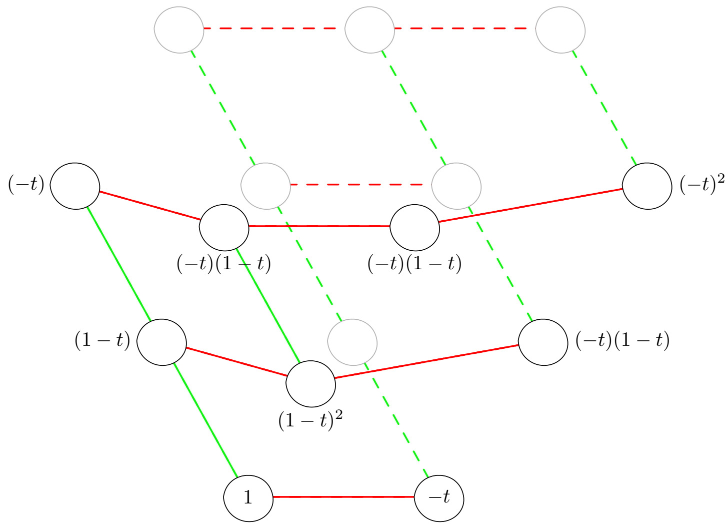

Figure 8: Gelfand-Tsetlin coefficients on $\mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,0)}$.

Tsetlin coefficients.

v	$\mathrm{wt}(v)$	$G^{(1,\lambda+\rho)}(v)$	$G^{(1,\mu)}(v)$
v_*	$(0,3,1)$	$-t$	1
v_1	$(1,2,1)$	$(-t)(1-t)$	$1-t$
v_2	$(2,1,1)$	$(-t)(1-t)$	$1-t$
v_3	$(3,0,1)$	$(-t)(-t)$	$-t$

Figure 9 shows the vertices labeled within $\mathcal{C}_{(3,0)} \subseteq \mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,0)} \subseteq \mathcal{C}_{(3,1,0)}$.

We see that if $\mu = (3,0)$ we have $d(\mu) = 3$ and $d(\lambda+\rho) = 4$. Further, $G^{(n,\lambda\lambda+\rho)}(v) = G^{(n,\mu)}(v) \cdot G^{(n,\lambda+\rho)}(v_*)$ holds for $v \in \mathcal{C}_{(3,0)}$.

It is useful to bear this example in mind while reading the following sections.

6.3.2 Branching rule for the Demazure crystal $\mathcal{C}^{(w)}_{\lambda+\rho}$

In this section, the goal is to prove part (i) of Proposition 6.3.3. We explain which components of the decomposition (6.3.2) belong to $\mathcal{C}^{(w)}_{\lambda+\rho}$ for a given w as in (6.1.2). We also characterize the elements, and in particular the lowest element, in such a component.

Recall from Proposition 3.3.2 that under the bijection between vertices of $\mathcal{C}_{\lambda+\rho}$ and

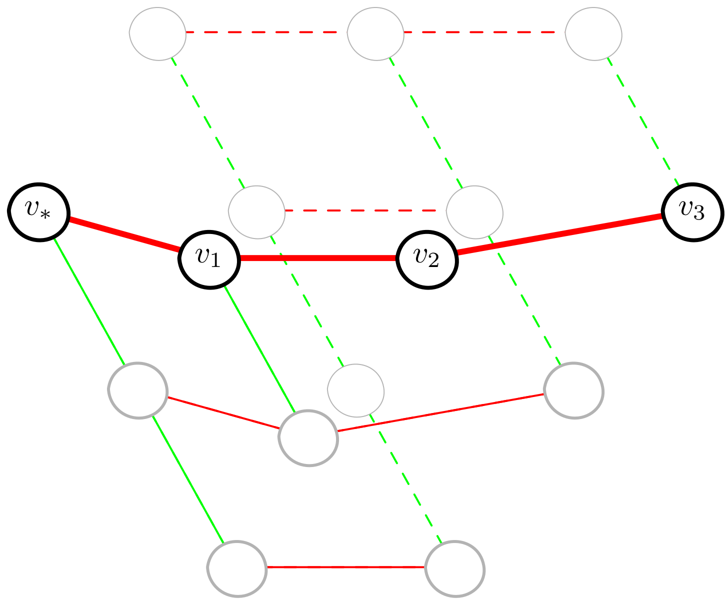

Figure 9: $\mathcal{C}_{(3,0)}$ within $\mathcal{C}^{(\sigma_1\sigma_2)}_{(3,1,0)}$.

Gelfand-Tsetlin patterns of top row $\lambda+\rho$, a vertex v corresponds to a pattern $\mathfrak{T}(v)$ if $BZL(v) = \Gamma(\mathfrak{T}(v))$. Furthermore, recall that for any vertex v, the entries of the BZL-array

$$BZL(v) = \Gamma(\mathfrak{T})_* = \begin{bmatrix} b_{\binom{r}{2}+1} & b_{\binom{r}{2}+2} & \cdots & b_{\binom{r+1}{2}} \\ & b_{\binom{r-1}{2}+1} & \cdots & b_{\binom{r}{2}} \\ & \ddots & & \\ & & b_2 & b_3 \\ & & & b_1 \end{bmatrix}$$

are lengths of segments of the BZL-path of v. If

$$w_0^{(r)} = \sigma_1\sigma_2\sigma_1\cdots\sigma_{r-1}\cdots\sigma_1\sigma_r\cdots\sigma_1 = \sigma_{\Omega_1}\sigma_{\Omega_2}\cdots\sigma_{\Omega_{\binom{r+1}{2}}},$$

then the j-th segment of the BZL-path of v consists of b_j steps on edges labeled by Ω_j. And for any j we have

$$f_{\Omega_j}^{b_j} f_{\Omega_{j-1}}^{b_{j-1}} \cdots f_{\Omega_1}^{b_1} v \neq 0,$$

$$f_{\Omega_j}^{b_j+1} f_{\Omega_{j-1}}^{b_{j-1}} \cdots f_{\Omega_1}^{b_1} v = 0,$$

holds for every $1 \leq j \leq \binom{r+1}{2}$. (Here f_{Ω_j} is the Ω_j-th Kashiwara operator.)

Now let $\mathcal{C}_\mu$ be a component in the decomposition (6.3.2). It has the structure of a highest weight crystal of type A_{r-1}, hence it has a lowest element. Call this element v_*. The following lemma characterizes the elements of $\mathcal{C}_\mu$.

Lemma 6.3.5. *The following are equivalent for an element $v \in \mathcal{C}_{\lambda+\rho}$.*

(i) $v \in \mathcal{C}_\mu$

(ii) The second row, $(a_{11}, a_{12}, \dots, a_{1r})$ *of* $\mathfrak{T}(v)$ *is* μ*.*

(iii) If b_i are the entries of $BZL(v) = \Gamma(\mathfrak{T}(v))$, then

$$f_{\Omega_{\binom{r}{2}}}^{b_{\binom{r}{2}}} f_{\Omega_{\binom{r}{2}-1}}^{b_{\binom{r}{2}-1}} \cdots f_{\Omega_1}^{b_1} v = v_*,$$

i.e. the end of the $\binom{r}{2}$-th segment of the BZL-path of v ends in v_*.

Proof. The equivalence of (i) and (ii) is included in the statement of Proposition 6.3.1. The equivalence with (iii) is easy to see from (i) of 3.3.2, since

$$w_0^{(r)} = w_0^{(r-1)} \sigma_r \cdots \sigma_1$$

and $(w_0^{(r-1)}) = \begin{pmatrix} r \\ 2 \end{pmatrix}$. □

Note that (iii) in particular means that for $v = v_*$, $b_j = 0$ if $j \leq \binom{r}{2}$. That is, all nonzero entries in $BZL(v_*)$ are in the first row.

Lemma 6.3.5 also implies that for any μ and w as in (6.1.2), i.e. $w = w_0^{(r-1)} \cdot \sigma_r \cdots \sigma_{r-k}$, the Demazure-crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$ is either disjoint from $\mathcal{C}_\mu$ or contains $\mathcal{C}_\mu$. The following lemma characterizes the components in (6.3.2) that belong to $\mathcal{C}_{\lambda+\rho}^{(w)}$. It is a straightforward consequence of Lemma 6.3.5 and our definitions.

Lemma 6.3.6. *For a weight $\mu = (\mu_1, \dots, \mu_r)$ that interleaves with $\lambda + \rho$ and $w = w_0^{(r-1)} \cdot \sigma_r \cdots \sigma_{r-k}$, the following statements about $\mathcal{C}_{\lambda+\rho}^{(w)}$ and $\mathcal{C}_\mu$ are equivalent.*

(i) $\mathcal{C}_\mu$ has at least one vertex in $\mathcal{C}_{\lambda+\rho}^{(w)}$.

(ii) $\mathcal{C}_\mu$ *is contained in* $\mathcal{C}^{(w)}_{\lambda+\rho}$.

(iii) *For any* $v \in \mathcal{C}_\mu$*, if* b_i *are the entries of* $BZL(v) = \Gamma(\mathfrak{T}(v))$*, then* $b_j = 0$ *for any* $j > \binom{r}{2} + k + 1$.

(iv) *If* v_* *is the lowest element in* $\mathcal{C}_\mu$*, then*

$$BZL(v_*) = \Gamma(\mathfrak{T}(v_*)) = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \cdots & \Gamma_{1,k+1} & 0 & \cdots & 0 \\ & 0 & & \cdots & & & 0 \\ & & & \ddots & & & \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix}. \tag{6.3.8}$$

(v) *For any* $v \in \mathcal{C}_\mu$*, if* $(\Gamma_{11}, \Gamma_{12}, \dots, \Gamma_{1r})$ *is the first row of* $BZL(v) = \Gamma(\mathfrak{T}(v))$*, then we have* $\Gamma_{1,j} = 0$ *for* $j > k + 1$.

(vi) *We have* $\mu_j = \lambda_{j+1} + r - j$ *for* $j > k + 1$.

6.3.3 Weights on the Demazure crystal $\mathcal{C}^{(w)}_{\lambda+\rho}$

Recall that $\mathbf{x} = (x_1, \dots, x_r, x_{r+1})$, $\mathbf{y} = (x_1, \dots, x_r)$ and $d(\mu)$ denotes the sum of the components of a weight μ. The next goal is to prove (6.3.5), i.e. that for any element $v \in \mathcal{C}_\mu$ of a component of $\mathcal{C}^{(w)}_{\lambda+\rho}$ we have

$$\mathbf{x}^{\mathrm{wt}(v)} = \mathbf{y}^{\mathrm{wt}_\mu(v)} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)}.$$

We start by computing the weight of the lowest element, $v_* \in \mathcal{C}_\mu \subseteq \mathcal{C}^{(w)}_{\lambda+\rho}$. Recall that by (3.2.5) and Proposition 3.3.2,

$$\mathrm{wt}(v) = \mathrm{wt}(\mathfrak{T}(v)) = (d_r, d_{r-1} - d_r, \dots, d_0 - d_1) \tag{6.3.9}$$

where d_i is the sum of the elements in the i-th row of the pattern $\mathfrak{T}(v)$ (3.2.4).

By (iv) in Lemma 6.3.6 for v_* we have $\Gamma_{ij} = a_{ij} - a_{i-1,j} + \Gamma_{i,j+1} = 0$ if $1 < i$. Thus $a_{i,j} = a_{1,j}$ for every $0 < i$. This means that the sum of entries in the i-th row of $\mathfrak{T}(v_*)$, $(0 < i)$ is

$$d_i = \sum_{j=i}^{r} a_{ij} = \sum_{j=i}^{r} a_{1j} = \sum_{j=i}^{r} a_{0j} + \sum_{j=i}^{r}(a_{1j} - a_{0j}) = \sum_{j=i}^{r}(\lambda_{j+1} + r - j) + \Gamma_{1i}.$$

This is sufficient to compute the components of $\mathrm{wt}(v_*)$ in (6.3.9). We have

$$d_r = \lambda_{r+1} + \Gamma_{1,r};$$

if $0 < i < r$ then

$$d_i - d_{i+1} = \lambda_{i+1} + r - i + \Gamma_{1,i} - \Gamma_{1,i+1};$$

and

$$d_1 - d_0 = a_{00} - \Gamma_{11} = \lambda_1 + r - \Gamma_{11}.$$

Thus by (6.3.9) we have

$$\mathrm{wt}(v_*) = (\lambda_{r+1}+\Gamma_{1,r}, \lambda_r+1+\Gamma_{1,r-1}-\Gamma_{1,r}, \dots, \lambda_2+r-1+\Gamma_{11}-\Gamma_{1,2}, \lambda_1+r-\Gamma_{11}). \quad (6.3.10)$$

Note that

$$\mathrm{wt}(v_*) - w_0(\rho) = (\lambda_{r+1} + \Gamma_{1,r}, \lambda_r + \Gamma_{1,r-1} - \Gamma_{1,r}, \dots, \lambda_2 + \Gamma_{11} - \Gamma_{1,2}, \lambda_1 - \Gamma_{11}). \quad (6.3.11)$$

We also express this in terms of the entries of $\mu = (\mu_1, \dots, \mu_r)$. Recall that

$$(a_{11}, a_{12}, \dots, a_{1r}) = (\mu_1, \dots, \mu_r),$$

and further

$$d_i = \sum_{j=i}^{r} a_{ij} = \sum_{j=i}^{r} a_{1j} = \sum_{j=i}^{r} \mu_j.$$

So we have $d_r = a_{1,r} = \mu_r$, $d_{i-1} - d_i = a_{1,i-1} = \mu_{i-1}$ for $1 < i < r$ and $d_0 - d_1 = a_{00} - \Gamma_{11} = \lambda_1 + r - \Gamma_{11}$. So

$$\mathrm{wt}(v_*) = (a_{1r}, a_{1,r-1}, \dots, a_{11}, a_{00} - \Gamma_{11}) = (\mu_r, \mu_{r-1}, \dots \mu_1, \lambda_1 + r - \Gamma_{11}). \quad (6.3.12)$$

Note that the first r components of $\mathrm{wt}(v_*)$ form the vector $w_0^{(r-1)}(\mu)$. The last component is

$$d_0 - d_1 = \lambda_1 + r - \Gamma_{11} = d(\lambda + \rho) - d(\mu).$$

If $\mathbf{x} = (x_1, \ldots, x_r, x_{r+1})$ and $\mathbf{y} = (x_1, \ldots, x_r)$, we can write

$$\mathbf{x}^{\mathrm{wt}(v_*)} = \mathbf{y}^{w_0^{(r-1)}(\mu)} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)}. \tag{6.3.13}$$

Now we turn to the weight of an arbitrary $v \in \mathcal{C}_\mu$.

