Stable Basis and Quantum Cohomology of Cotangent Bundles of Flag Varieties

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Abstract

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The stable envelope for symplectic resolutions, constructed by Maulik and Okounkov, is a key ingredient in their work on quantum cohomology and quantum K-theory of Nakajima quiver varieties. In this thesis, we study the various aspects of the cohomological stable basis for the cotangent bundle of flag varieties. We compute its localizations, use it to calculate the quantum cohomology of the cotangent bundles, and relate it to the Chern–Schwartz–MacPherson class of Schubert cells in the flag variety.

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To my parents.

Chapter 1

Introduction

1.1 Overview of the Main Results

In [MO2012], Maulik and Okounkov defined stable envelope for symplectic resolutions ([Kal2009]), with the aim of studying the quantum cohomology of Nakajiama quiver varieties. Symplectic resolutions, by definition, are smooth holomorphic symplectic varieties (X, ω) endowed with a proper resolution of singularities

$$X \to \operatorname{Spec} H^0(X, \mathcal{O}_X).$$

Important examples include cotangent bundles of flag varieties, hypertoric varieties, Hilbert schemes of points on the plane, and more generally, Nakajima quiver varieties.

In this thesis, we will focus on the stable basis for the cotangent bundles of the flag varieties. We study their localizations, use them to compute the quantum cohomology of the cotangent bundles, and relate them to the Chern–Schwartz–MacPherson classes of Schubert varieties.

1.1.1 Localization Formulas

One of the most useful techniques in equivariant cohomology is the Atiyah–Bott localization formula ([AB1984]). This formula transforms global computations into local ones, which are much easier to handle. In order to apply it, we need to know the local information of the cohomology classes we are interested in. Our first main result is of this kind for the stable basis of the cotangent bundles of the flag varieties.

Let G be a complex semisimple linear algerbaic group, with a Borel subgroup B. Then $T^*(G/B)$ is the resolution of the nilpotent cone $\mathcal{N}$ in the Lie algebra $\mathfrak{g}$. This is the so-called Springer resolution (see [CG2010]), which is ubiquitous in geometric representation theories. As a cotangent bundle, it is naturally equipped with a symplectic form. Thus it is a symplectic resolution. More generally, for any parabolic subgroup P, $T^*(G/P)$ is also a symplectic resolution.

Let A be a maximal torus contained in B, and let $\mathbb{C}^*_\hbar$ act on $T^*(G/B)$ by dilating the cotangent fiber by a weight of $-\hbar$, and act trivially on the base G/B. Then the stable basis $\{\mathrm{stab}_-(y)|y\in W\}$, which depends on a choice of the Weyl chamber in Lie A, is a basis for the localized equivariant cohomology $H^*_{A\times\mathbb{C}^*_\hbar}(T^*(G/B))_{loc}$ (see Section 2.2 for the definition). The torus fixed loci of $T^*(G/B)$ is in one-to-one correspondence with the Weyl group W. For any equivariant cohomology class $\gamma\in H^*_T(T^*(G/B))$, let $\gamma|_w$ denote the restriction of γ to the fixed point corresponding to $w\in W$. The following is our first main result.

Theorem 1.1.1. [Su2017] *Let $y=\sigma_1\sigma_2\cdots\sigma_l$ be a reduced expression for $y\in W$. Then*

$$\mathrm{stab}_-(w)|_y=(-1)^{l(y)}\prod_{\alpha\in R^+\setminus R(y)}(\alpha-\hbar)\sum_{\substack{1\leq i_1<i_2<\cdots<i_k\leq l\\ w=\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}}}\hbar^{l-k}\prod_{j=1}^{k}\beta_{i_j}, \tag{1.1}$$

where σ_i is the simple reflection associated to a simple root α_i, $\beta_i=\sigma_1\cdots\sigma_{i-1}\alpha_i$, and $R(y)=\{\beta_i|1\leq i\leq l\}$. Furthermore, the sum in Equation (1.1) does not depend on the reduced expression for y.

We also have a formula for the opposite Weyl chamber, see Theorem 2.4.9. This should be seen as a analogue of Billey's formula for localization for Schubert varieties ([Bil1999]), which can be obtained from the theorem via a limiting process (see Section 2.6). The formula is also generalized to $T^*(G/P)$ in Corollaries 2.5.3 and 2.5.6. When $G=\mathrm{GL}(n)$, this is also obtained by Rimányi, Tarasov and Varchenko[RTV2015].

The idea of the proof is to use the graded affine Hecke algebra $\mathcal{H}_\hbar$ action. See Section 3.2.1 for the definition. By a theorem of Lusztig [Lus1988], there is an isomorphism

$$H_*^{\mathbb{G}\times\mathbb{C}^*}(T^*(G/B)\times_{\mathcal{N}}T^*(G/B))\simeq\mathcal{H}_\hbar,$$

where $T^*(G/B)\times_{\mathcal{N}}T^*(G/B)$ is called the Steinberg variety. The left hand side acts on cohomology of $T^*(G/B)$ via convolution (see [CG2010]). For simple generators of $\mathcal{H}_\hbar$, it is easy to compute how it acts on the stable basis and the fixed point basis (Lemma 2.4.2). This gives a recursive formula for the localization of stable basis (Corollary 2.4.3), which leads to Theorem 1.1.1.

In various examples, the stable bases turn out to be very interesting objects. In the cotengent bundle case, it is the characteristic cycles of $\mathcal{D}$-modules, see [Gin1986] and [MO2012, Remark 3.5.3].

In the case of Hilbert schemes of points on $\mathbb{C}^2$, it corresponds to Schur functions if we identify the equivariant cohomology ring of Hilbert schemes with the symmetric functions, while the fixed point basis corresponds to Jack symmetric functions, see e.g. [MO2012], [Nak1999]. The transition matrix between these two bases was obtained in [She2013]. See [BFN2014] for a sheaf-theoretic approach to the stable envelopes.

1.1.2 Quantum cohomology

The motivation to calculate the localization of stable basis is to compute the quantum connection (see Section 3.1.2) of $T^*(G/P)$. The case when P is a Borel subgroup, i.e. the Springer resolution case ([CG2010]), is solved by Braverman, Maulik and Okounkov by using an elegant reduction to rank one argument ([BMO2011]). It turns out the quantum connection of the Springer resolution is isomorphic to the affine Knizhnik–Zamolodchikov connection of Cherednik and Matsuo (see Theorem 3.2.1), whose monodromy is computed by Cherednik ([Che2005]). Combing these, Braverman, Maulik and Okounkov confirm a conjecture of Bezrukavnikov relating the monodromy of quantum connection and derived equivalences (see [BMO2011, Section 1.10], [BR2012, Bez2006]).

In the general parabolic case, it seems hard to the author to apply directly the method in [BMO2011], since he does not know how to compute the rank one fibers (see [BMO2011, Section 5.3]) uniformally. Due to an idea of Okounkov, we can first compute, via virtual localizations (see [GP1999]), the $T := A \times \mathbb{C}^*$-equivariant quantum multiplications in terms of the stable basis, and then use the relation between $A \times \mathbb{C}^*$-equivariant cohomology and $G \times \mathbb{C}^*$-equivariant cohomology to get the quantum multiplications in the latter setting.

The stable basis, being a basis for the torus equivariant cohomology after localization, enjoys many good properties (see Theorem 2.2.1), which makes the virtual localization computation much easier than using the other basis, such as the fixed point basis. To be more specific, with the stable basis, we can reduce the calculation to A-equivariant localization. I.e., we can let $\hbar$ equal to 0. Then a result of Okounkov and Pandharipande ([OP2010]) asserts that only some of the fixed components of the moduli space have non-trivial contributions (see Section 3.1.3). This simplifies the calculation dramatically.

The quantum multiplication in terms of stable basis is described by the following theorem, which may be seen as a quantum Chevalley formula ([FW2004, Theorem 10.1]) in the cotangent bundle case. See Section 2.3 for the meaning of the notations.

Theorem 1.1.2. [Su2016] *The quantum multiplication by D_λ in $H_T^*(T^*(G/P))$ is given by:*

$$D_\lambda * \mathrm{stab}_+(\bar{y}) = y(\lambda)\,\mathrm{stab}_+(\bar{y}) - \hbar \sum_{\alpha\in R^+, y\alpha\in R^-} (\lambda,\alpha^\vee)\,\mathrm{stab}_+(\overline{y\sigma_\alpha})$$
$$- \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\left(\mathrm{stab}_+(\overline{y\sigma_\alpha}) + \prod_{\beta\in R_P^+}\frac{\sigma_\alpha\beta}{\beta}\,\mathrm{stab}_+(\bar{y})\right),$$

where y is a minimal representative in yW_P, and $d(\alpha)$ is defined by Equation 3.3.

Using the localization formulas for the stable basis and the following relation

$$H^*_{G\times\mathbb{C}^*}(T^*(G/P)) \simeq H_T^*(T^*(G/P))^W \simeq (\mathrm{sym}\,\mathfrak{a}^*)^{W_P}[\hbar],$$

we obtain the quantum multiplication formula for the $G\times\mathbb{C}^*$-equivariant cohomology.

Theorem 1.1.3. [Su2016] *Under the isomorphism $H^*_{G\times\mathbb{C}^*}(T^*\mathcal{P}) \simeq (\mathrm{sym}\,\mathfrak{a}^*)^{W_P}[\hbar]$, the operator of quantum multiplication by D_λ is given by*

$$D_\lambda * f = \lambda f + \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\left(\frac{\tilde{\sigma}_\alpha(f\prod_{\beta\in R_P^+}(\beta-\hbar))}{\prod_{\beta\in R_P^+}(\beta-\hbar)} - \frac{\prod_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod_{\beta\in R_P^+}\beta} f\right),$$

where $f\in(\mathrm{sym}\,\mathfrak{a}^*)^{W_P}[\hbar]$ and $\tilde{\sigma}_\alpha$ is an element in the graded affine Hekce algebra $\mathcal{H}_\hbar$. Therefore, the $G\times\mathbb{C}^*$-equivariant quantum connection is conjugate to the following one

$$\nabla_\lambda = \frac{d}{d\lambda} - x_\lambda - \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\tilde{\sigma}_\alpha + \cdots,$$

where $\cdots$ denotes some scalar determined by the condition $\nabla_\lambda 1 = -\lambda$.

This confirms a conjecture of Professor Braverman (through private communication), which is also expected by many other experts. In particular, this gives another proof for the quantum connection formula for the Springer resolutions.

For the other examples of symplectic resolutions, such as hypertoric varieties, resolutions of Slodowy slices, Hilbert schemes of points on $\mathbb{C}^2$, more generally, Nakajima varieties ([Nak1998]), and resolutions of slices in the affine Grassmannian ([MV2007]), their quantum cohomologies were studied in [MS2013, BMO2011, OP2010, MO2012, Vis2016] respectively.

In [LT2017], Lam and Templier proved Rietsch's mirror conjecture ([Rie2008]) for G/P when P is maximal and minuscule. One of the steps in the proof is to identify, via the quantum Chevalley formula, the quantum connection of G/P with Frenkel–Gross connection ([FG2009]) for minuscule representation of G. The Frenkel–Gross connection on $\mathbb{P}^1 \setminus \{0, \infty\}$ (see Equation 3.13) is the complex analogue of the Kloosterman sheaf, which is constructed by Heinloth, Ngô and Yun via geometric Langlands techniques ([HNY2013]).

In Section 3.3.3, we construct a regular connection ∇ on the trivial principle G bundle on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ when G is simply-laced (see Equation (3.17)). We show that when P is maximal and minuscule, the quantum connection for $T^*(G/P)$ is isomorphic to ∇ applied to the corresponding highest weight representation (see Theorem 3.3.6 for the precise statement). When $G = \mathrm{GL}(n, \mathbb{C})$, we show this connection is rigid in the sense of Katz (Theorem 3.3.8).

1.1.3 Chern–Schwartz–MacPherson classes

Pulled back to the zero section, the stable basis becomes some interesting classes in the cohomology of flag varieties, which is the so-called Chern–Schwartz–MacPherson classes.

According to a conjecture attributed to Deligne and Grothendieck, there is a unique natural transformation c_* from the constructible functions on a variety X to the homology of X, such that if X is smooth, then $c_*(1_X) = c(TX) \cap [X]$.

The functor was constructed by MacPherson using Chern–Mather class and local Euler obstructions ([Mac1974]). The class $c_*(1_X)$ was shown to coincide with a class defined earlier by Schwartz ([Sch1965]). For a constructible subvariety $W \subset X$, the class $c_{SM}(W) := c_*(1_W)$ is called the Chern–Schwartz–MacPherson (CSM) class of W.

In [AM2015], Aluffi and Mihalcea computed the image of the CSM classes of Schubert varieties under certain Demazure–Lusztig type operator (see Theorem 4.2.1), and obtained a recursive formula for the CSM classes. Combing with the recursive formulas in Corollary 2.4.4 for the stable basis, it is easy to deduce the following formula.

Theorem 1.1.4. [AMSS] *Let $\iota : G/B \to T^*(G/B)$ be the inclusion of the zero section. Then*

$$\iota^*(\mathrm{stab}_+(w))|_{\hbar=1} = (-1)^{\dim G/B} c_{SM}(X_w^\circ),$$

and

$$\iota^*(\mathrm{stab}_-(w))|_{\hbar=1} = (-1)^{\dim G/B} c_{SM}(Y_w^\circ),$$

where $X_w^\circ = BwB/B \subset G/B$ is the Schubert cell associated to $w \in W$, and $Y_w^\circ = B^- wB/B \subset G/B$ is the opposite Schubert cell.

These are also obtained by Rimányi and Varchenko ([RV2015]). In the above theorem, there is a specialization $\hbar = 1$. This leads us to define homogeneous version of the CSM classes, and Theorems 1.1.4, 4.2.1 can be easily generalized (see Section 4.3.2).

The stable basis for opposite chambers are dual to each other (see Remark 2.2.2(2)). So it is natural to consider the dual classes of the CSM classes (See Section 4.4). There are two approaches to it. The first one is to alternate the coefficients of the different degrees of the original classes. The second one is to use the relation between the cohomology pairings on the cotangent bundle and on the zero section. From these two approaches, we obtain a relation between the CSM classes and its alternating class (see Equation (4.4)).

Recall the stable basis is equal to the characteristic cycle of some constructible function on G/B. Then Theorem 1.1.4 is a relation between pullback of characteristic cycle of constructible functions and CSM classes, both of which are well defined for any smooth varieties. This inspired us to find a new formula for the CSM classes in terms of the characteristic cycles (see Theorem 4.5.1), whose proof involves the *shadow* construction of Aluffi ([Alu2004]) and is not presented in this thesis. There is also a closely related approach by Ginzburg ([Gin1986]). Using Theorem 4.5.1, we were able to reprove the index formula (see Theorem 4.5.2).

1.2 Structure of the thesis

The thesis is organized as follows. In Chapter 2, we introduce the stable basis for the cotangent bundle of flag varieties $T^*(G/P)$, prove the localization formulas, and deduce Billey's formula through a limiting process. In Chapter 3.1, we compute the torus equivariant quantum multiplication by divisors in terms of the stable basis, and deduce the $G \times \mathbb{C}^*$-equivariant quantum connection in Chapter 3.2. In Chapter 3.3, we construct a connection with regular singularities on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and show it is isomorphic to the quantum connection of $T^*(G/P)$ when P is maximal and minuscule. Moreover, we show this connection is rigid when G is $\mathrm{GL}(n)$. In Chapter 4, we relate the stable basis and CSM classes for Schubert varieties, and construct the dual classes of the CSM classes. Finally, we give a general formula for CSM classes in terms of characteristic cycles of constructible functions.

1.3 Notations

Throughout this thesis, let G denote a complex semisimple group, with a maximal torus A and a Borel subgroup B containing A. Let B^- dentoe the opposite Borel subgroup containg the maximal torus A. Let R^+ be the positive roots consisting of all the roots in B, and let Δ be the set of simple roots. Let P be a parabolic subgroup of G containing the fixed Borel subgroup B. Let W_P denote the corresponding Weyl subgroup of the Levi factor of P, Δ_P be the simple roots in P, and $R_P^{\pm}$ be the roots in $R^{\pm}$ spanned by Δ_P. Let $\mathcal{B}$ denote the full flag variety G/B, and $\mathcal{P}$ denote the partial flag variety. Let $\mathbb{C}_\hbar^*$ act on $T^*(G/P)$ by dilating the cotangent fiber by a weight of $\operatorname{Lie}\mathbb{C}_\hbar^*$ equaling $-\hbar$ and act trivially on the base G/P. Let T denote the product $A \times \mathbb{C}_\hbar^*$. Let $<$ denote the usual Bruhat order on the Weyl group W, i.e. $y < w$ if $ByB/B \subset \overline{BwB/B}$. We also use it to denote the Bruhat order on W/W_P.

Chapter 2

Cohomological Stable Basis

In this chapter, we prove our first main Theorem 1.1.1, and deduce Billey's formula by taking a limit. The main reference for this chapter is [Su2017].

2.1 Preliminaries on equivariant localization

The most important technique in this chapter is the following Atiyah–Bott localization formula.

Theorem 2.1.1. [AB1984] *Let a torus A acting on a smooth projective variety X with fixed components $X^A = \sqcup_j F_j$. Then for any equivariant cohomology class $\gamma \in H_A^*(X)$, we have*

$$\gamma = \sum_j \iota_{j,*} \frac{\iota_j^* \gamma}{e^A(N_j)},$$

where $\iota_j : F_j \hookrightarrow X$ denotes the inclusion of the j-th fixed component, and $e^A(N_j)$ is the A-equivariant Euler class of the normal bundle of F_j in X.

Therefore, we can define integration on a non-proper variety X whose torus A fixed loci is proper as follows:

$$\int_{[X]^A} : H_A^*(X) \to \operatorname{Frac} H_A^*(pt), \quad \gamma \mapsto \sum_j \int_{[X_j]^A} \frac{\iota_j^* \gamma}{e^A(N_j)},$$

where $\operatorname{Frac} H_A^*(pt)$ is the fraction field of the polynomial ring $H_A^*(pt) \simeq \mathbb{Q}[\operatorname{Lie} A]$. We can also define a pairing on $H_A^*(X)$ as follows: for any $\gamma_1, \gamma_2 \in H_A^*(X)$, define $\langle \gamma_1, \gamma_2 \rangle_X = \int_{[X]^A} \gamma_1 \cup \gamma_2$. More generally, we can define equivariant non-proper pushforward under the condition that the fibers have proper torus fixed component. See [Liu2013] for more details.

2.2 Stable basis for $T^*\mathcal{B}$

In this section, we apply the construction in [MO2012] to define the stable basis for $T^*\mathcal{B}$.

2.2.1 Fixed point set

The A-fixed points of $T^*\mathcal{B}$ is in one-to-one correspondence with the Weyl group W. Let wB denote the fixed point corresponds to $w \in W$, and let ι_w denote the inclusion of wB into $T^*\mathcal{B}$. By Theorem 2.1.1, there is a basis $\{\iota_{w,*}1 | w \in W\}$ for the localized cohomology $H_T^*(T^*\mathcal{B})_{loc} := H_T^*(T^*\mathcal{B}) \otimes_{H_T^*(pt)} \mathrm{Frac}\, H_T^*(pt)$, which is the so called fixed point basis. And we can use equivariant localization to define a pairing $\langle \cdot, \cdot \rangle$ on $H_T^*(T^*\mathcal{B})$. For any $\gamma \in H_T^*(T^*\mathcal{B})$, let $\gamma|_w$ denote the restriction of γ to the fixed point wB.

2.2.2 Chamber decomposition

The cocharacters

$$\sigma : \mathbb{C}^* \to A$$

form a lattice. Let

$$\mathfrak{a}_{\mathbb{R}} = \mathrm{cochar}(A) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Define the torus roots α_i to be the A-weights occurring in the normal bundle to $(T^*\mathcal{B})^A$. Then the root hyperplanes $\alpha_i^\perp$ partition $\mathfrak{a}_{\mathbb{R}}$ into finitely many chambers

$$\mathfrak{a}_{\mathbb{R}} \setminus \bigcup \alpha_i^\perp = \coprod \mathfrak{C}_i.$$

It is easy to see in this case that the torus roots are just the roots for G. Let $+$ denote the chamber such that all roots in R^+ take positive value on it, and let $-$ denote the opposite chamber.

