Probabilistic Approaches to Partial Differential Equations with Large Random Potentials

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ABSTRACT

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The thesis is devoted to an analysis of the equation

$$\partial_t u_\varepsilon(t,x) = \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon^\gamma}V(\frac{x}{\varepsilon})u_\varepsilon(t,x)$$

in high dimensions $d \geq 3$, where i is the imaginary unit and V is some stationary random field. The strength γ is chosen so that the large, highly oscillatory, random potential is producing non-trivial effects in the asymptotic limit $\varepsilon \to 0$. We prove either homogenization, i.e., $i\varepsilon^{-\gamma}V(\frac{x}{\varepsilon})$ is replaced by some deterministic constant, or convergence to a stochastic partial differential equation, i.e., $i\varepsilon^{-\gamma}V(\frac{x}{\varepsilon})$ is replaced by some stochastic noise, depending on the correlation property of the random field V. When the limit is deterministic, we provide estimates of the error between the heterogeneous and homogenized solutions when certain mixing assumption of V is satisfied. We also prove a central limit type of result when V is Gaussian or Poissonian type of potentials. Lower dimensional and time-dependent cases are also treated. Most of the ingredients in the analysis are probabilistic, including a Feynman-Kac representation, a Brownian motion in random scenery, the Kipnis-Varadhan's method, and a quantitative martingale central limit theorem. The result presented here is a summary of [14, 16, 13, 15].

Contents

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To My Parents

Chapter 1

Introduction

Partial differential equations (PDE) with small scales abound in physics and applied science. A general form is written as

$$P(x, \frac{x}{\varepsilon}, \partial_x) u_\varepsilon = 0,$$

where P is some differential operator with heterogeneous coefficients depending on both slow and fast scales. The separation of the scales is described by a parameter $\varepsilon \ll 1$. A natural mathematical question would be to understand the asymptotic behavior of u_ε as $\varepsilon \to 0$, which is closely related to the framework of *homogenization theory*. A typical example is the elliptic equation in divergence form

$$-\nabla \cdot a(\frac{x}{\varepsilon}) \nabla u_\varepsilon(x) = f(x) \tag{1.1}$$

in some bounded domain with appropriate boundary conditions. Here u_ε stands for certain physical quantities, e.g., the temperature distribution, while the highly oscillatory matrix coefficient $a(\frac{x}{\varepsilon})$, where a is assumed to be either *periodic* or *random*, describes physical properties of the underlying *medium*, e.g., the heat conductivity distribution. Solving (1.1) amounts to measuring the stationary temperature distribution given such highly heterogeneous medium and the source f. Heuristically, homogenization theory shows that when ε is small, solving u_ε is close to solving the same equation with a *homogeneous* medium described by some *deterministic* constant matrix a_{hom}. In other words, for small ε, the homogeneous medium almost takes the same effects as the heterogeneous medium. The mathematical formulation would be $u_\varepsilon \to u_{hom}$ in some appropriate sense as $\varepsilon \to 0$, where u_{hom} solves

$$-\nabla \cdot a_{hom} \nabla u_{hom}(x) = f(x). \tag{1.2}$$

In most cases, the underlying medium is only known approximately and usually modeled as a random medium with certain statistical property. Stochastic homogenization thus becomes important from both theoretical and numerical points of view. Deriving the limiting equation with a homogeneous medium helps to capture the main feature of the physical model. It is natural to ask the following questions:

1. Is the model homogenizable and how do we derive the homogenized limit? Some necessary assumptions should be made on the heterogeneous coefficients so that appropriate limiting procedures can be carried out to obtain the convergence as the small scale goes to zero.

2. How much error is induced when we use a homogenized model to replace the original one? Proving convergence is a qualitative result, and deriving the convergence rate and describing the error is the *quantitative* aspect of homogenization. It is as important as, but in general more difficult than proving the convergence.

3. What would happen if we can not obtain any homogenized limiting model? In the

random setting, sometimes it is possible for the stochasticity to remain in the asymptotic limit, i.e., the uncertainties survives. It is crucial to understand the physical mechanism of convergence to homogenized, deterministic models or to stochastic models.

The goal of the thesis is to present an example of PDE with highly heterogeneous random coefficients, for which we can provide a relatively complete answer to the above questions. More specifically, we analyze an equation with a large highly oscillatory random potential of the form $\partial_t u_\varepsilon(t,x) = \frac{1}{2}\Delta u_\varepsilon(t,x) + i\varepsilon^{-\alpha}V(x/\varepsilon)u_\varepsilon(t,x)$, where i is the imaginary unit and $V(x)$ is a stationary random field. We show that for a general class of *short-range-correlated* random fields, u_ε converges to a deterministic limit, i.e., homogenization occurs, while for a large class of *long-range-correlated* random fields, we obtain a convergence to stochastic partial differential equations (SPDE). In addition, we provide an optimal error estimate in the homogenization setting under some mixing assumptions on the random fields. We also derive the asymptotic distribution of the renormalized error when $V(x)$ is Gaussian or Poissonian.

There is a large body of literature in the area of stochastic homogenization, starting from the work of Kozlov [23] and Papanicolaou-Varadhan [28] for equations of the form (1.1). Error estimates are less well-understood, see the recent work on the discrete setting by Gloria, Otto and their coauthors [11, 12, 25], and the nonlinear case by Caffarelli and Souganidis [8].

From a probabilistic point of view, stochastic homogenization is a law of large numbers type result, and ergodicity is usually a necessary assumption on the random medium in general. In addition, since second order differential operators serve as the infinitesimal generators of diffusion processes, by probabilistic representation, the solution to PDE can be expressed as the average with respect to certain random process. For the equation with random coefficients, let ω denote a particular realization of the random environment, we thus obtain a family of stochastic processes indexed by ω, and the convergence of the corresponding PDE may be recast as a problem of weak convergence of random motion in random environment. A good introduction is [22].

In the following, we first describe the model of *random walk on random conductance* related to (1.1), then we give an overview of our work on the parabolic equation with large random potentials. The relevant stochastic process there is the so-called *Brownian motion in random scenery.*

1.1 Random walk on random conductance

We consider equations of the type (1.1) in the discrete and parabolic setting. Assuming $\{a(x), x \in \mathbb{Z}^d\}$ is a family of diagonal matrix, and $u(t,x), t \in \mathbb{R}_+, x \in \mathbb{Z}^d$ solves the equation

$$\partial_t u(t,x) = -\nabla^* a(x) \nabla u(t,x) \tag{1.3}$$

with initial condition $u(0,x) = f(x)$. ∇ denotes the discrete gradient operator

$$\nabla f(x) = (f(x+e_1) - f(x), \ldots, f(x+e_d) - f(x)),$$

where $\{e_i\}_{i=1}^d$ is the canonical basis of $\mathbb{Z}^d$. ∇^* denotes the adjoint operator of ∇ in the standard $l^2(\mathbb{Z}^d)$, i.e., for $F = (f_1, \ldots, f_d)$,

$$\nabla^* F = \sum_{i=1}^d (f_i(x - e_i) - f_i(x)).$$

It is straightforward to link (1.3) with a random conductance model. We assume the coefficient matrix satisfies a uniform elliptic condition, i.e., $\forall x \in \mathbb{Z}^d, C^{-1} I \leq a(x) \leq CI$ almost surely for some $C > 0$. Since $a(x)$ is diagonal, we assign each edge $(x, x+e_i)$ a positive conductance, denoted by $\omega_{x,x+e_i}$, with the value $a_{ii}(x)$. The conductance model on $\mathbb{Z}^d$ is then denoted by $\omega = \{\omega_e\}$, and clearly (1.3) can be rewritten as

$$\partial_t u(t,x) = \sum_{y \sim x} \omega_{x,y}(u(t,y) - u(t,x)) := L^\omega u(t,x), \tag{1.4}$$

where $y \sim x$ indicates y is connected to x by one edge. The operator L^ω, which depends on the random conductance, turns out to be the infinitesimal generator of a random walk on the random conductance ω. For fixed realization of ω, the random walk X_t jumps from x to its nearest neighbor y with rate $\omega_{x,y}$. The solution can be written as

$$u(t,x) = \mathbb{E}_x^\omega\{f(X_t)\}, \tag{1.5}$$

where the expectation $\mathbb{E}_x^\omega$ refers to the average with respect to the random walk X_t starting from x for the fixed conductance ω.

To consider the large time, long distance asymptotic limit of $u(t,x)$, we assume a parabolic scaling and denote $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$ for $t \in \mathbb{R}_+, x \in \varepsilon\mathbb{Z}^d$. With the initial condition replaced by $f(.) \to f(\varepsilon.)$, the equation for u_ε is

$$\partial_t u_\varepsilon(t,x) = -\nabla_\varepsilon^* a(\frac{x}{\varepsilon})\nabla_\varepsilon u_\varepsilon(t,x)$$

with initial condition $u_\varepsilon(0,x) = f(x)$, where ∇_ε is the discrete gradient operator on $\varepsilon\mathbb{Z}^d$. We expect that as $\varepsilon \to 0$, the conductance averages into some constant and the equation is being homogenized. By the probabilistic representation (1.5), the solution is

$$u_\varepsilon(t,x) = \mathbb{E}^\omega_{x/\varepsilon}\{f(\varepsilon X_{t/\varepsilon^2})\}.$$

If we assume that ω is translation-invariant, then $u_\varepsilon(t,x)$ has the same distribution as

$$u_\varepsilon(t,x) = \mathbb{E}^\omega_0\{f(x+\varepsilon X_{t/\varepsilon^2})\},$$

where X_t is the random walk on random conductance ω starting from the origin. We observe that except for a discretization error when $x \notin \varepsilon\mathbb{Z}^d$, a convergence of $u_\varepsilon(t,x)$ is directly related to the convergence of the random walk on random conductance X_t, which is diffusively scaled. It turns out that $\varepsilon X_{t/\varepsilon^2}$ converges weakly to some Brownian motion B_t with covariance matrix A [24, 9, 21], which implies $u_\varepsilon(t,x)$ converges in probability to $u_{hom}(t,x)$ solving

$$\partial_t u_{hom}(t,x) = \nabla \cdot a_{hom} \nabla u_{hom}(t,x) \tag{1.6}$$

with initial condition $u_{hom}(t,x) = f(x)$ and the homogenized matrix $a_{hom} = \frac{1}{2}A$. By quantifying the convergence rate of $\varepsilon X_{t/\varepsilon^2}$ to B_t, convergence rate in the homogenization can be obtained correspondingly, see [26].

Remark 1.1. The weak convergence in [24, 9, 21] refers to weak convergence in measure. It is improved in [32] as weak convergence a.s., i.e., quenched weak convergence.

1.2 Heat equation with random potentials and Brownian motion in random scenery

The model of random walk on random conductance is truly a random motion in random environment in the sense that the environment does affect the random path chosen by the walker. Mathematically it is equivalent with a $\omega-$dependent infinitesimal generator, which is not the case for our model of heat equation with random potentials. The equation we consider is of the form

$$\partial_t u_\varepsilon(t,x,\omega) = \frac{1}{2}\Delta u_\varepsilon(t,x,\omega) + i\frac{1}{\varepsilon^\gamma}V(\frac{x}{\varepsilon},\omega)u_\varepsilon(t,x,\omega), \tag{1.7}$$

for $t \in \mathbb{R}_+, x \in \mathbb{R}^d, d \geq 3$ and ω denotes a particular realization sampled from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By a Feynman-Kac formula, the solution is written as

$$u_\varepsilon(t,x,\omega) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^\gamma}\int_0^t V(\frac{x+B_s}{\varepsilon})ds)\}, \tag{1.8}$$

where $f \in \mathcal{C}_b(\mathbb{R}^d)$ is the initial condition and $\mathbb{E}_B$ denotes the expectation with respect to the Brownian motion. By the stationarity of V and the scaling property of Brownian motion, we observe that a key quantity to analyze is $\varepsilon^{2-\gamma}\int_0^{t/\varepsilon^2} V(B_s)ds$. In other words, asymptotic limit of u_ε is determined by the weak convergence of a Brownian motion in random scenery. The randomness in the equation appears only in the lower order term, so we have a "deterministic" Brownian motion, which is not affect by the random realization.

The corresponding discrete version is the Kesten-Spitzer model of random walk in random scenery [20] of the form $W_n = \sum_{i=1}^n \xi_{S_k}$. Here, $S_k = X_1 + \ldots + X_k$ is a random walk on $\mathbb{Z}$ with i.i.d. increments and $\xi_n, n \in \mathbb{Z}$, are i.i.d. and independent of X_i. When X_i and ξ_i

belong to the domain of attraction of certain stable laws, then after proper scaling $a(n)^{-1}W_{[nt]}$ converges weakly as $n \to \infty$ to a self-similar process with stationary increment. Non-stable limits may appear in that case. Assuming moreover that ξ_i has zero mean and finite variance, it is shown in [7] that $(n \log n)^{-\frac{1}{2}} W_{[nt]}$ converges weakly to a Brownian motion when $d = 2$. When $d \geq 3$, the argument contained in [20] essentially proves that $n^{-\frac{1}{2}} W_{[nt]}$ converges weakly to a Brownian motion.

For the continuous case, the invariance principle of $\varepsilon \int_0^{t/\varepsilon^2} V(B_s)ds$ is given by the Kipnis-Varadhan method [21] when $d \geq 3$ for the most general class of random field V, with only the necessary assumptions of stationarity, ergodicity and finiteness of the asymptotic variance. The convergence is in the sense of weak convergence in measure, which we will explain later in detail. In our context, the limit is Brownian motion, which can be viewed as some deterministic measure (the Wiener measure), so this type of weak convergence is essentially a law of large number result and fits well in our framework of homogenization. The analysis of Kipnis-Varadhan is based on the construction of a corrector function and a martingale decomposition, which constitute most of the main ingredients in proving homogenization, error estimate, and a further central limit result. Another tool we need is the quantitative martingale central limit theorem developed by Mourrat [26]. To derive error estimate, we need to quantify the Wasserstein distance between continuous martingales and Brownian motions. Since continuous martingales are time-changed Brownian motions, heuristically it is equivalent with estimating the difference between values a Brownian motion taking at a certain random time and a fixed deterministic time, hence reduces to the difference between quadratic variations. To obtain central limit result, we need to refine the error and separate those main contributions, which requires a refined quantitative martingale central limit theorem. The results are presented in Chapter 2 and a summary of [13] and part of

[14].

For $d = 2$, a logarithm scaling factor appears and it is not clear whether Kipnis-Varadhan can be applied. A type of piecewise constant potentials is treated in [31] and an annealed convergence to Brownian motion is obtained. In their case, the Brownian motion in random scenery is expressed as a weighted sum of i.i.d. random variables, and each weight describes the time that the Brownian motion spent in the corresponding "piece". The Lindeberg condition is verified to prove the central limit theorem. We consider two other types of simple yet important random fields, Gaussian and Poissonian, and prove the annealed invariance principle by the method of characteristic functions. The proof is pretty standard and involves estimations of integrals of Brownian functionals. Since the weak convergence is in the annealed sense, the proof of homogenization involves two steps, convergence of expectation, and convergence of second order moment. This is treated in Chapter 3 and most of the results come from [16].

A necessary condition for the invariance principle of Brownian motion in random scenery is a finite variance of the limiting Brownian motion. It turns out to be equivalent with certain integrability condition of the covariance function of random potentials, i.e., as long as the covariance function decays sufficiently fast, we obtain the invariance principle and homogenization. A natural question would be to investigate those long-range-correlated random potentials which do not satisfy the integrability condition. Homogenization does not occur and stochasticity survives in the asymptotic limit. We show that for a large class of long-range-correlated potentials chosen as functions of Gaussian fields, the limit of the Brownian motion in random scenery is something we call the Brownian motion in Gaussian noise, and the limiting equation is a SPDE with multiplicative Gaussian noise. To compare with the homogenization setting, an analogue here is the difference between central limit theorem for sum of i.i.d. random variables and limit theorem for sum of strongly dependent stationary random variables. The latter has been analyzed by Taqqu [33] and results show that either Gaussian or non-Gaussian law could appear as the limit depending on some Hermite rank, i.e., it is not universal as in central limit theorem and the limit depends on the specific correlation property. More details are explained in Chapter 4, and the result is part of [14].

Chapter 5 analyzes the time-dependent case, i.e., when we have a tempo-spatial random field $V(t,x)$ and the Brownian motion travels in a changing environment. For $V_\varepsilon(t,x) = \varepsilon^{-\gamma}V(t/\varepsilon^\alpha, x/\varepsilon)$ with $\gamma, \alpha > 0$, it turns out a key threshold is $\alpha = 2$. The temporal mixing from the random field dominates when $\alpha > 2$ while spatial mixing dominates when $\alpha < 2$. In other words, the mechanism for the convergence of Brownian motion in random scenery is $\alpha-$dependent. We prove in the short-ranged-correlated case that homogenization occurs with different homogenized constants for $\alpha > 2, \alpha = 2, \alpha \in (0,2)$ respectively. The difference is illustrated pretty clearly by the corresponding convergence of stochastic processes. The result comes from [15].

Since we have two sources of randomness, the random potential V and the Brownian motion B induced by the Feynman-Kac formula, we use $\mathbb{E}$ to denote the expectation with respect to the environment and $\mathbb{E}_B, \mathbb{E}_W$ the expectation with respect to independent Brownian motions B, W. Other notations will be introduced along the presentation of the main results.

1.3 Other work on equations with large random potentials

There are other work on equation with large random potentials, using probabilistic or analytic tools. We briefly discuss about them and illustrate the connection with our approach. A review and more references can be found in [4].

The case $d = 1$ has been analyzed by Pardoux, Piatnitski and Hairer. In [29], they analyzed the same type of equation with short-range-correlated real random potentials by probabilistic techniques, i.e., a Feynman-Kac representation and weak convergence of the Brownian motion in random scenery. The dimension plays an important role here. They applied the local time technique and expressed the corresponding process as a weighted integral of mixing potentials, where the weight is describe by the local time of the Brownian motion. In this way, the Brownian path is saved in the limit while the highly oscillatory mixing potential averages into a white noise. In mathematical language, for the potential of the form $\varepsilon^{-\frac{1}{2}}V(x/\varepsilon)$, they proved that

$$\frac{1}{\varepsilon^{\frac{1}{2}}}\int_0^t V(\frac{B_s}{\varepsilon})ds = \int_{\mathbb{R}} \frac{1}{\varepsilon^{\frac{1}{2}}} V(\frac{x}{\varepsilon}) L_t(x)dx \Rightarrow \int_{\mathbb{R}} L_t(x)W(dx),$$

where $L_t(x)$ is the local time of B_s and $W(dx)$ is some white noise independent of B_s. By the weak convergence result and some uniform integrability estimate, they pass to the limit of a one-dimensional SPDE with multiplicative noise in the Stratonovich sense. The result is similar to our long-range-correlated case in the following sense. When $d = 1$, the Brownian motion is recurrent, adding a lot of correlations to the additive functional by visiting the same location many times. In high dimensional case when $d \geq 3$, given a transient Brownian motion, the correlation is instead produced by the long-range-correlated random fields, i.e., B_t visits different locations at which V takes strongly correlated values. In both ways, the Brownian path is recorded in the limit and we get a limit for the Brownian motion in random scenery that could be formally written as $\int_0^t \dot{W}(B_s)ds$, where $\dot{W}$ is a random noise. Furthermore, the limiting SPDEs are of the same type, i.e., multiplicative noise in the Stratonovich sense. Later in [30], they considered the time-dependent potentials and proved homogenization when the temporal mixing is sufficiently strong (in our notation, $\alpha \geq 2$). The same probabilistic approach does not seem to cover the case $\alpha \in (0,2)$. In [17], they revisited the case of $\alpha \in (0,2)$ and proved homogenization by analytical method. It turns out invariance principle holds when $\alpha \geq 2$, i.e., the temporal mixing leads to a Brownian motion in the limit. When $\alpha \in (0,2)$, it is not clear whether the limit is still Brownian motion. Our results in Chapter 5 deal with high dimensions ($d \geq 3$) and the case of $\alpha \in (0,2)$ can be handled by Kipnis-Varadhan.

More general types of parabolic equations and Schrödinger equations with Gaussian potentials have been analyzed by Bal and his coauthors in a series of papers by the approach of diagram expansions [1, 2, 3, 35, 36]. The starting point is to iterate an integral equation to obtain a Duhamel expansion series. To show the infinite series defines a solution that converges, high order moments of the potential are calculated and the main contributions are separated by the analysis of diagrams. The results in high dimensions are consistent with those obtained in this thesis, i.e., short-range-correlation leads to homogenization while long-range-correlation implies convergence to SPDE. Low dimensional cases were considered as well (in our context, it only covers $d = 1$ whereas for the general elliptic operator of the form $(-\Delta)^{\frac{m}{2}}$, it covers $d < m$), and a convergence to SPDE is obtained, which is consistent with [29]. The method of diagrams provides a relatively complete picture about the solution to heterogeneous equations, e.g., the terms that remains in the asymptotic limit and constitutes the limiting solution, the terms that form the first order correction and so on. A central limit theorem is proved in [2] by considering those first order correction terms. The disadvantage of this method is the constraint of Gaussian potential. For other potentials like Poissonian, the calculation of high order moments leads to more terms compared with the Gaussian case, and more involved combinatorial techniques are required.

Other related work will be discussed along the presentation of our main results.

Chapter 2

Homogenization and Corrector Theory: $d \geq 3$

In this chapter, we build the major framework of homogenization and corrector theory using the approach of Kipnis-Varadhan [21]. We will first introduce the setup of *random medium*, then prove homogenization and corrector result for (1.7). Some technical lemmas are left in the appendix.

2.1 Homogenization and error estimate

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space associated with a group of measure-preserving transformations $\{\tau_x, x \in \mathbb{R}^d\}$, i.e, $\mathbb{P} \circ \tau_x = \mathbb{P}$ for all $x \in \mathbb{R}^d$. Furthermore, its action is ergodic and stochastically continuous, i.e.,

1. $\mathbb{P}\{A\} = 0$ or 1 for any event A such that $\mathbb{P}\{A \triangle \tau_x(A)\} = 0$ for all $x \in \mathbb{R}^d$ and

2. for any $\delta > 0$ and G bounded we have

$$\lim_{h \to 0} \mathbb{P}\{\omega : |G(\tau_h \omega) - G(\omega)| \geq \delta\} = 0.$$

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the above assumptions is called a random medium. The expectation on Ω is denoted by $\mathbb{E}$. The inner product and norm on $L^2(\Omega)$ are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively.

For any $f \in L^2(\Omega)$, let $(T_x f)(\omega) = f(\tau_{-x}\omega)$. The family $\{T_x, x \in \mathbb{R}^d\}$ forms a $d-$parameter group of unitary operators on $L^2(\Omega)$, and stochastic continuity implies that the group is strongly continuous. The generators of the group $\{T_x, x \in \mathbb{R}^d\}$ correspond to differentiation (in $L^2(\Omega)$) in the canonical directions e_k and are denoted by $\{D_k, k = 1, \ldots, d\}$.

Since $\{T_x, x \in \mathbb{R}^d\}$ is strongly continuous, by spectral theory we have that

$$T_x = \int_{\mathbb{R}^d} e^{i\xi x} U(d\xi),$$

with $U(d\xi)$ the associated projection valued measure. Let $L = \frac{1}{2}\sum_{k=1}^d D_k^2$ and $\mathbb{V} \in L^2(\Omega)$ satisfies $\int_\Omega \mathbb{V} d\mathbb{P} = 0$ and the following key assumption:

Assumption 2.1. $\langle -L^{-1}\mathbb{V}, \mathbb{V} \rangle < \infty$.

The random field we consider will be constructed by

$$V(x, \omega) = \mathbb{V}(\tau_{-x}\omega). \tag{2.1}$$

The measure-preserving property of τ_x implies $V(x, \omega)$ is stationary in x. Let $R(x) = \mathbb{E}\{V(x, \omega)V(0, \omega)\}$, whose Fourier transform is denoted by $\hat{R}(\xi) = \int_{\mathbb{R}^d} R(x) e^{-i\xi x} dx$.

By the spectral representation, we have

$$-L^{-1} = 2\int_{\mathbb{R}^d} |\xi|^{-2} U(d\xi).$$

Let $\hat{R}_{\mathbb{V}}(\xi)$ be the power spectrum associated with $\mathbb{V}$, i.e.,

$$\hat{R}_{\mathbb{V}}(\xi)d\xi = (2\pi)^d \langle U(d\xi)\mathbb{V}, \mathbb{V}\rangle,$$

then we obtain that

$$\langle -L^{-1}\mathbb{V}, \mathbb{V}\rangle = \langle 2\int_{\mathbb{R}^d} |\xi|^{-2} U(d\xi)\mathbb{V}, \mathbb{V}\rangle = \frac{2}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\hat{R}_{\mathbb{V}}(\xi)}{|\xi|^2} d\xi. \tag{2.2}$$

Therefore, the assumption that $\langle -L^{-1}\mathbb{V}, \mathbb{V}\rangle < \infty$ is equivalent with the integrability of $\hat{R}_{\mathbb{V}}(\xi)|\xi|^{-2}$. On the other hand, since

$$R(x) = \mathbb{E}\{V(x,\omega)V(0,\omega)\} = \langle T_x\mathbb{V}, \mathbb{V}\rangle = \int_{\mathbb{R}^d} e^{i\xi x}\langle U(d\xi)\mathbb{V}, \mathbb{V}\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi x}\hat{R}_{\mathbb{V}}(\xi)d\xi,$$

$\hat{R}(\xi) = \hat{R}_{\mathbb{V}}(\xi)$, and Assumption 2.1 is equivalent with the integrability of $\hat{R}(\xi)|\xi|^{-2}$. When $d \geq 3$, $|\xi|^{-2}$ is integrable around the origin, so as long as $R(x)$ does not decay too slowly, or equivalently $\hat{R}(\xi)$ does not blow up at the origin too quickly, Assumption 2.1 is satisfied.

Remark 2.2. For a more detailed setup of the random medium, we refer to e.g. [28] and [22, Chapter 9].

The equation we are interested in is

$$\partial_t u_\varepsilon(t,x,\omega) = \frac{1}{2}\Delta u_\varepsilon(t,x,\omega) + i\frac{1}{\varepsilon}V(\frac{x}{\varepsilon},\omega)u_\varepsilon(t,x,\omega), \tag{2.3}$$

with initial condition $u_\varepsilon(t,x,\omega) = f(x)$ for $f \in \mathcal{C}_b(\mathbb{R}^d)$, i.e., in (1.7) we choose $\gamma = 1$.

Defining

$$\sigma^2 = 2\langle \mathbb{V}, -L^{-1}\mathbb{V}\rangle = \frac{4}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi > 0,$$

and $u_{hom}(t,x)$ such that

$$\partial_t u_{hom}(t,x) = \frac{1}{2}\Delta u_{hom}(t,x) - \frac{1}{2}\sigma^2 u_{hom}(t,x) \tag{2.4}$$

with the same initial condition $u_0(0,x) = f(x)$, we have the following theorem:

Theorem 2.3 (*Homogenization*)**.** *Under Assumption 2.1, $u_\varepsilon(t,x) \to u_{hom}(t,x)$ in probability as $\varepsilon \to 0$.*

We are also interested in the convergence rate of $u_\varepsilon \to u_{hom}$. To give error estimate, one possible assumption we need is the following strongly mixing property of the random potential $V(x)$:

Assumption 2.4. $\mathbb{E}\{V^6(x)\} < \infty$ and there exists a mixing coefficient $\rho(r)$ decreasing in $r \in [0,\infty)$ such that for any $\beta > 0$, $\rho(r) \le C_\beta(1 \wedge r^{-\beta})$ for some $C_\beta > 0$ and the following bound holds

$$\mathbb{E}\{\phi_1(V)\phi_2(V)\} \le \rho(r)\sqrt{\mathbb{E}\{\phi_1^2(V)\}\mathbb{E}\{\phi_2^2(V)\}} \tag{2.5}$$

for any two compact sets K_1, K_2 with $d(K_1,K_2) = \inf_{x_1\in K_1, x_2\in K_2}\{|x_1 - x_2|\} \ge r$ and any random variables $\phi_1(V), \phi_2(V)$ with $\phi_i(V)$ being $\mathcal{F}_{K_i}-$measurable and $\mathbb{E}\{\phi_i(V)\} = 0$.

Remark 2.5. Under Assumption 2.4, we have $|R(x)| = |\mathbb{E}\{V(0)V(x)\}| \le C_\beta(1 \wedge |x|^{-\beta})$ for any $\beta > 0$. Note that

$$\sigma^2 = \frac{4}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi = \frac{1}{\pi^{\frac{d}{2}}}\Gamma(\frac{d}{2} - 1)\int_{\mathbb{R}^d} \frac{R(x)}{|x|^{d-2}} dx, \tag{2.6}$$

so the strongly mixing assumption implies finiteness of the homogenization constant.

The following is the result of convergence rate for strongly mixing potentials:

Theorem 2.6 (*Error estimate for strongly mixing potentials*)**.** *Under Assumption 2.4, if* $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$*, the following error estimates hold:*

$$\mathbb{E}\{|u_\varepsilon(t,x) - u_{hom}(t,x)|\} \leq (1+t)C_{d,f,\rho} \begin{cases} \sqrt{\varepsilon} & d=3, \\ \varepsilon\sqrt{|\log\varepsilon|} & d=4, \\ \varepsilon & d>4. \end{cases} \tag{2.7}$$

Remark 2.7. As suggested by the notation, $C_{d,f,\rho}$ only depends on the dimension, initial condition and mixing coefficient. If we follow the proof, it is easy to check that we only need to assume $\rho(r) \lesssim 1 \wedge r^{-\beta}$ for sufficiently large β, and the regularity assumption on f could be improved as well.

The solution to (2.3) is written as

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon}\int_0^t V(\frac{x+B_s}{\varepsilon})ds)\}, \tag{2.8}$$

with the Brownian motion B_s starting from the origin.

Remark 2.8. Note that for $u_\varepsilon(t,x)$ defined by (2.8) to be the classical solution to (2.3), we need to make certain regularity assumptions on the potential $V(x)$. However, it could be shown that the solution defined by the Feynman-Kac formula is indeed a weak solution to (2.3), see Lemma 2.29. Due to the possible unboundedness of $V(x)$, it is not clear that whether this weak solution is unique. We do not address this technical issue here but will focus on the probabilistic representations, therefore when we refer to the solution to (2.3), it is the weak solution given by the Feynman-Kac formula (2.8).

By the scaling property of Brownian motion,

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon\int_0^{t/\varepsilon^2} V(\frac{x}{\varepsilon}+B_s)ds)\}.$$

Since u_{hom} is deterministic, by the stationarity of V, the difference between the solutions to the heterogeneous and homogenized equations can be written as

$$\begin{aligned}&\mathbb{E}\{|u_\varepsilon(t,x)-u_{hom}(t,x)|\}\\ =&\mathbb{E}\{|\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds)\}-\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\}|\}.\end{aligned} \tag{2.9}$$

Now we look at $X_\varepsilon(t) := \varepsilon\int_0^{t/\varepsilon^2}\mathbb{V}(\tau_{-B_s}\omega)ds = \varepsilon\int_0^{t/\varepsilon^2}V(B_s)ds$. For $y_s^\omega = y_s := \tau_{-B_s}\omega$, by Lemma 2.30, it is a stationary, ergodic Markov process taking values in Ω with invariant measure $\mathbb{P}$. The generator of y_s is given by $L=\frac{1}{2}\sum_{k=1}^d D_k^2$, see [22, Proposition 9.8].

We define the corrector function Φ_λ for any $\lambda>0$ such that

$$(\lambda I - L)\Phi_\lambda = \mathbb{V}, \tag{2.10}$$

then the following proposition holds.

Proposition 2.9.

$$\Phi_\lambda = \int_{\mathbb{R}^d}\frac{1}{\lambda+\frac{1}{2}|\xi|^2}U(d\xi)\mathbb{V}. \tag{2.11}$$

Under Assumption 2.1, $\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle\to 0$ as $\lambda\to 0$.

Under Assumption 2.4,

$$\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle \lesssim \begin{cases} \sqrt{\lambda} & d=3,\\ \lambda|\log\lambda| & d=4,\\ \lambda & d>4. \end{cases} \tag{2.12}$$

If we define

$$\eta_k = \int_{\mathbb{R}^d} \frac{2i\xi_k}{|\xi|^2} U(d\xi)\mathbb{V} \tag{2.13}$$

for $k=1,\ldots,d$, $\sigma^2=\sum_{k=1}^d \|\eta_k\|^2$. Defining $\sigma_\lambda^2=\sum_{k=1}^d \|D_k\Phi_\lambda\|^2$, the following proposition holds.

Proposition 2.10. *Under Assumption 2.1, $D_k\Phi_\lambda\to\eta_k$ in $L^2(\Omega)$ as $\lambda\to 0$.*

Under Assumption 2.4,

$$|\sigma_\lambda^2-\sigma^2| \lesssim \begin{cases} \sqrt{\lambda} & d=3,\\ \lambda|\log\lambda| & d=4,\\ \lambda & d>4. \end{cases} \tag{2.14}$$

Proof of Proposition 2.9. First, we have

$$\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle = \int_{\mathbb{R}^d} \frac{\lambda}{\lambda+\frac{1}{2}|\xi|^2}\frac{\hat{R}(\xi)}{\lambda+\frac{1}{2}|\xi|^2}d\xi \lesssim \int_{\mathbb{R}^d}\frac{\lambda}{\lambda+|\xi|^2}\frac{\hat{R}(\xi)}{|\xi|^2}d\xi \tag{2.15}$$

Under Assumption 2.1, i.e., $\hat{R}(\xi)|\xi|^{-2}$ is integrable, by the dominated convergence theorem, $\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle\to 0$ as $\lambda\to 0$.

If Assumption 2.4 holds, $\hat{R}(\xi)$ is bounded, and we obtain by direct calculation:

$$\begin{aligned}\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle \lesssim&\lambda^{\frac{d}{2}-1}\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\frac{\hat{R}(\sqrt{\lambda}\xi)}{|\xi|^2}d\xi\\ \lesssim&\lambda^{\frac{d}{2}-1}\int_{\sqrt{\lambda}|\xi|<1}\frac{1}{1+|\xi|^2}\frac{1}{|\xi|^2}d\xi+\lambda\int_{|\xi|>1}\frac{\hat{R}(\xi)}{|\xi|^4}d\xi\\ \lesssim&\lambda^{\frac{d}{2}-1}\int_0^{\frac{1}{\sqrt{\lambda}}}\frac{r^{d-3}}{1+r^2}dr+\lambda,\end{aligned}\tag{2.16}$$

so when $d=3$, $\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle\lesssim\sqrt{\lambda}$. When $d=4$, $\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle\lesssim\lambda|\log\lambda|$. When $d>4$, $\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle\lesssim\lambda$.

□

Proof of Proposition 2.10. Since

$$\|D_k\Phi_\lambda-\eta_k\|^2=\int_{\mathbb{R}^d}\frac{\lambda^2\xi_k^2}{(\lambda+\frac{1}{2}|\xi|^2)^2\frac{1}{4}|\xi|^4}\hat{R}(\xi)d\xi\lesssim\int_{\mathbb{R}^d}\frac{\lambda^2}{\lambda^2+|\xi|^4}\frac{\hat{R}(\xi)}{|\xi|^2}d\xi,\tag{2.17}$$

and

$$\sigma_\lambda^2-\sigma^2=-16\int_{\mathbb{R}^d}\frac{(\lambda^2+\lambda|\xi|^2)}{|\xi|^2(2\lambda+|\xi|^2)^2}\hat{R}(\xi)d\xi,\tag{2.18}$$

we obtain the result as in the proof of Proposition 2.9. □

Now we are ready to prove the main theorems. We choose $\lambda=\varepsilon^2$ from now on.