Lemma 6.3.7. *Let $v \in \mathcal{C}_\mu$ be any element. Then $\mathrm{wt}(v) - \mathrm{wt}(v_*)$ is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{r-1}$. in particular, the last component of the vector $\mathrm{wt}(v) \in \mathbb{Z}^{r+1}$ agrees with the last component of $\mathrm{wt}(v_*) \in \mathbb{Z}^{r+1}$, i. e., $d(\lambda + \rho) - d(\mu) = \lambda_1 + r - \Gamma_{11}$.*

Proof. Let b_i denote the entries of $BZL(v)$. By (iii) in Proposition 3.3.2, we have

$$\mathrm{wt}(v) - \mathrm{wt}(v_{lowest}) = \sum_{i=1}^{\binom{r+1}{2}} b_i \cdot \alpha_{\Omega_i}.$$

Here v_{lowest} is the lowest element of $\mathcal{C}_{\lambda+\rho}$. It has weight $w_0(\lambda + \rho)$. Since the first $\binom{r}{2}$ entries of $BZL(v_*)$ are zero, and the entries in the top row agree with those of $BZL(v)$, we have

$$\mathrm{wt}(v) - \mathrm{wt}(v_*) = \sum_{i=1}^{\binom{r}{2}} b_i \cdot \alpha_{\Omega_i}.$$

Since $\Omega_i < r$ for $i \leq \binom{r}{2}$, this completes the proof. ∎

It is a direct consequence of (6.3.1) and Lemma 6.3.7 that we indeed have

$$\mathbf{x}^{\mathrm{wt}(v)} = \mathbf{y}^{\mathrm{wt}_\mu(v)} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)}.$$

This completes the proof of (6.3.5).

6.3.4 Gelfand-Tsetlin coefficients on the Demazure crystal $\mathcal{C}^{(w)}_{\lambda+\rho}$

Next, we examine Gelfand-Tsetlin coefficients on the components $\mathcal{C}_\mu \subseteq \mathcal{C}^{(w)}_{\lambda+\rho}$. Recall the definitions of decorations and Gelfand-Tsetlin coefficients from Section 3.2 and 3.4. We shall prove (6.3.6), i.e. that if $v \in \mathcal{C}_\mu$ is an element of such a component, then

$$G^{(n,\lambda+\rho)}(v) = G^{(n,\mu)}(v) \cdot G^{(n,\lambda+\rho)}(v_*).$$

where v_*, as above, denotes the lowest element of the component $\mathcal{C}_\mu$.

We would like to restrict our attention to weights μ that are strongly dominant, to weights $\mu = (\mu_1, \ldots, \mu_r)$, where $\mu_1 > \mu_2 > \cdots > \mu_r$. We can do this because of the following remark.

Remark 6.3.8. The statement (6.3.6) is trivial if μ is not strongly dominant. This is because by Remark 3.5.1, if a pattern is non-strict, then the corresponding Gelfand-Tsetlin coefficient is zero. By Lemma 6.3.5 (ii) is μ is not strongly dominant, then the pattern $\mathfrak{T}(v)$ is non-strict for any $v \in \mathcal{C}_\mu$, and hence $G^{(n,\lambda+\rho)}(v) = G^{(n,\lambda+\rho)}(v_*) = 0$.

So we may assume that $\mu_1 > \mu_2 > \cdots > \mu_r$. We have (by (6.3.8)) that for every entry $a_{i,j}$ of the pattern $\mathfrak{T}(v_*)$ below the top row $(1 \leq i)$, $a_{i,j} = a_{1,j} = \mu_j$. It follows that we have $a_{i-1,j-1} > a_{i,j} = a_{i-1,j}$, so every such entry is circled, but not boxed. By Definition 3.4.1 this implies

$$G^{(n,\lambda+\rho)}(\mathfrak{T}(v_*)) = \prod_{1\leq i\leq j\leq r} g^{n,\lambda+\rho}_{ij}(\mathfrak{T}(v_*)) = \prod_{1\leq j\leq r} g^{n,\lambda+\rho}_{1j}(\mathfrak{T}(v_*)) \cdot \prod_{2\leq i\leq j\leq r} 1.$$

Furthermore, by Lemma 6.3.6 (vi) we have $\mu_j = \lambda_{j+1} + r - j$ for $j > k+1$, so the entries in the first row are also circled after the first $k+1$. Thus we have

$$G^{(n,\lambda+\rho)}(\mathfrak{T}(v_*)) = \prod_{1\leq j\leq k+1} g^{n,\lambda+\rho}_{1j}(\mathfrak{T}(v_*)). \tag{6.3.14}$$

Now let $v \in \mathcal{C}_\mu$ be any element. By Lemma 6.3.5, the first row of $\Gamma(\mathfrak{T}(v))$ agrees with the first row of $\Gamma(\mathfrak{T}(v_*))$. This implies that

$$G^{(n,\lambda+\rho)}(\mathfrak{T}(v)) = \prod_{1\leq i\leq j\leq r} g^{n,\lambda+\rho}_{ij}(\mathfrak{T}(v)) = \prod_{1\leq j\leq k+1} g^{n,\lambda+\rho}_{ij}(\mathfrak{T}(v_*)) \cdot \prod_{2\leq i\leq j\leq r} g^{n,\lambda+\rho}_{ij}(\mathfrak{T}(v)). \tag{6.3.15}$$

Now recall that by Lemma 6.3.5 (ii), the top row of $\mathfrak{T}(v)$ is $\lambda+\rho$, and the row below that is μ. The definition of every factor $g_{i,j}(\mathfrak{T}(v))$ only depends on the entries of $\mathfrak{T}(v)$ in the positions (i,j), $(i-1,j-1)$ and $(i-1,j)$. This implies that for any $v\in\mathcal{C}_\mu$ and $2\le i\le j\le r$ we have

$$g_{ij}^{n,\lambda+\rho}(\mathfrak{T}(v)) = g_{i,j}^{n,\mu}(\mathfrak{T}(v)). \tag{6.3.16}$$

(Here we mean the index (i,j) as corresponding to the same position in $\mathfrak{T}(v)$ in both sides. That is, we think of the rows of the Γ-array parameterizing the crystal of type A_{r-1} as indexed from 2 to r.) From (6.3.14), (6.3.15) and (6.3.16) we conclude

$$G^{(n,\lambda+\rho)}(\mathfrak{T}(v)) = \prod_{1\le j\le k+1} g_{ij}^{n,\lambda+\rho}(\mathfrak{T}(v_*))\cdot\prod_{2\le i\le j\le r} g_{ij}^{n,\mu}(\mathfrak{T}(v)) = G^{(n,\lambda+\rho)}(\mathfrak{T}(v_*))\cdot G^{n,\mu}(\mathfrak{T}(v)).$$

This completes the proof of (6.3.6).

6.3.5 Notation for branching of the Demazure crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$

The branching of the Demazure crystal, as stated in Proposition 6.3.3 implies that when dealing with $\mathcal{C}_{\lambda+\rho}^{(w)}$, one can treat the components $\mathcal{C}_\mu$ as units. As a result, a lot of the computations will only involve $\lambda+\rho$ and μ. Thus we may restrict our attention to the top two rows of the Gelfand-Tsetlin patterns parameterizing $\mathcal{C}_\mu\subseteq\mathcal{C}_{\lambda+\rho}^{(w)}$.

Here we introduce some notation that makes use of this simplification. Recall that the first row of the Γ-array, $(\Gamma_{11},\dots,\Gamma_{1r})$ is the same for every element of the component $\mathcal{C}_\mu\subseteq\mathcal{C}_{\lambda+\rho}^{(w)}$. In most cases λ will be fixed, so it is more convenient to refine our notation based on this r-tuple. Lemma 6.3.10 justifies the choices made in the following definition.

Definition 6.3.9. Let $\lambda=(\lambda_1,\dots,\lambda_r,\lambda_{r+1})$, $\mu=(\mu_1,\dots,\mu_r)$, and $\Gamma=(\Gamma_{11},\dots,\Gamma_{1r})\in\mathbb{Z}^r$. (Set $\Gamma_{1,r+1}:=0$.) We call Γ *λ-admissible*

$$\Gamma_{1,j+1}\le\Gamma_{1,j}\le\Gamma_{1,j+1}+\lambda_j-\lambda_{j+1}+1 \text{ for every } 1\le j\le r. \tag{6.3.17}$$

We call Γ *(λ,k)-admissible* if

$$\Gamma \text{ is } \lambda\text{-admissible and } \Gamma_{i,j}=0 \text{ for } k+1<j. \tag{6.3.18}$$

We call Γ *non-strict* if

$$\Gamma_{1,j-1} = \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \text{ for at least one } 1 < j \leq r, \tag{6.3.19}$$

and *strict* if it is not non-strict. We define a weight function and Gelfand-Tsetlin coefficients for a Γ r-tuple that is λ-admissible:

$$\mathrm{wt}^{(\lambda)}(\Gamma) = (\lambda_{r+1} + \Gamma_{1,r}, \lambda_r + \Gamma_{1,r-1} - \Gamma_{1,r}, \ldots, \lambda_2 + \Gamma_{11} - \Gamma_{1,2}, \lambda_1 - \Gamma_{11}); \tag{6.3.20}$$

$$\begin{aligned} G_1^{(\lambda)}(\Gamma) = G_1^{(n,\lambda)}(\Gamma) &= \prod_{j=1}^{r} g_{1j}^{(\lambda)}(\Gamma) = \prod_{j=1}^{r} g_{1j}^{(n,\lambda)}(\Gamma), \\ g_{1j}^{(n,\lambda)}(\Gamma) &= \begin{cases} 1 & \Gamma_{1,j+1} = \Gamma_{1,j} < \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \\ h^\flat(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} < \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \\ g^\flat(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{6.3.21}$$

For convenience, we say Γ is associated to λ and μ and write $\Gamma = \Gamma(\lambda, \mu)$ if

$$\Gamma_{1,j} - \Gamma_{1,j+1} = \mu_j - (\lambda_{j+1} + r - j). \tag{6.3.22}$$

is satisfied. This is the case if Γ is the first row of an array $\Gamma(\mathfrak{T})$ of a pattern $\mathfrak{T}$ with top two rows $\lambda + \rho_r$ and μ.

Parts (i)-(v) of the following lemma justify the choices in Definition 6.3.9. Part (vi) will be convenient in later computations.

Lemma 6.3.10. *Let* $\lambda = (\lambda_1, \ldots, \lambda_r, \lambda_r + 1)$, $\mu = (\mu_1, \ldots, \mu_r)$ *and* $\Gamma = (\Gamma_{11}, \ldots, \Gamma_{1r}) = \Gamma(\lambda, \mu)$ *associated to* λ *and* μ *by* (6.3.22)*. Then the following statements hold.*

(i) The tuple Γ *is* λ*-admissible if and only if the weights* $\lambda + \rho_r$ *and* μ *interleave.*

(ii) Let w *be as in* (6.1.2)*. Then* Γ *is* (λ, k)*-admissible if and only if* $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$*.*

(iii) The tuple Γ *is strict if and only if* μ *is strongly dominant.*

(iv) Let v_* be the lowest element of a component $\mathcal{C}_\mu \subseteq \mathcal{C}^{(w)}_{\lambda+\rho}$. Then

$$\mathrm{wt}(v_*) - w_0^{(r)}(\rho_r) = \mathrm{wt}^{(\lambda)}(\Gamma). \tag{6.3.23}$$

(v) Let v_* be the lowest element of a component $\mathcal{C}_\mu \subseteq \mathcal{C}^{(w)}_{\lambda+\rho}$. Then

$$G^{(n,\lambda+\rho)}(v_*) = G_1^{(n,\lambda)}(\Gamma). \tag{6.3.24}$$

(vi) With the notation as above we have

$$\mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)} = \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r}. \tag{6.3.25}$$

Proof. Note first that the condition (6.3.22) is satisfied exactly if $\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r})$ is the first row of the array $\Gamma(\mathfrak{T})$, When $\mathfrak{T}$ is a pattern with top rows $\lambda+\rho$ and μ. With this observation, the proof is straightforward from the rest of this section. For (i), we have that by (6.3.22),

$$\lambda_j + r - j + 1 \geq \mu_j \geq \lambda_{j+1} + r - j \iff \lambda_j - \lambda_{j+1} + 1 \geq \Gamma_{1,j} - \Gamma_{1,j+1} \geq 0.$$

Again by (6.3.22) we have that Γ is (λ, k)-admissible if and only if for any $k+1 < j$ we have

$$\Gamma_{1,j} = \mu_j - (\lambda_{j+1} + r - j) + \Gamma_{1,j+1} = \Gamma_{1,j+1} = 0 \iff \mu_j = \lambda_{j+1} + r - j.$$

By part (i) of Proposition 6.3.3, this is equivalent to $\mathcal{C}_\mu \subseteq \mathcal{C}^{(w)}_{\lambda+\rho}$. This proves (ii). Part (iii) is true because

$$\mu_{j-1} = \mu_j \iff \mu_{j-1} = \lambda_j + r - j + 1 = \mu_j$$

and by (6.3.22)

$$\Gamma_{1,j-1} = \Gamma_{1,j} \iff \Gamma_{1,j-1} - \Gamma_{1,j} = \mu_{j-1} - (\lambda_j + r - j + 1) = 0;$$

$$\Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \iff \Gamma_{1,j} - \Gamma_{1,j+1} = \mu_j - (\lambda_{j+1} + r - j) = \lambda_j - \lambda_{j+1} + 1.$$

Part (iv) and part (v) follow from (6.3.11) and (6.3.14), respectively. Note that $\mathrm{wt}_\mu(v_*) = w_0^{(r-1)}(\mu)$ and $d(\lambda+\rho)-d(\mu) = \lambda_1+r-\Gamma_{11}$. Then from part (iv) and part (ii) of Proposition 6.3.3 we have

$$\begin{aligned}
\mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)} &= \mathbf{x}^{\mathrm{wt}(v_*)} \cdot \mathbf{x}^{-w_0^{(r)}(\rho_r)} = \mathbf{x}^{\mathrm{wt}(v_*)} \cdot \mathbf{y}^{w_0^{(r-1)}(\rho_{r-1})} \cdot x_{r+1}^{-r} \\
&= \mathbf{y}^{\mathrm{wt}_\mu(v_*)} \cdot \mathbf{y}^{w_0^{(r-1)}(\rho_{r-1})} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \\
&= \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r}.
\end{aligned}$$

□

6.4 Simplifications and proof of Theorem 6.2.1.

We are now ready to start the proof of Theorem 6.2.1. As discussed in Section 6.2, this will also prove Theorem 6.2.2. By Remark 6.3.4, it suffices to prove $IW_{r,k}$ for any pair of integers $0 \le k < r$. (Theorem 6.2.2 is the case $k = r-1$.)