2.2.3 Stable leaves

Let $\mathfrak{C}$ be a chamber. For any fixed point yB, define the stable leaf of yB by

$$\mathrm{Leaf}_{\mathfrak{C}}(yB) = \left\{ x \in T^*\mathcal{B} \,\middle|\, \lim_{z \to 0} \sigma(z) \cdot x = yB \right\}$$

where σ is any cocharacter in $\mathfrak{C}$; the limit is independent of the choice of $\sigma \in \mathfrak{C}$. Then it is easy to see that $\mathrm{Leaf}_+(yB) = T^*_{ByB/B}\mathcal{B}$, and $\mathrm{Leaf}_-(yB) = T^*_{B^-yB/B}\mathcal{B}$, where B^- is the opposite Borel subgroup.

Define a partial order on the fixed points as follows:

$$wB \preceq_{\mathfrak{C}} yB \quad \text{if} \quad \overline{\mathrm{Leaf}_{\mathfrak{C}}(yB)} \cap wB \neq \emptyset.$$

By the description of $\mathrm{Leaf}_+(yB)$, the order $\preceq_+$ is the same as the Bruhat order $\leq$, and $\preceq_-$ is the opposite order. Define the slope of a fixed point yB by

$$\mathrm{Slope}_{\mathfrak{C}}(yB) = \bigcup_{wB \preceq_{\mathfrak{C}} yB} \mathrm{Leaf}_{\mathfrak{C}}(wB).$$

2.2.4 Stable basis

For each $y \in W$, let $T_y^*\mathcal{B}$ and $T_y(T^*\mathcal{B})$ denote $T_{yB}^*\mathcal{B}$ and $T_{yB}(T^*\mathcal{B})$ respectively, and define $\epsilon_y = e^A(T_y^*\mathcal{B})$. Here, e^A denotes the A-equivariant Euler class. I.e., $e^A(T_y^*\mathcal{B})$ is the product of A-weights in the vector space $T_y^*\mathcal{B}$. Let N_y denote the tangent bundle of $T^*\mathcal{B}$ at the fixed point yB. The chamber $\mathfrak{C}$ gives a decomposition of the tangent bundle

$$N_y = N_{y,+} \oplus N_{y,-}$$

into A-weights which are positive and negative on $\mathfrak{C}$ respectively. Since $T = A \times \mathbb{C}^*$, $H_T^*(\mathrm{pt}) = H_A^*(\mathrm{pt})[\hbar]$. The sign in $\pm e(N_{y,-}) \in H_T^*(\mathrm{pt})$ is determined by the condition

$$\pm e(N_{y,-})|_{\hbar=0} = \epsilon_y.$$

The following theorem is Theorem 3.3.4 in [MO2012] applied to $T^*\mathcal{B}$.

Theorem 2.2.1 ([MO2012])**.** *There exists a unique map of $H_T^*(pt)$-modules*

$$\mathrm{stab}_{\mathfrak{C}} : H_T^*((T^*\mathcal{B})^A) \to H_T^*(T^*\mathcal{B})$$

such that for any $y \in W$, $\Gamma = \mathrm{stab}_{\mathfrak{C}}(1_y)$ satisfies:

1. $\mathrm{supp}\,\Gamma \subset \mathrm{Slope}_{\mathfrak{C}}(yB)$,

2. $\Gamma|_y = \pm e(N_{-,y})$, *with sign according to* ϵ_y,

3. $\Gamma|_w$ *is divisible by* $\hbar$, *for any* $wB \prec_{\mathfrak{C}} yB$,

where 1_y is the unit in $H_T^*(yB)$.

From here on, we let $\operatorname{stab}_{\mathfrak{C}}(y)$ denote $\operatorname{stab}_{\mathfrak{C}}(1_y)$.

Remark 2.2.2.

1. The map is defined by a Lagrangian correspondence between $(T^*\mathcal{B})^A \times T^*\mathcal{B}$, hence maps middle degree to middle degree.

2. From the characterization, the transition matrix from $\{\operatorname{stab}_{\mathfrak{C}}(y)|y \in W\}$ to the fixed point basis is a triangular matrix with nontrivial diagonal terms. Hence, after localization, $\{\operatorname{stab}_{\mathfrak{C}}(y)|y \in W\}$ form a basis for the localized cohomology, which is so-called the **stable basis**.

3. It is proved in [MO2012, Theorem 4.4.1] that $\{\operatorname{stab}_{\mathfrak{C}}(y)|y \in W\}$ and $\{(-1)^n \operatorname{stab}_{-\mathfrak{C}}(y)|y \in W\}$ are dual to each other, i.e.,

$$(\operatorname{stab}_{\mathfrak{C}}(y), (-1)^n \operatorname{stab}_{-\mathfrak{C}}(w)) = \delta_{y,w}.$$

Here $n = \dim_{\mathbb{C}} \mathcal{B}$.

2.3 Stable basis for $T^*\mathcal{P}$

A similar construction works for $T^*\mathcal{P}$. In this case, the fixed point set $(T^*\mathcal{P})^A$ corresponds to W/W_P ([BGG1973]). For any $y \in W$, let $\bar{y}$ denote the coset yW_P. Let yP denote the fixed point in $T^*\mathcal{P}$ corresponding to the coset $\bar{y} = yW_P$. Let $T^*_{\bar{y}}\mathcal{P}$ and $T_{\bar{y}}(T^*\mathcal{P})$ denote $T^*_{yP}\mathcal{P}$ and $T_{yP}(T^*\mathcal{P})$, respectively. Define $\epsilon_{\bar{y}} = e^A(T^*_{\bar{y}}\mathcal{P})$. The sign in $\pm e(N_{-,\bar{y}})$ is determined by $\pm e(N_{-,\bar{y}})|_{\hbar=0} = \epsilon_{\bar{y}}$. For any cohomology class $\alpha \in H^*_T(T^*\mathcal{P})$, let $\alpha|_{\bar{y}}$ denote the restriction of α to the fixed point yP. Then the theorem is

Theorem 2.3.1 ([MO2012])**.** *There exists a unique map of $H^*_T(pt)$-modules*

$$\operatorname{stab}_{\mathfrak{C}} : H^*_T((T^*\mathcal{P})^A) \to H^*_T(T^*\mathcal{P})$$

such that for any $\bar{y} \in W/W_P$, $\Gamma = \operatorname{stab}_{\mathfrak{C}}(1_{yP})$ satisfies:

1. $\operatorname{supp}\Gamma \subset \operatorname{Slope}_{\mathfrak{C}}(yP)$,

2. $\Gamma|_{\bar{y}} = \pm e(N_{-,\bar{y}})$, *with sign according to* $\epsilon_{\bar{y}}$,

3. $\Gamma|_{\bar{w}}$ *is divisible by* $\hbar$, *for any* $\bar{w} \prec_{\mathfrak{C}} \bar{y}$,

where 1_{yP} is the unit in $H^*_T(yP)$.

From here on, we let $\operatorname{stab}_{\mathfrak{C}}(\bar{y})$ denote $\operatorname{stab}_{\mathfrak{C}}(1_{yP})$.

Remark 2.3.2. 1. The Bruhat order on W/W_P is defined as follows:

$$yW_P < wW_P \quad \text{if} \quad \overline{ByP/P} \subset \overline{BwP/P}.$$

If the chamber $\mathfrak{C} = +$, then the order $\preceq_+$ is the Bruhat order on W/W_P. If the chamber $\mathfrak{C} = -$, the order is the opposite Bruhat order.

2. The stable basis for $H_T^*(T^*\mathcal{P})_{loc}$ is $\{\mathrm{stab}_{\mathfrak{C}}(\bar{y}) | \bar{y} \in W/W_P\}$.

3. By [MO2012, Theorem 4.4.1], $\{\mathrm{stab}_{\mathfrak{C}}(\bar{y}) | \bar{y} \in W/W_P\}$ and $\{(-1)^{\dim \mathcal{P}} \mathrm{stab}_{-\mathfrak{C}}(\bar{y}) | \bar{y} \in W/W_P\}$ are dual to each other.

From now on, we use $\mathrm{stab}_{\mathfrak{C}}(y)$ to denote the stable basis of $T^*\mathcal{B}$, and $\mathrm{stab}_{\mathfrak{C}}(\bar{y})$ to denote the stable basis of $T^*\mathcal{P}$.

2.4 Restriction formulas for the stable basis of $T^*\mathcal{B}$

In this section, we prove our first main Theorem 1.1.1 and an analogous one for the positive chamber (Theorem 2.4.9).

2.4.1 Proof of Theorem 1.1.1

Let Q be the quotient field of $H_T^*(pt)$, and $F(W, Q)$ be the functions from W to Q. Restriction to fixed points gives a map

$$H_T^*(T^*\mathcal{B}) \to H_T^*((T^*\mathcal{B})^T) = \bigoplus_{w \in W} H_T^*(wB)$$

and embeds $H_T^*(T^*\mathcal{B})$ into $F(W, Q)$ by the localization Theorem 2.1.1.

It is well-known that the diagonal G-orbits on $\mathcal{B} \times \mathcal{B}$ are indexed by the Weyl group (see [Chapter 3][CG2010]). For each simple root $\alpha \in \Delta$, let Y_α be the orbit corresponding to the reflection σ_α. Then

$$\overline{Y_\alpha} = \mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}$$

where $\mathcal{P}_\alpha = G/P_\alpha$ and P_α is the minimal parabolic subgroup corresponding to the simple root α and contains B. Let $T^*_{\overline{Y_\alpha}}(\mathcal{B} \times \mathcal{B})$ be the conormal bundle to $\overline{Y_\alpha}$. This is a Lagrangian correspondence in

$T^*\mathcal{B} \times T^*\mathcal{B}$,

$$\begin{tikzcd} T^*_{\overline{Y_\alpha}}(\mathcal{B} \times \mathcal{B}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & T^*\mathcal{B} \\ T^*\mathcal{B}. & \end{tikzcd}$$

Therefore, it defines a map

$$D_\alpha := \pi_{1,*}\pi_2^* : H_T^*(T^*\mathcal{B}) \to H_T^*(T^*\mathcal{B}),$$

via pullback and pushforward. Define an operator $A_0 : F(W,Q) \to F(W,Q)$ by the formula

$$(A_0\psi)(w) = \frac{\psi(w\sigma_\alpha) - \psi(w)}{w\alpha}(w\alpha - \hbar).$$

A similar operator is defined in [BGG1973]. Then we have the following important commutative diagram.

Proposition 2.4.1. *The diagram*

$$\begin{tikzcd} H_T^*(T^*\mathcal{B}) \arrow[r, hook] \arrow[d, "D_\alpha"'] & F(W,Q) \arrow[d, "A_0"] \\ H_T^*(T^*\mathcal{B}) \arrow[r, hook] & F(W,Q) \end{tikzcd}$$

commutes.

Proof. Since $H_T^*(T^*\mathcal{B})$ has a fixed point basis after localization, it suffices to show that the two paths around the diagram agree on the fixed point basis $\{\iota_{y*}(1)|y \in W\}$. Such an element gives to a function $\psi_y \in F(W,Q)$ characterized by

$$\psi_y(y) = e(T_yT^*\mathcal{B})$$

and $\psi_y(w) = 0$ for $w \neq y$.

Then

$$A_0(\psi_y)(y) = -\frac{y\alpha - \hbar}{y\alpha}\psi_y(y), \quad A_0(\psi_y)(y\sigma_\alpha) = \frac{y\alpha + \hbar}{y\alpha}\psi_y(y)$$

and

$$A_0(\psi_y)(w) = 0, \text{for } w \notin \{y, y\sigma_\alpha\}.$$

Going along the other way of the diagram, we have

$$D_\alpha(\iota_{y*}(1)) = \sum_{w\in W} \frac{\left(T^*_{\overline{Y_\alpha}}(\mathcal{B}\times\mathcal{B}), \iota_{y*}1\otimes\iota_{w*}1\right)}{e(T_wT^*\mathcal{B})}\iota_{w*}(1).$$

By the definition of Y_α, $\left(T^*_{\overline{Y_\alpha}}(\mathcal{B}\times\mathcal{B}), \iota_{y*}1\otimes\iota_{w*}1\right)$ is nonzero if and only if $w\in\{y, y\sigma_\alpha\}$. By localization,

$$\left(T^*_{\overline{Y_\alpha}}(\mathcal{B}\times\mathcal{B}), \iota_{y*}1\otimes\iota_{y*}1\right) = -\frac{y\alpha-\hbar}{y\alpha}e(T_yT^*\mathcal{B})$$

and

$$\left(T^*_{\overline{Y_\alpha}}(\mathcal{B}\times\mathcal{B}), \iota_{y*}1\otimes\iota_{y\sigma_\alpha*}1\right) = \frac{y\alpha-\hbar}{y\alpha}e(T_{y\sigma_\alpha}T^*\mathcal{B}).$$

Hence

$$D_\alpha(\iota_{y*}1) = -\frac{y\alpha-\hbar}{y\alpha}\iota_{y*}1 + \frac{y\alpha-\hbar}{y\alpha}\iota_{y\sigma_\alpha*}1.$$

Therefore,

$$D_\alpha(\iota_{y*}1)(y) = -\frac{y\alpha-\hbar}{y\alpha}\psi_y(y)$$

and

$$D_\alpha(\iota_{y*}1)(y\sigma_\alpha) = \frac{y\alpha-\hbar}{y\alpha}e(T_{y\sigma_\alpha}T^*\mathcal{B})$$

Since α is a simple root,

$$\begin{aligned}
e(T_{y\sigma_\alpha}T^*\mathcal{B}) &= \prod_{\beta\in R^+}(y\sigma_\alpha\beta-\hbar)(-y\sigma_\alpha\beta)\\
&= \prod_{\beta\in R^+\setminus\{\alpha\}}(y\sigma_\alpha\beta-\hbar)(-y\sigma_\alpha\beta)\cdot(-y\alpha-\hbar)y\alpha\\
&= \frac{y\alpha+\hbar}{y\alpha-\hbar}e(T_yT^*\mathcal{B}),
\end{aligned}$$

so we get

$$D_\alpha(\iota_{y*}1)(y\sigma_\alpha) = \frac{y\alpha+\hbar}{y\alpha}\psi_y(y).$$

Since $D_\alpha(\iota_{y*}1)$ and $A_0(\psi_y)$ take the same values on W,

$$D_\alpha(\iota_{y*}1) = A_0(\psi_y).$$

$\square$

The image of the stable basis under the operator D_α is given by the following lemma.

Lemma 2.4.2.

$$D_\alpha \operatorname{stab}_\pm(y) = -\operatorname{stab}_\pm(y) - \operatorname{stab}_\pm(y\sigma_\alpha).$$

Proof. We only prove for the + case; the − case can be proved similarly.

By Remark 2.2.2(3), the lemma is equivalent to

$$(D_\alpha \operatorname{stab}_+(y), (-1)^n \operatorname{stab}_-(w)) = \begin{cases} -1 & w \in \{y, y\sigma_\alpha\} \\ 0 & \text{otherwise.} \end{cases}$$

Since $T^*_{\overline{Y_\alpha}}(\mathcal{B}\times\mathcal{B})$ is a Steinberg correspondence in $T^*\mathcal{B}\times T^*\mathcal{B}$,

$$(D_\alpha \operatorname{stab}_+(y), (-1)^n \operatorname{stab}_-(w))$$

is a proper intersection number (see [MO2012, Section 3.2.6, 4.6]). Hence it lies in the nonlocalized coefficient ring $H_T^*(\mathrm{pt})$. A degree count shows it actually lies in $H_T^0(pt) = \mathbb{Q}$. So it is a constant. Therefore we can let $\hbar = 0$. Then Properties (1) and (3) of Theorem 2.2.1 shows $(\operatorname{stab}_\pm(y)|_w)|_{\hbar=0}$ is nonzero if and only if $y = w$. Using the second property, it is easy to calculate that the intersection number. □

Applying Proposition 2.4.1 to the stable basis $\{\operatorname{stab}_-(w)|w\in W\}$, we get

Corollary 2.4.3. *The stable basis* $\{\operatorname{stab}_-(w)|w \in W\}$ *are uniquely characterized by the following properties:*

1. $\operatorname{stab}_-(w)|_y = 0$, *unless* $y \geq w$.
2. $\operatorname{stab}_-(w)|_w = \prod_{\alpha\in R^+, w\alpha\in R^+}(w\alpha-\hbar)\prod_{\alpha\in R^+, w\alpha\in -R^+} w\alpha.$
3. *For any simple root* α, *and* $\ell(y\sigma_\alpha) = \ell(y)+1$,

$$\operatorname{stab}_-(w)|_{y\sigma_\alpha} = -\frac{\hbar}{y\alpha-\hbar}\operatorname{stab}_-(w)|_y - \frac{y\alpha}{y\alpha-\hbar}\operatorname{stab}_-(w\sigma_\alpha)|_y.$$

Proof. It is easy to see that $\{\operatorname{stab}_-(w)|w\in W\}$ satisfies these properties: (1) and (2) follow directly from Theorem 2.2.1, and (3) follows from Proposition 2.4.1 and Lemma 2.4.2.

To show these properties uniquely determines a cohomology class is equivalent to show these properties uniquely determine the values $\operatorname{stab}_-(w)|_y$. We argue by ascending induction on the length

$l(y)$ of y. Note that $\mathrm{stab}_-(w)|_1$ is determined by (1) and (2). Assume that $l(y\sigma_\alpha) = l(y) + 1$ for some simple root α. Then $\mathrm{stab}_-(w)|_{y\sigma_\alpha}$ is determined by $\mathrm{stab}_-(w)|_y$ and $\mathrm{stab}_-(w\sigma_\alpha)|_y$ by (3), which are known by the induction hypothesis. □

For the positive chamber, we get

Corollary 2.4.4. *The stable basis* $\{\mathrm{stab}_+(y)|y \in W\}$ *are uniquely characterized by the following properties:*

1. $\mathrm{stab}_+(y)|_w = 0$, *unless* $w \leq y$.
2. $\mathrm{stab}_+(y)|_y = \prod_{\alpha\in R^+, y\alpha<0} (y\alpha - \hbar) \prod_{\alpha\in R^+, y\alpha>0} y\alpha.$
3. *For any simple root* α, *and* $\ell(y\sigma_\alpha) = \ell(y) + 1$,

$$\mathrm{stab}_+(y\sigma_\alpha)|_w = -\frac{\hbar}{w\alpha}\,\mathrm{stab}_+(y)|_w - \frac{w\alpha - \hbar}{w\alpha}\,\mathrm{stab}_+(y)|_{w\sigma_\alpha}.$$

The proof is almost the same as the proof of Corollary 2.4.3, so we omit it.

We now prove Theorem 1.1.1. We show that the formula given in Theorem 1.1.1 does not depend on the reduced expression of y, and it satisfies the properties in Corollary 2.4.3.

Let Λ be the root lattice, and let $\mathcal{A}$ be the algebra over $\mathbb{Q}[\Lambda](\hbar)$ generated by $\{u_w|w \in W\}$, with relations

$$u_w u_y = u_{wy}, u_w f = f u_w,$$

where $f \in \mathbb{Q}[\Lambda](\hbar)$. For a reduced word $y = \sigma_1 \cdots \sigma_l$, define

$$R_{\alpha_1,\cdots,\alpha_l} = \prod_{i=1}^{l} \left(1 + \frac{\beta_i}{\hbar} u_{\sigma_i}\right),$$

where $\beta_i = \sigma_1 \cdots \sigma_{i-1}\alpha_i$. Expanding it, we have

$$R_{\alpha_1,\cdots,\alpha_l} = \frac{\prod_{i=1}^{l}(\hbar - \beta_i)}{\hbar^l \prod_{\alpha\in R^+}(\alpha - \hbar)} \sum_w \mathrm{stab}_-(w)|_{\sigma_1\cdots\sigma_l} u_w \tag{2.1}$$

where $\mathrm{stab}_-(w)|_{\sigma_1\cdots\sigma_l}$ is given by Theorem 1.1.1. We first prove

Proposition 2.4.5. $R_{\alpha_1,\cdots,\alpha_l}$ *does not depend on the reduced expression of* $y = \sigma_1 \cdots \sigma_l$. *Hence we can denote it by* R_y.

Remark 2.4.6.

1. With Equation (2.1), this shows that the sum in Equation (1.1) does not depend on the reduced expression of y.
2. In [Bil1999], Billey proves this case by case when the Weyl group is replaced by the nil-Coxeter group, which is defined by adding the relations $u_{\sigma_\alpha}^2 = 0$ for any simple root α.
3. The independence can also be proved easily as in [LZ2014].

Proof. Let $\sigma_1'\sigma_2'\cdots\sigma_l'$ be a different reduced expression for $\sigma_1\cdots\sigma_l$ that only differs in positions $p+1,\ldots,p+m$, with

$$\sigma_{p+1},\cdots,\sigma_{p+m} = \sigma_\alpha,\sigma_\beta,\sigma_\alpha,\sigma_\beta,\cdots$$

and

$$\sigma'_{p+1},\cdots,\sigma'_{p+m} = \sigma_\beta,\sigma_\alpha,\sigma_\beta,\sigma_\alpha,\cdots$$

for some simple roots α,β, and $m = m(\alpha,\beta) =$ order of $\sigma_\alpha\sigma_\beta$. It is well-known that every reduced expression can be obtained from any other by a series of transformations of this type.