By Itô's formula, the process of Brownian motion in random scenery can be decomposed as

$$X_\varepsilon(t)=\varepsilon\int_0^{t/\varepsilon^2}\mathbb{V}(\tau_{-B_s}\omega)ds=R_t^\varepsilon+M_t^\varepsilon,\tag{2.19}$$

where

$$R_t^\varepsilon : \;=\; \varepsilon \int_0^{t/\varepsilon^2} \lambda\Phi_\lambda(y_s)ds - \varepsilon\Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon\Phi_\lambda(y_0), \tag{2.20}$$

$$M_t^\varepsilon : \;=\; \varepsilon \int_0^{t/\varepsilon^2} \sum_{k=1}^d D_k\Phi_\lambda(y_s)dB_s^k. \tag{2.21}$$

Therefore, the error is decomposed correspondingly as $u_\varepsilon(t,x) - u_{hom}(t,x) = (i) + (ii)$, where

$$(i) \;=\; \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iR_t^\varepsilon + iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\}, \tag{2.22}$$

$$(ii) \;=\; \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\}. \tag{2.23}$$

We see (i) is caused by the residue R_t^ε, i.e., a measure of how close $X_\varepsilon(t)$ is to a martingale, while (ii) relates to convergence of martingale M_t^ε, i.e., a measure of how close the martingale is to a Brownian motion. Since f is bounded, we have the estimate $\mathbb{E}\{|(i)|\} \lesssim \mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|\}$. It is straightforward to check that

$$\mathbb{E}\{|(i)|\} \lesssim \mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|\} \lesssim \sqrt{\lambda\langle\Phi_\lambda, \Phi_\lambda\rangle}(1+t). \tag{2.24}$$

2.1.1 Convergence to a deterministic equation: ergodicity

For the martingale part, we rewrite

$$M_t^\varepsilon = \varepsilon \int_0^{t/\varepsilon^2} \sum_{k=1}^d (D_k\Phi_\lambda - \eta_k)(y_s)dB_s^k + \varepsilon \int_0^{t/\varepsilon^2} \sum_{k=1}^d \eta_k(y_s)dB_s^k := (I) + (II),$$

so

$$\begin{aligned}|(ii)| =&|\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i(II))\}| \\ &+|\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i(II))\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\}| \\ \lesssim&\mathbb{E}_B\{|(I)|\} + \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i(II))\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\}.\end{aligned} \tag{2.25}$$

We clearly have that

$$\mathbb{E}\mathbb{E}_B\{|(I)|\} \leq \sqrt{t\sum_{k=1}^d \|D_k\Phi_\lambda - \eta_k\|^2} \tag{2.26}$$

On the other hand, by the martingale central limit theorem [10], ergodicity of $\tau_{-B_s}\omega$, the fact that $\mathbb{E}\{\eta_k\} = 0$ for $k = 1, \ldots, d$, and $\sum_{k=1}^d \|\eta_k\|^2 = \sigma^2$, we have that for almost every $\omega \in \Omega$:

$$(\varepsilon B_{t/\varepsilon^2}, \varepsilon\int_0^{t/\varepsilon^2}\sum_{k=1}^d \eta_k(\tau_{-B_s}\omega)dB_s^k) \Rightarrow (W_t^1, \sigma W_t^2), \tag{2.27}$$

where W_t^1 is a $d-$dimensional Brownian motion and W_t^2 is an independent $1-$dimensional Brownian motion. While the above weak convergence is of process-level, we only need the result for fixed $t > 0$. Therefore,

$$\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i(II))\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\} \to 0 \tag{2.28}$$

as $\varepsilon \to 0$ almost surely.

To summarize, we have

$$\mathbb{E}\{|u_\varepsilon(t,x)-u_0(t,x)|\}\lesssim\sqrt{\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle}(1+t)+\sqrt{t\sum_{k=1}^d\|D_k\Phi_\lambda-\eta_k\|^2}$$
$$+\mathbb{E}\{|\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i(II))\}-\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\}|\}. \tag{2.29}$$

By Proposition 2.9 and 2.10, and the dominated convergence theorem, the proof of Theorem 2.3 is complete.

Remark 2.11. From the above discussion, the Brownian motion in random scenery $X_\varepsilon(t)=\varepsilon\int_0^{t/\varepsilon^2}V(B_s,\omega)ds$ satisfies an invariance principle. The above proof only indicates a convergence for fixed $t>0$, while by the proof in [21], it is the weak convergence in measure in the space of continuous functions. We explain it as follows. Taking the space $\mathcal{C}([0,T])$ for $T>0$, for each fixed ω, $X_\varepsilon(t)$ induces a probability measure on $\mathcal{C}([0,T])$, denoted by π_ε^ω, a random probability measure. [21] proves that $\pi_\varepsilon^\omega\Rightarrow\pi$ in probability for the Wiener measure π on $\mathcal{C}([0,T])$, i.e., $\mathbb{P}(\omega:d(\pi_\varepsilon^\omega,\pi)>\delta)\to 0$ as $\varepsilon\to 0$, where $d(.,.)$ is some distance in the space of probability measures on $\mathcal{C}([0,T])$ to induce the weak convergence. This type of convergence is equivalent with the fact that for any bounded continuous functional f on $\mathcal{C}([0,T])$, $\mathbb{E}_{\pi_\varepsilon^\omega}\{f(.)\}\to\mathbb{E}_\pi\{f(.)\}$ in $\mathbb{P}-$probability. We can view it as a law of large numbers type result since the limit π is a *deterministic* probability measure.

2.1.2 Error estimates: mixing property

Let $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}dx$, we can write (ii) as

$$\begin{aligned}(ii) =&\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\} \\ =&\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i(\varepsilon\xi\cdot B_{t/\varepsilon^2}+M_t^\varepsilon)} - e^{i\varepsilon\xi\cdot B_{t/\varepsilon^2}-\frac{1}{2}\sigma^2 t}\}d\xi,\end{aligned} \tag{2.30}$$

where $\varepsilon\xi\cdot B_{t/\varepsilon^2} + M_t^\varepsilon = \varepsilon\int_0^{t/\varepsilon^2}\sum_{k=1}^d(\xi_k + D_k\Phi_\lambda(y_s))dB_s^k$ is a continuous, square-integrable martingale for almost every $\omega\in\Omega$. The estimation of $\mathbb{E}_B\{e^{i(\varepsilon\xi\cdot B_{t/\varepsilon^2}+M_t^\varepsilon)} - e^{i\varepsilon\xi\cdot B_{t/\varepsilon^2}-\frac{1}{2}\sigma^2 t}\}$ reduces to a control of the Wasserstein distance between $\varepsilon\xi\cdot B_{t/\varepsilon^2}+M_t^\varepsilon$ and $\varepsilon\xi\cdot B_{t/\varepsilon^2}+\sigma W_t$, where W_t is an independent Brownian motion from B_t. This is given by the following result.

Proposition 2.12 (Theorem 3.2, equation (3.3) [26])**.** *If M_t is a continuous martingale and W_t is a standard Brownian motion, then*

$$d_{1,k}(M_1, W_1) \le (1\vee k)\mathbb{E}\{|\langle M\rangle_1 - 1|\}, \tag{2.31}$$

with the distance $d_{1,k}$ defined as

$$d_{1,k}(X,Y) = \sup\{|\mathbb{E}\{f(X)-f(Y)\}| : f\in C_b^2(\mathbb{R}), \|f'\|_\infty \le 1, \|f''\|_\infty \le k\}. \tag{2.32}$$

Since $\sigma_\lambda^2 = \sum_{k=1}^d\langle D_k\Phi_\lambda, D_k\Phi_\lambda\rangle$, by Proposition 2.12 we have for almost every $\omega\in\Omega$:

$$\begin{aligned}&|\mathbb{E}_B\{e^{i(\varepsilon\xi\cdot B_{t/\varepsilon^2}+M_t^\varepsilon)}\} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t}| \\ \le&\left(1\vee\frac{1}{\sqrt{(|\xi|^2+\sigma_\lambda^2)t}}\right)\mathbb{E}_B\{|\varepsilon^2\int_0^{t/\varepsilon^2}\sum_{k=1}^d(\xi_k+D_k\Phi_\lambda(y_s))^2ds - (|\xi|^2+\sigma_\lambda^2)t|\}.\end{aligned} \tag{2.33}$$

Now we can write $|(ii)| \le (I) + (II)$, where

$$(I) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)| \left(1 \vee \frac{1}{\sqrt{(|\xi|^2+\sigma_\lambda^2)t}}\right) \mathbb{E}_B\{|\varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d (\xi_k + D_k\Phi_\lambda(y_s))^2 ds - (|\xi|^2+\sigma_\lambda^2)t|\} d\xi,$$
$$(II) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)| |e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t} - e^{-\frac{1}{2}(|\xi|^2+\sigma^2)t}| d\xi.$$

First, we have

$$(II) \lesssim |\sigma_\lambda^2 - \sigma^2| t. \tag{2.34}$$

Secondly, for (I) we have

$$\begin{aligned}(I) =& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)| \left(1 \vee \frac{1}{\sqrt{(|\xi|^2+\sigma_\lambda^2)t}}\right) \mathbb{E}_B\{|\varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d (\xi_k + D_k\Phi_\lambda(y_s))^2 ds - (|\xi|^2+\sigma_\lambda^2)t|\} d\xi \\ =& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)| \left(1 \vee \frac{1}{\sqrt{(|\xi|^2+\sigma_\lambda^2)t}}\right) \mathbb{E}_B\{|\varepsilon^2 \int_0^{t/\varepsilon^2} Z_{\lambda,\xi}(B_s) ds|\} d\xi,\end{aligned} \tag{2.35}$$

where

$$Z_{\lambda,\xi}(x) := \sum_{k=1}^d (\xi_k + \int_{\mathbb{R}^d} \partial_{x_k} G_\lambda(x-y)V(y)dy)^2 - |\xi|^2 - \sigma_\lambda^2,$$

with G_λ the Green's function of $\lambda - \frac{1}{2}\Delta$. Note that we have used the fact that

$$D_k\Phi_\lambda(\tau_{-x}\omega) = \int_{\mathbb{R}^d} \partial_{x_k} G_\lambda(x-y)V(y)dy.$$

Clearly $Z_{\lambda,\xi}$ has zero mean; and by the ergodic theorem, we expect $\varepsilon^2 \int_0^{t/\varepsilon^2} Z_{\lambda,\xi}(B_s)ds$ to be small. This is quantified by the following control of the variance of Brownian motion in random scenery.

Lemma 2.13. *If V is a mean zero, stationary random field with covariance function $R(x)$, and B_s is Brownian motion independent from V, then*

$$\mathbb{E}\mathbb{E}_B\{\left(\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds\right)^2\} \lesssim t\int_{\mathbb{R}^d}\frac{|R(x)|}{|x|^{d-2}}dx. \tag{2.36}$$

Proof. By direct calculation, we have

$$\begin{aligned}
\mathbb{E}\mathbb{E}_B\{\left(\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds\right)^2\} =&2\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^s\int_{\mathbb{R}^d}R(x)\frac{1}{(2\pi u)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2u}}dxduds\\
=&2\varepsilon^2\int_0^\infty du(\frac{t}{\varepsilon^2}-u)1_{u<\frac{t}{\varepsilon^2}}\int_{\mathbb{R}^d}R(x)\frac{1}{(2\pi u)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2u}}dx\\
=&\varepsilon^2\int_0^\infty d\lambda(\frac{t}{\varepsilon^2}-\frac{|x|^2}{2\lambda})1_{\frac{|x|^2}{2\lambda}<\frac{t}{\varepsilon^2}}\lambda^{\frac{d}{2}-2}e^{-\lambda}\int_{\mathbb{R}^d}\frac{1}{\pi^{\frac{d}{2}}}R(x)\frac{1}{|x|^{d-2}}dx\\
\lesssim& t\int_{\mathbb{R}^d}\frac{|R(x)|}{|x|^{d-2}}dx.
\end{aligned} \tag{2.37}$$

□

Now we write $Z_{\lambda,\xi}(x) = Z_{1,\lambda,\xi}(x) + Z_{2,\lambda,\xi}(x)$ with

$$Z_{1,\lambda,\xi}(x) = \sum_{k=1}^d\left(\int_{\mathbb{R}^d}\partial_{x_k}G_\lambda(x-y)V(y)dy\right)^2 - \sigma_\lambda^2, \tag{2.38}$$

$$Z_{2,\lambda,\xi}(x) = 2\sum_{k=1}^d\xi_k\int_{\mathbb{R}^d}\partial_{x_k}G_\lambda(x-y)V(y)dy. \tag{2.39}$$

Since $\sigma_\lambda^2 = \sum_{k=1}^d\langle D_k\Phi_\lambda, D_k\Phi_\lambda\rangle$, we have $\mathbb{E}\{Z_{i,\lambda,\xi}(x)\} = 0, i = 1, 2$. Therefore, Lemma 2.13

implies

$$\mathbb{E}\{(I)\} \lesssim \frac{\varepsilon}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)| \left(1 \vee \frac{1}{\sqrt{(|\xi|^2+\sigma_\lambda^2)t}}\right) \sqrt{t \int_{\mathbb{R}^d} \frac{|\mathcal{R}_{1,\lambda,\xi}(x)|+|\mathcal{R}_{2,\lambda,\xi}(x)|}{|x|^{d-2}} dx d\xi} \quad (2.40)$$

where $\mathcal{R}_{i,\lambda,\xi}(x) := \mathbb{E}\{Z_{i,\lambda,\xi}(0)Z_{i,\lambda,\xi}(x)\}, i=1,2$. By recalling (2.24) and (2.34), we have

$$\begin{aligned} &\mathbb{E}\{|u_\varepsilon(t,x)-u_{hom}(t,x)|\} \\ \lesssim &\left(\sqrt{\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle}+|\sigma_\lambda^2-\sigma^2|+\varepsilon\int_{\mathbb{R}^d}\hat{f}(\xi)\sqrt{\int_{\mathbb{R}^d}\frac{|\mathcal{R}_{1,\lambda,\xi}(x)|+|\mathcal{R}_{2,\lambda,\xi}(x)|}{|x|^{d-2}}dx d\xi}\right)(1+t). \end{aligned} \quad (2.41)$$

Proposition 2.14. *Under Assumption 2.4, there exist a constant $c>0$ and a sufficiently large $\beta>0$ such that*

$$|\mathcal{R}_{1,\lambda,\xi}(x)|+|\mathcal{R}_{2,\lambda,\xi}(x)| \lesssim (1+|\xi|)^2\left(\lambda^{\frac{d}{2}-1}e^{-c\sqrt{\lambda}|x|}+1\wedge\frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}}+1\wedge\frac{1}{|x|^\beta}\right). \quad (2.42)$$

Proof. We assume a positive function $\Psi(x) \lesssim 1\wedge|x|^{-\beta}$ for $\beta>0$ sufficiently large.

We first consider $\mathcal{R}_{1,\lambda,\xi}(x)$. By denoting $\phi_\lambda(x)=\int_{\mathbb{R}^d}G_\lambda(x-y)V(y)dy$, for any $m,n=1,\dots,d$, we have

$$\begin{aligned} &\mathbb{E}\{(\partial_{x_m}\phi_\lambda(0))^2(\partial_{x_n}\phi_\lambda(x))^2\} \\ =&\int_{\mathbb{R}^{4d}}\partial_{x_m}G_\lambda(y_1)\partial_{x_m}G_\lambda(z_1)\partial_{x_n}G_\lambda(y_2)\partial_{x_n}G_\lambda(z_2)\mathbb{E}\{V(-y_1)V(-z_1)V(x-y_2)V(x-z_2)\}dy_1dy_2dz_1dz_2 \\ =&\int_{\mathbb{R}^{4d}}\partial_{x_m}G_\lambda(y_1)\partial_{x_m}G_\lambda(z_1)\partial_{x_n}G_\lambda(y_2)\partial_{x_n}G_\lambda(z_2)R(y_1-z_1)R(y_2-z_2)dy_1dy_2dz_1dz_2+I_{mn} \\ =&\|D_m\Phi_\lambda\|^2\|D_n\Phi_\lambda\|^2+I_{mn}, \end{aligned} \quad (2.43)$$

where I_{mn} are remainders in the calculation of fourth moment. By Lemma 2.31, we obtain

$$|I_{mn}| \leq 2 \int_{\mathbb{R}^{4d}} |\partial_m G_\lambda(y_1)\partial_m G_\lambda(z_1)\partial_n G_\lambda(y_2)\partial_n G_\lambda(z_2)|\Psi(x-y_1+y_2)\Psi(x-z_1+z_2)dy_1dy_2dz_1dz_2. \tag{2.44}$$

Since G_λ is the Green's function of $\lambda - \frac{1}{2}\Delta$, by scaling property, $G_\lambda(x) = \lambda^{\frac{d}{2}-1}G_1(\sqrt{\lambda}x)$. The estimate $|\nabla G_1(x)| \lesssim e^{-\rho|x|}|x|^{1-d}$ holds for some $\rho > 0$ [34]. Therefore, by change of variables, we have

$$|I_{mn}| \lesssim \left(\frac{1}{\lambda}\int_{\mathbb{R}^{2d}} \frac{e^{-\rho|y|}}{|y|^{d-1}}\frac{e^{-\rho|z|}}{|z|^{d-1}}\Psi(x - \frac{y-z}{\sqrt{\lambda}})dydz\right)^2. \tag{2.45}$$

Since $\sigma_\lambda^4 = \sum_{m,n=1}^d \|D_m\Phi_\lambda\|^2\|D_n\Phi_\lambda\|^2$, we derive the following estimate

$$|\mathcal{R}_{1,\lambda,\xi}(x)| \lesssim \left(\frac{1}{\lambda}\int_{\mathbb{R}^{2d}} \frac{e^{-\rho|y|}}{|y|^{d-1}}\frac{e^{-\rho|z|}}{|z|^{d-1}}\Psi(x - \frac{y-z}{\sqrt{\lambda}})dydz\right)^2. \tag{2.46}$$

Now we consider $\mathcal{R}_{2,\lambda,\xi}(x)$. Similary, we obtain that

$$\begin{aligned} |\mathcal{R}_{2,\lambda,\xi}(x)| =& |4\sum_{m,n=1}^d \xi_m\xi_n \int_{\mathbb{R}^{2d}} \partial_m G_\lambda(y)\partial_n G_\lambda(z)R(x-y+z)dydz| \\ \lesssim& |\xi|^2\frac{1}{\lambda}\int_{\mathbb{R}^{2d}} \frac{e^{-\rho|y|}}{|y|^{d-1}}\frac{e^{-\rho|z|}}{|z|^{d-1}}|R|(x - \frac{y-z}{\sqrt{\lambda}})dydz. \end{aligned} \tag{2.47}$$

Since $\Psi(x) \lesssim 1 \wedge |x|^{-\beta}$ for $\beta > 0$ sufficiently large, by Lemma 2.33, we obtain

$$|\mathcal{R}_{1,\lambda,\xi}(x)| + |\mathcal{R}_{2,\lambda,\xi}(x)| \lesssim (1+|\xi|)^2\left(\lambda^{\frac{d}{2}-1}e^{-c\sqrt{\lambda}|x|} + 1 \wedge \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}} + 1 \wedge \frac{1}{|x|^\beta}\right) \tag{2.48}$$

for some constant $c > 0$, and $\beta > 0$ sufficiently large. The proof is complete. □

By defining $F_\lambda(x) := \lambda^{\frac{d}{2}-1}e^{-c\sqrt{\lambda}|x|} + 1 \wedge \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}} + 1 \wedge \frac{1}{|x|^\beta}$, Proposition 2.14 and (2.41) lead to

$$\begin{aligned} &\mathbb{E}\{|u_\varepsilon(t,x) - u_0(t,x)|\} \\ \lesssim &\left(\sqrt{\lambda\langle\Phi_\lambda, \Phi_\lambda\rangle} + |\sigma_\lambda^2 - \sigma^2| + \varepsilon\sqrt{\int_{\mathbb{R}^d} \frac{F_\lambda(x)}{|x|^{d-2}}dx}\right)(1+t). \end{aligned} \tag{2.49}$$

We also see that for the initial condition f, the only requirement is $|\hat{f}(\xi)|(1+|\xi|)$ being integrable.

Now we only need the following lemma to complete the proof.

Lemma 2.15.

$$\int_{\mathbb{R}^d} \frac{F_\lambda(x)}{|x|^{d-2}}dx \lesssim \begin{cases} \lambda^{-\frac{1}{2}} & d=3, \\ |\log\lambda| & d=4, \\ 1 & d>4. \end{cases} \tag{2.50}$$

Proof. Note that $1 \wedge \frac{1}{|x|^\beta}$ gives a term of order 1 since β could be sufficiently large. We first look at

$$\int_{\mathbb{R}^d} \frac{1}{|x|^{d-2}}\lambda^{\frac{d}{2}-1}e^{-c\sqrt{\lambda}|x|}dx = \lambda^{\frac{d}{2}-2}\int_{\mathbb{R}^d} \frac{e^{-c|y|}}{|y|^{d-2}}dy \lesssim \lambda^{\frac{d}{2}-2}. \tag{2.51}$$

Now we only have to deal with $1 \wedge \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}}$.

$$\int_{\mathbb{R}^d} \frac{1}{|x|^{d-2}} 1 \wedge \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}}dx \le \int_{|x|<1} \frac{1}{|x|^{d-2}}dx + \int_{|x|>1} \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{2d-4}}dx. \tag{2.52}$$

When $d > 4$, RHS is bounded. When $d \le 4$,

$$\int_{|x|>1} \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{2d-4}}dx = \lambda^{\frac{d-4}{2}} \int_{\sqrt{\lambda}}^\infty \frac{e^{-cr}}{r^{d-3}}dr, \tag{2.53}$$

which concludes the proof. □

By Proposition 2.9 and 2.10 and the fact that $\lambda = \varepsilon^2$, we have under Assumption 2.4

$$\sqrt{\lambda\langle\Phi_\lambda, \Phi_\lambda\rangle} + |\sigma_\lambda^2 - \sigma^2| \lesssim \begin{cases} \sqrt{\varepsilon} & d = 3, \\ \varepsilon\sqrt{|\log\varepsilon|} & d = 4, \\ \varepsilon & d > 4. \end{cases} \tag{2.54}$$

By combining Lemma 2.15 and (2.49), the proof of Theorem 2.6 is complete.

Remark 2.16. The strongly mixing property is used in the calculation of fourth order moments in the proof of Proposition 2.14. When the fourth order moments can be calculated directly, e.g. in the Gaussian case, error estimates can be derived using the same approach for long-range-correlated random fields, see [14, Theorem 2.9].

2.2 A central limit theorem when V is Gaussian or Poissonian and $d = 3$

In this section, we will go beyond the error estimate in Theorem 2.6 and prove a central limit result. The error $u_\varepsilon - u_{hom}$ can be decomposed into two parts, the bias $\mathbb{E}\{u_\varepsilon\} - u_{hom}$ and the random fluctuation $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$. It turns out that when $d \geq 4$, the deterministic bias dominates, and to derive central limit results, we need to consider $(u_\varepsilon - \mathbb{E}\{u_\varepsilon\})/\varepsilon^\gamma$ for appropriate $\gamma > 0$. This is not the subject here. Instead, we focus on the case $d = 3$ and analyze the asymptotic distribution of $(u_\varepsilon - u_{hom})/\varepsilon^{\frac{1}{2}}$. We still work in the framework of Kipnis-Varadhan, except that more accurate estimations of errors are derived. We first make assumptions about the random fields then present the main theorem.

Assumption 2.17. V is assumed to be Gaussian or Poissonian, and

- when V is Gaussian, for any $\alpha > 0$, there exists $C_\alpha > 0$ such that the covariance function satisfies $|R(x)| \leq C_\alpha(1 \wedge |x|^{-\alpha})$.

- when V is Poissonian, $V(x) = \int_{\mathbb{R}^d} \varphi(x-y)\omega(dy) - c_\varphi$, where the shape function φ is continuous, compactly supported and satisfies $\int_{\mathbb{R}^d} \varphi(x)dx = c_\varphi$, and $\omega(dy)$ is the Poissonian point process with Lebesgue measure dy as its intensity. Then $R(x) = \int_{\mathbb{R}^d} \varphi(x+y)\varphi(y)dy$ is compactly supported, and $\hat{R}(\xi) = |\hat{\varphi}(\xi)|^2$.

For both potentials, we further assume $\hat{R}(0) > 0$. In particular it implies $c_\varphi \neq 0$ in the Poissonian case since there we have $\hat{R}(0) = c_\varphi^2$.

Theorem 2.18. *Let $d = 3$, under Assumption 2.17, we have*

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \Rightarrow v(t,x) \tag{2.55}$$

weakly with $v(t,x)$ solving the following SPDE with additive white noise and zero initial condition:

$$\partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) - \frac{1}{2}\sigma^2 v(t,x) + i\sqrt{\hat{R}(0)}u_{hom}(t,x)\dot{W}(x). \tag{2.56}$$

The weak convergence is in the following sense:

1. *The distribution of $\varepsilon^{-\frac{1}{2}}(u_\varepsilon(t,x) - u_{hom}(t,x)) \Rightarrow v(t,x)$ weakly for fixed (t,x).*

2. *The distribution of $\varepsilon^{-\frac{1}{2}} \int_{\mathbb{R}^d}(u_\varepsilon(t,x) - u_{hom}(t,x))g(x)dx \Rightarrow \int_{\mathbb{R}^d} v(t,x)g(x)dx$ weakly for any fixed t and test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.*

Remark 2.19. Since $\hat{R}(0) > 0$, it is clear that $v(t,x)$ is a Gaussian random field, so Theorem 2.18 is indeed a central limit type of result. In (2.56), the homogenization constant $\frac{\sigma^2}{2}$ appears as a potential term, which comes from the average of $\frac{i}{\varepsilon}V(\frac{x}{\varepsilon})$. The spatial white noise $\dot{W}(x)$ also results from the highly oscillatory potential. Therefore, one main difficulty in the proof is to separate the effects of $\frac{1}{\varepsilon}V(\frac{x}{\varepsilon})$ on those two parts and prove some type of asymptotic independence. This is formulated as Proposition 2.23 below, and the proof forces us to assume a Gaussian or Poissonian structure of V in order to carry out some explicit calculations.

Recall the error decomposition $u_\varepsilon(t,x) - u_{hom}(t,x) = (i) + (ii)$, where

$$(i) \quad = \quad \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iR_t^\varepsilon + iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\}, \quad (2.57)$$

$$(ii) \quad = \quad \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\}. \quad (2.58)$$

On the one hand, by Proposition 2.9, $(i) \sim R_t^\varepsilon \sim O(\varepsilon^{\frac{1}{2}})$. On the other hand, by Proposition 2.12 and the proof of Theorem 2.6, $(ii) \sim O(\varepsilon^{\frac{1}{2}})$. Therefore, to analyze $(u_\varepsilon - u_0)/\varepsilon^{\frac{1}{2}}$, we need to refine the error and separate those main contributions.

2.2.1 Refining the corrector

Recall that the solution to (2.3) is written as

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon\int_0^{t/\varepsilon^2} V(\frac{x}{\varepsilon}+B_s)ds)\}. \quad (2.59)$$

Define $y_s := \tau_{-\frac{x}{\varepsilon}-B_s}\omega$ as the environmental process taking values in Ω. Let Φ_λ solve the corrector equation

$$(\lambda - L)\Phi_\lambda = \mathbb{V} \quad (2.60)$$

with $\lambda = \varepsilon^2$ and $L = \frac{1}{2}\sum_{k=1}^d D_k^2$. Then by Itô's formula, the process $X_t^\varepsilon := \varepsilon \int_0^{t/\varepsilon^2} \mathbb{V}(y_s)ds$ can be decomposed as $X_t^\varepsilon = R_t^\varepsilon + M_t^\varepsilon$ with

$$R_t^\varepsilon := \varepsilon \int_0^{t/\varepsilon^2} \lambda\Phi_\lambda(y_s)ds - \varepsilon\Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon\Phi_\lambda(y_0), \tag{2.61}$$

$$M_t^\varepsilon := \varepsilon \int_0^{t/\varepsilon^2} \sum_{k=1}^d D_k\Phi_\lambda(y_s)dB_s^k. \tag{2.62}$$

Remark 2.20. In Theorem 2.18, we have a process-level weak convergence, so in the definition of y_s here we keep the factor of x/ε without using the stationarity of V.

By Proposition 2.9, we have

$$\mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|^2\} \lesssim \lambda\langle\Phi_\lambda, \Phi_\lambda\rangle \lesssim \sqrt{\lambda}. \tag{2.63}$$

Recall that $\sigma_\lambda^2 = \sum_{k=1}^d \int_\Omega (D_k\Phi_\lambda)^2 \mathbb{P}(d\omega)$, then by Proposition 2.10,

$$|\sigma_\lambda^2 - \sigma^2| \lesssim \sqrt{\lambda}. \tag{2.64}$$

The error is then decomposed into three parts, $u_\varepsilon(t,x) - u_{hom}(t,x) = (i) + (ii) + (iii)$ with

$$(i) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iX_t^\varepsilon}\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iM_t^\varepsilon}\}, \tag{2.65}$$

$$(ii) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iM_t^\varepsilon}\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma_\lambda^2 t}\}, \tag{2.66}$$

$$(iii) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma_\lambda^2 t}\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma^2 t}\}, \tag{2.67}$$

and we have

$$\mathbb{E}\{|(i)|\} \lesssim \mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|\} \lesssim \varepsilon^{\frac{1}{2}}, \tag{2.68}$$

$$|(iii)| \lesssim |\sigma_\lambda^2 - \sigma^2| \lesssim \varepsilon \ll \varepsilon^{\frac{1}{2}}. \tag{2.69}$$

(ii) is analyzed through the quantitative martingale central limit theorem. In Fourier domain, it is written as

$$(ii) = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{i\xi\cdot x} \mathbb{E}_B\{e^{i\varepsilon\xi\cdot B_{t/\varepsilon^2} + iM_t^\varepsilon} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t}\} d\xi. \tag{2.70}$$

Define $\tilde{M}_t^\varepsilon := \varepsilon\xi \cdot B_{t/\varepsilon^2} + M_t^\varepsilon$, then Proposition 2.12 implies

$$\mathbb{E}\{|(ii)|\} \lesssim \int_{\mathbb{R}^d} |\hat{f}(\xi)| \mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2 + \sigma_\lambda^2)t|\} d\xi. \tag{2.71}$$

Since $\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t = \varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d \left(D_k\Phi_\lambda(y_s)^2 - \sigma_\lambda^2\right) ds + 2\varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k \Phi_\lambda(y_s) ds$, by a second moment estimate in Proposition 2.14 and Lemma 2.15, we have

$$\mathbb{E}\mathbb{E}_B\{|\varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d \left(D_k\Phi_\lambda(y_s)^2 - \sigma_\lambda^2\right) ds|^2\} \lesssim \varepsilon^2 |\log \varepsilon| \ll \varepsilon, \tag{2.72}$$

$$\mathbb{E}\mathbb{E}_B\{|2\varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k \Phi_\lambda(y_s) ds|^2\} \lesssim \varepsilon, \tag{2.73}$$

so

$$\mathbb{E}\mathbb{E}_B\{|\langle \tilde{M}^\varepsilon\rangle_t - (|\xi|^2 + \sigma_\lambda^2)t|^2\} \lesssim \varepsilon, \tag{2.74}$$

which implies $\mathbb{E}\{|(ii)|\} \lesssim \varepsilon^{\frac{1}{2}}$. Therefore, to analyze the asymptotic distribution of $\varepsilon^{-\frac{1}{2}}(u_\varepsilon(t,x) - u_0(t,x))$, we need to refine (i) and (ii) to separate those contributions of order $\varepsilon^{\frac{1}{2}}$.

Remark 2.21. When applying a refined quantitative martingale central limit theorem to analyze (ii), we will use (2.74) frequently.

For (i), using the fact that $|e^{ix} - 1 - ix| \lesssim |x|^2$, we have that

$$\mathbb{E}\{|\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{iX_t^\varepsilon}\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})(1+iR_t^\varepsilon)e^{iM_t^\varepsilon}\}|\} \lesssim \varepsilon. \tag{2.75}$$

so we have $\mathbb{E}\{|(i) - v_{1,\varepsilon}|\} \ll \varepsilon^{\frac{1}{2}}$ with

$$v_{1,\varepsilon} := \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})iR_t^\varepsilon e^{iM_t^\varepsilon}\}. \tag{2.76}$$

2.2.2 A refined quantitative martingale central limit theorem

In this section, we analyze (ii). By the expression in (2.70), the goal is reduced to an estimation of $\mathbb{E}_B\{e^{i\varepsilon\xi\cdot B_{t/\varepsilon^2}+iM_t^\varepsilon} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t}\}$ and separating its main contribution of order $\varepsilon^{\frac{1}{2}}$. The following is a simply modified quantitative martingale central limit theorem we need.

Proposition 2.22. *Assume M_t is a continuous martingale and W_t is a Brownian motion, then for any $f \in \mathcal{C}_b(\mathbb{R})$ with up to third order bounded and continuous derivatives, we have*

$$|\mathbb{E}\{f(M_1) - f(W_1) - \frac{1}{2}f''(M_\tau)(\langle M\rangle_1 - 1)\}| \le C\mathbb{E}\{|\langle M\rangle_1 - 1|^{\frac{3}{2}}\}, \tag{2.77}$$

where $\tau = \sup\{s \in [0,1] | \langle M\rangle_s \le 1\}$ and the constant C only depends on the bound of f'''.

Proof. Since M_t is continuous, the quadratic variation process $\langle M\rangle_t$ is continuous as well. It is clear that τ is a stopping time, and we construct $\tilde{M}_t$ on $[0,2]$ as

$$\tilde{M}_t = \begin{cases} M_t & t \in [0,\tau], \\ M_\tau & t \in (\tau, 1], \\ M_\tau + b_{t-1} & t \in (1, 2-\langle M\rangle_\tau], \\ M_\tau + b_{1-\langle M\rangle_\tau} & t \in (2-\langle M\rangle_\tau, 2], \end{cases} \tag{2.78}$$

where b is an independent Brownian motion.

Clearly $\tilde{M}_t$ is a continuous martingale and $\langle \tilde{M}\rangle_2 = 1$, so $\tilde{M}_2 \sim N(0,1)$, which implies $\mathbb{E}\{f(M_1) - f(W_1)\} = \mathbb{E}\{f(M_1) - f(\tilde{M}_2)\}$. We write

$$f(M_1) - f(\tilde{M}_2) = f(M_1) - f(M_\tau) - (f(\tilde{M}_2) - f(M_\tau)). \tag{2.79}$$

For the first term, we have

$$\begin{aligned} &|\mathbb{E}\{f(M_1) - f(M_\tau) - (M_1 - M_\tau)f'(M_\tau) - \frac{1}{2}(M_1 - M_\tau)^2 f''(M_\tau)\}| \\ =&|\mathbb{E}\{f(M_1) - f(M_\tau) - \frac{1}{2}(\langle M\rangle_1 - \langle M\rangle_\tau) f''(M_\tau)\}| \leq C\mathbb{E}\{|M_1 - M_\tau|^3\}. \end{aligned} \tag{2.80}$$

For the second term, we have $\tilde{M}_2 = M_\tau + b_{1-\langle M\rangle_\tau}$, so

$$\begin{aligned} &|\mathbb{E}\{f(\tilde{M}_2) - f(M_\tau) - b_{1-\langle M\rangle_\tau} f'(M_\tau) - \frac{1}{2} b^2_{1-\langle M\rangle_\tau} f''(M_\tau\}| \\ =&|\mathbb{E}\{f(\tilde{M}_2) - f(M_\tau) - \frac{1}{2}(1-\langle M\rangle_\tau) f''(M_\tau\}| \leq C\mathbb{E}\{|b_{1-\langle M\rangle_\tau}|^3\} \leq C\mathbb{E}\{(1-\langle M\rangle_\tau)^{\frac{3}{2}}\}. \end{aligned} \tag{2.81}$$

Note that $\mathbb{E}\{|M_1 - M_\tau|^3\} \leq C\mathbb{E}\{(\langle M\rangle_1 - \langle M\rangle_\tau)^{\frac{3}{2}}\} \leq C\mathbb{E}\{|\langle M\rangle_1 - 1|^{\frac{3}{2}}\}$ and the same bound holds for $\mathbb{E}\{(1-\langle M\rangle_\tau)^{\frac{3}{2}}\}$. The proof is complete. □

For almost every $\omega \in \Omega$, $\tilde{M}^\varepsilon_t$ is a continuous, square-integrable martingale, we apply

Proposition 2.22 with $f = e^{ix}$ and obtain for almost every ω that

$$
\begin{aligned}
&|\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t} + \frac{1}{2}e^{i\tilde{M}_\tau^\varepsilon}(\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t)\}| \\
\lesssim &\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|^{\frac{3}{2}}\}
\end{aligned}
\tag{2.82}
$$

where $\tau := \sup\{s \in [0,t]| : \varepsilon^2 \int_0^{s/\varepsilon^2} \sum_{k=1}^d (\xi_k + D_k\Phi_\lambda(y_s))^2 ds \le (|\xi|^2 + \sigma_\lambda^2)t\}$.