The proof is by induction on r. We start from the statement of $IW_{r,k}$, and simplify it step by step. The technical ingredients will be stated as lemmas or propositions along the way. To make the main ideas of the proof more transparent, the technical proofs are presented later, in Section 6.5. The last ingredient is the proof of Proposition 7.0.4, which forms the contents of Chapter 7. The induction argument is summarized in Section 6.4.1.

To set up the induction, we show that both sides of $IW_{r,k}$ can be rewritten to have elements reminiscent of the two sides of Tok_{r-1}. The main tool in rewriting the right hand side of $IW_{r,k}$ is Proposition 6.3.3, Definition 6.3.9 and Lemma 6.3.10. The following lemma relates the left-hand side of $IW_{r,k}$ to that of Tok_{r-1}. It is really just a statement about the Bruhat-order. We omit the proof.

Lemma 6.4.1. *Let $w = w_0^{(r-1)} \cdot \sigma_r \cdots \sigma_{r-k}$. Then*

$$\sum_{u \le w} \mathcal{T}_u = \left(\sum_{u \le w_0^{(r-1)}} \mathcal{T}_u \right) \cdot (1 + \mathcal{T}_r + \mathcal{T}_r\mathcal{T}_{r-1} + \cdots + \mathcal{T}_r \cdots \mathcal{T}_{r-k}).$$

Note that the operator

$$\left(\sum_{u\le w_0^{(r-1)}} \mathcal{T}_u\right) = \Delta_t^{(r-1)} \cdot \mathcal{D}_{w_0^{(r-1)}} \tag{6.4.1}$$

is exactly the one appearing on the left-hand side of Tok_{r-1}. (The equality holds by Theorem 4.2.1.)

As for the right hand side of $IW_{r,k}$, we have by part (iv) of Proposition 6.3.3 that

$$\mathbf{x}^{-w_0(\rho)} \cdot \sum_{v\in\mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v)\cdot \mathbf{x}^{\mathrm{wt}(v)} = \sum_{\mu} G^{(n,\lambda+\rho)}(v_*)\cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \\ \cdot\left(\mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})}\cdot \sum_{v\in\mathcal{C}_\mu} G^{(n,\mu)}(v)\cdot \mathbf{y}^{\mathrm{wt}_\mu(v)}\right). \tag{6.4.2}$$

Here the sum is over all $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ that interleave with $\lambda+\rho$ and $\mu_j = \lambda_{j+1}+r-j$ for $j > k+1$.

Now let $\nu = (\nu_1, \nu_2, \dots, \nu_r)$ such that $\nu_i + r - i = \mu_i$, or $\nu + \rho_{r-1} = \mu$. Then ν is dominant if and only if μ is strongly dominant. The expression in parentheses on the right hand side of (6.4.2) can be written as

$$\mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})}\cdot \sum_{v\in\mathcal{C}_\mu} G^{(n,\mu)}(v)\cdot \mathbf{y}^{\mathrm{wt}_\mu(v)} = \mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})}\cdot \sum_{v\in\mathcal{C}_{\nu+\rho_{r-1}}} G^{(n,\nu+\rho_{r-1})}(v)\cdot \mathbf{y}^{\mathrm{wt}_{\nu+\rho}(v)} \tag{6.4.3}$$

Note that if ν is dominant, then the right hand side is exactly the right hand side of Tok_{r-1}. That is, if ν is dominant and Tok_{r-1} holds, then we have

$$\left(\sum_{u\le w_0^{(r-1)}} \mathcal{T}_u\right)\mathbf{y}^{w_0^{(r-1)}\nu} = \mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})}\cdot \sum_{v\in\mathcal{C}_{\nu+\rho_{r-1}}} G^{(n,\nu+\rho_{r-1})}(v)\cdot \mathbf{y}^{\mathrm{wt}_{\nu+\rho}(v)}.$$

Now in (6.4.2) the expressions (6.4.3) are multiplied by a power of x^{r+1}. This, as an operator, commutes with $\mathcal{T}_1, \dots, \mathcal{T}_{r-1}$, so it commutes with T_u for any $u \le w_0^{(r-1)}$. This means that if Tok_{r-1} holds, we can rewrite the right hand side of (6.4.2) and of $IW_{r,k}$

as the operator in (6.4.1) acting on a polynomial. This is the content of the following proposition. It is more conveniently phrased with the terminology of Definition 6.3.9. The argument above gives the main idea of the proof: the rest is keeping track of notation and handling the case when μ is not strongly dominant. The details are included in Section 6.5.1.

Proposition 6.4.2. *Assume $IW_{r-1,r-2}(= Tok_{r-1})$ holds. Then we have*

$$\mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}^{(w)}_{\lambda+\rho}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)} = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \sum_{\substack{\Gamma = (\Gamma_{11},\ldots,\Gamma_{1r}) \\ \Gamma\ (\lambda,k)-\text{admissible}}} G_1^{\lambda}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)} \quad (6.4.4)$$

Lemma 6.4.1 and Proposition 6.4.2 together produce both sides of $IW_{r,k}$ as the operator in (6.4.1) applied to a polynomial. The fact that the "inputs" are the same up to annihilation by this operator is the statement that we will call $M_{r,k}$. The next proposition phrases the statement $M_{r,k}$ explicitly for any $0 \leq k < r$.

Proposition 6.4.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_r, \lambda_{r+1})$ be any dominant weight, $0 \leq k < r$ integers. Then we have*

$$(1 + \mathcal{T}_r + \mathcal{T}_r\mathcal{T}_{r-1} + \cdots + \mathcal{T}_r \cdots \mathcal{T}_{r-k})\mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma = (\Gamma_{11},\ldots,\Gamma_{1r}) \\ \Gamma\ (\lambda,k)-\text{admissible}}} G_1^{\lambda}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)} \quad (6.4.5)$$

Here $\equiv$ means that the difference of the left and right hand side is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. Call this statement (that (6.4.5) holds for any λ dominant weight) $\mathbf{M_{r,k}}$.

The statement $M_{r,k}$ lends itself to an obvious simplification. On the left hand side, there is a sum of $k+1$ strings of Demazure-Lusztig operators. The statement $N_{r,k}$ involves only one of them.

Proposition 6.4.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_r, \lambda_{r+1})$ be any dominant weight, $0 \leq k < r$ integers. Then we have*

$$(\mathcal{T}_r \cdots \mathcal{T}_{r-k})\mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma = (\Gamma_{11},\ldots,\Gamma_{1r}) \\ \Gamma\ (\lambda,k)-\text{admissible} \\ \Gamma_{1,k+1} \neq 0}} G_1^{\lambda}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)} \quad (6.4.6)$$

Here $\equiv$ means that the difference of the left and right hand side is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. Call this statement (that (6.4.6) holds for any λ dominant weight) $\mathbf{N_{r,k}}$.

Remark 6.4.5. Note that in both $M_{r,k}$ and $N_{r,k}$, λ is not required to be effective, i.e. it we may have negative components. We may however always assume that it is effective. We may replace λ by $\kappa = (\lambda_1 + K, \ldots, \lambda_r + K, \lambda_{r+1} + K)$ to make it effective. This is because as an operator, multiplication by $(x_1 \cdot x_2 \cdots x_{r+1})^K$ commutes with $\mathcal{T}_i$ and $\mathcal{D}_i$ for any $1 \le i \le r$, and

$$\begin{aligned} \mathbf{x}^{w_0(\kappa)} =& \mathbf{x}^{w_0(\lambda)} \cdot (x_1 \cdot x_2 \cdots x_{r+1})^K, \\ \mathbf{x}^{\mathrm{wt}^\kappa(\Gamma)} =& \mathbf{x}^{\mathrm{wt}^\lambda(\Gamma)} \cdot (x_1 \cdot x_2 \cdots x_{r+1})^K. \end{aligned}$$

The following (straightforward) lemma is proved in Section 6.5.2.

Lemma 6.4.6. *Proposition 6.4.4 implies Proposition 6.4.3. That is, we have*

$$\forall r,k \quad N_{r,k} \quad \Longrightarrow \quad \forall r,k \quad M_{r,k}. \tag{6.4.7}$$

As a last step in the sequence of replacing Theorem 6.2.1 with simpler statements, we note that in the statement $N_{r,k}$, the parameter k is the interesting one. This is the content of Lemma 6.4.7 below. The proof is a simple renaming of variables; see Section 6.5.3.

Lemma 6.4.7. *If $N_{k+1,k}$ is true, then $N_{r,k}$ is true for every $r > k$. In fact, $N_{k+1,k}$ implies a slightly stronger statement than $N_{r,k}$: the difference of the left-hand side and the right-hand side is annihilated not just by $\mathcal{D}_{w_0^{(r-1)}}$, but by the Demazure-operator corresponding to the long word in the group $\langle \sigma_{r-k}, \sigma_{r-k+1}, \ldots, \sigma_{r-1} \rangle$.*

The statement $N_{k+1,k}$ will be proved in Chapter 7 as Proposition 7.0.4. We are now ready to summarize the above in the proof of Theorem 6.2.1.

6.4.1 The proof of Theorem 6.2.1

By Proposition 7.0.4 (proved in Chapter 7), we have that $N_{k+1,k}$ holds for any nonnegative k. By Lemma 6.4.7, this implies that $N_{r,k}$ holds for any pair of integers $0 \le k < r$, i.e.

Proposition 6.4.4 is true. By Lemma 6.4.6, this proves Proposition 6.4.3, i. e. $M_{r,k}$ for any pair of integers $0 \leq k < r$.

We prove $IW_{r,k}$ for any pair of integers $0 \leq k < r$ by induction on r.

To start, notice that both $M_{1,0}$ and $IW_{1,0}$ state that if $\lambda_1 \geq \lambda_2$, then

$$\begin{aligned}(1+T_1)x_1^{\lambda_2}x_2^{\lambda_1} &= \sum_{\Gamma_{11}=0}^{\lambda_1-\lambda_2+1} G_1^{(\lambda)}(\Gamma_{11}) \cdot x_1^{\lambda_2+\Gamma_{11}} x_2^{\lambda_2-\Gamma_{11}} \\ &= \frac{1}{x_2}\sum_{\mathfrak{T}} G(\mathfrak{T}) \cdot \mathbf{x}^{wt(\mathfrak{T})},\end{aligned} \tag{6.4.8}$$

where the sum is over all Gelfand-Tsetlin patterns $\mathfrak{T}$ of the form

$$\mathfrak{T} = \begin{pmatrix} \lambda_1+1 & & \lambda_2 \\ & \lambda_2+\Gamma_{11} & \end{pmatrix}.$$

Thus $IW_{1,0}$ is the same as $M_{1,0}$, and in particular, $IW_{r,k}$ is true if $r = 1$.

Now let $r > 1$, $0 \leq k < r$ and assume that $IW_{r-1,r-2} = Tok_{r-1}$ is true. We know $M_{r,k}$ holds, hence

$$(1+\mathcal{T}_r+\mathcal{T}_r\mathcal{T}_{r-1}+\cdots+\mathcal{T}_r\cdots\mathcal{T}_{r-k})\mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma=(\Gamma_{11},\ldots,\Gamma_{1r}) \\ \Gamma\ (\lambda,k)-\text{admissible}}} G_1^{\lambda}(\Gamma)\cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)}, \tag{6.4.9}$$

i.e. the difference of the two sides of (6.4.9) is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. By Theorem 4.2.1, the difference is then also annihilated by

$$\Delta_t^{(r-1)} \cdot \mathcal{D}_{w_0^{(r-1)}} = \sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u.$$

That is, we have

$$\begin{aligned}&\left(\sum_{u\leq w_0^{(r-1)}} \mathcal{T}_u\right)(1+\mathcal{T}_r+\mathcal{T}_r\mathcal{T}_{r-1}+\cdots+\mathcal{T}_r\cdots\mathcal{T}_{r-k})\mathbf{x}^{w_0(\lambda)} \\ &\qquad = \left(\sum_{u\leq w_0^{(r-1)}} \mathcal{T}_u\right) \sum_{\substack{\Gamma=(\Gamma_{11},\ldots,\Gamma_{1r}) \\ \Gamma\ (\lambda,k)-\text{admissible}}} G_1^{\lambda}(\Gamma)\cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)}.\end{aligned} \tag{6.4.10}$$

Rewriting the left hand side of (6.4.10) by Lemma 6.4.1, and the right hand side by Proposition 6.4.2 we arrive at

$$\left(\sum_{u\leq w_0^{(r)}} \mathcal{T}_u\right)\mathbf{x}^{w_0(\lambda)} = \mathbf{x}^{-w_0(\rho)}\cdot \sum_{v\in\mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v)\cdot \mathbf{x}^{\mathrm{wt}(v)}. \tag{6.4.11}$$

This is exactly the statement $IW_{r,k}$.

Thus $IW_{r,k}$ is true for any pair of integers $0 \leq k < r$. By Remark 6.3.4, this completes the proof of Theorem 6.2.1. □

6.5 Proof of Lemmas

In this section we include a few technical proofs. The meaning of the statements sometimes depends on n, this should be kept in mind even when not referred to explicitly.

The following lemmas describe simple properties of the Demazure operators, and will be of use in our computations.

Lemma 6.5.1. *We have the following two facts about Demazure operators annihilating polynomials.*

(i) A polynomial f is annihilated by $\mathcal{D}_i = \mathcal{D}_{\sigma_i}$ if and only if $\sigma_i(x_{i+1}^n \cdot f) = x_{i+1}^n \cdot f$.

(ii) If $\mathcal{D}_w(g) = 0$ for some w in the Weyl group W, and w_0 is the long element of W, then $\mathcal{D}_{w_0} g = 0$.

Proof. The proof of (i) is obvious from the definition of $\mathcal{D}_i$:

$$\mathcal{D}_i(f) = \frac{f - \frac{x_i^n}{x_{i+1}^n}\cdot\sigma_i(f)}{1-\frac{x_i^n}{x_{i+1}^n}} = 0) \quad\Longleftrightarrow\quad f = \frac{x_i^n}{x_{i+1}^n}\cdot\sigma_i(f) \quad\Longleftrightarrow\quad x_{i+1}^n\cdot f = \sigma_i(x_{i+1}^n\cdot f).$$

For (ii), let $u = w_0 w^{-1}$, so that $w_0 = u\cdot w$. Since w_0 is the longest element, we have $\ell(w_0) = \ell(u) + \ell(w)$, and as a consequence $\mathcal{D}_{w_0} = \mathcal{D}_u \circ \mathcal{D}_w$. Thus $\mathcal{D}_{w_0} g = \mathcal{D}_u(\mathcal{D}_w g) = \mathcal{D}_u(0) = 0$. □

The following lemma shows a symmetric monomial with respect to σ_i.