Since $(1+\sigma_i\sigma u) = \sigma_i(1+\sigma u)$, we have

$$R_{\alpha_1,\cdots,\alpha_l} = R_{\alpha_1,\cdots,\alpha_i}\sigma_1\sigma_2\cdots\sigma_i R_{\alpha_i+1,\cdots,\alpha_l}.$$

Hence,

$$R_{\alpha_1,\cdots,\alpha_l} = R_{\alpha_1,\cdots,\alpha_p}\sigma_1\cdots\sigma_p R_{\alpha,\beta,\alpha,\ldots}\sigma_1\cdots\sigma_p\sigma_\alpha\sigma_\beta\sigma_\alpha\cdots R_{\alpha_{p+m+1},\cdots,\alpha_l},$$

so we only need to prove

$$R_{\alpha,\beta,\alpha,\cdots} = R_{\beta,\alpha,\beta,\cdots}.$$

We show it case by case. We use letter α_i for α, and α_j for β.

1. $m=2$. Then $\sigma_i\alpha_j = \alpha_j, \sigma_j\alpha_i = \alpha_i$. Therefore,

$$\begin{aligned}
R_{\alpha_i,\alpha_j} &= \left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right)\\
&= 1+\frac{\alpha_i}{\hbar}u_{\sigma_i}+\frac{\alpha_j}{\hbar}u_{\sigma_j}+\frac{\alpha_i\alpha_j}{\hbar^2}u_{\sigma_i\sigma_j}\\
&= R_{\alpha_j,\alpha_i}.
\end{aligned}$$

2. $m=3$. Then

$$\sigma_i\alpha_j = \sigma_j\alpha_i = \alpha_i+\alpha_j.$$

Therefore,

$$\begin{aligned}
R_{\alpha_i,\alpha_j,\alpha_i} =& \left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\sigma_i\sigma_j\alpha_i}{\hbar}u_{\sigma_i}\right)\\
=&1+\frac{\alpha_i\alpha_j}{\hbar^2}+\frac{\alpha_i+\alpha_j}{\hbar}(u_{\sigma_i}+u_{\sigma_j})\\
&+\frac{\alpha_i(\alpha_i+\alpha_j)}{\hbar^2}u_{\sigma_i\sigma_j}+\frac{\alpha_j(\alpha_i+\alpha_j)}{\hbar^2}u_{\sigma_j\sigma_i}+\frac{\alpha_i\alpha_j(\alpha_i+\alpha_j)}{\hbar^3}u_{\sigma_i\sigma_j\sigma_j}\\
=&R_{\alpha_j,\alpha_i,\alpha_j}.
\end{aligned}$$

3. $m=4$. Without loss of generality, assume α_i is the short root. Then

$$\sigma_i\sigma_j\sigma_i\alpha_j=\alpha_j, \qquad \sigma_j\sigma_i\sigma_j\alpha_i=\alpha_i,$$

$$\sigma_i\sigma_j\alpha_i=\sigma_j\alpha_i=\alpha_i+\alpha_j, \qquad \sigma_j\sigma_i\alpha_j=\sigma_i\alpha_j=2\alpha_i+\alpha_j.$$

Therefore,

$$\begin{aligned}
R_{\alpha_i,\alpha_j,\alpha_i,\alpha_j} &= \left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\sigma_i\sigma_j\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\sigma_i\sigma_j\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right)\\
&=\left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{2\alpha_i+\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\alpha_i+\alpha_j}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\alpha_j}{\hbar}u_{\sigma_j}\right)
\end{aligned}$$

and

$$\begin{aligned}
R_{\alpha_j,\alpha_i,\alpha_i,\alpha_j} &= \left(1+\frac{\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\sigma_j\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\sigma_j\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\sigma_j\sigma_i\sigma_j\alpha_i}{\hbar}u_{\sigma_i}\right)\\
&=\left(1+\frac{\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\alpha_i+\alpha_j}{\hbar}u_{\sigma_i}\right)\left(1+\frac{2\alpha_i+\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right).
\end{aligned}$$

Due to Billey's calculation in [Bil1999], we only have to compare the coefficients of $1=u_{\sigma_i}^2=u_{\sigma_j}^2, u_{\sigma_i}=u_{\sigma_j}^2u_{\sigma_i}=u_{\sigma_i}u_{\sigma_j}^2$, and $u_{\sigma_j}=u_{\sigma_i}^2u_{\sigma_j}=u_{\sigma_j}u_{\sigma_i}^2$. It is easy to see this by a direct calculation.

4. $m=6$. Without loss of generality, assume α_i is the short root. Then

$$\sigma_i\sigma_j\sigma_i\sigma_j\sigma_i\alpha_j=\alpha_j, \qquad \sigma_j\sigma_i\sigma_j\sigma_i\sigma_j\alpha_i=\alpha_i,$$

$$\sigma_i\sigma_j\sigma_i\sigma_j\alpha_i=\sigma_j\alpha_i=\alpha_i+\alpha_j, \qquad \sigma_j\sigma_i\sigma_j\sigma_i\alpha_j=\sigma_i\alpha_j=3\alpha_i+\alpha_j,$$

$$\sigma_i\sigma_j\sigma_i\alpha_j=\sigma_j\sigma_i\alpha_j=3\alpha_i+2\alpha_j, \qquad \sigma_j\sigma_i\sigma_j\alpha_i=\sigma_i\sigma_j\alpha_i=2\alpha_i+\alpha_j.$$

Therefore,

$$\begin{aligned}
R_{\alpha_i,\alpha_j,\alpha_i,\alpha_j,\alpha_i,\alpha_j} &= \left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\sigma_i\sigma_j\alpha_i}{\hbar}u_{\sigma_i}\right) \\
&\quad \left(1+\frac{\sigma_i\sigma_j\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\sigma_i\sigma_j\sigma_i\sigma_j\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\sigma_i\sigma_j\sigma_i\sigma_j\sigma_i\alpha_j}{\hbar}u_{\sigma_j}\right) \\
&= \left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right)\left(1+\frac{3\alpha_i+\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{2\alpha_i+\alpha_j}{\hbar}u_{\sigma_i}\right) \\
&\quad \left(1+\frac{3\alpha_i+2\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\alpha_i+\alpha_j}{\hbar}u_{\sigma_i}\right)\left(1+\frac{\alpha_j}{\hbar}u_{\sigma_j}\right)
\end{aligned}$$

and

$$\begin{aligned}
R_{\alpha_j,\alpha_i,\alpha_j,\alpha_i,\alpha_j,\alpha_i} &= \left(1+\frac{\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\alpha_i+\alpha_j}{\hbar}u_{\sigma_i}\right)\left(1+\frac{3\alpha_i+2\alpha_j}{\hbar}u_{\sigma_j}\right) \\
&\quad \left(1+\frac{2\alpha_i+\alpha_j}{\hbar}u_{\sigma_i}\right)\left(1+\frac{3\alpha_i+\alpha_j}{\hbar}u_{\sigma_j}\right)\left(1+\frac{\alpha_i}{\hbar}u_{\sigma_i}\right).
\end{aligned}$$

Similarly to Case (3), we can show the coefficients of the corresponding terms are the same.

$\square$

Next we prove

Proposition 2.4.7. *The formula in Theorem 1.1.1 satisfies the properties in Corollary 2.4.3.*

Proof.

1. Property (1) follows from [Spr2009, Proposition 8.5.5].

2. Property (2) follows from the fact

$$\{y\alpha|\alpha \in R^+, y\alpha \in -R^+\} = \{-\beta_i|1 \le i \le l\}.$$

3. Suppose $y = \sigma_1 \cdots \sigma_l$ is reduced and $l(y\sigma_\alpha) = l(y)+1$. Then by definition

$$R_{\alpha_1,\cdots,\alpha_l,\alpha} = R_{\alpha_1,\cdots,\alpha_l} yR_\alpha,$$

where $yR_\alpha = (1+\frac{y\alpha}{\hbar}u_{\sigma_\alpha})$. Using Equation (2.1), we get

$$\frac{\hbar - y\alpha}{\hbar}\operatorname{stab}_-(w)|_{y\sigma_\alpha} = \operatorname{stab}_-(w)|_y + \frac{y\alpha}{\hbar}\operatorname{stab}_-(w\sigma_\alpha)|_y,$$

which is precisely property (3) in corollary 2.4.3.

□

This finishes the proof of Theorem 1.1.1. We give a corollary of it, which will be used in Chapter 3.

Corollary 2.4.8.

$$\operatorname{stab}_-(w)|_y \equiv \begin{cases} (-1)^{l(y)+1}\dfrac{\hbar \prod\limits_{\alpha\in R^+}\alpha}{y\beta} & \pmod{\hbar^2} \quad \text{if } w = y\sigma_\beta < y \text{ for some } \beta \in R^+, \\ \\ 0 & \pmod{\hbar^2} \quad \text{otherwise.} \end{cases}$$

Proof. Theorem 1.1.1 implies that $\operatorname{stab}_-(w)|_y \pmod{\hbar^2}$ is nonzero if and only if $w = \sigma_1\cdots\hat{\sigma_i}\cdots\sigma_l$ for some i. Then $w = y\sigma_\beta$ with $\beta = \sigma_l\cdots\sigma_{i+1}\alpha_i$ and $\beta_i = -y\beta$. And every element $w = y\sigma_\beta$ such that $w < y$ is of the form $\sigma_1\cdots\hat{\sigma_i}\cdots\sigma_l$ for some i. Putting these into Equation (1.1) gives the desired result. □

2.4.2 Restriction formula for the positive chamber

For the positive chamber, we have the following localization formulas.

Theorem 2.4.9. *Let $y = \sigma_1\sigma_2\cdots\sigma_l$ be a reduced expression for $y \in W$, and $w \le y$. Then*

$$\operatorname{stab}_+(y)|_w = \sum_{\substack{1\le i_1<i_2<\cdots<i_k\le l\\ w=\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}}} (-1)^l \prod_{j=1}^{k} \frac{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}-\hbar}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}} \tag{2.2}$$

$$\frac{\hbar^{l-k}}{\prod\limits_{j=0}^{k}\prod\limits_{i_j<r<i_{j+1}}\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_r}\prod_{\alpha\in R^+}\alpha. \tag{2.3}$$

Furthermore, the sum in Equation (2.2) does not depend on the reduced expression for y.

Let us consider the semidirect product $Q \rtimes W$, where Q is the quotient field of $H^*_T(\mathrm{pt})$. Let u_w denote the element w in the Weyl group W. The action of W on Q is induced from the action of W on the Lie algebra of the maximal torus A, and W acts trivially on $\hbar$. For example, for any root α, $u_w\frac{\hbar+\alpha}{\alpha} = \frac{\hbar+w\alpha}{w\alpha}u_w$.

For any reduced decomposition $\sigma_{\alpha_1}\cdots\sigma_{\alpha_l}$ of y, we define the following two functions

$$\xi_{\alpha_1,\alpha_2,\cdots,\alpha_l}(w) = \sum_{\substack{1\le i_1<i_2<\cdots<i_k\le l\\ w=\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}}} \prod_{j=1}^{k} \frac{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}-\hbar}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}} \frac{\hbar^{l-k}}{\prod_{j=0}^{k}\prod_{i_j<r<i_{j+1}}\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_r} \in Q,$$

and

$$R_{\alpha_1,\alpha_2,\cdots,\alpha_l} := \prod_{i=1}^{l}\left(\frac{\hbar}{\alpha_i}+\frac{\hbar+\alpha_i}{\alpha_i}u_{\sigma_i}\right) \in Q\rtimes W.$$

To expand $R_{\alpha_1,\alpha_2,\cdots,\alpha_l}$, we need to find the coefficient of u_w for every $w\in W$. For any subsequence $i_1<i_2<\cdots<i_k$ of $1<2<\cdots<l$ such that $u_w=u_{\sigma_{i_1}}u_{\sigma_{i_2}}\ldots u_{\sigma_{i_k}}$, we get the following term

$$\frac{\hbar}{\alpha_1}\frac{\hbar}{\alpha_2}\cdots\frac{\hbar+\alpha_{i_1}}{\alpha_{i_1}}u_{\sigma_{i_1}}\frac{\hbar}{\alpha_{i_1+1}}\cdots\frac{\hbar+\alpha_{i_2}}{\alpha_{i_2}}u_{\sigma_{i_2}}\cdots$$
$$\frac{\hbar+\alpha_{i_{k-1}}}{\alpha_{i_{k-1}}}u_{\sigma_{i_{k-1}}}\frac{\hbar}{\alpha_{i_{k-1}+1}}\cdots\frac{\hbar}{\alpha_{i_k-1}}\frac{\hbar+\alpha_{i_k}}{\alpha_{i_k}}u_{\sigma_{i_k}}\frac{\hbar}{\alpha_{i_k+1}}\cdots\frac{\hbar}{\alpha_l}.$$

Since $u_w\frac{\hbar+\alpha}{\alpha}=\frac{\hbar+w\alpha}{w\alpha}u_w$, the above term is equal to

$$\prod_{j=1}^{k}\frac{\hbar+\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_{j-1}}\alpha_{i_j}}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_{j-1}}\alpha_{i_j}}\frac{\hbar^{l-k}}{\prod_{j=0}^{k}\prod_{i_j<r<i_{j+1}}\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_r}u_w$$
$$=\prod_{j=1}^{k}\frac{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}-\hbar}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}}\frac{\hbar^{l-k}}{\prod_{j=0}^{k}\prod_{i_j<r<i_{j+1}}\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_r}u_w.$$

Therefore,

$$R_{\alpha_1,\alpha_2,\cdots,\alpha_l}=\sum_{w\in W}\xi_{\alpha_1,\alpha_2,\cdots,\alpha_l}(w)u_w. \tag{2.4}$$

Similarly to Proposition 2.4.5, we have

Proposition 2.4.10. *$R_{\alpha_1,\alpha_2,\cdots,\alpha_l}$ does not depend on the choice of the reduced expression for y.*

This can be checked case by case as in Proposition 2.4.5. We omit the details. With Equation (2.4), this shows that the sum in Theorem 2.4.9 does not depend on the reduced expression for y.

Finally, we prove

Proposition 2.4.11. *The formula in Theorem 2.4.9 satisfies the properties in Corollary 2.4.4.*

Proof.

1. Property (1) follows from [Spr2009, Proposition 8.5.5].

2. Property (2) follows from the fact

$$\{y\alpha | \alpha \in R^+, y\alpha \in -R^+\} = \{-\beta_i | 1 \leq i \leq l\}.$$

3. Suppose $y = \sigma_1 \cdots \sigma_l$ is reduced and $l(y\sigma_\alpha) = l(y) + 1$. Then by definition

$$R_{\alpha_1,\alpha_2,\cdots,\alpha_l,\alpha} = R_{\alpha_1,\alpha_2,\cdots,\alpha_l} \left(\frac{\hbar}{\alpha} + \frac{\hbar + \alpha}{\alpha} u_{\sigma_\alpha} \right),$$

   Equation (2.4) gives

$$\begin{aligned} \sum_w \xi_{\alpha_1,\alpha_2,\cdots,\alpha_l,\alpha}(w) u_w &= \sum_{w'} \xi_{\alpha_1,\alpha_2,\cdots,\alpha_l}(w') u_{w'} \left(\frac{\hbar}{\alpha} + \frac{\hbar + \alpha}{\alpha} u_{\sigma_\alpha} \right) \\ &= \sum_w \left(\frac{\hbar}{w\alpha} \xi_{\alpha_1,\alpha_2,\cdots,\alpha_l}(w) + \frac{w\alpha - \hbar}{w\alpha} \xi_{\alpha_1,\alpha_2,\cdots,\alpha_l}(w\sigma_\alpha) \right) u_w. \end{aligned}$$

   Hence

$$\xi_{\alpha_1,\alpha_2,\cdots,\alpha_l,\alpha}(w) = \frac{\hbar}{w\alpha} \xi_{\alpha_1,\alpha_2,\cdots,\alpha_l} + \frac{w\alpha - \hbar}{w\alpha} \xi_{\alpha_1,\alpha_2,\cdots,\alpha_l}(w\sigma_\alpha)$$

   which is precisely property (3).

□

As in Corollary 2.4.8, modulo $\hbar^2$, we get

Corollary 2.4.12.

$$\mathrm{stab}_+(y)|_w \equiv \begin{cases} (-1)^{l(y)+1} \dfrac{\hbar \prod\limits_{\alpha \in R^+} \alpha}{y\beta} \pmod{\hbar^2} & \textit{if } w = y\sigma_\beta < y \textit{ for some } \beta \in R^+, \\ \\ 0 \pmod{\hbar^2} & \textit{otherwise.} \end{cases}$$

This follows from the proof of Corollary 2.4.8 and Theorem 2.4.9.

2.5 Restriction of stables basis for $T^*\mathcal{P}$

In this section, we extend Theorems 1.1.1 and 2.4.9 to $T^*\mathcal{P}$ case. In type A, these formulas were also obtained in [She2013] via abelianization and in [RTV2015] using weight functions.

We need the following lemma from [BGG1973].

Lemma 2.5.1. *Each coset W/W_P contains exactly one element of minimal length, which is characterized by the property that it maps the simple roots in P into R^+.*

Let π be the projection map from $\mathcal{B}$ to $\mathcal{P}$, and Γ_π be its graph. Then the conormal bundle to Γ_π in $\mathcal{B} \times \mathcal{P}$ is a Lagrangian submanifold of $T^*(\mathcal{B} \times \mathcal{P})$,

$$\begin{tikzcd} T^*_{\Gamma_\pi}(\mathcal{B} \times \mathcal{P}) \arrow[r, "p_1"] \arrow[d, "p_2"'] & T^*\mathcal{B} \\ T^*\mathcal{P}, & \end{tikzcd}$$

where p_2 is proper.

Let

$$D_1 = p_{2*}p_1^* : H_T^*(T^*\mathcal{B}) \to H_T^*(T^*\mathcal{P}),$$

and

$$D_2 = p_{1*}p_2^* : H_T^*(T^*\mathcal{P}) \to H_T^*(T^*\mathcal{B})_{loc}$$

be the maps induced by the correspondence $T^*_{\Gamma_\pi}(\mathcal{B} \times \mathcal{P})$. The image for the second map is the localized cohomology since p_1 is not proper. The pushforward p_{1*} is defined using torus localization (see the discussion at the beginning of this chapter).

Recall we have an embedding of $H_T^*(T^*\mathcal{B})$ into $F(W, Q)$ by restricting every cohomology class to fixed points. Since the fixed point set $(T^*\mathcal{P})^T$ is in one-to-one correspondence with W/W_P, we can embed $H_T^*(T^*\mathcal{P})$ into $F(W/W_P, Q)$. Recall we use $\bar{y}$ to denote the coset yW_P, and use $\gamma|_{\bar{y}} \in H_T^*(pt)$ to denote the restriction of $\gamma \in H_T^*(T^*\mathcal{P})$ to the fixed point yP.

Define a map

$$A_1 : F(W, Q) \to F(W/W_P, Q)$$

as follows: for any $\psi \in F(W, Q)$,

$$A_1(\psi)(\bar{w}) = \sum_{\bar{z}=\bar{w}} \frac{\psi(z)}{\prod\limits_{\alpha \in R_P^+} (-z\alpha)}.$$

Then as Proposition 2.4.1, we have

Proposition 2.5.2. *The diagram*

$$\begin{tikzcd}
H_T^*(T^*\mathcal{B}) \arrow[r, hook] \arrow[d, "D_1"'] & F(W,Q) \arrow[d, "A_1"] \\
H_T^*(T^*\mathcal{P}) \arrow[r, hook] & F(W/W_P,Q)
\end{tikzcd}$$

commutes.

Proof. The proof is almost the same as that of Proposition 2.4.1. We show the two paths agree on the fixed point basis.