Therefore we have

$$
\begin{aligned}
&\mathbb{E}\{|(ii) - \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{-\frac{1}{2}e^{i\tilde{M}_\tau^\varepsilon}(\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t)\}d\xi|\} \\
\lesssim &\int_{\mathbb{R}^d} |\hat{f}(\xi)|\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|^{\frac{3}{2}}\}d\xi \\
\lesssim &\int_{\mathbb{R}^d} |\hat{f}(\xi)| \left(\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|^2\}\right)^{\frac{3}{4}} d\xi \lesssim \varepsilon^{\frac{3}{4}} \ll \varepsilon^{\frac{1}{2}}.
\end{aligned}
\tag{2.83}
$$

Next, we consider

$$
\begin{aligned}
&\mathbb{E}\{|\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{\frac{1}{2}\left(e^{i\tilde{M}_\tau^\varepsilon} - e^{i\tilde{M}_t^\varepsilon}\right)(\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t)\}d\xi|\} \\
\le &\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}|\hat{f}(\xi)|\frac{1}{2}\sqrt{\mathbb{E}\mathbb{E}_B\{|\tilde{M}_\tau^\varepsilon - \tilde{M}_t^\varepsilon|^2\}}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|^2\}}d\xi \\
\le &\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}|\hat{f}(\xi)|\frac{1}{2}\sqrt{\mathbb{E}\mathbb{E}_B\{\langle\tilde{M}^\varepsilon\rangle_t - \langle\tilde{M}^\varepsilon\rangle_\tau\}}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|^2\}}d\xi \\
\le &\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}|\hat{f}(\xi)|\frac{1}{2}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|\}}\sqrt{\mathbb{E}\mathbb{E}_B\{|\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t|^2\}}d\xi \\
\lesssim &\varepsilon^{\frac{3}{4}} \ll \varepsilon^{\frac{1}{2}}.
\end{aligned}
\tag{2.84}
$$

On the other hand,

$$
\langle\tilde{M}^\varepsilon\rangle_t - (|\xi|^2+\sigma_\lambda^2)t = 2\varepsilon^2\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds + \varepsilon^2\int_0^{t/\varepsilon^2}(\sum_{k=1}^d D_k\Phi_\lambda(y_s)^2 - \sigma_\lambda^2)ds, \tag{2.85}
$$

together with the fact that

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|\mathbb{E}\mathbb{E}_B\{|\varepsilon^2 \int_0^{t/\varepsilon^2} (\sum_{k=1}^d D_k\Phi_\lambda(y_s)^2 - \sigma_\lambda^2)ds|\}d\xi \lesssim \varepsilon|\log\varepsilon|^{\frac{1}{2}} \ll \varepsilon^{\frac{1}{2}}, \tag{2.86}$$

we have obtained $\mathbb{E}\{|(ii) - v_{2,\varepsilon}|\} \ll \varepsilon^{\frac{1}{2}}$, where

$$v_{2,\varepsilon} = -\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon}\varepsilon^2 \int_0^{t/\varepsilon^2} \sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds\}d\xi. \tag{2.87}$$

By writing $v_{1,\varepsilon}$ in Fourier domain as well, we have proved that

$$\begin{aligned}\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \approx & v_{1,\varepsilon} + v_{2,\varepsilon} \\ = & \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x}\varepsilon^{-\frac{1}{2}}\mathbb{E}_B\{e^{i\tilde{M}_t^\varepsilon}\left(iR_t^\varepsilon - \varepsilon^2\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds\right)\}d\xi,\end{aligned} \tag{2.88}$$

where " $\approx$ " means that as $\varepsilon \to 0$, the difference goes to zero in probability.

2.2.3 Proof of the CLT

Now we are ready to prove the main theorem. Recall that $\tilde{M}_t^\varepsilon = \varepsilon\xi \cdot B_{t/\varepsilon^2} + M_t^\varepsilon$, and $X_t^\varepsilon = R_t^\varepsilon + M_t^\varepsilon$, so

$$\tilde{M}_t^\varepsilon = \varepsilon\xi \cdot B_{t/\varepsilon^2} + X_t^\varepsilon - R_t^\varepsilon. \tag{2.89}$$

By (2.63) and (2.73), $\mathbb{E}\mathbb{E}_B\{(\varepsilon^{-\frac{1}{2}}R_t^\varepsilon)^2\}$ and $\mathbb{E}\mathbb{E}_B\{(\varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s)ds)^2\}$ are both bounded, thus since R_t^ε is small as in (2.63), we can replace $\tilde{M}_t^\varepsilon$ by $\varepsilon\xi \cdot B_{t/\varepsilon^2} + X_t^\varepsilon$ in (2.88) and obtain

$$\begin{aligned}&\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \\ \approx&\mathbb{E}_B\{\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}}e^{iX_t^\varepsilon}\left(\varepsilon^{-\frac{1}{2}}iR_t^\varepsilon - \varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds\right)d\xi\}.\end{aligned} \tag{2.90}$$

Let

$$Y_t^\varepsilon := i\varepsilon^{-\frac{1}{2}}\left(\varepsilon\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds - \varepsilon\Phi_\lambda(y_{t/\varepsilon^2}) + \varepsilon\Phi_\lambda(y_0)\right) - \varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds, \tag{2.91}$$

so $\mathbb{E}\mathbb{E}_B\{|Y_t^\varepsilon|^2\}$ is uniformly bounded, and we have

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \approx \mathbb{E}_B\{\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}}e^{iX_t^\varepsilon}Y_t^\varepsilon d\xi\}. \tag{2.92}$$

We show the interaction between X_t^ε and Y_t^ε goes to zero in the following sense:

Proposition 2.23.

$$\mathbb{E}_B\{\int_{\mathbb{R}^d}\hat{f}(\xi)e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}}(e^{iX_t^\varepsilon} - e^{-\frac{1}{2}\sigma^2 t})Y_t^\varepsilon d\xi\} \to 0 \tag{2.93}$$

in probability as $\varepsilon \to 0$.

By the above Proposition, the rescaled corrector can be written as

$$\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \approx \mathbb{E}_B\{\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x + i\varepsilon\xi\cdot B_{t/\varepsilon^2}}e^{-\frac{1}{2}\sigma^2 t}Y_t^\varepsilon d\xi\}. \tag{2.94}$$

Now we rewrite

$$
\begin{aligned}
&\mathbb{E}_B\{\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\hat{f}(\xi)e^{i\xi\cdot x+i\varepsilon\xi\cdot B_{t/\varepsilon^2}}e^{-\frac{1}{2}\sigma^2 t}\varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2}\sum_{k=1}^d \xi_k D_k\Phi_\lambda(y_s)ds\} \\
=&-i\sum_{k=1}^d \mathbb{E}_B\{\partial_{x_k} f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma^2 t}\varepsilon^{\frac{3}{2}}\int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s)ds\} \\
=&-i\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma^2 t}\varepsilon^{\frac{1}{2}}\sum_{k=1}^d\int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s)dB_s^k\},
\end{aligned}
\tag{2.95}
$$

where the last equality comes from a simple application of the duality relation in Malliavin calculus [27]. For the sake of convenience, we present some standard facts about Malliavin calculus in the appendix.

To summarize, we have

$$
\begin{aligned}
&\frac{u_\varepsilon(t,x)-u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} \\
\approx&\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma^2 t}i\varepsilon^{-\frac{1}{2}}\left(\varepsilon\int_0^{t/\varepsilon^2}\lambda\Phi_\lambda(y_s)ds-\varepsilon\Phi_\lambda(y_{t/\varepsilon^2})+\varepsilon\Phi_\lambda(y_0)\right)\} \\
+&\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma^2 t}i\varepsilon^{\frac{1}{2}}\sum_{k=1}^d\int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s)dB_s^k\} \\
=&\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})e^{-\frac{1}{2}\sigma^2 t}i\varepsilon^{-\frac{1}{2}}\varepsilon\int_0^{t/\varepsilon^2}\mathbb{V}(y_s)ds\} \\
=&\mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t}i\frac{1}{\varepsilon^{\frac{3}{2}}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds\},
\end{aligned}
\tag{2.96}
$$

which combines with the following Proposition to complete the proof of Theorem 2.18.

Proposition 2.24. $\mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t}i\varepsilon^{-\frac{3}{2}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds\} \Rightarrow v(t,x)$ *in the sense of Theorem 2.18, and* $v(t,x)$ *solves the following SPDE with additive white noise* $\dot{W}(x)$ *and zero* *initial condition:*

$$\partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) - \frac{1}{2}\sigma^2 v(t,x) + i\sqrt{\hat{R}(0)}u_{hom}(t,x)\dot{W}(x). \tag{2.97}$$

Remark 2.25. To combine (2.96) and Proposition 2.24 to prove Theorem 2.18, we need to note that the statistical error caused in (2.96) is $x-$independent, i.e.,

$$\mathbb{E}\{|\frac{u_\varepsilon(t,x) - u_{hom}(t,x)}{\varepsilon^{\frac{1}{2}}} - \mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t} i\frac{1}{\varepsilon^{\frac{3}{2}}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds\}|\} \le C_\varepsilon \tag{2.98}$$

for some $x-$independent constant $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

The proof of Proposition 2.23 is technical and presented in [13, Section 5]. We give a proof of Proposition 2.24.

First, we define $v_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t} i\varepsilon^{-\frac{3}{2}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds\}$ and prove the following lemma.

Lemma 2.26. $v_\varepsilon(t,x)$ *solves the following equation*

$$\partial_t v_\varepsilon(t,x) = \frac{1}{2}\Delta v_\varepsilon(t,x) - \frac{1}{2}\sigma^2 v_\varepsilon(t,x) + iu_{hom}(t,x)\frac{1}{\varepsilon^{\frac{3}{2}}}V(\frac{x}{\varepsilon}) \tag{2.99}$$

with zero initial condition.

Proof. By Feynman-Kac formula, we can write the solution to (2.99) as

$$v_\varepsilon(t,x) = \mathbb{E}_B\{\int_0^t e^{-\frac{1}{2}\sigma^2 s} iu_{hom}(t-s, x+B_s)\frac{1}{\varepsilon^{\frac{3}{2}}}V(\frac{x+B_s}{\varepsilon})ds\}. \tag{2.100}$$

Since u_{hom} solves the homogenized equation (2.4), $u_{hom}(t,x) = \mathbb{E}_W\{f(x+W_t)e^{-\frac{1}{2}\sigma^2 t}\}$, and we have

$$
\begin{aligned}
v_\varepsilon(t,x) =&\mathbb{E}_B\mathbb{E}_W\{\int_0^t e^{-\frac{1}{2}\sigma^2 s} i f(x+B_s+W_{t-s}) e^{-\frac{1}{2}\sigma^2(t-s)}\frac{1}{\varepsilon^{\frac{3}{2}}}V(\frac{x+B_s}{\varepsilon})ds\}\\
=&\mathbb{E}_B\mathbb{E}_W\{\int_0^t f(x+B_s+W_{t-s}) e^{-\frac{1}{2}\sigma^2 t} i\frac{1}{\varepsilon^{\frac{3}{2}}}V(\frac{x+B_s}{\varepsilon})ds\} \\
=&\mathbb{E}_B\{f(x+B_t)e^{-\frac{1}{2}\sigma^2 t} i\frac{1}{\varepsilon^{\frac{3}{2}}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds\}.
\end{aligned}
\tag{2.101}
$$

□

Since v_ε solves (2.99) with zero initial condition, the solution may be written as

$$
v_\varepsilon(t,x) = i\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)u_{hom}(s,y)\frac{1}{\varepsilon^{\frac{3}{2}}}V(\frac{y}{\varepsilon})dyds, \tag{2.102}
$$

where $\mathcal{G}_{t-s}(x-y) = e^{-\frac{1}{2}\sigma^2(t-s)}q_{t-s}(x-y)$.

We first show that for fixed (t,x), $v_\varepsilon(t,x)\Rightarrow v(t,x)$ in distribution.

The solution to the limiting SPDE (5.88) can be written as

$$
v(t,x) = i\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sqrt{\hat{R}(0)}u_{hom}(s,y)W(dy)ds, \tag{2.103}
$$

with $W(dy)$ the Wiener integral.

Let

$$
\begin{aligned}
var_\varepsilon : &= \int_0^t\int_0^t\int_{\mathbb{R}^{2d}}\mathcal{G}_{t-s}(x-y)\mathcal{G}_{t-u}(x-z)u_{hom}(s,y)u_{hom}(u,z)\frac{1}{\varepsilon^3}R(\frac{y-z}{\varepsilon})dydzdsdu,\\
var : &= \hat{R}(0)\int_0^t\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-z)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z)u_{hom}(u,z)dzdsdu.
\end{aligned}
$$

Lemma 2.27. *$var_\varepsilon\to var$.*

Proof. By change of variables, we have

$$var_\varepsilon = \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} \mathcal{G}_{t-s}(x-z-\varepsilon w)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z+\varepsilon w)u_{hom}(u,z)R(w)dwdzdsdu. \tag{2.104}$$

For fixed $s, u \in (0,t)$,

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \mathcal{G}_{t-s}(x-z-\varepsilon w)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z+\varepsilon w)u_{hom}(u,z)R(w)dwdz\\ \to&\hat{R}(0)\int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-z)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z)u_{hom}(u,z)dz\end{aligned} \tag{2.105}$$

by the dominated convergence theorem. Since u_0 is bounded, we have

$$|\int_{\mathbb{R}^{2d}} \mathcal{G}_{t-s}(x-z-\varepsilon w)\mathcal{G}_{t-u}(x-z)u_{hom}(s,z+\varepsilon w)u_{hom}(u,z)R(w)dwdz| \lesssim \frac{1}{(2t-s-u)^{\frac{d}{2}}}, \tag{2.106}$$

which is integrable in $[0,t]^2$ since $d=3$. Thus again by the dominated convergence theorem, the proof is complete. □

If V is Gaussian, then $v_\varepsilon(t,x)$ is Gaussian. Since both the mean and variance converge, we have $v_\varepsilon(t,x) \Rightarrow v(t,x)$ in distribution.

If V is Poissonian, i.e., $V(x) = \int_{\mathbb{R}^d} \varphi(x-y)\omega(dy) - c_\varphi$, then

$$\begin{aligned}v_\varepsilon(t,x) =& i\int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)u_{hom}(s,y)\frac{1}{\varepsilon^{\frac{3}{2}}}\varphi(\frac{y}{\varepsilon}-z)dyds\right)\omega(dz)\\ &-i\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)u_{hom}(s,y)\frac{1}{\varepsilon^{\frac{3}{2}}}c_\varphi dyds\end{aligned} \tag{2.107}$$

is Poissonian as well, and we have the following lemma.

Lemma 2.28.

$$\mathbb{E}\{\exp(\theta v_\varepsilon(t,x))\} \to \mathbb{E}\{\exp(\theta v(t,x))\} \tag{2.108}$$

as $\varepsilon \to 0$.

Proof. Let $f_\varepsilon(z) = \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_{hom}(s,y) \frac{1}{\varepsilon^{\frac{3}{2}}} \varphi(\frac{y}{\varepsilon} - z) dyds$, then

$$\begin{aligned}\mathbb{E}\{\exp(\theta v_\varepsilon(t,x))\} =& \exp(\int_{\mathbb{R}^d} (e^{i\theta f_\varepsilon(z)} - 1)dz) \exp(-i\theta \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_{hom}(s,y) \frac{1}{\varepsilon^{\frac{3}{2}}} c_\varphi dyds) \\ =& \exp(\int_{\mathbb{R}^d} \sum_{k=2}^{\infty} \frac{1}{k!} (i\theta)^k f_\varepsilon(z)^k dz),\end{aligned} \tag{2.109}$$

since $\int_{\mathbb{R}^d} \varphi(z)dz = c_\varphi$.

When $k = 2$, $\int_{\mathbb{R}^d} f_\varepsilon(z)^2 dz = var_\varepsilon$, so by Lemma 2.27, $\int_{\mathbb{R}^d} f_\varepsilon(z)^2 dz \to var$.

When $k \geq 3$, note that $\mathcal{G}_{t-s}(x-y) \leq q_{t-s}(x-y)$ and u_0 is bounded, so we have $|f_\varepsilon(z)| \lesssim \int_0^t \int_{\mathbb{R}^d} q_s(x-y) \frac{1}{\varepsilon^{\frac{3}{2}}} |\varphi|(\frac{y}{\varepsilon} - z) dyds$, which implies

$$\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \lesssim \frac{1}{\varepsilon^{\frac{3k}{2}}} \int_{\mathbb{R}^d} \int_{[0,t]^k} \int_{\mathbb{R}^{kd}} \prod_{i=1}^k q_{s_i}(x - y_i) |\varphi|(\frac{y_i}{\varepsilon} - z) dydsdz. \tag{2.110}$$

In the Fourier domain, by change of variables and integration in z, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \lesssim& \frac{1}{\varepsilon^{\frac{3k}{2}}} \int_{[0,t]^k} \int_{\mathbb{R}^{kd}} \prod_{i=1}^k |\mathcal{F}\{|\varphi|\}(\xi_i)| e^{-\frac{|\xi_i|^2}{2\varepsilon^2} s_i} \delta(\sum_{i=1}^k \xi_i = 0) d\xi ds \\ =& \frac{1}{\varepsilon^{\frac{3k}{2}}} \int_{[0,t]^k} \int_{\mathbb{R}^{(k-1)d}} |\mathcal{F}\{|\varphi|\}(-\sum_{i=2}^k \xi_i)| e^{-\frac{|\sum_{i=2}^k \xi_i|^2}{2\varepsilon^2} s_1} \prod_{i=2}^k |\mathcal{F}\{|\varphi|\}(\xi_i)| e^{-\frac{|\xi_i|^2}{2\varepsilon^2} s_i} d\xi ds.\end{aligned} \tag{2.111}$$

Changing variables $\xi_2 \to \varepsilon \xi_2$, $s_i \to \varepsilon^2 s_i, i \geq 3$, and since $|\mathcal{F}\{|\varphi|\}|$ is uniformly bounded, we have

$$\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \lesssim \varepsilon^{\frac{k}{2}-1} \int_{[0,t]^2} \int_{[0,t/\varepsilon^2]^{k-2}} \int_{\mathbb{R}^{(k-1)d}} e^{-\frac{1}{2}|\xi_2+\frac{1}{\varepsilon}\sum_{i=3}^k \xi_i|^2 s_1} e^{-\frac{1}{2}|\xi_2|^2 s_2} \prod_{i=3}^k |\mathcal{F}\{|\varphi|\}(\xi_i)| e^{-\frac{|\xi_i|^2}{2} s_i} d\xi ds. \tag{2.112}$$

Clearly $\int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi_2+\frac{1}{\varepsilon}\sum_{i=3}^k \xi_i|^2 s_1} e^{-\frac{1}{2}|\xi_2|^2 s_2} d\xi_2 \lesssim |s_1+s_2|^{-\frac{d}{2}}$, which is integrable in $[0,t]^2$ when $d=3$. Now we only have to integrate in $s_i, i \geq 3$ and use the fact that $\mathcal{F}\{|\varphi|\}(\xi)|\xi|^{-2}$ is integrable to conclude that $\int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \leq C^k \varepsilon^{\frac{k}{2}-1}$, so

$$\sum_{k\geq 3} \frac{1}{k!} |\theta|^k \int_{\mathbb{R}^d} |f_\varepsilon(z)|^k dz \to 0 \tag{2.113}$$

as $\varepsilon \to 0$. The proof is complete. □

We still need to show that for any test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} v_\varepsilon(t,x) g(x) dx \Rightarrow \int_{\mathbb{R}^d} v(t,x) g(x) dx \tag{2.114}$$

weakly, but the discussion is the same as in Lemma 2.27 and 2.28, thus the proof of Proposition 2.24 is complete.

2.3 Appendix

2.3.1 Technical lemmas

Lemma 2.29. *Consider the equation $\partial_t u = \frac{1}{2}\Delta u + iV(x)u$ with initial condition $u(0,x) = f(x) \in \mathcal{C}_b(\mathbb{R}^d)$. Let us define $u(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\int_0^t V(x+B_s)ds)\}$. If V has locally* *bounded sample path almost surely, we have for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \int_0^t \int_{\mathbb{R}^d} u(s,x)\frac{1}{2}\Delta\varphi(x)dxds + i\int_0^t \int_{\mathbb{R}^d} u(s,x)V(x)\varphi(x)dxds, \tag{2.115}$$

i.e., the Feynman-Kac solution $u(t,x)$ is a weak solution almost surely.

Proof. Fixing any $\delta, M > 0$, define

$$V_{\delta,M}(x) = \int_{\mathbb{R}^d} \phi_\delta(x-y)V(y)1_{|y|<M}dy, \tag{2.116}$$

where ϕ_δ is a family of compactly supported mollifier. Fixing the realization, since $V(y)1_{|y|<M}$ is bounded, $V_{\delta,M}$ is bounded, and we have $V_{\delta,M}(x) \to V(x)1_{|x|<M}$ almost everywhere as $\delta \to 0$. In addition, $V_{\delta,M}$ is smooth, so for the equation $\partial_t u_{\delta,M} = \frac{1}{2}\Delta u_{\delta,M} + iV_{\delta,M}u_{\delta,M}$ with initial condition $u_{\delta,M}(0,x) = f(x)$, we have its classical solution given by the Feynman-Kac formula

$$u_{\delta,M}(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\int_0^t V_{\delta,M}(x+B_s)ds)\}, \tag{2.117}$$

and if we first let $\delta \to 0$, then $M \to \infty$, $u_{\delta,M}(t,x) \to u(t,x)$ by the dominated convergence theorem. Since $u_{\delta,M}$ is also a weak solution, we have

$$\begin{aligned} \int_{\mathbb{R}^d} u_{\delta,M}(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \int_0^t \int_{\mathbb{R}^d} u_{\delta,M}(s,x)\frac{1}{2}\Delta\varphi(x)dxds \\ &\quad + i\int_0^t \int_{\mathbb{R}^d} u_{\delta,M}(s,x)V_{\delta,M}(x)\varphi(x)dxds. \end{aligned} \tag{2.118}$$

Let $\delta \to 0, M \to \infty$, we complete the proof. □

Lemma 2.30. *y_s^ω is a stationary, ergodic Markov process taking values in Ω, reversible with respect to the invariant measure $\mathbb{P}$.*

Proof. Since τ_x is a group of transformations, we have $\tau_{-B_t}\omega = \tau_{-(B_t-B_s)}\tau_{-B_s}\omega$, and by the independence of increments of Brownian motion, y_s^ω is Markov.

Now we show y_s^ω is reversible with respect to π by proving

$$\int_{A_1} \mathbb{P}(d\omega)\mathbb{P}_B(y_t^\omega \in A_2) = \int_{A_2} \mathbb{P}(d\omega)\mathbb{P}_B(y_t^\omega \in A_1) \tag{2.119}$$

for any $A_1, A_2 \in \mathcal{F}$. Actually, we have

$$\begin{aligned}
\int_{A_1} \mathbb{P}(d\omega)\mathbb{P}_B(y_t^\omega \in A_2) &= \int_\Omega \int_{\mathbb{R}^d} 1_{\omega\in A_1} 1_{\tau_{-x}\omega\in A_2} q_t(x) dx \mathbb{P}(d\omega), \\
\int_{A_2} \mathbb{P}(d\omega)\mathbb{P}_B(y_t^\omega \in A_1) &= \int_\Omega \int_{\mathbb{R}^d} 1_{\omega\in A_2} 1_{\tau_{-x}\omega\in A_1} q_t(x) dx \mathbb{P}(d\omega).
\end{aligned}$$

Using measure-preserving property of τ_x and the fact that $q_t(x) = q_t(-x)$, (2.119) is proved.

Since y_s^ω is reversible with respect to $\mathbb{P}$, $\mathbb{P}$ is an invariant measure. Furthermore, y_s^ω starts from its invariant measure, so it is stationary.

For ergodicity, we only need to show that if $A \in \mathcal{F}$ such that $\mathbb{P}_B(y_s^\omega \in A) = 1_{\omega\in A}$ for all $s \geq 0$, then $\mathbb{P}(A) = 0$ or 1. Actually, $\mathbb{P}_B(y_s^\omega \in A) = \int_{\mathbb{R}^d} 1_{\tau_{-x}\omega\in A} q_s(x) dx = 1_{\omega\in A}$ implies $1_{\tau_{-x}\omega\in A} = 1_{\omega\in A}$ for all $x \in \mathbb{R}^d$, since $q_s(x) > 0, \forall x \in \mathbb{R}^d$. By the ergodicity of τ_x, we have $\mathbb{P}(A) = 0$ or 1. □

Lemma 2.31. *Let $x_i \in \mathbb{R}^d, i = 1, \ldots, 4$, then under Assumption 2.4*

$$\begin{aligned}
&|\mathbb{E}\{V(x_1)V(x_2)V(x_3)V(x_4)\} - R(x_1 - x_2)R(x_3 - x_4)| \\
\leq &\Psi(|x_1 - x_3|)\Psi(|x_2 - x_4|) + \Psi(|x_1 - x_4|)\Psi(|x_2 - x_3|),
\end{aligned} \tag{2.120}$$

where $\Psi(r) \lesssim 1 \wedge r^{-\beta}$ for any $\beta > 0$.

Proof. The proof can be found in Lemma 2.3. [17], where $\mathbb{E}\{V^6(x)\} < \infty$ is used. □

Lemma 2.32.

$$\int_{\mathbb{R}^d} \frac{e^{-\rho|x-y|}}{|x-y|^{d-1}} \frac{e^{-\rho|y|}}{|y|^{d-1}} dy \lesssim e^{-\rho|x|}(1 + \frac{1}{|x|^{d-2}}). \tag{2.121}$$

Proof. See [5] Lemma A.1. □

The result in Lemma 2.33 is of convolution type. We prove it by the domain decomposition method. Here are some notations appearing in the proof. If we denote $B(z,r) = \{y : |y-z| \leq r\}$, then $\forall x \in \mathbb{R}^d$, let $\rho = |x| > 0$, $A_1 = \{z : |z| < |z-x|\}$, $A_2 = \{z : |z| \geq |z-x|\}$, and define $(I) = B(0,\rho) \cap A_1$, $(II) = B(x,\rho) \cap A_2$, $(III) = \mathbb{R}^d \setminus ((I) \cup (II))$.

$(I), (II), (III)$ appears in the proof of Lemma 2.33, and we will estimate the integral in each of them respectively. Ψ is assumed to be some positive function such that $\Psi(x) \lesssim 1 \wedge |x|^{-\alpha}$ for any $\alpha > 0$.

Lemma 2.33.

$$\frac{1}{\lambda}\int_{\mathbb{R}^{2d}} \frac{e^{-\rho|y|}}{|y|^{d-1}} \frac{e^{-\rho|z|}}{|z|^{d-1}} \Psi(x - \frac{y-z}{\sqrt{\lambda}}) dydz \lesssim \lambda^{\frac{d}{2}-1} e^{-c\sqrt{\lambda}|x|} + 1 \wedge \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}} + 1 \wedge \frac{1}{|x|^{\beta}} \tag{2.122}$$

for some $c > 0$ and sufficiently large $\beta > 0$.

Proof. By Lemma 2.32, we have

$$\frac{1}{\lambda}\int_{\mathbb{R}^2 d} \frac{e^{-\rho|y|}}{|y|^{d-1}} \frac{e^{-\rho|z|}}{|z|^{d-1}} \Psi(x - \frac{y-z}{\sqrt{\lambda}}) dydz \lesssim (i) + (ii), \tag{2.123}$$

where

$$(i) \quad = \quad \lambda^{\frac{d}{2}-1}\int_{\mathbb{R}^d} e^{-\rho\sqrt{\lambda}|y|}(1\wedge\frac{1}{|x-y|^{\alpha}})dy, \tag{2.124}$$

$$(ii) \quad = \quad \int_{\mathbb{R}^d} e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|y|^{d-2}}(1\wedge\frac{1}{|x-y|^{\alpha}})dy. \tag{2.125}$$

We have used $\Psi(x) \lesssim 1 \wedge \frac{1}{|x|^{\alpha}}$ for α sufficiently large. $(i), (ii)$ will be estimated separately but in the same way.

First of all, we clearly have that $(i) \lesssim \lambda^{\frac{d}{2}-1}$ and $(ii) \lesssim 1$. Now we assume $|x| \gg 1$ and divide $\mathbb{R}^d$ into three parts, $(I), (II), (III)$.

For (i), we have that when $|y-x| \le 1$, $\int_{|y-x|\le 1} e^{-\rho\sqrt{\lambda}|y|}dy \lesssim e^{-\rho\sqrt{\lambda}|x|}$. In region (I), we have $|y-x| \ge \frac{|x|}{2}$, so

$$\int_I e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|x-y|^{\alpha}}dy \lesssim \frac{1}{|x|^{\alpha-d}}.$$

In region (II), $|y| \ge \frac{|x|}{2}$, so

$$\int_{II} 1_{|x-y|>1}e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|x-y|^{\alpha}}dy \lesssim e^{-\rho\sqrt{\lambda}|x|/2}.$$

In region (III), $|x-y| \ge |y|/2$, so

$$\int_{III} e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|x-y|^{\alpha}}dy \lesssim \int_{\mathbb{R}^d} 1_{|y|>|x|}\frac{1}{|y|^{\alpha}}dye^{-\rho\sqrt{\lambda}|x|} \lesssim e^{-\rho\sqrt{\lambda}|x|}.$$

Therefore, in summary, we have

$$\int_{\mathbb{R}^d} e^{-\rho\sqrt{\lambda}|y|}(1\wedge\frac{1}{|x-y|^{\alpha}})dy \lesssim 1\wedge(e^{-c\sqrt{\lambda}|x|}+\frac{1}{|x|^{\beta}}) \tag{2.126}$$

for $c = \rho/2 > 0$ and β sufficiently large.

For (ii), when $|y-x| \leq 1$,

$$\int_{|y-x|\leq 1} e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|y|^{d-2}} \lesssim e^{-\rho\sqrt{\lambda}|x|}\frac{1}{|x|^{d-2}}.$$

In region (I), by a similar discussion, we have

$$\int_{(I)} e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|y|^{d-2}}dy\frac{1}{|x|^{\alpha}} \lesssim \frac{1}{|x|^{\alpha-2}}.$$

In region (II), $e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|y|^{d-2}} \lesssim e^{-\rho\sqrt{\lambda}|x|/2}\frac{1}{|x|^{d-2}}$, so

$$\int_{(II)} e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|y|^{d-2}}\frac{1}{|x-y|^{\alpha}}1_{|x-y|>1}dy \lesssim e^{-\rho\sqrt{\lambda}|x|/2}\frac{1}{|x|^{d-2}}.$$

In region (III), we have

$$\int_{(III)} e^{-\rho\sqrt{\lambda}|y|}\frac{1}{|y|^{d-2}}\frac{1}{|x-y|^{\alpha}}dy \lesssim e^{-\rho\sqrt{\lambda}|x|}\frac{1}{|x|^{d-2}}.$$

The proof is complete. □

2.3.2 Duality relation in Malliavin calculus

Let $H = \oplus^d L^2([0,t])$ and B_t a standard Brownian motion in $\mathbb{R}^d$. Take the isonormal Gaussian space $\{W(h)\}$ on H defined as $W(h) = \sum_{k=1}^d \int_0^t \tilde{h}_k(s)dB_s^k$ when $h = (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_d) \in H$, then B_t is written as

$$B_t = (W(h_1), W(h_2), \ldots, W(h_d))$$

with $h_i \in \oplus^d L^2([0,t])$, and only its $i-$ component is non-zero and equal to $1_{[0,t]}$.

Let $F = f(W(h_1), W(h_2), \ldots, W(h_d))$ for any test function f, and $G = (g_1(B_s), g_2(B_s), \ldots, g_d(B_s))$.

Then the Skorohod integral for G is defined as

$$\delta(G) = \sum_{k=1}^{d} \int_0^t g_k(B_s) dB_s^k. \tag{2.127}$$

The duality relation reads

$$\mathbb{E}\{F\delta(G)\} = \mathbb{E}\{\langle DF, G\rangle_H\}, \tag{2.128}$$

with D the Malliavin derivative operator.

We have $DF = \sum_{k=1}^{d} \partial_k f(B_t) h_k$, so

$$\langle DF, G\rangle_H = \sum_{k=1}^{d} \partial_k f(B_t) \int_0^t g_k(B_s) ds. \tag{2.129}$$

Therefore, (2.128) implies

$$\sum_{k=1}^{d} \mathbb{E}\{f(B_t) \int_0^t g_k(B_s) dB_s^k\} = \sum_{k=1}^{d} \mathbb{E}\{\partial_k f(B_t) \int_0^t g_k(B_s) ds\}. \tag{2.130}$$

Chapter 3

Low Dimensional Cases

In this chapter, we consider the lower dimensional short-range-correlated case for equation (1.7), focusing on $d = 2$. When $d = 1$, the result in [29] shows that the limit is a SPDE.

We first prove an invariance principle for Brownian motion in Gaussian or Poissonian scenery when $d = 2$ by the method of characteristic functions. When $\hat{R}(\xi)|\xi|^{-2}$ is integrable, it is implied by Kipnis-Varadhan [21], so we focus on the case when this is not the case, e.g., when $\hat{R}(0) > 0$. Since the weak convergence is in the annealed sense, we prove homogenization through the analysis of first and second moments.

3.1 An invariance principle for Brownian motion in Gaussian or Poissonian scenery when $d = 2$

The Gaussian and Poissonian potentials are denoted by $V_g(x)$ and $V_p(x)$, respectively, throughout the chapter.

For the Gaussian case, we assume $V_g(x)$ is stationary with zero mean and the covariance function $R_g(x) = \mathbb{E}\{V_g(x+y)V_g(y)\}$ is continuous and compactly supported. The power spectrum

$$\hat{R}_g(\xi) = \int_{\mathbb{R}^d} R_g(x)e^{-i\xi x}dx,$$

and by Bochner's theorem $\hat{R}_g(0) = \int_{\mathbb{R}^d} R_g(x)dx \geq 0$.

For the Poissonian case, we assume

$$V_p(x) = \int_{\mathbb{R}^d} \phi(x-y)\omega(dy) - c_p, \tag{3.1}$$

where $\omega(dx)$ is a Poissonian field in $\mathbb{R}^d$ with the $d-$dimensional Lebesgue measure as its intensity measure and ϕ is a continuous, compactly supported shape function such that $\int_{\mathbb{R}^d} \phi(x)dx = c_p$. It is straightforward to check that $V_p(x)$ is stationary and has zero mean, and its covariance function

$$R_p(x) = \mathbb{E}\{V_p(x+y)V_p(y)\} = \int_{\mathbb{R}^d} \phi(x+z)\phi(z)dz \tag{3.2}$$

is continuous and compactly supported as well. The power spectrum

$$\hat{R}_p(\xi) = \int_{\mathbb{R}^d} R_p(x)e^{-i\xi x}dx,$$

and since $\int_{\mathbb{R}^d} \phi(x)dx = c_p$, we have $\hat{R}_p(0) = c_p^2 \geq 0$.

In the Poissonian case, the random field $V_p(x)$ is mixing in the sense of Assumption 2.4, i.e., for two Borel sets $A, B \subset \mathbb{R}^d$, let $\mathcal{F}_A$ and $\mathcal{F}_B$ denote the sub-σ algebras generated by the field $V_p(x)$ for $x \in A$ and $x \in B$, respectively. Then there exists a positive and decreasing function $\varphi(r)$ such that

$$|Cor(\eta, \zeta)| \leq \varphi(2d(A,B)) \tag{3.3}$$

for all square integrable random variables η and ζ that are $\mathcal{F}_A$ and $\mathcal{F}_B$ measurable, respectively. The multiplicative factor 2 is only here for convenience. Actually, when $|x|$ is sufficiently large, $V_p(x+y)$ is independent of $V_p(y)$ and so the mixing coefficient $\varphi(r)$ can be chosen as a positive, decreasing function with compact support in $[0,\infty)$. We will use this in the estimation of the fourth moment of $V_p(x)$.

The following theorems are our main results, with a major contribution for $d = 2$.

Theorem 3.1. *Let $B_t, t \geq 0$ be a $d-$dimensional standard Brownian motion independent of the stationary random potential $V(x)$, which is chosen to be either Gaussian or Poissonian, and $R(x) = \mathbb{E}\{V(x+y)V(y)\}$ be the covariance function, $\hat{R}(0) = \int_{\mathbb{R}^d} R(x)dx$. Define $X_n(t) = a(n)^{-1}\int_0^{nt} V(B_s)ds$ with the scaling factor*

$$a(n) = \begin{cases} n^{\frac{3}{4}} & d = 1, \\ (n\log n)^{\frac{1}{2}} & d = 2, \\ n^{\frac{1}{2}} & d \geq 3. \end{cases}$$

Then we have that $X_n(t)$ converges weakly in $\mathcal{C}([0,1])$ to $\sigma_d Z_t$ with the following representations:

When $d = 1$, then $Z_t = \int_{\mathbb{R}} L_t(x)W(dx)$, where $L_t(x)$ is the local time of B_t and $W(dx)$ is a $1-$dimensional white noise independent of $L_t(x)$; $\sigma_d = \sqrt{\hat{R}(0)}$.

When $d = 2$, then Z_t is a standard Brownian motion; $\sigma_d = \sqrt{\hat{R}(0)/\pi}$.