Lemma 6.5.2. *$\sigma_i(x_i^{a+1}x_{i+1}^{a+n}) = x_i^{a+1}x_{i+1}^{a+n}$ for every n.*

Proof. This can be proved by a simple computation. By (2.4.2) we have

$$\sigma_i(x_i^{a+1}x_{i+1}^{a+n}) = \frac{x_i^{a+n}\cdot x_{i+1}^{a+1}}{1-t^n\cdot\left(\frac{x_i}{x_{i+1}}\right)^n}\cdot\left[\left(\frac{x_i}{x_{i+1}}\right)^{-r_n(n-1)}\cdot(1-t^n)\right.$$
$$\left.-t^n\cdot g_n\cdot\left(\frac{x_i}{x_{i+1}}\right)^{1-n}\cdot\left(1-\left(\frac{x_i}{x_{i+1}}\right)^n\right)\right]$$

Here $-r_n(n-1) = 1-n$ and $g_n = -1$ for any n, hence

$$\begin{aligned}
\sigma_i(f) &= \frac{x_i^{a+n}\cdot x_{i+1}^{a+1}}{1-t^n\cdot\left(\frac{x_i}{x_{i+1}}\right)^n}\cdot\left[\left(\frac{x_i}{x_{i+1}}\right)^{1-n}\cdot(1-t^n)+t^n\cdot\left(\frac{x_i}{x_{i+1}}\right)^{1-n}\cdot\left(1-\left(\frac{x_i}{x_{i+1}}\right)^n\right)\right]\\
&= \frac{x_i^{a+n}\cdot x_{i+1}^{a+1}}{1-t^n\cdot\left(\frac{x_i}{x_{i+1}}\right)^n}\cdot\left(\frac{x_i}{x_{i+1}}\right)^{1-n}\cdot\left[(1-t^n)+t^n\cdot\left(1-\left(\frac{x_i}{x_{i+1}}\right)^n\right)\right]\\
&= \frac{x_i^{a+n}\cdot x_{i+1}^{a+1}}{1-t^n\cdot\left(\frac{x_i}{x_{i+1}}\right)^n}\cdot\left(\frac{x_i}{x_{i+1}}\right)^{1-n}\cdot\left[1-t^n\cdot\left(\frac{x_i}{x_{i+1}}\right)^n\right]\\
&= x_i^{a+n}\cdot x_{i+1}^{a+1}\cdot\left(\frac{x_i}{x_{i+1}}\right)^{1-n}\\
&= x_i^{a+1}x_{i+1}^{a+n}.
\end{aligned}$$

This completes the proof. □

Corollary 6.5.3. *If $\beta = (\beta_1,\dots,\beta_{r+1})$ and $\beta_i = \beta_{i+1}+1$, then $\mathcal{D}_i(\mathbf{x}^\beta) = 0$.*

Proof. This is a trivial consequence of Lemmas 6.5.1 and 6.5.2, as the action of σ_i only involves the exponents of x_i and x_{i+1}. □

6.5.1 Proof of Proposition 6.4.2

We prove that if Tok_{r-1} (equivalently, $IW_{r-1,r-2}$) holds, then

$$\mathbf{x}^{-w_0(\rho)}\cdot\sum_{v\in\mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v)\cdot\mathbf{x}^{\mathrm{wt}(v)} = \left(\sum_{u\le w_0^{(r-1)}}\mathcal{T}_u\right)\sum_{\substack{\Gamma=(\Gamma_{11},\dots,\Gamma_{1r})\\ \Gamma\ (\lambda,k)-\text{admissible}}} G_1^\lambda(\Gamma)\cdot\mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)}. \quad (6.5.1)$$

By Proposition 6.3.3 part (iv), we have

$$\mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}^{(w)}_{\lambda+\rho}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)} = \sum_{\mu} G^{(n,\lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \\ \cdot \left(\mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})} \cdot \sum_{v \in \mathcal{C}_\mu} G^{(n,\mu)}(v) \cdot \mathbf{y}^{\mathrm{wt}_\mu(v)} \right). \tag{6.5.2}$$

Here the sum is over all $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ that interleave with $\lambda+\rho$ and $\mu_j = \lambda_{j+1}+r-j$ for $j > k+1$. We claim that

$$\mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})} \cdot \sum_{v \in \mathcal{C}_\mu} G^{(n,\mu)}(v) \cdot \mathbf{y}^{\mathrm{wt}_\mu(v)} = \left(\sum_{u \le w_0^{(r-1)}} \mathcal{T}_u \right) \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})}. \tag{6.5.3}$$

Since μ interleaves with $\lambda+\rho$, it is dominant. We distinguish between two cases according to whether μ is strongly dominant or not.

If μ is strongly dominant, then $(\mu - \rho_{r-1})$ is dominant. In this case (6.5.3) is the statement Tok_{r-1} $(IW_{r-1,r-2})$ for the weight $\mu-\rho_{r-1})$, hence it is true by the assumption that Tok_{r-1} holds.

We show that if μ is not strongly dominant, then both sides of (6.5.3) are zero. The left hand side is zero by Remark 6.3.8. We show that the operator on the right hand side of (6.5.3) annihilates the monomial $\mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})}$. Since μ is not strongly dominant, we have $\mu_j = \mu_{j+1}$ for some $1 \le j \le r-1$. Let $i = r+1-(j+1) = r-j$, $i+1 = r-j+1$. Then in the monomial $\mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})}$, x_i appears with exponent $\mu_{j+1}-r+j+1 = \mu_j-r+j+1$, while x_{i+1} appears with exponent $\mu_j - r + j$. Thus, by Corollary 6.5.3, we have that for at least one index $1 \le i = r-j \le r-1$,

$$\mathcal{D}_i \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})} = 0.$$

The long element in the Weyl group generated by $\sigma_1, \dots, \sigma_{r-1}$ is $w_0^{(r-1)}$. Thus by Lemma 6.5.1, we have

$$\mathcal{D}_{w_0^{(r-1)}} \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})} = 0.$$

Now by Theorem 4.2.1, the operator on the left hand side of (6.5.3) is $\Delta_t^{(r-1)} \cdot \mathcal{D}_{w_0^{(r-1)}}$. Thus the right hand side of (6.5.3) is indeed zero if μ is not strongly dominant.

Having established (6.5.3), we have that

$$\mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)} = \sum_{\mu} G^{(n,\lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \cdot \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})}. \tag{6.5.4}$$

The operator

$$\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u$$

commutes with multiplication by constants and powers of x_{r+1}. This means that

$$\mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)} = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \sum_{\mu} G^{(n,\lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})}. \tag{6.5.5}$$

By Lemma 6.3.10, we have that if $\Gamma = \Gamma(\lambda, \mu) = (\Gamma_1, \ldots, \Gamma_r)$ as in Definition 6.3.9, then Γ is (λ, k) admissible exactly if μ appears in the summation in (6.5.2), furthermore

$$G^{(n,\lambda+\rho)}(v_*) = G_1^{(n,\lambda)}(\Gamma)$$

and

$$\mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} = \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)}.$$

Thus we may rewrite (6.5.5) further as

$$\mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\mathrm{wt}(v)} = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \sum_{\substack{\Gamma=(\Gamma_{11},\ldots,\Gamma_{1r}) \\ \Gamma\ (\lambda,k)-\text{admissible}}} G_1^{(n,\lambda)}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)}. \tag{6.5.6}$$

This is exactly (6.5.1); the proof of Proposition 6.4.2 is complete.

6.5.2 Proof of Lemma 6.4.6

We prove 6.4.7, that is,

$$\forall r,k \;\; N_{r,k} \quad \Longrightarrow \quad \forall r,k \;\; M_{r,k}.$$

Note that by Definition 6.3.9, if Γ is (λ,k)-admissible, we have $\Gamma_{11} \geq \Gamma_{1,2} \geq \cdots \geq \Gamma_{1,k+1} \geq \Gamma_{1,k+2} = 0$. Let l be the smallest integer such that $\Gamma_{1,l+2} = 0$. Then it is the only $0 \leq l \leq k$ such that Γ is (λ,l)-admissible and $\Gamma_{1,l+1} \neq 0$. This implies that

$$N_{r,0} + N_{r,1} + \cdots + N_{r,k-1} + N_{r,k} = M_{r,k}.$$

Note that both $M_{r,0}$ and $N_{r,0}$ state the obvious $\mathbf{x}^{w_0(\lambda)} = \mathbf{x}^{w_0(\lambda)}$.

6.5.3 Proof of Lemma 6.4.7

Recall that the goal is to prove that $N_{k+1,k}$ implies $n_{r,k}$ for every $r \geq k+1$.

Let λ and $\mathbf{x} = (x_1, x_2, \ldots, x_{r+1})$ be as before, but, for the extent of this proof, take $\nu = (\lambda_1, \ldots, \lambda_{k+2})$, $\mathbf{y} = (x_{r-k}, x_{r-k+1}, \ldots, x_{r+1})$, and let $w_0^{(k+1)}$ denote the long word in $\langle \sigma_{r-k}, \sigma_{r-k+1}, \ldots, \sigma_r \rangle$ and $w_0^{(k)}$ be the long word in $\langle \sigma_{r-k}, \sigma_{r-k+1}, \ldots, \sigma_{r-1} \rangle$. Assume $N_{k+1,k}$ is true. Then, with notation as above, we have

$$(\mathcal{T}_r \cdots \mathcal{T}_{r-k}) \mathbf{y}^{w_0^{(k+1)}(\nu)} \equiv \sum_{\substack{\Gamma = (\Gamma_{1,1}, \ldots, \Gamma_{1,k+1}) \\ \Gamma \;\; (\nu,k)-\text{admissible} \\ \Gamma_{1,k+1} \neq 0}} G_1^{(\nu)}(\Gamma) \cdot \mathbf{y}^{\mathrm{wt}^{(\nu)}(\Gamma)}. \tag{6.5.7}$$

(Recall that $\equiv$ means that the difference of the two sides of (6.5.7) are annihilated by the operator $\mathcal{D}_{w_0^{(k)}}$.) We will show that the statement $N_{r,k}$ for $\mathbf{x}$ and λ is the same (6.5.7), except both sides are multiplied by the same monomial.

On the left-hand side, we have

$$\mathbf{y}^{w_0^{(k+1)}(\nu)} = x_{r-k}^{\lambda_{k+2}} \cdot x_{r-k+1}^{\lambda_{k+1}} \cdots x_{r+1}^{\lambda_1} = \frac{\mathbf{x}^{w_0(\lambda)}}{x_1^{\lambda_{r+1}} \cdots x_{r-k-1}^{\lambda_{k+3}}}. \tag{6.5.8}$$

Multiplication by the monomial $x_1^{\lambda_{r+1}} \cdots x_{r-k-1}^{\lambda_{k+3}}$ commutes with $\mathcal{T}_i$ when $r-k \leq i \leq r$. This implies that by (6.5.8),

$$x_1^{\lambda_{r+1}} \cdots x_{r-k-1}^{\lambda_{k+3}} \cdot (\mathcal{T}_r \cdots \mathcal{T}_{r-k}) \mathbf{y}^{w_0^{(k+1)}(\nu)} = (\mathcal{T}_r \cdots \mathcal{T}_{r-k}) \mathbf{x}^{w_0^{(k+1)}(\lambda)}. \tag{6.5.9}$$

Now we turn to the right hand side. Let $\Gamma = (\Gamma_{1,1}, \ldots, \Gamma_{1,k+1})$, and let Γ' be the vector we get by attaching $r-k-1$ zeros to the end,

$$\Gamma' = (\Gamma_{1,1}, \ldots, \Gamma_{1,k+1}, \Gamma_{1,k+2}, \ldots \Gamma_{1,r}) = (\Gamma_{1,1}, \ldots, \Gamma_{1,k+1}, 0, \ldots, 0). \tag{6.5.10}$$

Notice that Γ is (ν, k)-admissible if and only if Γ' is (λ, k)-admissible, and every (λ, k)-admissible vector Γ' can be produced from a (ν, k) -admissible vector Γ this way. Moreover, by Definition 6.3.9 we have that for Γ and Γ' related as in (6.5.10),

$$G_1^{(\lambda)}(\Gamma') = G_1^{(\nu)}(\Gamma) \cdot \prod_{j=k+2}^{r} g_{1j}^{(\lambda)}(\Gamma') = G_1^{(\nu)}(\Gamma) \cdot \prod_{j=k+2}^{r} 1 = G_1^{(\nu)}(\Gamma); \tag{6.5.11}$$

and

$$\mathbf{y}^{\mathrm{wt}^{(\nu)}(\Gamma)} = \frac{\mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma')}}{x_1^{\lambda_{r+1}} \cdots x_{r-k-1}^{\lambda_{k+3}}}. \tag{6.5.12}$$

This implies that

$$x_1^{\lambda_{r+1}} \cdots x_{r-k-1}^{\lambda_{k+3}} \cdot \sum_{\substack{\Gamma=(\Gamma_{1,1},\ldots,\Gamma_{1,k+1}) \\ \Gamma\ (\nu,k)-\text{admissible} \\ \Gamma_{1,k+1}\neq 0}} G_1^{(\nu)}(\Gamma) \cdot \mathbf{y}^{\mathrm{wt}^{(\nu)}(\Gamma)} = \sum_{\substack{\Gamma'=(\Gamma_{1,1},\ldots,\Gamma_{1,r}) \\ \Gamma\ (\lambda,k)-\text{admissible} \\ \Gamma_{1,k+1}\neq 0}} G_1^{(\lambda)}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma')}. \tag{6.5.13}$$

Multiplication by the monomial $x_1^{\lambda_{r+1}} \cdots x_{r-k-1}^{\lambda_{k+3}}$ commutes with $\mathcal{D}_i$ when $r-k \le i \le r$, hence with $\mathcal{D}_{w_0^{(k)}}$. Thus by (6.5.7), (6.5.9) and (6.5.13), we have

$$(\mathcal{T}_r \cdots \mathcal{T}_{r-k})\mathbf{x}^{w_0^{(k+1)}(\lambda)} \equiv \sum_{\substack{\Gamma'=(\Gamma_{1,1},\ldots,\Gamma_{1,r}) \\ \Gamma\ (\lambda,k)-\text{admissible} \\ \Gamma_{1,k+1}\neq 0}} G_1^{(\lambda)}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma')}.$$

This is the statement $N_{r,k}$ for λ and $\mathbf{x}$. This completes the proof; proving Proposition 7.0.4 will be sufficient to prove Proposition 6.4.4.

Chapter 7

Crystal Description of a Single String of Demazure-Lusztig Operators

In Chapter 6, the proof of Theorem 6.2.1 and Theorem 6.2.2 was reduced to describing the action of the string of Demazure-Lusztig operators $\mathcal{T}_r \dots \mathcal{T}_1$ on a monomial. The description was phrased as the statement $N_{r,r-1}$. This chapter consists of the proof of the statement $N_{r,r-1}$. We recall the statement in Proposition 7.0.4 below.