By the definition of A_1,

$$\begin{aligned}
A_1(\iota_{y*}1)(\bar{w}) &= \delta_{\bar{y},\bar{w}} \frac{e(T_y T^*\mathcal{B})}{\prod\limits_{\alpha\in R_P^+}(-y\alpha)} \\
&= \delta_{\bar{y},\bar{w}} e(T_{\bar{y}}T^*\mathcal{P}) \prod_{\alpha\in R_P^+}(y\alpha-\hbar).
\end{aligned}$$

By localization,

$$\begin{aligned}
D_1(\iota_{y*}1) &= \sum_{\bar{w}} \frac{(D_1(\iota_{y*}1),\iota_{\bar{w}*}1)}{e(T_{\bar{w}}T^*\mathcal{P})}\iota_{\bar{w}*}1 \\
&= \prod_{\alpha\in R_P^+}(y\alpha-\hbar)\iota_{\bar{y}*}1,
\end{aligned}$$

where $\iota_{\bar{y}}$ is the inclusion of the fixed point yP into $T^*\mathcal{P}$. Hence

$$\begin{aligned}
D_1(\iota_{y*}1)|_{\bar{w}} &= \delta_{\bar{y},\bar{w}} e(T_{\bar{y}}T^*\mathcal{P}) \prod_{\alpha\in R_P^+}(y\alpha-\hbar) \\
&= A_1(\iota_{y*}1)(\bar{w})
\end{aligned}$$

as desired. $\square$

If we apply this proposition to the stable basis, we get the following corollary.

Corollary 2.5.3. *The restriction formula of the stable basis of $T^*\mathcal{P}$ is given by*

$$\mathrm{stab}_\pm(\bar{y})|_{\bar{w}} = \sum_{\bar{z}=\bar{w}} \frac{\mathrm{stab}_\pm(y)|_z}{\prod\limits_{\alpha\in R_P^+} z\alpha}.$$

Proof. As in Lemma 2.4.2,

$$(D_1(\mathrm{stab}_\pm(y)), \mathrm{stab}_\mp(\bar{w}))$$

is a constant, so we can let $\hbar = 0$. Then $\mathrm{stab}_+(\bar{y})|_{\bar{w}}$ is nonzero if and only if $\bar{y} = \bar{w}$. A simple localization gives

$$D_1(\mathrm{stab}_\pm(y)) = (-1)^k \,\mathrm{stab}_\pm(\bar{y}),$$

where $k = \dim \mathcal{B} - \dim \mathcal{P} = |R_P^+|$. Applying Proposition 2.5.2 to $\mathrm{stab}_\pm(y)$ yields the result. $\square$

As in the $T^*\mathcal{B}$ case, modulo $\hbar^2$ we get

Corollary 2.5.4. *Let y be a minimal representative of the coset yW_P. Then*

$$\mathrm{stab}_+(\bar{y})|_{\bar{w}} \equiv \begin{cases} (-1)^{l(y)+1} \dfrac{\hbar \prod\limits_{\alpha \in R^+} \alpha}{y\beta \prod\limits_{\alpha \in R_P^+} y\sigma_\beta \alpha} \pmod{\hbar^2} & \textit{if } \bar{w} = \overline{y\sigma_\beta} \textit{ and } y\sigma_\beta < y \textit{ for some } \beta \in R^+, \\ \\ 0 \pmod{\hbar^2} & \textit{otherwise.} \end{cases}$$

Proof. Assume $y = \sigma_1\sigma_2 \cdots \sigma_l$ is a reduced decomposition. Because of Corollary 2.4.12 and Corollary 2.5.3, we only have to show: if $i < j$, then

$$\overline{\sigma_1 \cdots \hat{\sigma_i} \cdots \sigma_l} \neq \overline{\sigma_1 \cdots \hat{\sigma_j} \cdots \sigma_l}.$$

Assume the contrary. Then there exists an element $w \in W_P$ such that

$$\sigma_1 \cdots \hat{\sigma_i} \cdots \sigma_l = \sigma_1 \cdots \hat{\sigma_j} \cdots \sigma_l w.$$

Then

$$y = \sigma_1 \cdots \sigma_i \cdots \sigma_l = \sigma_1 \cdots \hat{\sigma_i} \cdots \hat{\sigma_j} \cdots \sigma_l w,$$

which is contradictory to the fact that y is minimal. $\square$

Using the map D_2, we can get another restriction formula for the stable basis of $T^*\mathcal{P}$. Define a map

$$A_2 : F(W/W_P, Q) \to F(W, Q)$$

as follows: for any $\psi \in F(W/W_P, Q)$,

$$A_2(\psi)(z) = \psi(\bar{z}) \prod_{\alpha \in R_P^+} (z\alpha - \hbar).$$

Then we have the following commutative diagram.

Proposition 2.5.5. *The diagram*

$$\begin{tikzcd} H_T^*(T^*\mathcal{P}) \arrow[r, hook] \arrow[d, "D_2"'] & F(W/W_P, Q) \arrow[d, "A_2"] \\ H_T^*(T^*\mathcal{B})_{loc} \arrow[r, hook] & F(W, Q) \end{tikzcd}$$

commutes.

Proof. The proof is almost the same as in Proposition 2.4.1. We show the two paths agree on the fixed point basis.

By the definition of A_2,

$$A_2(\iota_{\bar{y}*}1)(w) = \delta_{\bar{y},\bar{w}} e(T_{\bar{y}} T^*\mathcal{P}) \prod_{\alpha \in R_P^+} (w\alpha - \hbar).$$

By localization,

$$\begin{aligned} D_2(\iota_{\bar{y}*}1) &= \sum_{w} \frac{(D_2(\iota_{\bar{y}*}1), \iota_{w*}1)}{e(T_w T^*\mathcal{B})} \iota_{w*}1 \\ &= \sum_{\bar{w}=\bar{y}} \frac{1}{\prod_{\alpha \in R_P^+} (-w\alpha)} \iota_{w*}1. \end{aligned}$$

Hence

$$\begin{aligned} D_2(\iota_{\bar{y}*}1)|_w &= \delta_{\bar{y},\bar{w}} \frac{e(T_w T^*\mathcal{B})}{\prod_{\alpha \in R_P^+} (-w\alpha)} \\ &= A_2(\iota_{\bar{y}*}1)(w) \end{aligned}$$

as desired. $\square$

If we apply this diagram to the stable basis, we get

Corollary 2.5.6. *We have the restriction formula for the stable basis of $T^*\mathcal{P}$*

$$\operatorname{stab}_\pm(\bar{y})|_{\bar{z}} = \frac{\sum_{\bar{w}=\bar{y}} \operatorname{stab}_\pm(w)|_z}{\prod\limits_{\alpha\in R_P^+}(z\alpha-\hbar)}.$$

Proof. As in Lemma 2.4.2,

$$(D_2(\operatorname{stab}_\pm(\bar{y})), \operatorname{stab}_\mp(w))$$

is a constant, so we can let $\hbar = 0$. A simple localization calculation gives

$$D_2(\operatorname{stab}_\pm(\bar{y})) = \sum_{\bar{w}=\bar{y}} \operatorname{stab}_\pm(w).$$

Applying Proposition 2.5.5 to $\operatorname{stab}_\pm(y)$ yields the result. □

Modulo $\hbar^2$, we get

Corollary 2.5.7. *Let y be a minimal representative of the coset yW_P. Then*

$$\operatorname{stab}_-(\bar{w})|_{\bar{y}} \equiv \begin{cases} (-1)^{l(y)+1}\dfrac{\hbar\prod\limits_{\alpha\in R^+}\alpha}{y\beta\prod\limits_{\alpha\in R_P^+} y\alpha} \pmod{\hbar^2} & \text{if } \bar{w}=\overline{y\sigma_\beta} \text{ and } y\sigma_\beta < y \text{ for some } \beta\in R^+, \\ \\ 0 \pmod{\hbar^2} & \text{otherwise.} \end{cases}$$

Proof. This follows directly from Corollary 2.4.8 and Corollary 2.5.6 and the proof of Corollary 2.5.4. □

2.6 Restriction of Schubert varieties

In [Bil1999], Billey gave a restriction formula for Schubert varieties in G/B, and Tymoczko generalized it to G/P in [Tym2009]. In this section, we will deduce Billey's formula from Theorem 1.1.1 by a limiting process and generalize it to G/P case in two ways.

Let us first recall Billey's formula. Let B^- be the opposite Borel subgroup to B. Then $\overline{B^-wB/B}$ is the Schubert variety in G/B of dimension $\dim G/B - l(w)$, and as $w \in W$ varies, $[\overline{B^-wB/B}]$ form a basis of $H_A^*(G/B)$. The formula is

Theorem 2.6.1 ([Bil1999])**.** *Let $y = \sigma_1\sigma_2\cdots\sigma_l$ be a reduced decomposition. Then we have*

$$[\overline{B^- wB/B}]|_y = \sum_{\substack{1\le i_1<i_2<\cdots<i_k\le l\\ w=\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}\, reduced}} \beta_{i_1}\cdots\beta_{i_k}$$

where $\beta_i = \sigma_1\cdots\sigma_{i-1}\alpha_i$.

Proof. By the construction of the stable basis, we have

$$[\overline{B^- wB/B}]|_y = \pm \lim_{\hbar\to\infty} \frac{\mathrm{stab}_-(w)|_y}{(-\hbar)^{\dim \overline{B^- wB/B}}}.$$

The sign only depends on w, and can be determined by substituting $y = w$ as follows: the left hand side is

$$[\overline{B^- wB/B}]|_w = \prod_{\alpha\in R^+\cap wR^-} \alpha,$$

whereas the limit on the right is

$$\begin{aligned}\lim_{\hbar\to\infty} \frac{\mathrm{stab}_-(w)|_w}{(-\hbar)^{n-l(w)}} &= \lim_{\hbar\to\infty} \frac{\prod\limits_{\alpha\in R^+, w\alpha>0} (w\alpha-\hbar) \prod\limits_{\alpha\in R^+, w\alpha<0} w\alpha}{(-\hbar)^{n-l(w)}}\\ &= \prod_{\alpha\in R^+, w\alpha<0} w\alpha\\ &= (-1)^{l(w)}[\overline{B^- wB/B}]|_w.\end{aligned}$$

Hence the sign is $(-1)^{l(w)}$. Now the formula follows from Theorem 1.1.1. □

Remark 2.6.2. The proof of Theorems 1.1.1 and 2.4.9 is inspired by Billey's proof of Theorem 2.6.1. Using Theorem 2.4.9 we can also get a restriction formula for $[\overline{ByB/B}]|_w$.

A similar limiting process for $T^*\mathcal{P}$ yields the restriction formula for Schubert varieties in G/P. Recall that if w is minimal, then $\overline{B^-\bar{w}P/P}$ is the Schubert variety in G/P of dimension $\dim G/P - l(w)$, and as w runs through the minimal elements they form a basis of $H_A^*(G/P)$.

Theorem 2.6.3 ([Tym2009])**.** *Let y, w be minimal representatives of yW_P, wW_P respectively. Then we have*

$$[\overline{B^-\bar{w}P/P}]|_{\bar{y}} = [\overline{B^- wB/B}]|_y.$$

Remark 2.6.4. Tymoczko's generalization in [Tym2009] does not require y to be minimal. We will give two proofs for it. The first proof only works for minimal y, while the second works for any y.

Proof. Similarly to the proof of Theorem 2.6.1, we have

$$[\overline{B^{-}\bar{w}P/P}]|_{\bar{y}} = (-1)^{l(w)} \lim_{\hbar\to\infty} \frac{\mathrm{stab}_{-}(\bar{w})|_{\bar{y}}}{(-\hbar)^{m-l(w)}},$$

where $m = \dim G/P$. Then the formula follows from Corollary 2.5.6. □

We give another much simpler proof using a commutative diagram. Define a map

$$A_3 : F(W/W_P, Q) \to F(W, Q)$$

as follows: for any $\psi \in F(W/W_P, Q)$,

$$A_3(\psi)(z) = \psi(\bar{z}).$$

Then we have the following commutative diagram.

Proposition 2.6.5. *The diagram*

$$\begin{tikzcd} H_A^*(G/P) \arrow[r, hook] \arrow[d, "\pi^*"'] & F(W/W_P, Q) \arrow[d, "A_3"] \\ H_A^*(G/B) \arrow[r, hook] & F(W, Q) \end{tikzcd}$$

commutes, where π is the projection from G/B onto G/P.

Proof. We check on the fixed point basis.

$$\pi^*\iota_{\bar{y}*}1|_w = \iota_w^*\pi^*\iota_{\bar{y}*}1 = (\pi\circ\iota_w)^*\iota_{\bar{y}*}1 = \iota_{\bar{w}}^*\iota_{\bar{y}*}1 = \delta_{\bar{y},\bar{w}}e(T_{\bar{y}}G/P).$$

By definition of A_3,

$$A_3(\iota_{\bar{y}*}1)(w) = \iota_{\bar{y}*}1|_{\bar{w}} = \delta_{\bar{y},\bar{w}}e(T_{\bar{y}}G/P),$$

as desired. □

Since $\pi^*([\overline{B^{-}\bar{w}P/P}]) = [\overline{B^{-}wB/B}]$ if w is minimal (see [FA2007]), Proposition 2.6.5 gives

$$A_3([\overline{B^{-}\bar{w}P/P}])(y) = [\overline{B^{-}\bar{w}P/P}]|_{\bar{y}} = \pi^*([\overline{B^{-}\bar{w}P/P}])|_y = [\overline{B^{-}wB/B}]|_y,$$

which is just Theorem 2.6.3 without any conditions on y.

Chapter 3

Quantum Cohomology

In this chapter, we are going to use the stable basis studied in Chapter 2 to calculate the quantum cohomology of the cotangent bundle of the flag varieties $T^*(G/P)$, generalizing the result of Braverman–Maulik–Okounkov ([BMO2011]). We will first calculate the T-equivariant quantum multiplication by divisors in terms of the stable basis (Theorem 1.1.2), and then deduce the $G\times\mathbb{C}^*$ quantum multiplications (Theorem 1.1.3) using the restriction formulas for the stable basis in Chapter 2. The main reference for this chapter is [Su2016]. In the end, we construct a regular connection on the trivial principle G-bundle on $\mathbb{P}^1\setminus\{0,1,\infty\}$, such that when G is simply laced and P is minuscule, it is isomorphic to the quantum connection for $T^*(G/P)$. In the case $G=GL(n)$, we prove this connection is rigid. This last part is not contained in [Su2016].

As in the last chapter, we will use $T^*\mathcal{P}$ to denote $T^*(G/P)$. Recall that $T^*\mathcal{P}$ is an example of symplectic resolution ([Kal2009]), which is a smooth algebraic variety X with a holomorphic symplectic form ω and the affinization map

$$X\to X_0=\operatorname{Spec}H^0(X,\mathcal{O}_X)$$

is projective and birational. Conjecturally all the symplectic resolutions of the form T^*M for a smooth algebraic variety M are of the form $T^*\mathcal{P}$, see [Kal2009]. In [Fu2003], Fu proved that every symplectic resolution of a normalization of a nilpotent orbit closure in a semisimple Lie algebra $\mathfrak{g}$ is isomorphic to $T^*\mathcal{P}$ for some parabolic subgroup P in G.

3.1 T-equivariant quantum cohomology of $T^*\mathcal{P}$

In this section, we compute the T-equivariant quantum multiplication by divisors in terms of the stable basis.

Any divisors in $H_T^*(T^*\mathcal{P})$ is of the following form. If λ is a character of maximal torus, which vanishes on all $\alpha^\vee \in \Delta_P^\vee$, it determines a one-dimensional representation $\mathbb{C}_\lambda$ of P. Define a line bundle

$$\mathcal{L}_\lambda = G \times_P \mathbb{C}_\lambda$$

on G/P. Pulling it back to $T^*\mathcal{P}$, we get a line bundle on $T^*\mathcal{P}$, which will still be denoted by $\mathcal{L}_\lambda$. Let $D_\lambda := c_1(\mathcal{L}_\lambda) \in H^2(T^*\mathcal{P})$. Our goal is to determine the quantum multiplication by D_λ's. The quantum multiplication consists of the classical multiplication and the purely quantum part. We will deal with the calssical cup product first. To furthur simplify notations, we will use X to denote $T^*\mathcal{P}$ in this chapter.

3.1.1 Classical part

Let m denote the dimension of G/P. Since $\{\mathrm{stab}_+(\bar{y})\}$ and $\{(-1)^m \mathrm{stab}_-(\bar{y})\}$ are dual bases, we only need to calculate

$$(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{w})) = \sum_{\bar{w} \leq \bar{z} \leq \bar{y}} \frac{D_\lambda|_{\bar{z}} \cdot \mathrm{stab}_+(\bar{y})|_{\bar{z}} \cdot (-1)^m \mathrm{stab}_-(\bar{w})|_{\bar{z}}}{e(T_{\bar{z}}X)}. \tag{3.1}$$

This will be zero if $\bar{y} < \bar{w}$. Assume y is a minimal representative. Due to the proof in [MO2012, Theorem 4.4.1], the resulting expression lies in the nonlocalized coefficient ring. A degree count shows that it is in $H_T^2(pt)$. There are two cases.

1. Case $\bar{y} = \bar{w}$

   There is only one term in the sum of the right hand side of Equation (3.1). Hence,

$$(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{y})) = \frac{D_\lambda|_{\bar{y}} \cdot \mathrm{stab}_+(\bar{y})|_{\bar{y}} \cdot (-1)^m \mathrm{stab}_-(\bar{y})|_{\bar{y}}}{e(T_{\bar{y}}X)} = y(\lambda).$$

2. Case $\bar{y} \neq \bar{w}$

   Notice that $(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{w})) \in H_T^2(\mathrm{pt})$, and it is 0 if $\hbar = 0$, because every summand in Equation (3.1) is divisible by $\hbar$. Hence, it is a constant multiple of $\hbar$. So in Equation (3.1), only the terms $\bar{z} = \bar{y}$ and $\bar{z} = \bar{w}$ have contribution since all other terms are divisible by $\hbar^2$. Therefore,

$$\begin{aligned}(D_\lambda \cup \operatorname{stab}_+(\bar{y}), (-1)^m \operatorname{stab}_-(\bar{w})) &= y(\lambda)\frac{\operatorname{stab}_-(\bar{w})|_{\bar{y}}}{\operatorname{stab}_-(\bar{y})|_{\bar{y}}} + w(\lambda)\frac{\operatorname{stab}_+(\bar{y})|_{\bar{w}}}{\operatorname{stab}_+(\bar{w})|_{\bar{w}}} \\ &= y(\lambda)\frac{\hbar \text{ part of } \operatorname{stab}_-(\bar{w})|_{\bar{y}}}{\prod\limits_{\alpha\in R^+\setminus R_P^+} y\alpha} + w(\lambda)\frac{\hbar \text{ part of } \operatorname{stab}_+(\bar{y})|_{\bar{w}}}{\prod\limits_{\alpha\in R^+\setminus R_P^+} w\alpha},\end{aligned}$$

where the first equality follows from $(\operatorname{stab}_+(\bar{y}), \operatorname{stab}_-(\bar{y})) = (-1)^m e(T_{\bar{y}}X)$.

Corollaries 2.5.4 and 2.5.7 show this is zero if $\bar{w} \neq \overline{y\sigma_\beta}$ for any $\beta \in R^+$ with $y\sigma_\beta < y$. However, if $\bar{w} = \overline{y\sigma_\beta}$ for such a β, then since $(-1)^{l(y\sigma_\beta)} = (-1)^{l(y)+1}$, we have

$$\begin{aligned}&(D_\lambda \cup \operatorname{stab}_+(\bar{y}), (-1)^m \operatorname{stab}_-(\bar{w})) \\ =&y(\lambda)(-1)^{l(y)+1}\frac{\hbar \prod\limits_{\alpha\in R^+}\alpha}{y\beta \prod\limits_{\alpha\in R^+} y\alpha} + y\sigma_\beta(\lambda)(-1)^{l(y)+1}\frac{\hbar \prod\limits_{\alpha\in R^+}\alpha}{y\beta \prod\limits_{\alpha\in R^+} y\sigma_\beta\alpha} \\ =&-\frac{\hbar}{y\beta}y(\lambda) + \frac{\hbar}{y\beta}y\sigma_\beta(\lambda) \\ =&-\hbar(\lambda, \beta^\vee).\end{aligned}$$

Notice that for any $\beta \in R^+$, $y\sigma_\beta < y$ is equivalent to $y\beta \in R^-$. To summarize, we get

Theorem 3.1.1. *Let y be a minimal representative. Then the classical multiplication is given by*

$$D_\lambda \cup \operatorname{stab}_+(\bar{y}) = y(\lambda)\operatorname{stab}_+(\bar{y}) - \hbar \sum_{\alpha\in R^+, y\alpha\in R^-} (\lambda, \alpha^\vee)\operatorname{stab}_+(\overline{y\sigma_\alpha}).$$

3.1.2 Preliminaries on quantum cohomology

Now we deal with the purely quantum multiplication.

Let us recall that the operator of quantum multiplication by $\alpha \in H_T^*(X)$ has the following matrix elements

$$(\alpha * \gamma_1, \gamma_2) = \sum_{\beta\in H_2(X,\mathbb{Z})} q^\beta \langle \alpha, \gamma_1, \gamma_2\rangle^X_{0,3,\beta},$$

where $(\cdot,\cdot)$ denotes the standard inner product on cohomology and the quantity in angle brackets is a 3-point, genus 0, degree β equivariant Gromov–Witten invariant of X. Setting $q = 0$, we get the usual cup product in cohomology. This deformed product makes $H_T^*(X)$ into a commutative associative algebra.