When $d \geq 3$, then Z_t is a standard Brownian motion;

$$\sigma_d = \sqrt{\frac{1}{\pi^{\frac{d}{2}}}\Gamma(\frac{d}{2}-1)\int_{\mathbb{R}^d}\frac{R(x)}{|x|^{d-2}}dx}.$$

We note that when $d \geq 3$, then $\sigma_d = \sqrt{4(2\pi)^{-d}\int_{\mathbb{R}^d}\hat{R}(\xi)|\xi|^{-2}d\xi}$. Since $\hat{R}(\xi) \geq 0$, $\sigma_d > 0$ in both cases and the limit is nontrivial. When $d \leq 2$, the limit is nontrivial if $\hat{R}(0) \neq 0$. In the degenerate case when $\hat{R}(0) = 0$, for instance if $c_p = 0$ in the Poissonian case in $d = 1, 2$, then the limit obtained in the previous theorem is trivial. The scaling factor $a(n)$ should be chosen smaller to obtain a nontrivial limit. We have the following result by borrowing Kipnis-Varadhan [21]:

Theorem 3.2. *In $d = 1, 2$, let $B_t, t \geq 0$ be a $d-$dimensional standard Brownian motion independent of the stationary random potential $V(x)$, which is chosen to be either Gaussian or Poissonian, and $R(x) = \mathbb{E}\{V(x+y)V(y)\}$ with $\hat{R}(\xi)|\xi|^{-2}$ integrable. Define $X_n(t) = n^{-\frac{1}{2}}\int_0^{nt} V(B_s)ds$. Then we have $X_n(t)$ converges weakly in $\mathcal{C}([0,1])$ to σW_t, with W_t a standard Brownian motion and $\sigma = \sqrt{4(2\pi)^{-d}\int_{\mathbb{R}^d}\hat{R}(\xi)|\xi|^{-2}d\xi}$.*

Remark 3.3. In the degenerate case, the scaling factor $n^{-\frac{1}{2}}$ is the same as in $d \geq 3$. In $d = 1$, the limiting processes are different for the non-degenerate and degenerate cases. Since $\hat{R}(\xi) \in L^1$, for $\hat{R}(\xi)|\xi|^{-2}$ to be integrable, we only need to assume that $\hat{R}(\xi) \lesssim |\xi|^{\alpha}$ at the origin with $\alpha > 1$ when $d = 1$ and $\alpha > 0$ when $d = 2$.

We will refer to Theorem 3.1 and 3.2 as non-degenerate and degenerate cases respectively in the following sections. Since Theorem 3.2 is implied by [21], we only present a proof of Theorem 3.1. The proof contains convergence of finite dimensional distributions and tightness result. We point out that the case $d \geq 3$ is also implied by [21], but we present a different proof, which turns to be useful in other settings, e.g., it is used in the proof of Proposition 2.23 in Chapter 2. Since all the weak convergences are in the annealed sense, we use $\mathbb{E}$ to denote the expectation in the product probability space.

3.1.1 Convergence of finite dimensional distributions

We first prove the weak convergence of finite dimensional distributions through the estimation of characteristic functions.

For any $0 = t_0 < t_1 < \ldots < t_N \leq 1$ and $\alpha_i \in \mathbb{R}, i = 1, \ldots, N$, by considering $Y_N := \sum_{i=1}^N \alpha_i (X_n(t_i) - X_n(t_{i-1}))$, we have the following explicit expressions.

In the Gaussian case, since $Y_N = \sum_{i=1}^N \alpha_i a(n)^{-1} \int_{nt_{i-1}}^{nt_i} V_g(B_s) ds$, we obtain

$$\begin{aligned}\mathbb{E}\{\exp(i\theta Y_N)\} =& \mathbb{E}\{\mathbb{E}\{\exp(i\theta Y_N)|B\}\} \\ =& \mathbb{E}\{\exp(-\frac{1}{2}\theta^2 \sum_{i,j=1}^N \alpha_i \alpha_j \frac{1}{a(n)^2} \int_{nt_{i-1}}^{nt_i} \int_{nt_{j-1}}^{nt_j} R_g(B_s - B_u) ds du)\}.\end{aligned} \tag{3.4}$$

Since $\mathbb{E}\{\exp(i\theta Y_N)|B\}$ is bounded by 1, to prove convergence of $\mathbb{E}\{\exp(i\theta Y_N)\}$, we only need to prove the convergence in probability of $a(n)^{-2} \int_{nt_{i-1}}^{nt_i} \int_{nt_{j-1}}^{nt_j} R_g(B_s - B_u) ds du$.

In the Poissonian case, we write

$$Y_N = \int_{\mathbb{R}^d} \left(\sum_{i=1}^N \alpha_i \frac{1}{a(n)} \int_{nt_{i-1}}^{nt_i} \phi(B_s - y) ds \right) \omega(dy) - \frac{c_p}{a(n)} \sum_{i=1}^N \alpha_i (nt_i - nt_{i-1}),$$

and straightforward calculations lead to

$$\mathbb{E}\{\exp(i\theta Y_N)\} = \mathbb{E}\{\exp(\int_{\mathbb{R}^d} (e^{i\theta F_n(y)} - 1) dy)\} \exp(-i\theta \frac{c_p}{a(n)} \sum_{i=1}^N \alpha_i (nt_i - nt_{i-1})),$$

where $F_n(y) := \sum_{i=1}^N \alpha_i a(n)^{-1} \int_{nt_{i-1}}^{nt_i} \phi(B_s - y) ds$. Since $\int_{\mathbb{R}^d} \phi(x) dx = c_p$, we obtain

$$\mathbb{E}\{\exp(i\theta Y_N)\} = \mathbb{E}\{\mathbb{E}\{\exp(i\theta Y_N)|B\}\} = \mathbb{E}\{\exp(\int_{\mathbb{R}^d} \sum_{k=2}^\infty \frac{1}{k!} (i\theta F_n(y))^k dy)\}. \tag{3.5}$$

Similarly, $\mathbb{E}\{\exp(i\theta Y_N)|B\}$ is bounded by 1, so it suffices to show the convergence in probability of $\int_{\mathbb{R}^d}\sum_{k=2}^{\infty}\frac{1}{k!}(i\theta F_n(y))^k dy$. When $k=2$,

$$\int_{\mathbb{R}^d} F_n(y)^2 dy = \sum_{i,j=1}^{N} \alpha_i\alpha_j a(n)^{-2}\int_{nt_{i-1}}^{nt_i}\int_{nt_{j-1}}^{nt_j} R_p(B_s-B_u)dsdu \tag{3.6}$$

is the conditional variance of Y_N given B_s. We will see that the proof of the Poissonian case implies the Gaussian case.

Poissonian case $d=1$

When $d=1$, since V_p is mixing, [29, Theorem 2] implies the result. We give a different proof using characteristic functions.

First of all, by scaling property of Brownian motion, we do not distinguish between $F_n(y)=\sum_{i=1}^{N}\alpha_i n^{-\frac{3}{4}}\int_{nt_{i-1}}^{nt_i}\phi(B_s-y)ds$ and $F_n(y)=\sum_{i=1}^{N}\alpha_i n^{\frac{1}{4}}\int_{t_{i-1}}^{t_i}\phi(\sqrt{n}B_s-y)ds$. Using the second representation, we have

$$F_n(y)=\sum_{i=1}^{N}\alpha_i n^{\frac{1}{4}}\int_{t_{i-1}}^{t_i}\phi(\sqrt{n}B_s-y)ds=\sum_{i=1}^{N}\alpha_i n^{\frac{1}{4}}\int_{\mathbb{R}}\phi(\sqrt{n}x-y)(L_{t_i}(x)-L_{t_{i-1}}(x))dx,$$

where $L_t(x)$ is the local time of B_s. So

$$\int_{\mathbb{R}^d} F_n(y)^2 dy=\sum_{i,j=1}^{N}\alpha_i\alpha_j\sqrt{n}\int_{\mathbb{R}^2}R_p(\sqrt{n}(z-x))(L_{t_i}(x)-L_{t_{i-1}}(x))(L_{t_j}(z)-L_{t_{j-1}}(z))dxdz. \tag{3.7}$$

We have the following two propositions.

Proposition 3.4. $\int_{\mathbb{R}} F_n(y)^2 dy \to \hat{R}_p(0)\sum_{i,j=1}^{N}\alpha_i\alpha_j\int_{\mathbb{R}}(L_{t_i}(x)-L_{t_{i-1}}(x))(L_{t_j}(x)-L_{t_{j-1}}(x))dx$ *almost surely.*

Proposition 3.5. $\int_{\mathbb{R}} \sum_{k=3}^{\infty} \frac{1}{k!}(i\theta F_n(y))^k dy \to 0$ *almost surely.*

Proof of Proposition 3.4. For any $i, j = 1, \dots, N$, we consider

$$\begin{aligned}(i) :=& \sqrt{n} \int_{\mathbb{R}^2} R_p(\sqrt{n}(z-x))(L_{t_i}(x) - L_{t_{i-1}}(x))(L_{t_j}(z) - L_{t_{j-1}}(z))dxdz \\ =& \int_{\mathbb{R}^2} R_p(y)(L_{t_i}(x) - L_{t_{i-1}}(x))(L_{t_j}(x + \frac{y}{\sqrt{n}}) - L_{t_{j-1}}(x + \frac{y}{\sqrt{n}}))dydx,\end{aligned}$$

and because $L_t(x)$ is continuous with compact support almost surely, we have

$$(i) \to \hat{R}_p(0) \int_{\mathbb{R}} (L_{t_i}(x) - L_{t_{i-1}}(x))(L_{t_j}(x) - L_{t_{j-1}}(x))dx,$$

which completes the proof. □

Proof of Proposition 3.5. For a fixed realization of B_s, we have

$$|F_n(y)| \lesssim n^{\frac{1}{4}} \int_{|x|<M} |\phi(\sqrt{n}x - y)|dx,$$

where M is a constant depending on the realization and thus

$$\begin{aligned}&|\int_{\mathbb{R}} F_n(y)^k dy| \\ \lesssim& n^{\frac{k}{4}} \int_{\mathbb{R}} \int_{[-M,M]^k} \prod_{i=1}^{k} |\phi(\sqrt{n}x_i - y)|dxdy = n^{\frac{k}{4}} \int_{\mathbb{R}} \int_{[-M,M]^k} \prod_{i=1}^{k} |\phi(\sqrt{n}(x_i - x_1) + y)|dxdy \\ \leq& \frac{1}{n^{\frac{k-2}{4}}} \int_{\mathbb{R}^k} \int_{[-M,M]} |\phi(y)| \prod_{i=2}^{k} |\phi(x_i - \sqrt{n}x_1 + y)|dxdy \lesssim \frac{1}{n^{\frac{k-2}{4}}}.\end{aligned}$$

Since $\sum_{k=3}^{\infty} \frac{1}{k!} \frac{|\theta|^k}{n^{\frac{k-2}{4}}} \to 0$ as $n \to \infty$, the proof is complete. □

Recalling (3.5), by Proposition 3.4 and 3.5, we have proved the almost sure convergence of the exponents. Therefore, by the Lebesgue dominated convergence theorem we have

$$\begin{aligned}\mathbb{E}\{\exp(i\theta Y_N)\} =&\mathbb{E}\{\exp(\int_{\mathbb{R}^d}\sum_{k=2}^{\infty}\frac{1}{k!}(i\theta F_n(y))^k dy)\}\\ \to&\mathbb{E}\{\exp(-\frac{1}{2}\theta^2\hat{R}_p(0)\sum_{i,j=1}^{N}\alpha_i\alpha_j\int_{\mathbb{R}}(L_{t_i}(x)-L_{t_{i-1}}(x))(L_{t_j}(x)-L_{t_{j-1}}(x))dx)\}\\ =&\mathbb{E}\{\exp(i\theta\sigma_d\sum_{i=1}^{N}\alpha_i(Z_{t_i}-Z_{t_{i-1}}))\}\end{aligned} \tag{3.8}$$

when $Z_t = \int_{\mathbb{R}} L_t(x)W(dx)$.

Poissonian case $d \geq 2$

When $d \geq 2$, the local time does not exist, and to prove the convergence of the conditional variance of $X_n(t)$ given B_s, we need to calculate fourth moments. First, we define

$$\mathbb{V}_n = \mathbb{E}\{X_n(t)^2 | B_s, s \in [0,t]\} = \frac{1}{a(n)^2}\int_0^{nt}\int_0^{nt} R_p(B_s - B_u)dsdu \tag{3.9}$$

so that $\mathbb{E}\{X_n(t)^2\} = \mathbb{E}\{\mathbb{V}_n\}$. The following two lemmas show that the conditional variance converges in probability.

Lemma 3.6. $\mathbb{E}\{\mathbb{V}_n\} \to \sigma_d^2 t$ *as* $n \to \infty$.

Lemma 3.7. $\mathbb{E}\{\mathbb{V}_n^2\} \to \sigma_d^4 t^2$ *as* $n \to \infty$.

In the proofs, we deal with $d = 2$ and $d \geq 3$ in different ways. For the latter, we only use the fact that $\hat{R}_p(\xi)|\xi|^{-2}$ is integrable and so the proof also applies in the degenerate case. Both $R_p(x)$ and $\hat{R}_p(\xi)$ are even functions, a fact that we will use frequently in the proof.

Proof of Lemma 3.6. We first consider the case $d = 2$. For fixed x, by change of variables

$\lambda = \frac{|x|^2}{2u}$, we have

$$\begin{aligned}\mathbb{E}\{\mathbb{V}_n\} =& \frac{2}{a(n)^2}\int_0^{nt}\int_0^s\int_{\mathbb{R}^d} R_p(x)\frac{1}{(2\pi u)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2u}}dxduds \\ =& \frac{n}{a(n)^2}\int_0^t\int_{\mathbb{R}^d}\int_{\frac{|x|^2}{2ns}}^\infty R_p(x)\frac{1}{\pi^{\frac{d}{2}}}\lambda^{\frac{d}{2}-2}e^{-\lambda}\frac{1}{|x|^{d-2}}d\lambda dxds.\end{aligned} \tag{3.10}$$

Since $a(n) = (n\log n)^{\frac{1}{2}}$, by integrations by parts in λ, we have

$$\mathbb{E}\{\mathbb{V}_n\} = \frac{1}{\log n}\int_0^t\int_{\mathbb{R}^d}\frac{1}{\pi}R_p(x)\left(e^{-\frac{|x|^2}{2ns}}\log\frac{2ns}{|x|^2} + \int_{\frac{|x|^2}{2ns}}^\infty e^{-\lambda}\log\lambda d\lambda\right)dxds \to \frac{t}{\pi}\hat{R}_p(0) \tag{3.11}$$

by the Lebesgue dominated convergence theorem.

Consider now the case $d \geq 3$. Then, $a(n) = n^{\frac{1}{2}}$ and by Fourier transform, we have

$$\begin{aligned}\mathbb{E}\{\mathbb{V}_n\} =& \frac{1}{(2\pi)^d n}\int_{[0,nt]^2}\int_{\mathbb{R}^d}\hat{R}_p(\xi)e^{-\frac{1}{2}|\xi|^2|s-u|}d\xi dsdu \\ =& \frac{4}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{R}_p(\xi)}{|\xi|^2}\int_0^t(1-e^{-\frac{1}{2}|\xi|^2 ns})dsd\xi \to \frac{4t}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{R}_p(\xi)}{|\xi|^2}d\xi\end{aligned} \tag{3.12}$$

as $n \to \infty$. □

Proof of Lemma 3.7. By symmetry of $R(x)$, we write

$$\mathbb{V}_n^2 = a(n)^{-4}\int_{[0,nt]^4} R_p(B_{s_1} - B_{s_2})R_p(B_{s_3} - B_{s_4})ds = 8((i) + (ii) + (iii)),$$

where

$$(i) = \frac{1}{a(n)^4} \int_{0<s_1<s_2<s_3<s_4<nt} R_p(B_{s_1} - B_{s_2}) R_p(B_{s_3} - B_{s_4}) ds, \tag{3.13}$$

$$(ii) = \frac{1}{a(n)^4} \int_{0<s_1<s_3<s_2<s_4<nt} R_p(B_{s_1} - B_{s_2}) R_p(B_{s_3} - B_{s_4}) ds, \tag{3.14}$$

$$(iii) = \frac{1}{a(n)^4} \int_{0<s_1<s_3<s_4<s_2<nt} R_p(B_{s_1} - B_{s_2}) R_p(B_{s_3} - B_{s_4}) ds. \tag{3.15}$$

We consider first the case $d = 2$.

(i): for fixed x, y, by change of variables $u_1 = \frac{s_1}{n}, u_3 = \frac{s_3 - s_2}{n}, \lambda_2 = \frac{|x|^2}{2(s_2 - s_1)}, \lambda_4 = \frac{|y|^2}{2(s_4 - s_3)}$, we have

$$\mathbb{E}\{(i)\} = \frac{n^2}{a(n)^4} \int_{\mathbb{R}_+^4} \int_{\mathbb{R}^{2d}} \frac{R_p(x)}{|x|^{d-2}} \frac{R_p(y)}{|y|^{d-2}} \frac{1}{4\pi^d} \lambda_2^{\frac{d}{2}-2} e^{-\lambda_2} \lambda_4^{\frac{d}{2}-2} e^{-\lambda_4}$$

$$1_{0 \le u_1 + u_3 \le t; \frac{|x|^2}{2\lambda_2} + \frac{|y|^2}{2\lambda_4} \le n(t - u_1 - u_3)} dx dy d\lambda_2 d\lambda_4 du_1 du_3.$$

We define

$$f(c) = \frac{1}{(\log n)^2} \int_{\mathbb{R}_+^2} \frac{1}{\lambda_2} e^{-\lambda_2} \frac{1}{\lambda_4} e^{-\lambda_4} 1_{\frac{|x|^2}{2\lambda_2} \le cn(t - u_1 - u_3); \frac{|y|^2}{2\lambda_4} \le cn(t - u_1 - u_3)} d\lambda_2 d\lambda_4$$

for $c > 0$. Using integrations by parts, $f(c) \to 1$ as $n \to \infty$ as long as $x, y \neq 0, u_1 + u_3 < t$. Moreover, $f(c) \lesssim (1 + |\log c(t - u_1 - u_3)| + |\log |x||)(1 + |\log c(t - u_1 - u_3)| + |\log |y||)$. On the other hand, we note that

$$f(\frac{1}{2}) \le \frac{1}{(\log n)^2} \int_{\mathbb{R}_+^2} \frac{1}{\lambda_2} e^{-\lambda_2} \frac{1}{\lambda_4} e^{-\lambda_4} 1_{\frac{|x|^2}{2\lambda_2} + \frac{|y|^2}{2\lambda_4} \le n(t - u_1 - u_3)} d\lambda_2 d\lambda_4 \le f(1),$$

so by the Lebesgue dominated convergence theorem, we have $\mathbb{E}\{(i)\} \to \frac{t^2}{8\pi^2} \hat{R}_p(0)^2$.

(ii): by a similar change of variables as for (i), we have

$$\mathbb{E}\{(ii)\} = \frac{n^2}{a(n)^4}\int_{\mathbb{R}_+^4}\int_{\mathbb{R}^{3d}} \frac{R_p(x-z)}{|x|^{d-2}}\frac{R_p(y-z)}{|y|^{d-2}}\frac{1}{4\pi^d}\lambda_2^{\frac{d}{2}-2}e^{-\lambda_2}\lambda_4^{\frac{d}{2}-2}e^{-\lambda_4}$$

$$1_{0\le u_1+u_3\le t;\frac{|x|^2}{2\lambda_2}+\frac{|y|^2}{2\lambda_4}\le n(t-u_1-u_3)}q_{nu_3}(z)dxdydzd\lambda_2d\lambda_4du_1du_3.$$

By a change of variables and integration by parts in λ_2, λ_4, we have

$$\begin{aligned}&|\mathbb{E}\{(ii)\}|\\ \lesssim& \left(\frac{n}{\log n}\right)^2\int_{[0,t]^2}\int_{\mathbb{R}^6}|R_p(\sqrt{n}(x-z))R_p(\sqrt{n}(y-z))|q_{u_3}(z)\left(e^{-\frac{|x|^2}{2u}}\log\frac{2u}{|x|^2}+\int_{\frac{|x|^2}{2u}}^{\infty}\log\lambda e^{-\lambda}d\lambda\right)\\ &\left(e^{-\frac{|y|^2}{2u}}\log\frac{2u}{|y|^2}+\int_{\frac{|y|^2}{2u}}^{\infty}\log\lambda e^{-\lambda}d\lambda\right)dudu_3dxdydz.\end{aligned}$$

Note that $e^{-\frac{|x|^2}{2u}}\log\frac{2u}{|x|^2}+\int_{\frac{|x|^2}{2u}}^{\infty}\log\lambda e^{-\lambda}d\lambda \lesssim 1+|\log u|+|\log|x||$. By Lemma 3.16, we have

$$\begin{aligned}&\frac{n}{\log n}\int_{\mathbb{R}^2}|R_p(\sqrt{n}(x-z))|\left(e^{-\frac{|x|^2}{2u}}\log\frac{2u}{|x|^2}+\int_{\frac{|x|^2}{2u}}^{\infty}\log\lambda e^{-\lambda}d\lambda\right)dx\\ \lesssim&\frac{1}{\log n}\left(1+|\log u|+|\log|z||+\log n1_{|z|<\frac{2}{\sqrt{n}}}\right).\end{aligned}$$

The integral in y is controlled in the same way and we obtain

$$|\mathbb{E}\{(ii)\}| \lesssim \int_{[0,t]^2}\int_{\mathbb{R}^2}\frac{1}{(\log n)^2}\left(1+|\log u|+|\log|z||+\log n1_{|z|<\frac{2}{\sqrt{n}}}\right)^2 q_{u_3}(z)dzdudu_3.$$

So $|\mathbb{E}\{(ii)\}| \to 0$ as $n\to\infty$.

(iii): by a similar change of variables as for (i) and by symmetry of $R(x)$, we have

$$\mathbb{E}\{(iii)\} = \frac{n^2}{a(n)^4} \int_{\mathbb{R}_+^4} \int_{\mathbb{R}^{3d}} \frac{R_p(x-y+z)}{|x|^{d-2}} \frac{R_p(y)}{|y|^{d-2}} \frac{1}{4\pi^d} \lambda_2^{\frac{d}{2}-2} e^{-\lambda_2} \lambda_4^{\frac{d}{2}-2} e^{-\lambda_4}$$

$$1_{0 \le u_1+u_3 \le t; \frac{|x|^2}{2\lambda_2} + \frac{|y|^2}{2\lambda_4} \le n(t-u_1-u_3)} q_{nu_3}(z) dx dy dz d\lambda_2 d\lambda_4 du_1 du_3.$$

After integrations by parts in λ_2, λ_4, we have

$$\begin{aligned} &|\mathbb{E}\{(iii)\}| \\ \lesssim &\left(\frac{n}{\log n}\right)^2 \int_{[0,t]^2} \int_{\mathbb{R}^6} |R_p(\sqrt{n}(x-y+z)) R_p(\sqrt{n}y)| q_{u_3}(z) \left(e^{-\frac{|x|^2}{2u}} \log \frac{2u}{|x|^2} + \int_{\frac{|x|^2}{2u}}^\infty \log \lambda e^{-\lambda} d\lambda \right) \\ &\left(e^{-\frac{|y|^2}{2u}} \log \frac{2u}{|y|^2} + \int_{\frac{|y|^2}{2u}}^\infty \log \lambda e^{-\lambda} d\lambda \right) du du_3 dx dy dz. \end{aligned}$$

Note that $e^{-\frac{|x|^2}{2u}} \log \frac{2u}{|x|^2} + \int_{\frac{|x|^2}{2u}}^\infty \log \lambda e^{-\lambda} d\lambda \lesssim 1 + |\log u| + |\log |x||$. By applying Lemma 3.16 to the integral in x, we have

$$\begin{aligned} &\frac{n}{\log n} \int_{\mathbb{R}^2} |R_p(\sqrt{n}(x-(y-z)))| \left(e^{-\frac{|x|^2}{2u}} \log \frac{2u}{|x|^2} + \int_{\frac{|x|^2}{2u}}^\infty \log \lambda e^{-\lambda} d\lambda \right) dx \\ \lesssim &\frac{1}{\log n} \left(1 + |\log u| + |\log |y-z|| + \log n 1_{|y-z| < \frac{2}{\sqrt{n}}} \right). \end{aligned}$$

So

$$|\mathbb{E}\{(iii)\}| \lesssim \frac{n}{(\log n)^2} \int_{[0,t]^2} \int_{\mathbb{R}^4} \left(1 + |\log u| + |\log |y-z|| + \log n 1_{|y-z| < \frac{2}{\sqrt{n}}} \right) |R_p(\sqrt{n}y)| q_{u_3}(z)$$

$$(1 + |\log u| + |\log |y||) dy dz du_3 du.$$

Since $|R_p(\sqrt{n}y)| \lesssim 1 \wedge |\sqrt{n}y|^{-\alpha}$ for some $\alpha > 2$, by Lemma 3.16, we know $\mathbb{E}\{(iii)\} \to 0$ as $n \to \infty$.

We now consider the case $d \geq 3$.

(i): after Fourier transform and changing of variables $u_i = s_i - s_{i-1}$ for $i = 1, 2, 3, 4$ with $s_0 = 0$, we derive

$$\begin{aligned}\mathbb{E}\{(i)\} =& \frac{1}{(2\pi)^{2d} n^2} \int_{\mathbb{R}_+^4} \int_{\mathbb{R}^{2d}} 1_{\sum_{i=1}^4 u_i \leq nt} \hat{R}_p(\xi_1) \hat{R}_p(\xi_2) e^{-\frac{1}{2}|\xi_1|^2 u_2} e^{-\frac{1}{2}|\xi_2|^2 u_4} d\xi_1 d\xi_2 du \\ =& \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{2d}} \hat{R}_p(\xi_1) \hat{R}_p(\xi_2) 1_{0 \leq u_1 + u_3 \leq t} F_n(\frac{1}{2}|\xi_1|^2, \frac{1}{2}|\xi_2|^2, t - u_1 - u_3) d\xi_1 d\xi_2 du,\end{aligned}$$

where $F_n(a, b, t) := \int_{\mathbb{R}_+^2} 1_{0 \leq s+u \leq nt} e^{-as} e^{-bu} ds du$ for $a \geq 0, b \geq 0$. It is straightforward to check that $abF_n(a, b, t)$ is uniformly bounded and $F_n(a, b, t) \to \frac{1}{ab}$ as $n \to \infty$. Thus,

$$\begin{aligned}\mathbb{E}\{(i)\} \to& \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{2d}} \hat{R}_p(\xi_1) \hat{R}_p(\xi_2) 1_{0 \leq u_1 + u_3 \leq t} \frac{4}{|\xi_1|^2 |\xi_2|^2} d\xi_1 d\xi_2 du_1 du_3 \\ =& \frac{2t^2}{(2\pi)^{2d}} \left(\int_{\mathbb{R}^d} \frac{\hat{R}_p(\xi)}{|\xi|^2} d\xi \right)^2.\end{aligned}$$

(ii): similarly we have

$$\begin{aligned}&\mathbb{E}\{(ii)\} \\ =& \frac{1}{(2\pi)^{2d} n^2} \int_{\mathbb{R}_+^4} \int_{\mathbb{R}^{2d}} 1_{\sum_{i=1}^4 u_i \leq nt} \hat{R}_p(\xi_1) \hat{R}_p(\xi_2) e^{-\frac{1}{2}|\xi_1|^2 u_2} e^{-\frac{1}{2}|\xi_1+\xi_2|^2 u_3} e^{-\frac{1}{2}|\xi_2|^2 u_4} d\xi_1 d\xi_2 du \\ \lesssim& t \int_{\mathbb{R}^{2d}} \int_0^t \frac{\hat{R}_p(\xi_1) \hat{R}_p(\xi_2)}{|\xi_1|^2 |\xi_2|^2} e^{-\frac{1}{2}|\xi_1+\xi_2|^2 n u_3} du_3 d\xi_1 d\xi_2 \to 0\end{aligned}$$

as $n \to \infty$.

(iii): by the same change of variables, we obtain

$$
\begin{aligned}
&\mathbb{E}\{(iii)\}\\
=&\frac{1}{(2\pi)^{2d}n^2}\int_{\mathbb{R}_+^4}\int_{\mathbb{R}^{2d}} 1_{\sum_{i=1}^4 u_i\leq nt}\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)e^{-\frac{1}{2}|\xi_1|^2u_2}e^{-\frac{1}{2}|\xi_1+\xi_2|^2u_3}e^{-\frac{1}{2}|\xi_1|^2u_4}d\xi_1 d\xi_2 du\\
\lesssim&\frac{t}{n}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}_+^2} 1_{u_3+u_4\leq nt}\frac{\hat{R}_p(\xi_1)}{|\xi_1|^2}\hat{R}_p(\xi_2)e^{-\frac{1}{2}|\xi_1+\xi_2|^2u_3}e^{-\frac{1}{2}|\xi_1|^2u_4}du_3du_4d\xi_1d\xi_2\\
\lesssim&\frac{t}{n}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}_+^2} 1_{u_3+u_4\leq nt}\frac{\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)}{|\xi_1|^2|\xi_2|^2}(|\xi_1+\xi_2|^2+|\xi_1|^2)e^{-\frac{1}{2}|\xi_1+\xi_2|^2u_3}e^{-\frac{1}{2}|\xi_1|^2u_4}du_3du_4d\xi_1d\xi_2\\
\leq&t\int_{\mathbb{R}^{2d}}\int_0^t\int_{\mathbb{R}_+}\frac{\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)}{|\xi_1|^2|\xi_2|^2}|\xi_1+\xi_2|^2e^{-\frac{1}{2}|\xi_1+\xi_2|^2u_3}e^{-\frac{1}{2}|\xi_1|^2nu_4}du_3du_4d\xi_1d\xi_2\\
+&t\int_{\mathbb{R}^{2d}}\int_0^t\int_{\mathbb{R}_+}\frac{\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)}{|\xi_1|^2|\xi_2|^2}|\xi_1|^2e^{-\frac{1}{2}|\xi_1|^2u_4}e^{-\frac{1}{2}|\xi_1+\xi_2|^2nu_3}du_4du_3d\xi_1d\xi_2\to 0
\end{aligned}
$$

as $n\to\infty$.

To summarize, we have shown that $\mathbb{E}\{\mathbb{V}_n^2\}\to\sigma_d^4t^2$. The proof is complete. □

Remark 3.8. The proof of Lemma 3.7 only requires $R(x)$ to be symmetric, bounded, and to satisfy certain integrability condition. In particular, if $R(x)$ is compactly supported, then the result holds. This will be used in the proof of tightness.

The following lemma proves that those cross terms appearing in the conditional variance vanish in the limit. When $d\geq 3$, as in the proof of Lemma 3.6 and 3.7, we use the Fourier transform and the integrability of $\hat{R}_p(\xi)|\xi|^{-2}$ so that the proof also applies to the degenerate case.

Lemma 3.9. $\frac{1}{a(n)^2}\int_{nt_{i-1}}^{nt_i}\int_{nt_{j-1}}^{nt_j}R_p(B_s-B_u)dsdu\to 0$ *in probability when* $i\neq j$.

Proof. Assume $i>j$,

Consider the case $d = 2$. For fixed x, u, by change of variables $\lambda = \frac{|x|^2}{2(s-u)}$, we have

$$\frac{1}{a(n)^2}\int_{nt_{i-1}}^{nt_i}\int_{nt_{j-1}}^{nt_j}\mathbb{E}\{|R_p(B_s - B_u)|\}dsdu$$
$$=\frac{n}{a(n)^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}_+^2}1_{(\frac{|x|^2}{2n(t_i-u)},\frac{|x|^2}{2n(t_{i-1}-u)})}(\lambda)1_{(t_{j-1},t_j)}(u)\frac{1}{2\pi^{\frac{d}{2}}}\frac{|R_p(x)|}{|x|^{d-2}}\lambda^{\frac{d}{2}-2}e^{-\lambda}d\lambda dudx.$$

Recalling that $a(n) = \sqrt{n\log n}$, an integration by parts leads to

$$\int_{\mathbb{R}}1_{(\frac{|x|^2}{2n(t_i-u)},\frac{|x|^2}{2n(t_{i-1}-u)})}(\lambda)\lambda^{-1}e^{-\lambda}d\lambda \lesssim 1+\log n+|\log(t_i-u)|+|\log(t_{i-1}-u)|+|\log|x||,$$

and $\frac{1}{\log n}\int_{\mathbb{R}}1_{(\frac{|x|^2}{2n(t_i-u)},\frac{|x|^2}{2n(t_{i-1}-u)})}(\lambda)\lambda^{-1}e^{-\lambda}d\lambda \to 0$ as $n\to\infty$. We apply the dominated convergence theorem to conclude the proof.

Consider now the case $d \geq 3$ and

$$(i) := \frac{1}{n^2}\int_{\mathbb{R}_+^4}1_{s_1,s_2\in[nt_{i-1},nt_i]}1_{u_1,u_2\in[nt_{j-1},nt_j]}R_p(B_{s_1}-B_{u_1})R_p(B_{s_2}-B_{u_2})dsdu.$$

We show $\mathbb{E}\{(i)\} \to 0$ so the cross term goes to zero in probability. Actually, we have $(i) = 2((I)+(II))$, where

$$(I) = \frac{1}{n^2}\int_{\mathbb{R}_+^4}1_{nt_{j-1}\le u_2\le u_1\le nt_j}1_{\le nt_{i-1}\le s_2\le s_1\le nt_i}R_p(B_{s_1}-B_{u_1})R_p(B_{s_2}-B_{u_2})dsdu,$$
$$(II) = \frac{1}{n^2}\int_{\mathbb{R}_+^4}1_{nt_{j-1}\le u_1\le u_2\le nt_j}1_{nt_{i-1}\le s_2\le s_1\le nt_i}R_p(B_{s_1}-B_{u_1})R_p(B_{s_2}-B_{u_2})dsdu.$$

For (I) we have

$$\mathbb{E}\{(I)\} = \frac{1}{(2\pi)^{2d}n^2}\int_{\mathbb{R}^4_+}\int_{\mathbb{R}^{2d}} 1_{nt_{j-1}\le u_2\le u_1\le nt_j} 1_{\le nt_{i-1}\le s_2\le s_1\le nt_i} \hat{R}_p(\xi_1)\hat{R}_p(\xi_2)$$
$$e^{-\frac{1}{2}|\xi_1|^2(s_1-s_2)}e^{-\frac{1}{2}|\xi_1+\xi_2|^2(s_2-u_1)}e^{-\frac{1}{2}|\xi_2|^2(u_1-u_2)}d\xi_1 d\xi_2 ds du,$$

which implies $\mathbb{E}\{(I)\} \lesssim (t_j - t_{j-1})\int_{\mathbb{R}^{2d}}\int_{t_{i-1}-t_j}^{t_i-t_{j-1}} \frac{\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)}{|\xi_1|^2|\xi_2|^2} e^{-\frac{1}{2}|\xi_1+\xi_2|^2 nu} du d\xi_1 d\xi_2 \to 0$ as $n \to \infty$. Similarly, for (II) we have

$$\mathbb{E}\{(II)\} = \frac{1}{(2\pi)^{2d}n^2}\int_{\mathbb{R}^4_+}\int_{\mathbb{R}^{2d}} 1_{nt_{j-1}\le u_1\le u_2\le nt_j} 1_{\le nt_{i-1}\le s_2\le s_1\le nt_i} \hat{R}_p(\xi_1)\hat{R}_p(\xi_2)$$
$$e^{-\frac{1}{2}|\xi_1|^2(s_1-s_2)}e^{-\frac{1}{2}|\xi_1+\xi_2|^2(s_2-u_2)}e^{-\frac{1}{2}|\xi_1|^2(u_2-u_1)}d\xi_1 d\xi_2 ds du,$$

so

$$\begin{aligned}
&\mathbb{E}\{(II)\}\\
\lesssim&\frac{1}{n}\int_{[nt_{j-1},nt_i]^2}\int_{\mathbb{R}^{2d}} \frac{\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)}{|\xi_1|^2|\xi_2|^2}(|\xi_1+\xi_2|^2+|\xi_1|^2)e^{-\frac{1}{2}|\xi_1+\xi_2|^2u_1}e^{-\frac{1}{2}|\xi_1|^2u_2}d\xi_1 d\xi_2 du_1 du_2\\
\lesssim&(t_i - t_{j-1})\int_{\mathbb{R}^{2d}}\int_{t_{j-1}}^{t_i} \frac{\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)}{|\xi_1|^2|\xi_2|^2}e^{-\frac{1}{2}|\xi_1+\xi_2|^2nu}du d\xi_1 d\xi_2\\
+&(t_i - t_{j-1})\int_{\mathbb{R}^{2d}}\int_{t_{j-1}}^{t_i} \frac{\hat{R}_p(\xi_1)\hat{R}_p(\xi_2)}{|\xi_1|^2|\xi_2|^2}e^{-\frac{1}{2}|\xi_1|^2nu}du d\xi_1 d\xi_2 \to 0
\end{aligned}$$

as $n \to \infty$. □

The two following propositions holds, so $\int_{\mathbb{R}^d}\sum_{k=2}^{\infty}\frac{1}{k!}(i\theta F_n(y))^k dy$ converges in probability.