Proposition 7.0.4. *Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_{r+1})$ be any dominant weight. Then we have*

$$(\mathcal{T}_r \cdots \mathcal{T}_1)\mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma=(\Gamma_{11},\dots,\Gamma_{1r}) \\ \Gamma \ \lambda-\text{admissible} \\ \Gamma_{1,r}\neq 0}} G_1^{\lambda}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)}. \tag{7.0.1}$$

Here $\equiv$ means that the difference of the left and right hand side is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$.

Recall that the relevant notation has been introduced in Section 2.4, Section 6.1 and Section 6.3.5. In this chapter, we prefer v for denoting $v = t^n = q^{-1}$.

The proof is by induction on r and involves explicit computations.

To make the argument more transparent, we introduce notation to abbreviate both sides of the equation (7.0.1). On the left hand side, we have

$$\mathfrak{L}_r^{(\lambda)}(\mathbf{x}) := (\mathcal{T}_r \cdots \mathcal{T}_1)\mathbf{x}^{w_0(\lambda)} \tag{7.0.2}$$

while on the right hand side, we have

$$\mathfrak{R}_r^{\lambda}(\mathbf{x}) := \sum_{\substack{\Gamma=(\Gamma_{11},\ldots,\Gamma_{1r}) \\ \Gamma\ \lambda-\text{admissible} \\ \Gamma_{1,r}\neq 0}} G_1^{\lambda}(\Gamma) \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma)}. \tag{7.0.3}$$

We start by proving the statement for $r = 1$.

7.1 The case $r = 1$

For the extent of this section, let $\alpha = \alpha_1$, $\mathbf{x}^{\alpha} = \frac{x_1}{x_2}$, $\lambda = (\lambda_1, \lambda_2)$.

The goal is to show

$$\mathfrak{L}_1^{(\lambda)}(\mathbf{x}) = \mathcal{T}_1(x_1^{\lambda_2} x_2^{\lambda_1}) = \sum_{\Gamma_{11}=1}^{\lambda_1-\lambda_2+1} g_{11}^{(\lambda)}(\Gamma_{11}) \cdot x_1^{\lambda_2+\Gamma_{11}} x_2^{\lambda_1-\Gamma_{11}} = \mathfrak{R}_1^{\lambda}(\mathbf{x}). \tag{7.1.1}$$

We compute $\mathfrak{L}_1^{\lambda}(\mathbf{x})$ and $\mathfrak{R}_1^{\lambda}(\mathbf{x})$ separately.

Left hand side

We have

$$\mathcal{T}_1 = (1 - v \cdot \mathbf{x}^{n\alpha}) \cdot \frac{1 - \mathbf{x}^{n\alpha} \cdot \sigma_1}{1 - \mathbf{x}^{n\alpha}} - 1 = \frac{(1-v) \cdot \mathbf{x}^{n\alpha}}{1 - \mathbf{x}^{n\alpha}} - \frac{1 - v \cdot \mathbf{x}^{n\alpha}}{1 - \mathbf{x}^{n\alpha}} \cdot \mathbf{x}^{n\alpha} \cdot \sigma_1, \tag{7.1.2}$$

where for $f = \mathbf{x}^{\beta}$, $\beta = (\beta_1, \ldots, \beta_{r+1})$ we have (2.4.2)

$$\sigma_i(f) = \frac{\sigma_i.f}{1 - v \cdot \mathbf{x}^{n\alpha}} \cdot \left[\mathbf{x}^{-r_n(\beta_{i+1}-\beta_i)\alpha} \cdot (1-v) \ - v \cdot g_{1+\beta_{i+1}-\beta_i} \cdot \mathbf{x}^{\alpha - n\alpha} \cdot (1 - \mathbf{x}^{n\alpha})\right]. \tag{7.1.3}$$

Substituting (7.1.3) into (7.1.2) and simplifying, we get

$$\mathcal{T}_1(f) = \frac{(1-v) \cdot \mathbf{x}^{n\alpha}}{1 - \mathbf{x}^{n\alpha}} \left(f - \mathbf{x}^{-r_n(\beta_{i+1}-\beta_i)\alpha} \cdot \sigma_i.f\right) + \left(v \cdot g_{1+\beta_{i+1}-\beta_i} \cdot \mathbf{x}^{\alpha}\right) \cdot (\sigma_i.f). \tag{7.1.4}$$

Applying $\mathcal{T}_1$ to $\mathbf{x}^{w_0^{(1)}\lambda} = x_1^{\lambda_2} x_2^{\lambda_1}$ (so $\beta_1 = \lambda_2$, $\beta_2 = \lambda_1$), we have

$$\mathfrak{L}_1^{(\lambda)}(\mathbf{x}) = \frac{(1-v)\cdot \mathbf{x}^{n\alpha}}{1-\mathbf{x}^{n\alpha}} \left(x_1^{\lambda_2} x_2^{\lambda_1} - \mathbf{x}^{-r_n(\lambda_1-\lambda_2)\alpha} \cdot x_1^{\lambda_1} x_2^{\lambda_2}\right) + v \cdot g_{1+\lambda_1-\lambda_2} \cdot x_1^{\lambda_1+1} x_2^{\lambda_2-1}. \quad (7.1.5)$$

Right hand side

By Definition 6.3.9, if $\Gamma = (\Gamma_{11})$ is λ-admissible and $\Gamma_{11} \neq 0$, then

$$G_1^{(\lambda)}(\Gamma) = g_{11}^{(n,\lambda)}(\Gamma_{11}) = \begin{cases} h^\flat(\Gamma_{11}) & \text{if } 0 < \Gamma_{11} < \lambda_1 - \lambda_2 + 1; \\ g^\flat(\Gamma_{11}) & \text{if } 0 < \Gamma_{11} = \lambda_1 - \lambda_2 + 1. \end{cases}$$

Substituting into the right hand side of (7.1.1), we get

$$\mathfrak{R}_1^\lambda(\mathbf{x}) = x_1^{\lambda_2} x_2^{\lambda_1} \cdot \sum_{\Gamma_{11}=1}^{\lambda_1-\lambda_2} h^\flat(\Gamma_{11}) \cdot \left(\frac{x_1}{x_2}\right)^{\Gamma_{11}} + g^\flat(\lambda_1 - \lambda_2 + 1) \cdot x_1^{\lambda_1+1} x_2^{\lambda_2-1}. \quad (7.1.6)$$

According to Claim 2.3.1, we have

$$h^\flat(a) = \begin{cases} 0 & n \nmid a; \\ 1-v & n \mid a; \end{cases} \quad \text{and} \quad g^\flat(a) = v \cdot g_a. \quad (7.1.7)$$

Substituting (7.1.7) into (7.1.6) and writing $\lambda_1 - \lambda_2 = kn + r_n(\lambda_1 - \lambda_2)$, we have

$$\begin{aligned} \mathfrak{R}_1^\lambda(\mathbf{x}) &= x_1^{\lambda_2} x_2^{\lambda_1} \cdot (1-v) \cdot \sum_{j=1}^{k} \mathbf{x}^{n\alpha} + v \cdot g_{1+\lambda_1-\lambda_2} \cdot x_1^{\lambda_1+1} x_2^{\lambda_2-1} \\ &= x_1^{\lambda_2} x_2^{\lambda_1} \cdot (1-v) \cdot \mathbf{x}^{n\alpha} \cdot \frac{1-\mathbf{x}^{kn\alpha}}{1-\mathbf{x}^{n\alpha}} + v \cdot g_{1+\lambda_1-\lambda_2} \cdot x_1^{\lambda_1+1} x_2^{\lambda_2-1} \\ &= \frac{(1-v)\cdot \mathbf{x}^{n\alpha}}{1-\mathbf{x}^{n\alpha}} \cdot x_1^{\lambda_2} x_2^{\lambda_1} \cdot (1 - \mathbf{x}^{(\lambda_1-\lambda_2-r_n(\lambda_1-\lambda_2))\alpha}) + v \cdot g_{1+\lambda_1-\lambda_2} \cdot x_1^{\lambda_1+1} x_2^{\lambda_2-1} \\ &= \frac{(1-v)\cdot \mathbf{x}^{n\alpha}}{1-\mathbf{x}^{n\alpha}} \cdot (x_1^{\lambda_2} x_2^{\lambda_1} - \mathbf{x}^{-r_n(\lambda_1-\lambda_2)\alpha} \cdot x_1^{\lambda_1} x_2^{\lambda_2}) + v \cdot g_{1+\lambda_1-\lambda_2} \cdot x_1^{\lambda_1+1} x_2^{\lambda_2-1}. \end{aligned} \quad (7.1.8)$$

Comparing (7.1.5) to the last line of (7.1.8) we see that (7.1.1) holds. This completes the base case of the induction.

7.2 Notation and conventions

Before discussing the induction step, we fix a few conventions. These will allow us to relate the statements $N_{r,r-1}$, $N_{r-1,r-2}$ and $N_{r-2,r-3}$ with more transparency. Proposition 7.0.4 is phrased with notation introduced in Definition 6.3.9. In addition we make use of the following conventions.

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{r+1}), \quad &\mathbf{x} &= (x_1, \ldots, x_{r-1}, x_r, x_{r+1}); \\ \mu &= (\lambda_2, \lambda_3, \ldots, \lambda_{r+1}), \quad &\mathbf{y} &= (x_1, \ldots, x_{r-1}, x_r); \\ \nu &= (\lambda_3, \ldots, \lambda_{r+1}), \quad &\mathbf{z} &= (x_1, \ldots, x_{r-1}). \end{aligned} \tag{7.2.1}$$

Furthermore, let

$$\begin{aligned} \Gamma' &= (\Gamma_{11}, \Gamma_{12}, \Gamma_{13}, \ldots, \Gamma_{1r}); \\ \Gamma &= (\Gamma_{12}, \Gamma_{13}, \ldots, \Gamma_{1r}); \\ \Gamma_0 &= (\Gamma_{13}, \ldots, \Gamma_{1r}). \end{aligned} \tag{7.2.2}$$

With this notation, we have that

$$\Gamma' \text{ is } \lambda-\text{admissible if and only if } \begin{cases} \Gamma \text{ is } \mu-\text{admissible and} \\ \Gamma_{12} \le \Gamma_{11} \le \Gamma_{12} + \lambda_1 - \lambda_2 + 1; \end{cases} \tag{7.2.3}$$

and

$$\begin{aligned} G^{(\lambda)}(\Gamma') &= g_{11}^{(\lambda)}(\Gamma') \cdot G_1^{(\mu)}(\Gamma), \\ \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma')} &= \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^{\Gamma_{11}} \cdot x_{r+1}^{\lambda_1 - \Gamma_{11}}. \end{aligned} \tag{7.2.4}$$

Similarly,

$$\Gamma \text{ is } \mu-\text{admissible if and only if } \begin{cases} \Gamma_0 \text{ is } \nu-\text{admissible and} \\ \Gamma_{13} \le \Gamma_{12} \le \Gamma_{13} + \lambda_2 - \lambda_3 + 1; \end{cases} \tag{7.2.5}$$

and

$$\begin{aligned} G^{(\mu)}(\Gamma) &= g_{12}^{(\mu)}(\Gamma) \cdot G_1^{(\nu)}(\Gamma_0), \\ \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} &= \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12}}. \end{aligned} \tag{7.2.6}$$

Notice that in indexing the μ-admissible vector Γ, we write $g_{12}^{(\mu)}(\Gamma)$ for the Gelfand-Tsetlin coefficient corresponding to the first entry, Γ_{12}. Equations (7.2.4) and (7.2.6) are a direct consequence of the notation introduced. In particular, the relationship between the monomials is true even if Γ' (or Γ) is not λ-admissible (respectively, μ-admissible).

We will make use of the following function on pairs of (positive) integers:

$$\delta(A,B) = \begin{cases} h^\flat(A) & \text{if } A < B \\ h^\flat(A) - 1 & \text{if } A = B \\ 0 & A > B \end{cases} \tag{7.2.7}$$

We are now ready to tackle the induction step.

7.3 Simplifying the induction step

We assume that $N_{k+1,k}$ holds for $k < r-1$. The goal is to prove

$$\mathfrak{L}_r^{(\lambda)}(\mathbf{x}) \equiv \mathfrak{R}_r^{(\lambda)}(x), \text{ i.e. } \mathcal{D}_{w_0^{(r-1)}}\left(\mathfrak{L}_r^{(\lambda)}(\mathbf{x}) - \mathfrak{R}_r^{(\lambda)}(\mathbf{x})\right) = 0. \tag{7.3.1}$$

Claim 7.3.1. *It suffices to show that*

$$\mathfrak{R}_r^{(\lambda)}(x) \equiv \mathcal{T}_r\left(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right), \tag{7.3.2}$$

that is,

$$\mathcal{D}_{w_0^{(r-1)}}\left(\mathcal{T}_r\left(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right) - \mathfrak{R}_r^{(\lambda)}(\mathbf{x})\right) = 0. \tag{7.3.3}$$

Proof. This is clear if we show that

$$\mathcal{D}_{w_0^{(r-1)}}\left(\mathfrak{L}_r^{(\lambda)}(\mathbf{x}) - T_r\left(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right)\right) = 0.$$

Notice that since multiplication by $x_{r+1}^{\lambda_1}$ commutes with the operators $T_1, \ldots, T_{r-1}$, we have

$$\begin{aligned}\mathfrak{L}_r^{(\lambda)}(\mathbf{x}) &= (\mathcal{T}_r\mathcal{T}_{r-1}\cdots\mathcal{T}_1)\mathbf{x}^{w_0^{(r)}(\lambda)} \\ &= \mathcal{T}_r(\mathcal{T}_{r-1}\cdots\mathcal{T}_1)\left(\mathbf{y}^{w_0^{(r-1)}(\mu)} \cdot x_{r+1}^{\lambda_1}\right) \\ &= \mathcal{T}_r\left(x_{r+1}^{\lambda_1} \cdot \mathfrak{L}_{r-1}^{(\mu)}(\mathbf{y})\right).\end{aligned} \tag{7.3.4}$$

Now $N_{r-1,r-2}$ states that

$$\mathcal{D}_{w_0^{(r-2)}}\left(\mathfrak{L}_{r-1}^{(\mu)}(\mathbf{y})-\mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right)=0. \tag{7.3.5}$$

Multiplication by $x_{r+1}^{\lambda_1}$ and $\mathcal{T}_r$ commute with $\mathcal{D}_{w_0^{(r-2)}}$, hence

$$\mathcal{D}_{w_0^{(r-2)}}\left(\mathcal{T}_r\left(x_{r+1}^{\lambda_1}\cdot\mathfrak{L}_{r-1}^{(\mu)}(\mathbf{y})-\mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right)\right)=0. \tag{7.3.6}$$

Since $w_0^{(r-2)}\leq w_0^{(r-1)}$, by Lemma 6.5.1 and (7.3.4), we have

$$\mathcal{D}_{w_0^{(r-1)}}\left(\mathfrak{L}_r^{(\lambda)}(\mathbf{x})-T_r\left(x_{r+1}^{\lambda_1}\cdot\mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right)\right)=0.$$