Recall the well known divisor equation for Gromov–Witten invariants. If α is a divisor and $\beta \neq 0$, we have

$$\langle \alpha, \gamma_1, \gamma_2 \rangle^X_{0,3,\beta} = (\alpha, \beta) \langle \gamma_1, \gamma_2 \rangle^X_{0,2,\beta},$$

where (α, β) is the usual pairing between $H^2(X)$ and $H_2(X)$.

Using quantum multiplication by divisos, we can define a flat connection on the trivial vector bundle on $H^2(X)$ with fiber $H^*(X)$, which is the so-called quantum connection. It is defined as follows. Take any divisor $\lambda \in H^2(X)$, define

$$\nabla_\lambda = \frac{d}{d\lambda} - \lambda *,$$

where $\frac{d}{d\lambda}$ only acts on q^β by the following formula

$$\frac{d}{d\lambda}(q^\beta) = (\lambda, \beta) q^\beta,$$

and $\lambda *$ denotes the quantum multiplication by the divisor λ.

Since X has a everywhere-nondegenerate holomorphic symplectic form, the usual non-equivariant virtual fundamental class on $\overline{M}_{g,n}(X, \beta)$ vanishes for $\beta \neq 0$. However, we can modify the standard obstruction theory to get a reduced virtual fundamental class $[\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}}$, whose virtual dimension increases by 1 (see [BMO2011] or [OP2010]). The virtual fundamental class $[\overline{M}_{0,2}(X, \beta)]^{\mathrm{vir}}$ has expected dimension

$$K_X \cdot \beta + \dim X + 2 - 3 = \dim X - 1.$$

Hence the reduced virtual class has dimension $\dim X$, and for any $\beta \neq 0$,

$$[\overline{M}_{0,2}(X, \beta)]^{\mathrm{vir}} = -\hbar \cdot [\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}}, \tag{3.2}$$

where $\hbar$ is the weight of the symplectic form under the $\mathbb{C}^*-$action.

Therefore, the computation of the quantum connection is reduced to compute the following generating function of correspondences between X and itself,

$$\sum_{\beta \in H_2(X)} q^\beta \cdot ev_* [\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}},$$

where $ev : \overline{M}_{0,2}(X, \beta) \to X \times X$ is the evaluation map. Applying $\frac{d}{d\lambda}$ to the above generating function, we get the quantum multiplication by the divisor λ. The image of the evaluation map lies in the Steinberg variety $X \times_{X_0} X$, since the affinization map $X \to X_0$ contracts rational curves. Thus these reduced virtual fundamental class gives a lot of correspondence, which are key ingredients in many constructions in geometric representation theory (see [CG2010]). This explains why the quantum connection formula for $T^*(G/B)$ (Theorem 3.2.1) is expressed in terms of the elements in the graded affine Hecke algebra. See [Oko2017] for many other relations between enumerative geometry and geometric representation theory.

In computations with virtual fundament class, there is a useful technique called virtual localiztion (see [GP1999] or [HKK+2003]). It turns out in the virtual localization, only some of the fixed component have non-trivial contributions. Those component are called unbroken components in [OP2010].

3.1.3 Unbroken curves

Broken curves was introduced in [OP2010]. Let $f : C \to X$ be an A-fixed point of $\overline{M}_{0,2}(X, \beta)$ such that the domain is a chain of rational curves

$$C = C_1 \cup C_2 \cup \cdots \cup C_k,$$

with the marked points lying on C_1 and C_k respectively.

We say f is an unbroken chain if at every node $f(C_i \cap C_{i+1})$ of C, the weights of the two branches are opposite and nonzero. Note that all the nodes are fixed by A.

More generally, if (C, f) is an A-fixed point of $\overline{M}_{0,2}(X, \beta)$, we say that f is an unbroken map if it satisfies one of the three conditions:

1. f arises from a map $f : C \to X^A$,
2. f is an unbroken chain, or
3. the domain C is a chain of rational curves

$$C = C_0 \cup C_1 \cup \cdots C_k$$

such that C_0 is contracted by f, the marked points lie on C_0, and the remaining components form an unbroken chain.

Broken maps are A-fixed maps that do not satisfy any of these conditions.

Okounkov and Pandharipande proved the following Theorem in [OP2010, Section 3.8.3].

Theorem 3.1.2 ([OP2010]). *Every map in a given connected component of $\overline{M}_{0,2}(X,\beta)^A$ is either broken or unbroken. Only unbroken components contribute to the A-equivariant localization of reduced virtual fundamental class.*

3.1.4 Unbroken curves in X

Any $\alpha \in R^+ \setminus R_P^+$ defines an SL_2 subgroup $G_{\alpha^\vee}$ of G and hence a rational curve

$$C_\alpha := G_{\alpha^\vee} \cdot [P] \subset G/P \subset X.$$

This is the unique A-invariant rational curve connecting the fixed points $\bar{1}$ and $\bar{\sigma}_\alpha$, because any such rational curve has tangent weight at $\bar{1}$ in $R^- \setminus R_P^-$, and the following lemma in [FW2004, Section 4].

Lemma 3.1.3 ([FW2004]). *Let α, β be two roots in $R^+ \setminus R_P^+$. Then $\bar{\sigma}_\alpha = \bar{\sigma}_\beta$ if and only if $\alpha = \beta$.*

If C is an irreducible A-invariant rational curve in X, C must lie in G/P, and it connects two fixed points $\bar{y}$ and $\bar{w}$. Then its y^{-1}-translate $y^{-1}C$ is still an A-invariant curve, which connects fixed points $\bar{1}$ and $\overline{y^{-1}w}$. So $y^{-1}C = C_\alpha$ for a unique $\alpha \in R^+ \setminus R_P^+$, and $\overline{y^{-1}w} = \bar{\sigma}_\alpha$. Hence the tangent weight of C at $\bar{y}$ is $-y\alpha$. Therefore, we have

Lemma 3.1.4. *There are two kinds of unbroken curves C in X:*

1. *C is a multiple cover of rational curve branched over two different fixed points,*

2. *C is a chain of two rational curve $C = C_0 \cup C_1$, such that C_0 is contracted to a fixed point, the two marked points lie on C_0, and C_1 is a multiple cover of rational curve branched over two different fixed points.*

For any $\alpha \in \Delta \setminus \Delta_P$, define $\tau(\sigma_\alpha) := \overline{B\sigma_\alpha P/P}$. Then

$$\{\tau(\sigma_\alpha) | \alpha \in \Delta \setminus \Delta_P\}$$

form a basis of $H_2(X, \mathbb{Z})$. Let $\{\omega_\alpha | \alpha \in \Delta\}$ be the fundamental weights of the root system. For any $\alpha \in R^+ \setminus R_P^+$, define degree $d(\alpha)$ of α by

$$d(\alpha) = \sum_{\beta \in \Delta \setminus I} (\omega_\beta, \alpha^\vee)\tau(\sigma_\beta). \tag{3.3}$$

Lemma 3.1.5 ([FW2004]). *The degree of $[C_\alpha]$ is $d(\alpha)$, and $d(\alpha) = d(w\alpha)$ for any $w \in W_P$.*

3.1.5 Purely quantum part

Let $D_\lambda *_q$ denote the purely quantum multiplication. We want to calculate

$$(-1)^m(D_\lambda *_q \operatorname{stab}_+(\bar{y}), \operatorname{stab}_-(\bar{w})) = -\sum_{\beta \text{ effective}} (-1)^m \hbar q^\beta$$
$$(D_\lambda, \beta)(ev_*[\overline{M}_{0,2}(X,\beta)]^{\text{red}}, \operatorname{stab}_+(\bar{y}) \boxtimes \operatorname{stab}_-(\bar{w})),$$

where ev is the evaluation map from $\overline{M}_{0,2}(X,\beta)$ to $X \times X$. The image of the map ev lies in $X \times_{X_0} X$, where X_0 is the affinization of X. The $-$ sign appears because of Equation (3.2). Since

$$\dim[\overline{M}_{0,2}(X,\beta)]^{\text{red}} = \dim X,$$

$ev_*[\overline{M}_{0,2}(X,\beta)]^{\text{red}}$ will be a linear combination of the irreducible component of the Steinberg variety $X \times_{X_0} X$. Therefore, the product

$$(ev_*[\overline{M}_{0,2}(X,\beta)]^{\text{red}}, \operatorname{stab}_+(\bar{y}) \boxtimes \operatorname{stab}_-(\bar{w}))$$

lies in the nonlocalized coefficient ring (see [MO2012, Section 3.2.6, 4.6]). A degree count shows it must be a constant.

Therefore, we can let $\hbar = 0$. I.e., we can calculate it in A-equivariant chomology. Then by the definition of the stable basis,

$$\operatorname{stab}_\pm(\bar{y})|_{\bar{w}} = \delta_{\bar{y},\bar{w}} e^A(T^*_{\bar{y}}\mathcal{P}).$$

And we only need to compute the contribution from the unbroken components.

As in the classical multiplication, there are two cases depending whether the two fixed points $\bar{y}$ and $\bar{w}$ are the same or not.

1. Case $\bar{y} \neq \bar{w}$

   By virtual localization, Theorem 3.1.2 and Lemma 3.1.4,

$$(ev_*[\overline{M}_{0,2}(X,\beta)]^{\text{red}}, \operatorname{stab}_+(\bar{y}) \boxtimes \operatorname{stab}_-(\bar{w}))$$

   is nonzero if and only if $\bar{w} = \overline{y\sigma_\alpha}$ for some $\alpha \in R^+ \setminus R_P^+$. Only the first kind of unbroken curves have contribution to $(ev_*[\overline{M}_{0,2}(X,\beta)]^{\text{red}}, \operatorname{stab}_+(\bar{y}) \boxtimes \operatorname{stab}_-(\overline{y\sigma_\alpha}))$, and only restriction to the fixed point $(\bar{y}, \overline{y\sigma_\alpha})$ is nonzero in the localization of the product. The A-invariant rational curve $y[C_\alpha]$ connects the two fixed points $\bar{y}$ and $\overline{y\sigma_\alpha}$, and it is the unique one. Because, if $y[C_\beta]$ is also such a curve, then $\overline{y\sigma_\alpha} = \overline{y\sigma_\beta} = \bar{w}$. Hence $\alpha = \beta$ by Lemma 3.1.3. Therefore,

$$(-1)^m (D_\lambda *_q \operatorname{stab}_+(\bar{y}), \operatorname{stab}_-(\overline{y\sigma_\alpha})) = -\sum_{k>0} (-1)^m \hbar q^{k\cdot d(\alpha)} (D_\lambda, k\cdot d(\alpha))$$
$$(ev_*[\overline{M}_{0,2}(X, k\cdot d(\alpha))]^{\mathrm{red}}, \operatorname{stab}_+(\bar{y}) \boxtimes \operatorname{stab}_-(\overline{y\sigma_\alpha})).$$

Let f be an unbroken map of degree k from $C = \mathbb{P}^1$ to $y[C_\alpha]$. Then

$$\operatorname{Aut}(f) = \mathbb{Z}/k.$$

By virtual localization,

$$k(ev_*[\overline{M}_{0,2}(X, k\cdot d(\alpha))]^{\mathrm{red}}, \operatorname{stab}_+(\bar{y}) \boxtimes \operatorname{stab}_-(\overline{y\sigma_\alpha})) = \frac{e(T^*_{\bar{y}}\mathcal{P}) e(T^*_{\overline{y\sigma_\alpha}}\mathcal{P}) e'(H^1(C, f^*TX))}{e'(H^0(C, f^*TX))}.$$

Here e' is the product of nonzero A-weights.

We need [MO2012, Lemma 11.1.3].

Lemma 3.1.6 ([MO2012])**.** *Let A be a torus and let $\mathcal{T}$ be an A-equivariant bundle on $C = \mathbb{P}^1$ without zero weights in the fibers $\mathcal{T}_0$ and $\mathcal{T}_\infty$. Then*

$$\frac{e'(H^0(\mathcal{T} \oplus \mathcal{T}^*))}{e'(H^1(\mathcal{T} \oplus \mathcal{T}^*))} = (-1)^{\deg \mathcal{T} + rk\mathcal{T} + z} e(\mathcal{T}_0 \oplus \mathcal{T}_\infty)$$

where $z = \dim H^1(\mathcal{T} \oplus \mathcal{T}^*)^A$, i.e., z counts the number of zero weights in $H^1(\mathcal{T} \oplus \mathcal{T}^*)$.

Since

$$f^*TX = \mathcal{T} \oplus \mathcal{T}^* \quad \text{with} \quad \mathcal{T} = f^*T\mathcal{P},$$

Lemma 3.1.6 gives

$$\begin{aligned} &k(ev_*[\overline{M}_{0,2}(X, k\cdot d(\alpha))]^{\mathrm{red}}, \operatorname{stab}_+(\bar{y}) \boxtimes \operatorname{stab}_-(\overline{y\sigma_\alpha})) \\ =&\frac{e(T^*_{\bar{y}}\mathcal{P}) e(T^*_{\overline{y\sigma_\alpha}}\mathcal{P}) e'(H^1(C, f^*TX))}{e'(H^0(C, f^*TX))} \\ =&(-1)^{\deg \mathcal{T} + rk\mathcal{T} + z}. \end{aligned}$$

We now study the vector bundle $\mathcal{T} = f^*T\mathcal{P}$. First of all, $rk\mathcal{T} = \dim \mathcal{P}$. By localization,

$$\begin{aligned}\deg \mathcal{T} &= k\left(\frac{\sum_{\gamma\in R^+\setminus R_P^+}(-y\gamma)}{-y\alpha} + \frac{\sum_{\gamma\in R^+\setminus R_P^+}(-y\sigma_\alpha\gamma)}{y\alpha}\right)\\ &= k\sum_{\gamma\in R^+\setminus R_P^+}(\gamma,\alpha^\vee) = k(2\rho-2\rho_P,\alpha^\vee)\\ &= 2k\sum_{\beta\in\Delta\setminus I}(\omega_\beta,\alpha^\vee)\end{aligned}$$

is an even number, where ρ is the half sum of the positive roots, ρ_P is the half sum of the positive roots in R_P^+, and ω_β are the fundamental weights.

The vector bundle $\mathcal{T}$ splits as a direct sum of line bundles on C

$$\mathcal{T} = \bigoplus_i \mathcal{L}_i,$$

so

$$\bigoplus_i \mathcal{L}_i|_0 = \bigoplus_{\gamma\in R^+\setminus R_P^+}\mathfrak{g}_{-y\gamma},$$

where $\mathfrak{g}_{-y\gamma}$ are the root subspaces of $\mathfrak{g}$. Suppose $\mathcal{L}_i|_0 = \mathfrak{g}_{-y\gamma}$. Since $y\sigma_\alpha y^{-1}$ maps y to $y\sigma_\alpha$, we have

$$\mathcal{L}_i|_\infty = \mathfrak{g}_{-y\sigma_\alpha\gamma}.$$

Hence there is only one zero weight in $H^1(\mathcal{T}\oplus\mathcal{T}^*)$, occurring in $H^1(\mathcal{L}_i\oplus\mathcal{L}_i^*)$, where $\mathcal{L}_i|_0 = \mathfrak{g}_{-y\alpha}$. I.e., $\mathcal{L}_i$ is the tangent bundle of C.

Therefore $z = 1$ and we have

Lemma 3.1.7.

$$(-1)^m(D_\lambda *_q \mathrm{stab}_+(\bar{y}), \mathrm{stab}_-(\overline{y\sigma_\alpha})) = \sum_{k>0}\hbar q^{k\cdot d(\alpha)}(D_\lambda, d(\alpha)) = -\hbar\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}(\lambda,\alpha^\vee).$$

Proof. We only need to show

$$(D_\lambda, d(\alpha)) = -(\lambda,\alpha^\vee).$$

By definition and localization,

$$
\begin{aligned}
(D_\lambda, d(\alpha)) &= \sum_{\beta\in\Delta\setminus\Delta_P} (\omega_{\beta,\alpha^\vee}) \int_{\tau(\sigma_\beta)} c_1(\mathcal{L}_\lambda) = \sum_{\beta\in\Delta\setminus\Delta_P} (\omega_{\beta,\alpha^\vee}) \left(\frac{\lambda}{-\beta} + \frac{\sigma_\beta\lambda}{\beta}\right) \\
&= -\sum_{\beta\in\Delta\setminus\Delta_P} (\omega_{\beta,\alpha^\vee})(\lambda, \beta^\vee) = -\sum_{\beta\in\Delta} (\omega_{\beta,\alpha^\vee})(\lambda, \beta^\vee) \\
&= -(\lambda, \alpha^\vee).
\end{aligned}
$$

□

2. Case $\bar{y} = \bar{w}$

In this case, only the second kind of unbroken curves have contribution to $(D_\lambda *_q \mathrm{stab}_+(\bar{y}), \mathrm{stab}_-(\bar{y}))$. Let $C = C_0 \cup C_1$ be an unbroken curve of the second kind with C_0 contracted to the fixed point $\bar{y}$, and C_1 is a cover of the rational curve yC_α of degree k, where $\alpha \in R^+ \setminus R_P^+$. Let p denote the node of C, and let f be the map from C to X. Then the corresponding decorated graph Γ has two vertices, one of them has two marked tails, and there is an edge of degree k connecting the two vertices. Hence the automorphism group of the graph is trivial. The virtual normal bundle ([HKK+2003]) is

$$
e(N_\Gamma^{\mathrm{vir}}) = \frac{e'(H^0(C, f^*TX))}{e'(H^1(C, f^*TX))} \frac{-y\alpha/k}{y\alpha/k} = -\frac{e'(H^0(C, f^*TX))}{e'(H^1(C, f^*TX))}, \tag{3.4}
$$

where $e'(H^0(C, f^*TX))$ denotes the nonzero A-weights in $H^0(C, f^*TX)$. Consider the normalization exact sequence resolving the node of C:

$$
0 \to \mathcal{O}_C \to \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_1} \to \mathcal{O}_p \to 0.
$$

Tensoring with f^*TX and taking cohomology yields:

$$
\begin{aligned}
0 \to H^0(C, f^*TX) \to H^0(C_0, f^*TX) \oplus H^0(C_1, f^*TX) \to T_{\bar{y}}X \\
\to H^1(C, f^*TX) \to H^1(C_0, f^*TX) \oplus H^1(C_1, f^*TX) \to 0.
\end{aligned}
$$

Since C_0 is contracted to $\bar{y}$, $H^0(C_0, f^*TX) = T_{\bar{y}}X$ and $H^1(C_0, f^*TX) = 0$. Therefore, as virtual representations, we have

$$
H^0(C, f^*TX) - H^1(C, f^*TX) = H^0(C_1, f^*TX) - H^1(C_1, f^*TX).
$$

Due to Equation (3.4) and the analysis in the last case, we get

$$\begin{aligned} e(N_\Gamma^{\mathrm{vir}}) &= -\frac{e'(H^0(C_1, f^*TX))}{e'(H^1(C_1, f^*TX))} \\ &= (-1)^m e(T_{\bar{y}}\mathcal{P}) e(T_{\overline{y\sigma_\alpha}}\mathcal{P}). \end{aligned}$$

Then by virtual localization formula, we have

$$\begin{aligned} (-1)^m (D_\lambda *_q \operatorname{stab}_+(\bar{y}), \operatorname{stab}_-(\bar{y})) &= -\hbar \sum_{\alpha\in R^+\setminus R_P^+, k>0} (D_\lambda, d(\alpha)) q^{k\cdot d(\alpha)} \frac{e(T^*_{\bar{y}}\mathcal{P})^2}{e(T_{\bar{y}}\mathcal{P}) e(T_{\overline{y\sigma_\alpha}}\mathcal{P})} \\ &= \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod\limits_{\beta\in R^+\setminus R_P^+} y\beta}{\prod\limits_{\beta\in R^+\setminus R_P^+} y\sigma_\alpha\beta} \\ &= \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod\limits_{\beta\in R^+} y\beta}{\prod\limits_{\beta\in R^+} y\sigma_\alpha\beta} \frac{\prod\limits_{\beta\in R_P^+} y\sigma_\alpha\beta}{\prod\limits_{\beta\in R_P^+} y\beta} \\ &= -\hbar \cdot y \left(\sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod\limits_{\beta\in R_P^+} \sigma_\alpha\beta}{\prod\limits_{\beta\in R_P^+} \beta} \right). \end{aligned}$$

Here we have used

$$\prod_{\beta\in R^+} y\beta = (-1)^{l(y)} \prod_{\beta\in R^+} \beta, \quad \text{and} \quad (-1)^{l(y\sigma_\alpha)} = (-1)^{l(y)+l(\sigma_\alpha)} = (-1)^{l(y)+1}.$$

Notice that for any root $\gamma \in R_P^+$, σ_γ preserves $R^+ \setminus R_P^+$. For any $\alpha \in R^+ \setminus R_P^+$, $d(\sigma_\gamma(\alpha)) = d(\alpha)$, $(\lambda, \alpha^\vee) = (\lambda, \sigma_\gamma(\alpha)^\vee)$ and $\prod\limits_{\beta\in R_P^+} \sigma_\gamma\beta = -\prod\limits_{\beta\in R_P^+} \beta$. Hence,

$$\begin{aligned} &\sigma_\gamma \left(\sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod_{\beta\in R_P^+} \sigma_\alpha\beta \right) \\ &= \sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod_{\beta\in R_P^+} \sigma_{\sigma_\gamma\alpha}\sigma_\gamma\beta \\ &= -\sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod_{\beta\in R_P^+} \sigma_\alpha\beta. \end{aligned}$$

Therefore $\sum\limits_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod\limits_{\beta\in R_P^+} \sigma_\alpha\beta$ is divisible by $\prod\limits_{\beta\in R_P^+} \beta$. But they have the same degree, so

$$\sum_{\alpha\in R^+\backslash R_P^+}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\frac{\prod_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod_{\beta\in R_P^+}\beta} \tag{3.5}$$

is a scalar.