Proposition 3.10. $\int_{\mathbb{R}^d} F_n(y)^2 dy \to \sum_{i=1}^{N}\alpha_i^2\sigma_d^2(t_i - t_{i-1})$ *in probability.*

Proposition 3.11. $\int_{\mathbb{R}^d}\sum_{k=3}^{\infty}\frac{1}{k!}(i\theta F_n(y))^k dy \to 0$ *in probability.*

Proof of Proposition 3.10. Note that

$$\int_{\mathbb{R}^d} F_n(y)^2 dy = \sum_{i,j=1}^{N} \alpha_i \alpha_j \frac{1}{a(n)^2} \int_{nt_{i-1}}^{nt_i} \int_{nt_{j-1}}^{nt_j} R_p(B_s - B_u) ds du,$$

when $i = j$, Lemma 3.6 and 3.7 lead to

$$\frac{1}{a(n)^2} \int_{nt_{i-1}}^{nt_i} \int_{nt_{i-1}}^{nt_i} R_p(B_s - B_u) ds du \to \sigma_d^2 (t_i - t_{i-1})$$

in probability as $n \to \infty$.

When $i \neq j$, by Lemma 3.9, we have

$$\frac{1}{a(n)^2} \int_{nt_{i-1}}^{nt_i} \int_{nt_{j-1}}^{nt_j} R_p(B_s - B_u) ds du \to 0$$

in probability as $n \to \infty$. The proof is complete. □

Proof of Proposition 3.11. We will use C for possibly different constants in the following estimation. Recall that $F_n(y) = \sum_{i=1}^{N} \alpha_i \frac{1}{a(n)} \int_{nt_{i-1}}^{nt_i} \phi(B_s - y) ds$, so we have $|F_n(y)| \leq C \frac{1}{a(n)} \int_0^n |\phi(B_s - y)| ds$, and thus

$$\int_{\mathbb{R}^d} \mathbb{E}\{|F_n(y)|^k\} dy \leq C^k \int_{\mathbb{R}^d} \frac{1}{a(n)^k} \int_{[0,n]^k} \mathbb{E}\{\prod_{i=1}^{k} |\phi(B_{s_i} - y)|\} ds dy. \tag{3.16}$$

From now on, we use RHS to denote the RHS of (3.16). By change of variables $u_i = s_i - s_{i-1}$

for $i = 1, \dots, k$ with $s_0 = 0$, and $\lambda_i = \frac{|x_i|^2}{2u_i}$ for $i = 2, \dots, k$ when x_i is fixed, we have

$$\begin{aligned} RHS =& \frac{C^k k!}{a(n)^k} \int_{\mathbb{R}^{(k+1)d}} \int_{\mathbb{R}^k_+} 1_{\sum_{i=1}^k u_i \leq n} |\phi|(y)|\phi|(x_2 + y) \dots |\phi|(\sum_{i=2}^k x_i + y) \prod_{i=1}^k q_{u_i}(x_i) du dx dy \\ =& \frac{C^k k!}{a(n)^k} \int_{\mathbb{R}^{kd}} \int_{\mathbb{R}^k_+} 1_{u_1 + \sum_{i=2}^k \frac{|x_i|^2}{2\lambda_i} \leq n} |\phi|(y) \frac{|\phi|(x_2 + y)}{|x_2|^{d-2}} \dots \frac{|\phi|(\sum_{i=2}^k x_i + y)}{|x_k|^{d-2}} \\ & \prod_{i=2}^k \frac{1}{2\pi^{\frac{d}{2}}} \lambda_i^{\frac{d}{2}-2} e^{-\lambda_i} du_1 d\lambda dx dy. \end{aligned}$$

When $d \geq 3$, note that $\int_{\mathbb{R}^d} |\phi|(y + x)|y|^{2-d} dy$ is uniformly bounded in x, so after integration in $x_k, \dots, x_2, y$ and $\lambda_2, \dots, \lambda_k$, we have $RHS \leq C^k k! n^{-\frac{k}{2}+1}$ where the factor n comes from the integration in u_1. This leads to

$$\mathbb{E}\{|\int_{\mathbb{R}^d} \sum_{k=3}^\infty \frac{1}{k!} (i\theta F_n(y))^k dy|\} \leq \sum_{k=3}^\infty |C\theta|^k \frac{1}{n^{\frac{k}{2}-1}} \to 0$$

as $n \to \infty$.

When $d = 2$, we have

$$RHS \leq C^k \frac{nk!}{a(n)^k} \int_{\mathbb{R}^{kd}} \int_{\mathbb{R}^{k-1}_+} |\phi|(y)|\phi|(x_2 + y) \dots |\phi|(\sum_{i=2}^k x_i + y) \prod_{i=2}^k \frac{1}{2\pi} 1_{\lambda_i \geq \frac{|x_i|^2}{2n}} \frac{1}{\lambda_i} e^{-\lambda_i} d\lambda dx dy. \tag{3.17}$$

By integration by parts, we have $\frac{1}{(\log n)^{k-1}} \int_{\mathbb{R}^{k-1}_+} \prod_{i=2}^k \frac{1}{2\pi} 1_{\lambda_i \geq \frac{|x_i|^2}{2n}} \frac{1}{\lambda_i} e^{-\lambda_i} d\lambda \lesssim \prod_{i=2}^k (1 + |\log |x_i||)$. Since ϕ is compactly supported, we know that $x_i, i = 2, \dots, k$ are uniformly bounded. After integration in $x_k, \dots, x_2, y$, we have $RHS \leq C^k k! \left(\frac{\log n}{n}\right)^{\frac{k}{2}-1}$. So

$$\mathbb{E}\{|\int_{\mathbb{R}^d} \sum_{k=3}^\infty \frac{1}{k!} (i\theta F_n(y))^k dy|\} \leq \sum_{k=3}^\infty |C\theta|^k \left(\frac{\log n}{n}\right)^{\frac{k}{2}-1} \to 0$$

as $n \to \infty$. The proof is complete. $\square$

Remark 3.12. In (3.17), if we choose $a(n) = n^{\frac{1}{2}}$ instead of $a(n) = (n \log n)^{\frac{1}{2}}$, by the same calculation we still have

$$\mathbb{E}\{|\int_{\mathbb{R}^d} \sum_{k=3}^{\infty} \frac{1}{k!}(i\theta F_n(y))^k dy|\} \leq \sum_{k=3}^{\infty} |C\theta|^k \frac{\log n^{k-1}}{n^{\frac{k}{2}-1}} \to 0, \tag{3.18}$$

and this could be used in the proof for the degenerate Poissonian case when $d = 2$.

Recall (3.5), by using Propositions 3.10 and 3.11 and the Lebesgue dominated convergence theorem, we have proved

$$\begin{aligned}\mathbb{E}\{\exp(i\theta Y_N)\} = \mathbb{E}\{\exp(\int_{\mathbb{R}^d} \sum_{k=2}^{\infty} \frac{1}{k!}(i\theta F_n(y))^k dy)\} \to & \mathbb{E}\{\exp(-\frac{1}{2}\theta^2 \sum_{i=1}^{N} \alpha_i^2 \sigma_d^2 (t_i - t_{i-1}))\} \\ = & \mathbb{E}\{\exp(i\theta\sigma_d \sum_{i=1}^{N} \alpha_i (W_{t_i} - W_{t_{i-1}}))\}\end{aligned} \tag{3.19}$$

when W_t is a standard Brownian motion.

Gaussian case

When $d = 1$, by Proposition 3.4, we have

$$\begin{aligned}\mathbb{E}\{\exp(i\theta Y_N)\} = & \mathbb{E}\{\exp(-\frac{1}{2}\theta^2 \sum_{i,j=1}^{N} \alpha_i\alpha_j \frac{1}{a(n)^2} \int_{nt_{i-1}}^{nt_i} \int_{nt_{j-1}}^{nt_j} R_g(B_s - B_u) ds du)\} \\ \to & \mathbb{E}\{\exp(-\frac{1}{2}\theta^2 \hat{R}_g(0) \sum_{i,j=1}^{N} \alpha_i\alpha_j \int_{\mathbb{R}} (L_{t_i}(x) - L_{t_{i-1}}(x))(L_{t_j}(x) - L_{t_{j-1}}(x)) dx)\} \\ = & \mathbb{E}\{\exp(i\theta\sigma_d \sum_{i=1}^{N} \alpha_i (Z_{t_i} - Z_{t_{i-1}}))\},\end{aligned} \tag{3.20}$$

when $Z_t = \int_{\mathbb{R}} L_t(x)W(dx)$.

When $d \geq 2$, by Proposition 3.10, we have

$$\begin{aligned}\mathbb{E}\{\exp(i\theta Y_N)\} =&\mathbb{E}\{\exp(-\frac{1}{2}\theta^2 \sum_{i,j=1}^{N} \alpha_i\alpha_j \frac{1}{a(n)^2}\int_{nt_{i-1}}^{nt_i}\int_{nt_{j-1}}^{nt_j} R_g(B_s - B_u)dsdu)\} \\ \to&\mathbb{E}\{\exp(-\frac{1}{2}\theta^2 \sum_{i=1}^{N} \alpha_i^2\sigma_d^2(t_i - t_{i-1}))\} = \mathbb{E}\{\exp(i\theta\sigma_d \sum_{i=1}^{N} \alpha_i(W_{t_i} - W_{t_{i-1}}))\},\end{aligned} \tag{3.21}$$

when W_t is a standard Brownian motion.

3.1.2 Tightness

Proposition 3.13. *$X_n(t)$ is tight in $\mathcal{C}([0,1])$.*

Proof. Since $X_n(t) = a(n)^{-1}\int_0^{nt} V(B_s)ds$, then $X_n(0) = 0$. To prove tightness of X_n by [6, Theorem 12.3], we only need to show

$$\mathbb{E}\{|X_n(t) - X_n(s)|^\beta\} \leq C|t-s|^{1+\delta} \tag{3.22}$$

for some constant $\beta, C, \delta > 0$.

Consider $d = 1$. $\mathbb{E}\{|X_n(t) - X_n(s)|^2\} = n^{-\frac{3}{2}} \int_{[0,n(t-s)]^2} \mathbb{E}\{R(B_{u_1} - B_{u_2})\}du_1du_2$. Since R

is bounded and compactly supported, we have

$$\begin{aligned}\mathbb{E}\{|X_n(t)-X_n(s)|^2\} \leq & \frac{C}{n^{\frac{3}{2}}}\int_0^{n(t-s)}\int_0^{n(t-s)} \mathbb{P}(|B_{u_1}-B_{u_2}|\leq C)du_1du_2 \\ = & C\sqrt{n}\int_0^{t-s}\int_0^{t-s}\mathbb{P}(|N|\leq \frac{C}{\sqrt{n|u_1-u_2|}})du_1du_2 \\ = & C\int_0^{t-s}\int_0^{t-s}\int_{\mathbb{R}} 1_{|x|<\frac{C}{\sqrt{|u_1-u_2|}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{|x|^2}{2n}}dxdu_1du_2 \\ \leq & C\int_0^{t-s}\int_0^{t-s}\frac{1}{\sqrt{|u_1-u_2|}}du_1du_2 \leq C(t-s)^{\frac{3}{2}}.\end{aligned}$$

For the case $d\geq 2$, we calculate the 4−th moment of $X_n(t)-X_n(s)$. When $V(x)=V_g(x)$ is Gaussian, we have

$$\mathbb{E}\{|X_n(t)-X_n(s)|^4\} = \frac{1}{a(n)^4}\int_{[0,n(t-s)]^4}\sum_{\{\tau_i\}=\{u_i\}}\mathbb{E}\{R_g(B_{\tau_1}-B_{\tau_2})R_g(B_{\tau_3}-B_{\tau_4})\}du.$$

When $V(x)=V_p(x)$ is Poissonian, by Lemma 3.17, we have

$$\mathbb{E}\{|X_n(t)-X_n(s)|^4\} \leq \frac{C}{a(n)^4}\int_{[0,n(t-s)]^4}\sum_{\{\tau_i\}=\{u_i\}}\mathbb{E}\{\varphi^{\frac{1}{2}}(|B_{\tau_1}-B_{\tau_2}|)\varphi^{\frac{1}{2}}(|B_{\tau_3}-B_{\tau_4}|)\}du.$$

where φ is a bounded, compactly supported function. The proof of Lemma 3.7 applies to $\varphi^{\frac{1}{2}}(|x|)$ replacing $R_p(x)$ in light of Remark 3.8. Since $\mathbb{E}\{\mathbb{V}_n^2\}\leq Ct^2$, in both cases we have $\mathbb{E}\{|X_n(t)-X_n(s)|^4\}\leq C(t-s)^2$.

In (3.22), when $d=1$, we choose $\beta=2,\delta=\frac{1}{2}$ while when $d\geq 2$, we choose $\beta=4,\delta=1$. The proof is complete. □

3.2 Homogenization when $d = 2$

When $x \in \mathbb{R}^d$, the equation we consider is of the form

$$\partial_t u_\varepsilon(t,x) = \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon\sqrt{|\log\varepsilon|}}V(\frac{x}{\varepsilon})u_\varepsilon(t,x) \tag{3.23}$$

with V Gaussian or Poissonian as chosen in Section 3.1 satisfying $\hat{R}(0) > 0$. The initial condition $u_\varepsilon(0,x) = f(x) \in \mathcal{C}_b(\mathbb{R}^d)$.

The following is the main result.

Theorem 3.14. *Let u_{hom} solve the equation*

$$\partial_t u_{hom}(t,x) = \frac{1}{2}\Delta u_{hom}(t,x) - \frac{\hat{R}(0)}{2\pi}u_{hom}(t,x) \tag{3.24}$$

with initial condition $u_{hom}(0,x) = f(x)$, then we have $u_\varepsilon(t,x) \to u_{hom}(t,x)$ in probability.

By Feynman-Kac formula, we have

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(\frac{i}{\varepsilon\sqrt{|\log\varepsilon|}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds)\}. \tag{3.25}$$

The stationarity of V and scaling property of Brownian motion implies $u_\varepsilon(t,x)$ is of the same distribution as

$$\tilde{u}_\varepsilon(t,x) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\int_0^{t/\varepsilon^2} V(B_s)ds)\}. \tag{3.26}$$

We first show $\mathbb{E}\{\tilde{u}_\varepsilon(t,x)\}$ converges as $\varepsilon \to 0$. By Theorem 3.1, we have

$$\frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \int_0^{t/\varepsilon^2} V(B_s)ds \Rightarrow \sqrt{\frac{\hat{R}(0)}{\pi}} W_t \tag{3.27}$$

as $\varepsilon \to 0$, where W is a standard Brownian motion. If we follow the proof, it is straightforward to check that

$$(\varepsilon B_{t/\varepsilon^2}, \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \int_0^{t/\varepsilon^2} V(B_s)ds) \Rightarrow (B_t, \sqrt{\frac{\hat{R}(0)}{\pi}} W_t) \tag{3.28}$$

with B, W independent Brownian motions. Actually, by computing the characteristic function we have

$$\mathbb{E}\mathbb{E}_B\{e^{i\theta_1 \cdot \varepsilon B_{t/\varepsilon^2} + i\theta_2 \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \int_0^{t/\varepsilon^2} V(B_s)ds}\} = \mathbb{E}_B\{e^{i\theta_1 \cdot \varepsilon B_{t/\varepsilon^2}} e^{F_\varepsilon(\theta_2, B)}\} \tag{3.29}$$

for some functional F_ε depending on the Brownian path. We have proved in Proposition 3.10 and 3.11 that

$$F_\varepsilon(\theta_2, B) \to -\frac{\hat{R}(0)}{2\pi} \theta_2^2 t \tag{3.30}$$

in probability as $\varepsilon \to 0$. Therefore, it is clear that

$$\mathbb{E}\mathbb{E}_B\{e^{i\theta_1 \cdot \varepsilon B_{t/\varepsilon^2} + i\theta_2 \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \int_0^{t/\varepsilon^2} V(B_s)ds}\} \to e^{-\frac{1}{2}|\theta_1|^2 t} e^{-\frac{\hat{R}(0)}{2\pi} \theta_2^2 t}, \tag{3.31}$$

which proves (3.28) and leads to

$$\mathbb{E}\{\tilde{u}_\varepsilon(t,x)\} \to \mathbb{E}_B\{f(x+B_t)\} e^{-\frac{\hat{R}(0)}{2\pi} t}. \tag{3.32}$$

Now we consider the convergence of $\mathbb{E}\{|\tilde{u}_\varepsilon(t,x)|^2\}$. By introducing independent Brownian motions B^1, B^2, we obtain

$$\mathbb{E}\{|\tilde{u}_\varepsilon(t,x)|^2\} = \mathbb{E}\mathbb{E}_B\{f(x+\varepsilon B^1_{t/\varepsilon^2})\bar{f}(x+\varepsilon B^1_{t/\varepsilon^2})e^{i\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\int_0^{t/\varepsilon^2}(V(B^1_s)-V(B^2_s))ds}\}. \tag{3.33}$$

The problem reduces to the weak convergence of $(\varepsilon B^1_{t/\varepsilon^2}, \varepsilon B^1_{t/\varepsilon^2}, \frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\int_0^{t/\varepsilon^2}(V(B^1_s)-V(B^2_s))ds)$. We first deal with the Poissonian case.

By the same calculation as in Section 3.1, we have

$$\mathbb{E}\{e^{i\theta\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\int_0^{t/\varepsilon^2}(V(B^1_s)-V(B^2_s))ds}\} = e^{\int_{\mathbb{R}^d}\sum_{k=2}^\infty \frac{1}{k!}(i\theta F_\varepsilon(y))^k dy}, \tag{3.34}$$

where $F_\varepsilon(y) := \frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\int_0^{t/\varepsilon^2}(\phi(B^1_s-y)-\phi(B^2_s-y))ds$. For $k\geq 3$, we have

$$\begin{aligned}|\int_{\mathbb{R}^d}\sum_{k=3}^\infty \frac{1}{k!}(i\theta F_\varepsilon(y))^k dy| \leq & \int_{\mathbb{R}^d}\sum_{k=3}^\infty \frac{1}{k!}|\theta|^k 2^{k-1}\left(\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\int_0^{t/\varepsilon^2}|\phi|(B^1_s-y)ds\right)^k dy \\ &+ \int_{\mathbb{R}^d}\sum_{k=3}^\infty \frac{1}{k!}|\theta|^k 2^{k-1}\left(\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\int_0^{t/\varepsilon^2}|\phi|(B^2_s-y)ds\right)^k dy,\end{aligned} \tag{3.35}$$

which converges to zero in probability by the proof of Proposition 3.11. For $k=2$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}F_\varepsilon(y)^2dy = & \sum_{i=1,2}\frac{\varepsilon^2}{|\log\varepsilon|}\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B^i_s-B^i_u)dsdu \\ & -2\frac{\varepsilon^2}{|\log\varepsilon|}\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B^1_s-B^2_u)dsdu.\end{aligned} \tag{3.36}$$

By the proof of Proposition 3.10, $\frac{\varepsilon^2}{|\log\varepsilon|}\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}R(B^i_s-B^i_u)dsdu \to \frac{\hat{R}(0)}{\pi}t$ in probability. In addition, we prove the following lemma, which implies $e^{\int_{\mathbb{R}^d}\sum_{k=2}^\infty\frac{1}{k!}(i\theta F_\varepsilon(y))^k dy} \to e^{-\frac{\hat{R}(0)}{\pi}t}$ in probability and thus

$$(\varepsilon B^1_{t/\varepsilon^2}, \varepsilon B^1_{t/\varepsilon^2}, \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \int_0^{t/\varepsilon^2} (V(B^1_s) - V(B^2_s))ds) \Rightarrow (B^1_t, B^2_t, \sqrt{\frac{2\hat{R}(0)}{\pi}} W_t) \tag{3.37}$$

for independent Brownian motions B^1, B^2, W.

Lemma 3.15. $\frac{\varepsilon^2}{|\log \varepsilon|} \int_0^{t/\varepsilon^2} \int_0^{t/\varepsilon^2} R(B^1_s - B^2_u) ds du \to 0$ *in probability.*

Proof. We can assume R is positive and compactly supported. By explicit calculation, we have

$$\begin{aligned}
&\mathbb{E}_B\{\frac{\varepsilon^2}{|\log \varepsilon|} \int_0^{t/\varepsilon^2} \int_0^{t/\varepsilon^2} R(B^1_s - B^2_u) ds du\} \\
=&\frac{\varepsilon^2}{|\log \varepsilon|} \int_0^{t/\varepsilon^2} \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^d} R(x) q_{s+u}(x) dx ds du \\
\le&\frac{\varepsilon^2}{|\log \varepsilon|} \int_0^{2t/\varepsilon^2} \int_{\mathbb{R}^d} R(x) s q_s(x) dx ds \\
=&\frac{\varepsilon^2}{|\log \varepsilon|} \int_0^{2t/\varepsilon^2} \int_{\mathbb{R}^d} R(x) \frac{1}{2\pi} e^{-\frac{|x|^2}{2s}} dx ds \lesssim \frac{1}{|\log \varepsilon|} \to 0.
\end{aligned} \tag{3.38}$$

□

Now we conclude the proof by noting that

$$\begin{aligned}
\mathbb{E}\{|\tilde{u}_\varepsilon(t,x)|^2\} &= \mathbb{E}\mathbb{E}_B\{f(x + \varepsilon B^1_{t/\varepsilon^2}) \bar{f}(x + \varepsilon B^1_{t/\varepsilon^2}) e^{i \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \int_0^{t/\varepsilon^2} (V(B^1_s) - V(B^2_s)) ds}\} \\
&\to \mathbb{E}\mathbb{E}_B\{f(x + B^1_t) \bar{f}(x + B^2_t)\} e^{-\frac{\hat{R}(0)}{\pi} t}.
\end{aligned} \tag{3.39}$$

3.3 Appendix

Lemma 3.16. *Assume $d=2, \alpha>2, c>0$. Then we have the following inequalities*

$$\int_{\mathbb{R}^2}\left(1\wedge\frac{1}{|\sqrt{n}y|^\alpha}\right)|\log|y||dy \lesssim \frac{1}{n}+\frac{1}{n}\log n, \tag{3.40}$$

$$\int_{\mathbb{R}^2}\left(1\wedge\frac{1}{|\sqrt{n}(x-y)|^\alpha}\right)|\log|y||dy \lesssim \frac{1}{n}+\frac{1}{n}|\log|x||+\frac{1}{n}\log n 1_{|x|<\frac{2}{\sqrt{n}}}, \tag{3.41}$$

$$\int_{|x-y|<\frac{c}{\sqrt{n}}}|\log|y||dy \lesssim \frac{1}{n}+\frac{1}{n}|\log|x||+\frac{1}{n}\log n 1_{|x|<\frac{2c}{\sqrt{n}}}, \tag{3.42}$$

$$\int_{|x-y|<\frac{c}{\sqrt{n}}}\left(1\wedge\frac{1}{|\sqrt{n}y|^\alpha}\right)|\log|y||dy \lesssim \frac{1}{n}+\frac{1}{n}|\log|x||+\frac{1}{n}\log n 1_{|x|<\frac{2c}{\sqrt{n}}}, \tag{3.43}$$

$$\begin{aligned}&\int_{\mathbb{R}^2}|\log|x-y||\left(1\wedge\frac{1}{|\sqrt{n}y|^\alpha}\right)|\log|y||dy\\ \lesssim&\frac{1}{n}+\frac{1}{n}\log n+\frac{1}{n}|\log|x||+\frac{1}{n}\log n|\log|x||+\frac{1}{n}(\log|x|)^2+\frac{1}{n}(\log n)^2 1_{|x|<\frac{1}{\sqrt{n}}}.\end{aligned} \tag{3.44}$$

Proof. For the first inequality, we have

$$\int_{\mathbb{R}^2}\left(1\wedge\frac{1}{|\sqrt{n}y|^\alpha}\right)|\log|y||dy=-\int_0^{\frac{1}{\sqrt{n}}} r\log r dr+\frac{1}{n^{\frac{\alpha}{2}}}\int_{\frac{1}{\sqrt{n}}}^\infty\frac{1}{r^{\alpha-1}}|\log r|dr,$$

and by integrations by parts, we have $-\int_0^{\frac{1}{\sqrt{n}}} r\log r dr\lesssim\frac{1}{n}\log n$ and $\frac{1}{n^{\frac{\alpha}{2}}}\int_{\frac{1}{\sqrt{n}}}^\infty\frac{1}{r^{\alpha-1}}|\log r|dr\lesssim \frac{1}{n}(1+\log n)$.

For the other inequalities, they are all in the form of convolutions and are proved in a similar way. We only present the proof for the second one. We have

$$\int_{\mathbb{R}^2}\left(1\wedge\frac{1}{|\sqrt{n}(x-y)|^\alpha}\right)|\log|y||dy=(I)+(II),$$

where

$$(I) = \int_{|x-y|<\frac{1}{\sqrt{n}}} |\log|y||dy,$$
$$(II) = \frac{1}{n^{\frac{\alpha}{2}}} \int_{|x-y|>\frac{1}{\sqrt{n}}} \frac{1}{|x-y|^{\alpha}} |\log|y||dy.$$

Let $\rho = |x|$, and define $B(x,r) = \{y : |x-y| \le r\}$, $(i) = \{y : |y| < |y-x|\}, (ii) = \{y : |y| \ge |y-x|\}$. We divide $\mathbb{R}^d$ into three disjoint parts, $A_1 = B(0,\rho)\bigcap(i)$, $A_2 = B(x,\rho)\bigcap(ii)$, and $A_3 = \mathbb{R}^d\backslash(A_1\bigcup A_2)$.

For (I), when $y \in A_1$, $|y-x| \ge \frac{\rho}{2}$, so $\rho \le \frac{2}{\sqrt{n}}$ and $\int_{A_1}|\log|y||dy \le \int_0^\rho |\log r|rdr = \rho^2(\frac{1}{4} - \frac{1}{2}\log\rho)$, so we have $\int_{A_1}|\log|y||dy \lesssim \frac{1}{n}(1+\log n)1_{\rho\le\frac{2}{\sqrt{n}}}$. When $y \in A_2$, $2\rho \ge |y| \ge \frac{\rho}{2}$, so $|\log|y|| \lesssim 1 + |\log\rho|$, thus $\int_{A_2}|\log|y||dy \lesssim \frac{1}{n}(1+|\log\rho|)$. When $y \in A_3$, $\rho \le |y| \le 2|y-x| \le \frac{2}{\sqrt{n}}$, so $\int_{A_3}|\log|y||dy \le \int_\rho^{\frac{2}{\sqrt{n}}} r|\log r|dr \lesssim \frac{1}{n}(1+\log n1_{\rho\le\frac{2}{\sqrt{n}}})$. Therefore, we have shown that

$$(I) \lesssim \frac{1}{n}(1+|\log\rho| + \log n1_{\rho\le\frac{2}{\sqrt{n}}}).$$

For (II), similarly, when $y \in A_1$, $|x-y| \ge \frac{\rho}{2}$, so if $\rho > 1$, $\frac{1}{n^{\frac{\alpha}{2}}}\int_{A_1}\frac{1}{|x-y|^\alpha}|\log|y||dy \lesssim \frac{1}{n^{\frac{\alpha}{2}}\rho^\alpha}\int_0^\rho r|\log r|dr \lesssim \frac{1}{n}$, else if $\rho \in (\frac{2}{\sqrt{n}}, 1]$, we have $\frac{1}{n^{\frac{\alpha}{2}}}\int_{A_1}\frac{1}{|x-y|^\alpha}|\log|y||dy \lesssim \frac{1}{n}(1+|\log\rho|)$, and for the last case $\rho \le \frac{2}{\sqrt{n}}$, we have $\frac{1}{n^{\frac{\alpha}{2}}}\int_{A_1}\frac{1}{|x-y|^\alpha}|\log|y||dy \lesssim \int_{|y|<\rho}|\log|y||dy \lesssim \frac{1}{n}(1+\log n1_{\rho\le\frac{2}{\sqrt{n}}})$. When $y \in A_2$, $|\log|y|| \lesssim 1+|\log\rho|$, so $\frac{1}{n^{\frac{\alpha}{2}}}\int_{A_2}\frac{1}{|x-y|^\alpha}|\log|y||dy \lesssim \frac{1}{n}(1+|\log\rho|)$. When $y \in A_3$, $|\log|y|| \lesssim 1+|\log|x-y||$, so we only need to estimate $\frac{1}{n^{\frac{\alpha}{2}}}\int_{\rho\vee\frac{1}{\sqrt{n}}}^\infty \frac{1}{r^{\alpha-1}}(1+|\log r|)dr$. Following the same discussion as in A_1, and considering the different cases $\rho > 1$, $1 \ge \rho > \frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{n}} \ge \rho$, we can show $\frac{1}{n^{\frac{\alpha}{2}}}\int_{\rho\vee\frac{1}{\sqrt{n}}}^\infty \frac{1}{r^{\alpha-1}}(1+|\log r|)dr \lesssim \frac{1}{n}(1+|\log\rho|+$

$\log n 1_{\rho\le\frac{1}{\sqrt{n}}})$. Therefore, we have obtained that

$$(II) \lesssim \frac{1}{n}(1 + |\log \rho| + \log n 1_{\rho\le\frac{2}{\sqrt{n}}}).$$

The proof is complete. □

Lemma 3.17. *Let $V(x)$ be a mean zero stationary random field with $\mathbb{E}\{V(x)^6\} < \infty$ satisfying the mixing property (3.3) with positive, non-increasing mixing coefficient φ. Then we have*

$$|\mathbb{E}\{V(x_1)V(x_2)V(x_3)V(x_4)\}| \le C \sum_{\{y_k\}=\{x_k\}} \varphi^{\frac{1}{2}}(|y_1 - y_2|)\varphi^{\frac{1}{2}}(|y_3 - y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}. \quad (3.45)$$

Thus, (3.45) holds for the Poissonian potential $V(x) = \int_{\mathbb{R}^d} \phi(x-y)\omega(dy) - c_p$ when ϕ is continuous and compactly supported, and the mixing coefficient φ could be chosen as some continuous, compactly supported function as well.

Proof. Let y_1 and y_2 be two points in $\{x_k\}_{1\le k\le 4}$ such that $d(y_1,y_2) \ge d(x_i,x_j)$ for all $1 \le i,j \le 4$ and such that $d(y_1,\{y_3,y_4\}) \le d(y_2,\{y_3,y_4\})$, where $\{y_k\}_{1\le k\le 4} = \{x_k\}_{1\le k\le 4}$. We assume $d(y_3,y_1) \le d(y_4,y_1)$. Therefore by (3.3), we have

$$\mathcal{E} := |\mathbb{E}\{V(x_1)V(x_2)V(x_3)V(x_4)\}| \lesssim \varphi(2|y_1-y_3|)(\mathbb{E}\{V(y_1)^2\})^{\frac{1}{2}}(\mathbb{E}\{(V(y_2)V(y_3)V(y_4))^2\})^{\frac{1}{2}}.$$

The last two terms are bounded by $\mathbb{E}\{V(x)^6\}^{\frac{1}{6}}$ and $\mathbb{E}\{V(x)^6\}^{\frac{1}{2}}$ respectively. Because $\varphi(r)$ is decaying in $(0,\infty)$, we have $\mathcal{E} \lesssim \varphi(|y_1-y_3|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. On the other hand, if y_4 is (one of) the closest point(s) to y_2, the same argument shows that $\mathcal{E} \lesssim \varphi(|y_2-y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. Otherwise, y_3 is the closest point to y_2, and we find $\mathcal{E} \lesssim \varphi(2|y_2-y_3|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. However, by construction, we have

$$|y_2 - y_4| \leq |y_1 - y_2| \leq |y_1 - y_3| + |y_2 - y_3| \leq 2|y_2 - y_3|,$$

so we still have $\mathcal{E} \lesssim \varphi(|y_2 - y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. To summarize, we have

$$\mathcal{E} \lesssim \varphi^{\frac{1}{2}}(|y_1 - y_3|)\varphi^{\frac{1}{2}}(|y_2 - y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}},$$

and this completes the proof. □

Chapter 4

Convergence to Stochastic Partial Differential Equations

For the homogenization result (Theorem 2.3) to hold, a key assumption is the finiteness of the asymptotic variance, i.e., Assumption 2.1, or equivalently an integrability of $\hat{R}(\xi)|\xi|^{-2}$. Since $\int_{\mathbb{R}^d} \hat{R}(\xi)|\xi|^{-2}d\xi \sim \int_{\mathbb{R}^d} R(x)|x|^{2-d}dx$, we observe that as long as $R(x)$ decays slower than $|x|^{-2}$ in the infinity, Assumption 2.1 fails to hold. In this chapter, we will show that for a large class of random fields constructed as functionals of Gaussian fields, we instead obtain the result of convergence to SPDE when homogenization does not occur.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The following is our assumption on random coefficient $V(x) = V(x, \omega)$ with $\omega \in \Omega$ labeling the particular realization.

Assumption 4.1. $V(x) = \Phi(g(x))$, where

- $g(x)$ is a stationary Gaussian field with zero mean and unit variance. The auto-covariance function $R_g(x) = \mathbb{E}\{g(0)g(x)\}$ satisfies that $|R_g(x)| \lesssim \prod_{i=1}^d \min(1, |x_i|^{-\alpha_i})$

with $\alpha_i \in (0,1)$ and $R_g(x) \sim c_d \prod_{i=1}^d |x_i|^{-\alpha_i}$ as $\min_{i=1,\ldots,d} |x_i| \to \infty$. $\alpha := \sum_{i=1}^d \alpha_i \in (0,2)$.

- Φ is a deterministic function with Hermite rank 1, i.e., $\int_{\mathbb{R}} \Phi^2(x) \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx < \infty$ and if we define $V_k = \mathbb{E}\{\Phi(g) H_k(g)\}$ with $H_k(x) = (-1)^n \exp(x^2/2) \frac{d^n}{dx^n} \exp(-x^2/2)$ the $k-$th Hermite polynomial, then $V_0 = 0, V_1 \neq 0$.

We will see later that $R(x) = \mathbb{E}\{V(0)V(x)\} \sim V_1^2 c_d \prod_{i=1}^d |x_i|^{-\alpha_i}$, and since $\alpha = \sum_{i=1}^d \alpha_i < 2$, $R(x)|x|^{2-d}$ is not integrable, hence homogenization does not occur. We consider the equation when $d \geq 3$

$$\partial_t u_\varepsilon(t,x,\omega) = \frac{1}{2} \Delta u_\varepsilon(t,x,\omega) + i \frac{1}{\varepsilon^{\alpha/2}} V(\frac{x}{\varepsilon}, \omega) u_\varepsilon(t,x,\omega), \tag{4.1}$$

with initial condition $u_\varepsilon(0,x,\omega) = f(x)$ for $f \in \mathcal{C}_b(\mathbb{R}^d)$, i.e., in (1.7), we choose $\gamma = \frac{\alpha}{2} < 1$. The following is the result of convergence to SPDE.

Theorem 4.2. *Under Assumption 4.1, we have for fixed (t,x), $u_\varepsilon(t,x) \to u_{spde}(t,x)$ in distribution as $\varepsilon \to 0$, with u_{spde} solving the SPDE with multiplicative Gaussian noise:*

$$\partial_t u_{spde} = \frac{1}{2} \Delta u_{spde} + i V_1 \sqrt{c_d} \dot{W} u_{spde}, \tag{4.2}$$

where $\dot{W}(x)$ is a generalized Gaussian random field with covariance function $\mathbb{E}\{\dot{W}(x)\dot{W}(y)\} = \prod_{i=1}^d |x_i - y_i|^{-\alpha_i}$.

Remark 4.3. The proof of Theorem 4.2 also holds for $d = 1, 2$. When $d = 2$, since $\alpha_1, \alpha_2 \in (0,1)$, $\alpha = \alpha_1 + \alpha_2 \in (0,2)$ is automatically satisfied. When $d = 1$, we have $\alpha = \alpha_1 \in (0,1)$.

First, we recall that the $n-$th order Hermite polynomial is defined as

$$H_n(x) = (-1)^n \exp(\frac{x^2}{2})\frac{d^n}{dx^n}\exp(-\frac{x^2}{2}), \tag{4.3}$$

and it has the property that

$$\mathbb{E}\{H_m(X)H_n(Y)\} = \begin{cases} n!(\mathbb{E}\{XY\})^n & m = n, \\ 0 & m \neq n, \end{cases} \tag{4.4}$$

if $X, Y \sim N(0,1)$ and are jointly Gaussian.