This completes the proof of the claim. □

We set out to prove (7.3.2). To this end, we first rewrite

$$\mathcal{T}_r\left(x_{r+1}^{\lambda_1}\cdot\mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right).$$

By the conventions (7.2.6), and the fact that $\mathcal{T}_r$ commutes with multiplication by $x_1,\ldots,$ x_{r-1}, we have

$$\begin{aligned}\mathcal{T}_r\left(x_{r+1}^{\lambda_1}\cdot\mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right)&=\mathcal{T}_r\sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1,r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}G_1^{(\mu)}(\Gamma)\cdot\mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)}x_{r+1}^{\lambda_1}\\ &=\mathcal{T}_r\sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1,r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}G_1^{(\mu)}(\Gamma)\cdot\mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}\cdot x_{r-1}^{\Gamma_{12}}\cdot x_r^{\lambda_2-\Gamma_{12}}x_{r+1}^{\lambda_1}\\ &=\sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1,r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}G_1^{(\mu)}(\Gamma)\cdot\mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}\cdot x_{r-1}^{\Gamma_{12}}\cdot\mathcal{T}_r(x_r^{\lambda_2-\Gamma_{12}}x_{r+1}^{\lambda_1}).\end{aligned} \tag{7.3.7}$$

Since $N_{1,0}$ was proved in Section 7.1, $N_{r,0}$ is true by Lemma 6.4.7. Thus we have

$$\mathcal{T}_r(x_r^{\lambda_2-\Gamma_{12}}x_{r+1}^{\lambda_1})=\sum_{\Gamma_{11}=1}^{\lambda_1-(\lambda_2-\Gamma_{12})+1}g_{11}^{(\lambda_1,\lambda_2-\Gamma_{12})}(\Gamma_{11})x_r^{\lambda_2+\Gamma_{11}-\Gamma_{12}}x_{r+1}^{\lambda_1-\Gamma_{11}}. \tag{7.3.8}$$

Recall that by Definition 6.3.9, in particular (6.3.21), and (7.2.7), $\Gamma = (\Gamma_{11}, \ldots, \Gamma_{1r})$, we have

	$g_{11}^{(\lambda_1, \lambda_2 - \Gamma_{12})}(\Gamma_{11})$	$g_{11}^{(\lambda)}(\Gamma)$	$\delta(\Gamma_{11}, \Gamma_{12})$
$0 < \Gamma_{11} < \Gamma_{12}$	$h^\flat(\Gamma_{11})$	0	$h^\flat(\Gamma_{11})$
$\Gamma_{11} = \Gamma_{12}$	$h^\flat(\Gamma_{11})$	1	$h^\flat(\Gamma_{11}) - 1$
$\Gamma_{12} < \Gamma_{11} < \Gamma_{12} + \lambda_1 - \lambda_2 + 1$	$h^\flat(\Gamma_{11})$	$h^\flat(\Gamma_{11})$	0
$\Gamma_{11} = \Gamma_{12} + \lambda_1 - \lambda_2 + 1$	$g^\flat(\Gamma_{11})$	$g^\flat(\Gamma_{11})$	0

Thus

$$g_{11}^{(\lambda_1, \lambda_2 - \Gamma_{12})}(\Gamma_{11}) - g_{11}^{(\lambda)}(\Gamma) = \delta(\Gamma_{11}, \Gamma_{12}). \tag{7.3.9}$$

Now we may substitute (7.3.8) and (7.3.9) into the last line of (7.3.7), and use the conventions (7.2.3), (7.2.4) to get the following.

$$\begin{aligned}
\mathcal{T}_r\left(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})\right) = & \sum_{\substack{\Gamma = (\Gamma_{12}, \ldots, \Gamma_{1,r}) \\ \Gamma\ \mu-\text{admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\mu)}(\Gamma) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \\
& \cdot \sum_{\Gamma_{11}=1}^{\lambda_1 - (\lambda_2 - \Gamma_{12}) + 1} \left(g_{11}^{(\lambda)}(\Gamma) + \delta(\Gamma_{11}, \Gamma_{12})\right) x_r^{\lambda_2 + \Gamma_{11} - \Gamma_{12}} x_{r+1}^{\lambda_1 - \Gamma_{11}}. \\
= & \sum_{\substack{\Gamma' = (\Gamma_{11}, \Gamma_{12}, \ldots, \Gamma_{1,r}) \\ \Gamma\ \lambda-\text{admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\lambda)}(\Gamma') \cdot \mathbf{x}^{\mathrm{wt}^{(\lambda)}(\Gamma')} \\
& + \sum_{\substack{\Gamma = (\Gamma_{12}, \ldots, \Gamma_{1,r}) \\ \Gamma\ \mu-\text{admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^{\Gamma_{11}} x_{r+1}^{\lambda_1 - \Gamma_{11}} \cdot \sum_{1 \le \Gamma_{11}} \delta(\Gamma_{11}, \Gamma_{12}). \\
= & \ \mathfrak{R}_r^{(\lambda)}(x) \\
& + \sum_{1 \le \Gamma_{11}} x_{r+1}^{\lambda_1 - \Gamma_{11}} \cdot \sum_{\substack{\Gamma = (\Gamma_{12}, \ldots, \Gamma_{1,r}) \\ \Gamma\ \mu-\text{admissible} \\ \Gamma_{1,r} \neq 0}} \delta(\Gamma_{11}, \Gamma_{12}) \cdot G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^{\Gamma_{11}}.
\end{aligned} \tag{7.3.10}$$

Thus proving (7.3.2) is equivalent to showing that

$$\mathcal{D}_{w_0^{(r-1)}}\left(\sum_{1\le\Gamma_{11}} x_{r+1}^{\lambda_1-\Gamma_{11}}\cdot\sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1,r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}\delta(\Gamma_{11},\Gamma_{12})\cdot G_1^{(\mu)}(\Gamma)\cdot\mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)}\cdot x_r^{\Gamma_{11}}\right)=0 \quad (7.3.11)$$

Notice that since multiplication by x_{r+1} commutes with $\mathcal{D}_{w_0^{(r-1)}}$, (7.3.11) is equivalent to showing

$$\mathcal{D}_{w_0^{(r-1)}}\left(\sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1,r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}\delta(\Gamma_{11},\Gamma_{12})\cdot G_1^{(\mu)}(\Gamma)\cdot\mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)}\cdot x_r^{\Gamma_{11}}\right)=0 \quad (7.3.12)$$

holds for every $1\le\Gamma_{11}$.

This motivates the following notation. Let a be a positive integer, and μ, $\mathbf{y}$ as in Section 7.2.

$$F_{\mu,a}(\mathbf{y})=\sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}\delta(a,\Gamma_{12})\cdot G_1^{(\mu)}(\Gamma)\cdot\mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)}\cdot x_r^a. \quad (7.3.13)$$

Now we may phrase yet another statement, similar to $N_{r,r-1}$.

Proposition 7.3.2. *Let $\mu=(\lambda_2,\lambda_3,\ldots,\lambda_{r+1})$ be any dominant weight. Then for any positive integer a we have*

$$F_{\mu,a}(\mathbf{y})\equiv 0,\ \ \textit{i.e.},\ \mathcal{D}_{w_0^{(r-1)}}F_{\mu,a}(\mathbf{y})=0. \quad (7.3.14)$$

Call this statement (that (7.3.14) holds for any dominant weight μ and positive integer a) $\mathbf{F_r}$.

The computations in the present section amount to the following lemma.

Lemma 7.3.3. *If $N_{r-1,r-2}$ holds, then $N_{r,r-1}$ (for λ, $\mathbf{x}$ as above) is equivalent to the statement*

$$\forall a \quad F_{\mu,a}(\mathbf{y})\equiv 0,$$

or, equivalently,

$$\forall a \quad \mathcal{D}_{w_0^{(r-1)}} F_{\mu,a}(\mathbf{y}) = 0.$$

□

Now to complete the induction step, it remains to prove Proposition 7.3.2, i.e. that $F_{\mu,a}(\mathbf{y})$ is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. This is the content of Section 7.4. We distinguish between the cases where a is divisible by n or not. The case when it is *not* (Section 7.4.1), is significantly easier to handle.

7.4 Proof of Proposition 7.3.2

7.4.1 The non-divisible case

The goal is to prove that if $n \nmid a$, then $\mathcal{D}_{w_0^{(r-1)}} F_{\mu,\Gamma_{11}}(\mathbf{y}) = 0$.

Recall that by Claim 2.3.1, $n \nmid a$ and hence

$$h^\flat(a) = 0. \tag{7.4.1}$$

By (7.2.7), this means that since $n \nmid$, we have $\delta(a, \Gamma_{12}) = 0$ unless $a = \Gamma_{12}$, and $\delta(a, \Gamma_{12}) = -1$. Thus in this case we have

$$F_{\mu,a}(\mathbf{y}) = - \sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1r}) \\ \Gamma \ \mu-\text{admissible} \\ \Gamma_{12}=a \\ \Gamma_{1,r}\neq 0}} G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^a. \tag{7.4.2}$$

We will show that each term in the summation is either itself zero, or is annihilated by a Demazure-Lusztig operator corresponding to a simple reflection.

Fix a term $\Gamma = (\Gamma_{12}, \ldots, \Gamma_{1r})$, and take $\Gamma_{1,r+1} := 0$ and $\Gamma_{11} = a$. Then $\Gamma_{11} = \Gamma_{12}$ and $\Gamma_{1r} > \Gamma_{1,r+1}$. Let j be the smallest index such that $\Gamma_{1j} > \Gamma_{1,j+1}$ $(2 \leq j \leq r)$. In this case

$$a = \Gamma_{11} = \cdots = \Gamma_{1,j-1} = \Gamma_{1,j},$$

so $n \nmid \Gamma_{1,j}$ and by (7.4.1), $h^\flat(\Gamma_{1,j}) = 0$. By (6.3.21), we have

$$g_{1j}^{(\mu)}(\Gamma) = \begin{cases} h^\flat(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} < \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1; \\ g^\flat(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1. \end{cases}$$

This means that $G_1^{(\mu)}(\Gamma) = 0$ unless $\Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1$. We show that in the latter case $\mathcal{D}_{r-j+1}$ annihilates the corresponding term. Observe that $\mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^{\Gamma_{11}}$ has a factor of

$$x_{r-j+1}^{\lambda_{j+1}+\Gamma_{1,j}-\Gamma_{1,j+1}} x_{r-j+2}^{\lambda_j+\Gamma_{1,j-1}-\Gamma_{1,j}}$$

(and no other factors of x_{r-j+1} or r_{r-j+2}). Since $\Gamma_{1,j-1} = \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1$, this reduces to

$$x_{r-j+1}^{\lambda_j+1} x_{r-j+2}^{\lambda_j}.$$

This is annihilated by $\mathcal{D}_{r-j+1}$ by Corollary 6.5.3. Since $2 \leq j \leq r$, $1 \leq r-j+1 \leq r-1$. This implies that (by Proposition 6.5.1 part (ii)) $\mathcal{D}_{w_0^{(r-1)}}$ annihilates all nonzero terms. This completes the proof.

7.4.2 The divisible case

From now on we may assume that a is divisible by $\mathbf{n}$. Since $\delta(a, \Gamma_{12})$ will appear repeatedly in the computations below, we introduce the following shorthand.

$$\delta_a(\Gamma_{12}) = \delta(a, \Gamma_{12}) = \begin{cases} 1 - v & \text{if } a < \Gamma_{12}; \\ -v & \text{if } a = \Gamma_{12}; \\ 0 & a > \Gamma_{12}. \end{cases} \tag{7.4.3}$$

The goal is to prove $\mathbf{F_r}$:

$$\mathcal{D}_{w_0^{(r-1)}} F_{\mu,a}(\mathbf{y}) = \mathcal{D}_{w_0^{(r-1)}} \sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1r}) \\ \Gamma\ \mu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} \delta_a(\Gamma_{12}) \cdot G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^a = 0. \tag{7.4.4}$$

Proposition 7.3.3 has the following implication. Since, as part of the inductive hypothesis, we assume that $N_{r-1,r-2}$ and $N_{r-2,r-3}$ are true, we may use the statement $\mathbf{F_{r-1}}$:

$$\mathcal{D}_{w_0^{(r-2)}} F_{\nu,\Gamma_{12}}(\mathbf{z}) = 0 \tag{7.4.5}$$

for any Γ_{12}. This is true even when $r = 2$, since in this case, $F_{\nu,\Gamma_{12}}(\mathbf{z})$ is itself zero.

The strategy to prove (7.4.4) is the following. Using the conventions introduced in Section 7.2, in particular (7.2.5) and (7.2.6), we will rewrite the sum defining $F_{\mu,a}(\mathbf{y})$ into smaller pieces according to Γ_0. Then we break the sum up into two pieces. One piece will be annihilated by $\mathcal{D}_{w_0^{(r-2)}}$ as a consequence of $\mathbf{F_{r-1}}$, the other, we show, is annihilated by $\mathcal{D}_{r-1}$. By Lemma 6.5.1, this implies that $F_{\mu,a}(\mathbf{y})$ is indeed annihilated by $\mathcal{D}_{w_0^{(r-1)}}$.

Breaking up the sum

Remark 7.4.1. By (7.2.6), we have

$$\mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} = \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12}}.$$

Note that the exponent of x_{r-1} in $\mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}$ is $\lambda_3 - \Gamma_{13}$. This means that $\mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3}$ contains no factors of x_{r-1} or x_r. In particular, multiplication by $\mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3}$ commutes with $\mathcal{D}_{r-1}$.

We are now ready to start the computation.

$$
\begin{aligned}
F_{\mu,a}(\mathbf{y}) &= \sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} \delta(a,\Gamma_{12}) \cdot G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^a \\
&= \sum_{\substack{\Gamma=(\Gamma_{12},\ldots,\Gamma_{1r})\\ \Gamma\ \mu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2-\Gamma_{12}} \cdot x_r^a \\
&= \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} \sum_{\Gamma_{12}=\Gamma_{13}}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2-\Gamma_{12}} \cdot x_r^a \\
&= \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3} \\
&\quad \cdot \sum_{\Gamma_{12}=\Gamma_{13}}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot x_{r-1}^{\lambda_3-\Gamma_{13}+\Gamma_{12}} \cdot x_r^{\lambda_2-\Gamma_{12}+a} \\
&= \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3} \\
&\quad \cdot \sum_{\Gamma_{12}=1}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot x_{r-1}^{\lambda_3-\Gamma_{13}+\Gamma_{12}} \cdot x_r^{\lambda_2-\Gamma_{12}+a}
\end{aligned}
\tag{7.4.6}
$$

In the last step one can change the lower bound of the summation over Γ_{12}, because $g_{12}^{(\mu)}(\Gamma) = 0$ if $\Gamma_{12} < \Gamma_{13}$. (Moreover, $\Gamma_{13} \geq \Gamma_{1r} \geq 1$. If $r = 2$, then there is no change at all.)