To summarize, we get

Theorem 3.1.8. *The purely quantum multiplication by D_λ in $H_T^*(T^*\mathcal{P})$ is given by:*

$$D_\lambda *_q \operatorname{stab}_+(\bar{y}) = -\hbar\sum_{\alpha\in R^+\backslash R_P^+}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\operatorname{stab}_+(\overline{y\sigma_\alpha}) - \hbar\sum_{\alpha\in R^+\backslash R_P^+}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\frac{\prod_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod_{\beta\in R_P^+}\beta}\operatorname{stab}_+(\bar{y}).$$

Remark 3.1.9.

1. The scalar

$$-\hbar\sum_{\alpha\in R^+\backslash R_P^+}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\frac{\prod_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod_{\beta\in R_P^+}\beta}$$

can also be determined by the condition

$$D_\lambda *_q 1 = 0.$$

2. The element y is not necessarily a minimal representative.

3. The Theorem is also true if we replace all the stab_+ by stab_-.

3.1.6 Quantum multiplications

Combining Theorem 3.1.1 and Theorem 3.1.8, we get our second main Theorem 1.1.2. Taking $I=\emptyset$, we get the quantum multiplication by D_λ in $H_T^*(T^*\mathcal{B})$.

Theorem 3.1.10. *The quantum multiplication by D_λ in $H_T^*(T^*\mathcal{B})$ is given by:*

$$\begin{aligned} D_\lambda * \operatorname{stab}_+(y) =& y(\lambda)\operatorname{stab}_+(y) - \hbar\sum_{\alpha\in R^+, y\alpha\in -R^+}(\lambda,\alpha^\vee)\operatorname{stab}_+(y\sigma_\alpha) \\ &-\hbar\sum_{\alpha\in R^+}(\lambda,\alpha^\vee)\frac{q^{\alpha^\vee}}{1-q^{\alpha^\vee}}(\operatorname{stab}_+(y\sigma_\alpha)+\operatorname{stab}_+(y)). \end{aligned}$$

3.1.7 Calculation of the scalar in type A

We can define an equivalence relation on $R^+ \setminus R_P^+$ as follows

$$\alpha \sim \beta \quad \text{if} \quad d(\alpha) = d(\beta).$$

Then $w(\alpha) \sim \alpha$ for any $w \in W_P$. We have

$$\begin{aligned} &\sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}} \frac{\prod\limits_{\beta \in R_P^+} \sigma_\alpha \beta}{\prod\limits_{\beta \in R_P^+} \beta} \\ &= \sum_{\alpha \in (R^+ \setminus R_P^+)/\sim} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}} \sum_{\alpha' \sim \alpha} \frac{\prod\limits_{\beta \in R_P^+} \sigma_{\alpha'} \beta}{\prod\limits_{\beta \in R_P^+} \beta}. \end{aligned}$$

It is easy to see that

$$\sum_{\alpha' \sim \alpha} \frac{\prod\limits_{\beta \in R_P^+} \sigma_{\alpha'} \beta}{\prod\limits_{\beta \in R_P^+} \beta}$$

is a constant, which will be denoted by $C_P(\alpha)$.

In this section, we will determine the constant $C_P(\alpha)$ when G is of type A. We will first calculate this number in $T^*Gr(k,n)$ case, and the general case will follow easily. Now let $G = SL(n, \mathbb{C})$ and let x_i be the function on the Lie algebra of the diagonal torus defined by $x_i(t_1, \cdots, t_n) = x_i$.

$T^*Gr(k,n)$ case

Let P be a parabolic subgroup containing the upper triangular matrices such that $T^*(G/P)$ is $T^*Gr(k,n)$. Then

$$R_P^+ = \{x_i - x_j | 1 \le i < j \le k, \text{ or } k < i < j \le n\}, \quad R \setminus R_P^+ = \{x_i - x_j | 1 \le i \le k < j \le n\}$$

and all the roots in $R \setminus R_P^+$ are equivalent. The number $C_P(\alpha)$ will be denoted by C_P. By definition,

$$C_P = \frac{\sum\limits_{1 \le r \le k < s \le n} (rs) \left(\prod\limits_{1 \le i < j \le k} (x_i - x_j) \prod\limits_{1+k \le p < q \le n} (x_p - x_q) \right)}{\prod\limits_{1 \le i < j \le k} (x_i - x_j) \prod\limits_{1+k \le p < q \le n} (x_p - x_q)}, \tag{3.6}$$

where (rs) means the transposition of x_r and x_s.

Observe that

$$\prod_{1\le i<j\le k}(x_j-x_i)\prod_{1+k\le p<q\le n}(x_q-x_p)=\det\begin{pmatrix} 1 & x_1 & \cdots & x_1^{k-1} & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 1 & x_k & \cdots & x_k^{k-1} & & & & \\ & & & & 1 & x_{k+1} & \cdots & x_{k+1}^{n-k-1} \\ & & & & \vdots & \vdots & \ddots & \\ & & & & 1 & x_n & \cdots & x_n^{n-k-1} \end{pmatrix}.$$

Then it is easy to see that the coefficient of $x_2x_3^2\cdots x_k^{k-1}x_{k+2}x_{k+3}^2\cdots x_n^{n-k-1}$ in

$$\sum_{1\le r\le k<s\le n}(rs)\left(\prod_{1\le i<j\le k}(x_j-x_i)\prod_{1+k\le p<q\le n}(x_q-x_p)\right)$$

is $\min(k,n-k)$, since only when $s-r=k$, $(rs)\left(\prod_{1\le i<j\le k}(x_j-x_i)\prod_{1+k\le p<q\le n}(x_q-x_p)\right)$ has the term $x_2x_3^2\cdots x_k^{k-1}x_{k+2}x_{k+3}^2x_n^{n-k-1}$, and the coefficient is 1. Hence

Proposition 3.1.11.

$$C_P=\min(k,n-k).$$

General case

Let $\lambda=(\lambda_1,\cdots,\lambda_N)$ be a partition of n with $\lambda_1\ge\cdots\ge\lambda_N$. Let

$$\mathcal{F}_\lambda=\{0\subset V_1\subset V_2\cdots\subset V_N|\dim V_i/V_{i-1}=\lambda_i\}$$

be the partial flag variety, and let P be the corresponding parabolic subgroup. Then

$$R_P^+=\{x_i-x_j|\lambda_1+\cdots+\lambda_p<i<j\le\lambda_1+\cdots+\lambda_{p+1},\ \text{for some } p \text{ between } 0 \text{ and } N-1\}.$$

Two positive roots x_i-x_j and x_k-x_l are equivalent if and only if there exist $1\le p<q\le N$ such that

$$\lambda_1+\cdots+\lambda_p<i,k\le\lambda_1+\cdots+\lambda_{p+1},\lambda_1+\cdots+\lambda_q<j,l\le\lambda_1+\cdots+\lambda_{q+1}.$$

So the set $(R^+\setminus R_P^+)/\sim$ has representatives

$$\{x_{\lambda_1+\cdots+\lambda_p}-x_{\lambda_1+\cdots+\lambda_q}|1\le p<q\le N\}.$$

The same analysis as in the last case gives

Proposition 3.1.12. *For any* $1 \le p < q \le N$*,*

$$C_P(x_{\lambda_1+\cdots+\lambda_p} - x_{\lambda_1+\cdots+\lambda_q}) = \lambda_q.$$

3.2 $G \times \mathbb{C}^*$-equivariant quantum multiplication

Let $\mathbb{G} = G \times \mathbb{C}^*$. In this section, we will first get the $\mathbb{G}$-equivariant quantum multiplication formula in $T^*\mathcal{B}$, which is the main result of [BMO2011]. Then we show the quantum multiplication formula in $T^*\mathcal{P}$ is conjugate to the conjectured formula given by Braverman .

3.2.1 $T^*\mathcal{B}$ case

Let us recall the result from [BMO2011] first. Let $\mathfrak{a}$ be the Lie algebra of the maximal torus A. Then

$$H^*_{\mathbb{G}}(T^*\mathcal{B}) \simeq H^*_T(T^*\mathcal{B})^W \simeq H^*_T(pt) \simeq \operatorname{sym}\mathfrak{a}^*[\hbar].$$

The isomorphism is determined as follows: for any $\beta \in H^*_{\mathbb{G}}(T^*\mathcal{B})$, lift it to $H^*_T(T^*\mathcal{B})$, and then restrict it to the fixed point B. Similarly, we have

$$H^*_{\mathbb{G}}(T^*\mathcal{P}) \simeq H^*_T(T^*\mathcal{P})^W \simeq (\operatorname{sym}\mathfrak{a}^*)^{W_P}[\hbar].$$

Let us recall the definition of the graded affine Hecke algebra $\mathcal{H}_\hbar$ ([Lus1988]). It is generated by the symbols x_λ for $\lambda \in \mathfrak{a}^*$, Weyl elements $\tilde{w}$ and a central element $\hbar$ such that

1. x_λ depends linearly on $\lambda \in \mathfrak{a}^*$;

2. $x_\lambda x_\mu = x_\mu x_\lambda$;

3. the $\tilde{w}$'s form the Weyl group inside $\mathcal{H}_t$;

4. for any $\alpha \in \Delta$, $\lambda \in \mathfrak{a}^*$, we have

$$\tilde{\sigma}_\alpha x_\lambda - x_{\tilde{\sigma}_\alpha(\lambda)}\tilde{\sigma}_\alpha = \hbar(\alpha^\vee, \lambda).$$

According to [Lus1988], we have a natural isomorphism

$$H_*^{\mathbb{G}}(T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}) \simeq \mathcal{H}_\hbar,$$

where $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$. Since the left handside acts on $H^*_{\mathbb{G}}(T^*\mathcal{B})$ by convolution ([CG2010]), $\mathcal{H}_\hbar$ acts on $\operatorname{sym}\mathfrak{a}^*[\hbar]$. The action is defined as follows: x_λ acts by multiplication by λ, and for every simple root α, the action of $\tilde{\sigma}_\alpha$ is defined by

$$\tilde{\sigma}_\alpha f = (\frac{\hbar}{\alpha} + \frac{\alpha - \hbar}{\alpha}\sigma_\alpha)f$$

where $f \in \operatorname{sym}\mathfrak{a}^*[\hbar]$, and $\sigma_\alpha f$ is the usual Weyl group action on $\operatorname{sym}\mathfrak{a}^*[\hbar]$.

Having introduced the above notations, we can state the main Theorem of [BMO2011].

Theorem 3.2.1 ([BMO2011])**.** *The operator of quantum multiplication by D_λ in $H^*_{\mathbb{G}}(T^*\mathcal{B})$ is equal to*

$$x_\lambda + \hbar \sum_{\alpha\in R^+} (\lambda, \alpha^\vee) \frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} (\tilde{\sigma}_\alpha - 1).$$

We can now deduce Theorem 3.2.1 from Theorem 3.1.10 and Theorem 2.4.9. The classical multiplication is obvious. We only show that the purely quantum part matches. Let $f \in \operatorname{sym}\mathfrak{a}^*[\hbar]$ correspond to $\gamma \in H^*_{\mathbb{G}}(T^*\mathcal{B})$. We also let γ denote the lift in $H^*_T(T^*\mathcal{B})$. Then $\gamma|_w = w(f)$ for any $w \in W$. Since the stable and unstable basis are dual basis up to $(-1)^n$, where $n = \dim\mathcal{B}$, we have

$$\gamma = \sum_y (-1)^n (\gamma, \operatorname{stab}_+(y)) \operatorname{stab}_-(y).$$

Due to Theorem 3.1.10, we have

$$D_\lambda *_q \gamma = -\hbar \sum_{\alpha\in R^+} (\lambda, \alpha^\vee) \frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} \sum_y (\gamma, (-1)^n \operatorname{stab}_+(y))(\operatorname{stab}_-(y\sigma_\alpha) + \operatorname{stab}_-(y)).$$

Notice that $\operatorname{stab}_-(y)|_1 = \delta_{y,1} e(T_1^*\mathcal{B})$. Restricting to the fixed point 1, we get

$$\begin{aligned} D_\lambda *_q \gamma|_1 = & -\hbar \sum_{\alpha\in R^+} (\lambda, \alpha^\vee) \frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} \gamma|_1 \\ & - \hbar \sum_{\alpha\in R^+} (\lambda, \alpha^\vee) \frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} (\gamma, (-1)^n \operatorname{stab}_+(\sigma_\alpha)) e(T_1^*\mathcal{B}). \end{aligned}$$

Hence we only need to show

$$-(\gamma, (-1)^n \operatorname{stab}_+(\sigma_\alpha))e(T_1^*\mathcal{B}) = \tilde{\sigma}_\alpha f. \tag{3.7}$$

To prove this, we need the following lemma.

Lemma 3.2.2. *If $w = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}$, then*

$$\prod_{j=1}^{k} \frac{\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_{j-1}}\alpha_{i_j} - \hbar}{\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_j}\alpha_{i_j} - \hbar} = \frac{e(T_1^*\mathcal{B})}{e(T_w^*\mathcal{B})}.$$

Proof. If $w = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}$ is reduced, then this follows from the fact

$$\{w\beta | \beta \in R^+, w\beta \in R^-\} = \{\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_j}\alpha_{i_j} | 1 \le j \le l\}.$$

If $w = (\sigma_\alpha\sigma_\beta)^{m(\alpha,\beta)} = 1$ for some simple roots α and β, where $m(\alpha,\beta)$ is the order of $\sigma_\alpha\sigma_\beta$, we can check it case by case easily. If $w = \sigma_\alpha^2$, then it it trivial. In general, w will be a composition of these three cases. □

If $\sigma_\alpha = \sigma_{\alpha_1}\cdots\sigma_{\alpha_l}$ is a reduced decomposition, then

$$\tilde{\sigma}_\alpha f = \prod_{i=1}^{l}(\frac{\hbar}{\alpha_i} + \frac{\alpha_i - \hbar}{\alpha_i}\sigma_{\alpha_i})f.$$

Expanding this and using Theorem 2.4.9, Lemma 3.2.2 and the fact $(-1)^{l(\sigma_\alpha)} = -1$, we get

$$\tilde{\sigma}_\alpha(f) = \sum_w \frac{\operatorname{stab}_+(\sigma_\alpha)|_w wf}{e(T_w T^*\mathcal{B})}(-1)^{1+n}e(T_1^*\mathcal{B}) = -(\gamma, (-1)^n \operatorname{stab}_+(\sigma_\alpha))e(T_1^*\mathcal{B}), \tag{3.8}$$

which is precisely Equation (3.7).

3.2.2 $T^*\mathcal{P}$ case

In the parabolic case, Professor Braverman suggests (through private communication) that the quantum multiplication should be

$$D_\lambda * = x_\lambda + \hbar \sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}}\tilde{\sigma}_\alpha + \cdots, \tag{3.9}$$

where $\cdots$ is some scalar. Recall we have

$$H^*_{\mathbb{G}}(T^*\mathcal{P}) \simeq H^*_T(T^*\mathcal{P})^W \simeq (\operatorname{sym}\mathfrak{a}^*)^{W_P}[\hbar].$$

It is easy to see that classical multiplication by D_λ is given by multiplication by λ.

Now we do the similar calculation as in the $T^*\mathcal{B}$ case. We need the following restriction formula from Corollar 2.5.3:

$$\operatorname{stab}_\pm(\bar{y})|_{\bar{w}} = \sum_{\bar{z}=\bar{w}} \frac{\operatorname{stab}_\pm(y)|_z}{\prod\limits_{\alpha\in R_P^+} z\alpha}. \tag{3.10}$$

Take any $\gamma \in H^*_{\mathbb{G}}(T^*\mathcal{P})$, and assume it corresponds to $f \in (\operatorname{sym}\mathfrak{a}^*)^{W_P}[\hbar]$. We still let γ denote the corresponding lift in $H^*_T(T^*\mathcal{P})$. Then $\gamma|_{\bar{y}} = yf$. Recall m denotes the dimension of $\mathcal{P}$. Then we have

$$\gamma = \sum_{\bar{y}} (-1)^m (\gamma, \operatorname{stab}_+(\bar{y}))\operatorname{stab}_-(\bar{y}).$$

By Theorem 3.1.8,

$$\begin{aligned} D_\lambda *_q \gamma = &\sum_{\bar{y}} (\gamma, (-1)^m \operatorname{stab}_+(\bar{y}))(-\hbar) \sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \operatorname{stab}_-(\overline{y\sigma_\alpha}) \\ &- \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod\limits_{\beta\in R_P^+} \sigma_\alpha\beta}{\prod\limits_{\beta\in R_P^+} \beta}\gamma. \end{aligned}$$

Notice that

$$\operatorname{stab}_-(\overline{y\sigma_\alpha})|_{\bar{1}} = \begin{cases} e(T^*_{\bar{1}}\mathcal{P}) & \text{if} \quad \overline{y\sigma_\alpha} = \bar{1}; \\ 0 & \text{otherwise} . \end{cases}$$

Restricting $D_\lambda *_q \gamma$ to the fixed point $\bar{1}$, we get

$$\begin{aligned} D_\lambda *_q \gamma|_{\bar{1}} = &-\hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} (\gamma, (-1)^m \operatorname{stab}_+(\bar{\sigma}_\alpha)) e(T^*_{\bar{1}}\mathcal{P}) \\ &- \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod\limits_{\beta\in R_P^+} \sigma_\alpha\beta}{\prod\limits_{\beta\in R_P^+} \beta} f \end{aligned}$$

Due to restriction formula (3.10) and Equation (3.8), we have

$$(\gamma, (-1)^m \operatorname{stab}_+(\bar{\sigma}_\alpha)) e(T_1^*\mathcal{P}) = -\frac{\tilde{\sigma}_\alpha(f \prod_{\beta \in R_P^+}(\beta - \hbar))}{\prod_{\beta \in R_P^+}(\beta - \hbar)}.$$

Hence, we obtain Theorem 1.1.3.

Since

$$\hbar \sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}} \frac{\prod_{\beta \in R_P^+} \sigma_\alpha \beta}{\prod_{\beta \in R_P^+} \beta} \tag{3.11}$$

is a scalar, the quantum multiplication formula in Theorem 1.1.3 is conjugate to the conjectured formula (3.9) by the function

$$\prod_{\beta \in R_P^+} (\beta - \hbar).$$

This factor comes from geometry as follows. Let π be the projection map from $\mathcal{B}$ to $\mathcal{P}$, and Γ_π be its graph. Then the conormal bundle to Γ_π in $\mathcal{B} \times \mathcal{P}$ is a Lagrangian submanifold of $T^*(\mathcal{B} \times \mathcal{P})$.

$$\begin{tikzcd} T^*_{\Gamma_\pi}(\mathcal{B} \times \mathcal{P}) \arrow[r, "p_1"] \arrow[d, "p_2"'] & T^*\mathcal{B} . \\ T^*\mathcal{P} & \end{tikzcd}$$

Let $D = p_{1*} p_2^*$ be the map from $H^*_{\mathbb{G}}(T^*\mathcal{P})$ to $H^*_{\mathbb{G}}(T^*\mathcal{B})$ induced by this correspondence. Then under the isomorphisms

$$H^*_{\mathbb{G}}(T^*\mathcal{B}) \simeq \operatorname{sym} \mathfrak{a}^*[\hbar] \quad \text{and} \quad H^*_{\mathbb{G}}(T^*\mathcal{P}) \simeq (\operatorname{sym} \mathfrak{a}^*)^{W_P}[\hbar],$$

the map becomes multiplicaiton by the above factor (see the proof in Corollary 2.5.5). The scalar in the conjectured formula (3.9) is just the one in Equation (3.11). By the calculation in the Subsection 3.1.7, it is not equal to

$$\hbar \sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}}$$

in general. It can also be determined by the condition $D_\lambda *_q 1 = 0$.