Under Assumption 4.1, we can expand V in Hermite polynomials

$$V(x) = \Phi(g(x)) = \sum_{n=0}^{\infty} \frac{V_n}{n!} H_n(g(x)), \tag{4.5}$$

with $V_n = \mathbb{E}\{H_n(g(x))\Phi(g(x))\}$. By the assumption $V_0 = 0, V_1 \neq 0$, we have

$$\begin{aligned} R(x) = \mathbb{E}\{V(0)V(x)\} =& \mathbb{E}\{\Phi(g(0))\Phi(g(x))\} = \sum_{n=0}^{\infty} \frac{V_n^2}{(n!)^2}\mathbb{E}\{H_n(g(0))H_n(g(x))\} \\ =& \sum_{n=0}^{\infty} \frac{V_n^2}{n!} R_g(x)^n = V_1^2 R_g(x) + \sum_{n=2}^{\infty} \frac{V_n^2}{n!} R_g(x)^n. \end{aligned} \tag{4.6}$$

Since $\sum_{n=0}^{\infty} \frac{V_n^2}{n!} < \infty$, $R(x) \sim V_1^2 R_g(x)$ as $|x| \to \infty$. Since $R_g(x) \sim c_d \prod_{i=1}^d |x_i|^{-\alpha_i}$, we have $R(x) \sim V_1^2 c_d \prod_{i=1}^d |x_i|^{-\alpha_i}$ as $\min_{i=1,\ldots,d} |x_i| \to \infty$.

The assumption of $V_1 \neq 0$ is crucial for the appearance of Gaussian noise in the limiting equation, and it turns out that by this assumption we can reduce the possibly non-Gaussian case to Gaussian case, namely $V(x) = g(x)$, so conditioning on B, $X_\varepsilon(t) := \frac{1}{\varepsilon^{\alpha/2}} \int_0^t V(\frac{B_s}{\varepsilon})ds$ is Gaussian, and we can prove its weak convergence by proving convergence of the conditional mean and variance. Before that, following [18] we define the Brownian motion in Gaussian noise, then the solution to the limiting SPDE (4.2) is constructed by a Feynman-Kac formula.

4.1 Brownian motion in Gaussian noise

We define the formally-written random variable $\int_0^t \dot{W}(B_s)ds = \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s)W(dx)ds$, where $W(dx)$ is the generalized Gaussian random field independent of Brownian motion B_t. We use $\mathbb{E}$ to denote the expectation with respect to $W(dx)$, and assume that the covariance function $\mathbb{E}\{W(dx)W(dy)\} = \frac{1}{\prod_{i=1}^d |x_i - y_i|^{\alpha_i}} dxdy$.

Proposition 4.4. *Assume $\sum_{i=1}^d \alpha_i < 2$ and define $Y_\varepsilon(t) = \int_0^t \int_{\mathbb{R}^d} q_\varepsilon(x - B_s)W(dx)ds$. Then $Y_\varepsilon(t)$ converges in L^2 as $\varepsilon \to 0$ to some random variable $Y(t)$, denoted as*

$$Y(t) = \int_0^t \dot{W}(B_s)ds = \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s)W(dx)ds.$$

When conditioning on B, then Y_t is a Gaussian random variable with zero mean and variance

$$\mathbb{E}\{Y(t)^2\} = \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} dsdu, \tag{4.7}$$

where $B_i(s)$ denotes the $i-th$ component of B_s.

Proof. We first point out that the RHS of (4.7) is almost surely finite, and this comes from the fact that $\sum_{i=1}^d \alpha_i < 2$ and

$$\mathbb{E}_B\{\int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} dsdu\} = \int_0^t \int_0^t \frac{1}{|s-u|^{\sum_{i=1}^d \alpha_i/2}} dsdu \prod_{i=1}^d \int_{\mathbb{R}^d} |x|^{-\alpha_i} q_1(x)dx, \tag{4.8}$$

Secondly, we calculate

$$\mathbb{E}\mathbb{E}_B\{Y_\varepsilon^2(t)\} = \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} \mathbb{E}_B\{q_\varepsilon(x-B_s)q_\varepsilon(y-B_u)\} \frac{1}{\prod_{i=1}^d |x_i-y_i|^{\alpha_i}} dxdydsdu. \tag{4.9}$$

By Lemma 4.12, $\int_{\mathbb{R}^{2d}} q_\varepsilon(x-B_s)q_\varepsilon(y-B_u)\frac{1}{\prod_{i=1}^d |x_i-y_i|^{\alpha_i}}dxdy \to \frac{1}{\prod_{i=1}^d |B_i(s)-B_i(u)|^{\alpha_i}}$ as $\varepsilon \to 0$. By Lemma 4.13 and the dominated convergence theorem, we have the convergence

$$\mathbb{E}\mathbb{E}_B\{Y_\varepsilon^2(t)\} \to \int_0^t \int_0^t \mathbb{E}_B\{\frac{1}{\prod_{i=1}^d |B_i(s)-B_i(u)|^{\alpha_i}}\}dsdu. \tag{4.10}$$

Similarly, we can show $\mathbb{E}\mathbb{E}_B\{Y_{\varepsilon_1}(t)Y_{\varepsilon_2}(t)\} \to \int_0^t\int_0^t \mathbb{E}_B\{\frac{1}{\prod_{i=1}^d |B_i(s)-B_i(u)|^{\alpha_i}}\}dsdu$ as $\varepsilon_1, \varepsilon_2 \to 0$. Thus, we have shown that $\{Y_\varepsilon(t)\}$ is a Cauchy sequence in L^2, since

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbb{E}\mathbb{E}_B\{(Y_{\varepsilon_1}(t)-Y_{\varepsilon_2}(t))^2\} = 0.$$

The limit is then denoted as $Y(t) = \int_0^t \dot{W}(B_s)ds = \int_0^t \int_{\mathbb{R}^d} \delta(x-B_s)W(dx)ds$.

Next, we consider the conditional distribution. Since $Y_\varepsilon(t) \to Y(t)$ in L^2, there exists a subsequence ε_k such that $Y_{\varepsilon_k}(t) \to Y(t)$ almost surely. Note that $W(dx)$ and B_t are independent, so the probability space is the product space. Then we know that conditioning on the Brownian motion, $Y_{\varepsilon_k}(t) \to Y(t)$ almost surely as $k \to \infty$, and this leads to convergence in distribution. Given B, $Y_\varepsilon(t)$ is Gaussian with variance

$$\begin{aligned}\mathbb{E}\{Y_\varepsilon^2(t)\} &= \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} q_\varepsilon(x-B_s)q_\varepsilon(y-B_u)\frac{1}{\prod_{i=1}^d |x_i-y_i|^{\alpha_i}}dxdydsdu \\ &\to \int_0^t\int_0^t \frac{1}{\prod_{i=1}^d |B_i(s)-B_i(u)|^{\alpha_i}}dsdu.\end{aligned} \tag{4.11}$$

The proof is complete. □

Remark 4.5. If we define $Y^i(t) = \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s^i) W(dx) ds$ for independent Brownian motions B^1, B^2, the same proof implies that $Y^1(t), Y^2(t)$ are jointly Gaussian with covariance function given by $\mathbb{E}\{Y^1(t)Y^2(t)\} = \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i^1(s) - B_i^2(u)|^{\alpha_i}} ds du$ when conditioning on B^1, B^2.

Remark 4.6. From the proof of Proposition 4.4, we see that the distribution of $\int_0^t \int_{\mathbb{R}^d} \delta(x - B_s) W(dx) ds$ does not depend on the starting point of Brownian motion.

With random variable $\int_0^t \int_{\mathbb{R}^d} \delta(x - B_s) W(dx) ds$, we can formally write the solution to the SPDE

$$\partial_t u = \frac{1}{2} \Delta u + i \dot{W} u \tag{4.12}$$

with initial condition $u(0, x) = f(x)$ by Feynman-Kac formula as

$$u(t, x) = \mathbb{E}_B \{ f(x + B_t) \exp(i \int_0^t \int_{\mathbb{R}^d} \delta(y - x - B_s) W(dy) ds) \}. \tag{4.13}$$

We point that the $u(t, x)$ defined as above coincides with the usual definition of weak solution to SPDE (4.12):

Definition 4.7. A random field $u(t, x)$ is a weak solution to (4.12) if for any $\mathcal{C}^\infty$ function ϕ with compact support we have

$$\int_{\mathbb{R}^d} u(t, x)\phi(x) dx = \int_{\mathbb{R}^d} f(x)\phi(x) dx + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u(s, x) \Delta\phi(x) dx ds + i \int_{\mathbb{R}^d} \int_0^t u(s, x)\phi(x) ds W(dx). \tag{4.14}$$

Proposition 4.8. *If* $\sum_{i=1}^d \alpha_i < 2$, $u(t, x)$ *is a weak solution to* (4.12).

The proof is a direct adaption of Theorem 4.3 in [18], and we do not present it here.

4.2 Convergence of moments

First we reduce $V(x) = \Phi(g(x))$ to the Gaussian case by the following lemma:

Lemma 4.9. *In the annealed sense,* $Y_\varepsilon(t) := \frac{1}{\varepsilon^{\alpha/2}} \int_0^t (\Phi(g(\frac{B_s}{\varepsilon})) - V_1 g(\frac{B_s}{\varepsilon}))ds \to 0$ *in probability as* $\varepsilon \to 0$.

Proof. Since $\Phi(g) - V_1 g = \sum_{n=2}^\infty \frac{V_n}{n!} H_n(g)$ and $\sum_{n=0}^\infty \frac{V_n^2}{n!} < \infty$, we have conditioning on B that

$$\begin{aligned}\mathbb{E}\{Y_\varepsilon(t)^2\} =& \frac{1}{\varepsilon^\alpha} \int_0^t \int_0^t \sum_{n=2}^\infty \frac{V_n^2}{n!} R_g(\frac{B_s - B_u}{\varepsilon})^n dsdu \\ \leq& \frac{C}{\varepsilon^\alpha} \int_0^t \int_0^t R_g(\frac{B_s - B_u}{\varepsilon})^2 dsdu\end{aligned} \tag{4.15}$$

for some constant C. Since R_g is bounded and satisfies $|R_g(x)| \lesssim \prod_{i=1}^d |x_i|^{-\alpha_i}$, we have

$$\begin{aligned}\mathbb{E}\{Y_\varepsilon(t)^2\} \leq& C \sup_{|x| \geq M} |R_g(x)| \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} 1_{|B_s - B_u| > M\varepsilon} dsdu \\ &+ \frac{C}{\varepsilon^\alpha} \int_0^t \int_0^t 1_{|B_s - B_u| \leq M\varepsilon} dsdu,\end{aligned} \tag{4.16}$$

which leads to

$$\mathbb{E}\mathbb{E}_B\{Y_\varepsilon(t)^2\} \leq C \sup_{|x| \geq M} |R_g(x)| + \frac{C}{\varepsilon^\alpha} \int_0^t \int_0^t \mathbb{E}_B\{1_{|B_s - B_u| \leq M\varepsilon}\} dsdu. \tag{4.17}$$

By Lemma 4.14, first let $\varepsilon \to 0$, then $M \to \infty$, the proof is complete. □

Now we can prove the weak convergence of $X_\varepsilon(t) := \frac{1}{\varepsilon^{\frac{\alpha}{2}}} \int_0^t V(\frac{B_s}{\varepsilon}) ds$.

Proposition 4.10. *For fixed* $t > 0$, *in the annealed sense* $X_\varepsilon(t) \Rightarrow V_1 \sqrt{c_d} \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s) W(dx) ds$ *as* $\varepsilon \to 0$.

Proof. By writing $X_\varepsilon(t) = \frac{1}{\varepsilon^{\alpha/2}}\int_0^t(\Phi(g(\frac{B_s}{\varepsilon})) - V_1 g(\frac{B_s}{\varepsilon}))ds + \frac{1}{\varepsilon^{\alpha/2}}\int_0^t V_1 g(\frac{B_s}{\varepsilon})ds$ and Lemma 4.9, we only need to show the weak convergence of $\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V_1 g(\frac{B_s}{\varepsilon})ds$.

By conditioning on B, we calculate the characteristic function

$$\mathbb{E}\{\exp(i\theta\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V_1 g(\frac{B_s}{\varepsilon})ds)\} = \exp(-\frac{V_1^2\theta^2}{2\varepsilon^\alpha}\int_0^t\int_0^t R_g(\frac{B_s - B_u}{\varepsilon})dsdu), \tag{4.18}$$

and since $\frac{1}{\varepsilon^\alpha}\int_0^t\int_0^t R_g(\frac{B_s-B_u}{\varepsilon})dsdu \to c_d\int_0^t\int_0^t \frac{1}{\prod_{i=1}^d |B_i(s)-B_i(u)|^{\alpha_i}}dsdu$ almost surely, we only need to apply the dominated convergence theorem to derive

$$\begin{aligned}\mathbb{E}\mathbb{E}_B\{\exp(i\theta\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V_1 g(\frac{B_s}{\varepsilon})ds)\} \to & \mathbb{E}_B\{\exp(-\frac{1}{2}\theta^2 V_1^2 c_d\int_0^t\int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}}dsdu)\} \\ = & \mathbb{E}\mathbb{E}_B\{\exp(i\theta V_1\sqrt{c_d}\int_0^t\int_{\mathbb{R}^d}\delta(x - B_s)W(dx)ds)\}\end{aligned} \tag{4.19}$$

as $\varepsilon \to 0$. □

Remark 4.11. Proposition 4.10 shows that as $\varepsilon \to 0$, for the Brownian motion in random scenery $\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\int_0^t V(\frac{B_s}{\varepsilon})ds$, its annealed weak convergence limit is to replace $\frac{1}{\varepsilon^{\frac{\alpha}{2}}}V(\frac{x}{\varepsilon})$ with the corresponding Gaussian noise $\dot{W}(x)$. This is different from the short-range-correlated case when we have an invariance principle with a Brownian motion as the limit. We also emphasize that the weak convergence here is in the annealed sense, i.e., in the product probability space. We do need the averaging from the random potential here. Actually, if V is purely Gaussian, we will have a weak convergence for fixed Brownian path.

Now we are ready to prove the main theorem.

Proof of theorem 4.2. For fixed (t,x), we let

$$Z_\varepsilon := u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds)\}, \quad (4.20)$$

$$Z_0 := u_0(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(iV_1\sqrt{c_d}\int_0^t\int_{\mathbb{R}^d}\delta(y-x-B_s)W(dy)ds)\}, \quad (4.21)$$

and claim that $\forall m,n\in\mathbb{N}$, $\mathbb{E}\{Z_\varepsilon^m\overline{Z_\varepsilon^n}\}\to\mathbb{E}\{Z_0^m\overline{Z_0^n}\}$.

Actually, we have

$$\mathbb{E}\{Z_\varepsilon^m\overline{Z_\varepsilon^n}\} = \mathbb{E}\mathbb{E}_B\{\prod_{j=1}^m f(x+B_t^j)\prod_{j=m+1}^{m+n}\overline{f(x+B_t^j)}\exp(\frac{i}{\varepsilon^{\alpha/2}}\int_0^t(\sum_{j=1}^m V(\frac{x+B_s^j}{\varepsilon})-\sum_{j=m+1}^{m+n}V(\frac{x+B_s^j}{\varepsilon}))ds)\}, \quad (4.22)$$

where $B_t^j, j=1,\ldots,N=m+n$ are independent Brownian motions. Since $V(x)$ is stationary and all relevant functions are bounded and continuous, to prove the convergence of $\mathbb{E}\{Z_\varepsilon^m\overline{Z_\varepsilon^n}\}$, we only need to prove the annealed weak convergence of

$$W_\varepsilon := \sum_{j=1}^N \alpha_j B_t^j + \sum_{j=1}^N \beta_j\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V(\frac{B_s^j}{\varepsilon})ds \quad (4.23)$$

for $\alpha_j,\beta_j\in\mathbb{R}$. We write $W_\varepsilon = (i)+(ii)+(iii)$ with

$$(i) = \sum_{j=1}^N \alpha_j B_t^j, \quad (4.24)$$

$$(ii) = \sum_{j=1}^N \beta_j\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V_1 g(\frac{B_s^j}{\varepsilon})ds, \quad (4.25)$$

$$(iii) = \sum_{j=1}^N \beta_j\frac{1}{\varepsilon^{\alpha/2}}\int_0^t(\Phi(g(\frac{B_s^j}{\varepsilon}))-V_1 g(\frac{B_s^j}{\varepsilon}))ds, \quad (4.26)$$

$(iii) \to 0$ in probability by Lemma 4.9, and for $(i) + (ii)$, we calculate

$$\begin{aligned}&\mathbb{E}\mathbb{E}_B\{\exp(i\theta_1(i) + i\theta_2(ii))\}\\ =&\mathbb{E}_B\{\exp(i\theta_1 \sum_{j=1}^{N} \alpha_j B_t^j) \exp(-\frac{1}{2} V_1^2 \theta_2^2 \sum_{i,j=1}^{N} \beta_i \beta_j \frac{1}{\varepsilon^\alpha} \int_0^t \int_0^t R_g(\frac{B_s^i - B_u^j}{\varepsilon}) dsdu)\},\end{aligned} \tag{4.27}$$

and by the same proof as in Proposition 4.10, we have

$$\frac{1}{\varepsilon^\alpha} \int_0^t \int_0^t R_g(\frac{B_s^i - B_u^j}{\varepsilon}) dsdu \to \int_0^t \int_0^t \frac{c_d}{\prod_{k=1}^d |B_k^i(s) - B_k^j(u)|^{\alpha_k}} dsdu \tag{4.28}$$

almost surely. Therefore, we see that

$$(i) + (ii) \Rightarrow \sum_{j=1}^{N} \alpha_j B_t^j + V_1 \sqrt{c_d} \sum_{j=1}^{N} \beta_j \int_0^t \int_{\mathbb{R}^d} \delta(y - x - B_s^j) W(dy) ds \tag{4.29}$$

in distribution in light of Remark 4.5, so

$$W_\varepsilon \Rightarrow \sum_{j=1}^{N} \alpha_j B_t^j + V_1 \sqrt{c_d} \sum_{j=1}^{N} \beta_j \int_0^t \int_{\mathbb{R}^d} \delta(y - x - B_s^j) W(dy) ds$$

in distribution. Thus the claim is proved.

Note that $|Z_\varepsilon|, |Z_0|$ are uniformly bounded, if we let $Z_\varepsilon = Z_{\varepsilon,1} + iZ_{\varepsilon,2}, Z_0 = Z_{0,1} + iZ_{0,2}$, the corresponding real and imaginary parts are uniformly bounded as well. From the fact that $\mathbb{E}\{Z_\varepsilon^m \overline{Z_\varepsilon^n}\} \to \mathbb{E}\{Z_0^m \overline{Z_0^n}\}$, we know $\forall m,n \in \mathbb{N}$, $\mathbb{E}\{Z_{\varepsilon,1}^m Z_{\varepsilon,2}^n\} \to \mathbb{E}\{Z_{0,1}^m Z_{0,2}^n\}$. So

$$\begin{aligned}\mathbb{E}\{\exp(i\theta_1 Z_{\varepsilon,1} + i\theta_2 Z_{\varepsilon,2})\} =& \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{E}\{(i\theta_1 Z_{\varepsilon,1} + i\theta_2 Z_{\varepsilon,2})^k\} \\ \to& \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{E}\{(i\theta_1 Z_{0,1} + i\theta_2 Z_{0,2})^k\} = \mathbb{E}\{\exp(i\theta_1 Z_{0,1} + i\theta_2 Z_{0,2})\},\end{aligned} \tag{4.30}$$

which completes the proof. □

4.3 Appendix

Lemma 4.12. *When $\alpha \in (0,1)$, $\int_{\mathbb{R}^2} q_\varepsilon(x) q_\varepsilon(y) \frac{1}{|z+x-y|^\alpha} dxdy \to \frac{1}{|z|^\alpha}$ as $\varepsilon \to 0$ for $z \neq 0$.*

Proof. By change of variables, we write

$$\begin{aligned}\int_{\mathbb{R}^2} q_\varepsilon(x) q_\varepsilon(y) \frac{1}{|z+x-y|^\alpha} dxdy =& \int_{\mathbb{R}^2} q_\varepsilon(w+y-z) q_\varepsilon(y) \frac{1}{|w|^\alpha} dydw \\ =& \left(\int_{|w|<\frac{|z|}{2}} + \int_{|w|>\frac{|z|}{2}} \right) q_\varepsilon(w+y-z) q_\varepsilon(y) \frac{1}{|w|^\alpha} dydw \\ =& (i) + (ii),\end{aligned} \tag{4.31}$$

and since

$$(ii) = \int_{|\sqrt{\varepsilon} w + z| > \frac{|z|}{2}} q(w+y) q(y) \frac{1}{|\sqrt{\varepsilon} w + z|^\alpha} dydw, \tag{4.32}$$

by the dominated convergence theorem, we have $(ii) \to \frac{1}{|z|^\alpha}$ as $\varepsilon \to 0$. For (i), we write

$$(i) = \left(\int_{|w|<\frac{|z|}{2}, |y|>\frac{|z|}{4}} + \int_{|w|<\frac{|z|}{2}, |y|<\frac{|z|}{4}} \right) q_\varepsilon(w+y-z) q_\varepsilon(y) \frac{1}{|w|^\alpha} dydw. \tag{4.33}$$

For the first term, use $q_\varepsilon(|z|/4)$ to bound $q_\varepsilon(y)$, then integrate in y, w; for the second term, use $q_\varepsilon(|z|/4)$ to bound $q_\varepsilon(w+y-z)$, then integrate in y,w. Since $q_\varepsilon(|z|/4)\to 0$ as $\varepsilon\to 0$, we have $(i)\to 0$. The proof is complete. $\square$

Lemma 4.13. *Assume $\alpha\in(0,1)$, then $\int_{\mathbb{R}^2} q_{\varepsilon_1}(x_1+y_1)q_{\varepsilon_2}(x_2+y_2)|y_1-y_2|^{-\alpha}dy_1dy_2 \le C|x_1-x_2|^{-\alpha}$ for some uniform constant C.*

Proof. See Lemma A.2. in [18]. $\square$

Lemma 4.14. *When $d\ge 3$ and $\alpha\in(0,2)$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\alpha}\int_0^t\int_0^t \mathbb{P}(|B_s-B_u|\le\varepsilon)dsdu=0. \tag{4.34}$$

Proof. By explicit calculation, we have

$$\begin{aligned}
&\frac{1}{\varepsilon^\alpha}\int_0^t\int_0^t \mathbb{P}(|B_s-B_u|<\varepsilon)dsdu\\
=&\frac{1}{(\pi)^{\frac{d}{2}}\varepsilon^\alpha}\int_0^t\int_{|x|<\varepsilon}\int_{\frac{|x|^2}{2s}}^\infty \lambda^{\frac{d}{2}-2}e^{-\lambda}\frac{1}{|x|^{d-2}}d\lambda dxds\\
=&\frac{1}{(\pi)^{\frac{d}{2}}\varepsilon^\alpha}\int_0^\infty\int_{\mathbb{R}^d}\int_0^\infty 1_{|x|<\varepsilon}1_{|x|^2<2\lambda s}1_{s<t}\lambda^{\frac{d}{2}-2}e^{-\lambda}\frac{1}{|x|^{d-2}}d\lambda dxds\\
=&\frac{1}{(\pi)^{\frac{d}{2}}\varepsilon^\alpha}\int_0^\infty d\lambda\int \lambda^{\frac{d}{2}-2}e^{-\lambda}\left(\lambda s 1_{\lambda<\frac{\varepsilon^2}{2s}}+\frac{1}{2}\varepsilon^2 1_{\lambda>\frac{\varepsilon^2}{2s}}\right)1_{s<t}ds\\
=&\frac{1}{(\pi)^{\frac{d}{2}}\varepsilon^\alpha}\int_0^\infty d\lambda\lambda^{\frac{d}{2}-2}e^{-\lambda}\left(\frac{\lambda t^2}{2}1_{\frac{\varepsilon^2}{2\lambda}>t}+\frac{\varepsilon^2 t}{2}1_{\frac{\varepsilon^2}{2\lambda}<t}-\frac{\varepsilon^4}{8\lambda}1_{\frac{\varepsilon^2}{2\lambda}<t}\right)=(i)+(ii)+(iii).
\end{aligned} \tag{4.35}$$

We check that $(i)\sim\varepsilon^{d-\alpha}$, and $(ii)\sim\varepsilon^{2-\alpha}$, $(iii)\sim\varepsilon^{4-\alpha}+\varepsilon^{d-\alpha}$, so the proof is complete. $\square$

Chapter 5

Time-dependent Potentials: $d \geq 3$

The equation we consider in this chapter is of the form

$$\partial_t u_\varepsilon(t,x) = \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon^\gamma}V(\frac{t}{\varepsilon^\alpha}, \frac{x}{\varepsilon})u_\varepsilon(t,x) \tag{5.1}$$

with initial condition $u_\varepsilon(0,x) = f(x) \in \mathcal{C}_b(\mathbb{R}^d)$ and some $\gamma, \alpha > 0$. The size of the potential is chosen large enough to produce some non-trivial effects on the asymptotic limit. By a Feynman-Kac formula, a key object to analyze is the Brownian motion in tempo-spatial random scenery, i.e., the random process of the form $\int_0^t V_\varepsilon(s, B_s)ds$. If $V_\varepsilon(t,x) \sim V(t/\varepsilon^\alpha, x/\varepsilon)$, by a simple change of variables, we observe a threshold of $\alpha = 2$, which separates the effects of the random mixings generated by temporal and spatial variables. In other words, depending on whether $\alpha > 2$ or $\alpha < 2$, the averaging of $\int_0^t V_\varepsilon(s, B_s)ds$ is induced by the temporal or spatial mixing of V, respectively. As a result, the ways we prove weak convergence of $\int_0^t V_\varepsilon(s, B_s)ds$ are forced to be different correspondingly. When $\alpha > 2$, it is a standard proof of central limit theorem for functions of mixing processes [6, Chapter 4] after freezing the Brownian motion. When $\alpha \leq 2$, the spatial mixing dominates. We make use of the Brownian motion again by the Kipnis-Varadhan's method, i.e., constructing the corrector function and applying martingale decomposition. When $\alpha = 2$, a ergodicity suffices to pass to the limit; when $\alpha < 2$, a quantitative martingale central limit theorem is applied. As $\alpha \to \infty$, the spatial mixing tends to zero, so heuristically the Brownian motion remains in the weak convergence limit of $\int_0^t V_\varepsilon(s, B_s)ds$, leading to a stochastic equation.

Before presenting the main result, we make some assumptions on $V(t,x)$. It is similar with the setup of Chapter 2 except that we have a time parameter to deal with.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a random medium associated with a group of measure-preserving, ergodic transformations $\{\tau_{(t,x)}, t \in \mathbb{R}, x \in \mathbb{R}^d\}$. Let $\mathbb{V} \in L^2(\Omega)$ with $\int_\Omega \mathbb{V}(\omega)\mathbb{P}(d\omega) = 0$. Define $V(t,x,\omega) = \mathbb{V}(\tau_{(-t,-x)}\omega)$. The inner product and norm of $L^2(\Omega)$ are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively.

By defining $T_{(t,x)}$ on $L^2(\Omega)$ as $(T_{(t,x)}f)(\omega) = f(\tau_{(-t,-x)}\omega)$ and assuming it is strongly continuous in $L^2(\Omega)$, we obtain the spectral resolution

$$T_{(t,x)} = \int_{\mathbb{R}^d} e^{i\xi_0 t + i\xi \cdot x} U(d\xi_0, d\xi), \tag{5.2}$$

where $\xi = (\xi_1, \ldots, \xi_d)$ and $U(d\xi_0, d\xi)$ is the associated projection valued measure. Let $\{D_k, k = 0, \ldots, d\}$ be the $L^2(\Omega)$ generator of $T_{(t,x)}$.

Let $\hat{R}(\xi_0, \xi)$ be the power spectrum associated with $\mathbb{V}$, i.e.,

$$\hat{R}(\xi_0, \xi) d\xi_0 d\xi = (2\pi)^{d+1} \langle U(d\xi_0, d\xi)\mathbb{V}, \mathbb{V} \rangle,$$

we obtain

$$R(t,x) = \mathbb{E}\{V(t,x)V(0,0)\} = \langle T_{(t,x)}\mathbb{V}, \mathbb{V}\rangle = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} e^{i\xi_0 t + i\xi\cdot x} \hat{R}(\xi_0, \xi) d\xi_0 d\xi. \quad (5.3)$$

For any set $S \subseteq \mathbb{R}^{d+1}$, define the $\sigma-$algebras: $\mathcal{F}_S = \sigma(V(s,x) : (s,x) \in S)$, and we assume the mixing property of V as follows.

Assumption 5.1. *V is uniformly bounded. There exists a function $\varphi(r) : [0,\infty) \to [0,\infty)$ such that $\forall n > 0$, $\varphi(r) \leq C_n(1 \wedge r^{-n})$ for some C_n, and*

$$\sup_{A\in\mathcal{F}_{S_1}, B\in\mathcal{F}_{S_2}, \mathbb{P}(B)>0, dist(S_1,S_2)\geq r} |\mathbb{P}(A|B) - \mathbb{P}(A)| \leq \varphi(r), \quad (5.4)$$

where $dist(S_1, S_2)$ is the Euclidean distance between S_1, S_2.

In the following, we denote

$$\sigma(V(s,x) : s \leq t, x \in \mathbb{R}^d) = \mathcal{F}_t, \quad (5.5)$$

$$\sigma(V(s,x) : s \geq t, x \in \mathbb{R}^d) = \mathcal{F}^t. \quad (5.6)$$

By [6, Page 170, Lemma 1], the above mixing property implies that

$$|\mathbb{E}\{XY\} - \mathbb{E}\{X\}\mathbb{E}\{Y\}| \leq 2\varphi^{\frac{1}{2}}(r) \left(\mathbb{E}\{X^2\}\mathbb{E}\{Y^2\}\right)^{\frac{1}{2}}, \quad (5.7)$$

if X is $\mathcal{F}_{S_1}-$measurable and Y is $\mathcal{F}_{S_2}-$measurable with $dist(S_1, S_2) \geq r$, i.e., the mixing properties in Assumption 5.1 is stronger than Assumption 2.4. For example, we have $|R(t,x)| \lesssim 1 \wedge (|t|^2 + |x|^2)^{-n}$ for any $n > 0$.

The following is the main result.

Theorem 5.2 ($\alpha \in [0,\infty)$: *homogenization*)**.** *Under Assumption 5.1, let*

$$\begin{aligned}
\partial_t u_\varepsilon(t,x) &= \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon^{\frac{\alpha}{2}\vee 1}} V(\frac{t}{\varepsilon^\alpha},\frac{x}{\varepsilon})u_\varepsilon(t,x), && (5.8)\\
\partial_t u_{hom}(t,x) &= \frac{1}{2}\Delta u_{hom}(t,x) - \rho(\alpha)u_{hom}(t,x), && (5.9)
\end{aligned}$$

with initial condition $u_\varepsilon(0,x) = u_{hom}(0,x) = f(x)$ *and*

$$\rho(\alpha) = \begin{cases} \int_0^\infty R(t,0)dt & \alpha \in (2,\infty), \\ \int_0^\infty \mathbb{E}_B\{R(t,B_t)\}dt & \alpha = 2, \\ \int_0^\infty \mathbb{E}_B\{R(0,B_t)\}dt & \alpha \in [0,2). \end{cases} \quad (5.10)$$

Then $u_\varepsilon(t,x) \to u_{hom}(t,x)$ *in probability as* $\varepsilon \to 0$.

When $\alpha > 2$, by a change of parameter $\varepsilon^\alpha \mapsto \varepsilon$, we have $\frac{1}{\varepsilon^{\frac{\alpha}{2}}}V(\frac{t}{\varepsilon^\alpha},\frac{x}{\varepsilon}) \mapsto \frac{1}{\sqrt{\varepsilon}}V(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\frac{1}{\alpha}}})$, so $\alpha = \infty$ corresponds to the case when V has no micro-structure in the spatial variable, and the potential is of the form $\frac{1}{\sqrt{\varepsilon}}V(\frac{t}{\varepsilon},x)$. We obtain a transition from homogenization to convergence to SPDE in the following theorem.

Theorem 5.3 ($\alpha = \infty$: *convergence to SPDE*)**.** *Under Assumption 5.1, let*

$$\begin{aligned}
\partial_t u_\varepsilon(t,x) &= \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\sqrt{\varepsilon}} V(\frac{t}{\varepsilon},x)u_\varepsilon(t,x), && (5.11)\\
\partial_t u_{spde}(t,x) &= \frac{1}{2}\Delta u_{spde}(t,x) + i\dot{W}(t,x)\circ u_{spde}(t,x), && (5.12)
\end{aligned}$$

with initial condition $u_\varepsilon(0,x) = u_{spde}(0,x) = f(x)$ *and Gaussian noise* $\dot{W}(t,x)$ *of covariance structure* $\mathbb{E}\{\dot{W}(t,x)\dot{W}(s,y)\} = \delta(t-s)\int_{\mathbb{R}} R(t,x-y)dt$. *Then* $u_\varepsilon(t,x) \Rightarrow u_{spde}(t,x)$ *in distribution as* $\varepsilon \to 0$.

Remark 5.4. The product $\circ$ in the limiting SPDE is in the Stratonovich's sense.

The solution to (5.1) is written by Feynman-Kac formula as

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^\gamma}\int_0^t V(\frac{t-s}{\varepsilon^\alpha},\frac{x+B_s}{\varepsilon})ds)\}. \tag{5.13}$$

Since Theorem 5.2 and 5.3 are both results for fixed (t,x), by stationarity of V, $u_\varepsilon(t,x)$ has the same distribution as

$$\begin{aligned}\tilde{u}_\varepsilon(t,x) =& \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^\gamma}\int_0^t V(\frac{-s}{\varepsilon^\alpha},\frac{x+B_s}{\varepsilon})ds)\}\\ =& \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^\gamma}\int_0^t \tilde{V}(\frac{s}{\varepsilon^\alpha},\frac{x+B_s}{\varepsilon})ds)\},\end{aligned} \tag{5.14}$$

where $\tilde{V}(s,x) := V(-s,x)$. Since $\tilde{V}$ and V has the same covariance function and mixing property we need, from now on we will write our solution to (5.1) for simplicity as

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^\gamma}\int_0^t V(\frac{s}{\varepsilon^\alpha},\frac{x+B_s}{\varepsilon})ds)\}. \tag{5.15}$$

5.1 Homogenization by temporal mixing

When $\alpha > 2$, $\gamma = \frac{\alpha}{2}$. By Feynman-Kac formula, the solution is written as

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\varepsilon^{-\alpha/2}\int_0^t V(s/\varepsilon^\alpha,(x+B_s)/\varepsilon)ds)\}. \tag{5.16}$$

By the scaling property of Brownian motion, stationarity of V, and a change of parameter $\varepsilon^\alpha \mapsto \varepsilon^2$, we have

$$u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon\int_0^{t/\varepsilon^2} V(s,\varepsilon^\beta B_s)ds)\}, \tag{5.17}$$

with $\beta = 1-\frac{2}{\alpha} \in (0,1)$. Since $\beta > 0$, the spatial mixing from V for the process $\varepsilon \int_0^{t/\varepsilon^2} V(s, \varepsilon^\beta B_s) ds$ is small.

Let $\sigma = \sqrt{2\int_0^\infty R(t,0)dt}$. The goal in this section is to prove that $u_\varepsilon(t,x) \to u_{hom}(t,x)$ in probability with u_0 satisfying

$$\partial_t u_{hom}(t,x) = \frac{1}{2}\Delta u_{hom}(t,x) - \frac{1}{2}\sigma^2 u_{hom}(t,x). \tag{5.18}$$

The result comes from the following two propositions.

Proposition 5.5.

$$(\varepsilon B_{t/\varepsilon^2}, \varepsilon \int_0^{t/\varepsilon^2} V(s, \varepsilon^\beta B_s) ds) \Rightarrow (N_t^1, \sigma N_t^2),$$

where $N_t^1 \sim N(0, tI_d)$, independent from $N_t^2 \sim N(0,t)$. The weak convergence $\Rightarrow$ is in the annealed sense.

Proposition 5.6. *For independent Brownian motions B_t^1, B_t^2,*

$$(\varepsilon B^1_{t/\varepsilon^2}, \varepsilon B^2_{t/\varepsilon^2}, \varepsilon \int_0^{t/\varepsilon^2} V(s, \varepsilon^\beta B_s^1) ds - \varepsilon \int_0^{t/\varepsilon^2} V(s, \varepsilon^\beta B_s^2) ds) \Rightarrow (N_t^1, N_t^2, \sqrt{2}\sigma N_t^3),$$

where $N_t^1, N_t^2 \sim N(0, tI_d)$, $N_t^3 \sim N(0,t)$, and they are independent. The weak convergence $\Rightarrow$ is in the annealed sense.

By Proposition 5.5, we have $\mathbb{E}\{u_\varepsilon(t,x)\} \to u_{hom}(t,x)$; by Proposition 5.6, we have $\mathbb{E}\{|u_\varepsilon(t,x)|^2\} \to |u_{hom}(t,x)|^2$. So $u_\varepsilon(t,x) \to u_{hom}(t,x)$ in $L^2(\Omega)$.

We only prove Proposition 5.6. The proof of Proposition 5.5 is similar with some simplifications.