The part corresponding to a fixed $\Gamma_0 = (\Gamma_{13}, \ldots, \Gamma_{1r})$ is

$$
f_{a,\Gamma_0}(x_{r-1}, x_r) := \sum_{\Gamma_{12}=1}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}}. \tag{7.4.7}
$$

This is not quite annihilated by $\mathcal{D}_{r-1}$. We define a modification that is. Let

$$h^{(\mu,\Gamma_0)}(\Gamma_{12}) := g_{12}^{(\mu)}(\Gamma) + \delta(\Gamma_{12},\Gamma_{13}) = \begin{cases} h^\flat(\Gamma_{12}) & \text{if } \Gamma_{12} < \Gamma_{13}+\lambda_2-\lambda_3+1; \\ g^\flat(\Gamma_{12}) & \text{if } \Gamma_{12} = \Gamma_{13}+\lambda_2-\lambda_3+1; \\ 0 & \text{if } \Gamma_{12} > \Gamma_{13}+\lambda_2-\lambda_3+1. \end{cases} \tag{7.4.8}$$

Then f'_{a,Γ_0} is the polynomial with $h^{(\mu,\Gamma_0)}(\Gamma_{12})$ in place of $g_{12}^{(\mu)}(\Gamma)$:

$$f'_{a,\Gamma_0}(x_{r-1},x_r) := \sum_{\Gamma_{12}=1}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot h^{(\mu,\Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}}. \tag{7.4.9}$$

The following Lemma will be proved in Section 7.5.

Lemma 7.4.2.

$$\mathcal{D}_{r-1} f'_{a,\Gamma_0} = 0. \tag{7.4.10}$$

We turn our attention to the piece annihilated by $\mathcal{D}_{w_0^{(r-2)}}$ next.

Consequences of $\mathbf{F_{r-1}}$

We shall multiply the equation (7.4.5) by an element of $\mathbb{C}[x_r]$ to produce something similar to $\mathbf{F_{r-1}}$. Rewriting it by the definition gives

$$\mathcal{D}_{w_0^{(r-2)}} \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r}) \\ \Gamma\ \nu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} \delta(\Gamma_{1,2},\Gamma_{13}) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} = 0 \tag{7.4.11}$$

This holds for any $0 < \Gamma_{1,2}$. The operator $\mathcal{D}_{w_0^{(r-2)}}$ is linear and commutes with multiplication by x_r. This implies that for any positive integer a, we have

$$\mathcal{D}_{w_0^{(r-2)}} \sum_{0<\Gamma_{1,2}} \delta_a(\Gamma_{12}) \cdot x_r^{\lambda_2-\Gamma_{12}+a} \cdot \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r}) \\ \Gamma\ \nu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} \delta(\Gamma_{1,2},\Gamma_{13}) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} = 0. \tag{7.4.12}$$

We may change the order of summation to arrive at

$$\mathcal{D}_{w_0^{(r-2)}} \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r}) \\ \Gamma\ \nu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} \sum_{0<\Gamma_{1,2}} \delta_a(\Gamma_{12}) \cdot \delta(\Gamma_{1,2},\Gamma_{13}) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2-\Gamma_{12}+a} = 0. \tag{7.4.13}$$

Recall that by (7.2.6), we have

$$\mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} = \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2-\Gamma_{12}}.$$

Thus (7.4.13) may be rewritten as

$$\mathcal{D}_{w_0^{(r-2)}} \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r}) \\ \Gamma\ \nu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} \sum_{0<\Gamma_{1,2}} \delta_a(\Gamma_{12}) \cdot \delta(\Gamma_{1,2},\Gamma_{13}) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)} \cdot x_r^a = 0. \tag{7.4.14}$$

This looks rather similar to the polynomial annihilated in (7.4.4). Notice the factor $\delta(\Gamma_{1,2},\Gamma_{13})$, which also appears in the definition of $h^{(\mu,\Gamma_0)}(\Gamma_{12})$ (7.4.8).

Putting the parts together.

Notice that by (7.4.7), (7.4.9) and (7.4.8), and since $\delta(\Gamma_{12},\Gamma_{13}) = 0$ for $\Gamma_{12} > \Gamma_{13}$, we have

$$f'_{a,\Gamma_0}(x_{r-1},x_r) - f_{a,\Gamma_0}(x_{r-1},x_r) = \sum_{0<\Gamma_{12}} \delta_a(\Gamma_{12}) \cdot \delta(\Gamma_{12},\Gamma_{13}) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}} \tag{7.4.15}$$

With this observation, we are ready to put the result of Lemma 7.4.2 and (7.4.14) together. Continuing from (7.4.6), we have

$$\begin{aligned} F_{\mu,a}(\mathbf{y}) &= \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r}) \\ \Gamma\ \nu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3} \cdot f_{a,\Gamma_0}(x_{r-1},x_r) \\ &= \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r}) \\ \Gamma\ \nu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3} \cdot f'_{a,\Gamma_0}(x_{r-1},x_r) \\ &\quad - \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r}) \\ \Gamma\ \nu-\text{admissible} \\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3} \cdot \big(f'_{a,\Gamma_0}(x_{r-1},x_r) - f_{a,\Gamma_0}(x_{r-1},x_r)\big) \end{aligned} \tag{7.4.16}$$

Now, by (7.4.15) and (7.2.6), we may rewrite this as

$$\begin{aligned}F_{\mu,a}(\mathbf{y}) &= \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0)\cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}\cdot x_{r-1}^{\Gamma_{13}-\lambda_3}\cdot f'_{a,\Gamma_0}(x_{r-1},x_r)\\
&\quad -\sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0)\cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}\cdot \sum_{0<\Gamma_{12}} \delta_a(\Gamma_{12})\cdot\delta(\Gamma_{12},\Gamma_{13})\cdot x_{r-1}^{\Gamma_{12}}x_r^{\lambda_2+a-\Gamma_{12}}\\
&= \sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0)\cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}\cdot x_{r-1}^{\Gamma_{13}-\lambda_3}\cdot f'_{a,\Gamma_0}(x_{r-1},x_r)\\
&\quad -\sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}\sum_{0<\Gamma_{12}} \delta_a(\Gamma_{12})\cdot\delta(\Gamma_{12},\Gamma_{13})\cdot G_1^{(\nu)}(\Gamma_0)\cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)}\cdot x_r^a
\end{aligned}\tag{7.4.17}$$

Here both terms are annihilated by $\mathcal{D}_{w_0^{(r-1)}}$.

In the first term, we have that $\mathcal{D}_{r-1}$ commutes with multiplication by $\mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}\cdot x_{r-1}^{\Gamma_{13}-\lambda_3}$ by Remark 7.4.1. Thus by Lemma 7.4.2, and part (ii) of Lemma 6.5.1 we have

$$\mathcal{D}_{w_0^{(r-1)}}\sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}} G_1^{(\nu)}(\Gamma_0)\cdot \mathbf{z}^{\mathrm{wt}^{(\nu)}(\Gamma_0)}\cdot x_{r-1}^{\Gamma_{13}-\lambda_3}\cdot f'_{a,\Gamma_0}(x_{r-1},x_r)=0. \tag{7.4.18}$$

The second term is annihilated by $\mathcal{D}_{w_0^{(r-2)}}$ according to (7.4.14), so, again by part (ii) of Lemma 6.5.1,

$$\mathcal{D}_{w_0^{(r-1)}}\sum_{\substack{\Gamma_0=(\Gamma_{13},\ldots,\Gamma_{1r})\\ \Gamma\ \nu-\text{admissible}\\ \Gamma_{1,r}\neq 0}}\sum_{0<\Gamma_{12}} \delta_a(\Gamma_{12})\cdot\delta(\Gamma_{12},\Gamma_{13})\cdot G_1^{(\nu)}(\Gamma_0)\cdot \mathbf{y}^{\mathrm{wt}^{(\mu)}(\Gamma)}\cdot x_r^a=0 \tag{7.4.19}$$

Thus, by (7.4.18) and (7.4.19),

$$\mathcal{D}_{w_0^{(r-1)}}F_{\mu,a}(\mathbf{y})=0,$$

and the proof of $\mathbf{F_{r-1}}$ is complete.

7.5 Proof of Lemma 7.4.2

We prove Lemma 7.4.2

$$\mathcal{D}_{r-1} f'_{a,\Gamma_0}(x_{r-1}, x_r) = \mathcal{D}_{r-1} \sum_{\Gamma_{12}=1}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot h^{(\mu,\Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}} = 0 \tag{7.5.1}$$

By Lemma 6.5.1 proving (7.5.1) is equivalent to showing that $x_r^n \cdot f'_{a,\Gamma_0}$ is symmetric under the action of σ_{r-1}. Since $\delta_a(\Gamma_{12}) = 0$ if $a > \Gamma_{12}$, we have

$$x_r^n \cdot f'_{a,\Gamma_0} = \sum_{\Gamma_{12}=a}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot h^{(\mu,\Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}+n}. \tag{7.5.2}$$

The proof is a straightforward computation. The strategy is as follows. By (7.4.8), $h^{(\mu,\Gamma_0)}$ depends on the residue of Γ_{12} modulo n. We write

$$(\Gamma_{13} + \lambda_2 - \lambda_3 + 1) - a = nk + u, \quad 1 \leq u \leq n. \tag{7.5.3}$$

We define

$$P_{k,u}(\mathbf{x}) = \frac{x_r^n \cdot f'_{a,\Gamma_0}}{(1-v) \cdot (x_{r-1}x_r)^{\lambda_3+a-\Gamma_{13}}}. \tag{7.5.4}$$

Since by (2.4.2) σ_{r-1} commutes with multiplication by $(x_{r-1}x_r)$, proving (7.5.1) is equivalent to showing that $\sigma_{r-1}(P_{k,u}(\mathbf{x})) = P_{k,u}(\mathbf{x})$. In what follows, we write σ for σ_{r-1} and α for α_{r-1}. To prove that $P_{k,u}(\mathbf{x})$ is invariant under σ, we show that

$$P_{k,u}(\mathbf{x}) = \frac{1 - v \cdot \mathbf{x}^{-n\alpha}}{\mathbf{x}^{-n\alpha} - 1} \cdot x_r^{nk+u+n-1} + \frac{1 - v \cdot \mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} \cdot \sigma(x_r^{nk+u+n-1}). \tag{7.5.5}$$

This is sufficient by Lemma 2.2.2.

We are now ready to start the computation. By (7.5.2), (7.5.3) and (7.5.4) we have

$$\begin{aligned} P_{k,u}(\mathbf{x}) =& \frac{x_r^n}{(1-v) \cdot (x_{r-1}x_r)^{\lambda_3+a-\Gamma_{13}}} \cdot f'_{a,\Gamma_0} \\ =& \frac{1}{1-v} \cdot \sum_{\Gamma_{12}=a}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot h^{(\mu,\Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\Gamma_{12}-a} x_r^{\Gamma_{13}+\lambda_2-\lambda_3+n-\Gamma_{12}} \\ =& \frac{1}{1-v} \cdot x_r^{nk+u+n-1} \cdot \sum_{\Gamma_{12}=a}^{a+nk+u} \delta_a(\Gamma_{12}) \cdot h^{(\mu,\Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\Gamma_{12}-a} x_r^{-(\Gamma_{12}-a)} \end{aligned} \tag{7.5.6}$$

Recall that since $n|a$, we have

$$\delta_a(\Gamma_{12}) = \begin{cases} -v & a = \Gamma_{12}; \\ 1-v & a < \Gamma_{12}; \end{cases}$$

$$h^{(\mu,\Gamma_0)}(\Gamma_{12}) = \begin{cases} h^\flat(\Gamma_{12}) = h^\flat(\Gamma_{12}-a) & \text{if } \Gamma_{12} < \Gamma_{13}+\lambda_2-\lambda_3+1 = a+nk+u; \\ g^\flat(\Gamma_{12}) = g^\flat(\Gamma_{12}-a) & \text{if } \Gamma_{12} = \Gamma_{13}+\lambda_2-\lambda_3+1 = a+nk+u. \end{cases}$$

Furthermore, by Claim 2.3.1 we have $g^\flat(\Gamma_{12}-a) = v\cdot g_{\Gamma_{12}-a}$, $h^\flat(\Gamma_{12}-a)=0$ if $n\nmid\Gamma_{12}-a$ and $h^\flat(\Gamma_{12}-a) = 1-v$ if $n|\Gamma_{12}-a$. We proceed by substituting into (7.5.6).

$$\begin{aligned} P_{k,u}(\mathbf{x}) =& \frac{1}{1-v}\cdot x_r^{nk+u+n-1}\cdot\sum_{\Gamma_{12}=a}^{a+nk+u}\delta_a(\Gamma_{12})\cdot h^{(\mu,\Gamma_0)}(\Gamma_{12})\cdot\mathbf{x}^{(\Gamma_{12}-a)\alpha} \\ =& \frac{1}{1-v}\cdot x_r^{nk+u+n-1} \\ &\cdot\left((-v)\cdot(1-v)+\sum_{i=1}^{k}(1-v)^2\cdot\mathbf{x}^{in\alpha}+(1-v)\cdot v\cdot g_u\cdot\mathbf{x}^{(nk+u)\alpha}\right) \\ =& x_r^{nk+u+n-1}\cdot\left((-v)+\sum_{i=1}^{k}(1-v)\cdot\mathbf{x}^{in\alpha}+v\cdot g_u\cdot\mathbf{x}^{(nk+u)\alpha}\right) \\ =& x_r^{nk+u+n-1}\cdot\left((-v)+(1-v)\cdot\frac{\mathbf{x}^{(nk+n)\alpha}-\mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha}-1}+v\cdot g_u\cdot\mathbf{x}^{(nk+u)\alpha}\right) \end{aligned} \tag{7.5.7}$$

To rewrite this in the form of (7.5.5), note that by the definition of the Chinta-Gunnells action in type A (2.4.2), we have

$$\begin{aligned} \frac{1-v\cdot\mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha}-1}\cdot\sigma(x_r^{nk+u+n-1}) =& \frac{x_{r-1}^{nk+u+n-1}}{\mathbf{x}^{n\alpha}-1}\cdot\big(\mathbf{x}^{-r_n(nk+u+n-1)\alpha}\cdot(1-v) \\ & -v\cdot g_{nk+u+n}\cdot\mathbf{x}^{(1-n)\alpha}\cdot(1-\mathbf{x}^{n\alpha})\big) \\ =& x_{r-1}^{nk+u+n-1}\cdot\left(\frac{\mathbf{x}^{(1-u)\alpha}\cdot(1-v)}{\mathbf{x}^{n\alpha}-1}+v\cdot g_u\cdot\mathbf{x}^{(1-n)\alpha}\right) \\ =& x_r^{nk+u+n-1}\cdot\left(\frac{\mathbf{x}^{(nk+n)\alpha}\cdot(1-v)}{\mathbf{x}^{n\alpha}-1}+v\cdot g_u\cdot\mathbf{x}^{(nk+u)\alpha}\right) \end{aligned} \tag{7.5.8}$$

Comparing (7.5.8) to (7.5.7) we see that

$$\begin{aligned} P_{k,u}(\mathbf{x}) - \frac{1 - v \cdot \mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} \cdot \sigma(x_r^{nk+u+n-1}) &= x_r^{nk+u+n-1} \cdot \left((-v) + (1 - v) \cdot \frac{-\mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} \right) \\ &= x_r^{nk+u+n-1} \cdot \frac{(-v) \cdot (\mathbf{x}^{-n\alpha} - 1) + (1 - v)}{\mathbf{x}^{-n\alpha} - 1} \\ &= x_r^{nk+u+n-1} \cdot \frac{1 - v \cdot \mathbf{x}^{-n\alpha}}{\mathbf{x}^{-n\alpha} - 1}. \end{aligned} \tag{7.5.9}$$

This completes the proof of (7.5.5), and thus of Lemma 7.4.2.