3.3 A regular connection on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

In this section, we first review the work of Lam and Templier relating the quantum connection of G/P to the other connections. Then we construct a regular connection ∇ on a trivial principle G bundle on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, such that it will be isomorphic to the quantum connection of $T^*(G/P)$, when G is simply laced and P is maximal and minuscule.

3.3.1 G/P case

Let us first recall the work of Lam and Templier. Let β be a simple root of G, such that the fundamental weight ϖ_β is a minuscule weight of G. I.e., $(\varpi_\beta, \alpha^\vee) \in \{0, 1\}$ for any positive root α. Let P be a maximal parabolic subgroup of G with simple roots $\Delta_P = \Delta \setminus \{\beta\}$. Then G/P is called a minuscule partial flag variety. Let $G^\vee$ and $P^\vee$ be the corresponding Langlands dual groups. Then the main theorem of Lam and Templier is the following, which proves the mirror conjecture of Rietsch ([Rie2008]) for the minuscule case. We refer the readers to their paper for the meaning of the notations.

Theorem 3.3.1. [LT2017] *The geometric crystal $\mathcal{D}$-module $Cr_{G^\vee, P^\vee}$ is isomorphic to the quantum cohomology $\mathcal{D}$-module for G/P.*

The idea of the proof is the following. They first use quantum Chevalley formula ([FW2004]) to identify the quantum $\mathcal{D}$-module of G/P with the Frenkel–Gross connection ([FG2009]) for the minuscule representation of G. Then by a theorem of Zhu ([Zhu2017]), the latter is isomorphic to the Kloosterman $\mathcal{D}$-module constructed by Heinloth, Ngô and Yun ([HNY2013]). Finally, they show directly that the Kloosterman $\mathcal{D}$-module is isomphic to the geometric crystal $\mathcal{D}$-module $Cr_{G^\vee, P^\vee}$. To summarize, they identify the following four $\mathcal{D}$-modules on $\mathbb{P}^1$.

$$\begin{tikzcd}
\text{quantum } \mathcal{D}\text{-module} \arrow[r, dash, "\text{Rietsch's conjecture}"] \arrow[d, dash, "\text{quantum Chevalley}"] & \text{crystal } \mathcal{D}\text{-module } Cr_{G^\vee/P^\vee} \arrow[d, dash] \\
\text{Frenkel–Gross connection for } G \arrow[r, dash, "\text{Zhu}"] & \text{Kloosterman } \mathcal{D}\text{-module for } G
\end{tikzcd} \tag{3.12}$$

Since we are going to generalize the first step to the cotangent bundle case, we give more details for the first step.

The cohomology $H^*(G/P)$ has a Schubert basis $\{Y_{\bar{y}} := [\overline{B^- yP/P}] | \bar{y} \in W/W_P\}$ indexed by the coset W/W_P. Since ϖ_β is minuscule, the weights for the highest weight representation V_{ϖ_β} of G are just the orbits of the minuscule weight ϖ_β under the Weyl group W action, and each weight space is one-dimensional (see [Gro2000]). So $\dim V_{\varpi_\beta} = |W/W_P|$. There is a weight basis $\{v_{w\varpi_\beta} | w \in W/W_P\}$ satisfying the following properties.

Lemma 3.3.2. *For any simple root β_i, let e_i and f_i denote the root vectors corresponding to the roots β_i and $-\beta_i$. Then*

$$e_i(v_{w\varpi_\beta}) = \begin{cases} v_{\sigma_i w\varpi_\beta} & \text{if } \langle w\lambda, \beta_i\rangle = -1 \\ 0 & \text{otherwise,} \end{cases} \qquad f_i(v_{w\varpi_\beta}) = \begin{cases} v_{\sigma_i w\varpi_\beta} & \text{if } \langle w\lambda, \beta_i\rangle = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This weight basis can be constructed directly form the highest weight vector by applying the operators f_i's.

Recall the Frenkel–Gross connection ∇_{FG} is a irregular connection on the trivial principle G bundle on $\mathbb{P}^1 \setminus \{0, \infty\}$, which is defined as follows

$$\nabla_{FG} := d + \sum_i f_i \frac{dq}{q} + x_\theta dq, \tag{3.13}$$

with x_θ an element in the hightest root space. The main result of [FG2009] is that ∇_{FG} is rigid (see Section 3.3.4 or [Kat2016]). This connection has an oper structure in the sense of Beilinson and Drinfeld. It is the characteristic 0 counterpart of a family of ℓ-adic sheaves, which parametrize a specific automorphic representation under the global Langlands correspondence and is constructed by Heinloth, Ngô and Yun ([HNY2013]). It provides an example of the geometric Langlands correspondence with wild ramification.

Recall the quantum connection of G/P is a connection on the trivial vector bundle on $H^2(G/P)$ with fiber $H^*(G/P)$. Since P is maximal, $H^2(G/P)$ has dimension 1. And it is easy to see from the quantum Chevalley formula that the quantum connection has singularities at $\{0, \infty\}$. Hence, the quantum connection and the Frenkel–Gross connection have the same base $\mathbb{P}^1 \setminus \{0, \infty\}$.

Then the first step Lam and Templier established is

Theorem 3.3.3. [LT2017] *Choose the isomorphism $H^*(G/P) \simeq V_{\varpi_\beta}$ sendind $Y_{\bar{w}}$ to $v_{w\varpi_\beta}$. Then the quantum connection of G/P is isomorphic to $\nabla_{FG}(V_{\varpi_\beta})$, i.e., the Frenkel–Gross connection applied to the representation V_{ϖ_β}.*

It is this theorem we want to generalize to the cotangent bundle case.

3.3.2 Cotangent bundle case

In this section, we assume our group is simply-laced, and P is a maximal minuscule parabolic subgroup as in the last section.

3.3.3 A connection on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$

Let us first define an analogue of the Frenkel–Gross connection ∇_{FG}.

We define the following connection on a trivial G principle bundle on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$.

$$\nabla' := d - \sum_{\alpha \text{ simple roots}} \varpi_\alpha h_\alpha \frac{dq}{q} + \hbar \sum_{\alpha>0} e_\alpha \frac{dq}{q} + \hbar \sum_{\alpha>0} (e_\alpha + f_\alpha) \frac{dq}{1-q}, \tag{3.14}$$

where e_α and f_α are root vectors and $h_\alpha = [e_\alpha, f_\alpha]$. According to Lurie, we can choose the e_α and f_α such that the following is true.

Lemma 3.3.4. [Lur2001, Theorem 3.4.1] *There is a basis $\{v_{w\varpi_\beta} | w \in W/W_P\}$ for V_{ϖ_β}, such that*

$$h_\alpha(v_{w\varpi_\beta}) = (w\varpi_\beta, \alpha^\vee) v_{w(\varpi_\beta)},$$

$$e_\alpha(v_{w\varpi_\beta}) = \begin{cases} v_{\sigma_\alpha w \varpi_\beta} & \textit{if } (w\varpi_\beta, \alpha^\vee) = -1 \\ 0 & \textit{otherwise} \end{cases},$$

and

$$f_\alpha(v_{w\varpi_\beta}) = \begin{cases} v_{\sigma_\alpha w \varpi_\beta} & \textit{if } (w\varpi_\beta, \alpha^\vee) = 1 \\ 0 & \textit{otherwise} \end{cases}.$$

If we take our base field to be $\operatorname{Frac} H^*_T(pt)$, then $\dim H^*_T(T^*(G/P))_{loc} = |W/W_P| = \dim V_{\varpi_\beta}$. Therefore, we can choose an isomorphism from $H^*_T(T^*(G/P))_{loc}$ to V_{ϖ_β} By sending $\operatorname{stab}_+(\bar{w})$ to $v_{w\varpi_\beta}$. Then we can show

Theorem 3.3.5. *The T-equivariant quantum connection for $T^*(G/P)$ is isomorphic to $\nabla'(V_{\varpi_\beta})$.*

Proof. The base for the quantum connection $\nabla^{quantum}$ has dimension one, and it has a canonical generator D_{ϖ_β}. Since $\dim H_2(T^*(G/P)) = 1$ with a canonical generator $d(\beta)$ (see Equation (3.3)), we can just use q to denote $q^{d(\beta)}$. Therefore, the quantum connection has the following formula

$$\nabla^{quantum} = d - \frac{dq}{q} \cdot D_{\varpi_\beta} * .$$

Notice that for any $\alpha \in R^+$, $(\varpi_\beta, \alpha^\vee) = 1$ if and only if $\alpha \in R^+ \setminus R_P^+$. So the quantum multiplication by D_{ϖ_β} in Theorem 1.1.2 takes the following form

$$\begin{aligned} D_{\varpi_\beta} * \operatorname{stab}_+(\bar{y}) =& y(\varpi_\beta) \operatorname{stab}_+(\bar{y}) - \hbar \sum_{\alpha \in R^+ \setminus R_P^+, y\alpha \in R^-} \operatorname{stab}_+(\overline{y\sigma_\alpha}) && (3.15) \\ & - \hbar \frac{q}{1-q} \sum_{\alpha \in R^+ \setminus R_P^+} \operatorname{stab}_+(\overline{y\sigma_\alpha}) - C_P \hbar \frac{q}{1-q} \cdot \operatorname{stab}_+(\bar{y}), && (3.16) \end{aligned}$$

where C_P is a scalar does not depending on q,

Since $y(\varpi_\beta) = \sum_{\alpha \ simple} (y(\varpi_\beta), \alpha^\vee) \varpi_\alpha$, the term $y(\varpi_\beta) \operatorname{stab}_+(\bar{y})$ in Equation (3.15) corresponding to the second summand in Equation (3.14).

By Lemma 3.3.4, we have

$$\begin{aligned} \sum_{\alpha > 0} e_\alpha v_{y\varpi_\beta} &= \sum_{\alpha > 0, (y\varpi_\beta, \alpha^\vee) = -1} v_{\sigma_\alpha y \varpi_\beta} \\ &= \sum_{\alpha > 0, (\varpi_\beta, y^{-1}\alpha^\vee) = -1} v_{y \sigma_{y^{-1}\alpha} \varpi_\beta} \\ &= \sum_{\alpha \in R^+ \setminus R_P^+, y\alpha < 0} v_{y\sigma_\alpha \varpi_\beta}, \end{aligned}$$

where in the last equality, we rename $-y^{-1}\alpha$ to be α. This shows the term $\hbar \sum\limits_{\alpha \in R^+ \setminus R_P^+, y\alpha \in R^-} \operatorname{stab}_+(\overline{y\sigma_\alpha})$ in Equation (3.15) matches the third summand in Equation (3.14).

Similarly, we can show

$$\sum_{\alpha > 0} (e_\alpha + f_\alpha) v_{y\varpi_\beta} = \sum_{\alpha \in R^+ \setminus R_P^+} v_{y\sigma_\alpha \varpi_\beta}.$$

Hence, the term $\hbar \frac{q}{1-q} \sum_{\alpha \in R^+ \setminus R_P^+} \operatorname{stab}_+(\overline{y\sigma_\alpha})$ in Equation (3.16) matches the last summand in Equation (3.14).

Therefore, we have

$$\nabla^{\text{quantum}} = \nabla'(V_{\varpi_\beta}) + \hbar C_P \frac{dq}{1-q}.$$

Let $g := (1-q)^{C_P \hbar}$. Then

$$g^{-1} \nabla^{\text{quantum}} g = \nabla'(V_{\varpi_\beta}).$$

This finishes the proof. $\square$

The second summand in Equation (3.14) is a weight-recording operator. It appears since we are considering T-equivariant quantum cohomology. If we instead consider $\mathbb{C}_\hbar^*$-equivariant quantum cohomology, i.e., if we let the A-equivariant parameters be 0, then we get the following connection:

$$\nabla := d + \hbar \sum_{\alpha>0} e_\alpha \frac{dq}{q} + \hbar \sum_{\alpha>0} (e_\alpha + f_\alpha) \frac{dq}{1-q}. \tag{3.17}$$

Notice that $\{\mathrm{stab}_+(\bar{y}) | \bar{y} \in W/W_P\}$ still form a basis for the localized cohomology $H_{\mathbb{C}^*}(T^*G/P)_{loc}$, since their pullback to the zero section form a basis for $H^*_{\mathbb{C}^*}(G/P)_{loc}$. And the quantum multiplication formula in $H_{\mathbb{C}^*}(T^*G/P)_{loc}$ can be obtained from Theorem 1.1.2 by setting the A-equivariant parameters to 0. Therefore, Theorem 3.3.5 implies

Theorem 3.3.6. *The $\mathbb{C}^*_\hbar$-equivariant quantum connection for $T^*(G/P)$ is isomorphic to $\nabla(V_{\varpi_\beta})$.*

This should be seen as an anologue of Theorem 3.3.3 in the cotangent bundle case. Notice that if we also let $\hbar$ equal 0, then the quantum connection for $T^*(G/P)$ will have no quantum part, as the non-equivariant vitual fundamentcal class vanishes due the existence of the symplectic form.

In [Yun2014a], Z. Yun also constructed certain interesting local systems on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to answer Serre's question about motivic Galois groups. There are some conjectural description about the monodromy of these connections in [Yun2014a, Section 5.5], which are proved in [Yun2016, Remark 4.19]. From the description, our connection ∇ has different local monodromies. So they are different local systems on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

3.3.4 Rigidity

From the definition of ∇ in Equation 3.17, ∇ is a regular connection with three singularities $\{0, 1, \infty\}$. The connetion has principal unipotent local monodromies at the poins 0 and ∞, and semisimple monodromy at 1. And it does not admit a structure of an oper. Recall the Frenkel–Gross connection is rigid. It is natural to ask whether our connection ∇ is rigid or not. We show ∇ is rigid when $G = \mathrm{GL}(n)$.

By Riemann–Hilbert correspondence, there is an equivalence between the category of flat connections on algebraic vector bundles on X with regular singularities and the category of local systems of finite dimensional complex vector spaces on X. So we will not distinguish these two.

Recall a G-local system $\mathcal{F}$ on an open curve $j : U \hookrightarrow \mathbb{P}^1$ is cohomogically rigid (see [Yun2014b, Definition 3.2.4]), if

$$\mathrm{Rig}(\mathcal{F}) := H^1(X, j_{!*} \mathrm{Ad}(\mathcal{F})) = 0,$$

where $j_{!*}$ is the middle extension functor and $\mathrm{Ad}(\mathcal{F})$ is the adjoint vector bundle associated to $\mathcal{F}$. There is also a notion of physically rigidity (see [Kat2016, Section 1.0]) defined as follows. We say that a local system $\mathcal{F}$ on U is physically rigid if for every local systam $\mathcal{G}$ on U such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy, then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic. The relation between these two notions is that cohomological rigid $\mathrm{GL}(n)$-local system are also physically rigid (see [Kat2016, Theorem 5.02]).

Besides, Katz shows ([Kat2016, Theorem 1.1.2]) an irreducible local system on U is physically rigid if and only if

$$\chi(\mathbb{P}^1, j_*\mathcal{E}nd\mathcal{F}) = 2.$$

By the Euler–Poincaré formula, we have

$$\chi(\mathbb{P}^1, j_*\mathcal{E}nd\mathcal{F}) = (2-k)n^2 + \sum_i \dim Z(A_i),$$

where $k = |\mathbb{P}^1 \setminus U|$, A_i is the local monodromy at the i-th point in $\mathbb{P}^1 \setminus U$, and $Z(A_i)$ is the centralizer of A_i in $\mathrm{GL}(n, \mathbb{C})$ (see the proof of [Kat2016, Theorem 1.1.2]).

In the case of $\mathrm{GL}(n, \mathbb{C})$, our regular connection ∇ has principle unipotent monodromy at 0 and ∞. Hence $\dim Z(A_0) = \dim Z(A_\infty) = n$. And it is easy to compute $\dim Z(A_1) = 1 + (n-1)^2$. Therefore,

$$\chi(\mathbb{P}^1, j_*\mathcal{E}nd\mathcal{F}) = (2-1)n^2 + n + n + 1 + (n-1)^2 = 2.$$

Hence, our connection is physically rigid.

We also have the following criterion for cohomological rigid G-local system.

Proposition 3.3.7. [Yun2014b, Proposition 3.2.7] *Let ρ be a G-local system on an open curve $U = X - S$. Let g_X be the genus of X. Then ρ is cohomologically rigid if and only if*

$$\frac{1}{2} \sum_{x \in S} a_x(Ad(\rho)) = \dim \mathfrak{g}/(\mathfrak{g})^{\pi_1(U,u)} - g_X \dim \mathfrak{g}.$$

Here $a_x(\mathrm{Ad}(\rho))$ is the Artin conductor of the action of the inertia group $\mathcal{I}_x$ on $\mathfrak{g}$.

Recall

$$a_x(\mathrm{Ad}(\rho)) := \dim(\mathfrak{g}/\mathfrak{g}^{\mathcal{I}_x}) + Sw(\mathrm{Ad}(\rho)),$$

where $Sw(\mathrm{Ad}(\rho))$ is the Swan conductor.

Now we can prove our connection is also cohomological rigid when $G = \mathrm{GL}(n, \mathbb{C})$. Since our connection is regular, the Swan conductor is 0. Therefore, by Proposition 3.3.7, ∇ is rigid if and only if

$$\dim \mathfrak{g} + 2\dim \mathfrak{g}^{\pi_1(U,u)} = \sum_{x\in\{0,1,\infty\}} \dim \mathfrak{g}^{\mathcal{I}_x} = \sum_{x\in\{0,1,\infty\}} \dim Z_{\mathfrak{g}}(A_x),$$

where A_x are the local monodromy around the singularity x. We already know $\dim Z_{\mathfrak{g}}(A_0) = \dim Z_{\mathfrak{g}}(A_\infty) = n$, and $\dim Z_{\mathfrak{g}}(A_1) = 1 + (n-1)^2$. So the above is equivalent to

$$\dim \mathfrak{g}^{\pi_1(U,u)} = 1.$$

It is easy to see from linear algebra that the above is true. Because of the following two points.

1. The invariant space $\mathfrak{g}^{\pi_1(U,u)}$ contains the scalar matrices.

2. For our connection ∇, the image of $\pi_1(U,u)$ in $\mathrm{GL}(n,\mathbb{C})$ contains the matrices $\exp(\sum_{\alpha>0} e_\alpha)$ and $\exp(\sum_{\alpha>0} f_\alpha)$. Only scalar matrices commute with both of them.

In conclusion, we get

Theorem 3.3.8. *For $G = \mathrm{GL}(n,\mathbb{C})$, ∇ is a cohomological rigid connection on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$.*

It would be interesting to consider the corresponding automorphic sheaf on the moduli space of rank n vector bundles on $\mathbb{P}^1$.

Chapter 4

Chern-Schwartz-MacPherson Classes

In this chapter, we study the Chern–Schwartz–MacPherson (CSM) class of flag variety and its relation with the stable basis. Using the duality between the stable basis, we find the dual classes of the CSM classes. Finally, we give a formula for the CSM class of a general variety in terms of the characteristic cycles of constructible functions. The main reference for this chapter is [AMSS]. However, in this thesis, we take a localization approach, which is different from the one in [AMSS].

4.1 Preliminaries on Chern-Schwartz-MacPherson classes

The conjectured Chern class theory for singular varieties associates a class $c_*(\varphi) \in H_*(X)$ to every constructible function φ on X, and satisfies $c_*(1_X) = c(TX) \cap [X]$ if X is a smooth compact complex variety and the map c_* is functorial. Funtoriality means that if $f : Y \to X$ is proper, then the following diamgram is commutative:

$$\begin{tikzcd}
\mathrm{CF}(Y) \arrow[r, "c_*"] \arrow[d, "f_*"] & H_*(Y) \arrow[d, "f_*"] \\
\mathrm{CF}(X) \arrow[r, "c_*"] & H_*(X),
\end{tikzcd}$$

where $\mathrm{CF}(X)$ denotes the constructible functions on X. The map $f_* : \mathrm{CF}(Y) \to \mathrm{CF}(X)$ is defined by the following formula

$$f_*(1_Z)(x) := \chi(f^{-1}(x) \cap Z),$$

where Z is a constructible subvariety in Y, and χ denotes the topological Euler characteristic.