Proof of Proposition 5.6. The goal is to show that for any $a, b \in \mathbb{R}^d, c \in \mathbb{R}$, as $\varepsilon \to 0$

$$\mathbb{E}\mathbb{E}_B\{e^{ia\cdot\varepsilon B^1_{t/\varepsilon^2}+ib\cdot\varepsilon B^2_{t/\varepsilon^2}+ic(\varepsilon\int_0^{t/\varepsilon^2}V(s,\varepsilon^\beta B^1_s)ds-\varepsilon\int_0^{t/\varepsilon^2}V(s,\varepsilon^\beta B^2_s)ds)}\}\to e^{-\frac{1}{2}|a|^2t-\frac{1}{2}|b|^2t-c^2\sigma^2t}. \quad (5.19)$$

We first consider the average with respect to the random environment. Let

$$X_\varepsilon(t)=\varepsilon\int_0^{t/\varepsilon^2}V(s,\varepsilon^\beta B^1_s)ds-\varepsilon\int_0^{t/\varepsilon^2}V(s,\varepsilon^\beta B^2_s)ds,$$

$\Delta t=\varepsilon^{-\gamma_1}+\varepsilon^{-\gamma_2}$, $0<\gamma_2<\gamma_1<2$ to be determined, and $N=[\frac{t}{\varepsilon^2\Delta t}]\sim t\varepsilon^{\gamma_1-2}$. Define the intervals $I_k=[(k-1)\Delta t,(k-1)\Delta t+\varepsilon^{-\gamma_1}]$ and $J_k=[(k-1)\Delta t+\varepsilon^{-\gamma_1},k\Delta t]$ for $k=1,\ldots,N$, we have

$$\begin{aligned}X_\varepsilon(t)=&\sum_{k=1}^N\int_{I_k}\varepsilon(V(s,\varepsilon^\beta B^1_s)-V(s,\varepsilon^\beta B^2_s))ds\\&+\sum_{k=1}^N\int_{J_k}\varepsilon(V(s,\varepsilon^\beta B^1_s)-V(s,\varepsilon^\beta B^2_s))ds\\&+\int_{N\Delta t}^{t/\varepsilon^2}\varepsilon(V(s,\varepsilon^\beta B^1_s)-V(s,\varepsilon^\beta B^2_s))ds:=(I)+(II)+(III).\end{aligned}\quad (5.20)$$

We show that for every realization of B^1_s, B^2_s, $\mathbb{E}\{|(II)|^2\}+\mathbb{E}\{|(III)|^2\}\to 0$. Take (II) for example, we have

$$\mathbb{E}\{|(II)|^2\}\lesssim\varepsilon^2\sum_{m=1}^N\sum_{n=1}^N\int_{J_m}\int_{J_n}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu. \quad (5.21)$$

Since $\int_{\mathbb{R}}\sup_{x\in\mathbb{R}^d}|R(t,x)|dt<\infty$, for those terms when $m=n$, we have an order of $\varepsilon^2N\varepsilon^{-\gamma_2}\sim\varepsilon^{\gamma_1-\gamma_2}\to 0$ as $\varepsilon\to 0$. For the terms when $m\neq n$, we have $|s-u|\geq\varepsilon^{-\gamma_1}$ if $s\in J_m, u\in J_n$, so $\sup_{x\in\mathbb{R}^d}|R(s-u,x)| \lesssim \varphi^{\frac{1}{2}}(\varepsilon^{-\gamma_1})$, and we can use the following crude bound

$$\varepsilon^2\sum_{m\neq n}\int_{J_m}\int_{J_n}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu \leq \varepsilon^2 N^2\varepsilon^{-2\gamma_2}\varphi^{\frac{1}{2}}(\varepsilon^{-\gamma_1}) \lesssim \varepsilon^{2\gamma_1-2\gamma_2+\gamma_1\lambda-2}, \tag{5.22}$$

if $\varphi^{\frac{1}{2}}(r) \lesssim r^{-\lambda}$. Since λ can be sufficiently large(e.g. $\lambda > 2/\gamma_1$), we have

$$\varepsilon^2\sum_{m\neq n}\int_{J_m}\int_{J_n}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu \to 0$$

as $\varepsilon \to 0$. Therefore, $\mathbb{E}\{|(II)|^2\} \to 0$. Similar discussion holds for (III). Now we have

$$\mathbb{E}\{|e^{icX_\varepsilon(t)} - e^{ic(I)}|\} \to 0, \tag{5.23}$$

so

$$\lim_{\varepsilon\to 0}|\mathbb{E}\mathbb{E}_B\{e^{ia\cdot\varepsilon B^1_{t/\varepsilon^2}+ib\cdot\varepsilon B^2_{t/\varepsilon^2}+icX_\varepsilon(t)}\} - \mathbb{E}\mathbb{E}_B\{e^{ia\cdot\varepsilon B^1_{t/\varepsilon^2}+ib\cdot\varepsilon B^2_{t/\varepsilon^2}+ic(I)}\}| = 0. \tag{5.24}$$

Next, we consider $(I) = \sum_{k=1}^N\int_{I_k}\varepsilon(V(s,\varepsilon^\beta B^1_s) - V(s,\varepsilon^\beta B^2_s))ds = \sum_{k=1}^N \varepsilon^{1-\frac{\gamma_1}{2}}Y^\varepsilon_k$, with

$$Y^\varepsilon_k := \int_{I_k}\varepsilon^{\frac{\gamma_1}{2}}(V(s,\varepsilon^\beta B^1_s) - V(s,\varepsilon^\beta B^2_s))ds, \tag{5.25}$$

Let $t_k = (k-1)\Delta t + \varepsilon^{-\gamma_1}, k = 1,\dots,N$, then

$$\mathbb{E}\{e^{ic(I)}\} = \mathbb{E}\{e^{ic\sum_{k=1}^N\varepsilon^{1-\frac{\gamma_1}{2}}Y^\varepsilon_k}\} = \mathbb{E}\{\mathbb{E}\{e^{ic\sum_{k=1}^N\varepsilon^{1-\frac{\gamma_1}{2}}Y^\varepsilon_k}|\mathcal{F}_{t_{N-1}}\}\}, \tag{5.26}$$

and

$$\mathbb{E}\{e^{ic\sum_{k=1}^N\varepsilon^{1-\frac{\gamma_1}{2}}Y^\varepsilon_k}|\mathcal{F}_{t_{N-1}}\} = e^{ic\sum_{k=1}^{N-1}\varepsilon^{1-\frac{\gamma_1}{2}}Y^\varepsilon_k}\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y^\varepsilon_N}|\mathcal{F}_{t_{N-1}}\}. \tag{5.27}$$

When freezing the Brownian motions, Y^ε_N is $\mathcal{F}^{(N-1)\Delta t}$-measurable. Since $(N-1)\Delta t - t_{N-1} =$

$\varepsilon^{-\gamma_2}$ and $e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_N^\varepsilon}$ is uniformly bounded by 1, we have by Lemma 5.17 that

$$|\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_N^\varepsilon}|\mathcal{F}_{t_{N-1}}\} - \mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_N^\varepsilon}\}| \le 2\varphi(\varepsilon^{-\gamma_2}) \to 0. \tag{5.28}$$

Therefore we obtain

$$\lim_{\varepsilon\to 0}\left(\mathbb{E}\{e^{ic\sum_{k=1}^N \varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\} - \mathbb{E}\{e^{ic\sum_{k=1}^{N-1}\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\}\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_N^\varepsilon}\}\right) = 0. \tag{5.29}$$

Iterating the above procedure, in the end we have

$$\lim_{\varepsilon\to 0}\left(\mathbb{E}\{e^{ic\sum_{k=1}^N \varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\} - \prod_{k=1}^N \mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\}\right) = 0. \tag{5.30}$$

Now we consider $\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\}$. By Talor expansion, we have $|e^{ix}-1-ix+\frac{1}{2}x^2| \le C|x|^3$, so

$$\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\} = 1-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}\mathbb{E}\{(Y_k^\varepsilon)^2\}+\varepsilon^{2-\gamma_1}O(\varepsilon^{1-\frac{\gamma_1}{2}}|\int_{I_k}\varepsilon^{\frac{\gamma_1}{2}}(V(s,\varepsilon^\beta B_s^1)-V(s,\varepsilon^\beta B_s^2))ds|^3). \tag{5.31}$$

Since V is uniformly bounded, we have

$$\varepsilon^{1-\frac{\gamma_1}{2}}|\int_{I_k}\varepsilon^{\frac{\gamma_1}{2}}(V(s,\varepsilon^\beta B_s^1)-V(s,\varepsilon^\beta B_s^2))ds|^3 \lesssim \varepsilon^{1-\frac{\gamma_1}{2}}\varepsilon^{-\frac{3\gamma_1}{2}} = \varepsilon^{1-2\gamma_1} \to 0, \tag{5.32}$$

if $\gamma_1 < \frac{1}{2}$. Then we obtain

$$\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\} = 1-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}\mathbb{E}\{(Y_k^\varepsilon)^2\} + o(\varepsilon^{2-\gamma_1}). \tag{5.33}$$

Now we have

$$\prod_{k=1}^{N}\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\}=\prod_{k=1}^{N}\left(1-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(\varepsilon^{2-\gamma_1})\right)=e^{\sum_{k=1}^N\log\left(1-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(\varepsilon^{2-\gamma_1})\right)}, \tag{5.34}$$

and we claim that

$$\prod_{k=1}^{N}\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\}\to e^{-c^2\sigma^2 t} \tag{5.35}$$

in probability. If this is true, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\mathbb{E}_B\{e^{ia\cdot\varepsilon B^1_{t/\varepsilon^2}+ib\cdot\varepsilon B^2_{t/\varepsilon^2}+icX_\varepsilon(t)}\}=&\lim_{\varepsilon\to 0}\mathbb{E}_B\{e^{ia\cdot\varepsilon B^1_{t/\varepsilon^2}+ib\cdot\varepsilon B^2_{t/\varepsilon^2}}\prod_{k=1}^{N}\mathbb{E}\{e^{ic\varepsilon^{1-\frac{\gamma_1}{2}}Y_k^\varepsilon}\}\}\\ =&\lim_{\varepsilon\to 0}\mathbb{E}_B\{e^{ia\cdot\varepsilon B^1_{t/\varepsilon^2}+ib\cdot\varepsilon B^2_{t/\varepsilon^2}}e^{-c^2\sigma^2 t}\}\\ =&e^{-\frac{1}{2}|a|^2t-\frac{1}{2}|b|^2t-c^2\sigma^2t},\end{aligned} \tag{5.36}$$

and the proof is complete.

To prove (5.35), we consider

$$\log\left(1-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(\varepsilon^{2-\gamma_1})\right)=\log\left(1-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}(\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(1))\right),$$

where $o(1)$ is uniformly bounded, independent of k, and $o(1)\to 0$ as $\varepsilon\to 0$. In addition,

$$\mathbb{E}\{(Y_k^\varepsilon)^2\}\lesssim\varepsilon^{\gamma_1}\int_{I_k^2}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu \tag{5.37}$$

is uniformly bounded. By Talor expansion, we have

$$\sum_{k=1}^{N}\log\left(1-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}(\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(1))\right)=\sum_{k=1}^{N}\left(-\frac{1}{2}c^2\varepsilon^{2-\gamma_1}(\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(1))\right)$$
$$+\sum_{k=1}^{N}O\left(\left(\frac{1}{2}c^2\varepsilon^{2-\gamma_1}(\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(1))\right)^2\right). \tag{5.38}$$

Clearly $\sum_{k=1}^{N}O\big(\big(\frac{1}{2}c^2\varepsilon^{2-\gamma_1}(\mathbb{E}\{(Y_k^\varepsilon)^2\}+o(1))\big)^2\big)\to 0$ in probability, and we only need to consider

$$(*)=\sum_{k=1}^{N}\varepsilon^{2-\gamma_1}\mathbb{E}\{(Y_k^\varepsilon)^2\}, \tag{5.39}$$

and prove $(*)\to 2\sigma^2 t$ in probability. Actually, we have

$$(*)-2\sigma^2 t=\sum_{k=1}^{N}\varepsilon^{2-\gamma_1}(\mathbb{E}\{(Y_k^\varepsilon)^2\}-2\sigma^2)+2\sigma^2(N\varepsilon^{2-\gamma_1}-t). \tag{5.40}$$

The second part goes to zero as $\varepsilon\to 0$. For the first part, we note that

$$\begin{aligned}\mathbb{E}\{(Y_k^\varepsilon)^2\}=&\varepsilon^{\gamma_1}\int_{I_k^2}R(s-u,\varepsilon^\beta(B_s^1-B_u^1))dsdu\\ &+\varepsilon^{\gamma_1}\int_{I_k^2}R(s-u,\varepsilon^\beta(B_s^2-B_u^2))dsdu\\ &-2\varepsilon^{\gamma_1}\int_{I_k^2}R(s-u,\varepsilon^\beta(B_s^1-B_u^2))dsdu.\end{aligned} \tag{5.41}$$

By Lemma 5.18, we have $\mathbb{E}_B\{(\varepsilon^{\gamma_1}\int_{I_k^2}R(s-u,\varepsilon^\beta(B_s^i-B_u^i))dsdu-\sigma^2)^2\}\to 0$ for $i=1,2$, and clearly it is independent of k. Now we only have to consider

$$\sum_{k=1}^{N}\varepsilon^{2-\gamma_1}\varepsilon^{\gamma_1}\int_{I_k^2}|R(s-u,\varepsilon^\beta(B_s^1-B_u^2))|dsdu\le\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}|R(s-u,\varepsilon^\beta(B_s^1-B_u^2))|dsdu. \tag{5.42}$$

Again by Lemma 5.18, $\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}|R(s-u,\varepsilon^\beta(B_s^1-B_u^2))|dsdu\to 0$ in probability. The proof is complete. $\square$

5.2 Homogenization by spatial mixing

When $\alpha\le 2$, we use a different approach, which makes use of the mixing from Brownian motions. It is very similar with the time-independent case in Chapter 2.

By Feynman-Kac formula, the solution is written as

$$u_\varepsilon(t,x)=\mathbb{E}_B\{f(x+B_t)\exp(i\varepsilon^{-1}\int_0^t V(s/\varepsilon^\alpha,(x+B_s)/\varepsilon)ds)\}. \tag{5.43}$$

By the scaling property of the Brownian motion and stationarity of V, we only need to consider

$$u_\varepsilon(t,x)=\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon\int_0^{t/\varepsilon^2} V(\varepsilon^{2-\alpha}s,B_s)ds)\}. \tag{5.44}$$

Recall that $V(t,x)=\mathbb{V}(\tau_{(-t,-x)}\omega)$, define the environmental process $y_s^\varepsilon=\tau_{(-\varepsilon^{2-\alpha}s,-B_s)}\omega$ taking values in Ω. By Lemma 5.19, for fixed $\varepsilon>0$, it is a stationary Markov process, ergodic with respect to the invariant measure $\mathbb{P}$. By a straightforward calculation, the generator is $L=\varepsilon^{2-\alpha}D_0+\frac{1}{2}\sum_{k=1}^d D_k^2$. Now the Brownian motion in random scenery can be rewritten as $X_\varepsilon(t):=\varepsilon\int_0^{t/\varepsilon^2}\mathbb{V}(y_s^\varepsilon)ds$, i.e., an additive functional of a stationary, ergodic Markov process. The Kipnis-Varadhan's method involves constructing a corrector function Φ_λ and applying a martingale decomposition as follows.

Define the corrector function Φ_λ with $\lambda=\varepsilon^2$ as

$$(\lambda-L)\Phi_\lambda=\mathbb{V}. \tag{5.45}$$

By Itô's formula, $X_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} \mathbb{V}(y_s^\varepsilon)ds = R_t^\varepsilon + M_t^\varepsilon$ with

$$R_t^\varepsilon = \varepsilon \int_0^{t/\varepsilon^2} \lambda \Phi_\lambda(y_s^\varepsilon)ds - \varepsilon \Phi_\lambda(y_{t/\varepsilon^2}^\varepsilon) + \varepsilon \Phi_\lambda(y_0^\varepsilon), \tag{5.46}$$

$$M_t^\varepsilon = \sum_{k=1}^d \varepsilon \int_0^{t/\varepsilon^2} D_k \Phi_\lambda(y_s^\varepsilon)dB_s^k. \tag{5.47}$$

To show the distribution of $X_\varepsilon(t)$ is close to a normal distribution, the idea is to prove R_t^ε is small and the martingale M_t^ε is close to a Brownian motion. We first have the following lemma.

Lemma 5.7.

$$\lambda \langle \Phi_\lambda, \Phi_\lambda \rangle \to 0 \tag{5.48}$$

as $\lambda \to 0$.

Proof. By spectral representation, Φ_λ is written as

$$\Phi_\lambda = \int_{\mathbb{R}^{d+1}} \frac{1}{\lambda + \frac{1}{2}|\xi|^2 - i\xi_0 \varepsilon^{2-\alpha}} U(d\xi_0, d\xi)\mathbb{V}, \tag{5.49}$$

so

$$\lambda \langle \Phi_\lambda, \Phi_\lambda \rangle = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{\lambda \hat{R}(\xi_0, \xi)}{(\lambda + \frac{1}{2}|\xi|^2)^2 + \varepsilon^{4-2\alpha}\xi_0^2} d\xi_0 d\xi. \tag{5.50}$$

Clearly $\frac{\lambda \hat{R}(\xi_0,\xi)}{(\lambda+\frac{1}{2}|\xi|^2)^2+\varepsilon^{4-2\alpha}\xi_0^2} \lesssim \frac{\hat{R}(\xi_0,\xi)}{|\xi|^2}$. Since $\hat{R}$ is bounded and integrable, and $\frac{1}{|\xi|^2}$ is integrable around the origin when $d \geq 3$, by the dominated convergence theorem, the proof is complete. □

By Lemma 5.7, we have

$$\mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|^2\} \lesssim \lambda \langle \Phi_\lambda, \Phi_\lambda \rangle \to 0 \tag{5.51}$$

as $\varepsilon \to 0$.

To deal with the martingale M_t^ε, we distinguish between $\alpha = 2$ and $\alpha < 2$.

5.2.1 $\alpha = 2$: ergodicity

Let $\eta_k = \int_{\mathbb{R}^{d+1}} \frac{i\xi_k}{\frac{1}{2}|\xi|^2 - i\xi_0} U(d\xi_0, d\xi)\mathbb{V} \in L^2(\Omega)$, $k = 1, \ldots, d$, we have

Lemma 5.8. *$D_k\Phi_\lambda \to \eta_k$ in $L^2(\Omega)$, $k = 1, \ldots, d$.*

Proof. Since

$$D_k\Phi_\lambda = \int_{\mathbb{R}^{d+1}} \frac{i\xi_k}{\lambda + \frac{1}{2}|\xi|^2 - i\xi_0} U(d\xi_0, d\xi)\mathbb{V}, \tag{5.52}$$

we have

$$\begin{aligned}\|D_k\Phi_\lambda - \eta_k\|^2 =& \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{\lambda^2 \xi_k^2 \hat{R}(\xi_0, \xi)}{((\lambda + \frac{1}{2}|\xi|^2)^2 + \xi_0^2)(\frac{1}{4}|\xi|^4 + \xi_0^2)} d\xi_0 d\xi \\ &\lesssim \int_{\mathbb{R}^{d+1}} \frac{\lambda^2}{(\lambda + \frac{1}{2}|\xi|^2)^2 + \xi_0^2} \frac{|\xi|^2 \hat{R}(\xi_0, \xi)}{\frac{1}{4}|\xi|^4 + |\xi_0|^2} d\xi_0 d\xi \to 0\end{aligned} \tag{5.53}$$

by the dominated convergence theorem. □

Let $M_t = \sum_{k=1}^d \varepsilon \int_0^{t/\varepsilon^2} \eta_k(y_s^\varepsilon) dB_s^k$ and

$$\sigma^2 = \sum_{k=1}^d \|\eta_k\|^2 = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{|\xi|^2 \hat{R}(\xi_0, \xi)}{\frac{1}{4}|\xi|^4 + \xi_0^2} d\xi_0 d\xi = 2\int_0^\infty \mathbb{E}_B\{R(t, B_t)\} dt,$$

and since u_{hom} solves $\partial_t u_{hom}(t, x) = \frac{1}{2}\Delta u_{hom}(t, x) - \frac{1}{2}\sigma^2 u_{hom}(t, x)$, i.e.,

$$u_{hom}(t, x) = \mathbb{E}_B\{f(x + \varepsilon B_{t/\varepsilon^2}) \exp(-\frac{1}{2}\sigma^2 t)\}, \tag{5.54}$$

we have the decomposition of the error

$$\begin{aligned}u_\varepsilon(t,x) - u_{hom}(t,x) =& \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iX_\varepsilon(t))\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} \\ &+ \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t)\} \\ &+ \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\} \\ :=& (I) + (II) + (III).\end{aligned} \tag{5.55}$$

Clearly $\mathbb{E}\{|(I)|\} \lesssim \mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|\} \lesssim \sqrt{\lambda\langle\Phi_\lambda,\Phi_\lambda\rangle} \to 0$, and

$$\mathbb{E}\{|(II)|\} \lesssim \sqrt{\mathbb{E}\mathbb{E}_B\{|M_t^\varepsilon - M_t|^2\}} = \sqrt{\sum_{k=1}^d \|D_k\Phi_\lambda - \eta_k\|^2 t} \to 0. \tag{5.56}$$

For (III), we note that $M_t = \sum_{k=1}^d \varepsilon \int_0^{t/\varepsilon^2} \eta_k(y_s^\varepsilon) dB_s^k$ is a square-integrable martingale for almost every $\omega \in \Omega$, and when $\alpha = 2$, $y_s^\varepsilon = \tau_{(s,B_s)}\omega$ is ergodic. By martingale central limit theorem, ergodicity of y_s^ε, and the fact that $\mathbb{E}\{\eta_k\} = 0, k = 1,\ldots,d$, we obtain that for almost every ω

$$(\varepsilon B_{t/\varepsilon^2}, M_t) \Rightarrow (W_t^1, \sigma W_t^2), \tag{5.57}$$

with W^1, W^2 independent Brownian motions. Thus $(III) \to 0$ almost surely. By the dominated convergence theorem we have $\mathbb{E}\{|(III)|\} \to 0$ as $\varepsilon \to 0$.

To summarize, $\mathbb{E}\{|u_\varepsilon(t,x) - u_{hom}(t,x)|\} \to 0$ as $\varepsilon \to 0$. The proof of the case $\alpha = 2$ is complete.

Remark 5.9. It is worth noting that for the case $\alpha = 2$, the mixing property in Assumption 5.1 is not used. All we need is the ergodicity and certain integrability condition of $\hat{R}(\xi_0, \xi)$.

5.2.2 $\alpha \in [0,2)$: quantitative martingale central limit theorem

In this regime, $y_s^\varepsilon = \tau_{(-\varepsilon^{2-\alpha}s,-B_s)}\omega$ is $\varepsilon-$dependent, so a ergodicity does not seem to establish (5.57). We apply a quantitative martingale central limit theorem instead. A fourth moment estimation appears in the proof, for which we use the mixing property of V.

We define $\sigma_\lambda^2 = \sum_{k=1}^d \|D_k\Phi_\lambda\|^2$, and

$$\sigma^2 = \frac{4}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{\hat{R}(\xi_0,\xi)}{|\xi|^2} d\xi_0 d\xi = 2\int_0^\infty \mathbb{E}_B\{R(0,B_t)\}dt, \tag{5.58}$$

the following lemma holds.

Lemma 5.10. $\sigma_\lambda^2 \to \sigma^2$ *as* $\lambda \to 0$.

Proof. Since

$$D_k\Phi_\lambda = \int_{\mathbb{R}^{d+1}} \frac{i\xi_k}{\lambda + \frac{1}{2}|\xi|^2 - i\xi_0\varepsilon^{2-\alpha}} U(d\xi_0, d\xi)\mathbb{V}, \tag{5.59}$$

we have

$$\sigma_\lambda^2 = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{|\xi|^2\hat{R}(\xi_0,\xi)}{(\lambda + \frac{1}{2}|\xi|^2)^2 + \xi_0^2\varepsilon^{4-2\alpha}} d\xi_0 d\xi. \tag{5.60}$$

Since $\alpha < 2$, by the dominated convergence theorem, the proof is complete. □

Now we decompose the error as

$$\begin{aligned} u_\varepsilon(t,x) - u_{hom}(t,x) =& \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iX_\varepsilon(t))\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} \\ &+ \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma_\lambda^2 t)\} \\ &+ \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma_\lambda^2)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma^2 t)\} \\ :=& (I) + (II) + (III). \end{aligned} \tag{5.61}$$

By the same discussion as before, we have $\mathbb{E}\{|(I)|\} \lesssim \mathbb{E}\mathbb{E}_B\{|R_t^\varepsilon|\} \lesssim \sqrt{\lambda\langle\Phi_\lambda, \Phi_\lambda\rangle} \to 0$, and $\mathbb{E}\{|(III)|\} \lesssim |\sigma_\lambda^2 - \sigma^2| \to 0$.

For (II), we rewrite it in Fourier domain as

$$\begin{aligned}(II) =&\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(iM_t^\varepsilon)\} - \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma_\lambda^2 t)\} \\ =&\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{f}(\xi)e^{i\xi\cdot x}\mathbb{E}_B\{e^{i\xi\cdot\varepsilon B_{t/\varepsilon^2}+iM_t^\varepsilon} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t}\}d\xi.\end{aligned} \tag{5.62}$$

$\xi\cdot\varepsilon B_{t/\varepsilon^2} + M_t^\varepsilon = \sum_{k=1}^d \varepsilon\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi_\lambda(y_s^\varepsilon))dB_s^k$ is a continuous and square-integrable martingale for almost every $\omega\in\Omega$, so the estimation of (II) boils down to a control of the Wasserstein distance between the martingale and a Brownian motion. By a quantitative martingale central limit theorem, Proposition 2.12, we obtain for some constant C that

$$|\mathbb{E}_B\{e^{i\xi\cdot\varepsilon B_{t/\varepsilon^2}+iM_t^\varepsilon} - e^{-\frac{1}{2}(|\xi|^2+\sigma_\lambda^2)t}\}| \le C\mathbb{E}_B\{|\sum_{k=1}^d\varepsilon^2\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi_\lambda(y_s^\varepsilon))^2 ds - (|\xi|^2+\sigma_\lambda^2)t|\}. \tag{5.63}$$

Let $\sum_{k=1}^d\varepsilon^2\int_0^{t/\varepsilon^2}(\xi_k + D_k\Phi_\lambda(y_s^\varepsilon))^2 ds - (|\xi|^2+\sigma_\lambda^2)t = (i) + (ii)$ with

$$(i) \quad = \quad 2\sum_{k=1}^d \xi_k\varepsilon^2\int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s^\varepsilon)ds, \tag{5.64}$$

$$(ii) \quad = \quad \varepsilon^2\int_0^{t/\varepsilon^2}\left(\sum_{k=1}^d(D_k\Phi_\lambda(y_s^\varepsilon))^2 - \sigma_\lambda^2\right)ds. \tag{5.65}$$

We will show that $\mathbb{E}\mathbb{E}_B\{|\varepsilon^2\int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s^\varepsilon)ds|\} \to 0$ and $\mathbb{E}\mathbb{E}_B\{|(ii)|\} \to 0$ as $\varepsilon \to 0$ in Lemma 5.11 and 5.12 below, which implies $\mathbb{E}\{|(II)|\} \to 0$ by the dominated convergence theorem if we assume $|\hat{f}(\xi)||\xi| \in L^1$. Therefore $\mathbb{E}\{|u_\varepsilon(t,x) - u_{hom}(t,x)|\} \to 0$, and the proof of the case $\alpha < 2$ is complete.

Lemma 5.11. $\mathbb{E}\mathbb{E}_B\{|\varepsilon^2 \int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s^\varepsilon)ds|\} \to 0,\ k = 1, \ldots, d.$

Proof. We recall that

$$D_k\Phi_\lambda = \int_{\mathbb{R}^{d+1}} \frac{i\xi_k}{\lambda + \frac{1}{2}|\xi|^2 - i\xi_0\varepsilon^{2-\alpha}} U(d\xi_0, d\xi)\mathbb{V}, \tag{5.66}$$

so

$$\langle T_{(t,x)}D_k\Phi_\lambda, D_k\Phi_\lambda\rangle = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{e^{i\xi_0\cdot t}e^{i\xi\cdot x}\xi_k^2\hat{R}(\xi_0,\xi)}{(\lambda + \frac{1}{2}|\xi|^2)^2 + \xi_0^2\varepsilon^{4-2\alpha}} d\xi_0 d\xi. \tag{5.67}$$

This leads to

$$\mathbb{E}\mathbb{E}_B\{D_k\Phi_\lambda(y_s^\varepsilon)D_k\Phi_\lambda(y_u^\varepsilon)\} = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{e^{i\xi_0\cdot\varepsilon^{2-\alpha}(s-u)}e^{-\frac{1}{2}|\xi|^2|s-u|}\xi_k^2\hat{R}(\xi_0,\xi)}{(\lambda + \frac{1}{2}|\xi|^2)^2 + \xi_0^2\varepsilon^{4-2\alpha}} d\xi_0 d\xi. \tag{5.68}$$

Therefore,

$$\begin{aligned}
(\mathbb{E}\mathbb{E}_B\{|\varepsilon^2 \int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s^\varepsilon)ds|\})^2 \le& \mathbb{E}\mathbb{E}_B\{\varepsilon^4 \int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2} D_k\Phi_\lambda(y_s^\varepsilon)D_k\Phi_\lambda(y_u^\varepsilon)dsdu\} \\
\lesssim& \varepsilon^4 \int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^{d+1}} \frac{e^{-\frac{1}{2}|\xi|^2|s-u|}\hat{R}(\xi_0,\xi)}{|\xi|^2} d\xi_0 d\xi dsdu \\
=& \int_0^t\int_0^t\int_{\mathbb{R}^{d+1}} \frac{e^{-\frac{1}{2\varepsilon^2}|\xi|^2|s-u|}\hat{R}(\xi_0,\xi)}{|\xi|^2} d\xi_0 d\xi dsdu \to 0
\end{aligned} \tag{5.69}$$

by the dominated convergence theorem. ☐

Lemma 5.12. $\mathbb{E}\mathbb{E}_B\{|(ii)|\} \to 0.$

Proof. Define $\mathcal{Z}_\lambda(t,x) = \sum_{k=1}^d (D_k\Phi_\lambda(\tau_{(-\varepsilon^{2-\alpha}t,-x)}\omega))^2 - \sigma_\lambda^2$, which is zero-mean and stationary. Then $(ii) = \varepsilon^2 \int_0^{t/\varepsilon^2} \mathcal{Z}_\lambda(s, B_s)ds$. Denote the covariance function of $\mathcal{Z}_\lambda$ by $\mathcal{R}_\lambda(t,x)$, Lemma 5.20 implies

$$\mathbb{E}\mathbb{E}_B\{|(ii)|^2\} \lesssim \varepsilon^2 \int_{\mathbb{R}^d} \frac{\sup_t |\mathcal{R}_\lambda(t,x)|}{|x|^{d-2}} dx, \tag{5.70}$$

so we only need to estimate $\sup_t |\mathcal{R}_\lambda(t,x)|$.

Clearly,

$$\mathcal{R}_\lambda(t,x) = \sum_{m,n=1}^{d} \mathbb{E}\{(D_m\Phi_\lambda(\tau_{(-\varepsilon^{2-\alpha}t,-x)}\omega))^2(D_n\Phi_\lambda(\tau_{(0,0)}\omega))^2\} - \sigma_\lambda^4. \tag{5.71}$$

Let $G_\lambda(t,x)$ be the Green's function of $\lambda - \varepsilon^{2-\alpha}\partial_t - \frac{1}{2}\Delta$, it is straightforward to check that

$$G_\lambda(t,x) = \varepsilon^{\alpha-2}e^{\varepsilon^\alpha t}q_{-t\varepsilon^{\alpha-2}}(x)1_{t<0}, \tag{5.72}$$

so $D_k\Phi_\lambda(\tau_{(-t,-x)}\omega) = \int_{\mathbb{R}^{d+1}} \partial_{x_k}G_\lambda(t-s,x-y)V(s,y)dsdy$, and we obtain

$$\begin{aligned} &\mathbb{E}\{(D_m\Phi_\lambda(\tau_{(-\varepsilon^{2-\alpha}t,-x)}\omega))^2(D_n\Phi_\lambda(\tau_{(0,0)}\omega))^2\} \\ =&\int_{\mathbb{R}^{4d+4}} \prod_{i=1}^{2}\partial_{x_m}G_\lambda(\varepsilon^{2-\alpha}t-s_i,x-y_i)\prod_{i=3}^{4}\partial_{x_n}G_\lambda(-s_i,-y_i)\mathbb{E}\{\prod_{i=1}^{4}V(s_i,y_i)\}dsdy. \end{aligned} \tag{5.73}$$

By Lemma 5.21, we have

$$\begin{aligned} &|\mathbb{E}\{\prod_{i=1}^{4}V(s_i,y_i)\} - R(s_1-s_2,y_1-y_2)R(s_3-s_4,y_3-y_4)| \\ \le&\Psi(s_1-s_3,y_1-y_3)\Psi(s_2-s_4,y_2-y_4) + \Psi(s_1-s_4,y_1-y_4)\Psi(s_2-s_3,y_2-y_3) \\ \le& g(y_1-y_3)g(y_2-y_4) + g(y_1-y_4)g(y_2-y_3) \end{aligned} \tag{5.74}$$

for $g(x) = \sup_t \Psi(t,x)$. In addition, we have $\int_{\mathbb{R}} |\partial_{x_k}G_\lambda(s,x)|ds = \int_0^\infty e^{-\lambda t}q_t(x)\frac{|x_k|}{t}dt = |\partial_{x_k}\mathcal{G}_\lambda(x)|$, where $\mathcal{G}_\lambda(x) = \int_0^\infty e^{-\lambda t}q_t(x)dt$ is the Green's function of $\lambda - \frac{1}{2}\Delta$. Therefore, by the fact that

$$\sigma_\lambda^4 = \sum_{m,n=1}^{d} \int_{\mathbb{R}^{4d+4}} \prod_{i=1}^{2} \partial_{x_m} G_\lambda(\varepsilon^{2-\alpha} t - s_i, x - y_i) \prod_{i=3}^{4} \partial_{x_n} G_\lambda(-s_i, -y_i) R(s_1 - s_2, y_1 - y_2) R(s_3 - s_4, y_3 - y_4)\} ds dy \tag{5.75}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^{4d+4}} \prod_{i=1}^{2} |\partial_{x_m} \mathcal{G}_\lambda(x - y_i)| \prod_{i=3}^{4} |\partial_{x_n} \mathcal{G}_\lambda(-y_i)| \left(g(y_1 - y_3) g(y_2 - y_4) + g(y_1 - y_4) g(y_2 - y_3) \right) dy \\ &\lesssim \left(\int_{\mathbb{R}^{2d}} |\partial_{x_m} \mathcal{G}_\lambda(y)| |\partial_{x_n} \mathcal{G}_\lambda(z)| g(x - y - z) dy dz \right)^2, \end{aligned} \tag{5.76}$$

we obtain

$$\sup_t |\mathcal{R}_\lambda(t,x)| \lesssim \left(\int_{\mathbb{R}^{2d}} \frac{e^{-c\sqrt{\lambda}|y|}}{|y|^{d-1}} \frac{e^{-c\sqrt{\lambda}|z|}}{|z|^{d-1}} g(x - y - z) dy dz \right)^2, \tag{5.77}$$

where we have also used the fact $|\partial_{x_k} \mathcal{G}_\lambda(y)| \le C e^{-c\sqrt{\lambda}|x|} |x|^{1-d}$ for some $c, C > 0$. Since $g(x) \le C_\beta (1 \wedge |x|^{-\beta})$ for any $\beta > 0$, by the same estimation as in the proof of Proposition 2.14, we have

$$\sup_t |\mathcal{R}_\lambda(t,x)| \lesssim \lambda^{\frac{d}{2}-1} e^{-c\sqrt{\lambda}|x|} + 1 \wedge \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{d-2}} + 1 \wedge \frac{1}{|x|^\beta} \tag{5.78}$$

for some constant $c > 0$ and $\beta > 0$ sufficiently large. By direct calculation as in Lemma 2.15, we have

$$\mathbb{E}\mathbb{E}_B\{|(ii)|^2\} \lesssim \varepsilon^2 \int_{\mathbb{R}^d} \frac{\sup_t |\mathcal{R}_\lambda(t,x)|}{|x|^{d-2}} dx \to 0. \tag{5.79}$$

The proof is complete. □

Remark 5.13. By a similar discussion as in Chapter 2, we can actually establish an error estimate for the case $\alpha \in [0, 2)$, which we do not present here.