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Appendix A

Proof of Lemma 2.1.1

This section is the proof of Lemma 2.1.1. Definition 2.2.1 and the crucial Lemma 2.2.2 both make use of the statement, which we recall in (A.0.2) below. Recall (from Chapter 2) that Λ is a sublattice containing the root system Φ. Take an integer-valued, Weyl group invariant quadratic form Q on Λ. This determines the bilinear form $B(.,.)$ by $B(x,y) = Q(x+y) - Q(x) - Q(y)$. The sublattice $\Lambda_0 \subseteq \Lambda$ is defined by

$$\Lambda_0 = \{\lambda \in \Lambda : B(\alpha, \lambda) \equiv 0 \bmod n \text{ for all simple roots } \alpha\}. \tag{A.0.1}$$

The goal is to prove that for every $\gamma \in \Phi$,

$$m(\gamma) \cdot \gamma \in \Lambda_0 \Longleftrightarrow \forall \alpha \in \Delta \; n | B(m(\gamma) \cdot \gamma, \alpha). \tag{A.0.2}$$

Recall that here

$$m(\gamma) = \frac{n}{\gcd(n, Q(\gamma))}.$$

The proof of (A.0.2) is case-by case according to the Cartan type of the root system Φ, with all simply laced cases handled together. In each non-simply laced case (B_r, C_r, F_4 and G_2) we recover the matrix of B in the basis Δ of simple roots, and complete the proof by a few simple divisibility arguments.

A.1 General remarks

We start by making a few remarks that shall be of use in every case. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be a set of simple roots. It is clear that it suffices to show (A.0.2) for positive roots γ. The form B is bilinear, hence for a fixed γ,

$$\begin{aligned} B(m(\gamma) \cdot \gamma, \alpha) =& m(\gamma) \cdot B(\gamma, \alpha) \\ =& \frac{n}{\gcd(n, Q(\gamma))} \cdot B(\gamma, \alpha), \end{aligned} \tag{A.1.1}$$

hence it suffices to check that for any positive root γ and simple root α,

$$Q(\gamma)|B(\gamma,\alpha) \tag{A.1.2}$$

holds.

Further, note that Q and B are both determined by an $r \times r$ symmetric matrix. The matrix corresponding to B in a fixed basis is twice the matrix corresponding to Q in the same basis, $Q(x) = \frac{1}{2} \cdot B(x,x)$. The following lemmas are easy facts about irreducible root systems.

Lemma A.1.1. *Let Φ be irreducible. Then all roots of a given length are conjugate under the Weyl group.*

Lemma A.1.2. *Let Φ be irreducible. Then Q on the vector space generated by Φ, Q and B are determined up to a scalar factor by the fact that Q is invariant under the action of the Weyl group.*

A.2 The simply laced cases.

First observe that if Φ is simply laced and thus all roots are of the same length,

$$\frac{1}{2}B(\gamma,\gamma) = Q(\gamma) = \mathbb{Q}(\alpha_1)$$

for any root γ. Thus by (A.1.2) it is sufficient to show that the matrix of B is a $Q(\alpha_1)$-multiple of an integer matrix. And this is true; the matrix of B in the basis Δ of simple roots is

$$\begin{aligned}[B]_\Delta =& \left(B(\alpha_i,\alpha_j)\right)_{\substack{1\le i\le r\\1\le j\le r}}\\ =& B(\alpha_1,\alpha_1)\cdot\left(\frac{B(\alpha_i,\alpha_j)}{B(\alpha_j,\alpha_j)}\right)_{\substack{1\le i\le r\\1\le j\le r}}\\ =& Q(\alpha_1)\cdot\left(\frac{2B(\alpha_i,\alpha_j)}{B(\alpha_j,\alpha_j)}\right)_{\substack{1\le i\le r\\1\le j\le r}},\end{aligned} \tag{A.2.1}$$

and in the last line of (A.2.1) we have the Cartan matrix with integer entries. (Note that by Lemma A.1.2, the Cartan matrix is independent of the choice of Q.)

This observation implies (A.1.2), thus the statement is true in the simply laced cases.

A.3 Type $\mathbf{B_r}$

In this case the roots are $\pm e_i$ and $\pm e_i \pm e_j$ with $1 \leq i \neq j \leq r$, a set of simple roots being

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \dots,\ \alpha_{r-1} = e_{r-1} - e_r,\ \alpha_r = e_r.$$

The roots $\pm e_i$ are the short roots. In the usual inner product, $\|\alpha_r\|^2 = 1$ and $\|\alpha_i\|^2 = 2$ for $i < r$, hence here $Q(\alpha_i) = 2Q(\alpha_r)$. If $j - i > 2$, then $\langle \alpha_i, \alpha_j \rangle = 0$, hence $B(\alpha_i, \alpha_j) = 0$. For any $i + 1 < r$,

$$\begin{aligned} B(\alpha_i, \alpha_{i+1}) =& Q(\alpha_i + \alpha_{i+1}) - Q(\alpha_i) - Q(\alpha_{i+1}) \\ =& - Q(e_i - e_{i+2}) \\ =& - 2Q(\alpha_r). \end{aligned}$$

Further,

$$B(\alpha_{r-1}, \alpha_r) = Q(e_{r-1}) - Q(e_{r-1} - e_r) - Q(e_r) = -2Q(\alpha_r).$$

From the above it follows that the matrix of B is

$$[B]_\Delta = Q(\alpha_r) \cdot \begin{pmatrix} 4 & -2 & 0 & & & \cdots & & 0 \\ -2 & 4 & -2 & 0 & & \cdots & & 0 \\ 0 & -2 & 4 & -2 & 0 & \cdots & & 0 \\ & & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & \cdots & -2 & 4 & -2 \\ 0 & 0 & 0 & & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

Since every element of $[B]_\Delta$ is divisible by $Q(\alpha_r)$, it follows that (A.1.2) holds if γ is a short root. (In this case $Q(\gamma) = Q(\alpha_r)$.) For the long roots $Q(\gamma) = 2Q(\alpha_r)$, hence (A.1.2) also holds for these: every element of $[B]_\Delta$ is an even multiple of $Q(\alpha_r)$, hence divisible by $Q(\gamma)$.

A.4 Type $\mathbf{C_r}$

In this case the roots are $\pm 2e_i$ and $\pm e_i \pm e_j$ with $1 \leq i \neq j \leq r$, a set of simple roots being

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \dots,\ \alpha_{r-1} = e_{r-1} - e_r,\ \alpha_r = 2e_r.$$

The short roots are of the form $\pm e_i \pm e_j$. In $[B]_\Delta$, the same positions will have nonzero elements as before. $Q(\alpha_r) = 2Q(\alpha_1)$. Moreover, if $i + 1 < r$, we have

$$\begin{aligned} B(\alpha_i, \alpha_{i+1}) =& Q(\alpha_i + \alpha_{i+1}) - Q(\alpha_i) - Q(\alpha_{i+1}) \\ =& Q(e_i - e_{i+2}) - Q(e_i - e_{i+2}) - Q(e_{i+1} - e_{i+2}) \\ =& - Q(\alpha_1) \end{aligned} \tag{A.4.1}$$

and similarly

$$\begin{aligned} B(\alpha_{r-1},\alpha_r) =& Q(e_{r-1}+e_r) - Q(e_{r-1}-e_r) - Q(2e_r) \\ =& Q(\alpha_1) - Q(\alpha_1) - 2Q(\alpha_1) \\ =& -2Q(\alpha_1). \end{aligned} \tag{A.4.2}$$

From (A.4.1) and (A.4.2) it follows that the matrix of B in this case is

$$[B]_\Delta = Q(\alpha_r)\cdot \begin{pmatrix} 2 & -1 & 0 & & & \cdots & & 0 \\ -1 & 2 & -1 & 0 & & \cdots & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & & 0 \\ & & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & & \cdots & 0 & -2 & 4 \end{pmatrix}.$$

As before, every element of $[B]_\Delta$ is divisible by $Q(\alpha_r)$, so (A.1.2) holds if γ is a short root. It remains to check (A.1.2) when $\gamma = 2e_i$ is a (positive) long root. In this case $Q(\gamma) = 2Q(\alpha_1)$, so we must check that $B(\gamma,\alpha_j)$ is an even multiple of $Q(\alpha_r)$ for every α_j simple root. This is trivial if i (or j) is equal to l. Otherwise notice that the decomposition of γ into a linear combination of simple roots is

$$\begin{aligned} \gamma =& 2e_i \\ =& 2(e_i - e_{i+1}) + 2(e_{i+1-e_{i+2}}) + \cdots + 2(e_{r-1}-e_r) + 2e_r \\ =& 2\cdot(\alpha_i + \cdots + \alpha_{r-1}) + \alpha_r. \end{aligned}$$

The coefficient of every α_h for $h < r$ is even, and $Q(\alpha_1)|B(\alpha_h,\alpha_j)$; $B(\alpha_r,\alpha_j)$ is an even multiple of $Q(\alpha_1)$. This implies (A.1.2) for γ.

A.5 Type $\mathbf{F_4}$

In this case $\Phi \subset \mathbb{R}^4$, and the roots are $\pm e_i \pm e_j$ ($1 \le i \ne j \le 4$, these 24 are long), $\pm e_i$ and $\frac{1}{2}\cdot(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ (these $16+8$ are short). Let

$$\Delta = \left\{ \alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}\cdot(e_1 - e_2 - e_3 - e_4), \right\}.$$

For the short roots $Q(\gamma) = Q(\alpha_4)$, for the long roots $Q(\gamma) = 2Q(\alpha_4)$. For the other elements of $[B]_\Delta$, we have

$$\begin{aligned}
B(\alpha_1, \alpha_2) =& Q(e_2 - e_4) - Q(e_2 - e_3) - Q(e_3 - e_4) = -2Q(\alpha_4);\\
B(\alpha_1, \alpha_3) =& B(\alpha_1, \alpha_4) = 0;\\
B(\alpha_2, \alpha_3) =& Q(e_3) - Q(e_3 - e_4) - Q(e_4) = -2Q(\alpha_4);\\
B(\alpha_2, \alpha_4) =& 0;\\
B(\alpha_3, \alpha_4) =& Q\left(\frac{1}{2} \cdot (e_1 - e_2 - e_3 + e_4)\right) - Q(e_4) - Q(\alpha_4) = -Q(\alpha_4).
\end{aligned} \tag{A.5.1}$$

(It is easy to see the cases where B is zero, since B is a constant multiple of a Euclidean inner product.) Thus the matrix of B is as follows:

$$[B]_\Delta = Q(\alpha_4) \cdot \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Again, the matrix of B is a $Q(\alpha_4)$-multiple of an integer matrix, hence (A.1.2) holds if γ is any of the short roots. Now let γ be a long root. Then $Q(\gamma) = 2Q(\alpha_4)$. We have

$$\gamma = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4,$$

hence

$$B(\gamma, \alpha) = c_1 \cdot B(\alpha_1, \alpha) + c_2 \cdot B(\alpha_2, \alpha) + c_3 \cdot B(\alpha_3, \alpha) + c_4 \cdot B(\alpha_4, \alpha).$$

Here $B(\alpha_1, \alpha)$ and $B(\alpha_2, \alpha)$ are even multiples of $Q(\alpha_4)$ for any α, hence

$$Q(\gamma) | c_1 \cdot B(\alpha_1, \alpha) + c_2 \cdot B(\alpha_2, \alpha).$$

Notice that the long roots are of the form $\pm e_i \pm e_j$, hence the their coordinates are integers that add up to an even number. The coordinates being integers implies c_4 is even. If $2|c_4$, then the sum of coordinates of $c_4\alpha_4$ is even. This implies that for $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ to have an even sum of coordinates, c_3 must be even, as well. Since $B(\alpha_3, \alpha)$ and $B(\alpha_4, \alpha)$ are divisible by $Q(\alpha_4)$, this implies that $Q(\gamma) = 2Q(\alpha)_4 | c_3 \cdot B(\alpha_3, \alpha) + c_4 \cdot B(\alpha_4, \alpha)$. That is, (A.1.2) holds for long roots as well.

A.6 Type $\mathbf{G_2}$

In this case

$$\Delta = \{\alpha = e_1 - e_2, \beta = (e_2 - e_3) - \alpha = -e_1 + 2e_2 - e_3\}.$$

Among the positive roots, α, $\alpha+\beta$ and $2\alpha+\beta$ are short, β, $3\alpha+\beta$ and $3\alpha+2\beta$ are long. In the usual inner product $\|\alpha\|^2 = 2$ and $\|\beta\|^2 = 6$, hence $Q(\beta) = 3Q(\alpha)$. Furthermore

$$B(\alpha, \beta) = Q(\alpha + \beta) - Q(\alpha) - Q(\beta) = -3Q(\alpha).$$

Thus the matrix of B is

$$[B]_\Delta = Q(\alpha) \cdot \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}.$$

Every element is an integer multiple of $Q(\alpha)$, so again (A.1.2) automatically holds if γ is a short root. If γ is a long root, then $Q(\gamma) = 3Q(\alpha)$, thus we must check that $B(\gamma, \beta)$ and $B(\gamma, \alpha)$ is divisible by $3Q(\alpha)$. This is clear in the case of $B(\gamma, \beta)$ or if $\gamma = \beta$, since $3Q(\alpha)|B(\beta, *)$ by the matrix above. On the other hand, if $\gamma \neq \beta$ is an other long root, then $\gamma = 3\alpha + k\beta$, hence

$$B(\gamma, \alpha) = 3 \cdot B(\alpha, \alpha) + k \cdot B(\beta, \alpha) = (6 - 3k) \cdot Q(\alpha).$$

Thus (A.1.2) holds in this case, too.