The existence of such a theory was established by MacPherson [Mac1974]. If X is a compact complex variety, then $c_*(1_X)$ is the same class defined by Schwartz [Sch1965] independently. This class is called the *Chern–Schwartz–MacPherson* (CSM) class of X. We denote the class $c_*(1_W)$ by $c_{SM}(W)$ for any constructible subvariety W of X.

The equivariant version is developed by Ohmoto in [Ohm2006]. Let X be a variety with a torus T-action. Ohmoto proved that there is a MacPherson transformation $c_*^T : \mathrm{CF}^T(X) \to H_*^T(X)$ from T-equivariant constructible function to equivariant homology of X, satisfying similar properties as above.

4.2 A formula of Aluffi and Mihalcea

In this section, we review a formula of Aluffi and Mihalcea for the CSM classes of Schubert cells ([AM2015]), which is the starting point for [AMSS].

Let $X := G/B = \mathcal{B}$ denote the flag variety. Recall A denotes the maxiaml torus of G inside the Borel subgroup B. For any $w \in W$, let $X(w)^\circ := BwB/B$ (resp. $Y(w)^\circ := B^- wB/B$) denote the corresponding Schubert cell (resp. opposite Schubert cell), with its closure $X(w) = \overline{BwB/B}$ (resp. $Y(w) := \overline{B^- wB/B}$) denoting the Schubert variety (resp. opposite Schubert variety). For any A-invariant constructible subvariety $Z \subset X$, the CSM class $c_{SM}(Z)$ lies in $H_*^A(Z)$. We push it forward along $Z \to X$, use Poincaré duality for $H_*^A(X)$, and consider $c_{SM}(Z)$ as a cohomology class in $H_A^*(X)$.

For each simple root α, we have the BGG operator ([BGG1973])

$$\partial_\alpha : H_*^A(X) \to H_{*+2}^A(X)$$

defined as follows. Let P_α be the corresponding subminimal parabolic subgroup containg B. Let π_α be the projection from G/B to G/P_α. Then

$$\partial_\alpha := \pi_\alpha^* \circ \pi_{\alpha,*}.$$

There is also a right action of W on X, inducing an action of W on $H_*^A(X)$ (see [Knu2003]). Therefore, we can define the following operator

$$\mathcal{T}_\alpha := \partial_\alpha - \sigma_\alpha : H_A^*(X) \to H_A^*(X). \tag{4.1}$$

Then we have

Theorem 4.2.1. [AM2015] *The operator $\mathcal{T}_\alpha$ acts on the CSM classes of Schubert cells as follows:*

$$\mathcal{T}_\alpha(c_{SM}(X(w)^\circ)) = c_{SM}(X(w\sigma_\alpha)^\circ). \tag{4.2}$$

In particular, if $w = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ is reduced, then

$$c_{SM}(X(w)^\circ) = \mathcal{T}_{i_k}\cdots\mathcal{T}_{i_1}c_{SM}(X(id)^\circ) = \mathcal{T}_{i_k}\cdots\mathcal{T}_{i_1}[X(id)],$$

and this does not depend on the choice of the reduced expression. There are explicit formulas for the action of these operators on the Schubert classes (see [AM2015]). Therefore, Theorem 4.2.1 gives an algorithm to compute all the CSM classes of Schubert cells.

4.3 CSM classes and stable basis

In this section, we give a proof proof for Theorem 1.1.4. All the results in this section can be easily generalized to the partial flag variety G/P case. We refer the interested reader to [AMSS] for this direction.

Recall both the fixed loci $\mathcal{B}^A$ and $(T^*\mathcal{B})^A$ are both indexed by the Weyl group. We use j_w to denote the inclusion of the fixed point wB into X. For any cohomology class γ in $H_A^*(\mathcal{B})$ or $H_A^*(T^*\mathcal{B})$, we use $\gamma|_w$ to denote the corresponding restriction of γ to the fixed point corresponding to w. We hope this will not cause confusion.

4.3.1 Proof of Theorem 1.1.4

By localization, $H_A^*(X)$ embeds into functions from the Weyl group W to $Q := \operatorname{Frac} H_A^*(pt)$. Define an operator

$$A_\alpha : F(W,Q) \to F(W,Q)$$

by the formula

$$(A_\alpha\psi)(w) = \frac{\psi(w\sigma_\alpha) - \psi(w)}{w\alpha} - \psi(w\sigma_\alpha).$$

Then we have the following commutative diagram.

Proposition 4.3.1. *The diagram*

$$\begin{tikzcd} H_A^*(X) \arrow[r, hook] \arrow[d, "\mathcal{T}_\alpha"'] & F(W,Q) \arrow[d, "A_\alpha"] \\ H_A^*(X) \arrow[r, hook] & F(W,Q) \end{tikzcd}$$

commutes.

Proof. By [FA2007, Proposition 4.1, Lectur14], the term ∂_α in Equation 4.1 accounts for the first term in A_α. For the remaining terms, we only need the following easy fact

$$\sigma_\alpha(j_{w*}1) = -j_{w\sigma_\alpha,*}1.$$

□

Applying this diagram the CSM classes, we get

Corollary 4.3.2. *The equivariant CSM classes $c_{SM}(X(y)^\circ)$ are uniquely characterized by the following properties:*

1. $c_{SM}(X(y)^\circ)|_w = 0$*, unless* $w \leq y$.
2. $c_{SM}(X(y)^\circ)|_y = (-1)^{\dim X} \prod_{\alpha\in R^+, y\alpha<0} (y\alpha - 1) \prod_{\alpha\in R^+, y\alpha>0} y\alpha.$
3. *For any simple root* α*, and* $\ell(y\sigma_\alpha) = \ell(y) + 1$,

$$c_{SM}(X(ys_\alpha)^\circ)|_w = -\frac{1}{w\alpha} c_{SM}(X(y)^\circ)|_w + \frac{1 - w\alpha}{w\alpha} c_{SM}(X(y)^\circ)|_{w\sigma_\alpha}.$$

Proof. The first property follows from the fact that $c_{SM}(X(y)^\circ)$ is supported on $\cup_{u\leq y} X(u)$. Since $X(y)^\circ$ is smooth, and $c_{SM}(X(y)^\circ)$ is understood as a class in $H_T^*(X)$ via pushforward along $X(y)^\circ \to X$,the second one follows from [Ohm2006, Theorem 1.1]. The last one follows directly from Theorem 4.2.1 and Corollary 4.3.1. The uniqueness can be obtained via induction on $\ell(y)$ as in Corollary 2.4.4. □

Comparing Corollaries 2.4.4 and 4.3.2, we get the first identity in Theorem 1.1.4. The second one can be proved similarly.

4.3.2 Homogeneous class

In Theorem 1.1.4, there is a specialization $\hbar = 1$. In fact, we can upgrade it to a homogeneous version, so that no such specialization is needed.

Since the homology of the flag variety only has even degree component, any homology class $\gamma \in H^A_*(X)$ can be written as

$$\gamma = \sum_i \gamma_i,$$

with $\gamma_i \in H^A_{2i}(X)$. Recall the torus $\mathbb{C}^*_\hbar$ acts trivially on X, and the equivariant parameter $\hbar$ has cohomology degree 2. Therefore, we can define the following homogeneous class associated to γ:

$$\gamma^h := \sum_i \gamma_i \hbar^i \in H^T_0(X) = \sum_i H^A_{2i}(X)\hbar^i,$$

where $T = A \times \mathbb{C}^*_\hbar$ is the bigger torus.

The homogeneous version of Theorem 1.1.4 is

Theorem 4.3.3. *Let $\iota : G/B \to T^*(G/B)$ be the inclusion, we have*

$$\iota^*(\mathrm{stab}_+(w)) = (-1)^{\dim G/B} c^h_{SM}(X^\circ_w),$$

and

$$\iota^*(\mathrm{stab}_-(w)) = (-1)^{\dim G/B} c^h_{SM}(Y^\circ_w).$$

Proof. It follows immediately from Theorem 1.1.4 and the fact that $\iota^*(stab_\pm(w)) \in H^T_0(X)$ (see Remark 2.2.2(2)). □

We also have a homogeneous version of Theorem 4.2.1. Define the homogeneous version of the operator $\mathcal{T}_\alpha$ in Equation (4.1) as

$$\mathcal{T}^h_\alpha := \hbar\partial_\alpha - \sigma_\alpha : H^*_T(X) \to H^*_T(X).$$

This operator preserves homogeneous degree, and we have

Theorem 4.3.4. *The operator $\mathcal{T}^h_\alpha$ acts on the homogeneous CSM classes of Schubert cells as follows:*

$$\mathcal{T}^h_\alpha(c^h_{SM}(X(w)^\circ)) = c^h_{SM}(X(w\sigma_\alpha)^\circ). \tag{4.3}$$

We refer interested readers to [AMSS] for more about the homogeneous classes.

4.4 Dual CSM classes

In this section, we study the dual class of the CSM classes. There are two approaches to it. The first one is suggested by Prof. Aluffi. The second one comes from localization. Both proofs in this thesis depend on the duality between the stable basis. In [AMSS], a much more conceptual proof is given.

4.4.1 Alternating class

By the discussion in Section 4.3.2, we can write $c_{SM}(Y(w)^\circ) = \sum_i c_{SM}(Y(w)^\circ)_i$, with $c_{SM}(Y(w)^\circ)_i \in H^A_{2i}(X)$. Prof. Aluffi suggests to look at the following alternating class[1]

$$c^\vee_{SM}(Y(w)^\circ) := \sum_i (-1)^i c_{SM}(Y(w)^\circ)_i.$$

By Theorem 4.3.3, we have

$$c^\vee_{SM}(Y(w)^\circ) = (-1)^{\dim X} \iota^*(stab_-(w))|_{\hbar=-1}.$$

Then we have the following duality statement.

Theorem 4.4.1. *The paring between the CSM class and the alternating one is*

$$(c_{SM}(X(y)^\circ), c^\vee_{SM}(Y(w)^\circ)) = (-1)^{\ell(y)+\dim X} \delta_{y,w} \prod_{\beta\in R^+} (1+\beta).$$

Proof. From the localization formula for $\mathrm{stab}_-(w)$ in Theorem 1.1.1, we have

$$\frac{\mathrm{stab}_-(w)|_{\hbar=-1,u}}{\mathrm{stab}_-(w)|_{\hbar=1,u}} = (-1)^{\ell(w)+\ell(u)} \frac{\prod_{\alpha\in R^+\setminus R(u)}(\alpha+1)}{\prod_{\alpha\in R^+\setminus R(u)}(\alpha-1)} = (-1)^{\ell(w)} \frac{\prod_{\alpha\in R^+}(\alpha+1)}{e(T^*_u G/B)|_{\hbar=1}}.$$

Here we have used the fact

$$R(u) = \{-u\beta \in R^+ | \beta \in R^+, u\beta \in R^-\}.$$

[1]The definition in this thesis is equal to the one in [AMSS] up to a sign.

Therefore, by localization on G/B, we have

$$\begin{aligned}(c_{SM}(X(y)^\circ), c^\vee_{SM}(Y(w)^\circ)) &= (i^*(\mathrm{stab}_+(y)|_{\hbar=1}, i^*(\mathrm{stab}_-(w))|_{\hbar=-1}) \\ &= \sum_u \frac{i^*(\mathrm{stab}_+(y))|_{\hbar=1,u} i^*(\mathrm{stab}_-(w))|_{\hbar=-1,u}}{e(T_uG/B)} \\ &= \sum_u \frac{\mathrm{stab}_+(y)|_{\hbar=1,u}\,\mathrm{stab}_-(w)|_{\hbar=-1,u}}{e(T_uG/B)} \\ &= (-1)^{\ell(w)} \prod_{\beta\in R^+}(\beta+1) \sum_u \frac{\mathrm{stab}_+(y)|_{\hbar=1,u}\,\mathrm{stab}_-(w)|_{\hbar=1,u}}{e(T^*_uG/B)|_{\hbar=1} e(T_uG/B)} \\ &= (-1)^{\ell(w)} \prod_{\beta\in R^+}(\beta+1)(\mathrm{stab}_+(y), \mathrm{stab}_-(w))_{T^*X}|_{\hbar=1} \\ &= (-1)^{\ell(w)+\dim X} \prod_{\beta\in R^+}(\beta+1).\end{aligned}$$

The last but one equality follows from the localization pairing on T^*X, while the last one follows from Remark 2.2.2(3). □

4.4.2 Another dual class

We need the following formula relating the pairing in $H^*(T^*Y)$ and the one in $H^*(Y)$ for a general smooth projective variety Y. Let $\mathbb{C}^*_\hbar$ act on T^*Y by dilating the cotangent fiber of T^*Y by a weight of $\mathrm{Lie}\,\mathbb{C}^*_\hbar$ equaling $-\hbar$. Let $\mathbb{C}^*_\hbar$ act trivially on Y. Let $\iota : Y \to T^*Y$ be the inclusion of zero section. We have

Lemma 4.4.2. *For any $\gamma_1, \gamma_2 \in H^*_{\mathbb{C}^*}(T^*Y)$, the following identity holds*

$$\langle \gamma_1, \gamma_2 \rangle_{T^*Y} = \langle \iota^*\gamma_1, \frac{\iota^*\gamma_2}{e^{\mathbb{C}^*}(T^*Y)} \rangle_Y.$$

Remark 4.4.3.

1. In fact, $e^{\mathbb{C}^*}(T^*Y) = \prod(-\hbar - x_i)$, where x_i are the first Chern classes of TY. Therefore $e^{\mathbb{C}^*}(T^*Y)|_{\hbar=1} = (-1)^{\dim Y} c(TY)$, where $c(TY)$ is the total Chern class of the tangent bundle of Y.
2. This is also true if Y admits a nontrivial torus action, provided $e^{\mathbb{C}^*}(T^*Y)$ is replaced by the full torus equivariant Euler class.

Proof. Since Y is the only fixed component of T^*Y under the torus $\mathbb{C}^*_\hbar$, localization gives

$$\begin{aligned}\langle\gamma_1,\gamma_2\rangle_{T^*Y} &= \int_{T^*Y}\gamma_1\cup\gamma_2\\ &= \int_Y \frac{\iota^*\gamma_1\cup\iota^*\gamma_1}{e^{\mathbb{C}^*}(N_Y)}\\ &= \langle\iota^*\gamma_1, \frac{\iota^*\gamma_2}{e^{\mathbb{C}^*}(T^*X)}\rangle_Y,\end{aligned}$$

where N_Y in the second line is the normal bundle of Y in T^*Y, which is the same as the cotangent bundle T^*Y on Y. □

Now let us consider the flag variety case. Remark 2.2.2(3), Theorem 1.1.4, and Lemma 4.4.2 gives

Theorem 4.4.4. *We have the following duality relation*

$$\left(c_{SM}(X(y)^\circ), \frac{c_{SM}(Y(w)^\circ)}{e^T(T^*X)|_{\hbar=1}}\right) = (-1)^{\dim X}\delta_{y,w}.$$

Combing with Theorem 4.4.1, we get

$$\frac{c_{SM}(Y(w)^\circ)}{e^T(T^*X)|_{\hbar=1}} = (-1)^{\ell(w)}\frac{c^\vee_{SM}(Y(w)^\circ)}{\prod_{\beta\in R^+}(1+\beta)} \in H^*_A(X).$$

Specializing all the A-equivariant parameters to 0, this becomes

$$c_{SM}(Y(w)^\circ) = (-1)^{\dim X+\ell(w)}c(TX)c^\vee_{SM}(Y(w)^\circ) \in H^*(X). \tag{4.4}$$

This gives some constraint on the different components of $c_{SM}(Y(w)^\circ)$, which are conjectured to be effective by Aluffi and Mihalcea (see [AM2015]). We expect the above identity can shed some light on this conjecture.

4.5 CSM classes and characteristic cycles

Recall that for a smooth variety, we have the following commutative diagram (see [Gin1986]),

$$\begin{tikzcd} \text{constructible complexes} \arrow[d, "\chi"] & \text{holonomic modules} \arrow[l, "\mathrm{DR}"'] \arrow[d, "\mathrm{Ch}"] \\ \text{constructible functions} \arrow[r, "\mathrm{CC}"] & \text{Lagrangian cylces in } T^*X, \end{tikzcd} \tag{4.5}$$

where DR is the De Rham functor defined by

$$\mathrm{DR}(\mathcal{M}) = \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[\dim X],$$

χ is the stalk Euler characteristic

$$\chi(x, \mathcal{F}^\bullet) = \sum (-1)^k \dim \mathcal{H}^k(\mathcal{F}^\bullet)|_x,$$

CC is the characteristic cycle map associated to constructible functions, and Ch is the characteristic cycle map for $\mathcal{D}_X$-modules.

Let us consider the flag variety case. Let $\mathcal{M}_w$ be the Verma module with highest weight $-\rho - w\rho$, where ρ is half sum of the positive roots. Through localization ([BB1981]), we have the corresponding $\mathcal{D}_X$-module $\mathcal{M}_w := \mathcal{D}_X \otimes_{\mathcal{U}\mathfrak{g}} M_w$. Recall the following famous result ([BB1981, BK1981])

$$\mathrm{DR}(\mathcal{M}_w) = \mathbb{C}_{X(w)^\circ}[\ell(w)].$$

Combing with [MO2012, Remark 3.5.3], we get

$$\mathrm{stab}_+(w) = (-1)^{\dim X - \ell(w)} \mathrm{Ch}(\mathcal{M}_w).$$

The sign is determined by looking at the coefficients of the leading term $\overline{T_{X(w)^\circ} X}$ on both sides (see [AMSS] for another proof.).

Due to the commutative diagram 4.5, we get

$$\mathrm{Ch}(\mathcal{M}_w) = (-1)^{\ell(w)} \mathrm{CC}(1_{X(w)^\circ}).$$

Therefore,

$$\mathrm{CC}(1_{X(w)^\circ}) = (-1)^{\dim X} \mathrm{stab}_+(w).$$

Theorem 4.3.3 gives

$$\iota^*(\mathrm{CC}(1_{X(w)^\circ})) = c_*^h(1_{X(w)^\circ}) \in H_0^T(X). \tag{4.6}$$

It is this formula we generalize to general varieties.

Let X be a smooth projective variety with a torus A-action. Let $\mathbb{C}_\hbar^*$ act on T^*X as before. For any constructible function φ on X, define

$$c_*^h(\varphi) := \sum_i \hbar^i c_{*,i}(\varphi) \in H_0^{A\times\mathbb{C}^*}(X),$$

where $c_*(\varphi) = \sum_i c_{*,i}(\varphi)$, with $c_{*,i}(\varphi) \in H_{2i}(X)$. The following is one of the main theorems in [AMSS].

Theorem 4.5.1. [AMSS] *Let $\iota : X \to T^*X$ be the inclusion of the zero section. Then for any constructible function φ on X, we have*

$$\iota^*[CC(\varphi)] = c_*^h(\varphi) \in H_0^{A\times\mathbb{C}^*}(X).$$

The proof uses Aluffi's shadow construction ([Alu2004]). We refer the readers to [AMSS] for the details.

As an application, we give another proof for following index formula (see [Gin1986] for various generalizations).

Theorem 4.5.2. *For any constructible sheaf $\mathcal{F}$ on X,*

$$\chi(X,\mathcal{F}) = \mathrm{CC}(\mathcal{F}) \cdot [T_X^*X],$$

where the intersection on the right hand side is the usual non-equivariant intersection.

Proof. Since both sides are additive on $\mathcal{F}$, it suffices to prove it in the case $\mathcal{F}$ is a local system on X. Then

$$\chi(X,\mathcal{F}) = \operatorname{rank}\mathcal{F}\cdot\chi(X,\mathbb{C}_X) = \operatorname{rank}\mathcal{F}\cdot\chi(X),$$

where $\chi(X)$ is the topological Euler characteristic of X. For any constructible sheaf $\mathcal{F}$, we can define a constructible function $\chi_{\mathcal{F}}(x) := \dim \mathcal{F}|_x$. Therefore $\chi_{\mathcal{F}} = \operatorname{rank}\mathcal{F}\cdot 1_X$. By functoriality, $\deg c_*(1_X) = c_*^h(1_X)|_{\hbar=0} = \chi(X)$. Hence

$$\begin{aligned}\chi(X,\mathcal{F}) &= \operatorname{rank}\mathcal{F}\cdot\chi(X) = \operatorname{rank}\mathcal{F}\cdot\deg c_*(1_X)\\ &= c_*(\chi_{\mathcal{F}})|_{\hbar=0} = \iota^*[\mathrm{CC}(\chi_{\mathcal{F}})]|_{\hbar=0}\\ &= \mathrm{CC}(\mathcal{F})\cdot[T_X^*X].\end{aligned}$$

$\square$

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