5.3 Convergence to SPDE by temporal mixing

In this regime, $V_\varepsilon(t,x) = \frac{1}{\sqrt{\varepsilon}} V(\frac{t}{\varepsilon}, x)$, so heuristically it approaches a white noise in the temporal direction. We define a formally written random variable $\int_0^t \dot{W}(t-s, x+B_s)ds$, i.e., a Brownian motion in Gaussian noise, by standard mollification argument.

Let $\mathcal{R}(x) = \int_{\mathbb{R}} R(t,x)dt$ and define the Gaussian noise $W(dt,dx)$ on Ω with a formal covariance structure

$$\mathbb{E}\{\dot{W}(t,x)\dot{W}(s,y)\} = \delta(t-s)\mathcal{R}(x-y). \tag{5.80}$$

Let B_t be an independent Brownian motion.

We pick a mollifier $\phi_\varepsilon(t)q_\varepsilon(x)$ and define the stationary random field

$$W_\varepsilon(t,x) = \int_{\mathbb{R}^{d+1}} \phi_\varepsilon(t-s)q_\varepsilon(x-y)W(ds,dy), \tag{5.81}$$

where $\phi_\varepsilon(t) = \frac{1}{\varepsilon}1_{[-\varepsilon,0]}(t)$ and $q_\varepsilon(x)$ is the heat kernel with variance ε. $W_\varepsilon(t,x)$ is a well-defined Wiener integral and clearly

$$\mathbb{E}\{W_\varepsilon(t,x)W_\varepsilon(s,y)\} = \frac{\varepsilon-|t-s|}{\varepsilon^2}1_{|t-s|<\varepsilon}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\xi|^2\varepsilon}\hat{\mathcal{R}}(\xi)e^{i\xi\cdot(x-y)}d\xi. \tag{5.82}$$

Since $\mathbb{E}\{W_\varepsilon(t,x)^2\} < \infty$, $\int_0^t W_\varepsilon(t-s, x+B_s)ds$ is well-defined for every realization of B_s.

Proposition 5.14. *For almost every realization of B_s, $\int_0^t W_\varepsilon(t-s,x+B_s)ds$ is Cauchy in $L^2(\Omega)$, whose limit is denoted as $\int_0^t \dot{W}(t-s,x+B_s)ds \sim N(0,\mathcal{R}(0)t)$.*

Proof. We will show that for almost every realization of B_s,

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathbb{E}\{\int_0^t W_{\varepsilon_1}(t-s,x+B_s)ds\int_0^t W_{\varepsilon_2}(t-s,x+B_s)ds\} = \mathcal{R}(0)t, \tag{5.83}$$

so $\int_0^t W_\varepsilon(t-s,x+B_s)ds$ is a Cauchy sequence in $L^2(\Omega)$.

By direct calculation, we have

$$
\begin{aligned}
&\mathbb{E}\{\int_0^t W_{\varepsilon_1}(t-s,x+B_s)ds\int_0^t W_{\varepsilon_2}(t-s,x+B_s)ds\}\\
=&\int_0^t\int_0^t dsdu\int_{\mathbb{R}^{2d+1}}\phi_{\varepsilon_1}(t-s-r)q_{\varepsilon_1}(x+B_s-y)\phi_{\varepsilon_2}(t-u-r)q_{\varepsilon_2}(x+B_u-z)\mathcal{R}(y-z)drdydz\\
=&\int_0^t\int_0^t dsdu\int_{\mathbb{R}^{2d+1}}\phi_{\varepsilon_1}(s-r)\phi_{\varepsilon_2}(u-r)q_{\varepsilon_1}(y)q_{\varepsilon_2}(z)\mathcal{R}(y-z+B_{t-s}-B_{t-u})drdydz\\
=&\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-\frac{1}{2}|\xi|^2(\varepsilon_1+\varepsilon_2)}\hat{\mathcal{R}}(\xi)\left(\int_{\mathbb{R}}\int_0^t\int_0^t\phi_{\varepsilon_1}(s-r)\phi_{\varepsilon_2}(u-r)e^{i\xi\cdot(B_{t-s}-B_{t-u})}dsdudr\right)d\xi.
\end{aligned}
\tag{5.84}
$$

For fixed $\xi\in\mathbb{R}^d$, we consider

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_0^t\int_0^t\phi_{\varepsilon_1}(s-r)\phi_{\varepsilon_2}(u-r)e^{i\xi\cdot(B_{t-s}-B_{t-u})}dsdudr\\
=&\frac{1}{\varepsilon_1\varepsilon_2}\int_{\mathbb{R}}\int_0^t\int_0^t 1_{r-\varepsilon_1<s<r}1_{r-\varepsilon_2<u<r}e^{i\xi\cdot(B_{t-s}-B_{t-u})}dsdudr \qquad (5.85)\\
=&\int_0^{t+(\varepsilon_1\vee\varepsilon_2)}\left(\frac{1}{\varepsilon_1\varepsilon_2}\int_0^t\int_0^t 1_{r-\varepsilon_1<s<r}1_{r-\varepsilon_2<u<r}e^{i\xi\cdot(B_{t-s}-B_{t-u})}dsdu\right)dr.
\end{aligned}
$$

For almost every realization, B_s is continuous in $[0,t]$, so we have as $\varepsilon_1,\varepsilon_2\to 0$, $e^{i\xi\cdot(B_{t-s}-B_{t-u})}\to 1$ almost surely. Thus for fixed $r\in(0,t)$, $\frac{1}{\varepsilon_1\varepsilon_2}\int_0^t\int_0^t 1_{r-\varepsilon_1<s<r}1_{r-\varepsilon_2<u<r}e^{i\xi\cdot(B_{t-s}-B_{t-u})}dsdu\to 1$ almost surely, which implies $\int_{\mathbb{R}}\int_0^t\int_0^t\phi_{\varepsilon_1}(s-r)\phi_{\varepsilon_2}(u-r)e^{i\xi\cdot(B_{t-s}-B_{t-u})}dsdudr\to t$ almost surely. By the dominated convergence theorem, we have

$$
\mathbb{E}\{\int_0^t W_{\varepsilon_1}(t-s,x+B_s)ds\int_0^t W_{\varepsilon_2}(t-s,x+B_s)ds\}\to\mathcal{R}(0)t. \tag{5.86}
$$

Therefore, for almost every realization of B_s, we can define a random variable $\int_0^t\dot{W}(t-s,x+B_s)ds$ by $\int_0^t\dot{W}(t-s,x+B_s)ds=\lim_{\varepsilon\to 0}\int_0^t W_\varepsilon(t-s,x+B_s)ds$. For fixed B_s, $\int_0^t W_\varepsilon(t-s,x+$

$B_s)ds$ is a zero-mean Gaussian, and by the above calculation, we obtain $\mathbb{E}\{(\int_0^t W_\varepsilon(t-s,x+B_s)ds)^2\} \to \mathcal{R}(0)t$. So $\int_0^t \dot{W}(t-s,x+B_s)ds \sim N(0,\mathcal{R}(0)t)$ for almost every realization of B_s.

The proof is complete. $\square$

Remark 5.15. By the same proof as in Proposition 5.14, we can define $\int_0^t \dot{W}(t-s,x+f_s)ds, \int_0^t \dot{W}(t-s,x+g_s)ds$ for any two continuous paths f_s, g_s, and obtain that

$$\mathbb{E}\{\int_0^t \dot{W}(t-s,x+f_s)ds \int_0^t \dot{W}(t-s,x+g_s)ds\} = \int_0^t \mathcal{R}(f_s-g_s)ds. \tag{5.87}$$

In particular, we can choose $f_s = B_s$ and $g_s = W_s$ for independent Brownian motions B, W.

The solution to the SPDE with multiplicative noise in the Stratonovich's sense

$$\partial_t u_0 = \frac{1}{2}\Delta u_0 + i\dot{W} \circ u_0, \tag{5.88}$$

and initial condition $u_0 = f$ is then defined by Feynman-Kac formula as

$$u_0(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\int_0^t \dot{W}(t-s,x+B_s)ds)\}. \tag{5.89}$$

(5.88) is written in the Itô's form as

$$\partial_t u_0(t,x) = \frac{1}{2}\Delta u_0(t,x) - \frac{1}{2}\mathcal{R}(0)u_0(t,x) + i\dot{W}(t,x)u_0(t,x). \tag{5.90}$$

By [19, Theorem 3.1], the solution given by (5.89) is a mild solution to (5.90).

Our goal is to show that for fixed (t,x), $u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\sqrt{\varepsilon}}\int_0^t V(\frac{s}{\varepsilon}, x+$

$B_s)ds)\}$ converges in distribution to $u_0(t,x)$. Since $u_\varepsilon(t,x), u_0(t,x)$ are both bounded, it suffices to prove the convergence of moments.

Proposition 5.16. *For any $N_1, N_2 \in \mathbb{N}$, as $\varepsilon \to 0$,*

$$\mathbb{E}\{u_\varepsilon(t,x)^{N_1}\overline{u_\varepsilon(t,x)^{N_2}}\} \to \mathbb{E}\{u_0(t,x)^{N_1}\overline{u_0(t,x)^{N_2}}\}. \tag{5.91}$$

Proof. The proof is similar with Proposition 5.6. Wherever the argument can be directly copied here, we do not present the details.

First, by the scaling property of Brownian motion, a change of parameter $\sqrt{\varepsilon} \mapsto \varepsilon$, and the stationarity of V, we have

$$\begin{aligned}&\mathbb{E}\{u_\varepsilon(t,x)^{N_1}\overline{u_\varepsilon(t,x)^{N_2}}\}\\ =&\mathbb{E}\mathbb{E}_B\{\prod_{j=1}^{N_1+N_2} f(x+\varepsilon B^j_{t/\varepsilon^2})\exp(i\sum_{j=1}^{N_1}\varepsilon\int_0^{t/\varepsilon^2} V(s,\varepsilon B^j_s)ds - i\sum_{j=N_1+1}^{N_1+N_2}\varepsilon\int_0^{t/\varepsilon^2} V(s,\varepsilon B^j_s)ds)\},\end{aligned} \tag{5.92}$$

where $B^j, j=1,\ldots,N_1+N_2$ are independent Brownian motions. On the other hand, by the Feynman-Kac representation of u_0, we have

$$\begin{aligned}&\mathbb{E}\{u_0(t,x)^{N_1}\overline{u_0(t,x)^{N_2}}\}\\ =&\mathbb{E}\mathbb{E}_B\{\prod_{j=1}^{N_1+N_2} f(x+B^j_t)\exp(i\sum_{j=1}^{N_1}\int_0^t \dot{W}(t-s,x+B^j_s)ds - i\sum_{j=N_1+1}^{N_1+N_2}\int_0^t \dot{W}(t-s,x+B^j_s)ds)\}.\end{aligned}$$

Therefore, it boils down to show in the annealed sense that

$$\begin{aligned}
&(\varepsilon B^1_{t/\varepsilon^2},\dots,\varepsilon B^{N_1+N_2}_{t/\varepsilon^2},\varepsilon\int_0^{t/\varepsilon^2}V(s,\varepsilon B^1_s)ds,\dots,\varepsilon\int_0^{t/\varepsilon^2}V(s,\varepsilon B^{N_1+N_2}_s)ds)\\
\Rightarrow&(B^1_t,\dots,B^{N_1+N_2}_t,\int_0^t\dot W(t-s,x+B^1_s)ds,\dots,\int_0^t\dot W(t-s,x+B^{N_1+N_2}_s)ds).
\end{aligned}\tag{5.93}$$

By Proposition 5.14 and Remark 5.15, we have for any $a_j\in\mathbb{R}^d, b_j\in\mathbb{R}, j=1,\dots,N_1+N_2$ that

$$\begin{aligned}
&\mathbb{E}\mathbb{E}_B\{\exp(i\sum_{j=1}^{N_1+N_2}a_j\cdot B^j_t+i\sum_{j=1}^{N_1+N_2}b_j\int_0^t\dot W(t-s,x+B^j_s)ds)\}\\
=&\mathbb{E}_B\{\exp(i\sum_{j=1}^{N_1+N_2}a_j\cdot B^j_t)\exp(-\frac12\sum_{j_1,j_2=1}^{N_1+N_2}b_{j_1}b_{j_2}\int_0^t\mathcal{R}(B^{j_1}_s-B^{j_2}_s)ds)\}.
\end{aligned}\tag{5.94}$$

Let $X_\varepsilon(t):=\sum_{j=1}^{N_1+N_2}a_j\cdot\varepsilon B^j_{t/\varepsilon^2}+\sum_{j=1}^{N_1+N_2}b_j\varepsilon\int_0^{t/\varepsilon^2}V(s,\varepsilon B^j_s)ds$, and we prove $\mathbb{E}\mathbb{E}_B\{e^{iX_\varepsilon(t)}\}$ converges to the RHS of (5.94).

Let $\Delta t=\varepsilon^{-\gamma_1}+\varepsilon^{-\gamma_2}$, $0<\gamma_2<\gamma_1<2$ to be determined, and $N=[\frac{t}{\varepsilon^2\Delta t}]\sim t\varepsilon^{\gamma_1-2}$. Define the intervals $I_k=[(k-1)\Delta t,(k-1)\Delta t+\varepsilon^{-\gamma_1}]$ and $J_k=[(k-1)\Delta t+\varepsilon^{-\gamma_1},k\Delta t]$ for $k=1,\dots,N$. By the same discussion, we have

$$\lim_{\varepsilon\to0}\left(\mathbb{E}\mathbb{E}_B\{e^{iX_\varepsilon(t)}\}-\mathbb{E}\mathbb{E}_B\{\exp(i\sum_{j=1}^{N_1+N_2}a_j\cdot\varepsilon B^j_{t/\varepsilon^2}+i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\sum_{k=1}^N\int_{I_k}V(s,\varepsilon B^j_s)ds)\}\right)=0.$$

Then we consider $\mathbb{E}\{\exp(i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\sum_{k=1}^N\int_{I_k}V(s,\varepsilon B^j_s)ds)\}$. By the same discussion, we have for every realization of B^j_s that

$$\lim_{\varepsilon\to0}\left(\mathbb{E}\{\exp(i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\sum_{k=1}^N\int_{I_k}V(s,\varepsilon B^j_s)ds)\}-\prod_{k=1}^N\mathbb{E}\{\exp(i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\int_{I_k}V(s,\varepsilon B^j_s)ds)\}\right)=0.$$

Since

$$\prod_{k=1}^{N}\mathbb{E}\{\exp(i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\int_{I_k}V(s,\varepsilon B_s^j)ds)\}=\exp(\sum_{k=1}^{N}\log\mathbb{E}\{\exp(i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\int_{I_k}V(s,\varepsilon B_s^j)ds)\}),$$

by the same discussion, we have

$$\log\mathbb{E}\{\exp(i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\int_{I_k}V(s,\varepsilon B_s^j)ds)\}=-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2}b_mb_n\varepsilon^2\int_{I_k^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu+o(\varepsilon^{2-\gamma_1}),$$

where $\frac{o(\varepsilon^{2-\gamma_1})}{\varepsilon^{2-\gamma_1}}$ is independent of k, uniformly bounded and goes to zero as $\varepsilon\to 0$. Thus we obtain

$$\begin{aligned}&\prod_{k=1}^{N}\mathbb{E}\{\exp(i\sum_{j=1}^{N_1+N_2}b_j\varepsilon\int_{I_k}V(s,\varepsilon B_s^j)ds)\}\\ =&\exp(-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2}b_mb_n\sum_{k=1}^{N}\varepsilon^2\int_{I_k^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu+\sum_{k=1}^{N}o(\varepsilon^{2-\gamma_1})).\end{aligned}\tag{5.95}$$

Since $\sum_{k=1}^{N}o(\varepsilon^{2-\gamma_1})\to 0$ as $\varepsilon\to 0$, we have

$$\lim_{\varepsilon\to 0}\left(\mathbb{E}\mathbb{E}_B\{e^{iX_\varepsilon(t)}\}-\mathbb{E}_B\{e^{i\sum_{j=1}^{N_1+N_2}a_j\cdot\varepsilon B^j_{t/\varepsilon^2}}e^{-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2}b_mb_n\sum_{k=1}^{N}\varepsilon^2\int_{I_k^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu}\}\right)=0.\tag{5.96}$$

We claim that in (5.96), $\sum_{k=1}^{N}\varepsilon^2\int_{I_k^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu$ can be replaced by $\varepsilon^2\int_{[0,t/\varepsilon^2]^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu$. If this is true, we have

$$\lim_{\varepsilon\to 0}\left(\mathbb{E}\mathbb{E}_B\{e^{iX_\varepsilon(t)}\}-\mathbb{E}_B\{e^{i\sum_{j=1}^{N_1+N_2}a_j\cdot\varepsilon B^j_{t/\varepsilon^2}}e^{-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2}b_mb_n\varepsilon^2\int_{[0,t/\varepsilon^2]^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu}\}\right)=0.$$

By the scaling property of Brownian motion and a change of variables, we have

$$\begin{aligned}
&\mathbb{E}_B\{e^{i\sum_{j=1}^{N_1+N_2} a_j\cdot\varepsilon B^j_{t/\varepsilon^2}} e^{-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2} b_m b_n \varepsilon^2\int_{[0,t/\varepsilon^2]^2} R(s-u,\varepsilon B^m_s-\varepsilon B^n_u)dsdu}\}\\
=&\mathbb{E}_B\{e^{i\sum_{j=1}^{N_1+N_2} a_j\cdot B^j_t} e^{-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2} b_m b_n \varepsilon^{-2}\int_{[0,t]^2} R(\frac{s-u}{\varepsilon^2},B^m_s-B^n_u)dsdu}\}.
\end{aligned} \tag{5.97}$$

For almost every realization of B^j_s, we decompose $\varepsilon^{-2}\int_0^t\int_0^t R(\frac{s-u}{\varepsilon^2}, B^m_s - B^n_u)dsdu = (i)+(ii)$ with

$$(i) \quad = \quad \int_0^t ds\int_0^{s/\varepsilon^2} R(u, B^m_s - B^n_{s-u\varepsilon^2})du, \tag{5.98}$$

$$(ii) \quad = \quad \int_0^t du\int_0^{u/\varepsilon^2} R(s, B^n_u - B^m_{u-s\varepsilon^2})ds. \tag{5.99}$$

By the dominated convergence theorem, we have

$$(i) \to \int_0^t ds\int_0^\infty R(u, B^m_s - B^n_s)du = \frac{1}{2}\int_0^t \mathcal{R}(B^m_s - B^n_s)ds.$$

The same limit holds for (ii). So

$$\frac{1}{\varepsilon^2}\int_0^t\int_0^t R(\frac{s-u}{\varepsilon^2}, B^m_s - B^n_u)dsdu \to \int_0^t \mathcal{R}(B^m_s - B^n_s)ds \tag{5.100}$$

for almost every realization of B^j_s, which implies

$$\begin{aligned}
&\mathbb{E}_B\{e^{i\sum_{j=1}^{N_1+N_2} a_j\cdot B^j_t} e^{-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2} b_m b_n \varepsilon^{-2}\int_{[0,t]^2} R(\frac{s-u}{\varepsilon^2},B^m_s-B^n_u)dsdu}\}\\
\to&\mathbb{E}_B\{e^{i\sum_{j=1}^{N_1+N_2} a_j\cdot B^j_t} e^{-\frac{1}{2}\sum_{m,n=1}^{N_1+N_2} b_m b_n \int_0^t \mathcal{R}(B^m_s-B^n_s)ds}\}
\end{aligned} \tag{5.101}$$

and completes the proof.

What remains is to show that

$$|\sum_{k=1}^{N}\varepsilon^2\int_{I_k^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu-\varepsilon^2\int_{[0,t/\varepsilon^2]^2}R(s-u,\varepsilon B_s^m-\varepsilon B_u^n)dsdu|\to 0. \quad (5.102)$$

By direct calculation, we have

$$\begin{aligned} LHS \lesssim & \sum_{m\neq n}\varepsilon^2\int_{I_m\times I_n}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu+\sum_{m,n=1}^{N}\varepsilon^2\int_{J_m\times J_n}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu \\ & +\sum_{m,n=1}^{N}\varepsilon^2\int_{I_m\times J_n}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu+\varepsilon^2\int_{[0,t/\varepsilon^2]\times[N\Delta t,t/\varepsilon^2]}\sup_{x\in\mathbb{R}^d}|R(s-u,x)|dsdu \\ :=& (I)+(II)+(III)+(IV). \end{aligned} \quad (5.103)$$

We first assume $\varphi^{\frac{1}{2}}(r)\lesssim r^{-\lambda}$. For (I), when $m\neq n$ and $s\in I_m, u\in I_n$, we have $|s-u|\geq \varepsilon^{-\gamma_2}$ hence $\sup_{x\in\mathbb{R}^d}|R(s-u,x)|\lesssim\varphi^{\frac{1}{2}}(\varepsilon^{-\gamma_2})$, so $(I)\lesssim N^2\varepsilon^2\varepsilon^{-2\gamma_1}\varphi^{\frac{1}{2}}(\varepsilon^{-\gamma_2})\lesssim\varepsilon^{\lambda\gamma_2-2}$. If we choose λ sufficiently large (e.g., $\lambda>2/\gamma_2$), $(I)\to 0$ as $\varepsilon\to 0$. The discussion for (II) is contained in the proof of Proposition 5.6. For (III), if $|m-n|\leq 1$, we have a bound of order $N\varepsilon^2\varepsilon^{-\gamma_2}\sim\varepsilon^{\gamma_1-\gamma_2}\to 0$; if $|m-n|\geq 2$, by similar discussion, we have a bound of order $\varepsilon^2N^2\varepsilon^{-\gamma_1-\gamma_2}\varphi^{\frac{1}{2}}(\varepsilon^{-\gamma_1})\lesssim\varepsilon^{\lambda\gamma_1-2}\to 0$. For (IV), we have $(IV)\lesssim\varepsilon^{2-\gamma_1}\to 0$ as $\varepsilon\to 0$. The proof is complete. □

Let $u_\varepsilon(t,x)=u_{\varepsilon,1}+iu_{\varepsilon,2}$ and $u_0(t,x)=u_{0,1}+iu_{0,2}$, then Proposition 5.16 implies that for any $N_1,N_2\in\mathbb{N}$, we have $\mathbb{E}\{u_{\varepsilon,1}^{N_1}u_{\varepsilon,2}^{N_2}\}\to\mathbb{E}\{u_{0,1}^{N_1}u_{0,2}^{N_2}\}$. Since both $u_{\varepsilon,i}$ and $u_{0,i}$ are bounded for $i = 1, 2$, we obtain for any θ_1, θ_2 that

$$\begin{aligned}\mathbb{E}\{e^{i\theta_1 u_{\varepsilon,1}+i\theta_2 u_{\varepsilon,2}}\} &= \sum_{k=0}^{\infty}\frac{1}{k!}\mathbb{E}\{(i\theta_1 u_{\varepsilon,1}+i\theta_2 u_{\varepsilon,2})^k\} \\ &\to \sum_{k=0}^{\infty}\frac{1}{k!}\mathbb{E}\{(i\theta_1 u_{0,1}+i\theta_2 u_{0,2})^k\} = \mathbb{E}\{e^{i\theta_1 u_{0,1}+i\theta_2 u_{0,2}}\},\end{aligned} \tag{5.104}$$

which completes the proof of the case $\alpha = \infty$.

5.4 Appendix

Lemma 5.17. *Assume $|X| \leq C$ is $\mathcal{F}^{k+n}$ measurable, then*

$$|\mathbb{E}\{X|\mathcal{F}_k\} - \mathbb{E}\{X\}| \leq 2C\varphi(n). \tag{5.105}$$

Proof. First, we show (5.105) holds with $\mathcal{F}_k$ replaced by any $A \in \mathcal{F}_k$ with $\mathbb{P}(A) > 0$, i.e.,

$$|\mathbb{E}\{X|A\} - \mathbb{E}\{X\}| \leq 2C\varphi(n). \tag{5.106}$$

This can be done by approximation. Let $X = \sum_k c_k 1_{A_k}$ with $|c_k|$ uniformly bounded by C and $A_i \cap A_j = \emptyset$ when $i \neq j$, then $|\mathbb{E}\{X|A\} - \mathbb{E}\{X\}| \leq C \sum_k |\mathbb{P}(A_k|A) - \mathbb{P}(A_k)|$. Let A_+ be the union of A_k such that $\mathbb{P}(A_k|A) > \mathbb{P}(A_k)$ and A_- the union of the rest. Then we have

$$|\mathbb{E}\{X|A\} - \mathbb{E}\{X\}| \leq C\left(\mathbb{P}(A_+|A) - \mathbb{P}(A_+) + \mathbb{P}(A_-) - \mathbb{P}(A_-|A)\right) \leq 2C\varphi(n). \tag{5.107}$$

Next we assume $\|\mathbb{E}\{X|\mathcal{F}_k\} - \mathbb{E}\{X\}\|_\infty > 2C\varphi(n) + \varepsilon$ for some $\varepsilon > 0$, so there exists a set

$A \in \mathcal{F}_k$ such that $\mathbb{P}(A) > 0$ and $|(\mathbb{E}\{X|\mathcal{F}_k\} - \mathbb{E}\{X\})1_A(\omega)| > (2C\varphi(n) + \varepsilon)1_A(\omega)$. Without lose of generality, assume $(\mathbb{E}\{X|\mathcal{F}_k\} - \mathbb{E}\{X\})1_A(\omega) > (2C\varphi(n) + \varepsilon)1_A(\omega)$, possibly for some other A. Integrating on both sides, we obtain $\mathbb{E}\{X1_A\} - \mathbb{E}\{X\}\mathbb{P}(A) \geq (2C\varphi(n) + \varepsilon)\mathbb{P}(A)$, so

$$\mathbb{E}\{X|A\} - \mathbb{E}\{X\} \geq 2C\varphi(n) + \varepsilon, \tag{5.108}$$

which is a contradiction. The proof is complete. $\square$

Lemma 5.18. *For independent Brownian motions B_t, W_t and any $\gamma > 0, \beta \in (0,1)$, we have*

$$\varepsilon^\gamma \int_0^{\varepsilon^{-\gamma}} \int_0^{\varepsilon^{-\gamma}} R(s-u, \varepsilon^\beta (B_s - B_u)) ds du \to 2 \int_0^\infty R(t,0) dt \tag{5.109}$$

in L^2 and

$$\varepsilon \int_0^{t/\varepsilon^2} \int_0^{t/\varepsilon^2} |R(s-u, \varepsilon^\beta (B_s - W_u))| ds du \to 0 \tag{5.110}$$

in probability.

Proof. Let $\tilde{R}(t,\xi) = \int_{\mathbb{R}^d} R(t,x) e^{-i\xi\cdot x} dx$. For $\varepsilon^\gamma \int_0^{\varepsilon^{-\gamma}} \int_0^{\varepsilon^{-\gamma}} R(s-u, \varepsilon^\beta(B_s - B_u)) ds du$, by direct calculation, we have

$$\begin{aligned}
&\mathbb{E}_B\{\varepsilon^\gamma \int_0^{\varepsilon^{-\gamma}} \int_0^{\varepsilon^{-\gamma}} R(s-u, \varepsilon^\beta (B_s - B_u)) ds du\} \\
=&\varepsilon^\gamma \int_0^{\varepsilon^{-\gamma}} \int_0^{\varepsilon^{-\gamma}} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \tilde{R}(s-u,\xi) e^{-\frac{1}{2}|\xi|^2 \varepsilon^{2\beta}|s-u|} d\xi ds du \\
=&2\varepsilon^\gamma \int_0^{\varepsilon^{-\gamma}} ds \int_0^s \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \tilde{R}(u,\xi) e^{-\frac{1}{2}|\xi|^2 \varepsilon^{2\beta} u} d\xi du \\
=&2 \int_0^1 ds \int_0^{s/\varepsilon^\gamma} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \tilde{R}(u,\xi) e^{-\frac{1}{2}|\xi|^2 \varepsilon^{2\beta} u} d\xi du \\
\to&2 \int_0^1 ds \int_0^\infty \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \tilde{R}(u,\xi) d\xi du = 2\int_0^\infty R(u,0) du,
\end{aligned} \tag{5.111}$$

and

$$\begin{aligned}
&\mathbb{E}_B\{\left(\varepsilon^\gamma \int_0^{\varepsilon^{-\gamma}} \int_0^{\varepsilon^{-\gamma}} R(s-u,\varepsilon^\beta(B_s-B_u))dsdu\right)^2\} \\
=&4\varepsilon^{2\gamma}\int_{[0,\varepsilon^{-\gamma}]^4} 1_{s_1>u_1}1_{s_2>u_2}\mathbb{E}_B\{R(s_1-u_1,\varepsilon^\beta(B_{s_1}-B_{u_1}))R(s_2-u_2,\varepsilon^\beta(B_{s_2}-B_{u_2}))\}dsdu \\
=&4\varepsilon^{2\gamma}\int_{[0,\varepsilon^{-\gamma}]^4} 1_{s_1>u_1}1_{s_2>u_2}\mathbb{E}_B\{R(u_1,\varepsilon^\beta(B_{s_1}-B_{s_1-u_1}))R(u_2,\varepsilon^\beta(B_{s_2}-B_{s_2-u_2}))\}dsdu \\
=&4\int_0^1 ds_1\int_0^{s_1\varepsilon^{-\gamma}} du_1\int_0^1 ds_2\int_0^{s_2\varepsilon^{-\gamma}} du_2 \\
&\frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\tilde{R}(u_1,\xi_1)\tilde{R}(u_2,\xi_2)\mathbb{E}_B\{e^{i\xi_1\cdot\varepsilon^\beta(B_{s_1\varepsilon^{-\gamma}}-B_{s_1\varepsilon^{-\gamma}-u_1})}e^{i\xi_2\cdot\varepsilon^\beta(B_{s_2\varepsilon^{-\gamma}}-B_{s_2\varepsilon^{-\gamma}-u_2})}\}d\xi_1d\xi_2.
\end{aligned} \tag{5.112}$$

For fixed s_i, u_i, $\xi_1\cdot\varepsilon^\beta(B_{s_1\varepsilon^{-\gamma}}-B_{s_1\varepsilon^{-\gamma}-u_1})+\xi_2\cdot\varepsilon^\beta(B_{s_2\varepsilon^{-\gamma}}-B_{s_2\varepsilon^{-\gamma}-u_2})\to 0$ in probability, so we have $\mathbb{E}_B\{\left(\varepsilon^\gamma\int_0^{\varepsilon^{-\gamma}}\int_0^{\varepsilon^{-\gamma}}R(s-u,\varepsilon^\beta(B_s-B_u))dsdu\right)^2\}\to\left(2\int_0^\infty R(u,0)du\right)^2$.

Now we consider $\varepsilon\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}|R(s-u,\varepsilon^\beta(B_s-W_u))|dsdu$. Without lose of generality, we can assume R is positive, so

$$\begin{aligned}
&\mathbb{E}_B\{\varepsilon\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}|R(s-u,\varepsilon^\beta(B_s-W_u))|dsdu\} \\
=&\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\tilde{R}(s-u,\xi)e^{-\frac12|\xi|^2\varepsilon^{2\beta}(s+u)}d\xi dsdu \\
=&2\varepsilon^2\int_0^{t/\varepsilon^2}ds\int_0^s du\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\tilde{R}(u,\xi)e^{-\frac12|\xi|^2\varepsilon^{2\beta}(2s-u)}d\xi \\
\le&2\varepsilon^2\int_0^{t/\varepsilon^2}ds\int_0^s du\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\tilde{R}(u,\xi)e^{-\frac12|\xi|^2\varepsilon^{2\beta}s}d\xi \\
=&2\int_0^t ds\int_0^{s\varepsilon^{-2}}du\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\tilde{R}(u,\xi)e^{-\frac12|\xi|^2\varepsilon^{2\beta-2}s}d\xi\to 0
\end{aligned} \tag{5.113}$$

by the dominated convergence theorem since $\beta\in(0,1)$. The proof is complete. □

Lemma 5.19. *For fixed $\varepsilon>0$, $y_s^\varepsilon=\tau_{(-\varepsilon^{2-\alpha}s,-B_s)}\omega$ is a stationary Markov process, ergodic* *with respect to the invariant measure* $\mathbb{P}$.

Proof. First, since $y^\varepsilon_s = \tau_{(-\varepsilon^{2-\alpha}(s-u),-(B_s-B_u))} y^\varepsilon_u$ for any $u < s$, it is a Markov process. Next, for any $A \in \mathcal{F}$, we have

$$\begin{aligned} \mathbb{E}\mathbb{E}_B\{1_A(y^\varepsilon_s)\} &= \int_\Omega \int_{\mathbb{R}^d} 1_A(\tau_{(-\varepsilon^{2-\alpha}s,-x)}\omega) q_s(x) dx \mathbb{P}(d\omega) \\ &= \int_{\mathbb{R}^d} \int_\Omega 1_A(\tau_{(-\varepsilon^{2-\alpha}s,-x)}\omega) \mathbb{P}(d\omega) q_s(x) dx = \mathbb{P}(A), \end{aligned} \tag{5.114}$$

so $\mathbb{P}$ is an invariant measure and y^ε_s is stationary. Now if for some $A \in \mathcal{F}$, we have

$$\mathbb{E}_B\{1_A(y^\varepsilon_s)\} = \int_{\mathbb{R}^d} 1_A(\tau_{(-\varepsilon^{2-\alpha}s,-x)}\omega) q_s(x) dx = 1_A(\omega) \tag{5.115}$$

for all $s > 0$. Since $q_s(x) > 0$ for all $s > 0, x \in \mathbb{R}^d$, we obtain that $1_A(\tau_{(-\varepsilon^{2-\alpha}s,-x)}\omega) = 1_A(\omega)$ for all $s > 0, x \in \mathbb{R}^d$. Let $s \to 0$, by the strongly continuity of T_x, we obtain $1_A(\tau_{(0,-x)}\omega) = 1_A(\omega)$ for all $x \in \mathbb{R}^d$. Hence,

$$1_A(\tau_{(s,x)}\omega) = 1_A(\omega) \tag{5.116}$$

for all $s \le 0, x \in \mathbb{R}^d$. Let $\omega = \tau_{(-s,x)}\omega$, we have (5.116) holds for all $(s,x) \in \mathbb{R}^{d+1}$. Therefore, by the ergodicity of τ, we conclude that $\mathbb{P}(A) = 0$ or 1 and y^ε_s is ergodic with respect to $\mathbb{P}$. The proof is complete. □

Lemma 5.20. *If V is a mean zero, stationary random field with covariance function $R(t,x)$, and B_s is Brownian motion independent from V, then*

$$\mathbb{E}\mathbb{E}_B\{\left(\varepsilon \int_0^{t/\varepsilon^2} V(s,B_s) ds\right)^2\} \lesssim t \int_{\mathbb{R}^d} \frac{\sup_t |R(t,x)|}{|x|^{d-2}} dx. \tag{5.117}$$

Proof. By direct calculation, we have

$$
\begin{aligned}
&\mathbb{E}\mathbb{E}_B\{\left(\varepsilon\int_0^{t/\varepsilon^2} V(s,B_s)ds\right)^2\} \\
\lesssim&\varepsilon^2\int_0^{t/\varepsilon^2}\int_0^s\int_{\mathbb{R}^d}\sup_t|R(t,x)|\frac{1}{(2\pi u)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2u}}dxduds \\
\lesssim&\varepsilon^2\int_0^\infty du(\frac{t}{\varepsilon^2}-u)1_{u<\frac{t}{\varepsilon^2}}\int_{\mathbb{R}^d}\sup_t|R(t,x)|\frac{1}{(2\pi u)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2u}}dx \\
\lesssim&\varepsilon^2\int_0^\infty d\lambda(\frac{t}{\varepsilon^2}-\frac{|x|^2}{2\lambda})1_{\frac{|x|^2}{2\lambda}<\frac{t}{\varepsilon^2}}\lambda^{\frac{d}{2}-2}e^{-\lambda}\int_{\mathbb{R}^d}\frac{1}{\pi^{\frac{d}{2}}}\sup_t|R(t,x)|\frac{1}{|x|^{d-2}}dx \\
\lesssim&t\int_{\mathbb{R}^d}\frac{\sup_t|R(t,x)|}{|x|^{d-2}}dx.
\end{aligned}
\tag{5.118}
$$

□

Lemma 5.21. *Let $X_i=(t_i,x_i)\in\mathbb{R}^{d+1}, i=1,\ldots,4$, then under Assumption* **??**,

$$
\begin{aligned}
&|\mathbb{E}\{V(X_1)V(X_2)V(X_3)V(X_4)\}-R(X_1-X_2)R(X_3-X_4)| \\
\le&\Psi(X_1-X_3)\Psi(X_2-X_4)+\Psi(X_1-X_4)\Psi(X_2-X_3),
\end{aligned}
\tag{5.119}
$$

where $0\le\Psi(t,x)\le C_\beta 1\wedge(t^2+|x|^2)^{-\beta}$ for any $\beta>0$ and some contant C_β.

Proof. See [17, Lemma 2.3], where the mixing property is required. □

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